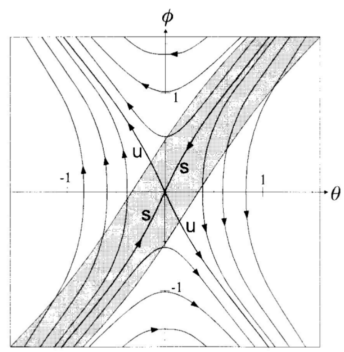
# PHY820/422 HW #6 — Due Monday 10/23/17 @ 5pm
# Scattering and linear oscillations

1. **Scattering off of two disks.** On the following figure identify the points that correspond to trajectories that leave the scattering region above and those that leave it below the two disks. Argue that the curve of trapped orbits is the separatrix between these two sets. Find a way to determine which region of the figure is forbidden to first hits.



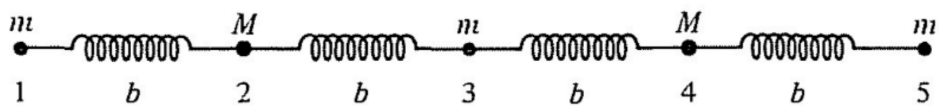

2. **Two vs three disks.** This is a question where you are expected to provide qualitative answers and plots. Consider the scattering off of the two disks that we discussed in class. For simplicity, let's fix $\theta_0 = \pi/4$ and vary $\phi_0$ (see the lecture notes or the textbook for the definition of the angles). Plot the qualitative features of the "dwell time", i.e. the time that the particle spends in the scattering region, as a function of the angle $\phi_0$. How does this compare to the scattering features for three disks in Fig. 4.11?

3. **Cantor sets and the Lyapunov exponent.**

   (a) A Cantor set can be formed by removing some fraction $1/f$ other than 1/3 of the intervals in each step. What is the fractal dimension of such a Cantor set? It will be useful to define the quantity $g = (f-1)/f$.

   (b) In the case of the scattering off of three disks, let's define two trajectories to be *equivalent of order* $n$ if they are the same $n$-string in our representation in terms of $l$ and $r$. What is the probability that two trajectories that are equivalent of the order $n$ would also be equivalent of the order $n+1$? (Assume that the "size" of the set $I_n$ is reduced by a factor $g$ upon the $n+1$-th bounce.) What is the probability that two trajectories will be separated in $m$ collisions? Expressing this probability as $e^{-\lambda m}$ the exponent $\lambda$ is called the Lyapunov exponent. What is the Lypaunov exponent in terms of $g$?

4. **Linear oscillations.** A 5-atom linear molecule is simulated by a configuration of masses and ideal springs that looks like the following diagram:




All force constants are equal. Find the eigenfrequencies and normal modes for longitudinal vibration. [Hint: transform the coordinates $x_i$ to $\xi_i$ defined by

$$x_1 = \frac{\xi_1 + \xi_5}{\sqrt{2}}, x_2 = \frac{\xi_2 + \xi_4}{\sqrt{2}}, \quad x_3 = \xi_3, \quad x_4 = \frac{\xi_2 - \xi_4}{\sqrt{2}}, \quad x_5 = \frac{\xi_1 - \xi_5}{\sqrt{2}}$$

The matrix $\Lambda$ takes a simpler form in the new basis.]

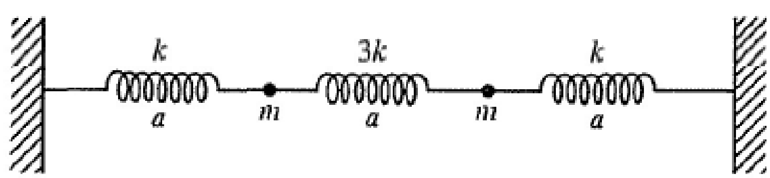
5. **Linear oscillations.**

   (a) Three equal mass points have equilibrium positions at the vertices of an equilateral triangle. They are connected by equal springs that lie along the arcs of the circle circumscribing the triangle. Mass points and springs are constrained to move only on the circle, so that, e.g., the potential energy of a spring is determined by the arc length covered. Determine the eigenfrequencies and normal modes of small oscillations in the plane. Identify physically any zero frequencies.

   (b) Suppose one of the springs has a change in force constant $\delta k$, the others remaining unchanged. To first order in $\delta k$ what are the changes in the eigenfrequencies and normal modes?

   (c) Suppose what is changed is the mass of one of the particles by an amount $\delta m$. Now how do the normal eigenfrequencies and normal modes change?

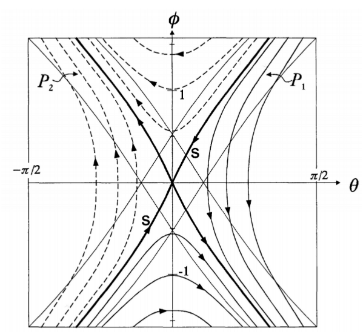
6. **Linear oscillations.** Two particles move in one dimension at the junction of three springs, as shown in the figure. The springs all have unstretched lengths equal to $a$, and the force constant and masses are shown.



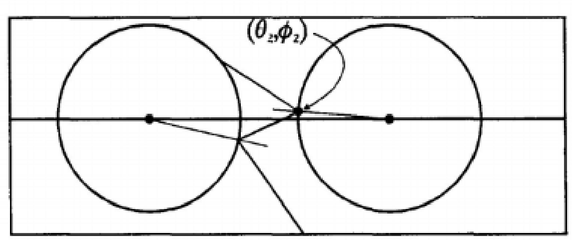

Find the eigenfrequencies and normal modes of the system.

**Solutions**

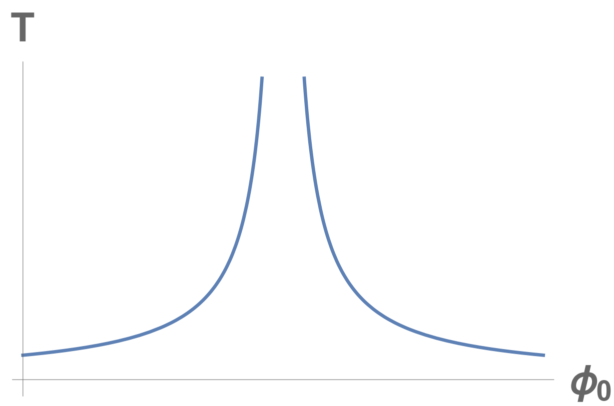
1. From the following figure we see that the solid trajectories are those that that leave above the two disks at $\theta > \frac{\pi}{2}$. Trajectories that leave below the two disks ($\theta < \frac{\pi}{2}$) are represented with dashed lines. $S$ labels the curve of trapped trajectories, those that do not leave the scattering region. $S$ is clearly the separatrix between the solid and dashed ones.




Let the first hit be on Disk L. The shaded strip $P_1$ on the previous figure shows values of $(\phi, \theta)$ on Disk L that correspond to a subsequent hit on Disk R. Each orbit coming off a $(\theta_1, \phi_1) \in P_1$ arrives at a $(\theta_2, \phi_2)$ on Disk R and then bounces on. Now reverse the arrows on the figure and reflect the figure about the vertical symmetry axis between the disks: this makes $(\theta_2, \phi_2)$ look just like a point in $P_1$. In other words, the values of $(\theta_2, \phi_2)$ fill a strip that looks just like $P_1$ but reversed. This is the strip $P_2$ on the first figure. If $(\theta_2, \phi_2)$ is also in $P_1$ the corresponding orbit goes on to make a subsequent hit on Disk L. This $(\theta_2, \phi_2)$ is not available to first hits, for any orbit reaching it must first have hit the other disk. Hence the region on the first figure that is unavailable to first hits is the diamond shape formed by the intersection of $P_1$ and $P_2$.




2. The dwell time $T$ diverges for the exceptional orbit at a particular value of $\phi_0$ where the corresponding orbit is trapped in the scattering region. The qualitative feature is shown below. This is qualitatively different from the scattering off of three disk (Fig. 4.11 of the textbook) where the dwell time diverges at infinitely many points, is self-similar, and whose pattern is irregular or chaotic.




3. The answers to both parts are given in the textbook attached in the next page. The short answers are provided here.

   (a) The fractal dimension is given by

   $$\frac{\log 2}{\log 2 + |\log g|} \tag{1}$$

   (b) The probability for two trajectories that are equivalent of the order $n$ would be equivalent of the order $n + 1$ is $g/2$. Hence the probability that two trajectories will be separated in $m$ collisions would be $(g/2)^m$, and the Lyapunov exponent is $\lambda = |\log(g/2)|$.

according to (4.23) the fractal dimension of the middle third Cantor set $C$ is

$$d_f = \frac{n \ln 2}{n \ln 3} \approx 0.631.$$

Sets of noninteger dimension are called *fractals*: $C$ is a fractal.

A Cantor set can be formed also by removing some fraction $1/f$ other than $1/3$ of the intervals in each step. The fractal dimension $d_f$ of such a Cantor set depends on $f$, and in general $d_f$ is $\ln 2/\ln\{2f/(f-1)\}$ (the number of intervals in each stage is still $2^n$). This result is often stated in terms of the ratio $g$ of the length of $I_{n+1}$ to that of $I_n$, which is $g = (f-1)/f < 1$. In these terms, the fractal dimension of such a Cantor is

$$d_f = \frac{\ln 2}{\ln 2 + |\ln g|}. \tag{4.24}$$

These considerations can be used for the further analysis of three-disk scattering. For the Cantor set obtained in three-disk scattering it is not easy to find the ratio $I_{n+1}/I_n$ (we use $I_n$ for both the set and its length). It is clear that it varies with $n$, and one may assume that as $n \to \infty$ it approaches some limit $g$ that depends only on $D$. This is because the longer the projectile bounces around among the disks, the more it loses any memory of how it got there in the first place, and $g$ is essentially the probability that the projectile, coming off in some random direction from one of the disks, will hit one of the others (the probability enters because the length of $I_n$ is a measure of the number of $n$th hits). This obviously depends on $D$. Thus physically $g$ is understood as the probability that after the $n$th collision the particle will make an $(n+1)$st. It follows that the probability that it will make $m$ in a row is $g^m \equiv e^{-\gamma m}$, where $\gamma \equiv |\ln g|$. As just mentioned, the length of $I_n$ is a measure of the number of trajectories in it. Each collision reduces the number of remaining trajectories by a factor of $g$, and the number in $I_{n+m}$ is lower by a factor of $e^{-\gamma m}$ than the number in $I_n$. In analogy with radioactive decay, $\gamma$ is a sort of lifetime for the set of trajectories. In computer simulations one can determine how the lengths of the $I_n$ depend on $n$ and thereby establish the lifetime and the fractal dimension of $I_\infty$.

Let us call two trajectories *equivalent of order n* if they both belong to the same $n$-string. Two trajectories that are equivalent of order $n$ will eventually belong to different $k$-strings, where $k > n$, and may therefore end up far apart, especially if one of them leaves the scattering region and the other goes on colliding with the disks. For this reason among others, three-disk scattering is called *chaotic* or *irregular*. What is the probability that two trajectories that are equivalent of order $n$ are also equivalent of order $n+1$? This depends on the length ratio of an $n$-string to an $(n+1)$-string; the ratio is $g/2$ (for $I_n/I_{n+1} = g$ and there are twice as many strings in $I_{n+1}$ as in $I_n$). Thus the probability that two orbits will be separated in one collision is $g/2$, and the probability that they will be separated in $m$ collisions is $(g/2)^m = e^{-\lambda m}$, where $\lambda = \gamma + \ln 2$. The number $\lambda$, called the *Lyapunov exponent* (Liapounoff, 1949), is a measure of the instability of the scattering system, as it shows how fast two trajectories that are initially close together will separate.

It follows from (4.24) and the defining equations for $\gamma$ and $\lambda$ that

$$\gamma = (1 - d_f)\lambda. \tag{4.25}$$

4. To find the normal modes and frequencies, we first obtain the matrix $\mathbf{\Lambda} = \mathbf{M}^{-1}\mathbf{K}$. The mass matrix is simply given by

$$\mathbf{M} = \begin{pmatrix} m & 0 & 0 & 0 & 0 \\ 0 & M & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & M & 0 \\ 0 & 0 & 0 & 0 & m \end{pmatrix} \tag{2}$$

while the matrix describing the spring constants is given by

$$\mathbf{K} = k \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \tag{3}$$

Therefore, we find for the matrix $\mathbf{\Lambda}$

$$\mathbf{\Lambda} = k \begin{pmatrix} \frac{1}{m} & -\frac{1}{m} & 0 & 0 & 0 \\ -\frac{1}{M} & \frac{2}{M} & -\frac{1}{M} & 0 & 0 \\ 0 & -\frac{1}{m} & \frac{2}{m} & -\frac{1}{m} & 0 \\ 0 & 0 & -\frac{1}{M} & \frac{2}{M} & -\frac{1}{M} \\ 0 & 0 & 0 & -\frac{1}{m} & \frac{1}{m} \end{pmatrix} \tag{4}$$

The transformation suggested in the problem amounts to transforming the matrix $\mathbf{\Lambda}$ to $\tilde{\mathbf{\Lambda}} = \mathbf{R}^T \mathbf{\Lambda} \mathbf{R}$ where

$$\mathbf{R} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{5}$$

from which we find

$$\tilde{\mathbf{\Lambda}} = k \begin{pmatrix} \frac{1}{m} & -\frac{1}{m} & 0 & 0 & 0 \\ -\frac{1}{M} & \frac{2}{M} & -\frac{\sqrt{2}}{M} & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{m} & \frac{2}{m} & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{M} & -\frac{1}{M} \\ 0 & 0 & 0 & -\frac{1}{m} & \frac{1}{m} \end{pmatrix} \tag{6}$$

This matrix is in a block-diagonal form in the sense that the first three degrees of freedom $(\xi_1, \xi_2, \xi_3)$ are decoupled from the last two ($\xi_4$ and $\xi_5$). The diagonalization of the former matrix is further simplified by noting that there must be a zero mode due to the motion of the center-of-mass coordinate. Therefore, in the end, one only need to diagonalize $2 \times 2$ matrices. However, having access to softwares of the kinf of Mathematica, we will not further pursue this line of argument, and rather directly compute the normal frequencies by diagonalizing the $5 \times 5$ matrix to find

$$\Big\{0, \sqrt{k\frac{-\sqrt{4m^2+M^2}+2m+M}{2mM}}, \sqrt{k\frac{-\sqrt{4m^2+M^2}+2m+3M}{2mM}}, \\ \sqrt{k\frac{\sqrt{4m^2+M^2}+2m+M}{2mM}}, \sqrt{k\frac{\sqrt{4m^2+M^2}+2m+3M}{2mM}}\Big\} \tag{7}$$

5. (a) For the three masses the kinetic energy clearly is given by

$$T = \frac{m}{2}(\dot{\eta}_1^2 + \dot{\eta}_2^2 + \dot{\eta}_3^2) \tag{8}$$

where $\eta_i$ is the displacement of the ith particle from its equilibrium position. The potential energy of the system is given by

$$V = \frac{k}{2}((\eta_1 - \eta_2)^2 + (\eta_1 - \eta_3)^2 + (\eta_2 - \eta_3)^2) \tag{9}$$

Thus, the Lagrangian for this system is given by

$$L = T - V = \frac{m}{2}(\dot{\eta}_1^2 + \dot{\eta}_2^2 + \dot{\eta}_3^2) - \frac{k}{2}((\eta_1 - \eta_2)^2 + (\eta_1 - \eta_3)^2 + (\eta_2 - \eta_3)^2) \tag{10}$$

The Euler-Lagrange equations for this system are given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\eta}_i} - \frac{\partial L}{\partial \eta_i} = m\ddot{\eta}_i + k(2\eta_i - \eta_j - \eta_k) = 0 \quad i, j, k = 1, 2, 3 \tag{11}$$

Rewritting this equations using matrix notation we have that

$$\hat{T}\ddot{\vec{\eta}} + \hat{V}\vec{\eta} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} \begin{pmatrix} \ddot{\eta}_1 \\ \ddot{\eta}_2 \\ \ddot{\eta}_3 \end{pmatrix} + \begin{pmatrix} 2k & -k & -k \\ -k & 2k & -k \\ -k & -k & 2k \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{12}$$

Since we anticipate a harmonic motion for this system around the equilibrium positions for each particle, we have that

$$\vec{\eta} = e^{i\omega t}\vec{A} = e^{i\omega t}\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \tag{13}$$

Substituting this solution into the previous equation we obtain

$$(-\omega^2\hat{T} + \hat{V})\vec{\eta} = \vec{0} \tag{14}$$

In order to obtain the eigenfrequencies of this system we require that

$$\det(-\omega^2\hat{T} + \hat{V}) = \begin{vmatrix} -m\omega^2 + 2k & -k & -k \\ -k & -m\omega^2 + 2k & -k \\ -k & -k & -m\omega^2 + 2k \end{vmatrix} = 0 \tag{15}$$

This condition reduces to the following cubic equation

$$\Omega^3 - 6\Omega^2 + 9\Omega = 0 \quad \Omega = \frac{m\omega^2}{k} \tag{16}$$

The solutions to this equation are $\omega_1 = 0$ and $\omega_2 = \omega_3 = \sqrt{\frac{3k}{m}}$. The normal mode corresponding to $\omega_1$ is obtained by substituting this value into the matrix equation and then solve for the components of $\vec{\eta}$. Thus, after substituting $\omega_1$ we obtain

$$\begin{pmatrix} 2k & -k & -k \\ -k & 2k & -k \\ -k & -k & 2k \end{pmatrix} \begin{pmatrix} A_{11} \\ A_{12} \\ A_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{17}$$

Thus we find that $A_{11} = A_{12} = A_{13}$. Thus the normal mode for $\omega_1$ is given by

$$\vec{\eta}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{18}$$

This is nothing but the collective motion of the center of mass.

For the repeated eigenfrequencies $\omega_2 = \omega_3$ we will be able to solve only for one normal mode using the matrix equation. The other normal mode must be determined by knowing that independent normal modes are perpendicular to each other. By substituting the value $\omega = \sqrt{\frac{3k}{m}}$ into the matrix equation we obtain

$$\begin{pmatrix} -k & -k & -k \\ -k & -k & -k \\ -k & -k & -k \end{pmatrix} \begin{pmatrix} A_{21} \\ A_{22} \\ A_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{19}$$

Thus we have that $A_{21} = -(A_{22} + A_{23})$. If we let $A_{23} = 0$ then we obtain the following normal mode

$$\vec{\eta}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \tag{20}$$

Since the third normal mode must be perpendicular to the other two then we can have

$$\vec{\eta}_3 = \vec{\eta}_1 \times \vec{\eta}_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \tag{21}$$

(b) Without loss of generality suppose that the the spring that connects particle 1 and 2 has a change in force constant $k \to k' = k + \delta k$. The new Lagrangian is given by

$$L=\frac{m}{2}(\dot\eta_1^2+\dot\eta_2^2+\dot\eta_3^2)-\frac{k}{2}((1+\epsilon)(\eta_1-\eta_2)^2+(\eta_1-\eta_3)^2+(\eta_2-\eta_3)^2)\quad \epsilon=\frac{\delta k}{k} \tag{22}$$

After finding the Euler-Lagrange equations and rewritting them into matrix form we can determine the eigenfrequencies of the system by solving

$$\begin{vmatrix} 2+\epsilon-\Omega & -(1+\epsilon) & -1 \\ -(1+\epsilon) & 2+\epsilon-\Omega & -1 \\ -1 & -1 & 2-\Omega \end{vmatrix} = \Omega(\Omega-3)(\Omega-3-2\epsilon)=0 \quad \Omega=\frac{m\omega^2}{k} \tag{23}$$

Thus the new eigenfrequencies are $\omega_1=0$, $\omega_2=\sqrt{\frac{3k}{m}}$, and $\omega_3=\sqrt{\frac{(3+2\epsilon)k}{m}}$. Following the same procedure as part (a) to find the normal modes for $\omega_1$ and $\omega_2$ we obtain

$$\vec\eta_1=\begin{pmatrix}1\\1\\1\end{pmatrix} \quad \vec\eta_2=\begin{pmatrix}1\\1\\-2\end{pmatrix} \tag{24}$$

Finally we obtain the normal mode for $\omega_3$ as

$$\vec\eta_1\times\vec\eta_2=\begin{pmatrix}-3\\3\\0\end{pmatrix}\equiv\begin{pmatrix}1\\-1\\0\end{pmatrix} \tag{25}$$

Comparing with the results from part (a) we see that the normal modes remain the same.

(c) Without loss of generality suppose that the mass of particle 1 changes $m\to m'=m+\delta m$. The new Lagrangian is given by

$$L=\frac{m}{2}((1+\mu)\dot\eta_1^2+\dot\eta_2^2+\dot\eta_3^2)-\frac{k}{2}((\eta_1-\eta_2)^2+(\eta_1-\eta_3)^2+(\eta_2-\eta_3)^2)\quad \mu=\frac{\delta m}{m} \tag{26}$$

The eigenfrequencies of the system will be obtained from

$$\begin{vmatrix} -(1+\mu)\Omega+2 & -1 & -1 \\ -1 & -\Omega+2 & -1 \\ -1 & -1 & -\Omega+2 \end{vmatrix} = -\Omega(\Omega-3)(\Omega(\mu+1)-(\mu+3))=0 \tag{27}$$

Thus the eigenfrequencies are given by $\omega_1=0$, $\omega_2=\sqrt{\frac{3k}{m}}$, and $\omega_3=\sqrt{\frac{k(\mu+3)}{m(\mu+1)}}$. Similarly to part (a), we obtain that

$$\vec\eta_1=\begin{pmatrix}1\\1\\1\end{pmatrix} \quad \vec\eta_3=\begin{pmatrix}\frac{-2}{\mu+1}\\1\\1\end{pmatrix} \quad \vec\eta_2=\vec\eta_1\times\vec\eta_3 \tag{28}$$

6. Let $r_1(t)$ and $r_2(t)$ be the position at any time of particles 1 and 2 respectively. Also we denote $x_1$ and $x_2$ as the displacements from the equilibrium positions of particles 1 and 2 respectively. Thus, we have that $r_1 = a + x_1$ and $r_2 = 2a + x_2$. The kinetic energy of the system is then given by

$$T = \frac{m}{2}(\dot{r}_1^2 + \dot{r}_2^2) = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) \tag{29}$$

The potential energy for this system is given by

$$V = \frac{k}{2}((r_1 - a)^2 + 3(r_2 - r_1 - a)^2 + (r_2 - 2a)^2) = \frac{k}{2}(4x_1^2 + 4x_2^2 - 6x_1x_2) \tag{30}$$

The Lagrangian of this system is given by

$$L = T - V = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \frac{k}{2}(4x_1^2 + 4x_2^2 - 6x_1x_2) \tag{31}$$

The Euler-Lagrange equations of motion are given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = m\ddot{x}_i + k(4x_i - 3x_j) = 0 \quad i, j = 1, 2 \tag{32}$$

Note that this system of equations can be written in the following form using matrix notation

$$\hat{T}\ddot{\vec{x}} + \hat{V}\vec{x} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} 4k & -3k \\ -3k & 4k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{33}$$

Since we already anticipate that the motion of this system will behave as linear combinations of harmonic oscillators then we guess that the solutions of the system will have the form

$$\vec{x} = \begin{pmatrix} Ae^{i\omega t} \\ Be^{i\omega t} \end{pmatrix} \tag{34}$$

Substituting this solution into the matrix equation we obtain

$$(-\omega^2\hat{T} + \hat{V}) \begin{pmatrix} A \\ B \end{pmatrix} = \vec{0} \tag{35}$$

The eigenfrequencies are obtained by solving the following quadratic equation

$$\det(\omega^2\hat{T} + \hat{V}) = \begin{vmatrix} m\omega^2 - 4k & 3k \\ 3k & m\omega^2 - 4k \end{vmatrix} = (m\omega^2 - 4k)^2 - 9k^2 = 0 \tag{36}$$

The solutions to this equation are $\omega_- = \sqrt{\frac{k}{m}}$ and $\omega_+ = \sqrt{\frac{7k}{m}}$. To find the normal modes corresponding to each eigenfrequency, we substitute the values of $\omega_-$ and $\omega_+$ into equation (8) and solve for $A$ and $B$. Thus, we have that

$$(-\omega_-^2 \hat{T} + \hat{V}) \begin{pmatrix} A_- \\ B_- \end{pmatrix} = \begin{pmatrix} -3k & 3k \\ 3k & -3k \end{pmatrix} \begin{pmatrix} A_- \\ B_- \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{37}$$

From the previous equation we conclude that $A_- = B_-$. Hence the normal mode for this case is given by

$$\vec{x}_- = \begin{pmatrix} A_- \\ A_- \end{pmatrix} \tag{38}$$

To normalize this eigenvector, we apply the following normalization condition

$$\vec{x}_-^T \hat{T} \vec{x}_- = \begin{pmatrix} A_-, & A_- \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} A_- \\ A_- \end{pmatrix} = 1 \Rightarrow A_- = \frac{1}{\sqrt{2m}} \tag{39}$$

$$\therefore \vec{x}_= \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{40}$$

Applying a similar procedure with $\omega_+$ we find that the its corresponding normal mode is given by

$$A_+ = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{41}$$