PHY820/422 HW #5 — Due Monday 10/16/17 @ 5pm
Noether theorem and Scattering

1. **Conserved quantity from translation invariance.** Consider a system consisting of N particles of masses m_i and subject to a "pairwise" potential

$$V(x_1, x_2, \cdots, x_N) = \sum_{i<j} V(x_i - x_j) \tag{1}$$

(We have restricted the sum over $i < j$ to avoid double counting and self-interactions when $i = j$.) The potential is an arbitrary function that only depends on the distance between two given particles. Now consider a spatial translation

$$x_i \to \tilde{x}_i = x_i + \epsilon \tag{2}$$

   (a) Write down the Lagrangian L explicitly. Is the Lagrangian invariant under the spatial translation?

   (b) Use the Noether theorem to derive the conserved quantity associated with the spatial translation. What is the physical interpretation of this conserved quantity.

2. **Conserved quantity from Galilean invariance.** Consider a system consisting of N particles defined in Problem 1. This time, however, consider a Galilean transformation under which

$$x_i \to \tilde{x}_i = x_i + ut \tag{3}$$

where u has the units of velocity. This is the familiar transformation that takes us to a new reference frame moving at the velocity u.

   (a) Write down the Lagrangian L. Is the Lagrangian invariant under Galilean transformation? Find the change of the Lagrangian.

   (b) For an "infinitesimal transformation" where u is small, it is sufficient to keep everything only to the first order in u (drop terms that are of a higher order in u). Find the change of the *action* under this infinitesimal transformation. Show that the change of the action can be written as a "boundary" term that only depends on the initial and the final time of the integral in the action.

   (c) Use (extension of) the Noether theorem to derive the conserved quantity. What is the physical interpretation of the conserved quantity that you have obtained.

3. **Rutherford scattering for a *repulsive* potential.** In class, we have derived the Rutherford scattering formula for an attractive potential. Obtain the scattering rate $\sigma(\Theta)$ for a repulsive Coulomb potential.

4. **Scattering off of a short-ranged potential.** A central force potential frequently encountered in nuclear physics is the *rectangular well*, defined by the potential

$$V(r) = \begin{cases} 0, & r > a \\ -V_0, & r \leq a \end{cases} \tag{4}$$

Show that the scattering produced by such a potential in classical mechanics is identical with the refreaction of light rays by a sphere of radius a and relative index of refraction

$$n = \sqrt{\frac{E + V_0}{E}} \tag{5}$$

(This equivalence demonstrates why it is possible to explain refraction phenomena both by Huygnes' waves and by Newton's mechanical corpuscles.) Show also that the differential cross section is

$$\sigma(\Theta) = \frac{n^2 a^2}{4\cos\left(\frac{\Theta}{2}\right)} \frac{\left(n\cos\left(\frac{\Theta}{2}\right) - 1\right)\left(n - \cos\left(\frac{\Theta}{2}\right)\right)}{\left(1 + n^2 - 2n\cos\left(\frac{\Theta}{2}\right)\right)^2} \tag{6}$$

What is the total cross section?

5. **An exercise in inverse scattering theory.** Find the central potential whose scattering cross section is given by

$$\sigma(\Theta) = \alpha\pi^2 \frac{\pi - \Theta}{(2\pi - \Theta)^2 \Theta^2 \sin\Theta} \tag{7}$$

[Answer: $V = K/r^2$, where the dependence of K on E and α will be found when the problem is solved.]

Solutions

1. The Lagrangian can be written as

$$L = \sum_i \frac{m_i}{2}\dot{x}_i^2 - \sum_{i<j} V(x_i - x_j) \tag{8}$$

Under spatial translation $x_i \to \tilde{x}_i = x_i + \epsilon$, the Lagrangian is invariant. This is because the potential term is invariant as $\tilde{x}_i - \tilde{x}_j = x_i - x_j$. The Kinetic term is also invariant since, constant ϵ, the time derivative does not change, that is, $d\tilde{x}_i/dt = dx_i/dt$. Now notice that the infinitesimal change in the position is given by $\delta x_i = \epsilon$. Hence, the conserved quantity of the Lagrangian is

$$\sum_i \frac{\partial L}{\partial \dot{x}_i} \cdot \delta x_i = \sum_i m_i \dot{x}_i \cdot \epsilon \tag{9}$$

It follows from this equation that the total momentum is conserved,

$$P = \sum_i m_i \dot{x}_i = \text{const} \tag{10}$$

2. The Lagrangian is the same as the first problem,

$$L = \sum_i \frac{m_i}{2}\dot{x}_i^2 - \sum_{i<j} V(x_i - x_j) \tag{11}$$

The Galilean transformation $x_i \to \tilde{x}_i = x_i + ut$ where u is a velocity can be understood as a transformation that moves the system to a new reference frame that is moving with velocity u. Under this transformation, the Lagrangian changes as

$$L \to \tilde{L} = L + u\sum_i m_i \dot{x}_i + \frac{u^2}{2}\sum_i m_i \tag{12}$$

For an infinitesimal transformation where u is small we can discard any term that is higher than the first order in u; therefore,

$$\tilde{L} \approx L + u\sum_i m_i \dot{x}_i \tag{13}$$

The change of the action under the same infinitesimal transformation is given by

$$S \to \tilde{S} = S + u\sum_i \int_{t_0}^{t_1} m_i \dot{x}_i dt \tag{14}$$

We then have

$$\tilde{S} = S + u \sum_i m_i x_i \Big|_{t_0}^{t_1} \tag{15}$$

In the last equality, we used the fact that $\int_{t_0}^{t_1} \dot{x} = x(t_1) - x(t_0)$. The last term in equation (15) is a boundary term $\Phi = u \sum_i m_i x_i$. We can write the change of the action as

$$\delta S = \Phi(t_1) - \Phi(t_0) \tag{16}$$

where $\Phi = u \sum_i m_i \dot{x}_i$.

Using the generalized form of Noether's theorem we find the conserved quantity as

$$\sum_i \frac{\partial L}{\partial \dot{x}_i} \delta x_i - \Phi \tag{17}$$

hence

$$\sum_i m_i \dot{x}_i u t - \sum_i m_i x_i u = -u[-P + MX_{\text{CM}}] = \text{const} \tag{18}$$

where we have defined the total mass $M = \sum_i m_i$, the total momentum $P = \sum_i m_i \dot{x}_i$, and the center of mass coordinate $X_{\text{CM}} = \sum_i m_i x_i / M$. Therefore, we conclude that the center of mass coordinate follows a uniform linear motion with the velocity $V = P/M$. In a way, we could have inferred this result from the conservation of the total momentum, and relating the momentum to that of the center of mass coordinate.

3. The scattering rate $\sigma(\Theta)$ for a repulsive Coulomb potential can be easily obtained from the scattering formula of an attractive potential by interchanging α with $-\alpha$.

4. We first show that the scattering of the particle is equivalent to refraction of light rays. To this end, note that the potential is constant inside and outside the sphere, hence the particle follows a straight line with a constant velocity in each region although it may have different velocities in the two regions. Now let's consider a particle incident on the surface that separates the two regions. In the outer region, the velocity is given by $v_1 = |\vec{v}_1| = \sqrt{2mE}$, while in the inner region where the potential is a nonzero constant $V = -V_0$, the velocity is given by $\frac{1}{2} m \vec{v}_2^2 - V_0 = E$, or equivalently $v_2 = |\vec{v}_2| = \sqrt{2m(E + V_0)}$. Now upon hitting the surface, the component of the velocity tangent to the surface, $\vec{v}_{\|}$, does not change; this is because the force is normal to the surface (note that $\vec{F} = -\nabla V(r)$ is along the radial direction, and in general is normal to a surface of a constant potential). The normal component of the velocity, however, changes as the particle enters the scattering region. Let θ_1 be the angle between the normal and incident velocity, $\sin\theta_1 = |v_{\|}|/v_1$. Similarly, we define θ_2 as the angle between the normal to the surface and the velocity in the inner region, $\sin\theta_2 = |\vec{v}_{\|}|/v_2$. Therefore, we find (using the fact that tangential components are identical)

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_2}{v_1} = \sqrt{\frac{E + V_0}{E}} \equiv n \tag{19}$$

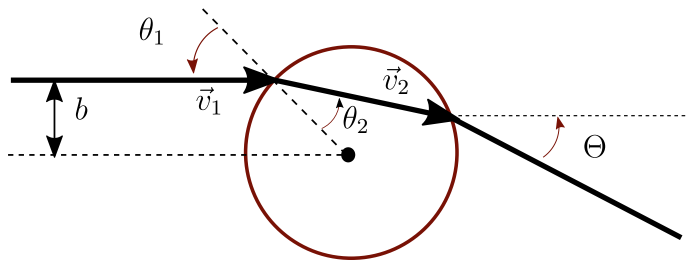

Figure 1: The circle defines the scattering region of the radius a. The incident (θ_1), refracted (θ_2), and the scattering (Θ) angles are shown in the figure.

This shows that the scattering is equivalent to that of light refraction.

Next we express the scattering angle in terms of the incident and transmitted angles as $\Theta = 2(\theta_1 - \theta_2)$. Exploiting the analogy with the light refraction, we have $\sin(\theta_1) = \frac{b}{a}$ and that $n = \frac{\sin(\theta_1)}{\sin(\theta_2)}$, hence we obtain that $\sin(\theta_2) = \frac{b}{na}$. Notice that here b is the impact parameter. By expressing the scattering angle Θ in terms of a and b we obtain

$$\Theta = 2(\arcsin(\frac{b}{a}) - \arcsin(\frac{b}{na})) \tag{20}$$

The differential cross section is given by

$$d\sigma(\Theta) = \frac{sds}{\sin(\Theta)d\Theta} \tag{21}$$

which can be manipulated into the following expression

$$d\sigma(\Theta) = \frac{sds}{\sin(\Theta)d\Theta} = \frac{d(s^2)/d\Theta}{2\sin(\Theta)} = \frac{1}{4\sin(\frac{\Theta}{2})\cos(\frac{\Theta}{2})}\frac{d(s^2)}{d\Theta} \tag{22}$$

In order to obtain an expression for the cross section in terms of the angles we have to solve for b^2 in terms of Θ. By substituting the expression of Θ in terms of θ_1,θ_2 and b into $\sin(\frac{\Theta}{2})$ and $\cos(\frac{\Theta}{2})$ we obtain that (after using many trigonometric identities)

$$\sin(\frac{\Theta}{2}) = \frac{b}{na^2}(\sqrt{n^2a^2 - b^2} - \sqrt{a^2 - b^2}) \tag{23}$$

$$\cos(\frac{\Theta}{2}) = \frac{1}{na^2}(\sqrt{a^2 - b^2}\sqrt{n^2a^2 - b^2} + b^2) \tag{24}$$

which is surprisingly independent of the incident and transmitted angles. By squaring the sine term we obtain that

$$\sin^2(\frac{\Theta}{2}) = \frac{b^2}{n^2a^4}(n^2a^2 - b^2 - 2\sqrt{n^2a^2 - b^2}\sqrt{a^2 - b^2} + a^2 - b^2) \tag{25}$$

From equation (24) we can identify the second term in equation (25) as

$$\sqrt{n^2a^2 - b^2}\sqrt{a^2 - b^2} = na^2 \cos(\frac{\Theta}{2}) - b^2 \tag{26}$$

hence we obtain that

$$\sin^2(\frac{\Theta}{2}) = \frac{b^2}{n^2a^2}(1 + n^2 - 2n\cos(\frac{\Theta}{2})) \tag{27}$$

yielding the following expression for b^2

$$b^2 = \frac{n^2a^2\sin^2(\frac{\Theta}{2})}{1 + n^2 - 2n\cos(\frac{\Theta}{2})} \tag{28}$$

Now we calculate the derivative of equation (28) with respect to Θ and obtain

$$\frac{ds^2}{d\Theta} = \frac{n^2a^2\sin(\frac{\Theta}{2})(n\cos(\frac{\Theta}{2}) - 1))(n - \cos(\frac{\Theta}{2}))}{(1 - 2n\cos(\frac{\Theta}{2}) + n^2)^2} \tag{29}$$

By substituting equation (29) into equation (22) we find that the differential cross section is given by

$$d\sigma(\Theta) = \frac{n^2a^2}{4\cos(\frac{\Theta}{2})}\frac{(n\cos(\frac{\Theta}{2}) - 1))(n - \cos(\frac{\Theta}{2}))}{(1 - 2n\cos(\frac{\Theta}{2}) + n^2)^2} \tag{30}$$

To obtain the total cross section we integrate equation (30) from $\Theta = 0$ up to its maximum value. This maximum value is obtained when the differential cross section is equal to zero. From equation (30) we see that this happens when $n\cos(\frac{\Theta}{2}) - 1 = 0 \Rightarrow \Theta_{\text{max}} = 2\arccos(\frac{1}{n})$. Thus, the total cross section is given by

$$\sigma = \int_0^{2\pi}\int_0^{2\arccos(\frac{1}{n})} \frac{n^2a^2}{4\cos(\frac{\Theta}{2})}\frac{(n\cos(\frac{\Theta}{2}) - 1))(n - \cos(\frac{\Theta}{2}))}{(1 - 2n\cos(\frac{\Theta}{2}) + n^2)^2}\sin(\Theta)d\Theta d\phi = \pi a^2 \tag{31}$$

5. We start by rewritting the expression of the total cross section given in the problem as

$$\sigma(\Theta) = \frac{\alpha}{\pi}\frac{1 - \frac{\Theta}{\pi}}{(2 - \frac{\Theta}{\pi})^2(\frac{\Theta}{\pi})^2\sin(\Theta)} \tag{32}$$

We now determine the impact parameter using

$$b^2 = 2\int_\Theta^\pi \sigma(\Theta)\sin(\Theta)d\Theta = \frac{2\alpha}{\pi}\int_\Theta^\pi \frac{1 - \frac{\Theta}{\pi}}{(2 - \frac{\Theta}{\pi})^2(\frac{\Theta}{\pi})^2}d\Theta \tag{33}$$

This integral can be computed explicitly as

$$b^2 = \frac{\alpha(\Theta - \pi)^2}{\pi^2 - (\Theta - \pi)^2} \tag{34}$$

Solving for Θ we obtain that

$$\Theta = \pi(1 - \frac{b}{\sqrt{b^2 + \alpha}}) \tag{35}$$

From here we find that

$$T = \int_y^\infty \frac{\Theta}{\pi\sqrt{b^2 - y}} db = \int_y^\infty \frac{(1 - \frac{b}{\sqrt{b^2+\alpha}})}{\sqrt{b^2 - y}} db \tag{36}$$

This integral can also be evaluated explicitly as

$$T = \frac{1}{2} \ln\left(\frac{\sqrt{b^2(b^2 - y^2)} + b^2 - \frac{y^2}{2}}{\sqrt{(b^2 + \alpha)(b^2 - y^2)} + b^2 - \frac{y^2}{2} + \frac{\alpha}{2}}\right)\Big|_y^{b^2 \to \infty} = \ln\left(\sqrt{1 + \frac{\alpha}{y^2}}\right) \tag{37}$$

From this equation together with the relation $r(y) = y \exp(T(y))$, we obtain $r = y\sqrt{1 + \frac{\alpha}{y^2}}$, hence $y^2 = r^2 - \alpha$. This allows us to determine the potential as

$$V = E\frac{\alpha}{r^2} \tag{38}$$

From here we identify $K = E\alpha$.