PHY820/422 HW #9 Due Monday 11/20/2017 @ 5pm
Canonical transformations & electromagnetism in Lagrangian and Hamiltonian formalisms

1. **Canonical transformation.** Consider a *complex* transformation

$$Q = \frac{m\omega q + ip}{\sqrt{2m\omega}}, \qquad P = i\frac{m\omega q - ip}{\sqrt{2m\omega}} = iQ^* \tag{1}$$

   (a) Show that this is a canonical transformation, and find its generating function.

   (b) Apply this canonical transformation to the harmonic oscillator whose Hamiltonian is

$$H = \frac{1}{2}m\omega^2 q^2 + \frac{p^2}{2m}. \tag{2}$$

   Solve for the equation of motion in terms of (Q, P), use that in turn to find the solution for q and p.

2. **Canonical transformation.** The Hamiltonian for a system has the form

$$H = \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right) \tag{3}$$

   (a) Find the equation of motion for q.

   (b) Find a canonical transformation that reduces H to the form of a harmonic oscillator. Show that the solution for the transformed variables is such that the equation of motion found in part (a) is satisfied.

3. **EM in the Lagrangian and Hamiltonian formalisms.** The Lagrangian of a particle moving in two dimensions (x, y) is given by

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - e\phi(x, y, t) + eA_x(x, y, t)\dot{x} + eA_y(x, y, t)\dot{y} \tag{4}$$

   where e is a constant, $\phi(x, y, t)$ is a "scalar" potential, and $\mathbf{A}(x, y, t) = (A_x, A_y)$ is a "vector" potential.

   (a) Derive the equations of motion and show that they describe the Lorentz force on a particle of charge e under the action of an electric field $\mathbf{E} = -\nabla\phi - \partial_t\mathbf{A}$ and a magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$.

   (b) The electromagnetic field is invariant under a gauge transformation of the scalar and vector potentials given by

$$\mathbf{A} \to \mathbf{A} + \nabla\psi(x, y, t), \quad \phi \to \phi - \frac{\partial\psi}{\partial t} \tag{5}$$

where ψ is arbitrary (but differentiable). What effect does this gauge transformation have on the Lagrangian of a particle moving in the electromagnetic field? Is the motion affected?

(c) For a constant magnetic field $\mathbf{B} = B\hat{z}$ along the z direction, we can choose $\phi = 0$ and $(A_x, A_y) = (-\frac{1}{2}By, \frac{1}{2}Bx)$ (Show this!). Write the Lagrangian with this choice of scalar and vector potentials. Compare the latter to the Lagrangian in a rotating coordinate frame (See old lecture notes). Use the Euler-Lagrange equations to find the equations of motion, and show that the orbits are circles of all possible radii centered everywhere in the plane, and the particle rotates around the circle at the rate $\omega = \frac{qB}{2m}$.

Next we will study the same problem in the Hamiltonian language.

(d) Obtain the Hamiltonian from the Lagrangian using the standard procedure. Show that the Hamiltonian takes the form

$$H = \frac{(\mathbf{p} - e\mathbf{A})^2}{2m} + e\phi \equiv \frac{(p_x - eA_x)^2 + (p_y - eA_y)^2}{2m} + e\phi \tag{6}$$

(e) For a constant magnetic field $\mathbf{B} = B\hat{z}$ along the z direction, we can also choose $\phi = 0$ and $(A_x, A_y) = (0, Bx)$ (Show this!). Write down the Hamiltonian with this choice of (scalar and vector) potential, derive the equations of motion, and show explicitly that they lead to the equations of motion of a charged particle in the presence of a magnetic field.

4. **Poisson brackets in the presence of magnetic field.** A charged particle moves in space with a constant magnetic field $\mathbf{B}$ such that

$$\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r} \tag{7}$$

(a) If v_j are the Cartesian components of the velocity of the particle, evaluate the Poisson brackets

$$\{v_i, v_j\}, \qquad i \neq j = 1, 2, 3 \tag{8}$$

(b) If p_i is the canonical momentum conjugate to x_i, also evaluate the Poisson brackets

$$\{x_i, v_j\}, \qquad \{p_i, v_j\} \tag{9}$$

5. * **Bonus** * A particle of mass m and charge e moves in the two-dimensional plane under the combined influence of the harmonic oscillator potential $\frac{1}{2}m\omega^2(x^2 + y^2)$ and a constant magnetic field $\mathbf{B} = B\hat{z}$ where B is constant [you can choose $\mathbf{A} = (0, Bx)$]. Use the canonical transformation

$$\begin{aligned} x' &= x\cos\alpha - \frac{p_y}{\beta}\sin\alpha, \quad p'_x = \beta y \sin\alpha + p_x\cos\alpha, \\ y' &= y\cos\alpha - \frac{p_x}{\beta}\sin\alpha, \quad p'_y = \beta x \sin\alpha + p_y\cos\alpha \end{aligned} \tag{10}$$

with $\tan(2\alpha) = m\omega/(eB)$ and β chosen for convenience to find the motion in general and in the two limits $B = 0$ and $B \to \infty$.