MF 803 Homework 6

Due: Wed, Dec 11th, 18:30 p.m.

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1. Covariance Matrix Decomposition:

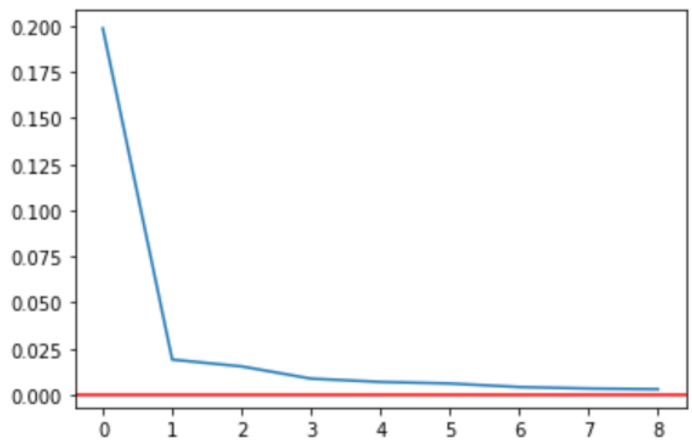
a. Download historical data for the sector ETFs from January 1st 2010 until today, after examining historical data (using the describe module in pandas, the result are shown below), no anomalies.

	XLB	XLE	XLF	XLI	XLK	XLP	XLU	XLV	XLY
count	2487	2487	2487	2487	2487	2487	2487	2487	2487
mean	45.58146	71.33592	18.80917	52.64645	43.24566	44.35439	43.48401	60.73368	70.33068
std	8.791203	10.06564	5.806229	15.65606	17.42712	10.33119	9.071938	21.16664	26.85574
min	28.05	49.38	9.16	26.9	20.29	25.45	28.09	27.96	28.59
0.25	37.765	64.895	13.04	37.07	28.93	34.305	35.525	37.54	44.515
0.5	46.19	70.14	18.53	53.49	40.16	47.51	42.92	66.84	69.81
0.75	52.535	77.085	24.135	66.655	55.545	53.535	50.815	77.09	89.39
max	64.09	101.29	30.17	82.31	87.5	61.65	64.93	97.46	124.48

b. The covariance matrix of daily returns for the sector ETFs are listed below,

	XLB	XLE	XLF	XLI	XLK	XLP	XLU	XLV	XLY
XLB	0.0372	0.0332	0.0306	0.0293	0.0252	0.0146	0.0117	0.0202	0.0251
XLE	0.0332	0.0471	0.0305	0.0286	0.0246	0.0139	0.0120	0.0198	0.0244
XLF	0.0306	0.0305	0.0393	0.0295	0.0256	0.0150	0.0114	0.0214	0.0262
XLI	0.0293	0.0286	0.0295	0.0302	0.0244	0.0144	0.0113	0.0197	0.0246
XLK	0.0252	0.0246	0.0256	0.0244	0.0293	0.0134	0.0100	0.0192	0.0240
XLP	0.0146	0.0139	0.0150	0.0144	0.0134	0.0142	0.0109	0.0125	0.0138
XLU	0.0117	0.0120	0.0114	0.0113	0.0100	0.0109	0.0196	0.0099	0.0103
XLV	0.0202	0.0198	0.0214	0.0197	0.0192	0.0125	0.0099	0.0223	0.0187
XLY	0.0251	0.0244	0.0262	0.0246	0.0240	0.0138	0.0103	0.0187	0.0267

c. The eigenvalues are listed below:
[0.19866122, 0.01912094, 0.01541953, 0.00879541, 0.00701684, 0.00615508, 0.00425379, 0.00344016, 0.00307203]



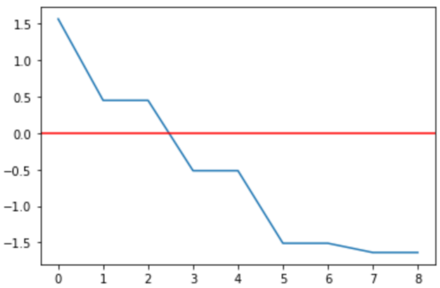
As shown in the figure, every eigenvalue is positive and no eigenvalue is negative or zero, since the covariance matrix should be positive semi-definite or even positive finite if it's full rank. If there is any negative eigenvalue, either the process of calculation or the data may go wrong. And if I take the threshold of 0.1, only one eigenvalue is statistically significant.

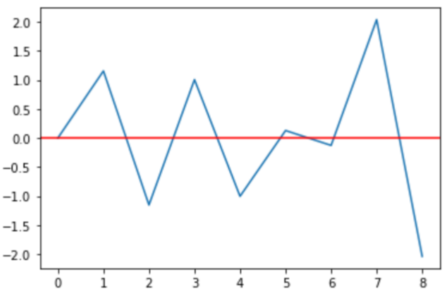
d. Use .shape to acquire the rows and columns of the covariance matrix, then use np.random.normal to generate a random matrix of the same size as the covariance matrix.

e. The eigenvalues of the random matrix are listed below,

[1.56851351+0.j,
0.45050015+1.15189324j,
0.45050015-1.15189324j,
-0.51399751+1.00277554j,
-0.51399751-1.00277554j,
-1.5098122+0.12777674j,
-1.5098122-0.12777674j,
-1.63665652+2.03504132j,
-1.63665652-2.03504132j]

Since the eigenvalues contain real number as well as imaginary number, we plot the real part and the imaginary part separately.





The structure of the eigenvalues of the random matrix is different from that of the historical covariance matrix, as the random matrix is not positive semi-definite, we can see negative eigenvalues in the result of the random matrix.

2. Portfolio Optimization:

a. The historical annualized returns for each sector ETFs are listed below,

	Annualized return
XLB	5.95%
XLE	0.13%
XLF	9.47%
XLI	10.96%
XLK	13.72%
XLP	8.64%
XLU	7.29%
XLV	11.40%
XLY	14.31%

b. In the light of the formula,

$$w = \frac{1}{2} C^{-1} R$$

the weights of the unconstrained mean-variance optimal portfolio by using the annualized returns as expected returns and the covariance matrix above are listed below,

	Weight
XLB	-2.0200
XLE	-2.2800
XLF	-0.9260
XLI	1.6800
XLK	1.2700
XLP	0.3820
XLU	0.6830
XLV	1.2000
XLY	3.5800

c. With the adjusted expected returns calculated by

$$\mathrm{E}[\mathrm{r}] = \mu + \sigma * Z, \qquad Z \sim \mathrm{N}(0,1)$$

the unconstrained mean variance optimal portfolio using the adjusted expected returns under different σ and the covariance matrix are listed below

	σ = 0.005	σ = 0.01	σ = 0.05	σ = 0.1
XLB	−1.790	−2.350	−3.100	−5.030
XLE	−2.060	−2.180	−0.706	−0.845
XLF	−0.454	−1.010	1.060	−0.917
XLI	1.280	1.700	−1.550	8.080
XLK	1.500	0.003	3.880	4.730
XLP	0.336	−0.820	−7.390	−14.400
XLU	0.743	0.531	2.540	5.720
XLV	1.260	2.360	8.960	2.060
XLY	2.640	4.960	−1.790	1.480

I use the sum of absolute difference between new weight and the weight calculated in (b) to estimate the stability, and I find as σ increases, the sum becomes larger meaning the weights are more unstable.

σ = 0.005	2.661
σ = 0.01	5.704
σ = 0.05	33.241
σ = 0.1	37.067

d. In the light of the formula

$$\tilde{\sum} = \delta \sum_{diag} + (1-\delta) \sum_{full}$$

we shall calculate the regularized covariance matrix.

e. Set $\delta = 1$, the rank of the regularized covariance matrix is 9 and its eigenvalues are

[0.01421923, 0.01958098, 0.02232867, 0.02674681, 0.02929398, 0.03018693, 0.0371544, 0.03930282, 0.04712118]

f. After trying different δ between 0 and 1, I find the calculated eigenvalues are still non-negative, every eigenvalue is positive and none of them are zero. Since after calculating the regularized covariance matrix, the matrix is still positive semi-definite.

g. Similarly, I repeat the exercise in (c) with the regularized covariance matrix for a few values of δ and still use the sum of absolute difference between new weight and the weight calculated in (b) to estimate the stability. Fixed σ, I notice the higher the δ, the more unstable the weights will be.

δ =0. 1	16. 726357
δ =0. 2	25. 233977
δ =0. 5	36. 939891
δ =0. 7	40. 48012