Canonical Transformation

Introduction: Recall one of the basic points of the Hamiltonian formalism is that if q^0 is cyclic $(\partial L/\partial q^0 = 0)$, Then p_0 is conserved. As a result, $\dot{p}_0 = \partial H/\partial q^0 = 0$ & H is also independent of q^0. Therefore q^0 and p_0 decouple from (q^β, p_β) for $\beta \neq 0$ and The dynamics has been "reduced" by 2 variables.

Wouldn't it be cool if one could find n cyclic coordinates and reduce the problem all the way down to zero variables?

(This line of ideas will be useful in Hamilton-Jacobi theory.)

But note this means $H = H(p_\alpha)$ only (no q's) which is almost never true to start with. Thus, to make in any generality, we will have to perform a coordinate transformation on phase space to define new $P_\alpha = P_\alpha(q,p)$ and thus new $Q_\alpha = Q_\alpha(q,p)$.

So let's warm up by thinking about such coordinate transformations.

Canonical transformations & generating functions

To simplify our lives, I only consider coordinate transformation that are NOT explicitly time-dependent.

$$Q^\alpha = Q^\alpha(q,p) \qquad \text{and} \qquad P_\alpha = P_\alpha(q,p)$$

But we want to preserve the structure of Hamilton's equations. Recall that

$$\dot{q}^\alpha = \{q^\alpha, H\}^{q,p} \quad , \quad \dot{p}^\alpha = \{p_\alpha, H\}^{q,p}$$

So we want

$$\dot{Q}^\beta = \{Q^\beta, H\}^{Q,P} \quad , \quad \dot{P}_\beta = \{P_\beta, H\}^{Q,P}$$

The superscripts denote phase space coordinates that go into the definition of Poisson Brackets:

$$\{A,B\}^{q,p} = \frac{\partial A}{\partial q^\alpha}\frac{\partial B}{\partial p_\alpha} - \frac{\partial A}{\partial p_\alpha}\frac{\partial B}{\partial q^\alpha}$$

for $A = A(q,p)$ and $B = B(q,p)$

and

$$\{A,B\}^{Q,P} = \frac{\partial A}{\partial Q^\alpha}\frac{\partial B}{\partial P_\alpha} - \frac{\partial A}{\partial P_\alpha}\frac{\partial B}{\partial Q^\alpha}$$

where $A = A(q(Q,P), p(Q,P))$ and $B = B(q(Q,P), p(Q,P))$

Aside:

For time-dependent coordinate transformations, canonical transformations are defined in a similar way except that the new coordinates (Q,P) evolve under a different Hamiltonian K that is not neccessarily the same as H.

Now let's find the condition on (Q,P) for a canonical transformation. Since we assumed that $Q = Q(q,p)$ and $P = P(q,p)$ do not explicitly depend on time, we have as usual

$$\dot{Q}^\alpha = \{Q^\alpha, H\}^{q,p}, \qquad \dot{P}_\alpha = \{P_\alpha, H\}^{q,p}$$

Comparing these equations to our definition of canonical transformations, we see that for $Q = Q(q,p)$ and $P = P(q,p)$ to define a canonical transformation, (Q,P) and (q,p) should define the same Poisson Bracket.

I.e., for any A and B, we should have

$$\{A,B\}^{q,p} = \{A,B\}^{Q,P}$$

This condition may seem abstract but to show that it is satisfied, we only need to show it for A & B being the phase space coordinates. For example, let's choose $A = Q^\alpha$ and $B = P_\beta$. Then,

$$\{Q^\alpha, P_\beta\}^{q,p} = \{Q^\alpha, P_\beta\}^{Q,P} = \delta^\alpha_\beta$$

and similarly $\{Q^\alpha, Q^\beta\}^{q,p} = 0$ and $\{P_\alpha, P_\beta\}^{q,p} = 0$

In short, canonical transformations are changes of coordinates of phase space $(q,p) \to (Q,P)$ with $Q^\alpha = Q^\alpha(q,p)$ and $P_\alpha = P_\alpha(q,p)$ such that

$$\{Q^\alpha, P_\beta\}^{q,p} = \delta^\alpha_\beta, \quad \{Q^\alpha, Q^\beta\}^{q,p} = 0, \quad \{P_\alpha, P_\beta\}^{q,p} = 0$$

For such transformations, the Hamilton's equations are unchanged, i.e.

$$\dot{A} = \{A, H\}^{Q,P} + \frac{\partial A}{\partial t}$$

for all $A = A(q,p,t)$.

Example (1): $Q = p$, $P = -q$

then $\{Q, P\} = 1$ canonical trasf. ✓

Example (2): $Q = \frac{1}{2} p^2$, $P = -\frac{q}{p}$

Then again $\{Q, P\} = 1$ Canonical transf. ✓

Let's look at canonical transformation from the Hamilton's variational principle. We have shown that Hamilton's equation of motion follow from

$$\delta \int_{t_0}^{t_1} \left(p_\alpha \dot{q}^\alpha - H(q,p) \right) dt = 0 . \qquad (*)$$

At the same time, Hamilton's equations are also satisfied in terms of Q & P which are related to q & p by a canonical transformation, so we must have

$$\delta \int_{t_0}^{t_1} \left(P_\alpha \dot{Q}^\alpha - H \right) dt = 0 \qquad (**)$$

with the Hamiltonian $H = H\left(q(Q,P), p(Q,P)\right)$ is now a function of Q and P.

The fact that both $(*)$ and $(**)$ hold of course doesn't mean that the integrands are the same because we have the freedom in adding a total time derivative dF/dt to the integrand without changing the equations of motion. Therefore,

$$p_\alpha \dot{q}^\alpha - \cancel{H} = P_\alpha \dot{Q}^\alpha - \cancel{H} + \frac{dF}{dt} \qquad (***)$$

where $F = F(q,p)$.

Two quick points:

(i) in general, F can also explicitly depend on time but for canonical transformations that do not explicitly depend on time, it turns out that F is not explicitly time-dependent

(ii) Notice that we have taken $F = F(q,P)$ a function of both q and P, and not just q as opposed to what we did for Lagrangians. This is because in Hamilton's version of variational principle, the variation of P is indep. of those of q. In particular, the variation of p at the end points is unimportant and can be set to $\delta p = 0$. With this choice, the variation of $\delta \int dt \frac{dF}{dt} = \delta F \Big|_{t_0}^{t_1}$ is zero since both δq and δp are zero at the endpoints.

The Hamiltonian in equation $(***)$ can be dropped and it can be written in a suggestive form

$$p_\alpha dq^\alpha - P_\alpha dQ^\alpha = dF \qquad (****)$$

The function $F = F(q,p)$ tell us how the two sets of coordinates are related to each other, and is called the generating function.

Example: $Q = p$, $\quad P = -q$

$$p\,dq - P\,dQ = p\,dq + q\,dp = d(qp)$$

$\rightsquigarrow \; F = qp$

Another example: $Q = \frac{1}{2}p^2$, $\quad P = -q/p$

also gives $F = qp$

The morale of these examples are that F does not uniquely define the canonical transformation but still leaves some ambiguity. But it turns out that it is possible to fully specify a canonical transformation.

Note that F is a function of the phase space with $2n$ coordinates. And, it can be written in terms of any $2n$ variables.

For example, we can take F as a function of 1 old variable and one new variable. For example, let's take

$$F = F(q, Q)$$

In the example we discussed before, we defined $Q = p$ or $\frac{1}{2}p^2$ so q and Q can be used instead of q and p to specify a point on the phase space. With this transformation, all functions can be written as a function of (q, Q). Equation $(\ast\ast\ast\ast)$ can be then written as

$$dF = \frac{\partial F}{\partial q^\alpha} dq^\alpha + \frac{\partial F}{\partial Q^\alpha} dQ^\alpha = p_\alpha dq^\alpha - P_\alpha dQ^\alpha$$

This gives p_α and P_α immediately as a function of (q, Q):

$$p_\alpha = \frac{\partial F}{\partial q^\alpha}, \qquad P_\alpha = -\frac{\partial F}{\partial Q^\alpha}$$

Example: Let's consider $F = qQ$

we then have

$$p = \frac{\partial F}{\partial q} = Q$$

and $$P = -\frac{\partial F}{\partial Q} = -Q$$