E&M II Homework 9

Due Nov 29, 2011

1. **Light from a Tank of Water**
   There are many situations in which we find ourselves looking through a pane of glass into a tank of water. Of course, the light from inside the tank has to make it through two boundaries worth of reflection. If the water has index of refraction n_w, glass has $n_g < n_w$, and air has $n_a < n_g$, then what are the minimum and maximum transmission coefficients according to Eq.(9.199)?

2. **'Massive' E&M Waves** !
   (Note: at a fundamental level, beyond the short-ranged scalar electric potential, adding a distance scale to electricity and magnetism is really just giving the photon - the fundamental force particle that mediates electromagnetic interactions - a mass. Thus, I'm dropping the 'Yukawa' name and calling it massive, but it's really just the same equations you dealt with in homework 2 and 8.)

   From homework 8, in the absence of charges and currents we concluded

$$\left[\mu_0\epsilon_0\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{1}{\lambda^2}\right]\phi = 0 \tag{1}$$

$$\left[\mu_0\epsilon_0\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{1}{\lambda^2}\right]\vec{A} = 0 \tag{2}$$

   where ϕ is the electric potential and $\vec{A}$ is the vector potential, which satisfy

$$\nabla \cdot \vec{A} = -\frac{1}{c^2}\frac{\partial \phi}{\partial t} \tag{3}$$

a) Show that there exists plane wave solutions for ϕ and $\vec{A}$ (please use complex notation). What is the dispersion relaiton (that is, what is the functional relationship between the angular frequency and the angular wave number: $\omega(k)$)? What must the relative phase between the $\vec{A}$ and ϕ waves be? What's the relationship between the amplitude of $\vec{A}$ and the amplitude of ϕ (this will involve k and ω)?

b) Using $\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \nabla \times \vec{A}$, what are the corresponding electric and magnetic fields (in terms of ω, k, and the scalar - i.e. electric -and vector potential amplitudes)? Show that because Maxwell's equations now have a length scale (λ) we can now have longitudinal polarizations of the electric field (that is, there can be a component of the electric field in the direction of propogation, $\hat{k}$).

c) One can step through the derivation in section 8.1.2 with the modified forms of Maxwell's equations to discover

$$u = u_0 + \frac{1}{2}\left[\frac{\epsilon_0}{\lambda^2}\phi^2 + \frac{1}{\mu_0\lambda^2}A^2\right] \tag{4}$$

and

$$\vec{S} = \vec{S}_0 + \frac{1}{\mu_0\lambda^2}\phi\vec{A} \tag{5}$$

where $\mu_0 = \frac{1}{2}\left[\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right]$ and $\vec{S}_0 = \frac{1}{\mu_0}\vec{E}\times\vec{B}$. Interestingly, this demonstrates that the potentials themselves are physical because they make measurable contributions to the energy density and the Poynting vector, which is a drastic change from our understanding of the potentials' role in the ordinary theory of electromagnetism. We might as well avoid mixed notation again and stick with just one physical quantity: Write the new time-averaged energy density and Poynting vector for this plane wave in terms of just ω, k, and the scalar and vector potential amplitudes.

d) One can additionally step through the derivations in section 8.2.1 and find both a modified form of the stress-energy tensor (not covered in the lectures) and the associated momentum density of the electromagnetic fields. It turns out the momentum density is exactly what one would expect given Eq.(8.30):

$$\mathscr{P}_{\text{em}} = \mu_0 \epsilon_0 \vec{S} = \mu_0 \epsilon_0 \left(\vec{S}_0 + \frac{1}{\mu_0 \lambda^2} \phi \vec{A} \right) \tag{6}$$

This last part asks two things:
1) At what speed does the massive E&M wave propagate?
2) Suppose a rectangular block of perfectly absorptive material is put in front of the propagating wave. It is initially at rest and has mass m. From the above equation for time-averaged momentum density and knowing the propagation speed of the waves, determine first the force acting on the block and then the velocity of the block as a function of time, $v(t)$, if the light strikes perpendicular to a surface of area A. For simplicity, ignore relativistic dynamics (that is, $p_{\text{block}} = m dv/dt$ and $F = dp/dt$) but do not ignore the effect of the speed of the block on the force acting on it. (Hint: as in section, determine the force acting on the block by considering the amount of impulse imparted to the block in a short amount of time via absorption of a small 'chunk' of the incoming wave.)

(For those Sci-Fi fans out there, after solving this problem you can make an extra quick calculation: given equivalent energy densities stored in both 'massless' - take the above results and let $\lambda \to \infty$ - and 'massive' electromagnetic plane waves, you can determine which would exert the greater force on a material. This will settle once and for all what the better Sci-Fi weapon is: laser beams or 'massive photon cannons' - though this clearly doesn't settle which sounds cooler to audiences. This isn't a required problem - simply one for those interested - but hint anyway: it would be easier to relate $\langle S \rangle$ to $\langle u_{\text{em}} \rangle$ instead of the potentials, since that's common to the fields produced by both the massive and massless weapons.)

3. **Circularly Polarized Plane Waves**

Consider the Earth's inosphere to be a dilute plasma of uniform dnesity (that is, the interactions between electrons and between electrons and ions can be ignored) with a static, uniform magnetic field $\vec{B}_E$ (the Earth's field) in the $+z$ direction (simply set up your coordinate axes appropriately). Consider the propagation of circularly polarized plane waves parallel to $\vec{B}_E$:

$$\vec{E}_{\pm} = E_0(\hat{x} \pm i\hat{y})e^{i(kz-\omega t)} \tag{7}$$

(The $-$ sign corresponds to right-handed circular polarization; the $+$ sign corresponds to the left-handed polarizations.) Deduce the time dependence $\vec{r}(t)$ of the position of an ionized electron (charge $-e$ and mass m_e) assuming the electron to be at rest on average. The electromagnetic wave should be weak enough that the motion is nonrelativistic (you may use the regular $F = \frac{dp}{dt}$, $p = m\frac{dv}{dt}$). Also, assume $E_0 = cB_0 \ll cB_E$.

4. **Evanescent Waves** !

(This problem is similar to 9.37 in the book, so you can refer there for extra guidance if you wish.) Recall that Snell's law predicts a critical angle for incoming plane waves passing the boundary from optical material 1 to optical material 2 that results in total internal reflection:

$$\theta_I = \theta_c = \sin^{-1}(n_2/n_1) \tag{8}$$

In intro physics, we never stopped to ask what happens when the incoming wave enters at an angle steeper than the critical angle: $\theta_I > \theta_c$. This is because there was never a second boundary after the first one, so such considerations weren't interesting. Let's, however, now consider what happens when there's a second boundary the refracted light can pass through. It turns out that in between the two boundarys, in material 2, the waves carry, on average, no energy in the z direction but they can reform on the other side of the second boundary and continue to propagate, albeit highly attenuated.

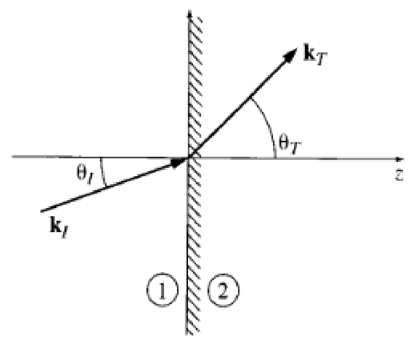

Figure 1:

a) Consider the refraction at the first boundary (see Figure 1). Show that

$$\vec{E}_T(\vec{r},t) = \vec{E}_{0_T} e^{-\kappa z} e^{i(kx-\omega t)} \tag{9}$$

where $\kappa \equiv \frac{\omega}{c}\sqrt{(n_1 \sin\theta_I)^2 - n_2^2}$ and $k \equiv \frac{\omega n_1}{c}\sin\theta_I$.

b) Use Eq.(9.109) to calculate the reflection coefficient for polarization parallel to the plane of incidence.

c) Show that the time-averaged Poynting vector in material 2 is 0.

d) Now insert the second boundary a distance D away from the first one such that the evanescent wave propagates from material 2 back to a second layer of material 1. What is the electric field on the other side of the boundary (back in material 1)? Show it has non-zero time-averaged Poynting vector.

5. **Origin of Electron Mass and Spin?**
Think of the electron as a uniformly charged spherical, hollow shell with charge e and (classical) radius R, spinning at a constant angular velocity ω.

a) What is the total amount of energy stored in the electron's electromagnetic field? (Don't forget, it's spinning, so it generates magnetic fields too! Feel free to lift the formula for the magnetic field from an earlier chapter.)

b) What is the total angular momentum contained in the fields?

c) There is a famous relationship between mass an energy: $E = mc^2$. Clearly, the energy stored in the fields should contribute something to the effective mass of the electron (if you glue something to a ball, the total mass is then the sum of the ball's mass and the mass of whatever was glued on). Suppose the entire mass of the electron is due to the electromagnetic fields. Since we know the mass of the electron, m_e, this could constrain the electron radius, R. Furthermore, the magnetic field here is that of a perfect dipole with magnetic moment depending on ω. We also know the spin-angular momentum of the electron, $L = \frac{1}{2}\hbar$. This constrains ω (choose ω such that the correct angular momentum is generated).
Determine R and ω in terms of e, m_e and $\hbar$.
Is this model plausible? (Yes or no, I just want to see a physically plausible justification for why it is or isn't.)