PHY820/422 Solutions to HW #1
Fundamentals of Mechanics: Warm Up and Review

1. José and Saletan, Chapter 1, problem 5

   Two masses m_1 and m_2 in a uniform gravitational field are connected by a spring of unstretched length *h* and spring constant *k*. The system is held by m_1 so that m_2 hangs down vertically, stretching the spring. At $t = 0$ both m_1 and m_2 are at rest, and m_1 is released, so that the system starts to fall. Set up a suitable coordinate system and describe the subsequent motion of m_1 and m_2.

2. José and Saletan, Chapter 1, problem 7

   Show that a one-dimensional particle subject to the force $F = -kx^{2n+1}$ where *n* is an integer, will oscillate with a period proportional to A^{-n}, where *A* is the amplitude. Pay special attention to the case of $n \leq 0$.

3. José and Saletan, Chapter 1, problem 15

   A particle of mass *m* moves in one dimension under the influence of the force

   $$F = -kx + \frac{a}{x^3}$$

   Find the equilibrium points, show that they are stable, and calculate the frequencies of oscillation about them. Show that the frequencies are independent of energy.

4. José and Saletan, Chapter 1, problem 21

   Draw the phase portrait for the system of Problem 15.

SOLUTIONS

1. Let $x_1(t)$ and $x_2(t)$ denote the vertical position of m_1 and m_2 respectively. The coordinate system that we will use is defined by having the positive vertical axis point downwards as shown below

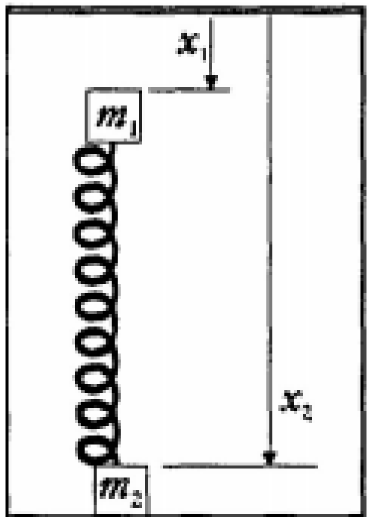

Figure 1: Coordinate system for the two-body system

From Newton's second law we find that the equations of motion for m_1 and m_2 are as follows

$$m_1\ddot{x}_1 = m_1 g + k(x_2 - x_1) \tag{1.1}$$

$$m_2\ddot{x}_2 = m_2 g - k(x_2 - x_1) \tag{1.2}$$

Note that these two equations are coupled. In the anticipation that we can treat the center of mass coordinate and relative coordinates separately, let us proceed as follows. First take equation (1.2) and multiply both sides by m_1; similarly, let's multiply the two sides of equation (1.1) by m_2. Finally let's subtract the two sides of the equations obtained to find

$$\mu\ddot{x} = -kx \quad \text{where} \quad \mu \equiv \frac{m_1 m_2}{m_1 + m_2} \quad \text{and} \quad x \equiv x_2 - x_1 \tag{1.3}$$

Next let's divide equations (1.1) and (1.2) by the total mass of the system $M = m_1 + m_2$ and then add them together to obtain

$$\ddot{y} = g \quad \text{where} \quad y \equiv \frac{m_1 x_1 + m_2 x_2}{M} \tag{1.4}$$

Instead of solving the two-body system for the motions of m_1 and m_2 directly by using equations (1.1) and (1.2), we have transformed the problem into solving for the motion of two one-body systems, namely the reduced mass system in equation (1.3) and the center of mass system in equation (1.4). Both equations (1.3) and (1.4) are second order linear ordinary differential equations with constant coefficients, the former being homogeneous and the latter being inhomogeneous. The general solutions to both equations are

$$x(t) = A\cos(\omega t + \phi) \quad \text{where} \quad \omega = \sqrt{\frac{k}{\mu}} \tag{1.5}$$

$$y(t) = \frac{1}{2}gt^2 + v_0 t + y_0 \tag{1.6}$$

Here A, ϕ, v_0, and y_0 are constant that can be determined by using the initial conditions of the two-body system

$$x_1(0) = 0 \quad x_2(0) = h + \frac{m_2 g}{k} \quad \dot{x}_1(0) = 0 \quad \dot{x}_2(0) = 0 \tag{1.7}$$

From this it follows that the initial conditions for the one-body systems are

$$x(0) = x_2(0) \quad y(0) = \frac{m_2}{M}x_2(0) \quad \dot{x}(0) = 0 \quad \dot{y}(0) = 0 \tag{1.8}$$

Applying these conditions into equations (1.5) and (1.6) we obtain

$$A = h + \frac{m_2 g}{k} \quad \phi = 0 \quad v_0 = 0 \quad y_0 = \frac{m_2(h + \frac{m_2 g}{k})}{M} \tag{1.9}$$

Thus the solutions for the two one-body systems are

$$x(t) = (h + \frac{m_2 g}{k})\cos(\omega t) \tag{1.10}$$

$$y(t) = \frac{1}{2}gt^2 + \frac{m_2(h + \frac{m_2 g}{k})}{M} \tag{1.11}$$

From the definitions of x and y in equations (1.3) and (1.4) we find that

$$x_2 = x + x_1 \Rightarrow y = \frac{m_2 x + M x_1}{M} \Rightarrow x_1 = y - \frac{m_2}{M}x \tag{1.12}$$

$$\therefore \quad x_1(t) = \frac{1}{2}gt^2 + \frac{m_2(h + \frac{m_2 g}{k})}{M}(1 - \cos(\omega t)) \tag{1.13}$$

$$\therefore \quad x_2(t) = \frac{1}{2}gt^2 + \frac{(h + \frac{m_2 g}{k})}{M}(m_2 + m_1\cos(\omega t)) \tag{1.14}$$

2. Starting with the case $n \geq 0$, for the given force

$$F = -kx^{2n+1} \tag{2.1}$$

we have that the potential is given by the integral of the force (with a minus sign) (see Fig. 2 for the potential for different values of n):

$$V = \frac{k}{2(n+1)} x^{2(n+1)} \tag{2.2}$$

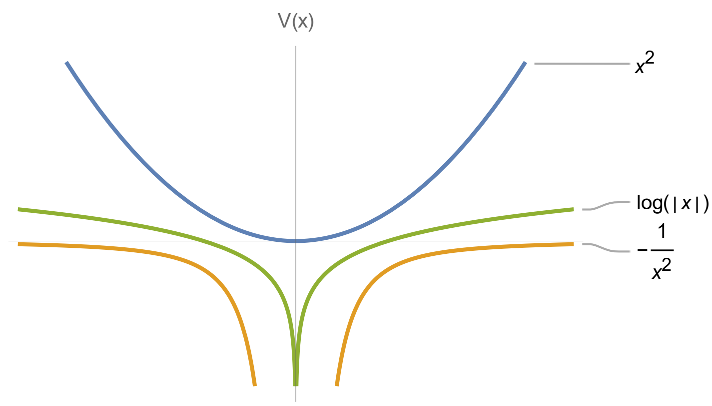

Figure 2: Plot of $V(x)$ for three different values of $n > -1$, $n = -1$, and $n < -1$. For $n \geq -1$, the motion is always bounded and the particle cannot escape to infinity. For $n < -1$, the motion is bound if the total energy is negative.

The period of the particle in the symmetric potential is then given by

$$P = \sqrt{2m} \int_{-A}^{A} \frac{dx}{\sqrt{E - V(x)}} \quad \text{where} \quad E = V(A) \tag{2.3}$$

Hence by substituting $V(x)$ and E in (2.3) with (2.2) and $E = \frac{k}{2(n+1)} A^{2(n+1)}$ respectively, we obtain

$$P = \sqrt{\frac{4m(n+1)}{k}} \int_{-A}^{A} \frac{dx}{\sqrt{A^{2(n+1)} - x^{2(n+1)}}}$$

$$= \sqrt{\frac{4m(n+1)}{k}} \frac{1}{A^{n+1}} \int_{-A}^{A} \frac{dx}{\sqrt{1 - (\frac{x}{A})^{2(n+1)}}} \quad \text{Let} \quad \frac{x}{A} = y \Rightarrow dx = Ady$$

$$\therefore P = \sqrt{\frac{4m(n+1)}{k}} A^{-n} \int_{-1}^{1} \frac{dy}{\sqrt{1 - y^{2(n+1)}}} \tag{2.4}$$

As we can see from (2.4), the integral is convergent and it is just a constant so we can conclude that $P \propto A^{-n}$.

Next we consider the case $n = -1$ because the potential function in this case looks different from a power law:

$$V = k \ln\left(\frac{|x|}{b}\right) \tag{2.5}$$

Here b is a constant reference point of the potential. For the period we obtain

$$\begin{aligned}
P =& \sqrt{2m} \int_{-A}^{A} \frac{dx}{\sqrt{k \ln\left(\frac{A}{b}\right) - k \ln\left(\frac{|x|}{b}\right)}} \\
=& \sqrt{\frac{2m}{k}} \int_{-A}^{A} \frac{dx}{\sqrt{-\ln\left(\frac{|x|}{A}\right)}} \quad \text{Let} \quad \frac{x}{A} = y \Rightarrow dx = A dy \\
\therefore P =& \sqrt{\frac{2m}{k}} A \int_{-1}^{1} \frac{dy}{\sqrt{-\ln(|y|)}}
\end{aligned} \tag{2.6}$$

Just as the previous case, the integral is convergent hence it is a constant and we conclude $P \propto A = A^{-(-1)}$ as desired. Finally we consider the case $n < -1$. The potential is given by

$$V = \frac{-k}{2(|n|-1)} x^{-2(|n|-1)} \tag{2.7}$$

The period is then given by

$$\begin{aligned}
P =& \sqrt{\frac{4m(|n|-1)}{k}} \int_{-A}^{A} \frac{dx}{\sqrt{-A^{-2(|n|-1)} + x^{-2(|n|-1)}}} \\
=& \sqrt{\frac{4m(|n|-1)}{k}} A^{|n|-1} \int_{-A}^{A} \frac{dx}{\sqrt{\left(\frac{x}{A}\right)^{-2(|n|-1)} - 1}} \quad \text{Let} \quad \frac{x}{A} = y \Rightarrow dx = A dy \\
\therefore P =& \sqrt{\frac{4m(|n|-1)}{k}} A^{|n|} \int_{-1}^{1} \frac{dy}{\sqrt{y^{-2(|n|-1)} - 1}}
\end{aligned} \tag{2.8}$$

Again, the integral is convergent and constant and thus we can conclude that $P \propto A^{|n|} = A^{-n}$.

A simpler solution: One can use dimensional analysis to solve the problem. We'd like to find the time period P which has the units of time; let's represent this by choosing the notation $[P] = T$ where T stands for the unit of time. The time period is a function of the parameters of the problem that include the mass m (we have $[m] = M$ with M the unit of mass), the amplitude A (we have $[A] = L$ with L the unit of length), and k whose units can be determined from the defnition of the force F. Note that $[F] = MLT^{-2}$ that simply follows from equating the units on the two sides of Newton's law $F = ma$. With the form of the force $F \propto kx^{2n+1}$, you can see that the unit of k has to be $[k] = [F]/L^{2n+1} \rightarrow ML^{-2n}T^{-2}$. Next you can convince yourself that the only combination of m, A, and k that has the units of time (to be identified with the period) is proportional to $\sqrt{m/k}\,A^{-n}$ as we wanted to show.

3. For the given force

$$F = -kx + \frac{a}{x^3} \tag{3.1}$$

We have that the potential is given by

$$V = \frac{kx^2}{2} + \frac{a}{2x^2} + C \tag{3.2}$$

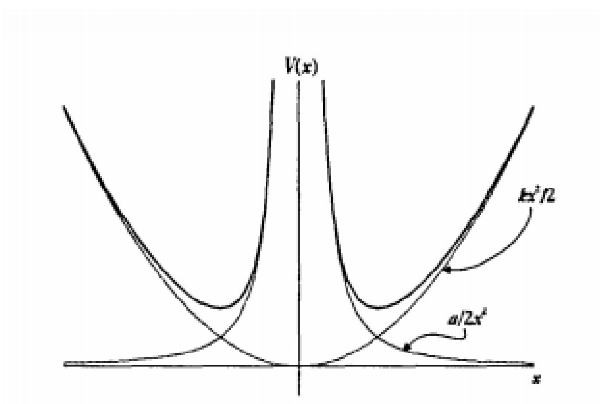

Figure 3: Plot of V, $\frac{kx^2}{2}$, and $\frac{a}{2x^2}$

The equilibrium points of the system are where $F = 0$ or $\frac{dV}{dx} = 0$. Applying these conditions either to (3.1) or (3.2) we find that the equilibrium points are $x = \pm(\frac{a}{k})^{\frac{1}{4}}$. Note that in order to have equilibrium points we must have $a > 0$ and $k > 0$ or $a < 0$ and $k < 0$. The condition to have a stable equilibrium point is that $\frac{d^2V}{dx^2} > 0$ at the given equilibrium point, thus

$$\frac{d^2V(\pm(\frac{a}{k})^{\frac{1}{4}})}{dx^2} = k + 3a(\frac{k}{a}) = 4k \tag{3.3}$$

From here we see that we only get stable equilibrium points if $a > 0$ and $k > 0$. To find the frequencies of oscillation we expand V in a Taylor series around $x_0 = (\frac{a}{k})^{\frac{1}{4}}$

$$V(q) = (ak)^{\frac{1}{2}} + \frac{1}{2} 4q^2 k + \cdots \quad \text{where} \quad q = x - x_0 \tag{3.4}$$

Thus the harmonic equation of motion around the equilibrium point is given by

$$m\ddot{q} = 4kq \tag{3.5}$$

From here wee see that the frequency is $\omega = \sqrt{\frac{4k}{m}}$

4. Only some of the arrows are drawn

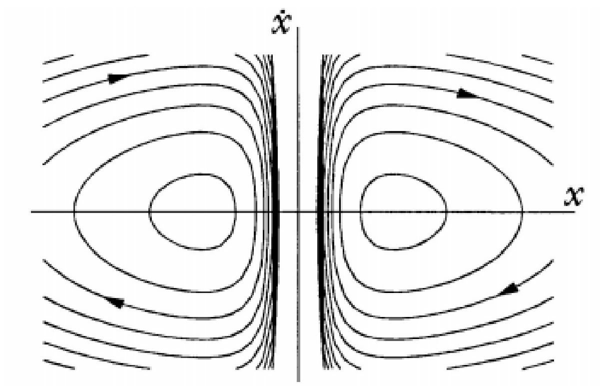

Figure 4: Phase portrait for the system in problem 3