PHY820/422 HW #2 — Due Monday 9/18/17 @ 5pm
Lagrangians: Formal proofs and some applications

1. José and Saletan, Chapter 2, problem 3

   Prove that if L' is defined by (2.36), then Eq.(2.28) implies Eq.(2.37).

   Hints: You can use the following two facts that we showed in class (if you are not sure, try to prove them again starting from Eq. (2.35) of the textbook):

   (I) d/dt and $\partial/\partial q^\alpha$ commute. In particular, we have (now formulated in primed coordinates)

   $$\frac{\partial}{\partial q'^\gamma}\dot{q}^\alpha = \frac{d}{dt}\left(\frac{\partial q^\alpha}{\partial q'^\gamma}\right) \tag{1}$$

   (II) Cancellation of dots in generalized coordinates:

   $$\frac{\partial \dot{q}^\alpha}{\partial \dot{q}'^\gamma} = \frac{\partial q^\alpha}{\partial q'^\gamma} \tag{2}$$

2. Goldstein (Ed. 2), Chapter 1, problem 14

   If L is a Lagrangian for a system satisfying Lagrange's equations, show by direct substitution that

   $$L' = L + \frac{dF(q^\alpha, t)}{dt}$$

   also satisfies Lagrange's equations where F is any arbitrary, but differentiable, function of its arguments.

   Hint: Use another result that we derived in class. For any function $F = F(q^\alpha, t)$, we have

   $$\frac{dF(q^\alpha, t)}{dt} = \frac{\partial F}{\partial q^\alpha}\,\dot{q}^\alpha + \frac{\partial F}{\partial t} \tag{3}$$

   Don't forget that repeated indices mean a summation (in this case, over α).

3. Goldstein (Ed. 2), Chapter 1, problem 16

   A Lagrangian for a particular physical system can be written as

   $$L' = \frac{m}{2}(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2) - \frac{K}{2}(ax^2 + 2bxy + cy^2)$$

   where a, b, and c are arbitrary constants but subject to the condition that $b^2 - ac \neq 0$. What is the physical system described by the above Lagrangian?

4. José and Saletan, Chapter 2, problem 9

   A double plane pendulum consists of a simple pendulum (mass m_1, length l_1) with another simple pendulum (mass m_2, length l_2) suspended from m_1, both constrained to move in the same vertical plane.

   (a) Write down the Lagrangian of this system in suitable coordinates.

   (b) Derive the Euler-Lagrange equations.

5. José and Saletan, Chapter 2, problem 10

   Consider a stretchable plane pendulum, that is, a mass m suspended from a spring of spring constant k and unstretched length l, constrained to move in a vertical plane. Define the angle from the vertical axis to be θ and the length stretch to be s. Write down the Lagrangian and obtain the Euler-Lagrange equations. Is energy to conserved? Why or why not? **Bonus point:** Use the method (to be covered on Wednesday) to obtain a constant of motion, and check if it is the same as or different from energy.

Solutions

1. By generalizing Eq.(2.36) in José and Saletan we have that L' is defined as

$$L'(q'^{\alpha}, \dot{q}'^{\alpha}, t) \equiv L(q^{\alpha}(q'^{\alpha}, t), \dot{q}^{\alpha}(q'^{\alpha}, \dot{q}'^{\alpha}, t), t) \tag{1.1}$$

   where q^{α} and q'^{α} are both α-dimensional vectors of generalized coordinates. Before proceeding any further, we make the assumption that the Jacobian determinant of the transformation between the prime and unprimed variables is non-zero, thus in this way we guarantee that if $q = q(q', t)$ and $\dot{q} = \dot{q}(q', \dot{q}', t)$ then there exists an inverse mapping $q' = q'(q, t)$ and $\dot{q}' = \dot{q}'(q, \dot{q}, t)$. From Eq.(2.28) we have that the equations of motion of L are given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^{\alpha}} - \frac{\partial L}{\partial q^{\alpha}} = 0 \tag{1.2}$$

   Note that by applying the chain rule to the partial derivatives of L we obtain (double greek indices indicate a sum)

$$\frac{\partial L}{\partial q^{\alpha}} = \frac{\partial L}{\partial q'^{\mu}}\frac{\partial q'^{\mu}}{\partial q^{\alpha}} + \frac{\partial L}{\partial \dot{q}'^{\mu}}\frac{\partial \dot{q}'^{\mu}}{\partial q^{\alpha}} \tag{1.3}$$

$$\frac{\partial L}{\partial \dot{q}^{\alpha}} = \frac{\partial L'}{\partial \dot{q}'^{\mu}}\frac{\partial \dot{q}'^{\mu}}{\partial \dot{q}^{\alpha}} = \frac{\partial L'}{\partial \dot{q}'^{\mu}}\frac{\partial q'^{\mu}}{\partial q^{\alpha}} \tag{1.4}$$

   Here we used the second hint to eliminate the time derivatives on the generalized coordinates in (1.4). By substituting both equations (1.3) and (1.4) into equation (1.2) we obtain

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}'^{\mu}}\frac{\partial q'^{\mu}}{\partial q^{\alpha}}\right) - \frac{\partial L}{\partial q'^{\mu}}\frac{\partial q'^{\mu}}{\partial q^{\alpha}} - \frac{\partial L}{\partial \dot{q}'^{\mu}}\frac{\partial \dot{q}'^{\mu}}{\partial q^{\alpha}} = 0 \tag{1.5}$$

By expanding the time derivative term we obtain

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}'^{\mu}}\right)\left(\frac{\partial q'^{\mu}}{\partial q^{\alpha}}\right) + \frac{\partial L}{\partial \dot{q}'^{\mu}}\frac{\partial \dot{q}'^{\mu}}{\partial q^{\alpha}} - \frac{\partial L}{\partial q'^{\mu}}\frac{\partial q'^{\mu}}{\partial q^{\alpha}} - \frac{\partial L}{\partial \dot{q}'^{\mu}}\frac{\partial \dot{q}'^{\mu}}{\partial q^{\alpha}} = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}'^{\mu}}\right)\left(\frac{\partial q'^{\mu}}{\partial q^{\alpha}}\right) - \frac{\partial L}{\partial q'^{\mu}}\frac{\partial q'^{\mu}}{\partial q^{\alpha}} = \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}'^{\mu}}\right) - \frac{\partial L}{\partial q'^{\mu}}\right)\left(\frac{\partial q'^{\mu}}{\partial q^{\alpha}}\right) = 0 \tag{1.6}$$

From equation (1.6) we see that either $\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}'^{\mu}}\right) - \frac{\partial L}{\partial q'^{\mu}}\right) = 0\ \forall\mu$, or $\left(\frac{\partial q'^{\mu}}{\partial q^{\alpha}}\right) = 0\ \forall\mu$. We know that the latter cannot be true since it would contradict the assumption we made about the Jacobian being non-zero.

$$\therefore \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}'^{\mu}}\right) - \frac{\partial L}{\partial q'^{\mu}}\right) = \left(\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}'^{\mu}}\right) - \frac{\partial L'}{\partial q'^{\mu}}\right) = 0 \quad \forall\mu \tag{1.7}$$

2. Given that L' is defined by

$$L' = L + \frac{dF(q^{\alpha}, t)}{dt} \tag{2.1}$$

where L satisfies the Lagrange equations of motion, then we have that

$$\frac{\partial L'}{\partial q^{\mu}} = \frac{\partial L}{\partial q^{\mu}} + \frac{\partial}{\partial q^{\mu}}\frac{dF}{dt} = \frac{\partial L}{\partial q^{\mu}} + \frac{\partial}{\partial q^{\mu}}\left(\frac{\partial F}{\partial q^{\alpha}}\dot{q}^{\alpha} + \frac{\partial F}{\partial t}\right) \tag{2.2}$$

$$\frac{\partial L'}{\partial \dot{q}^{\mu}} = \frac{\partial L}{\partial \dot{q}^{\mu}} + \frac{\partial}{\partial \dot{q}^{\mu}}\frac{dF}{dt} = \frac{\partial L}{\partial \dot{q}^{\mu}} + \frac{\partial}{\partial \dot{q}^{\mu}}\left(\frac{\partial F}{\partial q^{\alpha}}\dot{q}^{\alpha} + \frac{\partial F}{\partial t}\right) \tag{2.3}$$

Both equations (2.2) and (2.3) use the identity $\frac{dF}{dt} = \frac{\partial F}{\partial q^{\alpha}}\dot{q}^{\alpha} + \frac{\partial F}{\partial t}$. The last term in (2.3) can be further simplified by noting that $\frac{\partial F}{\partial \dot{q}^{\mu}} = 0 \quad \forall\mu$ since F is a function of q^{α} and t only, and $\frac{\partial F}{\partial q^{\alpha}}\frac{\partial \dot{q}^{\alpha}}{\partial \dot{q}^{\mu}} = \frac{\partial F}{\partial q^{\mu}}$ since $\frac{\partial \dot{q}^{\alpha}}{\partial \dot{q}^{\mu}} = \delta_{\alpha\mu}$ is the Kronecker delta. Hence (2.3) is simplified into

$$\frac{\partial L'}{\partial \dot{q}^{\mu}} = \frac{\partial L}{\partial \dot{q}^{\mu}} + \frac{\partial F}{\partial q^{\mu}} \tag{2.4}$$

By subtracting (2.2) to the time derivative of (2.4) we obtain

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}^{\mu}} - \frac{\partial L'}{\partial q^{\mu}} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^{\mu}} + \frac{d}{dt}\frac{\partial F}{\partial q^{\mu}} - \left(\frac{\partial L}{\partial q^{\mu}} + \frac{\partial}{\partial q^{\mu}}\left(\frac{\partial F}{\partial q^{\alpha}}\dot{q}^{\alpha} + \frac{\partial F}{\partial t}\right)\right) \tag{2.5}$$

Note that the first and third terms in (2.5) cancel since this is the Lagrange equation of motion of L with respect to the generalized coordinate q^μ, and we know satisfies the Lagrange equations of motion. Furthermore, by interchanging the orders of derivatives for the second term in (2.5) we obtain

$$\frac{d}{dt}\frac{\partial F}{\partial q^\mu} = \frac{\partial}{\partial q^\mu}\frac{dF}{dt} = \frac{\partial}{\partial q^\mu}(\frac{\partial F}{\partial q^\alpha}\dot{q}^\alpha + \frac{\partial F}{\partial t}) \tag{2.6}$$

Equation (2.6) is equal to the last two terms in equation (2.5) with opposite sign thus canceling each other

$$\therefore \frac{d}{dt}\frac{\partial L'}{\partial \dot{q}^\mu} - \frac{\partial L'}{\partial q^\mu} = 0 \quad \forall \mu \tag{2.7}$$

3. The Lagrange equations of motion for this system are

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m(a\ddot{x} + b\ddot{y}) + K(ax + by) = 0 \tag{3.1}$$

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{y}} - \frac{\partial L'}{\partial y} = m(b\ddot{x} + c\ddot{y}) + K(bx + cy) = 0 \tag{3.2}$$

The system being described by the Lagrangian is a two dimensional harmonic oscillator, although this is not evident from both equations (3.1) and (3.2). If we define the following two new variables

$$u_1 = ax + by \tag{3.3}$$

$$u_2 = bx + cy \tag{3.4}$$

then the Jacobian matrix for this transformation is given by

$$\begin{bmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \tag{3.5}$$

As we can see, the determinant of the Jacobian matrix in (3.5) is equal to $ac - b^2$. From our assumption that $b^2 - ac \neq 0$ we conclude that the transformation is invertible. This in turn allows us to use the result of problem 1, $\exists L(u_1, u_2)$ such that it is physically equivalent to $L'(x, y)$ with its equations of motion given by

$$m\ddot{u}_1 + Ku_1 = 0 \tag{3.6}$$

$$m\ddot{u}_2 + Ku_2 = 0 \tag{3.7}$$

If $b^2 = ac$ then it can be seen that both Euler Lagrange equations of motion for x and y are equal meaning that the physical system is a one dimensional harmonic oscillator.

4. (a) First we start this problem by finding suitable coordinates that describe the motions of m_1 and m_2. Note that the position of m_1 can be completely determined by the angle θ_1 by letting

$$x_1(t) = l_1 \sin(\theta_1(t)) \tag{4.1}$$

$$y_1(t) = l_1 \cos(\theta_2(t)) \tag{4.2}$$

The position of m_2 is dependent from the position of m_1, thus we have that

$$x_2(t) = x_1(t) + l_2 \sin(\theta_2(t)) = l_1 \sin(\theta_1(t)) + l_2 \sin(\theta_2(t)) \tag{4.3}$$

$$y_2(t) = y_1(t) + l_2 \cos(\theta_2(t)) = l_1 \cos(\theta_1(t)) + l_2 \cos(\theta_2(t)) \tag{4.4}$$

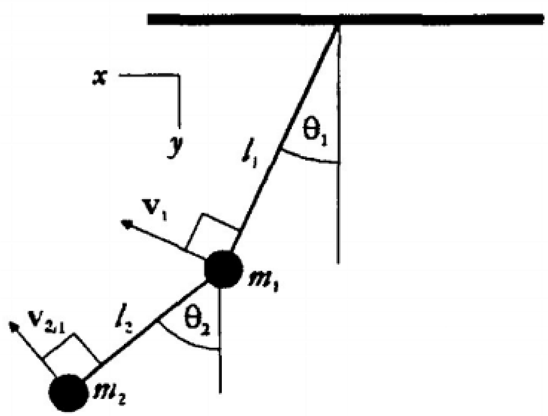

Figure 1: Double pendulum system

The kinetic energy of $m1$ and m_2 are given by

$$T_1 = \frac{m_1}{2}(\dot{x_1}^2 + \dot{y_1}^2) = \frac{m_1}{2}(l_1{}^2\dot{\theta_1}^2) \tag{4.5}$$

$$T_2 = \frac{m_2}{2}(\dot{x_2}^2 + \dot{y_2}^2) = \frac{m_2}{2}(l_1{}^2\dot{\theta_1}^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) + l_2^2\dot{\theta_2}^2) \tag{4.6}$$

The potential energy of m_1 and m_2 are given by

$$V_1 = -m_1gl_1\cos(\theta_1) \tag{4.7}$$

$$V_2 = -(m_2g(l_1\cos(\theta_1) + l_2\cos(\theta_2))) \tag{4.8}$$

The kinetic and potential energy of the system are then given by $T = T_1 + T_2$ and $V = V_1 + V_2$, and the Lagrangian of the system is

$$L = \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta_1}^2+\frac{m_2}{2}(l_2^2\dot{\theta_2}^2+2l_1l_2\dot{\theta_1}\dot{\theta_2}\cos(\theta_1-\theta_2))+(m_1+m_2)gl_1\cos(\theta_1)+m_2gl_2\cos(\theta_2)$$

(b) The Euler Lagrange equations of motion for θ_1 and θ_2 are obtained from

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta_1}} - \frac{\partial L}{\partial\theta_1} = 0 \tag{4.9}$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta_2}} - \frac{\partial L}{\partial\theta_2} = 0 \tag{4.10}$$

Hence the equations are given by

$$(m_1+m_2)l_1(l_1\ddot{\theta}_1+g\sin(\theta_1))+l_1l_2m_2(\ddot{\theta}_2\cos(\theta_1-\theta_2)+\dot{\theta_2}^2\sin(\theta_1-\theta_2)) = 0 \tag{4.11}$$

$$m_2l_2(l_2\ddot{\theta}_2 + g\sin(\theta_2) + l_1(\ddot{\theta}_1\cos(\theta_1-\theta_2) - \dot{\theta_1}^2\sin(\theta_1-\theta_2))) = 0 \tag{4.12}$$

5. Let the mass m be suspended at a distance $l = l_0 + s$ of the spring, where l_0 is the equilibrium length of the spring and s is the distance stretched of the spring due to the mass m. Then we have that the potential energy of the system is given by

$$V(s,\theta) = \frac{1}{2}ks^2 + mg(l_0+s)(1-\cos(\theta)) \tag{5.1}$$

Note that the term $1-\cos(\theta)$ in the gravitational potential energy takes into account the fact that there is no gravitational potential energy when $\theta = 0$. The kinetic energy of the system is given by

$$T = \frac{1}{2}m(\dot{s}^2 + (l_0+s)^2\dot{\theta}^2) \tag{5.2}$$

Hence, the Lagrangian of the system is given by

$$L = \frac{m}{2}(\dot{s}^2 + (l_0+s)^2\dot{\theta}^2) - \frac{k}{2}s^2 - mg(l_0+s)(1-cos(\theta)) \tag{5.3}$$

The Lagrange equations of motion for s and θ are then given by

$$m\ddot{s} + ks - m(l_0+s)\dot{\theta}^2 + mg(1-\cos(\theta)) = 0 \tag{5.4}$$

$$m(l_0+s)(2\dot{s}\dot{\theta} + (l_0+s)\ddot{\theta}) + mg(l_0+s)\sin(\theta) = 0 \tag{5.5}$$