PHY820/422 HW #8 — Due Monday 11/06/17 @ 5pm
Hamiltonians & Poisson Brackets

1. **Conserved quantities from Hamiltonian.** Given the Hamiltonian

$$H = xp_x - yp_y - ax^2 + by^2 \tag{1}$$

where a and b are constants show that the three functions

$$f_1 = \frac{p_y - by}{x}, \quad f_2 = xy, \quad f_3 = xe^{-t} \tag{2}$$

are constants of motions. Are they functionally independent? Do there exist other independent constants of the motion? Find them if they do exist. Show explicitly that the Poisson Bracket of any two conserved quantities itself is conserved.

2. **Conserved quantities from Hamiltonian.** For a one-dimensional system with the Hamiltonian

$$H = \frac{p^2}{2} - \frac{1}{2q^2} \tag{3}$$

   (a) Show that there is a constat of the motion

$$D = \frac{pq}{2} - Ht \tag{4}$$

   (b) As a generalization of part (a), for motion in a plane with the Hamiltonian

$$H = |\mathbf{p}|^n - ar^{-n} \tag{5}$$

   where $\mathbf{p}$ is the vector of the momenta conjugate to the Cartesian coordinates, show that there is constant of the motion

$$D = \frac{\mathbf{p} \cdot \mathbf{r}}{n} - Ht \tag{6}$$

   (c) The transformation $Q = \lambda q$, $P = p/\lambda$ is obviously canonical. However, the same transformation with t *time dilation*, $Q = \lambda q$, $P = p/\lambda$, $t' = \lambda^2 t$, is not. Show that, however, the equations of motion for p and q in the Hamiltonian in part (a) are invariant under the transformation. The constant of the motion is said to be associated with this invariance.

3. **Poisson Brackets.**

   (a) Prove that the Poisson bracket satisfies the Jacobi idenitity, i.e., that $\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0$ for any three dynamical variables A, B, and C.

(b) Prove that the Poisson bracket satisfies the Leibnitz rule, i.e., that $\{A, BC\} = B\{A, C\} + \{A, B\}C$.

4. **Poisson Brackets of angular and linear momentum.** Using the definition of angular momenta (see lecture notes), show the following relations

$$\{L_x, L_y\} = L_z, \quad \{L_y, L_z\} = L_x, \quad \{L_z, L_x\} = L_y,$$
$$\{p_x, L_z\} = -p_y, \quad \{p_y, L_z\} = p_x, \quad \{p_z, L_x\} = p_y, \quad \{p_z, L_y\} = -p_x$$

5. *** Bonus * Laplace-Rung-Lenz vector**. For the Kepler problem, there exists in addition to the angular momentum, $\mathbf{L}$, another conserved vector quantity, $\mathbf{A}$, the Laplace-Rung-Lenz vector defined as

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \frac{mk\mathbf{r}}{r} \tag{7}$$

(a) Using your favorite method, show that this vector is conserved. (My favorite method is using Poisson Brackets.)

(b) Use this fact to derive the orbit equation of the Kepler problem.

(c) Find the Poisson bracket of the components of this vector (A_x, A_y, A_z) with each other and with those of angular momentum, (L_x, L_y, L_z). Show explicitly that the Poisson Bracket of all conserved quantities are themselves conserved.

6. **Examples of canonical transformations.** Show that

(a) the following transformation is canonical

$$Q = \log\left(\frac{1}{q}\sin p\right), \quad P = q\cot p \tag{8}$$

(b) the following transformation is canonical for any choice of the constant α

$$Q = \arctan\frac{\alpha q}{p}, \quad P = \frac{\alpha q^2}{2}\left(1 + \frac{p^2}{\alpha^2 q^2}\right) \tag{9}$$

(c) the following transformation is canonical for any choice of α

$$Q = q\cos\alpha - p\sin\alpha, \quad P = q\sin\alpha + p\cos\alpha \tag{10}$$

What canonical transformations do $\alpha = 0, \pi/2$ represent?

7. **Gauge canonical transformation.** We saw that the equations of motion are unchanged when a total derivative $d\psi(q,t)/dt$ is added the Lagrangian. Such a transformation is also called a gauge transformation. How does the Hamiltonian change under such a gauge transformation? [Remark: The momenta p_α change to new momenta p'_α, and the transformation from (q, p) to (q, p') is a particular kind of canonical transformation called a gauge canonical transformation.]