PHY820/422 HW #5 — Due Monday 10/16/17 @ 5pm
Noether theorem and Scattering

1. **Conserved quantity from translation invariance.** Consider a system consisting of N particles of masses m_i and subject to a "pairwise" potential

$$V(x_1, x_2, \cdots, x_N) = \sum_{i<j} V(x_i - x_j) \tag{1}$$

(We have restricted the sum over $i < j$ to avoid double counting and self-interactions when $i = j$.) The potential is an arbitrary function that only depends on the distance between two given particles. Now consider a spatial translation

$$x_i \to \tilde{x}_i = x_i + \epsilon \tag{2}$$

   (a) Write down the Lagrangian L explicitly. Is the Lagrangian invariant under the spatial translation?

   (b) Use the Noether theorem to derive the conserved quantity associated with the spatial translation. What is the physical interpretation of this conserved quantity.

2. **Conserved quantity from Galilean invariance.** Consider a system consisting of N particles defined in Problem 1. This time, however, consider a Galilean transformation under which

$$x_i \to \tilde{x}_i = x_i + ut \tag{3}$$

where u has the units of velocity. This is the familiar transformation that takes us to a new reference frame moving at the velocity u.

   (a) Write down the Lagrangian L. Is the Lagrangian invariant under Galilean transformation? Find the change of the Lagrangian.

   (b) For an "infinitesimal transformation" where u is small, it is sufficient to keep everything only to the first order in u (drop terms that are of a higher order in u). Find the change of the *action* under this infinitesimal transformation. Show that the change of the action can be written as a "boundary" term that only depends on the initial and the final time of the integral in the action.

   (c) Use (extension of) the Noether theorem to derive the conserved quantity. What is the physical interpretation of the conserved quantity that you have obtained.

3. **Rutherford scattering for a *repulsive* potential.** In class, we have derived the Rutherford scattering formula for an attractive potential. Obtain the scattering rate $\sigma(\Theta)$ for a repulsive Coulomb potential.

4. **Scattering off of a short-ranged potential.** A central force potential frequently encountered in nuclear physics is the *rectangular well*, defined by the potential

$$V(r) = \begin{cases} 0, & r > a \\ -V_0, & r \leq a \end{cases} \tag{4}$$

Show that the scattering produced by such a potential in classical mechanics is identical with the refreaction of light rays by a sphere of radius a and relative index of refraction

$$n = \sqrt{\frac{E + V_0}{E}} \tag{5}$$

(This equivalence demonstrates why it is possible to explain refraction phenomena both by Huygnes' waves and by Newton's mechanical corpuscles.) Show also that the differential cross section is

$$\sigma(\Theta) = \frac{n^2 a^2}{4\cos\left(\frac{\Theta}{2}\right)} \frac{\left(n\cos\left(\frac{\Theta}{2}\right) - 1\right)\left(n - \cos\left(\frac{\Theta}{2}\right)\right)}{\left(1 + n^2 - 2n\cos\left(\frac{\Theta}{2}\right)\right)^2} \tag{6}$$

What is the total cross section?

5. **An exercise in inverse scattering theory.** Find the central potential whose scattering cross section is given by

$$\sigma(\Theta) = \alpha\pi^2 \frac{\pi - \Theta}{(2\pi - \Theta)^2 \Theta^2 \sin\Theta} \tag{7}$$

[Answer: $V = K/r^2$, where the dependence of K on E and α will be found when the problem is solved.]