Problem set # 7

Due: Wednesday, April 15, by 8 a.m.

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Problem 1: Simulation in the Heston Model: Suppose that the underlying security SPY evolves according to the Heston model. That is, we know its dynamics are dened by the following system of SDEs:

$$\begin{aligned} dS_t &= (r-q)S_t dt + \sqrt{\nu_t} S_t dW_t^1 \\ d\nu_t &= \kappa\left(\theta - \nu_t\right) dt + \sigma\sqrt{\nu_t} dW_t^2 \\ \operatorname{Cov}\left(dW_t^1, dW_t^2\right) &= \rho dt \end{aligned}$$

You know that the last closing price for SPY was 282. You also know that the dividend yield for SPY is 1.77% and the corresponding risk-free rate is 1.5%.

Using this information, you want to build a simulation algorithm to price a knock-out option on SPY, where the payoff is a European call option contingent on the option not being knocked out, and the knock-out is an upside barrier that is continuously monitored.
We will refer to this as an **up-and-out call**.

This payoff can be written as:

$$c_0 = \mathbb{E}\left[\left(S_T - K_1\right)^+ 1_{\{M_T < K_2\}}\right]$$

where MT is the maximum value of S over the observation period, and K1 < K2 are the strikes of the European call and the knock-out trigger respectively.

1. Find a set of Heston parameters that you believe govern the dynamics of SPY. You may use results from a previous Homework, do this via a new calibration, or some other empirical process. Explain how you got these and why you think they are reasonable.

From previous homework, I set the Heston parameters that I believe govern the dynamics of SPY.

$$\begin{aligned} \sigma &= 0.7 \\ v_0 &= 0.05 \\ \kappa &= 3.65 \\ \rho &= -0.8 \\ \theta &= 0.07 \end{aligned}$$

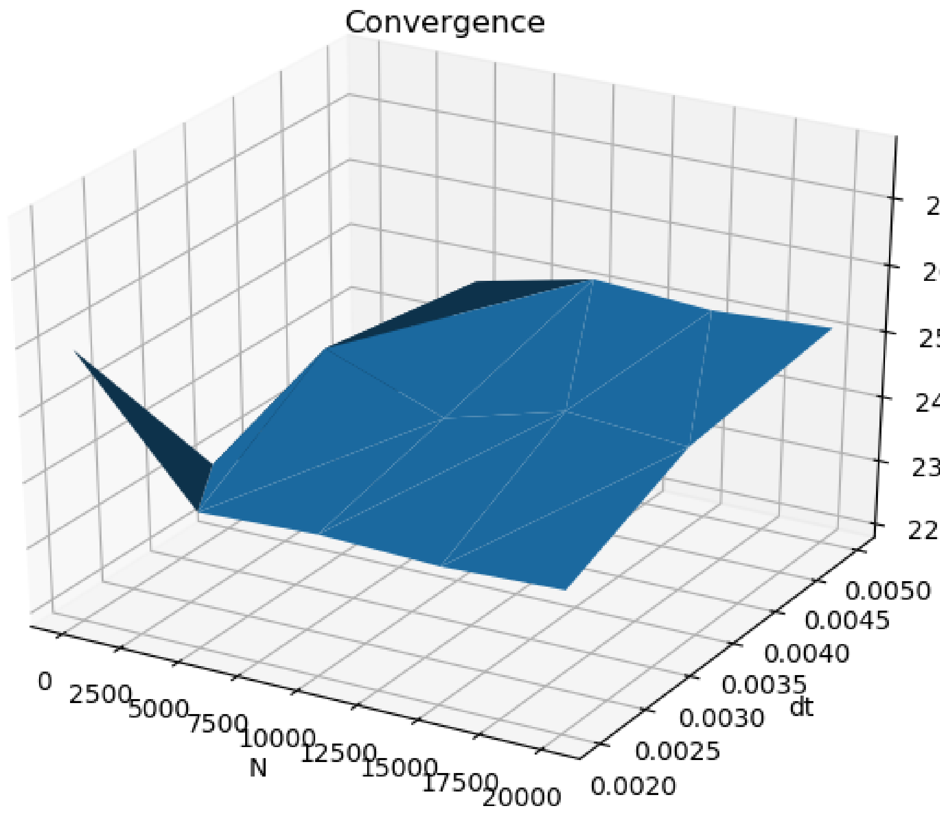
Using calibration I got these Heston parameters which enables the result of FFT pricing model to converge to the market price, so I think they are reasonable.

2. Choose a discretization for the Heston SDE. In particular, choose the time spacing, ΔT as well as the number of simulated paths, N. Explain why you think these choices will lead to an accurate result.

By Euler Discretization,

$$\begin{aligned} \hat{S}_{t_{j+1}} &= \hat{S}_{t_j} + \mu \hat{S}_{t_j} \Delta t + \sqrt{\nu_{t_j}} \hat{S}_{t_j} \sqrt{\Delta t} Z_j^1 \\ \hat{\nu}_{t_{j+1}} &= \hat{\nu}_{t_j} + \kappa\left(\theta - \nu_{t_j}\right) \Delta t + \sigma \sqrt{\nu_{t_j}} \sqrt{\Delta t} Z_j^2 \end{aligned}$$

where Z1 and Z2 are standard normal random variables with correlation ρ

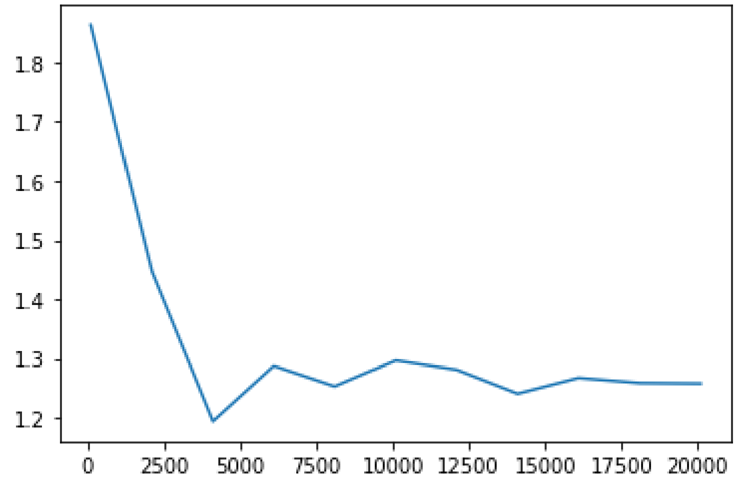


I choose N = 10000 and ΔT = 1/252, as shown above, when N is about 10000 and ΔT is about 1/300 ~ 1/200, the simulated price begin to converge so lead to an accurate result.

3. Write a simulation algorithm to price a European call with strike K = 285 and time to expiry T = 1. Calculate the price of this European call using FFT and comment on the difference in price.

The price of the European call using simulation algorithm is about **23.9** while the price of this European call using FFT is about **24.01**. If we run the code for several times, we will notice there is little difference in two prices since when we using simulation algorithm we need to handle negative volatilities which in some ways changes the density of volatility.

4. Update your simulation algorithm to price an up-and-out call with T = 1, K1 = 285 and K2 = 315. Try this for several values of N. How many do you need to get an accurate price?



As shown above, when N = 10000 I will get a relatively accurate price and the price of the up-and-out call which is about **1.26** .

5. Re-price the up-and-out call using the European call as a control variate. Try this for several values of N. Does this converge faster than before?

Using the European call as a control variate, we have

$$\tilde{\theta} = \mathbb{E}(h(X)) + c(Z - \mathbb{E}(Z))$$

and

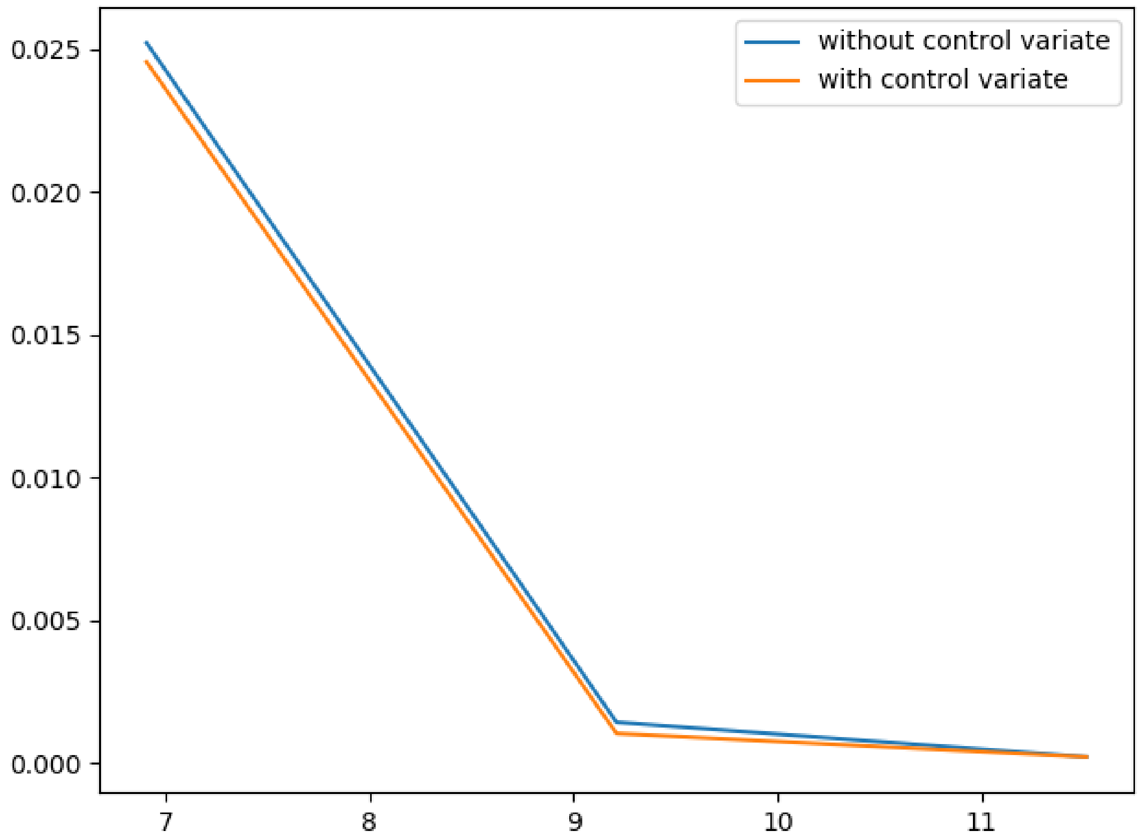
$$\text{var}(\tilde{\theta}) = \text{var}(\theta) - \frac{\text{cov}(\theta, Z)^2}{\text{var}(Z)}$$

by choosing c such that

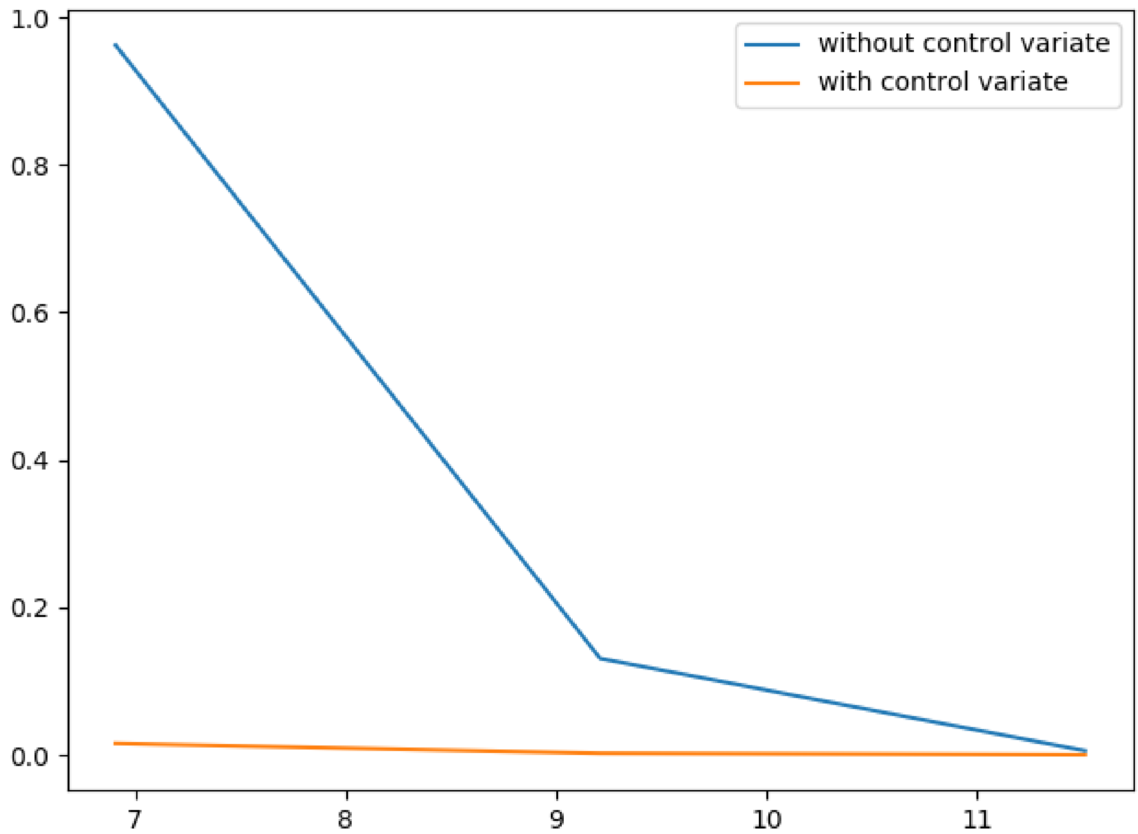
$$c^* = -\frac{\text{cov}(\theta, Z)}{\text{var}(Z)}$$

So the higher correlation between θ and Z, the lower the variance.

When K2 = 315, the payoff of up-and-out call is very different from European call, so the correlation between θ and Z is low that the effect of introducing control variate is not significant. The variance of two methods are shown above (axis are log value).



If we set K2 = 500, the payoff of up-and-out call is very close to European call, so the correlation between θ and Z is high that the effect of introducing control variate is significant. The variance of two methods are shown above (axis are log value).



However, I don't recognize that using control variate converges faster than before when N increases. As the variance of theta still dominate, the control variate just reduces significantly the variance at single N so we can acquire relative accurate price with small N then save computational cost.