PHY820/422 HW #7 — Due Monday 10/30/17 @ 5pm
Linear oscillations in long chains & Hamiltonians

1. **Finite chain with fixed boundary conditions.** We saw that for an infinite chain there is a wave-like solution for every wavevector k oscillating at a frequency $\omega(k)$ that is a particular function of k. In this problem, we want to study a finite chain of $N+1$ particles with fixed boundary conditions at the two endpoints $x_0 = x_{N+1} = 0$.

   (a) Using a superposition of running waves propagating to the left or to the right, construct a solution that satisfies the boundary condition $x_0 = 0$.

   (b) We still need to satisfy the boundary condition at the other end of the chain, $x_{N+1} = 0$, but there is no more freedom in choosing arbitrary superposition of running waves (convince yourself). Therefore, in this case, a solution will not exist for every wavevector k. Find "allowed" values of k for which both boundary conditions are satisfied.

2. **Finite chain with periodic boundary conditions.** A chain of $N+1$ particles interacting through springs (as in Section 4.2.3 of the textbook) is subject to the condition that $x_0 = x_{N+1}$; that is, we impose the periodic boundary conditions, or the chain is looped on itself (similar to Problem 5 of the previous homework). This means that there are just N independent particles.

   (a) Find the normal modes. (The normal modes are running waves.)

   (b) Consider the limit of a small wavevector ($k \to 0$), and expand the dispersion relation to find the speed of sound.

3. **Hamiltonian and Hamilton's equations.** The Lagrangian for a particle moving in three dimensions (in terms of x, y, and z coordinates) can be written as

$$L = a\dot{x}^2 + b\frac{\dot{y}}{x} + c\dot{x}\dot{y} + fy^2\dot{x}\dot{z} + g\dot{y} - k\sqrt{x^2+y^2} \tag{1}$$

   where a, b, c, f, g, and k are constants. What is the Hamiltonian? What quantities are conserved?

4. **Hamiltonian and Hamilton's equations.** The Hamiltonian of a particle moving in one dimension is described by

$$L(q,\dot{q}) = \frac{1}{2}m\dot{q}^2 + mu\dot{q} - V(q) \tag{2}$$

   where u is a constant.

   (a) Find the equation of motion from the Euler-Lagrange equation.

(b) Construct the Hamiltonian $H(q, p)$ and obtain the Hamilton's equations. Show that the two set of (Hamiltonian and Euler-Lagrange) equations are consistent.

5. **Hamiltonian and Hamilton's equations.** The Hamiltonian of a particle moving in one dimension is described by

$$H(p, q) = \frac{p^2}{2m} - up + V(q) \tag{3}$$

where u is a constant.

   (a) Find the equation of motion from Hamilton's equations. (Write an equation that involves only q and its time derivatives.)

   (b) Construct the Lagrangian $L(q, \dot{q})$ and derive the Euler-Lagrange equations. Show that the two set of (Hamiltonian and Euler-Lagrange) equations are consistent with each other.

6. **Conserved quantities in Lagrangian and Hamiltonian formalisms.** Consider a system with two degrees of freedom x and y

$$L = \dot{x}^2 + \frac{1}{x}\dot{y}^2 \tag{4}$$

   (a) Derive the equations of motion for both degrees of freedom. Identify the cyclic coordinate, and deduce the constant of motion.

   (b) Define a new Lagrangian $\tilde{L}$ that is obtained from the above Lagrangian by eliminating the cyclic coordinate and its velocity using the constant of motion. Derive the Euler-Lagrange equation for the remaining coordinate. Is this equation consistent with your result in the previous part. Why or why not?

   (c) Next construct the Hamiltonian $H(x, y, p_x, p_y)$ for the Lagrangian L.

   (d) Derive the equations of motion from the Hamilton's equations, and show that they are consistent with the Euler-Lagrange equations

   (e) This time, first identify the conserved momentum, and replace it by a constant in the Hamiltonian (if p_α is conserved for a certain α, replace it by $p_\alpha = a = \text{const}$). Now derive the Hamilton's equations of motion (treat the conserved momentum as a constant). Do you find the same equations of motion. Why or why not?

7. **Another exercise in Hamiltonian dynamics.** Formulate the double-pendulum problem (See Problem 4 of Homework 2) in terms of the Hamiltonian and the Hamilton's equations of motion.

8. ***Bonus* Lagrangian from Hamiltonian.** Given a Hamiltonian function $H(q, p, t)$, how does one obtain the corresponding Lagrangian? That is, (a) describe the inverse of the procedure that leads to the Hamiltonian. (b) Show that the Euler-Lagrange equations are derivable from Hamilton's equations.

Solutions

1. (a) The solution for each particle

$$\tilde{x}_n = C(k)e^{i\omega t}\sin(nka) \tag{5}$$

can be used to form the superposition of running waves that satisfies the boundary condition

$$x_0 = C(k)e^{i\omega t}\sin(0) = 0 \tag{6}$$

(b) In order to satisfy the boundary condition $x_{N+1} = 0$ we must have

$$x_{N+1} = C(k)e^{i\omega t}\sin((N+1)ka) = 0 \tag{7}$$

In order to avoid trivial solutions ($C(k) = 0$) we conclude that

$$\sin\left[(N+1)ka\right] = 0 \Rightarrow k_p = \frac{\pi p}{(N+1)a} \quad \text{where} \quad 0 \leq p \in \mathbb{Z} \tag{8}$$

(Negative values of p just result in a relative sign and do not produce independent functions.) Thus the wave vectors k can only take a discrete set of values. Given the peridocity of the sin function, it is sufficient to take these these values in the range $[-\pi/a, \pi/a]$.

2. (a) To find the normal modes we treat the chain as a loop and use the general solution

$$x_n(t,k) = C(k)e^{i(kna-\omega(k)t)} \tag{9}$$

From our condition $x_0 = x_N$ we have that

$$C(k)e^{-i\omega t} = C(k)e^{i(kNa-\omega t)} \Rightarrow e^{ikNa} = 1 \Rightarrow k = \frac{2\pi p}{Na} \equiv k_p \tag{10}$$

where p is an (positive, zero, or negative) integer. From the dispersion relation we have that

$$\omega_p^2 = 4\omega_0^2\sin^2\left(\frac{\pi p}{N}\right) \tag{11}$$

hence we can restrict p into the interval $(-\frac{N}{2}, \frac{N}{2})$ since the values of x_n will repeat outside this interval. Now we must consider the following two cases: N is even or odd. The general solution is given by

$$x_n(t) = \sum_{p=-R'}^{R} x_{n,p}(t) = \sum_{-R'}^{R} C_p e^{i(k_z na-\omega_z t)} \tag{12}$$

where if N is even then $R' = \frac{N}{2} - 1$, $R = \frac{N}{2}$, and if N is odd then $R' = R = \frac{N-1}{2}$. Note that the solution can be rewritten as

$$x_n(t) = C_0 + \sum_{p=1}^{\frac{N}{2}} C_p e^{i(k_z na - \omega_z t)} + \sum_{p=1}^{\frac{N}{2}-1} C_{-p} e^{i(-k_z na - \omega_z t)} \quad \text{N even} \tag{13}$$

$$x_n(t) = C_0 + \sum_{p=1}^{\frac{N-1}{2}} C_p e^{i(k_z na - \omega_z t)} + \sum_{p=1}^{\frac{N-1}{2}} C_{-p} e^{i(-k_z na - \omega_z t)} \quad \text{N odd} \tag{14}$$

From here we see that all particles move around the loop at the same constant speed. The two directions represent a degeneracy, that is, two solutions for each ω_p. Ignoring the zero mode, denote $C_z e^{ik_z np} = a_{n,p}\alpha_p$. Thus

$$x(t) = \sum_{p=1}^{\Gamma} a_p[\alpha_z e^{-i\omega_z t} + \alpha_p^* e^{i\omega_z t}] + \sum_{p=1}^{\Gamma} a_{-p}[\alpha_{-p} e^{-i\omega_z t} + \alpha_{-p}^* e^{i\omega_z t}] \tag{15}$$

The real part of the solution is then given by

$$x(t) = \sum_{p=1}^{\Gamma} a_z A_p \cos(k_z na - \omega_z t + \phi_p) + \sum_{p=1}^{\Gamma} a_{-p} A_{-p} \cos(k_z na + \omega_z t + \phi_{-p}) \tag{16}$$

The coefficients of each a_p and a_{-p} oscillates independently. These are the normal modes.

(b) From

$$x_n(t) = \sum_{p=-R'}^{R} x_{n,p}(t) = \sum_{-R'}^{R} C_p e^{i(k_z na - \omega_z t)} \tag{17}$$

we note that na is the distance x along the chain, so the dispersion relation for small k or, equivalently, small p, becomes $\omega_p \approx 2\omega_0 \pi p/N = \omega_0 a k_p$, and therefore of the pth wave is

$$v = \omega_0 a \tag{18}$$

This is indeed the same as the speed of sound that we found in an infinite chain. This simply means that the boundary conditions at the endpoints do not affect the speed of sound in (the middle of) the chain.

3. From the definition of the Hamiltonian

$$H(\vec{p}_q, \vec{q}) = \vec{p} \cdot \dot{\vec{q}}(\vec{p}_q, \vec{q}, t) - L(\vec{q}, \dot{\vec{q}}(\vec{p}_q, \vec{q}, t), t) \tag{19}$$

we must determine the Lagrangian and the generalized velocities in terms of the canonical momenta which are given by

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{20}$$

Since the given Lagrangian in this problem depends on x, y, and z, we will obtain three canonical momenta. Thus

$$p_x = \frac{\partial L}{\partial \dot{x}} = 2a\dot{x} + c\dot{y} + fy^2\dot{z} \quad p_y = \frac{\partial L}{\partial \dot{y}} = \frac{b}{x} + c\dot{x} + g \quad p_z = \frac{\partial L}{\partial \dot{z}} = fy^2\dot{x} \tag{21}$$

Note that the Lagrangian can now be rewritten as

$$L = a\dot{x}^2 + p_y\dot{y} + p_z\dot{z} - k\sqrt{x^2+y^2} \tag{22}$$

The Hamiltonian is then given by

$$p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - (a\dot{x}^2 + p_y\dot{y} + p_z\dot{z} - k\sqrt{x^2+y^2}) = \frac{p_xp_z}{fy^2} - \frac{ap_z^2}{f^2y^4} + k\sqrt{x^2+y^2} \tag{23}$$

where in the last equality was obtained after solving for $\dot{x}$ in terms of p_z. From Hamilton's equations of motion

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{24}$$

we can determine that p_z is a conserved quantity since the Hamiltonian does not depend on z hence $\dot{p}_z = 0$. Additionally since the Hamiltonian does not have explicit time dependence then it is also conserved.

4. (a) From the Euler-Lagrange equation of motion we obtain that

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = m\ddot{q} + \frac{dV}{dq} = 0 \tag{25}$$

(b) To obtain the Hamiltonian we first determine the canonical momentum of the system which is given by

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} + mu \Rightarrow \dot{q} = \frac{p}{m} - u \tag{26}$$

The Lagrangian in terms of the canonical momentum is

$$L = \frac{m}{2}(\frac{p}{m} - u)^2 + mu(\frac{p}{m} - u) - V(q) \tag{27}$$

hence, the Hamiltonian is given by

$$H = p\dot{q} - L = p(\frac{p}{m} - u) - [\frac{m}{2}(\frac{p}{m} - u)^2 + mu(\frac{p}{m} - u) - V(q)] \tag{28}$$

which can be simplified to

$$H = \frac{p^2}{2m} - pu + \frac{mu^2}{2} + V(q) \tag{29}$$

The equations of motion obtained from the Hamiltonian are

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} - u \quad \dot{p}_q = -\frac{\partial H}{\partial q} = \frac{dV}{dq} \tag{30}$$

Note that if we take a time derivative of the first equation and then substitute the expression of $\dot{p}_q$ we obtain

$$\ddot{q} = \frac{\dot{p}}{m} = -\frac{1}{m}\frac{dV}{dq} \Rightarrow m\ddot{q} + \frac{dV}{dq} = 0 \tag{31}$$

which is equivalent to the result we obtained using the Lagrangian and the Euler-Lagrange equation of motion

5. (a) From the given Hamiltonian we obtain the following equations of motion

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} - u \quad \dot{p} = -\frac{\partial H}{\partial q} = -\frac{dV}{dq} \tag{32}$$

Taking the time derivative of the first equation and then substituting the second equation we obtain

$$\ddot{q} = \frac{-1}{m}\frac{dV}{dq} \tag{33}$$

(b) The Lagrangian can be obtained from the Hamiltonian as

$$L = p\dot{q} - H = p\dot{q} - (\frac{p^2}{2m} - up + V(q)) \tag{34}$$

Since the Lagrangian is a function of the generalized position and velocities, we must find an expression of the canonical momentum p in terms of $\dot{q}$. From part (a) the first equation of motion obtained from the Hamiltonian gives us this relation, namely

$$p(\dot{q}) = m\dot{q} + mu \tag{35}$$

Thus, the Lagrangian is given by

$$L = m\dot{q}^2 + mu\dot{q} - \frac{(m\dot{q}+mu)^2}{2m} + mu\dot{q} + mu^2 - V(q) = \frac{m\dot{q}^2}{2} + mu\dot{q} + mu^2 - V(q) \tag{36}$$

The Euler-Lagrange equation of motion is given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = m\ddot{q} + \frac{dV}{dq} = 0 \Rightarrow \ddot{q} = \frac{-1}{m}\frac{dV}{dq} \tag{37}$$

which is the exact result we obtained using the Hamiltonian.

6. (a) From the Euler-Lagrange equations of motion we obtain the following two equations for each degree of freedom

$$2\ddot{x} + (\frac{\dot{y}}{x})^2 = 0 \quad 2\frac{d}{dt}(\frac{\dot{y}}{x}) = 0 \tag{38}$$

From the last equation we determine that $k = \frac{\dot{y}}{x}$ is a constant of motion, from which we find

$$2\ddot{x} + k^2 = 0 \tag{39}$$

We also identify y as the cyclic coordinate since the Lagrangian does not depend on it.

(b) The new Lagrangian $\tilde{L}$ is given by

$$\tilde{L} = \dot{x}^2 + \frac{\dot{y}^2}{x} = \dot{x}^2 + x(\frac{\dot{y}}{x})^2 = \dot{x}^2 + k^2 x \tag{40}$$

The Euler-Lagrange equation of motion is given by

$$2\ddot{x} - k^2 = 0 \tag{41}$$

which is not consistent with the results of the previous part

(c) To obtain the Hamiltonian we first determine the canonical momenta from the Lagrangian

$$p_x = \frac{\partial L}{\partial \dot{x}} = 2\dot{x} \quad p_y = \frac{\partial L}{\partial \dot{y}} = 2\frac{\dot{y}}{x} \tag{42}$$

Thus the Hamiltonian is given by

$$H = p_x\dot{x} + p_y\dot{y} - L = \frac{p_x^2}{2} + x\frac{p_y^2}{2} - (\frac{p_x^2}{4} + x\frac{p_y^2}{4}) = \frac{p_x^2 + xp_y^2}{4} \tag{43}$$

(d) The Equations of motion obtained from the Hamiltonian are

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{2} \quad \dot{p}_x = -\frac{\partial H}{\partial x} = -\frac{p_y^2}{4} \tag{44}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{x p_y}{2} \quad \dot{p}_y = -\frac{\partial H}{\partial y} = 0 \tag{45}$$

From these equations we determine that p_y is constant which implies that $\frac{\dot{y}}{x}$ must be constant. Additionally we also determine that $\ddot{x} = -\frac{p_y^2}{8} = \frac{-1}{2}(\frac{\dot{y}}{x})^2$. These are the exact results we obtained from the Euler-Lagrange equations from part (a).

(e) Having identified p_y as constant then equations of motion obtained from the Lagrangian are

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{2} \quad \dot{p}_x = -\frac{\partial H}{\partial x} = -\frac{p_y^2}{4} \tag{46}$$

which yields

$$\ddot{x} = -\frac{p_y^2}{8} \tag{47}$$

which is again consistent with the result in part (a).

7. The Lagrangian for the double pendulum system is given by

$$L = \frac{(m_1+m_2)}{2} l_1^2\dot{\theta}_1^2 + \frac{m_2}{2}(l_2^2\dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)) + (m_1+m_2) g l_1 \cos(\theta_1) + m_2 g l_2 \cos(\theta_2) \tag{48}$$

First we will obtain expressions for the canonical momenta p_1 and p_2

$$p_1 = \frac{\partial L}{\partial \dot{\theta}_1} = (m_1+m_2) l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \tag{49}$$

$$p_2 = \frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2) \tag{50}$$

As an intermediate step we can rewrite the Lagrangian as

$$L = \frac{p_1\dot{\theta}_1}{2} + \frac{p_2\dot{\theta}_2}{2} + (m_1+m_2) g l_1 \cos(\theta_1) + m_2 g l_2 \cos(\theta_2) \tag{51}$$

The Hamiltonian is given by

$$H = p_1\dot{\theta}_1 + p_2\dot{\theta}_2 - L = \frac{p_1\dot{\theta}_1}{2} + \frac{p_2\dot{\theta}_2}{2} - (m_1+m_2)gl_1\cos(\theta_1) - m_2gl_2\cos(\theta_2) \quad (52)$$

Now we need to solve for $\dot{\theta}_1$ and $\dot{\theta}_2$ in terms of the canonical momenta. Thus

$$\dot{\theta}_1 = (\frac{p_1}{(m_1+m_2)l_1^2} - \frac{p_2}{(m_1+m_2)l_1l_2}\cos(\theta_1-\theta_2))(1-\frac{m_2}{m_1+m_2}\cos^2(\theta_1-\theta_2))^{-1} \quad (53)$$

$$\dot{\theta}_2 = (\frac{p_2}{m_2l_2^2} - \frac{p_1}{(m_1+m_2)l_1l_2}\cos(\theta_1-\theta_2))(1-\frac{m_2}{m_1+m_2}\cos^2(\theta_1-\theta_2))^{-1} \quad (54)$$

Substituting these expressions into the Hamiltonian we find that

$$H = \frac{1}{2}(\frac{p_1^2}{(m_1+m_2)l_1^2} + \frac{p_2^2}{m_2l_2^2})(1-\frac{m_2}{m_1+m_2}\cos^2(\theta_1-\theta_2))^{-1} - (m_1+m_2)gl_1\cos(\theta_1) - m_2gl_2\cos(\theta_2) \quad (55)$$

The equations of motion are obtained from

$$\dot{\theta}_i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial \theta_i} \quad i = 1,2 \quad (56)$$

8. (a) Beginning from the definition of the Hamiltonian

$$H(\vec{q},\vec{p},t) = \vec{p}\cdot\dot{\vec{q}}(\vec{q},\vec{p},t) - L(\vec{q},\dot{\vec{q}}(\vec{q},\vec{p},t),t) \quad (57)$$

we have that

$$L(\vec{q},\dot{\vec{q}},t) = \vec{p}(\vec{q},\dot{\vec{q}},t)\cdot\dot{\vec{q}} - H(\vec{q},\vec{p}(\vec{q},\dot{\vec{q}},t),t) \quad (58)$$

where in the last step we have inverted the relation between the canonical momenta and the generalized velocities using

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad (59)$$

(b) Starting from

$$L(\vec{q},\dot{\vec{q}},t) = \vec{p}(\vec{q},\dot{\vec{q}},t)\cdot\dot{\vec{q}} - H(\vec{q},\vec{p}(\vec{q},\dot{\vec{q}},t),t) \quad (60)$$

we have that

$$\frac{\partial L}{\partial \dot{q}_i} = p_i + \frac{\partial \vec{p}}{\partial \dot{q}_i}\cdot\dot{\vec{q}} - \frac{\partial H}{\partial \dot{q}_i} = p_i + \frac{\partial \vec{p}}{\partial \dot{q}_i}\cdot\dot{\vec{q}} - \sum_j \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial \dot{q}_i} \equiv p_i \quad (61)$$

$$\frac{\partial L}{\partial q_i} = \frac{\partial \vec{p}}{\partial q_i} \cdot \dot{\vec{q}} - \frac{\partial H}{\partial q_i} - \sum_j \frac{\partial H}{\partial p_j} \frac{\partial p_j}{q_i} \equiv -\frac{\partial H}{\partial q_i} \tag{62}$$

The Euler-Lagrange equations of motion then give us

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \frac{dp_i}{dt} + \frac{\partial H}{\partial q_i} = 0 \tag{63}$$