PHY820/422 MidTerm Exam 1

1. Consider a particle of mass m moving along the x direction in an external potential $V(x - ut)$ that depends on both the position of the particle x and time t only through the combination $x - ut$; the parameter u is a constant.

   (a) Write down the Lagrangian $L = T - V$ explicitly and use the Euler-Lagrange equations to derive the equations of motion.

   (b) Is the sum of kinetic and potential energies, $E = T + V$, conserved? Explain in words (or in equations if you prefer).

   (c) Rewrite the Lagrangian in terms of a new coordinate $x' = x - ut$. (You should write both kinetic and potential energies in the new coordinate system.)

   (d) Using the form of the Lagrangian in the new coordinate system, find a constant of motion. (You can use the formulas you have learned in class, but you have to justify why they can be used.)

   (e) Rewrite the conserved quantity in the original coordinate system, and compare it to the energy E defined in part (b).

2. Consider two particles with masses m_1 and m_2 moving in one dimension. Let x_1 and x_2 describe the positions of the two particles. The first particle is subject to the potential $V_1(x_1) = \frac{1}{2}k_1 x_1^2$, while the second particle is subject to a different potential $V_2(x_2) = \frac{1}{2}k_2 x_2^2$. (Both k_1 and k_2 are positive.) The two particles are constrained such that their distance is always a constant a, that is, $x_2 - x_1 = a$.

   (a) Write down the equations of motion.

   (b) Find the equilibrium positions of the two particles.

   (c) Find the period of oscillations around equilibrium. Does it depend on the amplitude of the oscillation?

   For this question, you are free to use your preferred method.

Solutions

1. (a) The Lagrangian is

$$L = T - V = \frac{1}{2}m\dot{x}^2 - V(x - ut). \tag{1}$$

Using the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \tag{2}$$

we find the equation of motion

$$m\ddot{x} = -\partial_x V(x - ut) = -V'(x - ut) \tag{3}$$

where $V'(z) \equiv \partial V/\partial z$.

(b) The sum of Kinetic and potential energies is not conserved because the potential is time dependent. You can see this explicitly if you take the time derivative of the sum:

$$\begin{aligned}\frac{d}{dt}\left[\frac{1}{2}m\dot{x}^2 + V(x - ut)\right] &= m\dot{x}\ddot{x} + \dot{x}\partial_x V + \partial_t V(x - ut) = \partial_t V(x - ut) \\ &= -uV'(x - ut)\end{aligned} \tag{4}$$

In evaluating the time derivative we have used $df(x,t)/dt = \dot{\partial} f/\partial x + \partial f/\partial t$.

(c) Defining a new coordinate $x' = x - ut$, the position and velocities should be substituted in the new coordinate system from $x = x' + ut$ and $\dot{x} = \dot{x}' + u$. The Lagrangian then takes the form

$$L'(x', \dot{x}') = \frac{1}{2}m(\dot{x}'^2 + 2u\dot{x}' + u^2) - V(x') \tag{5}$$

(d) The Lagrangian L' does not depend on time t explicitly, that is, $\partial L'/\partial t = 0$, therefore, we can define a constant of motion as

$$W = \dot{x}'\frac{\partial L'}{\partial \dot{x}'} - L' = \frac{1}{2}m\dot{x}'^2 + V(x') \tag{6}$$

(e) In the original coordinate system, the latter can be written as

$$W = \frac{1}{2}m\left(\dot{x}^2 - 2u\dot{x} + u^2\right) + V(x - ut) \tag{7}$$

The constant term, $mu^2/2$, can be dropped because it doesn't affect the equations of motion. We then find that $W = E + mu\dot{x} + \text{const} = E + up + \text{const}$ where $p = m\dot{x}$ is the linear momentum. Note that E and p are not separately conserved; however their combination $E + up$ is indeed conserved.

2. (a) There are a number of ways to tackle this problem. The easiest method is to use the energy conservation

$$E = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 \tag{8}$$

Using the constraint $x_2 = x_1 + a$ together with its time derivative $\dot{x}_2 = \dot{x}_1$, we have

$$E = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + \frac{1}{2}kx_1^2 + \frac{1}{2}k_2(x_1 + a)^2 \equiv \frac{1}{2}m_{\text{tot}}\dot{x}_1^2 + V_{\text{eff}}(x_1) \tag{9}$$

where in the last equality we have defined the total mass $m_{tot} = m_1 + m_2$ and the effective potential $V_{\text{eff}}(x_1) = \frac{1}{2}kx_1^2 + \frac{1}{2}k_2(x_1 + a)^2$ governing the motion of the first particle. The equation of motion is given by

$$m_{\text{tot}}\ddot{x}_1 = -\frac{d}{dx_1}V_{\text{eff}}(x_1) \tag{10}$$

(b) The equilibrium position is obtained by setting $V'_{\text{eff}}(x_1) = 0$; we find the equilibrium position of the first particle as

$$\bar{x}_1 = -a\frac{k_2}{k_1 + k_2} \tag{11}$$

Using the constraint $x_2 = x_1 + a$, we also find

$$\bar{x}_2 = a\frac{k_1}{k_1 + k_2} \tag{12}$$

(c) To find the period of oscillations, we can expand the energy around the equilibrium point. Let $x_1(t) = \bar{x}_1 + \delta(t)$; also, $\dot{x} = \dot{\delta}$. We find the energy in terms of δ and $\dot{\delta}$ as

$$E = \frac{1}{2}m_{\text{tot}}\dot{\delta}^2 + \frac{1}{2}(k_1 + k_2)\delta^2 + \text{const} \tag{13}$$

In comparison with the energy of a harmonic oscillator, we find the frequency of oscillations as

$$\omega = \sqrt{\frac{k_1 + k_2}{m_1 + m_2}} \tag{14}$$

Just like the harmonic oscillator, the frequency does not depend on the amplitude.