MF 796
Computational Methods of Mathematical Finance
Take-Home Midterm

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Section: D1

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Please work alone. You may refer to standard books and you may reuse the code *from your own homeworks.* Each problem has more than one solution. Please provide only one and explain all your steps and choices.

Please upload this exam as you normally upload a homework, together with the code you write by 8am Thursday, April 9.

In the code, please mark clearly which (sub)problem each piece of code refers to.

If you need clarification of a question please email both of us simultaneously at sorets@bu.edu and cmk44@bu.edu.

In addition, we will be on Zoom (our usual class meeting) between 8am and 11am Wednesday in case there are questions about the exam.

Enjoy and good luck!

Eugene and Chris.

Problem 1 (25 points) Consider the following risk-neutral pricing formula for a European Digital Call:

$$c_0 = \tilde{\mathbb{E}}\left[e^{-rT} 1_{\{S_T > K\}}\right]$$

(a) (9 Points) Derive the pricing formula that can be used to price European Digital Call options using the characteristic function of a stochastic process.

NOTE: Your answer should have a single integral and be a function of the characteristic function and K.

(b) (4 Points) Describe two approaches for calculating the final integral and the trade-offs between them.

(c) (6 Points) Discuss how you would use this pricing formula to create an approximation for an American Digital put option. The payoff for an American Digital put option can be written as:

$$p_0 = \tilde{\mathbb{E}}\left[e^{-rT} 1_{\{M_T < K\}}\right]$$

where M_T is the minimum value for the asset over the observation period.

Comment on what factors would go into the accuracy of your approximation and whether the estimate would be reliable.

(d) (6 points) Consider the following strike spacing grid used in the FFT method:

$$k_m = \beta + (m-1)\Delta k \quad \text{for} \quad m = 1, \ldots, N = 2^n,$$

where β is defined as:

$$\beta = \log S_0$$

Is this a valid strike spacing grid for the FFT technique? Will the at-the-money strike, S_0 fall on the grid? If so, at which value of m will the at-the-money-strike be?

(a) $C_0 = \mathbb{E}[e^{-rT} \mathbb{1}\{S_T > K\}]$

$= e^{-rT} \int_K^{+\infty} f(S)\, dS$

we denote $s = \log(S_T)$, $k = \log(K)$

$C_0 = e^{-rT} \int_k^{+\infty} q(s)\, ds$

and we define a modified digital price with a damping factor α

Since it's a digital call, we require $\alpha > 0$

$\Rightarrow \tilde{C_0} = e^{\alpha k} C_0$

$= e^{\alpha k} \int_k^{+\infty} q(s)\, ds$

apply Fourier Transform

$\Psi(v) = \int_{-\infty}^{+\infty} e^{ivk}\, \tilde{C_0}\, dk$

$= e^{-rT} \int_{-\infty}^{+\infty} e^{(iv+\alpha)k} \left\{ \int_k^{+\infty} q(s) ds \right\} dk$

$= e^{-rT} \int_{-\infty}^{+\infty} q(s) \left\{ \int_{-\infty}^{s} e^{(iv+\alpha)k} dk \right\} ds$

$= e^{-rT} \int_{-\infty}^{+\infty} q(s) \frac{1}{iv+\alpha} e^{(iv+\alpha)s}\, ds$

$= \frac{e^{-rT}}{iv+\alpha} \int_{-\infty}^{+\infty} q(s) e^{i(v-\alpha i)s}\, ds$

we see the remaining integral is just the characteristic function of $q(s)$ evaluated at $v - \alpha i$

so $\Psi(v) = \frac{e^{-rT}}{iv+\alpha} \Phi(v - \alpha i)$

finally we use Fourier Inversion to extract the digital option price from the Fourier Transform derived

$C_0 = \frac{e^{-\alpha k}}{\pi} \int_0^{+\infty} e^{-ivk} \Psi(v)\, dv = \frac{e^{-\alpha k - rT}}{\pi} \int_0^{+\infty} e^{-ivk} \frac{1}{iv+\alpha} \Phi(v-\alpha i)\, dv$

(b) we can calculate the final integral via Quadrature or FFT.

If we were pricing a single strike, the Quadrature method is more efficient since we could calculate the integral in N operations per strike.

However, pricing N options via quadrature would be $O(N^2)$ while via Fast Fourier Transform is $O(N\log N)$ whatever pricing a single strike or N strikes.

(c) Since the reflection principle tells us that for each realization that finishes above the barrier there is an inverted path with equal probability that finishes below the barrier.

then we have

$$P(S_T<K) = P(S_T<K, M_T \geq K) + P(S_T<K, M_T<K)$$
$$\approx P(S_T \geq K, M_T \geq K) + P(M_T<K)$$
$$\approx 1 - P(S_T<K) + P(M_T<K)$$

so $$P(M_T<K) = 2P(S_T<K) - 1$$
$$= 1-2P(S_T \geq K)$$

Given a European digital call, we have $C_0 \approx P(S_T \geq K)$

So $\mathbb{E}[e^{+T}\mathbb{1}\{M_T \leq K\}] \approx 1-2C_0$

$$\approx 1 - 2\times\frac{e^{-\alpha k - rT}}{\pi(iv+\alpha)}\int_0^{+\infty} e^{-ivk}\,\Phi(v-\alpha i)\,dv$$

And drift, volatility, skewness would go into the accuracy of our approximation, since they influence the conditional distribution at K.

If the conditional distribution is symmetric at K, our estimate would be reliable. the European should by half the cost of the American.

(d) If we only used such a strike spacing grid for the FFT technique to pricing an at-the-money strike, this grid is valid. However, it falls in other situation since it only calculate price of option whose strikes are higher than S_0. Under these situations, this grid cannot be considered as a valid one. In that sense, the at-the-money strike S_0 will fall on the grid where $m=1$

Problem 2 (25 points) Consider the following risk neutral pricing formula for a European put option:

$$p_0 = \tilde{\mathbb{E}}\left[(K - S)^+\right]$$

For purposes of this problem you may assume that interest rates are equal to zero and thus ignore discounting terms.

(a) (5 Points) Construct a butterfly centered on some strike, K^*, using only put options. Please include the strike of each option as well as the units and whether you should be long or short each put.

(b) (8 Points) Using the Breeden-Litzenberger technique, derive the relationship between the risk-neutral density of S and put options.

(c) (6 points) Why must put option prices be convex with respect to strike in the absence of arbitrage? Suppose you noticed that this condition was violated. Construct a riskless portfolio that would take advantage of this.

(d) (6 points) Describe the structure that you would trade if you believe that the market was mispricing the CDF of S. You should assume only European Calls and Puts are traded in your answer.

(a) Butterfly centered on K^*:

Long one put at strike K^*-h

Short two put at strike K^*

Long one put at strike K^*+h

(b) By Breeden-Litzenberger technique,

$$p_0 = \mathbb{E}[(K-S)^+]$$
$$= \int_{-\infty}^{K} (K-S_T)\,\phi(S_T)\,dS_T$$

Differentiating the payoff function with respect to strike,

$$\frac{\partial p_0}{\partial K} = \int_{-\infty}^{K} \phi(S_T)\,dS_T = \Phi(K)$$

Differentiating again with respect to strike

$$\frac{\partial^2 p_0}{\partial K^2} = \phi(K)$$

Then we employ finite difference to obtain a numerical approximation to $\frac{\partial^2 p_0}{\partial K}$

$$\frac{\partial^2 p_0}{\partial K^2} \approx \frac{P_0(K-h) - 2P_0(K) + P_0(K+h)}{h^2}$$

$$\Rightarrow \quad \phi(K) \approx \frac{P_0(K-h) - 2P_0(K) + P_0(K+h)}{h^2}$$

(c) Since a negative probability in density function implies an arbitrage opportunity, if in the absence of arbitrage, we must have $\phi(K) \geq 0$ which means $\frac{\partial^2 p_0}{\partial K^2} \geq 0$. Thus put option prices must be convex with respect to strike, that $\frac{\partial^2 P_0}{\partial K^2}$ be positive at all strikes, in the absence of arbitrage.

If the condition was violated, we could construct such a riskless portfolio:

Long one put at strike K^*-h

Short two put at strike K^*

Long one put at strike K^*+h

Since $\phi(K) < 0$

implies $P_0(K-h)+P_0(K+h) < 2P_0(K)$, then by the time we construct the portfolio we can make profit since we short two put at K^* and long one put at K^*+h and one put at K^*-h.

If by maturity, S_T falled between K^*-h and K^*+h, we can make another profit during termination. Even if not, we have made money at the beginning without potential loss in the future. So the portfolio is riskless.

(d) $P_0 = \mathbb{E}[(K-S)^+]$

$= \int_{-\infty}^{K} (K-S_T)\,\phi(S_T)\,dS_T$

denote C_0 as pricing formula of European call option

$$C_0 = \mathbb{E}[(S-K)^+]$$
$$= \int_K^{+\infty} (S_T - K)\phi(S_T)\, dS_T$$
$$\Rightarrow \frac{\partial P_0}{\partial K} = \int_{-\infty}^{K} \phi(S_T)\, dS_T = \Phi(K)$$
$$\frac{\partial C_0}{\partial K} = \int_K^{+\infty} -\phi(S_T)\, dS_T = \Phi(K) - 1$$

By Breeden-Litzenberger technique,

$$\Phi(K) = \frac{\partial P_0}{\partial K} \simeq \frac{P_0(K+h) - P_0(K)}{h}$$
$$\Phi(K) - 1 = \frac{\partial C_0}{\partial K} \simeq \frac{C_0(K+h) - C_0(K)}{h}$$

So $P_0(K+h) \simeq P_0(K) + \Phi(K)h$

$C_0(K+h) \simeq C_0(K) + (\Phi(K) - 1)h$

If we belive the market was mispricing the CDF of S, that is $\Phi(K)$

Case 1:

if CDF of S. $\Phi(K)$, was overestimating then we long one call at lower strike and short one call at higher strike. Or we long one put at lower strike and short one put at higher strike. Then when the CDF was pricing normally, we closed out all positions.

Case 2:

if CDF of S, $\Phi(K)$ was underestimating

then we short one put at lower strike and long one put at higher strike.
Or we short one call at lower strike and long one call at higher strike.
Then when the CDF was pricing normally we closed out all positions.

Problem 3 (25 points) Consider an Asian option whose risk-neutral pricing formula can be written as:

$$c_0 = \tilde{\mathbb{E}}\left[e^{-rT}(\bar{S} - K)^+\right]$$

where $\bar{S}$ is the arithmetic average value of S over the observation period, r is equal to 0 and the asset pays no dividends. Additionally, assume that the current value of the asset is 100. Consider a three-month at-the-money spot Asian option ($K = 100$) on this asset.

Also assume that the market follows the dynamics in the Heston model which are defined as:

$$\begin{aligned} dS_t &= rS_t\,dt + \sigma_t S_t\,dW_t^1 \\ d\sigma_t^2 &= \kappa(\theta - \sigma_t^2)\,dt + \xi\sigma_t\,dW_t^2 \\ Cov(dW_t^1, dW_t^2) &= \rho\,dt \end{aligned}$$

(a) (3 points) List all parameters in the Heston model and describe intuitively what they represent.

(b) (6 points) Using the Heston model and the Euler discretization scheme for Monte-Carlo, compute an approximation for the Asian option assuming the following parameters:

$$\begin{aligned} \sigma_0 &= 0.05 \\ \kappa &= 1 \\ \theta &= 0.1 \\ \rho &= -0.5 \\ \xi &= 0.25 \\ \Delta t &= \frac{1}{252} \end{aligned}$$

(c) (3 points) Run your approximation with many different numbers of simlated paths. Estimate the number of draws that are needed for the price of the option to converge. Comment on the difference between this and the theoretical convergence rate.

(d) (5 points) Consider Z and $-\frac{1}{2}Z$ as antithetic variables, where Z is a standard normal random variable. Calculate the mean and variance of an estimator using these pairs of variables. Is this approach unbiased? Does it reduce the variance? By how much?

(e) (3 points) Describe what you think would be an ideal control variate for the Asian option priced above. Justify your answer with either a theoretical or empirical argument.

(f) (5 points) Reprice the Asian option above using the control variate of your choice. Does the simulated price converge faster? Why or why not?

(a) parameters in the Heston model:

r - drift, rate of return of the asset

k - speed of mean reversion of volatility

θ - level of mean reversion of volatility

ξ - volatility of the volatility

ρ - correlation between W_t^1 and W_t^2

(b) Based on Heston model, we derive the Euler discretization for Monte-Carlo,

$$\hat{S}_{t_{j+1}} = \hat{S}_{t_j} + r\hat{S}_{t_j}\Delta t + \hat{\sigma}_{t_j}\hat{S}_{t_j} Z_j^1$$

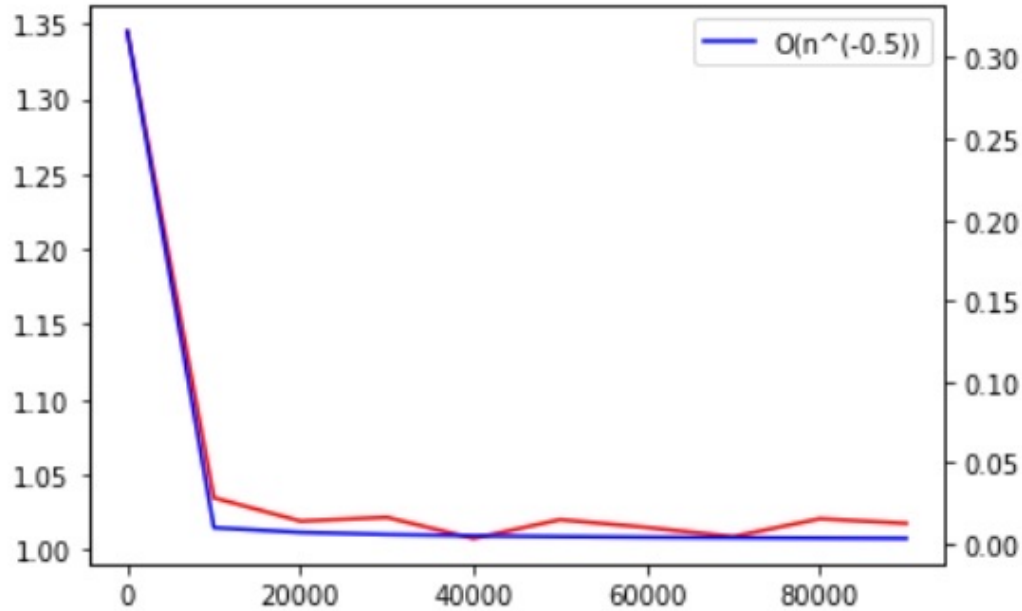
$$\hat{\sigma}^2_{t_{j+1}} = \hat{\sigma}^2_{t_j} + K(\theta - \hat{\sigma}^2_{t_j})\Delta t + \xi\,\hat{\sigma}_{t_j} Z_j^2$$

where $\mathrm{cov}(Z_j^1, Z_j^2) = \rho\Delta t$

since there is some positive probability that the volatility process will become negative, we here truncate the value at zero by applying a max function to the volatility process, then we can compute an approximation for the Asian option given those parameters

$C_0 \approx 1.0139$

(c) As shown below



I estimate the number of draws that needed for the price of the option to converge is about 1×10^5. The red line in the figure is the convergence rate of our approximation, the blue one is the theoretical convergence rate, $O(N^{-\frac{1}{2}})$. we can see the two are close to each other, our approximation does converge at the rate of the theoretical convergence.

(d) Given the pairs of variables z and $-\frac{1}{2}z$ we denote $X=z$ and $Y=-\frac{1}{2}z$

then $X \sim N(0,1)$

$Y \sim N(0,\frac{1}{4})$

further $\tilde{z} = X+Y$

$\mathbb{E}[\tilde{z}] = \mathbb{E}[X+Y] = \mathbb{E}[X]+\mathbb{E}[Y] = 0 = \mathbb{E}[z]$

then

$\mathrm{Var}(\tilde{Z}) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X, Y)$

since $\mathrm{Cov}(X, Y) = -\frac{1}{2}$

So $\mathrm{Var}(\hat{Z}) = 1 + \frac{1}{4} - \frac{1}{2} \times 2 = \frac{1}{4} < \mathrm{Var}(Z)$

In sum, since $\mathbb{E}[\tilde{Z}] = \mathbb{E}[Z]$, this approach is unbiased. And it does reduce the variance, from 1 to $\frac{1}{4}$, by about $\frac{3}{4}$ of Z's original variance.

(e) In the light of control variate

$$\theta = \mathbb{E}[h(x)]$$

where $h(x)$ is the payoff function of Asian Option.

then using a control variate,

$$\hat{\theta} = \mathbb{E}[h(x)] + c(Z - \mathbb{E}[Z])$$

this estimator is still biased

$$\mathbb{E}[\hat{\theta}] = \theta$$

and the variance of new estimator is

$$\mathrm{Var}(\hat{\theta}) = \mathrm{Var}(\mathbb{E}[h(x)] + c(Z - \mathbb{E}[Z]))$$

$$= \mathrm{Var}(\theta) + c^2\,\mathrm{Var}(Z) + 2c\,\mathrm{Cov}(\theta, Z)$$

by setting $c^* = -\frac{\mathrm{Cov}(\theta, Z)}{\mathrm{Var}(Z)}$

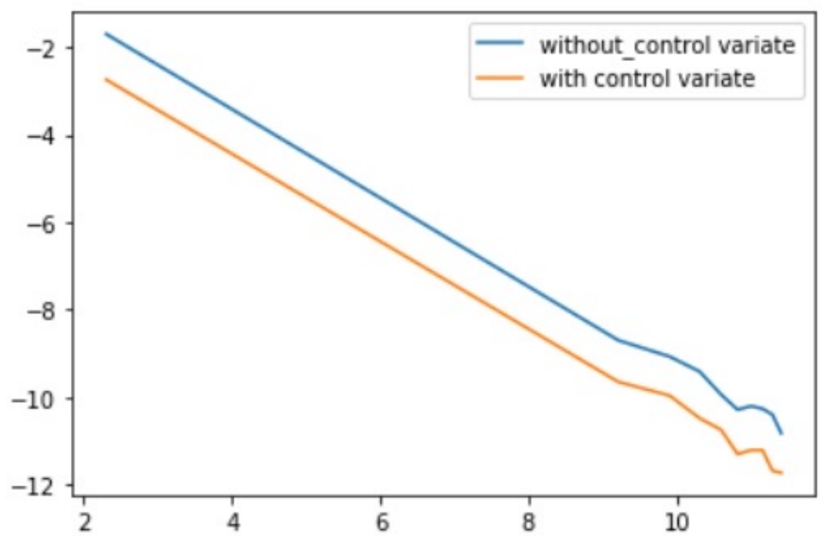
we have minimal variance

$$\mathrm{Var}(\hat{\theta}^*) = \mathrm{Var}(\theta) - \frac{\mathrm{Cov}(\theta, Z)^2}{\mathrm{Var}(Z)}$$

So we should choose Z whose expectation is known and $\frac{\mathrm{Cov}(\theta, Z)^2}{\mathrm{Var}(Z)}$ be as large as possible.

In that sense, I use the price of European call option as control variate Z. From empirical side, I run the simulation many times and notice absolute value of correlation between θ and Z is about 0.75, which I think is significant enough for the European call option price to be the control variate to reduce the variance of estimator.

(f)



As shown above, using the control variate of my choice, it does reduce the variance. However, the convergence rate stay same, since we still have the term $Var(\theta)$, which dominate the convergence rate.

Problem 4 (25 points) For this problem please refer to the file `DataForProblem4.csv` in the same folder as this test.

Each column refers to a security and each row represent a day in history.

Each cell contains that day's return of that security in decimals.

We also view each day as a possible state of the world and assume that all these states are equally likely.

Also assume that your total wealth is \$1 and no shorting is allowed.

Justify each of your answers by setting up the optimization problem you are planning to solve, explaining why you think this choice is the right one, and which method you are going to use to solve it.

(a) (3 Points) Using the history of the returns of the instruments Sec1 through Sec10 in `DataForProblem4.csv`, construct the fully-invested minimum variance portfolio.

(b) (3 Points) Find the fully-invested optimal portfolio, assuming your risk aversion coefficient is 0.5.

(c) (3 Points) Find the portfolio that would maximize the expected return.

(d) (3 Points) Find the fully-invested portfolio that would track the benchmark B1 best.

(e) (4 Points) In the rest of the problem, we explore an alternative way of constructing portfolios.

Let r_j^i denote the return of the i-th security in the j-th state of the world, i.e., in the $(j+1)$st row in `DataForProblem4.csv`.

Let

$$U(x) = \log(x) \qquad \forall x > 0 \tag{1}$$

be the utility of the amount of wealth x, i.e., what x dollars is worth to us. Note that U is twice differentiable and $U''(x) < 0$ for all x.

Denote by w_i the fraction of our wealth that we are planning to allocate to the ith security and by w_0 the fraction that we will leave in cash. Assuming that all states of the world are equally likely, write down the expression for the expected utility for the allocation vector $w = (w_0, w_1, \ldots, w_{10})$.

(f) (5 Points) Find the portfolio, i.e., the allocation vector w that maximizes the expected logarithmic utility.

(g) (4 Points) Are portfolios in (c), (b), and (f) the same?

Which portfolio would you choose if you were a mutual fund?

Which would you choose if you were managing your own money?

Each question is solved by python, please check code to see the detail. I only intend to post the formula and result below.

(a) fully-invested minimum variance portfolio

so we have,

$$\min w^T C w$$
$$\|w\| = 1$$
$$w \geq 0$$

then we solve,

$$w = [0, 0, 0.6906, 0.1451, 0, 0, 0.1643, 0, 0, 0]^T$$

$$w^T C w = 2.5958 \times 10^{-5}$$

(b) fully-invested optimal portfolio, where the risk aversion coefficient $a = 0.3$

so we have,

$$\max w^T R - a w^T C w$$
$$\|w\| = 1$$
$$w \geq 0$$

then we solve

$$w = [0, 0, 0.03395, 0, 0, 0, 0, 0.96605, 0, 0]^T$$

$$w^T R - a w^T C w = 15.8876 \times 10^{-5}$$

(c) the portfolio maximizing the expected return

So we have

$$\max \; w^T R$$
$$w \geq 0$$
$$0 \leq \|w\| \leq 1$$

then we solve

$$w = [0, 0, 0, 0, 0, 0, 0, 1, 0, 0]^T$$

$$w^T R = 41.03586 \times 10^{-5}$$

(d) fully-invested portfolio that track the benckmark B, best

So we have

$$\min \sum_{i=1}^{n} (w^T R^i - R^i_{benchmark})^2$$
$$\|w\| = 1$$
$$w \geq 0$$

then we solve

$$w = [0.007, 0, 0.158, 0.195, 0.005, 0.1039, 0.1775, 0.2378, 0.052, 0]^T$$

$$MSE = 3.36 \times 10^{-5}$$

(e) we denote $w = (w_0, w_1, \dots w_{10})$

and $r_j = (r_j^1, r_j^2, \dots r_j^{10})$

since w_0 be the fraction left in cash so we assume $r_j^0 = 0$

then $r_j^* = (1, 1+r_j^1, 1+r_j^2, \dots, 1+r_j^{10})$

given all states of the world are equally likely, $p = 1/251$

Thus

$$\mathbb{E}[U(x)] = \mathbb{E}[\log(x)] = \sum_{i=1}^{251} p \log(w(1+r_j^*)^T)$$

(f) $w = [\underset{w_0}{\underline{\underline{0.191}}}, 0, 0, 0, 0, 0, 0, 0, 0.809, 0, 0]^T$

(g) Portfolio in (c), (b), (f) are not the same. If I were a mutual fund, I choose portfolio in (b) which balance between return and risk.

If I were managing my own money, I choose portfolio in (c), to realize the maximal expected utility.