PHY820/422 HW #3 — Due Monday 9/25/17 @ 5pm
Lagrangians: More practice

1. José and Saletan, Chapter 2, problem 11 (simplified version)

   A wire is bent into the shape given by $z = A\rho^2$ and oriented vertically opening upward, in a uniform gravitational field g. Here, z and ρ are defined in cylindrical coordinates. The wire rotates at a constant angular velocity Ω about the vertical (z) axis, and a bead of mass m is free to slide on it without friction.

   (a) Find the equilibrium height of the bead on the wire.

   (b) Find the frequency of small vibrations about the equilibrium position(s).

   You are excepted to tackle part (b) in two different ways. (i) Expand the Euler-Lagrange equation around the equilibrium point(s) (ii) Construct a conserved quantity from the Lagrangian, and expand it around the equilibrium point(s) to find the frequency of small oscillations.

   Bonus point: Solve the same problem for a wire bent into the shaped $z = A\rho^n$ for a positive n.

2. Goldstein (Ed. 2), Chapter 1, Problem 18

   A particle of mass m moves in one dimension such that it has the Lagrangian

$$L = \frac{m^2\dot{x}^4}{12} + m\dot{x}^2 V(x) - V(x)^2$$

   where V is some differentiable function of x. Find the equation of motion for $x(t)$ and describe the physical nature of the system on the basis of this equation.

3. Jose and Saletan, Chapter 3, Problem 12(a).

   Describe the motion of the Lagrangian $L = \dot{q}_1\dot{q}_2 - \omega^2 q_1 q_2$. Describe the physical motion and write another Lagrangian (L') that produces the same equations of motion. Is it possible to relate the two Lagrangians by a total time derivative, i.e., $L - L' = dF/dt$ for a function $F(q_1, q_2, t)$?

   Hint: Read section 2.2.2 of the textbook.

SOLUTIONS

1. (a) Given that the particle is constrained to move in the wire with shape $z = A\rho^2$, we have that the kinetic and potential energy of the particle are given by

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{m}{2}(\dot{\rho}^2 + \rho^2\Omega^2 + 4A^2\rho^2\dot{\rho}^2) \tag{1.1}$$

$$V = mgz = mgA\rho^2 \tag{1.2}$$

where we have used the cylindrical coordinates defined as

$$x = \rho\cos(\Omega t), \quad y = \rho\sin(\Omega t), \quad z = z(\rho) = A\rho^2 \tag{1.3}$$

The Lagrangian for the system is then given by

$$L = T - V = \frac{m}{2}(\dot{\rho}^2 + \rho^2\Omega^2 + 4A^2\rho^2\dot{\rho}^2) - mgA\rho^2 \tag{1.4}$$

which yields the following Euler-Lagrange equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\rho}} - \frac{\partial L}{\partial\rho} = m\ddot{\rho} + 4mA^2(\rho^2\ddot{\rho} + 2\rho\dot{\rho}^2) - m\rho\Omega^2 - 4mA^2\rho\dot{\rho}^2 + 2mgA\rho = 0$$

$$= m[(1 + 4A^2\rho^2)\ddot{\rho} + 4A^2\rho\dot{\rho}^2 - \rho\Omega^2 + 2gA\rho] = 0 \tag{1.5}$$

The system will be at a point of equilibrium if both $\dot{\rho} = 0$ and $\ddot{\rho} = 0$, thus from (1.5) we have that at equilibrium the system must satisfy

$$-\rho\Omega^2 + 2gA\rho = 0 \tag{1.6}$$

Thus from (1.6) we can conclude that either $\rho = 0$ is an equilibrium point or any ρ is an equilibrium point only if the wire is rotating with the constant angular velocity $\Omega = \sqrt{2gA}$.

(b) Let ρ_0 be an equilibrium point of the bead, then by expanding (1.4) in a Taylor series around ρ_0 we obtain

$$L = \sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^nL(\rho_0)}{dx^n}(\rho - \rho_0)^n$$

$$= [\frac{m}{2}(\dot{\rho}^2 + \rho_0^2\Omega^2 + 4A^2\rho_0^2\dot{\rho}^2) - mgA\rho_0^2]$$

$$+m(\rho_0\Omega^2 + 4A^2\rho_0\dot{\rho}^2 - 2gA\rho_0)(\rho - \rho_0) + \frac{m}{2}(\Omega^2 + 4A^2\dot{\rho}^2 - 2gA)(\rho - \rho_0)^2 + \cdots$$

The above expression can be further simplified by noticing that for the first and second terms $\rho_0\Omega^2 - 2gA\rho_0 = 0$ since this is our requirement for an equilibrium point as we showed in (1.6). Furthermore since we are considering small vibrations we can then omit any terms in the expansion that have a higher than quadratic dependence between the terms ρ and $\dot{\rho}$. Thus we can then ignore the terms $4mA^2\rho_0\dot{\rho}^2(\rho-\rho_0)$ in the second term of the expansion and $2mA^2\dot{\rho}^2(\rho-\rho_0)^2$ in the third term of the expansion. We can now approximate the Lagrangian for small vibrations as

$$\begin{aligned} L &\approx \frac{m}{2}[\dot{\rho}^2+\rho_0^2\Omega^2+4A^2\rho_0^2\dot{\rho}^2-2gA\rho_0^2+(\Omega^2-2gA)(\rho-\rho_0)^2] \\ &= \frac{m}{2}[\dot{q}^2+\rho_0^2\Omega^2+4A^2\rho_0^2\dot{q}^2-2gA\rho_0^2+(\Omega^2-2gA)q^2] \end{aligned} \tag{1.7}$$

where we have defined $q=\rho-\rho_0$. The Euler-Lagrange equation of motion is then given by

$$(1+4A^2\rho_0^2)\ddot{q}-(\Omega^2-2gA)q=0 \tag{1.8}$$

If $\rho_0=0$ then the equation of motion reduces to that of a harmonic oscillator with frequency given by $\omega=\sqrt{2gA-\Omega^2}$. In the case where $\Omega^2=2gA$ then we do not observe oscillatory motion. Now consider the following conserved quantity obtained from the Lagrangian

$$E=\frac{\partial L}{\partial\dot{\rho}}\dot{\rho}-L \tag{1.9}$$

By substituting (1.4) into (1.9) we obtain

$$\begin{aligned} E &= m[\dot{\rho}^2+4A^2\rho^2\dot{\rho}^2]-\frac{m}{2}(\dot{\rho}^2+\rho^2\Omega^2+4A^2\rho^2\dot{\rho}^2)+mgA\rho^2 \\ E &= \frac{m}{2}[\dot{\rho}^2-\rho^2\Omega^2+4A^2\rho^2\dot{\rho}^2]+mgA\rho^2 \end{aligned} \tag{1.10}$$

Following a similar procedure as we did for the Lagrangian, we obtain that the constant of motion for small vibrations around rho_0 is given by

$$E=\frac{m}{2}\dot{q}^2+\frac{m}{2}(2gA-\Omega^2)q^2 \tag{1.11}$$

where $q=\rho-\rho_0$. As we can see, if $\Omega^2<2gA$ then (1.11) is the total energy of a harmonic oscillator with frequency $\omega=\sqrt{2gA-\Omega^2}$ which agrees with the result obtained from the Lagrangian.

Bonus

Here we will assume that $n>2$ since we already solved the case $n=2$. For the general case where $z=A\rho^n$ for $n>2$, we have that the kinetic and potential energy are given by

$$T=\frac{m}{2}[\dot{\rho}^2(1+n^2A^2\rho^{2n-2})+\Omega^2\rho^2] \tag{1.12}$$

$$V = mg\rho^n \tag{1.13}$$

Hence the Lagrangian is given by

$$L = T - V = \frac{m}{2}(1 + n^2A^2\rho^{2n-2})\dot{\rho}^2 + \frac{m}{2}\Omega^2\rho^2 - mgA\rho^n \tag{1.14}$$

The Euler-Lagrange equation of motion for this system is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\rho}} - \frac{\partial L}{\partial \rho} = m(1+n^2A^2\rho^{2n-2})\ddot{\rho} + m(n^2(n-1)A^2\rho^{2n-3})\dot{\rho}^2 - m\Omega^2\rho + nmgA\rho^{n-1} = 0$$

In equilibrium $\dot{\rho} = \ddot{\rho} = 0$ we have that the Lagrangian reduces to

$$-\Omega^2\rho + ngA\rho^{n-1} = 0 \tag{1.15}$$

from this condition we conclude that the equilibrium positions of the bead are either $\rho_0 = 0$ or $\rho_0 = (\frac{\Omega^2}{ngA})^{\frac{1}{n-2}}$. Next we expand the Lagrangian in a Taylor series around the equilibrium point ρ_0 and assuming small vibrations. In the anticipation that the vibrations around the equilibrium point will take a form similar to a harmonic oscillator, we expand things at most to the quadratic order. Let us define $\delta(t) = \rho(t) - \rho_0$ and consequently $\dot{\delta} = \dot{\rho}$. A little algebra yields the Lagrangian to the quadratic order in δ and $\dot{\delta}$ as

$$L = \text{const} + \frac{1}{2}m\left(\frac{\Omega^4\left(\frac{Agn}{\Omega^2}\right)^{-\frac{2}{n-2}}}{g^2} + 1\right)\dot{\delta}(t)^2 - \frac{1}{2}m(n-2)\Omega^2\delta(t)^2 \tag{1}$$

where the first term is just a constant independent of δ and $\dot{\delta}$ and does not affect the equations of motion. Now there are two ways to find the oscillations around the equilibrium point ($\delta = 0$): Either form the equation of motion, or simply compare the above Lagrangian to the Lagrangian for a harmonic oscillator, $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$. We know that the latter leads to an oscillation frequency $\omega = \sqrt{k/m}$. Comparing against the above expression, the frequency of small oscillations is given by

$$\omega = \sqrt{\frac{(n-2)\Omega^2}{\frac{\Omega^4\left(\frac{Agn}{\Omega^2}\right)^{-\frac{2}{n-2}}}{g^2} + 1}} \tag{1.22}$$

In a similar way one can define the conserved quantity $E = \frac{\partial L}{\partial \dot{\rho}}\dot{\rho} - L$ and then expand it around the equilibrium positions to arrive to the same results obtained from the expansion of the Lagrangian.

2. From the given Lagrangian we obtain

$$\frac{\partial L}{\partial \dot{x}} = \frac{m^2\dot{x}^3}{3} + 2m\dot{x}V(x) \tag{2.1}$$

$$\frac{\partial L}{\partial x} = m^2\dot{x}^2\frac{dV}{dx} - 2V(x)\frac{dV}{dx} \tag{2.2}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m^2\dot{x}^2\ddot{x} + 2m\ddot{x}V(x) + 2m\dot{x}^2\frac{dV}{dx} \tag{2.3}$$

where we have used the fact that $\frac{dV}{dx} = \dot{x}\frac{dV}{dx}$.We now obtain the following Euler-Lagrange equation

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= m^2\dot{x}^2\ddot{x} + 2m\ddot{x}V(x) + 2m\dot{x}^2\frac{dV}{dx} - m^2\dot{x}^2\frac{dV}{dx} + 2V(x)\frac{dV}{dx} = 0 \\ &= m^2\dot{x}^2\ddot{x} + 2m\ddot{x}V(x) + m\dot{x}^2\frac{dV}{dx} + 2V(x)\frac{dV}{dx} = 0\end{aligned}$$

$$(m\ddot{x} + \frac{dV}{dx})(m\dot{x}^2 + 2V(x)) = 0 \tag{2.4}$$

By identifying $F = -\frac{dV}{dx}$ as the force and $T = \frac{m\dot{x}^2}{2}$ as the kinetic energy, we can rewrite (2.4) as

$$(F - m\ddot{x})(T + V) = 0 \tag{2.5}$$

Thus we conclude that at all times either $F = ma$ or the total energy is zero.

3. From the given Lagrangian we obtain

$$\frac{\partial L}{\partial \dot{q}_i} = \dot{q}_j \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \ddot{q}_j \quad i \neq j \quad i,j = 1,2 \tag{3.1}$$

$$\frac{\partial L}{\partial q_i} = -\omega^2 q_j \quad i \neq j \quad i,j = 1,2 \tag{3.2}$$

Thus, the Euler-Lagrange equations of motion are given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \ddot{q}_j + \omega^2 q_j = 0 \quad i \neq j \quad i,j = 1,2 \tag{3.3}$$

The physical motion described by L is that of a two dimensional isotropic harmonic oscillator. Another Lagrangian that produces the same physical system but not the same equations of motion is

$$L' = \frac{1}{2}(\dot{q_1}^2 + \dot{q_2}^2 - \omega^2({q_1}^2 + {q_2}^2)) \tag{3.4}$$

In this case we can conclude that $\nexists F(q_1, q_2, t) : L - L' = \frac{dF}{dt}$ for the given L' in (3.4). We can prove this by contradiction. Suppose that there exists such an F that satisfies $L - L' = \frac{dF}{dt}$, then we have that

$$L - L' = \frac{1}{2}(-(\dot{q}_1 - \dot{q}_2)^2 + \omega^2(q_1 - q_2)^2) = \frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_{n=1}^{2} \frac{\partial F}{\partial q_n}\dot{q}_n \tag{3.5}$$

Then by taking the partial derivative of (3.5) with respect to $\dot{q}_1$ we obtain

$$\frac{\partial F}{\partial q_1} = \dot{q}_2 - \dot{q}_1$$

which contradicts the fact that F is independent of $\dot{q}_1$ and $\dot{q}_2$. However, this does not mean that $\nexists L'$ such that $L - L' = \frac{dF}{dt}$. We can build different L' that satisfy this relation by making guess choices on the function F. For example if we let $F = q_1 + q_2$ then by substituting this value of F and the Lagrangian L on the relation $L - L' = \frac{dF}{dt}$ we obtain

$$\dot{q}_1\dot{q}_2 - \omega^2 q_1 q_2 - L' = \dot{q}_1 + \dot{q}_2 \Rightarrow L' = \dot{q}_1\dot{q}_2 - \dot{q}_1 - \dot{q}_2 - \omega^2 q_1 q_2 \tag{3.6}$$

As we can see L' produces the same equations of motion as L. As another example, if we let $F = q_1 q_2$ then by substituting L and F in $L - L' = \frac{dF}{dt}$ we obtain

$$\dot{q}_1\dot{q}_2 - \omega^2 q_1 q_2 - L' = q_2\dot{q}_1 + q_1\dot{q}_2 \Rightarrow L' = \dot{q}_1\dot{q}_2 - q_2\dot{q}_1 - q_1\dot{q}_2 - \omega^2 q_1 q_2 \tag{3.7}$$

Again we see that L' produces the same equations of motion as L. As shown, the Lagrangians in (3.4), (3.6), and (3.7) produce the same physics as L, but the L' given in (3.4) does not produce the same equations of motion as L, that is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} \neq \frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_i} - \frac{\partial L'}{\partial q_i}$$

This why we cannot find a function $F(q_1, q_2, t)$ such that $L - L' = \frac{dF}{dt}$.