PHY820/422 HW #9 Due Monday 11/20/2017 @ 5pm
Canonical transformations & electromagnetism in Lagrangian and Hamiltonian formalisms

1. **Canonical transformation.** Consider a *complex* transformation

$$Q = \frac{m\omega q + ip}{\sqrt{2m\omega}}, \qquad P = i\frac{m\omega q - ip}{\sqrt{2m\omega}} = iQ^* \tag{1}$$

   (a) Show that this is a canonical transformation, and find its generating function.

   (b) Apply this canonical transformation to the harmonic oscillator whose Hamiltonian is

$$H = \frac{1}{2}m\omega^2 q^2 + \frac{p^2}{2m}. \tag{2}$$

   Solve for the equation of motion in terms of (Q, P), use that in turn to find the solution for q and p.

2. **Canonical transformation.** The Hamiltonian for a system has the form

$$H = \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right) \tag{3}$$

   (a) Find the equation of motion for q.

   (b) Find a canonical transformation that reduces H to the form of a harmonic oscillator. Show that the solution for the transformed variables is such that the equation of motion found in part (a) is satisfied.

3. **EM in the Lagrangian and Hamiltonian formalisms.** The Lagrangian of a particle moving in two dimensions (x, y) is given by

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - e\phi(x, y, t) + eA_x(x, y, t)\dot{x} + eA_y(x, y, t)\dot{y} \tag{4}$$

   where e is a constant, $\phi(x, y, t)$ is a "scalar" potential, and $\mathbf{A}(x, y, t) = (A_x, A_y)$ is a "vector" potential.

   (a) Derive the equations of motion and show that they describe the Lorentz force on a particle of charge e under the action of an electric field $\mathbf{E} = -\nabla\phi - \partial_t\mathbf{A}$ and a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$.

   (b) The electromagnetic field is invariant under a gauge transformation of the scalar and vector potentials given by

$$\mathbf{A} \to \mathbf{A} + \nabla\psi(x, y, t), \quad \phi \to \phi - \frac{\partial\psi}{\partial t} \tag{5}$$

where ψ is arbitrary (but differentiable). What effect does this gauge transformation have on the Lagrangian of a particle moving in the electromagnetic field? Is the motion affected?

(c) For a constant magnetic field $\mathbf{B} = B\hat{z}$ along the z direction, we can choose $\phi = 0$ and $(A_x, A_y) = (-\frac{1}{2}By, \frac{1}{2}Bx)$ (Show this!). Write the Lagrangian with this choice of scalar and vector potentials. Compare the latter to the Lagrangian in a rotating coordinate frame (See old lecture notes). Use the Euler-Lagrange equations to find the equations of motion, and show that the orbits are circles of all possible radii centered everywhere in the plane, and the particle rotates around the circle at the rate $\omega = \frac{eB}{m}$.

Next we will study the same problem in the Hamiltonian language.

(d) Obtain the Hamiltonian from the Lagrangian using the standard procedure. Show that the Hamiltonian takes the form

$$H = \frac{(\mathbf{p} - e\mathbf{A})^2}{2m} + e\phi \equiv \frac{(p_x - eA_x)^2 + (p_y - eA_y)^2}{2m} + e\phi \tag{6}$$

(e) For a constant magnetic field $\mathbf{B} = B\hat{z}$ along the z direction, we can also choose $\phi = 0$ and $(A_x, A_y) = (0, Bx)$ (Show this!). Write down the Hamiltonian with this choice of (scalar and vector) potential, derive the equations of motion, and show explicitly that they lead to the equations of motion of a charged particle in the presence of a magnetic field.

4. **Poisson brackets in the presence of magnetic field.** A charged particle moves in space with a constant magnetic field $\mathbf{B}$ such that

$$\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r} \tag{7}$$

(a) If v_j are the Cartesian components of the velocity of the particle, evaluate the Poisson brackets

$$\{v_i, v_j\}, \qquad i \neq j = 1, 2, 3 \tag{8}$$

(b) If p_i is the canonical momentum conjugate to x_i, also evaluate the Poisson brackets

$$\{x_i, v_j\}, \qquad \{p_i, v_j\} \tag{9}$$

5. * **Bonus** * A particle of mass m and charge e moves in the two-dimensional plane under the combined influence of the harmonic oscillator potential $\frac{1}{2}m\omega^2(x^2+y^2)$ and a constant magnetic field $\mathbf{B} = B\hat{z}$ where B is constant [you can choose $\mathbf{A} = (0, Bx)$]. Use the canonical transformation

$$\begin{aligned} x' &= x\cos\alpha - \frac{p_y}{\beta}\sin\alpha, \quad p_x' = \beta y \sin\alpha + p_x\cos\alpha, \\ y' &= y\cos\alpha - \frac{p_x}{\beta}\sin\alpha, \quad p_y' = \beta x \sin\alpha + p_y\cos\alpha \end{aligned} \tag{10}$$

with $\tan(2\alpha) = m\omega/(eB)$ and β chosen for convenience to find the motion in general and in the two limits $B = 0$ and $B \to \infty$.

Solutions

1. (a) In order to see if this transformation is canonical we just have to show that $\{Q, P\} = 1$. Thus, we have that

$$\{Q, P\} = \{\frac{m\omega q + ip}{\sqrt{2m\omega}}, i\frac{m\omega q - ip}{\sqrt{2m\omega}}\} = 2\frac{m\omega}{2m\omega}\{q, p\} = 1 \tag{11}$$

Since this canonical transformation is of type 1 (Q and q are independent), the generating function can be determined from the following expressions

$$\frac{\partial F}{\partial q} = p = i(m\omega q - \sqrt{2m\omega}Q) \tag{12}$$

$$-\frac{\partial F}{\partial Q} = P = i(\sqrt{2m\omega}q - Q) \tag{13}$$

Integrating the first equation with respect to q we find that

$$F(q, Q) = i(\frac{m\omega}{2}q^2 - \sqrt{2m\omega}Qq) + \phi(Q) \tag{14}$$

Now taking the partial derivative of the previous expression with respect to Q we find that

$$-i\sqrt{2m\omega}q + \phi'(Q) = -i(\sqrt{2m\omega}q - Q) \Rightarrow \phi(Q) = \frac{i}{2}Q^2 + c \tag{15}$$

where c is an arbitrary constant. Thus the generating function is given by

$$F = i(\frac{m\omega}{2}q^2 - \sqrt{2m\omega}Qq + \frac{1}{2}Q^2) + c \tag{16}$$

(b) Notice that the Hamiltonian can be written as

$$H = \frac{1}{2}m\omega^2 q^2 + \frac{p^2}{2m} = -i\omega QP \tag{17}$$

Hamilton's equations of motion under the new coordinates are

$$\dot{Q} = \frac{\partial H}{\partial P} = -i\omega Q \quad \dot{P} = -\frac{\partial H}{\partial Q} = i\omega P \tag{18}$$

which have the following solutions

$$Q(t) = Q_0 e^{-i\omega t} \quad P(t) = P_0 e^{i\omega t} \tag{19}$$

where Q_0 and P_0 are complex constants. From the definitions of Q and P we can see that q and p can be obtained from the real and imaginary parts of Q respectively. Thus we have that

$$q(t) = \sqrt{\frac{2}{m\omega}}\frac{Q(t)+Q^*(t)}{2} = \frac{1}{\sqrt{2m\omega}}(Q_0 e^{-i\omega t} + Q_0^* e^{i\omega t}) \tag{20}$$

In a similar way we determine that

$$p(t) = -i\sqrt{\frac{m\omega}{2}}(Q(t) - Q^*(t)) = -i\sqrt{\frac{m\omega}{2}}(Q_0 e^{-i\omega t} - Q_0^* e^{i\omega t}) \tag{21}$$

Both of these expressions can be rewritten in the following forms

$$q(t) = q_0\cos(\omega t) + \frac{p_0}{m\omega}\sin(\omega t) \quad p(t) = -m\omega q_0 \sin(\omega t) + p_0\cos(\omega t) \tag{22}$$

2. (a) For the given Hamiltonian the equations of motion are

$$\dot{q} = \frac{\partial H}{\partial p} = pq^4 \quad \dot{p} = -\frac{\partial H}{\partial q} = \frac{1}{q^3} - 2p^2q^3 \tag{23}$$

(b) The canonical transformation that will transform the Hamiltonian to that of a harmonic oscillator is found to be

$$Q = \frac{1}{\sqrt{m}}pq^2 \quad P = \frac{\sqrt{m}}{q} \tag{24}$$

Solving for q and p in terms of Q and P we find that

$$q = \frac{\sqrt{m}}{P} \quad p = \frac{QP^2}{\sqrt{m}} \tag{25}$$

Substituting these expressions into the original Hamiltonian we find that

$$H = \frac{1}{2}(\frac{P^2}{m} + mQ^2) \tag{26}$$

Hamilton's equations of motion under the canonical transformation are

$$\dot{Q} = \frac{\partial H}{\partial P} = \frac{P}{m} \quad \dot{P} = -\frac{\partial H}{\partial Q} = -mQ \tag{27}$$

If we substitute the expressions of $Q(q,p)$ and $P(q,p)$ we find that

$$\dot{P} = -\frac{\sqrt{m}}{q^2}\dot{q} = -\sqrt{m}pq^2 \Rightarrow \dot{q} = pq^4 \tag{28}$$

$$\dot{Q} = \frac{\dot{p}q^2 + 2pq\dot{q}}{\sqrt{m}} = \frac{P}{m} = \frac{1}{\sqrt{m}q} \Rightarrow \dot{p} = \frac{1}{q^3} - 2p^2q^3 \tag{29}$$

where in the last equation we substituted the expression of $\dot{q}$ that we found on the first equation. As we can see these are the same equations of motion that we obtained before applying the canonical transformation.

3. (a) Starting from the given Lagrangian we have that

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + eA_x(x,y,t) \quad \frac{\partial L}{\partial \dot{y}} = m\dot{y} + eA_y(x,y,t) \tag{30}$$

$$\frac{\partial L}{\partial x} = -e\frac{\partial \phi}{\partial x} + e(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y}) \quad \frac{\partial L}{\partial y} = -e\frac{\partial \phi}{\partial y} + e(\frac{\partial A_x}{\partial y}\dot{x} + \frac{\partial A_y}{\partial y}\dot{y}) \tag{31}$$

Taking the time derivative of the first set of partial derivatives we obtain the following

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} + e(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial t}) \tag{32}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = m\ddot{y} + e(\frac{\partial A_y}{\partial x}\dot{x} + \frac{\partial A_y}{\partial y}\dot{y} + \frac{\partial A_y}{\partial t}) \tag{33}$$

Here we used the fact that since both x and y depend on time we need to apply the chain rule when taking the total time derivate of the components of the vector potential $\vec{A}$. The Euler-Lagrange equations of motion are given by

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m\ddot{x} + e(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial t}) + e\frac{\partial \phi}{\partial x} - e(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y}) = 0 \tag{34}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = m\ddot{y} + e(\frac{\partial A_y}{\partial x}\dot{x} + \frac{\partial A_y}{\partial y}\dot{y} + \frac{\partial A_y}{\partial t}) + e\frac{\partial \phi}{\partial y} - e(\frac{\partial A_x}{\partial y}\dot{x} + \frac{\partial A_y}{\partial y}\dot{y}) = 0 \tag{35}$$

By recognizing the terms $E_i = -(\frac{\partial \phi}{\partial x_i} + \frac{\partial A_i}{\partial x_i})$ as the components of the electric fields we can now rewrite the equations of motion as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m\ddot{x} - eE_x + e\dot{y}(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}) = 0 \tag{36}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = m\ddot{y} - eE_y + e\dot{x}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) = 0 \tag{37}$$

Next we can recognize the term $\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$ as the z component of $\vec{B} = \nabla \times \vec{A}$ which is B_z. Thus the equations of motion can be written as

$$m\ddot{x} = e(E_x + \dot{y}B_z) \quad m\ddot{y} = e(E_y - \dot{x}B_z) \tag{38}$$

These equations of motion are the x and y components of the Lorentz force for a particle with charge e moving through an electric field $\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$ and magnetic field $\vec{B} = \nabla \times \vec{A}$ in the xy plane

$$\vec{F} = e(\vec{E} + \vec{v} \times \vec{B}) \tag{39}$$

(b) A gauge transformation of the electromagnetic potentials changes the action as

$$L \to L' = L + e\partial_t \psi + e\dot{x}\partial_x \psi + e\dot{y}\partial_y \psi = L + \frac{d\psi}{dt} \tag{40}$$

Therefore, the Lagrangian changes by a total time derivative, and thus the equations of motion are unaffected. One can also see this explicitly at the level of the equations of motion.

$$\vec{E}' = -(\nabla\phi' + \frac{\partial \vec{A}'}{\partial t}) = -(\nabla(\phi - \frac{\partial \psi}{\partial t}) + \frac{\partial}{\partial t}(\vec{A} + \nabla\psi)) = -(\nabla\phi + \frac{\partial \vec{A}}{\partial t}) = \vec{E} \tag{41}$$

$$\vec{B}' = \nabla \times \vec{A}' = \nabla \times (\vec{A} + \nabla\psi) = \nabla \times \vec{A} = \vec{B} \tag{42}$$

We have used the fact that the curl of the gradient of any scalar function is zero.

(c) Using the given scalar and vector potentials the Lagrangian will be given by

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}eB(x\dot{y} - y\dot{x}) \tag{43}$$

The last term has an obvious analogy to the form of the Lagrangian in a rotating frame, and one can anticipate that the solution to the equation of motion will involve some form of rotation. To see this, let's derive the Euler-Lagrange equations:

$$m\ddot{x} = eB\dot{y} \quad m\ddot{y} = -eB\dot{x} \tag{44}$$

To solve these equations of motion we define $z = x + iy$ and note that

$$m\ddot{z} = m\frac{d\dot{z}}{dt} = m\ddot{x} + im\ddot{y} = eB(\dot{y} - i\dot{x}) = -ieB\dot{z} \tag{45}$$

Solving this equation for $\dot{z}$ we obtain

$$\dot{z} = \dot{z}(0)e^{-\frac{ieBt}{m}} \tag{46}$$

Integrating this last equation we obtain an expression for $z(t)$

$$z(t) = z(0) + \frac{im}{eB}\dot{z}(0)e^{-\frac{ieBt}{m}} \tag{47}$$

To obtain the x and y components of the motion we write the last expression explicitly in terms of x and y

$$x(t) + iy(t) = x(0) + iy(0) + \frac{im}{eB}(\dot{x}(0) + i\dot{y}(0))(\cos(\frac{eBt}{m}) - i\sin(\frac{eBt}{m})) \tag{48}$$

Hence $x(t)$ will be given by the real part of the previous expression while $y(t)$ will be given by the imaginary part

$$x(t) = x(0) + \frac{m}{eB}(\dot{x}(0)\sin(\frac{eBt}{m}) - \dot{y}(0)\cos(\frac{eBt}{m})) \tag{49}$$

$$y(t) = y(0) + \frac{m}{eB}(\dot{x}(0)\cos(\frac{eBt}{m}) + \dot{y}(0)\sin(\frac{eBt}{m})) \tag{50}$$

Notice that the "center of motion" $(x(0), y(0))$ can be anywhere in the plane. Moreover, from this equation, it is easy to see that

$$(x(t) - x(0))^2 + (y(t) - y(0))^2 = \left(\frac{mv}{eB}\right)^2 \tag{51}$$

where $v = \sqrt{\dot{x}(0)^2 + \dot{y}(0)^2}$ is the magnitude of velocity which is constant. Hence the motion takes place on a circled of the radius $R = mv/(eB)$.

(d) To obtain the Hamiltonian we first need to determine the canonical momenta from the Lagrangian. We find out that

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + eA_x(x, y, t) \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + eA_y(x, y, t) \tag{52}$$

Thus we find that the velocities are given by

$$\dot{x} = \frac{1}{m}(p_x - eA_x(x, y, t)) \quad \dot{y} = \frac{1}{m}(p_y - eA_y(x, y, t)) \tag{53}$$

Substituting these expressions into the Lagrangian we obtain

$$L = \frac{1}{2m}[(p_x - eA_x)^2 + (p_y - eA_y)^2] - e\phi + \frac{e}{m}(A_x(p_x - eA_x) + A_y(p_y - eA_y)) \tag{54}$$

The Hamiltonian will be given by

$$H = \vec{p} \cdot \vec{v} - L \tag{55}$$

where the first term is given by

$$\vec{p} \cdot \vec{v} = \frac{p_x}{m}(p_x - eA_x) + \frac{p_y}{m}(p_y - eA_y) \tag{56}$$

When substituting the expressions for $\vec{p} \cdot \vec{v}$ and L into the Hamiltonian, if we combine the last two terms of the Lagrangian with the terms in $\vec{p} \cdot \vec{v}$ we obtain the following

$$\vec{p} \cdot \vec{v} - \frac{e}{m}(A_x(p_x - eA_x) + A_y(p_y - eA_y)) = \frac{1}{m}[(p_x - eA_x)^2 + (p_y - eA_y)^2] \tag{57}$$

thus the Hamiltonian is given by

$$H = \frac{1}{2m}[(p_x - eA_x)^2 + (p_y - eA_y)^2] + e\phi \tag{58}$$

(e) For the constant magnetic field along the z direction with vector potential $\vec{A} = Bx\hat{y}$ the Hamiltonian will be given by

$$H = \frac{1}{2m}(p_x^2 + (p_y - eBx)^2) \tag{59}$$

Hamilton's equation of motion are given by

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \quad \dot{p}_x = -\frac{\partial H}{\partial x} = \frac{eB}{m}(p_y - eBx) \tag{60}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y - eBx}{m} \quad \dot{p}_y = -\frac{\partial H}{\partial y} = 0 \tag{61}$$

Taking a time derivative to the fist equation in each set of Hamilton's equation we obtain that

$$\ddot{x} = \frac{\dot{p}_x}{m} = \frac{1}{m}\frac{eB}{m}(p_y - eBx) = \frac{eB}{m}\dot{y} \tag{62}$$

$$\ddot{y} = \frac{\dot{p}_y - eB\dot{x}}{m} = -\frac{eB}{m}\dot{x} \tag{63}$$

As we can see these are the same equations of motion that we obtained from the Lagrangian even though we had a different vector potential from the one we used with the Lagrangian. This is just a consequence of the gauge invariance of the electromagnetic fields. The equations of motion only care about the fields and not the potentials. We got the same equations of motion since we were considering the same electric and magnetic fields in both cases. If we let $\vec{A}' = (-\frac{1}{2}By, \frac{1}{2}Bx, 0)$ and $\vec{A} = (0, Bx, 0)$ then we can find a function ψ such that $\vec{A}' = \vec{A} + \nabla\psi$. It is easy to see that $\psi = -\frac{B}{2}xy$ satisfies this relation.

4. (a) First we must determine the expressions of the velocities in terms of the canonical momenta and the positions. In order to do this we must first determine the Lagrangian of the system which is given by

$$L = \frac{m}{2}(v_x^2 + v_y^2 + v_z^2) - e\phi + e\vec{v}\cdot\vec{A} = \frac{m}{2}(v_x^2 + v_y^2 + v_z^2) + \frac{e}{2}\vec{v}\cdot(\vec{B}\times\vec{r}) \tag{64}$$

Thus, we have that

$$p_i = \frac{\partial L}{\partial v_i} = mv_i + \frac{e}{2}(\vec{B}\times\vec{r})_i \Rightarrow v_i = \frac{1}{m}(p_i - \frac{e}{2}(\vec{B}\times\vec{r})_i) \tag{65}$$

Here the term $(\vec{B} \times \vec{r})_i$ refers to the ith component of the cross product between the magnetic field and the position vector which can be written as $(\vec{B} \times \vec{r})_i = \epsilon_{ijk} B^j r^k$ using Einstein's notation for tensors where ϵ_{ijk} is the Levi-Civita symbol. The Poisson brackets between the Cartesian components of the velocity will be given by

$$\{v_i, v_j\} = \{\frac{1}{m}(p_i - \frac{e}{2}\epsilon_{ilk} B^l r^k), \frac{1}{m}(p_j - \frac{e}{2}\epsilon_{jmn} B^m r^n)\} \tag{66}$$

which can be simplified into

$$\{v_i, v_j\} = \{\frac{p_i}{m}, -\frac{e}{2m}\epsilon_{jmn} B^m r^n\} - \{\frac{p_j}{m}, -\frac{e}{2m}\epsilon_{ilk} B^l r^k\} \tag{67}$$

Next we apply the following property of Poisson brackets

$$\{p_i, f(r^i, r^j, r^k)\} = -\frac{\partial f}{\partial r^i} \tag{68}$$

to obtain

$$\{v_i, v_j\} = \frac{e}{2m^2}(\epsilon_{jmn} B^m \frac{\partial r^n}{\partial r^i} - \epsilon_{ilk} B^l \frac{\partial r^k}{\partial r^j}) \tag{69}$$

Next we notice that the derivatives of the positions of the components will be zero if the indexes are different and will be one if the indexes are equal, that is, $\frac{\partial r^n}{\partial r^i} = \delta_i^n$ where δ_i^n is the Kronecker Delta symbol. Thus we find that

$$\{v_i, v_j\} = \frac{e}{2m^2}(\epsilon_{jmn} B^m \delta_i^n - \epsilon_{ilk} B^l \delta_j^k) = \frac{e}{2m^2}(\epsilon_{jmi} B^m - \epsilon_{ilj} B^j) \tag{70}$$

Since the Levi-Civita is zero if any of the indexes is repeated then we must have that $m = l = k$, hence

$$\{v_i, v_j\} = \frac{e}{2m^2}(\epsilon_{jki} B^k - \epsilon_{ikj} B^j) = \frac{e}{2m^2}(\epsilon_{ijk} B^k + \epsilon_{ijk} B^j) = \frac{e}{m^2}\epsilon_{ijk} B^k \tag{71}$$

where in the last equation we used the fact that $\epsilon_{jki} = \epsilon_{ijk}$ and $\epsilon_{ikj} = -\epsilon_{ijk}$. In particular we have that

$$\{v_1, v_2\} = \frac{eB_3}{m^2} \quad \{v_1, v_3\} = -\frac{eB_2}{m^2} \quad \{v_2, v_3\} = \frac{eB_1}{m^2} \tag{72}$$

(b) The Poisson bracket between the positions and the velocities is given by

$$\{r^i, v_j\} = \{r^i, \frac{1}{m}(p_j - \frac{e}{2}\epsilon_{jmn} B^m r^n)\} = \{r^i, \frac{p_j}{m}\} = \frac{1}{m}\delta_{ij} \tag{73}$$

The Poisson bracket between the canonical momenta and the velocities is given by

$$\{p_i, v_j\} = \{p_i, \frac{1}{m}(p_j - \frac{e}{2}\epsilon_{jmn} B^m r^n)\} = \{p_i, -\frac{e}{2m}\epsilon_{jmn} B^m r^n\} = \frac{e}{2m}\epsilon_{ijk} B^k \tag{74}$$

5. The Hamiltonian for this system can be obtained from the Hamiltonian of a two dimensional harmonic oscillator with the only difference being that the momentum along the y direction must be changed into the canonical momentum that takes into account the effects of the magnetic field, thus

$$H = \frac{1}{2m}(p_x^2 + (p_y - eBx)^2) + \frac{m\omega^2}{2}(x^2 + y^2) \tag{75}$$

Now if we apply the given canonical transformation we can start rewriting the Hamiltonian in the following way

$$2mH = (-\beta y's + p_x'c)^2 + \Omega^2(x'c + \frac{p_y'}{\beta}s)^2 + \Omega^2(y'c + \frac{p_x'}{\beta}s)^2 + (-x'(\beta s + eBc) + p_y'(c - \frac{eBs}{\beta}))^2 \tag{76}$$

where $\Omega = m\omega$ and $c = \cos(\alpha)$ and $s = \sin(\alpha)$. After expanding and rearranging some terms we find that

$$2mH = x'^2(\Omega^2 + 2\Omega eBsc + e^2B^2c^2) + p_x'^2 + y'^2\Omega^2 + p_y'^2(1 + \frac{e^2B^2s^2}{\Omega^2} - \frac{2eBsc}{\Omega}) + 2p_y'x'(eB(s^2 - c^2) + \frac{e^2B^2sc}{\Omega}) \tag{77}$$

The last term cancels out since we have that

$$\tan(2\alpha) = \frac{2sc}{c^2 - s^2} = \frac{2\Omega}{eB} \tag{78}$$

Note that the first and second terms in parenthesis can be written as

$$\Omega^2 + 1\Omega eBsc + e^2B^2c^2 = \Omega^2\cot^2(\alpha) \tag{79}$$

$$1 + \frac{e^2B^2s^2}{\Omega^2} - \frac{2eB}{\Omega}sc = \tan^2(\alpha) \tag{80}$$

The Hamiltonian can now be rewritten as

$$H = \frac{1}{2m}(p_x'^2 + \tan^2(\alpha)p_y'^2) + \frac{m\omega^2}{2}(\cot^2(\alpha)x'^2 + y'^2) \tag{81}$$

which is the Hamiltonian for an anisotropic harmonic oscillator with frequencies $\omega\cot(\alpha)$ and $\frac{\omega}{\tan(\alpha)}$. Taking the limit where the magnetic field vanished we find that

$$\omega\cot(\alpha) \to \omega \tag{82}$$

If the magnetic field goes to infinity we find that

$$\omega \cot(\alpha) \to \frac{eB}{\omega} \tag{83}$$