A new topic: Scattering

Scattering arises in many branches of physics. E.g. particle physics, condensed matter & astrophysics.

It is a standard topic in both classical & quantum mechanics

It is also an important topic for both theorists and expt.ists.

It contains the material that grows out and away from Lagrangian mechanics. It is intended to prepare the reader for topics to be discussed later: chaotic scattering off of hard disks is a particularly understandable example of chaos.

The general framework of scattering theory considers a flux of particles incident on a target, which contains of one or more "scattering centers" → points where the forces become large.

E.g. Consider a flux of hard small spheres of radius r incident on a hard sphere target of radius R.

Quantities of interest are the flux-density or "intensity"

$$I = \frac{\#\text{ of particles}}{(\text{area})\,(\text{time})}$$

In other words, I is the total number of particles crossing a unit area perpendicular to the beam per unit time.

The other quantity of interest is the <u>cross section</u> of the interaction. In the hard sphere case, the cross section per scattering center is the area over which an incident sphere can range & still have it interact with a single target sphere: $\sigma = \pi (R+r)^2$

If there are N targets, arranged sparsely enough that they do not hide each other, the total interaction cross section is $\Sigma = N\sigma$.

The average scattering rate (# of particles scattered from a beam per unit time) is

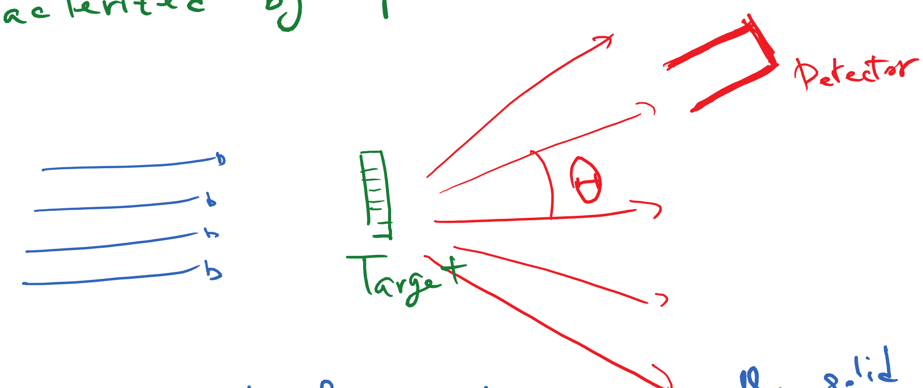
$$S = I\Sigma = IN\sigma$$

The total number of particles scattered over a long experiment (where fluctuations average out) is $S \cdot \underset{\text{time}}{\underset{\sim}{t}}$

The fact that scattering rate is proportional to the flux I and to the number of particles is intuitive.

The only factor in this equation that is intrinsic to the particular interaction is the cross section: For hard spheres, this means the size of the spheres. For arbitrary interactions other than hard core, this equation is taken as the definition of the total cross section σ (per particle) for the interaction.

In an experiment, usually a detector is placed at a location some distance from the target and records particle coming out in a fixed direction. Let that direction be characterized by spherical angles Θ and ϕ



- The detector itself occupies a small solid angle $d\Omega = \sin\Theta \, d\Theta \, d\phi$
- the number dS of particles detected (per unit time) is some fraction of S and depends on the angles Θ and ϕ.

The differential cross section is defined by the equation

$$dS = I \, N \, \sigma(\Theta, \phi) \, d\Omega$$

(In general $\sigma(\theta,\phi)$ also depends on the energy but we do not include that dependence in the notation.)

The total cross section σ_{tot} measures the total number of particles scattered

$$\sigma_{tot} = \int \sigma(\Theta, \phi)\, d\Omega$$

Consider the scattering by a single target $(N=1)$

If the target is symmetric around the beam axis, then $\sigma(\Theta, \phi)$ is independent of ϕ & we can integrate over ϕ to obtain

$$2\pi\, \sigma(\Theta) \sin\Theta\, d\Theta = \frac{dS}{I}$$

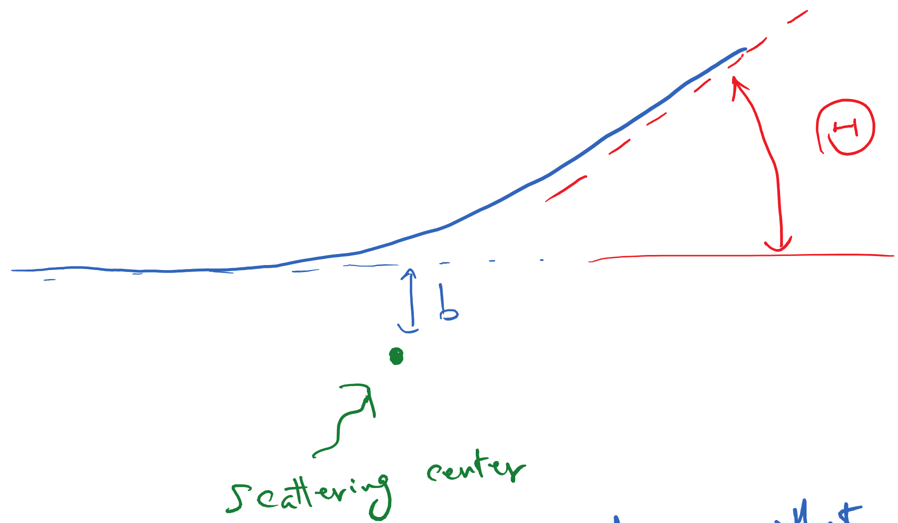
In practice, there are two ways to use this equation

(I) "Scattering theory": Use some $V(r)$ to calculate $\sigma(\theta)$ & predict $\frac{dS}{I}$ for an experiments

(II) "Inverse scattering theory": Measure $\frac{dS}{I}$ & calculate $\sigma(\theta)$. Then use this information to reconstruct $V(r)$

We will have something to say about both cases.

But first a useful concept in such calculations is that of the impact parameter b: the distance the particle would have passed from the scattering center if there were no interaction

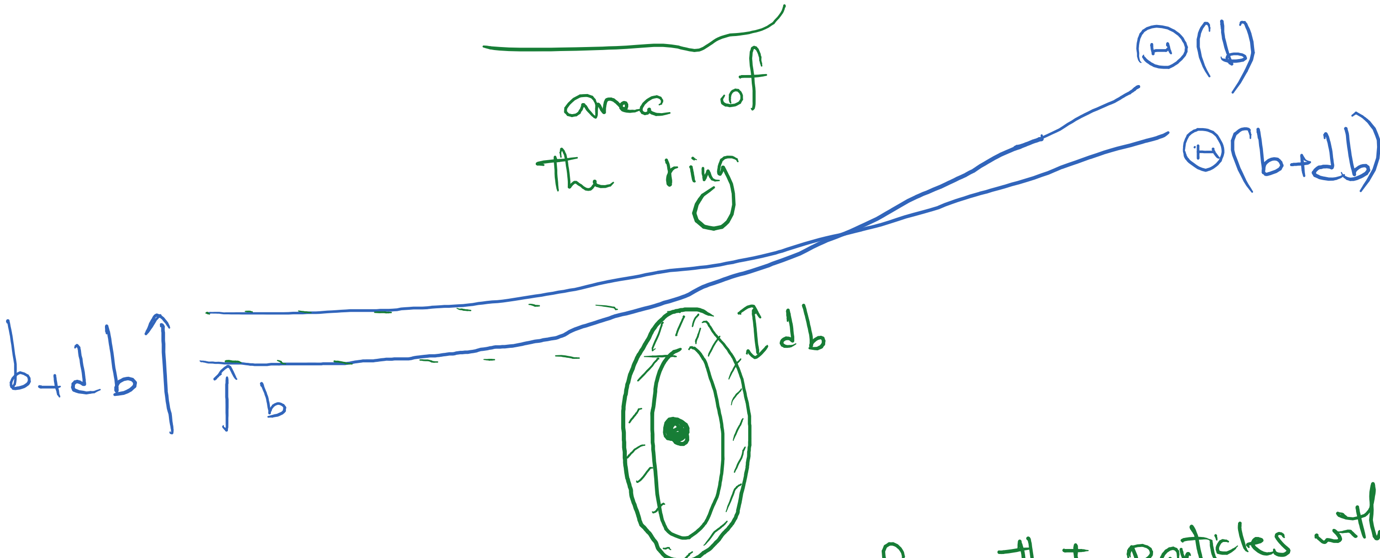


In general, Θ depends on b so that we have

$$\Theta = \Theta(b) \quad \text{or} \quad b = b(\Theta)$$

But both functions can be multi-valued.

$$\underbrace{2\pi\, b(\Theta)\, db}_{\text{area of the ring}} = -2\pi\, \sigma(\Theta)\, \sin\Theta\, d\Theta$$



The minus sign reflects the fact that particles with smaller b penetrate deeper into the potential and encounter stronger forces, and scatter more.

Let us consider scattering of an incident particle of mass m by a fixed scattering center w/ spherically symmetric potential $V(r)$

→ Then there is enough symmetry to solve the dynamics using energy methods.

$$E = \frac{1}{2} m \dot{r}^2 + \frac{\ell^2}{2mr^2} + V(r)$$

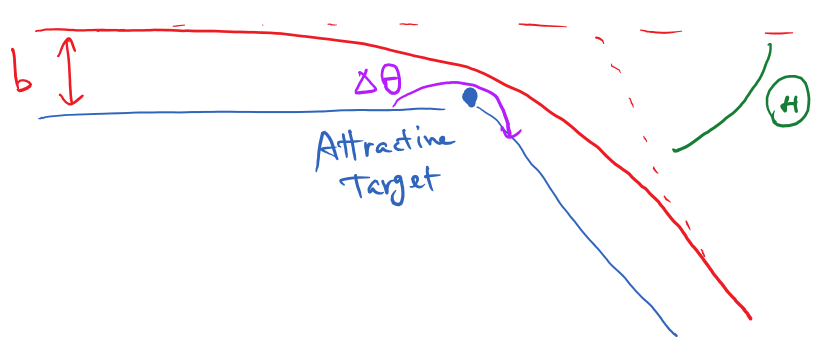
where $\ell = mr^2\dot{\theta}$ is the conserved angular momentum

Thus: $\dot{r} = \frac{dr}{dt} = \frac{d\theta}{dt}\frac{dr}{d\theta} = \frac{\ell}{mr^2}\frac{dr}{d\theta}$

$$\Longrightarrow \quad E = \frac{1}{2}\frac{\ell^2}{mr^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{\ell^2}{2mr^2} + V(r)$$

$$\Longrightarrow \quad \theta(r) = \pm\sqrt{\frac{\ell^2}{2m}} \int_\infty^r \frac{dr}{r^2\sqrt{E - V(r) - \ell^2/2mr^2}}$$

depending on the sign of $\frac{dr}{d\theta}$



natural to take $\theta < 0$ for an attractive potential.

$\Theta = \pi - \Delta\theta$

$\Delta\theta = 2 \times$ change in θ between $r = \infty$ & turning point (= zero of the square root in the denominator)

Let's also write the angular momentum ℓ in terms of the impact parameter b.

Note $E = E(r=\infty) = \frac{1}{2} m v_\infty^2$ $\rightsquigarrow$ $\ell = b\sqrt{2mE}$

and $\ell = \ell(r=\infty) = m v_\infty b$

$$\Longrightarrow \quad \Theta = \pi - 2b \int_{r_0}^{\infty} \frac{dr}{r\sqrt{r^2\left(1-\frac{V(r)}{E}\right) - b^2}}$$

r_0 : turning point

Now that we have $\Theta = \Theta(b)$ we can differentiate and compute the cross section.

$\boxed{\text{Problem Solved}}$

An example: An attractive Coulomb center

$$V(r) = -\frac{\alpha}{r}$$

Some algebra [Section 4.1 of the text book]

$$b(\Theta) = K \cot\frac{\Theta}{2} \qquad K \equiv \frac{\alpha}{m v^2}$$

From $b\, db = -\sigma(\Theta)\, d\Theta$ we find

Rutherford scattering cross section

$$\boxed{\sigma(\Theta) = \frac{\alpha^2}{4 m^2 v^2} \frac{1}{\sin^4\frac{\Theta}{2}}}$$

Inverse scattering

For spherically symmetric potentials, there is a general method to reconstruct $V(r)$ from $\Theta(b)$.

To see how this works, define $y(r) = r\sqrt{1-\frac{V(r)}{E}}$.

It turns out that it is easier to use $\Theta(b)$ to compute $y(r)$ and then invert above to find $V(r)$.

Let's first recall

$$\Theta = \pi - 2b\int_{r_0}^{\infty}\frac{dr}{r\sqrt{\underbrace{r^2\left(1-\frac{V(r)}{E}\right)}_{y^2(r)}-b^2}}$$

$$= \pi - 2b\int_{r_0}^{\infty}\frac{dr}{r\sqrt{y^2(r)-b^2}}$$

Assume that $y(r)$ is invertible, so that we can write $r=r(y)$

Also note that $y(r_0) = b$ because the lower limit is the root r_0 of the square root in the denominator.

Also, $y(r=\infty) = \infty$. Therefore

$$\pi - \Theta = 2b\int_b^{\infty}\frac{r'\,dy}{r\sqrt{y^2-b^2}} = 2b\int\frac{dy}{\sqrt{y^2-b^2}}\frac{d\ln r}{dy}$$

On the other hand,

$$\pi = 2b \int \frac{dy}{y\sqrt{y^2-b^2}} = 2b \int_b^\infty \frac{dy}{\sqrt{y^2-b^2}} \frac{d}{dy} \ln y$$

This is the result for $V=0$, where $\Theta = 0$ identically (and $r = y$)

Combining things, we find

$$\Theta(b) = 2b \int_b^\infty \frac{dy}{\sqrt{y^2-b^2}} \frac{d}{dy} \ln \frac{y}{r(y)}$$

Now it turns out to be useful to define

$$T(y) = \frac{1}{\pi} \int_y^\infty \frac{db}{\sqrt{b^2-y^2}} \Theta(b) \qquad (*)$$

To understand the relation between $T(y)$ & $y(r)$, note that

$$T(y) = \frac{1}{\pi} \int_y^\infty \frac{2b}{\sqrt{b^2-y^2}} \underbrace{\left[\int_b^\infty \frac{du}{\sqrt{u^2-b^2}} \frac{d}{du} \log \frac{u}{r(u)} \right]}_{\Theta(b)} db$$

Perform this integral by reversing order of integrations.

Note integration region is $u > b > y$

So $$T(y) = \frac{1}{\pi} \int_y^\infty \left\{ \frac{d}{du} \ln \frac{u}{r(u)} \right\} du \underbrace{\int_y^u \frac{2b\, db}{\sqrt{(b^2-y^2)(u^2-b^2)}}}_{=\pi}$$

independent of y and u

Thus, $T(y) = -\log \frac{y}{r(y)}$

$\rightsquigarrow \quad r(y) = y \exp\left(T(y)\right) \qquad (**)$

But $\frac{y}{r} = \sqrt{1 - \frac{V(r)}{E}}$

Therefore, $V(r) = E\left(1 - e^{-2T}\right) \qquad (***)$

$\Longrightarrow$ 1) Measure $\Theta(b)$ at any fixed E

2) Compute $T(y)$ (maybe numerically) using $(*)$

3) Invert $(**)$ (numerically) to find $y(r)$

4) Compute $V(r)$ using $(***)$

Cool, eh?