①

Friday, August 25, 2017 3:50 PM

Plan: I. Generalized coordinates

II. Review and derive Lagrange's equation

Last time, we reviewed some problems in Newtonian Mechanics. We used energy methods which are typically easier to work w/ than forces. One reason is that energy is scalar, & so it transforms simply between different coordinates

One example was the Kepler problem → used polar coordinates and also used the symmetry of the problem (rotational symmetry → conservation of angular momentum)

Using energy method → things are more tractable

This is more complicated if one thinks in terms of forces → Vectors transform in much more complicated ways.

Sadly, energy methods are useful for problems which are effectively one dimensional (Kepler)

(2)

Lagrangian dynamics provides one answer

It formulates all dynamics in terms of a scalar L, so that the changes of coordinates are straightforward

Historically, The Lagrangian came from a treatment of constrained systems. It was these constraints that motivated the introduction of novel coordinate systems. However, it was soon found to yield interesting insights into dynamics. For this reason, it is one of the two most preferred languages used in most of modern physics (the other is the Hamiltonian formulation, which we will treat later)

□ Historic context of constrained systems:

Consider N particles, each in 3-D

$$\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_N$$

w/ K constraints

$$f_I(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_N, t) = 0 \qquad I = 1, 2, \ldots, K$$

Examples: motion confined to plane, sphere, or surface

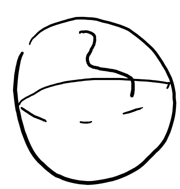
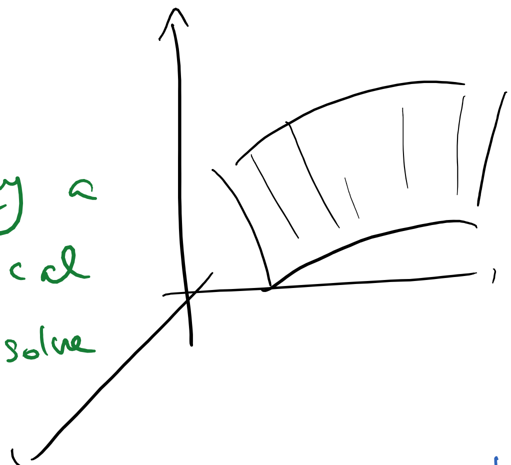


Fig. Particle, on a sphere's surface

③

These constraints are called holonomic because they do not depend on velocities

Let us think of the constraints as defining a $3N-K$ dimensional surface in $3N$-dimensional space



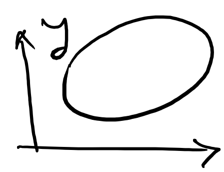
Due to the constraints, this is really a problem in terms of $3N-K$ dynamical variables. Let's imagine we can solve the constraints by expressing $\vec{X}_i$ in terms of $3N-K$ parameters q^α on the constraint hypersurface.

generalized coordinates

I.e. $\vec{X}_i = \vec{X}_i(q^1, \dots, q^{3N-K}; t) \longrightarrow f_I(\vec{x}_1, \dots, \vec{x}_N) = 0$

Example: $N=1$, constraint on an ellipse

$$\sum_{ab=1}^{2} g_{ab} X_a X_b = 1 \qquad \begin{cases} X_1 = x \\ X_2 = y \end{cases}$$



in 2D

Let the normalized eigenvectors of $\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ be $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ & $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

and the corresponding eigenvalues λ and $\tilde{\lambda}$

$$\Longrightarrow \quad g_{ab} = \lambda\, v_a v_b + \tilde{\lambda}\, w_a w_b$$

4

One way to solve these constraints are

$$x_a = \frac{\sin\theta}{\sqrt{\lambda}} v_a + \frac{\cos\theta}{\sqrt{\bar{\lambda}}} w_a$$

check that $\sum g_{ab} x_a x_b = \sin^2\theta + \cos^2\theta = 1$

If g_{ab} is independent of time, the problem is easy to solve

$$\dot{x}_a = \dot{\theta}\left(\frac{\cos\theta}{\sqrt{\lambda}} v_a - \frac{\sin\theta}{\sqrt{\bar{\lambda}}} w_a\right)$$

$$\Rightarrow \quad \frac{2E}{m} = v^2 = \dot{\theta}^2\left(\frac{\cos^2\theta}{\lambda} + \frac{\sin^2\theta}{\bar{\lambda}}\right) \qquad (*)$$

1b energy integral

$$\Rightarrow \quad v(t-t_0) = \pm\int_{\theta_0}^{\theta} d\theta \sqrt{\frac{\cos^2\theta}{\lambda} + \frac{\sin^2\theta}{\bar{\lambda}}}$$

or, to find $\ddot{x}_a$, compute $\frac{d}{dt}(*)$ to find $\ddot{\theta}$

What if the constraint is time dependent?

In the last example, consider $\sum_{a,b} g_{ab}(t)\, x^a x^b = 1$

Can we do things in a similar way?

5

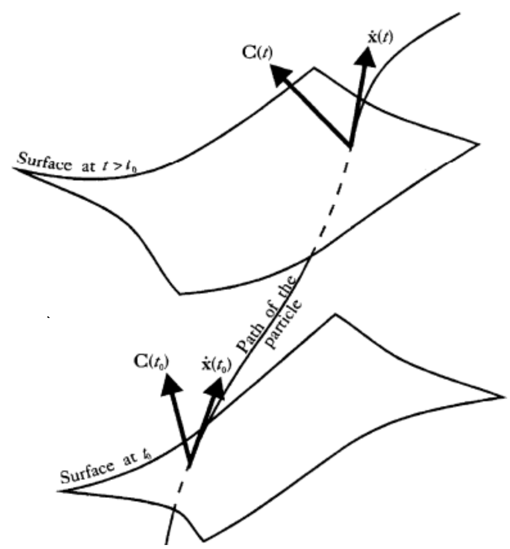
Constraints and Work

First suppose that the surface is not moving.

Since $\vec{C}$ (the constraint force) is normal to surface, it is always perpendicular to the velocity $\dot{\vec{x}}$ and thus $\vec{C} \cdot \dot{\vec{x}} = 0$; The rate at which work is done by The constraint force vanishes

On the other hand if the surface moves the velocity does not have to be tangent to the surface and in general $\vec{C} \cdot \dot{\vec{x}} \neq 0$

i.e., the surface can do work.



Without conservation of energy, should we just give up and go back to Newton's equations in a vector form ???

6

☐ Lagrange equation

Let's start from Newtonian equation of motion

$$\dot{\vec{P}}_i = m\ddot{\vec{x}}_i = \vec{F}_i + \vec{C}_i$$

Recall that $\vec{F}_i$ external forces

$\vec{C}_i$ constrain forces : $\vec{C}_i \perp$ surface

We wish to write an equation of motion in terms of q_α

☆ If we succeed, we have a description of dynamics where constraints are solved at the outset ☆

As noted before, it is simplest to work with scalars like energy ... or work. Let's think about the work that would be done by these forces under a virtual displacement $\delta\vec{x}_i$

$$\text{Work} = W = \vec{F}_i \cdot \delta\vec{x}_i + \vec{C}_i \cdot \delta\vec{x}_i$$

using notation of summing over repeated indices

It is useful to absorb the LHS of Newton's law and write

$$0 = (\vec{F}_i - \dot{\vec{P}}_i) \cdot \delta\vec{x}_i + \vec{C}_i \cdot \delta\vec{x}_i$$

Note that imposing the condition for all $\delta\vec{x}_i$ is equivalent to Newton's law (E.g., can take $\delta\vec{x}_i \propto \hat{x}$ or $\hat{y}$ or $\hat{z}$)

7

Untitled page

Now let's consider $\delta \vec{x}_i$ to be on the constraint surface. the constraint forces being normal to the surface, we have

$$\vec{C}_i \cdot \delta \vec{x}_i = 0 \implies$$

so that the net "virtual work" done by the forces of constraint vanishes at each time

[Virtual work ⟹ no time passes. There is no contribution even if the surface is moving. This is a convention we can make. It just means that we consider $\delta \vec{x}_i$ that preserve the constraints at a given time and keep track of only the corresponding components of the forces.]

$$\implies \quad 0 = \left(\vec{F}_i - \dot{\vec{P}}_i\right) \cdot \delta \vec{x}_i$$

when $\delta \vec{x}_i$ preserves the constraints.

I.e., when $\underbrace{\delta \vec{x}_i}_{t \text{ held constant}} = \left.\frac{\partial \vec{x}_i}{\partial q^\alpha}\right|_t \delta q^\alpha$ (Sum over α assumed)

Let's now rewrite $\vec{F}_i \cdot \delta \vec{x}_i = \vec{F}_i \cdot \frac{\partial \vec{x}_i}{\partial q^\alpha} \delta q^\alpha \equiv Q_\alpha \delta q^\alpha$

where $Q_\alpha \equiv \vec{F}_i \cdot \frac{\partial \vec{x}_i}{\partial q^\alpha}$ is called the generalized force

8

Now let's look at the Kinetic term

$$\sum_i \dot{\vec{p}}_i \cdot \delta \vec{x}_i = \sum_{i,\alpha} \dot{\vec{p}}_i \cdot \frac{\partial \vec{x}_i}{\partial q^\alpha} \delta q^\alpha$$

write these in terms of q^α to solve the constraints

$$\left(\begin{array}{c}\text{"integrate}\\ \text{by parts"}\end{array}\right) = \sum_{i,\alpha} \delta q^\alpha \left[\frac{d}{dt}\left(m_i \dot{\vec{x}}_i \cdot \frac{\partial \vec{x}_i}{\partial q^\alpha} \right) - m_i \dot{\vec{x}}_i \cdot \frac{d}{dt} \frac{\partial \vec{x}_i}{\partial q^\alpha} \right]$$

Let's check this ← expect $\frac{d}{dt}$ & $\frac{\partial}{\partial q^\alpha}$ to commute

$$\vec{x}_i = \vec{x}_i\left(q^1, \ldots, q^{3N-K}; t\right)$$

$$\longrightarrow \frac{\partial \vec{x}_i}{\partial q^\alpha} = \frac{\partial \vec{x}_i}{\partial q^\alpha}\left(q^1, \ldots, q^{3N-K}; t\right)$$

Next $\displaystyle \frac{d}{dt} \frac{\partial \vec{x}_i}{\partial q^\alpha} = \frac{\partial^2 \vec{x}_i}{\partial q^\alpha \partial q^\beta} \dot{q}^\beta + \frac{\partial^2 \vec{x}_i}{\partial q^\alpha \partial t}$

While $\displaystyle \frac{d}{dt} \vec{x}_i = \frac{\partial \vec{x}_i}{\partial q^\beta} \dot{q}^\beta + \frac{\partial \vec{x}_i}{\partial t}$

In general, for a function $g(q^\alpha, t)$.

$$\frac{d}{dt} g(q^\alpha, t) = \left.\frac{\partial g}{\partial q^\alpha}\right|_t \dot{q}^\alpha + \left.\frac{\partial g}{\partial t}\right|_{q^\alpha}$$

& $\displaystyle \frac{\partial}{\partial q^\alpha} \frac{d}{dt} \vec{x}_i = ?$

what is $\displaystyle \frac{\partial \dot{q}^\beta}{\partial q^\alpha}$???

This is up to us! What do we want to mean by $\frac{\partial}{\partial q^\alpha}$ when we act on a function of velocities? Useful to take $\frac{\partial \dot{q}^\beta}{\partial q^\alpha} = 0$

i.e., treat velocities as independent variables.

9

Then $\displaystyle \frac{\partial}{\partial q^\alpha}\bigg|_{t,\dot q^\beta} \frac{d}{dt}\vec{x}_i = \frac{\partial^2 \vec{x}_i}{\partial q^\alpha \partial q^\beta}\dot q^\beta + \frac{\partial^2 \vec{x}_i}{\partial q^\alpha \partial t}$

$$= \frac{d}{dt}\frac{\partial \vec{x}_i}{\partial q^\alpha}\bigg|_t \quad \text{as desired}$$

Note also that $\displaystyle \frac{\partial \dot{\vec{x}}_i}{\partial \dot q^\alpha} = \frac{\partial x_i}{\partial q^\alpha}$ "cancellation of dots"

Putting all the pieces together

$$\sum_i \vec{P}_i \cdot \delta\vec{x}_i = \sum_\alpha \delta q^\alpha \left[\frac{d}{dt}\left(\sum_i m_i \dot{\vec{x}}_i \cdot \frac{\partial \dot{\vec{x}}_i}{\partial \dot q^\alpha}\bigg|_{q^\beta, t} \right) - \sum_i m_i \dot{\vec{x}}_i \cdot \frac{\partial}{\partial q^\alpha}\bigg|_{\dot q^\beta, t} \dot{\vec{x}}_i \right]$$

$$= \sum_\alpha \delta q^\alpha \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q^\alpha}\bigg|_{q^\beta, t} \right) - \frac{\partial T}{\partial q^\alpha}\bigg|_{\dot q^\beta, t} \right]$$

where $T = \sum_i \frac{1}{2} m |\dot{\vec{x}}_i|^2$ is the total kinetic energy

We then find $$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q^\alpha}\right) - \frac{\partial T}{\partial q^\alpha} = Q_\alpha}$$

where we have combined the result with the generalized force

10

We now specialize further. Suppose that all non-constraint forces ($\vec{F}_i$) are conservative $\vec{F}_i = -\vec{\nabla}_i V$

Then $Q_\alpha = \vec{F}_i \cdot \frac{\partial \vec{x}_i}{\partial q^\alpha} = -\vec{\nabla}_i V \cdot \frac{\partial \vec{x}_i}{\partial q^\alpha}$

$$= -\frac{\partial}{\partial q^\alpha} V$$

Since $\frac{\partial V}{\partial \dot{q}^\alpha} = 0$, it is easy to absorb V into the LHS

If $L = T - V$, we have

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right) - \frac{\partial L}{\partial q^\alpha} = 0}$$

Euler-Lagrange equations

Remarkably, Lagrange equations are valid in one coordinate system as in another. This is an obvious advantage over Newton's equation whose form changes drastically going from one coord. system (e.g. Cartesian) to another (spherical).

See the discussion in the book around Eq. 2.35