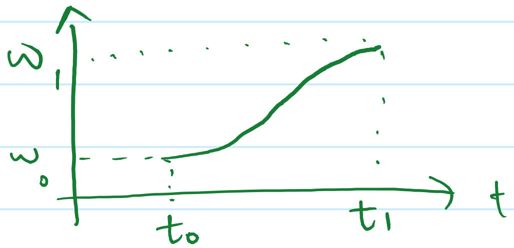
Adiabatic theorem

When the Hamiltonian depends on time, it can be really hard to solve even when we are dealing with one degree of freedom. This is because the energy is not conserved in a time-dependent Hamiltonian.

Example: Consider the harmonic oscillator with a time-dependent natural frequency:

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega(t)^2 q^2$$

$\omega(t)$ depends on time



Energy is clearly not conserved $\left(\frac{\partial H}{\partial t} \neq 0 \right)$.

But when time dependence is sufficiently slow there often exist dynamical variables that are almost constant. But first let's say more precisely what we mean by slow. While the change of frequency $\Delta\omega = \omega_1 - \omega_0$ can be arbitrarily large, the time $\Delta t = t_1 - t_0$ over which it changes occurs must be very long: $\Delta\omega / \Delta t$ should be small. This is called the adiabatic assumption.

In the example of the harmonic oscillator, ω changes and most likely also the energy E. But what may remain a constant of motion??

It turns out that while both E and ω change, if the change of ω is adiabatic their ratio remains the same!!

$$\frac{E(t)}{\omega(t)} = \text{Const.}$$

The argument that leads to this is called the adiabatic theorem. We will make a heuristic argument why this should hold. But let's consider the more general case of an arbitrary Hamiltonian (not necessarily harmonic oscillator) for one degree of freedom

$$H(q, p, \lambda(t))$$

where λ is some parameter in the Hamiltonian (like the frequency in the example of the harmonic oscillator) that changes slowly as a function of time.

The change of the Hamiltonian is

$$\frac{d}{dt} H = \underbrace{\frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p}} + \frac{\partial H}{\partial t}$$

Hamilton's equations

$$= 0 + \frac{\partial H}{\partial \lambda}\dot{\lambda}(t)$$

The Hamiltonian only explicitly depends on time through the dependence of λ on t.

The total change of the Hamiltonian in time is given by

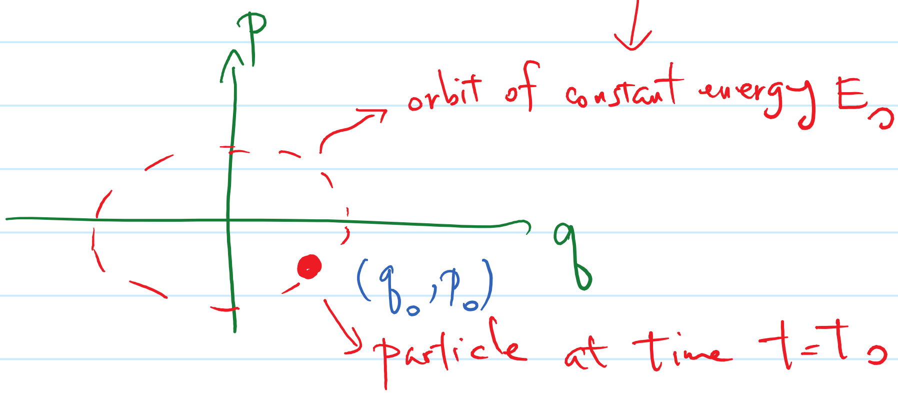
$$\Delta E \equiv \Delta H = \int_{t_0}^{t_1} \frac{\partial H}{\partial \lambda}\dot{\lambda}\, dt$$

The adiabatic assumption is that $\dot{\lambda}$ is small.

Let's start with a particle of initial energy E_0

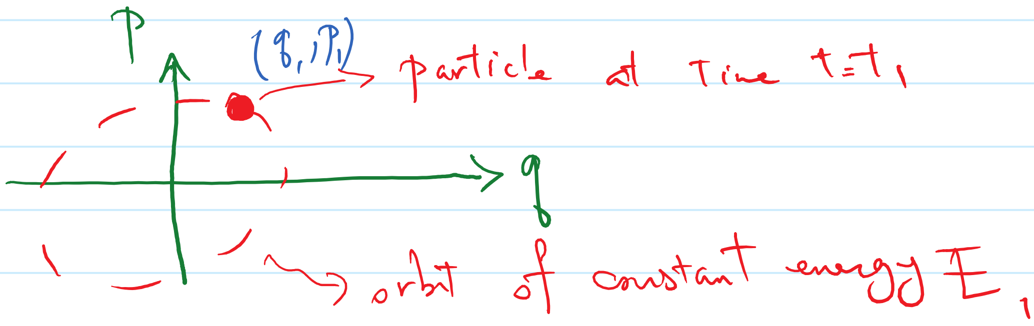
$$E_0 = H(q_0, p_0, \underbrace{\lambda_0}_{\text{Parameter at time } t=t_0})$$

In the phase-space coordinates



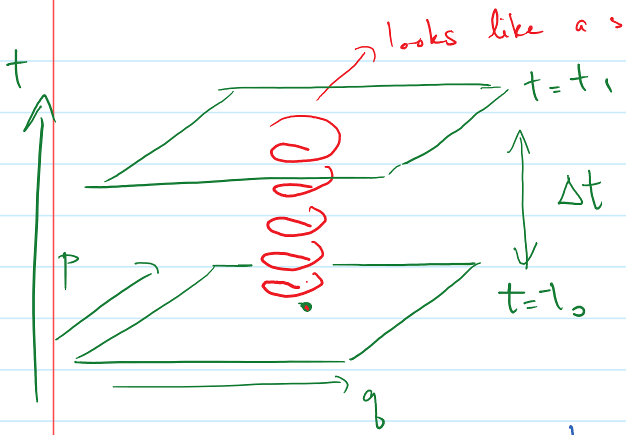
Fact 1. With adiabatic assumption, particles on the same orbit, i.e. with the same energy, at time $t=t_0$ end up on an orbit of the same energy E_1 at time t_1.

after time Δt



In other words, the final energy E_1 of the particle does not depend on where exactly the motion started on the orbit of energy E_0!

To see why this happens, let's look at motion as a function of time



The motion over a long time period Δt involves many cycles (think about the harmonic oscillator). Over each cycle, the parameter λ barely changes.

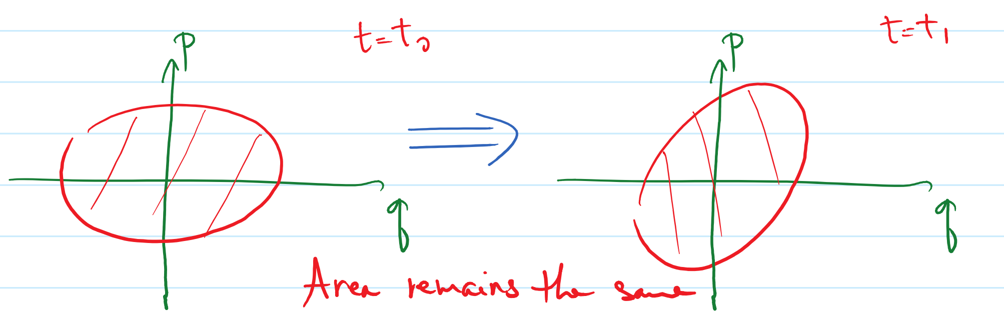
Therefore,

$$\Delta H = \int_{t_0}^{t_1} \frac{\partial H}{\partial \lambda} \dot{\lambda}\, dt$$

$$\approx \int_{t_0}^{t_1} \underbrace{\overline{\frac{\partial H}{\partial \lambda}}}_{\text{average over one cycle}} \dot{\lambda}\, dt$$

Replacing $\frac{\partial H}{\partial \lambda}$ by its average over one cycle completely washes out information about where exactly the orbit started from.

Summary: Starting from an orbit of constant energy at time $t = t_0$ we end up on a (different) orbit of constant energy at time t_1



Fact 2: Use Liouville's theorem:

The area enclosed by the orbit is the same!
This is the adiabatic invariant of motion

$$J = \frac{1}{2\pi} \int_{R(t)} p\, dq$$

$R(t)$: the region enclosed by an orbit of energy $E(t)$

$$= \text{area} = \text{constant}$$

> The "action" variable $J = \frac{1}{2\pi} \int p\, dq$ is an adiabatic invariant of motion, that is, it remains constant for sufficiently slow change of the Hamiltonian.

Example: Back to harmonic oscillator

$$H = \frac{1}{2m} p^2 + \frac{1}{2} m\, \omega(t)^2\, q^2$$

We showed in previous lecture that

$$J = \frac{E}{\omega}$$

($E = H$ is the value of energy)

Under adiabatic motion,

$$J = \frac{E(t)}{\omega(t)} = \text{Const}.$$

I.e., when $\omega(t)$ changes, so does $E(t)$ but remains proportial to $\omega(t)$.

The orbit in phase space is an ellipse whose shape can change, but whose area is the same.

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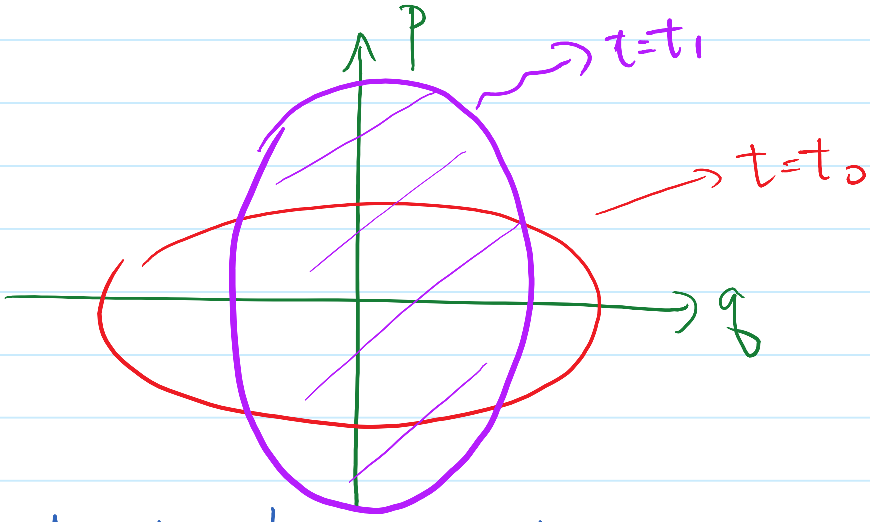

Although the shape may change as $\omega = \omega(t)$ changes as a function of time, the area remains the same.

→ There are many applications of adiabatic invariance across physics : motion of charged particles, motion of planets, ...

Finaly, there is a powerful extension of the theorem to Quantum Mechanics. In the old version of quantum mechanics, the Bohr-Sommerfeld rule stated that quantum states of energy are given by quantized values of the area in phase space in units of Planck constant

$$\frac{1}{2\pi} \oint p\, dq = \hbar \left(n + \frac{1}{2}\right)$$ where n is an integer

Together with (classical) adiabatic theorem, this suggests that the n-th quantum state of the initial Hamiltonian evolves to the n-th state of the final Hamiltonian if the motion occurs sufficiently slowly!