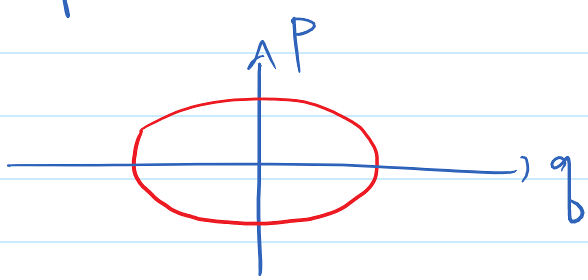
Action-angle variables : A qualitative introduction

To introduce action-angle variables, let's go back to the familiar example of harmonic oscillator. The Hamiltonian of the system is

$$H(p,q) = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$$

At an energy E, the system's orbit is an elipse in the phase space



On this orbit, the particle goes round and round in a periodic motion. It is simple to describe the periodic motion of a harmonic oscillator by an angle. To do this, let's write $H(p,q)=E$ as

$$p^2 + (m\omega q)^2 = 2mE$$

This equation is automatically satisfied by introducing the angle variable θ:

$$\begin{cases} p = \sqrt{2mE}\,\sin\phi \\ m\omega q = \sqrt{2mE}\,\cos\phi \end{cases}$$

We recall from an elementary discussion that the angle θ progresses linearly in time:

$$\phi = \omega t + \phi_0$$

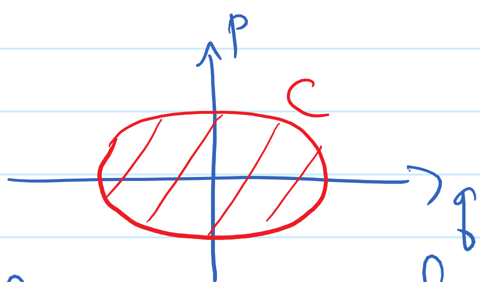
where ϕ_0 is a constant.

Notice that each orbit can be characterized by the value of energy E and the angle ϕ. More formally, we can make a canonical change of variables from (q, p) to (ϕ, J) where ϕ is the angle as defined above and J is a constant of motion that, for our simple example of harmonic oscillator, should be something like energy.

For this transformation to be canonical, we must have

$$\{\phi, J\} = 1$$

where the brackets define the Poisson bracket defined with respect to q & p. It turns out that the constant J should be identified with the area inside the orbit



$$J = \frac{\text{Shaded Area}}{2\pi}$$

This factor is chosen for convenience

Or, more formally, $J = \frac{1}{2\pi} \oint_C p \, dq$

where C denotes the closed orbit of motion.

This is a general definition of the "action" variable but let's see it explicitly for a harmonic oscillator

$$J = \text{Area} = \frac{E}{\omega} = \frac{H}{\omega}$$

Let's confirm that $\{\phi, J\} = 1$.

Notice that

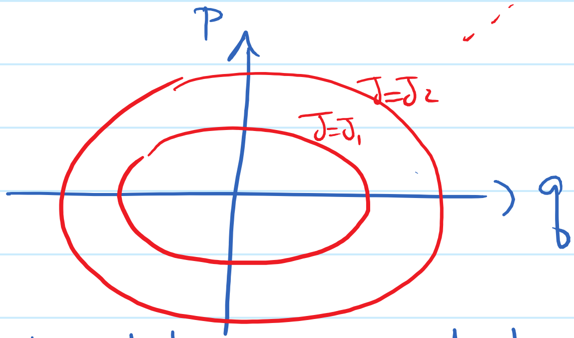
$$\dot{\phi} = \{\phi, H\} = \omega$$

which is simply the equation of motion for ϕ.

Therefore,

$$\left\{\phi, \frac{H}{\omega}\right\} = 1 \implies \{\phi, J\} = 1$$

In short, instead of describing the motion in the phase space from (q, p), we can represent different orbits by the action and angle variables



where different orbits correspond to different (constant) values of J while the motion is given by the angle

$$\phi = \omega t + \phi_0$$

You might think: "Oh well, this is a complicated way of thinking about simple physics."

The motivation is really to think about situations where we can describe a more complicated motion by "projecting" it into 2-dimensional phase space each of which look like what we just discussed.

Integrability: A system of n degrees of freedom with n q's an n p's is completely integrable if it can be described as the bounded motion in n submanifolds of the phase space.

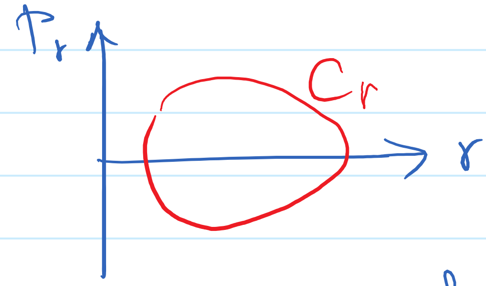
As an example, let's consider the central force problem

$$H = \frac{P_r^2}{2m} + \frac{P_\theta^2}{2mr^2} + V(r)$$

The motion takes place in the 4-dimensional phase space $(r, \theta, P_r, P_\theta)$. However, we know that P_θ is conserved so the motion in the (r, P_r) plane is completely determined by

$$E = \frac{P_r^2}{2m} + \frac{P_\theta^2}{2mr^2} + V(r)$$

which, for a bounded motion, looks like



A suitable change of coordinates from (r, P_r) to action-angle variables (ϕ_r, J_r) where

$$J_r = \frac{1}{2\pi}\oint_{C_r} P_r \, dr$$

and ϕ_r is similarly determined from $\{\phi_r, J_r\} = 1$ provides a mapping similar to that of harmonic oscillator, that is

$$J_r = \text{Const}$$

$$\text{and} \quad \phi_r = \omega_r t + \phi_r(0)$$

where ω_r is the frequency and $\phi_r(0)$ is the initial value of the angle.

A similar procedure can be made for θ and p_θ. The action variable in this case is simply p_θ itself because

$$J_\theta = \frac{1}{2\pi}\oint_{C_\theta} \underbrace{p_\theta}_{\text{constant}} d\theta = \frac{1}{2\pi} p_\theta (2\pi) = p_\theta$$

Finding the angle variable is a little more work, but it should be such that $\{\phi_\theta, J_\theta\} = \{\phi_\theta, p_\theta\} = 1$. Again the solution is

$$J_\theta = \text{const}$$
$$\phi_\theta = \omega_\theta t + \phi_\theta(0)$$

where ω_θ is a constant and $\phi_\theta(0)$ denotes the initial value of ϕ_θ.

Aside:

> For the Kepler problem, $V = \frac{k}{r}$, we have $\omega_r = \omega_\theta$ and the trajectories form a closed orbit in (r, θ). For a different central force problem, $V(r) \propto \frac{1}{r^n}$, the two frequencies are not the same $\omega_r \neq \omega_\theta$ and the orbit does not close on itself.

In general, for a completely integrable system, it is possible to transform canonically from (q^α, p) to new coordinates (ϕ^α, J). The new momenta J_α are constants of motion. In fact, the Hamiltonian only depends on J's:

$$H = H(J)$$

The coordinate ϕ^α describe the motion on each C_α, and are defined to increase by 2π each time C_α is traversed

$$\int_{C_\alpha} d\phi^\alpha = 2\pi$$

(C_α is the projection on the submanifold α)

Because H is a function only of the J_α, the $\phi^\alpha(t)$ are linear in time, that is,

$$\dot{\phi}^\alpha = \frac{\partial H}{\partial J_\alpha} = \omega^\alpha(J)$$

and since J's are constant, ω's are constants too and this integrates to

$$\phi^\alpha(t) = \omega^\alpha t + \phi^\alpha(0)$$

Liouville's integrability theorem

Now we briefly discuss a different notion of integrability due to Liouville. Liouville's criterion for integrability relies on finding n independent constants of motion in a system with n degrees of freedom rather than finding a "separable" coordinate system to define the action-angle variables.

The theorem states the system is completely integrable (in the sense defined before) if it has n independent constants that commute with each other

$$\{ I_\alpha , I_\beta \} = 0$$

in the sense of Poisson bracket.

Independence of the I_α is defined as follows. Let Σ_α be the $(2n-1)$ dimensional submanifold defined by equation

$$I_\alpha = c_\alpha = \text{const}$$

Write $\bar{I} = (c_1, c_2, \cdots, c_n)$ for a set of constant values of I_α and define the intersection of the Σ_α as

$$\Sigma_{\bar{I}} = \Sigma_1 \cap \Sigma_2 \cap \cdots \cap \Sigma_n$$

Then I_α are independent functions iff $\dim \Sigma_{\bar{I}} = n$.

In words, this means that each constant of motion reduces the dimension of the space of possible motion by 1. Therefor n constants of motion bring a $2n$-dimensional phase space to n dimension.