Solutions to PHY820/422 HW #4

1. Using the variational principle, the goal of this problem is to show something really simple: A free particle undergoes a uniform linear motion, or, more precisely, it moves from a position x_1 at time t_1 to a position x_2 at time t_2 at a constant speed $v = (x_2 - x_1)/(t_2 - t_1)$. The equation of motion is determined from the minimum of the action over all possible trajectories $x(t)$ such that $x(t_1) = x_1$ and $x(t_2) = x_2$. To treat the boundary conditions, let's consider a general trajectory of the form

$$x(t) = x_1 + \frac{x_2 - x_1}{t_2 - t_1}(t - t_1) + f(t) \tag{1}$$

where the function $f(t)$ has the property that $f(t_1) = f(t_2) = 0$ but is otherwise completely arbitrary.

   (a) We first need to define the space of all possible trajectories. To do so, let's define a complete set of functions as

$$f_n(t) = \sqrt{\frac{2}{t_2 - t_1}} \sin\left(\frac{n\pi(t - t_1)}{t_2 - t_1}\right), \qquad n = 1, 2, 3, \cdots \tag{2}$$

   Convince yourself that this is a complete basis. (An analogy with the quantum mechanical problem of a particle in a box may be useful.) Furthermore, show that it defines an orthonormal basis, i.e.,

$$\int_{t_1}^{t_2} f_n(t) f_m(t) dt = \delta_{nm} \tag{3}$$

   where the Kronecker delta function δ_{nm} is unity if $n = m$ and zero otherwise.

   (b) Consider a general trajectory $f(t) = \sum_{n=1,2,\cdots} a_n f_n(t)$ with arbitrary coefficients $a_1, a_2, \cdots$. Using the orthornomal property of these functions[1], construct the action $S = (m/2)\int_{t_1}^{t_2} \dot{x}^2 dt$ in terms of the coefficients a_n. Show that the minimum of the action is given by all $a_n = 0$, hence $f(t) = 0$ and particle moves at a constant speed.

2. A particle is free to move on a circle under the influence of no forces other than those than constrain it to the circle. It starts at an angle θ_1 at time t_1 and ends at another angle θ_2 at time t_2.

   (a) Show that there are a (countably) infinite number of physical paths $\theta(t)$ the particle can take in doing so; label different trajectories by n. Therefore, the variational principle in this case does not deterministically specify the motion.

   (b) **Bonus:** In quantum mechanics, on the other hand, there is no unique trajectory, but instead one should sum over all possible trajectories where each trajectory contributes a phase e^{iS_n} with S_n the action corresponding to the trajectory labeled by n. What quantum number does n represent?

[1] You also need the fact that $\int_{t_1}^{t_2} f_n(t) = 0$, that is, the function f_n is also orthogonal to a constant function.

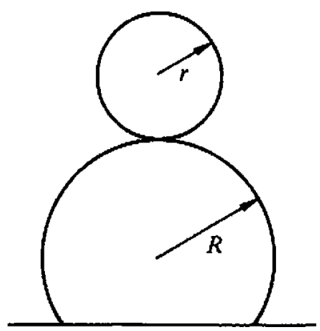
3. José and Saletan, Chapter 3, problem 22

   Use the Lagrange multiplier method to solve the following problem. A particle in a uniform gravitational field is free to move without friction on a paraboloid of revolution whose symmetry axis is vertical (opening upward). Obtain the force of constraint. Prove that for given energy and angular momentum about the symmetry axis there are a minimum and maximum height to which the particle will go.

   Note: Use cylindrical coordinates (z, ρ, ϕ) with z along the vertical axis, and ρ and ϕ the radial and (polar) angular coordinates. The paraboloid is defined by the equation $z = a\rho^2$.

4. Goldstein (Ed. 2), Chapter 2, Problem 12

   A uniform hoop of mass m and radius r rolls without slipping on a fixed cylinder of radius R as shown in the figure. The only external force is that of gravity. If the smaller cylinder starts rolling from rest on top of the bigger cylinder, find by method of Lagrange multipliers the point at which the hoop falls off the cylinder.



5. Follow up on the problem covered during class.

   In class, we discussed the problem of a bead that is constrained to a loop whose radius is changing linearly in time, $R(t) = ct$. Recall that the equations of motion are given by

$$\ddot{x} = \frac{c^2 - \dot{x}^2 - \dot{y}^2}{c^2 t^2} x$$
$$\ddot{y} = \frac{c^2 - \dot{x}^2 - \dot{y}^2}{c^2 t^2} y$$

   (a) Show that the constraint force is in the radial direction, and find its magnitude. Is the constraint force pointing towards or away from the center? Does your answer change if, instead of an expanding wire, the radius is decreasing in time at a constant rate? Explain.

   (b) Repeat this problem, using the method of Lagrange multipliers, for a loop whose radius is a general function of time $R = R(t)$. Derive the new equations of motion, determine the constraint forces, and discuss how your answers to part (a) may be different. Show that the constraint force is precisely the force needed to generate a radial acceleration $\ddot{R}(t) - v_\theta^2/R(t)$ with v_θ the angular velocity tangent to the loop.

Solutions

1. (a) You can see that the set of functions $f_n(x)$ form a complete basis by reminding yourself about the quantum mechanical problem of a free particle in an infinite square well of size L. The Hamiltonian for a free particle in quantum mechanics is $H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$. The infinite square well imposes the boundary conditions $\psi(x=0) = \psi(x=L) = 0$ and there you can show that a complete set of solutions are provided by $\psi_n = \sin(n\pi x/L)$ the reason being that they exhaust all possible solutions to the Hamiltonian that is a hermittian operator, and thus form a complete basis. The generalization to the problem here is straightforward. The fact that these functions are orthonormal is also straightforward to check.

   (b) To compute the Lagrangian of a free particle $L = \frac{1}{2}\dot{x}^2$, let us first compute $\dot{x}$ explicitly. Let $X \equiv x_2 - x_1$ and $T \equiv t_2 - t_1$, and $V \equiv X/T$. We have

$$\dot{x} = V + \sum_{n=1}^{\infty} \frac{n\pi}{T}\sqrt{\frac{2}{T}} \cos\left(\frac{n\pi(t-t_1)}{T}\right) \tag{4}$$

   Plugging this expression in the Lagrangian and using the fact that the conine functions are also orthogonal for different n and m and to a constant function, we find

$$S = \frac{1}{2}mV^2T + \sum_n \frac{n^2\pi^2}{2mT^2}a_n^2 \tag{5}$$

   Next we should minimize the action to find the equation of motion. Keep in mind that the only free parameters are a_ns. A minimum of the action is clearly given by choosing all $a_n = 0$ because each term in the sum contributes a strictly positive number for other choices of a_n. This proves what we wanted, namely that the minimum of the action for a free particle leads to a uniform linear motion.

2. (a) A free particle moving at a constant rate is described by the equation

$$\ddot{\theta} = 0 \tag{6}$$

   In other words, the velocity $\dot{\theta}$ has to be constant. In Euclidean space, there is only one trajectory that takes you from a given point A to another point B at a constant speed. On a circle, on the other hand, you can go around the circle a finite number of times before arriving at the final destination. Formally, different solutions can be characterized as

$$\theta(t) = \theta_1 + \frac{\theta_2 - \theta_1 + 2n\pi}{t_2 - t_1}(t - t_1) \tag{7}$$

   where n is an integer number which indicates how many times the particle rotates around the full circle before arriving at θ_2 at time t_2. Notice that at time t_2, the angle is $\theta = \theta_2 + 2n\pi$ but if you live on a circle, θ and $\theta + 2n\pi$ are the same thing! The path with $n = 0$ still gives the absolute minimum of the action (assuming that

$0 < \theta_2 - \theta_1 < \pi$); however, all other values of n provide other local minima of the action. Classical mechanics, being a deterministic theory, leads the particle to follow the path of the absolute minimum of the action. In quantum mechanics, on the other hand, the particle follows a superposition of different paths. The physical interpretation of each path is given in the next part.

(b) In quantum mechanics, n labels quantized angular momentum. Unlike classical mechanics where the angular momentum can take any value, in quantum mechanics it will be quantized.

3. The Lagrangian in cylindrical coordinates is given by

$$L = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - mgz \tag{8}$$

The system is subject to the following constraint

$$f(\rho, \phi, z) = a\rho^2 - z = 0 \tag{9}$$

describing a paraboloid. By multiplying the constraint f by a Lagrange multiplier λ and adding it to the Lagrangian, we obtain

$$\tilde{L} = L + \lambda f = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - mgz + \lambda(a\rho^2 - z) \tag{10}$$

The Euler-Lagrange equations of motion are then given by

$$m\ddot{\rho} - m\rho\dot{\phi}^2 - 2\lambda a\rho = 0 \tag{11}$$

$$m\ddot{z} + mg + \lambda = 0 \tag{12}$$

$$\frac{d}{dt}(m\rho^2\dot{\phi}) = 0 \Rightarrow m\rho^2\dot{\phi} = l \tag{13}$$

The last equation implies that angular momentum is conserved. The forces of constraint are obtained as

$$N_\rho = -\lambda\frac{\partial f}{\partial \rho} = -2\lambda a\rho \tag{14}$$

$$N_z = -\lambda\frac{\partial f}{\partial z} = \lambda \tag{15}$$

$$N_\phi = -\lambda\frac{\partial f}{\partial \phi} = 0 \tag{16}$$

Now we solve for the Lagrange multiplier λ by eliminating the accelerations and $\dot{\phi}$ from the Euler-Lagrange equations and the constraint equations. From equation (12) we obtain

$$\ddot{z} = \frac{\lambda}{m} - g = 2a(\dot{\rho}^2 + \rho\ddot{\rho}) \tag{17}$$

The last equality follows from taking the second-order time derivative of the constraint defined in equation (9). We now obtain the following expression for $\ddot{\rho}$

$$\ddot{\rho} = \frac{\lambda}{2ma\rho} - \frac{g}{2a\rho} - \frac{\dot{\rho}^2}{\rho} \tag{18}$$

By substituting equation (18) into equation (11) and then solving for λ we obtain

$$\lambda = \frac{1}{1 + 4a^2\rho^2}[m(g + 2a\dot{\rho}^2) + \frac{2al^2}{m\rho^2}] \tag{19}$$

Using the definition of λ obtained in equation (19) we can now calculate the constraint forces in terms of ρ and $\dot{\rho}$ by substituting λ into equations (14)-(16). To find the energy of the system we eliminate z and $\dot{\phi}$ in the definition of the Lagrangian by using the constraint equation and the definition of angular momentum we obtained

$$L = \frac{m}{2}(4a^2\rho^2 + 1)\dot{\rho}^2 - \frac{l^2}{2m\rho^2} - mga\rho^2 \tag{20}$$

The energy of the system is then given by

$$E = \dot{\rho}\frac{\partial L}{\partial \dot{\rho}} - L = \frac{m}{2}(4a^2\rho^2 + 1)\dot{\rho}^2 + \frac{l^2}{2m\rho^2} + mga\rho^2 \tag{21}$$

The turning points are given when $\dot{\rho} = 0$, hence we obtain

$$E = -\frac{l^2}{2m\rho^2} + mga\rho^2 = \frac{al^2}{2mz} + mgz \tag{22}$$

Solving for z in equation (22) we obtain

$$z = \frac{1}{2mg}[E \pm \sqrt{E^2 - 2l^2ag}] \tag{23}$$

These are the minimum and maximum heights that the particle will go for a given energy E and angular momentum l about the symmetry axis.

4. We will setup our system of coordinates with the origin coinciding with the center of the cylinder. In this case we can describe the motion of the hoop with by the angle ϕ. We will also need an additional angular coordinate to characterize the rotation of the hoop about its axis ψ. The kinetic energy of the hoop is then given by

$$T = \frac{m}{2}v^2 + \frac{1}{2}I\dot{\psi}^2 = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\phi}^2) + \frac{mr^2}{2}\dot{\psi}^2 \tag{24}$$

where $I = mr^2$ is the moment of inertia of the hoop. The potential energy is given by

$$V = mg\rho\sin(\phi) \tag{25}$$

Thus, the Lagrangian for this system is given by

$$L = T - V = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\phi}^2) + \frac{mr^2}{2}\dot{\psi}^2 - mg\rho\sin(\phi) \tag{26}$$

We now need to formulate equations for the constraints of the hoop. Since we know that the hoop lies on the surface of the cylinder and that the hoop is rolling without slipping, then we have that

$$f_1 = \rho - (R + r) = 0 \tag{27}$$

$$f_2 = (R + r)\phi + r\psi = 0 \tag{28}$$

To each constraint we associate a Lagrange multiplier λ_1 and λ_2 respectively. The first constraint is the one that will give us an expression for the normal force that the hoop experiences, thus allowing us to determine at what point does the hoop fall from the cylinder. We now define the Lagrangian including the constraints as

$$\tilde{L} = L + \lambda_1 f_1 + \lambda_2 f_2 \tag{29}$$

The Euler-Lagrange equations of motion are then given by

$$\frac{d}{dt}\frac{\partial\tilde{L}}{\partial\dot{\rho}} - \frac{\partial\tilde{L}}{\partial\rho} = m\ddot{\rho} - m\rho\dot{\phi}^2 + mg\sin(\phi) - \lambda_1 = 0 \tag{30}$$

$$\frac{d}{dt}\frac{\partial\tilde{L}}{\partial\dot{\phi}} - \frac{\partial\tilde{L}}{\partial\phi} = m\rho^2\ddot{\phi} + 2m\rho\dot{\rho}\dot{\phi} + mg\rho\cos(\phi) - \lambda_2(R + r) = 0 \tag{31}$$

$$\frac{d}{dt}\frac{\partial\tilde{L}}{\partial\dot{\psi}} - \frac{\partial\tilde{L}}{\partial\psi} = mr^2\ddot{\psi} - \lambda_2 r = 0 \tag{32}$$

Now by solving for ρ in equation (27) we can conclude that $\dot{\rho} = \ddot{\rho} = 0$. This in turn simplifies equations (30) and (31) into

$$-m(R + r)\dot{\phi}^2 + mg\sin(\phi) - \lambda_1 = 0 \tag{33}$$

$$m(R + r)^2\ddot{\phi} + mg(R + r)\cos\phi - \lambda_2(R + r) = 0 \tag{34}$$

By substituting the condition in equation (28) in equation (32) we obtain

$$-m(R+r)\ddot{\phi} = \lambda_2 \tag{35}$$

Using this expression for λ_2 in equation (34) we obtain

$$\ddot{\phi} = -\frac{g}{2(R+r)}\cos(\phi) \tag{36}$$

Note that if we multiply equation (37) by $\dot{\phi}$ then we obtain

$$\ddot{\phi}\dot{\phi} + \frac{g}{2(R+r)}\dot{\phi}\cos(\phi) = 0 \rightarrow \frac{d}{dt}[\frac{\dot{\phi}^2}{2} + \frac{g}{2(R+r)}\sin(\phi)] = 0 \tag{37}$$

Thus we can conclude that the following quantity is a constant

$$\frac{\dot{\phi}^2}{2} + \frac{g}{2(R+r)}\sin(\phi) = C \tag{38}$$

By solving for $\dot{\phi}^2$ and using this expression in equation (33) we obtain

$$\lambda_1 = mg(2\sin(\phi) - 1) \tag{39}$$

The hoop will fall off the cylinder when the normal force is zero, or equivalently, when λ_1 is zero. This happens when $\phi = \frac{\pi}{6}$.

5. (a) The only force acting on the particle is the constraint force $\mathbf{C}$. Comparing the equations of motion against the Newton's law $m\mathbf{a} = \mathbf{F}$, we identify the constraint force as

$$\mathbf{C} = \frac{c^2 - \dot{x}^2 - \dot{y}^2}{c^2t^2}\mathbf{r} \tag{40}$$

where $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ is the position vector. Therefore, the force is along the radial direction, $\mathbf{C} = C_r\hat{\mathbf{r}}$ where C_r is the component of the force along the radial direction, and is given by (note that $|\mathbf{r}| = R(t) = ct$)

$$C_r = \frac{c^2 - \dot{x}^2 - \dot{y}^2}{ct} \tag{41}$$

To find the (inward or outward) direction of the radial force, note that the magnitude of the velocity $v = \sqrt{\dot{x}^2 + \dot{y}^2}$ can be written in cylindrical coordinates $v = \sqrt{\dot{R}^2 + v_\theta^2} = \sqrt{c^2 + v_\theta^2}$ where v_θ is the angular velocity. Plugging the latter expression in the above equation for the radial component of the force yields

$$C_r = -\frac{v_\theta^2}{ct} = -\frac{v_\theta^2}{R(t)} \tag{42}$$

Therefore the constraint force simply provides the force for the acceleration of a particle rotating on a circle. The force does not depend on whether the size of the radius is increasing or decreasing as a function of time as long as it is changing at a constant rate.

(b) The constraint in this case is

$$f(x, y, t) = x^2 + y^2 - R(t)^2 = 0 \tag{43}$$

The Lagrangian including the Lagrange multiplier for the constraint is

$$\tilde{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \lambda(t)(x^2 + y^2 - R(t)^2) \tag{44}$$

The equations of motion that follow from the Lagrangian are given by

$$m\ddot{x} - 2\lambda x = 0, \tag{45}$$

$$m\ddot{y} - 2\lambda y = 0, \tag{46}$$

To use the constraint, let's take time derivatives of Eq. (43) to obtain

$$x\dot{x} + y\dot{y} - R\dot{R} = 0 \tag{47}$$

following from $\dot{f} = 0$ and

$$\dot{x}^2 + x\ddot{x} + \dot{y}^2 + y\ddot{y} - \dot{R}^2 - R\ddot{R} = 0 \tag{48}$$

following from $\ddot{f} = 0$. Substituting $\ddot{x}$ and $\ddot{y}$ from the equations of motion in the last equation, we find an equation in terms of $x, y, \dot{x}, \dot{y}$ from which we can solve λ as (also using the constraint $x^2 + y^2 = R(t)^2$)

$$\lambda = m\frac{\dot{R}^2 + R\ddot{R} - \dot{x}^2 - \dot{y}^2}{2R(t)^2} \tag{49}$$

Using the fact that $v^2 = \dot{x}^2 + \dot{y}^2 = \dot{R}^2 + v_\theta^2$, we can rewrite the last equation as

$$\lambda = m\frac{R\ddot{R} - v_\theta^2}{2R(t)^2} \tag{50}$$

Substituting λ in the equations of motion from the expression in the last equation, we find the constraint force ($m\ddot{\mathbf{r}} = \mathbf{C}$) as

$$\mathbf{C} = C_r\hat{\mathbf{r}} = m\frac{R\ddot{R} - v_\theta^2}{R(t)}\hat{\mathbf{r}} \tag{51}$$

You may recall that the expression on the right hand side is nothing but the acceleration along the radial direction in cylindrical coordinates. Therefore, the constraint force is again responsible to provide the radial acceleration on a trajectory $(R(t), \theta(t))$. Depending on $\ddot{R}$, the acceleration (or the force) can point towards or away from the center.