Problem set # 5

Due: Wednesday, March 4th, by 8am

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1. **Implementation of Breeden-Litzenberger:**

   You are given the following volatility data:

Expiry / Strike	1M	3M
10DP	32.25%	28.36%
25DP	24.73%	21.78%
40DP	20.21%	18.18%
50D	18.24%	16.45%
40DC	15.74%	14.62%
25DC	13.70%	12.56%
10DC	11.48%	10.94%

You also know that the current stock price is 100, the risk-free rate is 0, and the asset pays no dividends.
NOTE: The table of strikes is quoted in terms of Deltas, where the DP rows indicate "X Delta Puts" and the DC rows indicate "X Delta Calls".

(a) Using the table of quoted (Black-Scholes) Deltas and volatilities, extract a table of strikes corresponding to each option.

Expiry / Strike	1M	3M
10DP	89.138758	84.225674
25DP	95.542103	93.470685
40DP	98.700642	98.127960
50D	100.138720	100.338826
40DC	101.262273	102.141761
25DC	102.783751	104.532713
10DC	104.395838	107.422225

(b) Choose an interpolation scheme that define the volatility function for all strikes,

$$\sigma(K)$$

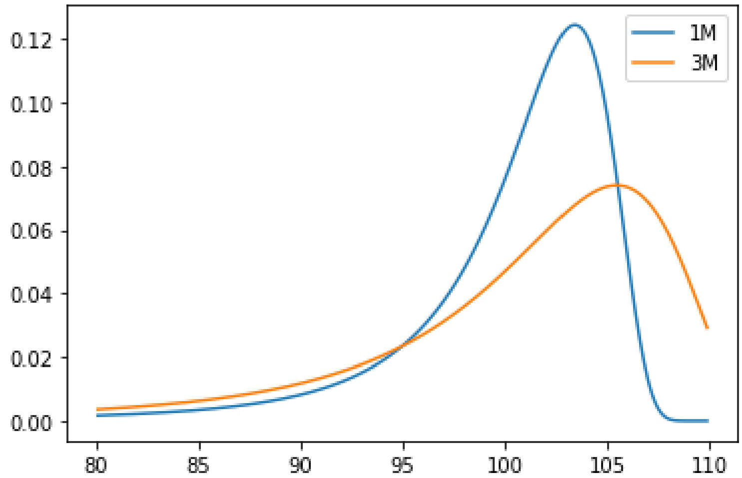
Using linear interpolation, the volatility function for all strikes is defined as

$$\begin{aligned}\sigma_{1M} &= 1.5577881380475331 - 0.013788352222217588 * K \\ \sigma_{3M} &= 0.9322345680108484 - 0.007673547897915936 * K\end{aligned}$$

(c) Extract the risk neutral density for 1 & 3 month options. Comment on the differences between the two distributions. Is it what you would expect?

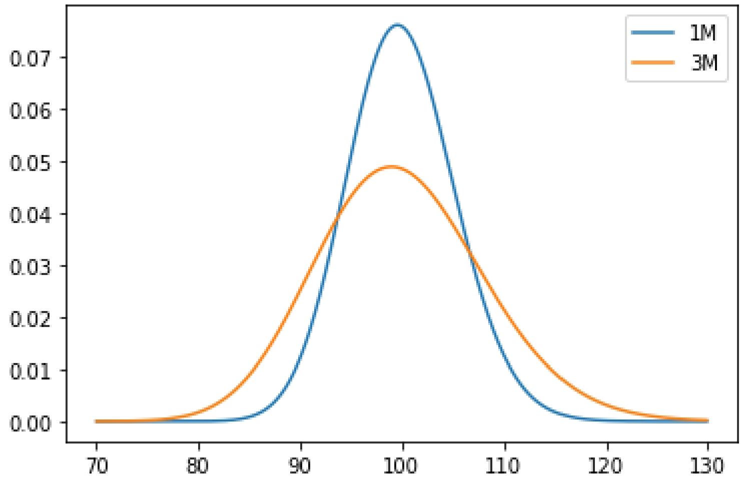
The risk neutral density is calculated by

$$\phi(\mathrm{K}) \approx \frac{C(K-h) - 2C(K) + C(K+h)}{h^2}$$



As shown from the figure, the risk neutral density for 1&3 month options are all skewed, because of the underlying volatility is skewed. Further, the risk neutral density for 3 month option has fatter tail in both side, which is consistent with our expectation as the longer the maturity, the higher probability that the price will raise is.

(d) Extract the risk neutral density for 1 & 3 month options using a constant volatility equal to the 50D volatility. Contrast these densities to the densities obtained above.



As shown from the figure, the risk neutral density for 1 & 3 month options using a constant volatility has no skewness compared to these densities obtained in (c) as the underlying volatility now has no skewness.

(e) Price the following European Options using the densities you constructed in (1c).

i. 1M European Digital Put Option with Strike 110.
ii. 3M European Digital Call Option with Strike 105.
iii. 2M European Call Option with Strike 100.

the price of 1M European Digital Put Option with Strike 110 is **0.9822320264947844.**

the price of 3M European Digital Call Option with Strike 105 is **0.3292756110096208.**

the price of 2M European Call Option with Strike 100 is **2.662407380974223.**

2. **Calibration of Heston Model:**

Recall that the Heston Model is dened by the following system of SDE's:

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{\nu_t} S_t dW_t^1 \\ d\nu_t &= \kappa(\theta - \nu_t) dt + \sigma\sqrt{\nu_t} dW_t^2 \\ \operatorname{Cov}\left(dW_t^1, dW_t^2\right) &= \rho dt \end{aligned}$$

Recall also that the characteristic function for the Heston Model is known to be:

$$\begin{aligned} \omega(u) &= \frac{\exp\left(iu\ln S_0 + iu(r-q)t + \sigma^{-2}\{\kappa\theta t(\kappa - i\rho\sigma u)\}\right)}{\left(\cosh\frac{\lambda t}{2} + \frac{\kappa - i\rho\sigma u}{\lambda}\sinh\frac{\lambda t}{2}\right)^{\frac{2\kappa\theta}{\sigma^2}}} \\ \Phi(u) &= \omega(u)\exp\left(\frac{-\left(u^2 + iu\right)\nu_0}{\lambda\coth\frac{\lambda t}{2} + \kappa - i\rho\sigma u}\right) \\ \lambda &= \sqrt{\sigma^2\left(u^2 + iu\right) + (\kappa - i\rho\sigma u)^2} \end{aligned}$$

See the attached spreadsheet for options data. r = 1.5%, q = 1.77%, S0 = 267.15.
Consider the given market prices and the following equal weight least squares minimization function:

$$\vec{p}_{\min} = \min_{\vec{p}}\left\{\sum_{\tau,K}\left(\tilde{c}(\tau, K, \vec{p}) - c_{\tau,K}\right)^2\right\}$$

$where\ \tilde{c}(\tau, K, \vec{p})\ is\ the\ FFT\ based\ model\ price\ of\ a\ call\ option\ with\ expiry\ \tau\ and\ strike\ K$

(a) Check the option prices for arbitrage. Are there arbitrage opportunities at the mid? How about after accounting for the bid-ask spread? Remove any arbitrage violations from the data.

To check the option prices for arbitrage. We shall examine

1. Call (put) prices that are monotonically decreasing (increasing) in strike.

```
expDays
49      True
140     True
203     True
Name: call_bid, dtype: bool
expDays
49      True
140     True
203     True
Name: call_ask, dtype: bool
expDays
49      True
140     True
203     True
Name: put_bid, dtype: bool
expDays
49      True
140     True
203     True
Name: put_ask, dtype: bool
```

2. Call (put) prices whose rate of change is greater than 0 (-1) and less than 1 (0)

```
expDays
49      True
140     True
203     True
Name: call_bid, dtype: bool
expDays
49      True
140     True
203     True
Name: call_ask, dtype: bool
expDays
49      True
140     True
203     True
Name: put_bid, dtype: bool
expDays
49      True
140     True
203     True
Name: put_ask, dtype: bool
```

3. Call and put prices that are convex with respect to changes in strike.

```
expDays
49      True
140     True
203     True
Name: call_bid, dtype: bool
expDays
49      True
140     True
203     True
Name: call_ask, dtype: bool
expDays
49      True
140     True
203     True
Name: put_bid, dtype: bool
expDays
49      True
140     True
203     True
Name: put_ask, dtype: bool
```

In sum there are no arbitrage opportunities either at the mid or for the bid-ask spread.

(b) Using the FFT Pricing code from your last homework, and the values of

$$\kappa, \theta, \sigma, \rho \; and \; \nu_0$$

that minimize the equal weight least squared pricing error. You may choose the starting point and upper and lower bounds of the optimization. You may also choose whether to calibrate to calls, puts, or some combination of the two.
NOTE: In R, the optimx function can be used to perform this optimization, which has a parameter named method which can be set to Nelder-Mead.
Note also that you are given data for multiple expiries, each of which should use the same parameter set, but will require a separate call to the FFT algorithm.

(c) Try several starting points and several values for the upper and lower bounds of your
parameters. Does the optimal set of parameters change? If so, what does this tell you
about the stability of your calibration algorithm?

Starting Points	Lower Bounds	Upper Bounds	Squared Error	Calibrated
(2,0.2,0.5,-1,0.1)	(0.01, 0, 0, -1, 0)	(5, 2, 1, 1, 1)	1.4437162948939728	(1.31084425, 0.04097418, 1, -0.79970322, 0.13428685])
(2,0.2,0.5,-1,0.1)	(0.01, 0, 0,-1,0)	(2.5, 1, 1 ,0.5 ,0.5)	1.4437161259	(1.31073329, 0.04097377, 1, -0.79972359, 0.13428111)
(0.5,0.2,0.2,0,0.2)	(0.01,0.01, 0,-1,0,1)	(5, 2, 2, 1 ,1)	1.0191942816913917	(1.56841636, 0.04373841, 2 , -0.79947253, 0.09275175)

As shown from the table, changing starting points and upper or lower bounds may change the optimal set of parameters, however, the change is trivial so I think the calibration algorithm is stable.

(d) Instead of applying an equal weight to each option consider the following function
which makes the weights inversely proportional to the quoted bid-ask spread:

$$\omega_{\tau,K} = \frac{1}{c_{\tau,K,\text{ ask}} - c_{\tau,K,\text{ bid}}}$$

$$\vec{p}_{\min} = \min_{\vec{p}} \left\{ \sum_{\tau,K} \left(\omega_{\tau,K}\,\tilde{c}(\tau,K,\vec{p}) - c_{\tau,K}\right)^2 \right\}$$

$where\ \tilde{c}(\tau,K,\vec{p})\ is\ the\ FFT\ based\ model\ price\ of\ a\ call\ option\ with\ expiry\ \tau\ and\ strike\ K$

Repeat the calibration with this objective function and comment on how this weighting affects the optimal parameters.

Starting Points	Lower Bounds	Upper Bounds	Squared Error	Calibrated
(2,0.2,0.5,-1,0.1)	(0.01, 0, 0, -1, 0)	(5, 2, 1, 1, 1)	9.31485887	(1.20968397, 0.03958377, 1, -0.8148759 , 0.12985453)
(2,0.2,0.5,-1,0.1)	(0.01, 0, 0,-1,0)	(2.5, 1, 1 ,0.5 ,0.5)	9.31485918	(1.20971253, 0.03958395, 1, -0.81487213, 0.12985742)
(0.5,0.2,0.2,0,0.2)	(0.01,0.01, 0,-1,0,1)	(5, 2, 2, 1 ,1)	6.6221233	(1.44985826, 0.04281099, 2, -0.81132511, 0.08875559)

Similarly, changing starting points and upper or lower bounds may change the optimal set of parameters, so the calibration algorithm is stable. However, the weighting affects the final squared error of model using optimal parameters but affect little on the optimal parameters.

3. **Hedging Under Heston Model:** Consider a 3 month European call with strike 275 on
the same underlying asset.

(a) Calculate this option's Heston delta using finite differences. That is, calculate a first
order central difference by shifting the asset price, leaving all other parameters constant
and re-calculating the FFT based Heston model price at each value of S0.

i. Compare this delta to the delta for this option in the Black-Scholes model. Are they different, and if so why? If they are different, which do you think is better and why? Which would you use for hedging?

By setting

$$\kappa, \theta, \sigma, \rho,\ \nu_0,\ S0,\ r,\ q,\ T,\ K = 3.51, 0.052, 1.17, -0.77, 0.034, 267.15, 0.015, 0.0177, 0.025, 275$$

The option's Heston delta is **0.4762603833210668** while the delta in the Black-Scholes model is **0.36687192939578234**. We notice that the two delta is different for the assumption in the Black-Scholes model is that the volatility is constant while the volatility in Heston model is dynamic. And volatility would definitely affect the value of delta. I think Heston delta is better for the effect of volatility on the option price is taken into consideration in the Heston model.

And I'd like to use Heston delta for hedging.

ii. How many shares of the asset do you need to ensure that a portfolio that is long
one unit of the call and short x units of the underlying is delta neutral?

Given one unit of call we can calculate its Heston delta, once we long one unit of the call, we need short Δ units of the underlying to be delta neutral.

(b) Calculate the vega of this option numerically via the following steps:
i. Calculate the Heston vega using finite differences. To do this, shift

$$\theta \ and \ \nu_0$$

by the same amount and calculate a first order central difference leaving all other
parameters constant and re-calculating the FFT based Heston model price at each
value of

$$\theta \ and \ \nu_0$$

Given

$$\frac{\partial C}{\partial \sigma} = S_t N'(d_1)\sqrt{T-t}$$

The Heston vega is **131.82303086316568**.

ii. Compare this vega to the vega for this option in the Black-Scholes model. Are they different, and if so why?

The vega in the Black-Scholes model is **50.293392366377915**. The vega in Heston model is different from the vega in Black-Scholes model. The negative calibrated value of the parameter ρ will affect the vega. As the ρ is negative, the underlying asset and the volatility has a negative relationship with each other when we use the Heston model. Therefore, given the same increase in the volatility, the option price obtained by Heston model will have a smaller increase than that in the Black-Scholes model.