Variational principle

You will have noticed the "weird" form of the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha} = 0$$

You might ask where this comes from?

I.e. is there a simpler way to encode this physics?

In fact, there is!

To motivate it, let's recall that L and $L + \frac{dF}{dt}$ both lead to the same Euler-Lagrange (EL) equations.

This suggests that the important property may be

$$S = \int_{t_0}^{t} L(\dot{q}^\alpha, q^\alpha, t)\, dt$$

which only changes by a boundary term under $L \to L + \frac{dF}{dt}$

Note that S depends on the entire $q(t) \Rightarrow$ "functional" of $q(t)$

We show this by using the notation $S[q(t)]$: means that S depends on the entire path $q(t)$

Square bracket

Note: this brings to mind other forms of variational principles which determine dynamics

E.g. (i) Straight line the shortest distance between 2 pts.

$$L = \int dl$$

(ii) Fermat's principle of least time in optics

$$T = \int n\, dl$$

refraction index

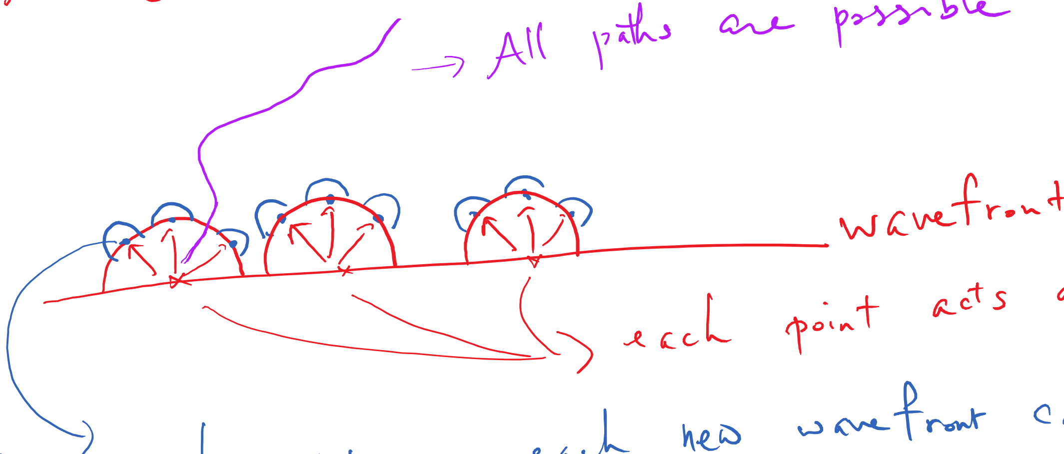
length

Before deriving this, let's just talk a minute about the interpretation of this result.

→ You may recall another famous minimization principle in physics, Fermat's principle of least time for light propagation.

Same structure: Fix $\vec{x}_0$ & $\vec{x}_1$, Then claim that if a ray of light travels from $\vec{x}_0$ to $\vec{x}_1$ it does so over the path which takes the least time.

Are you familiar w/ the connection to Huygen's principle?



Also each point on each new wavefront can be a source.

$\Longrightarrow$ All paths generated

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However, phase = ωt

$\Longrightarrow$ when you add up all contributions, find That many paths cancel each other out due to the phase difference.

Only remaining contributions come from paths w/ stationary phase (= time)

The deep explanation of the least action principle is much the same.

In QM, matter satisfies a wave eq (Schrödinger eqn)

$\Longrightarrow$ obeys something like Huygen's principle

$\Longrightarrow$ takes all paths.

For matter, the phase of The wavefunction is precisely the action S.

One sees this in two places in QM

1) The Feynman path integral

$$\langle x,t | x',t' \rangle = \int_{\substack{\text{paths from } x,t \\ \text{to } x',t'}} e^{iS}$$

next page $\longrightarrow$

(2) The WKB approximation where one finds $\psi \sim e^{iS}$

Note: WKB $\approx$ geometric optics approximation

Now I realize that you will not yet have studied these topics in our grad quantum sequence.

Nevertheless it is important to point out these connections as we go along. Q.M. is considered to be hard or counterintuitive. However, many aspects of QM are in fact directly related to those in classical mechanics.

In fact, one of the primary uses of the more abstract part of classical mechanics is to give insight into Q.M. & serve as a starting point for the semiclassical WKB approximation.

Now, back to our derivation...

We want to show $\delta S = 0 \iff$ Euler-Lagrange equations.

Note: Allow $\underline{\underline{\text{any}}}$ $q(t)$ at this stage.

Do not impose equation of motion.

Let us now compute variations: changes in S associated with changes in $\delta q^\alpha(t)$ [path taken between t_0 & t]

Let's be more explicit by what we mean: we want to compute the change of the action by changing the "path":

$$\delta S = S[q^\alpha(t)] - S[\underline{q}^\alpha(t)]$$

$[q^\alpha(t)]$: new path

notice the bar $[\underline{q}^\alpha(t)]$: old path

Similarly, we define

$$\delta q^\alpha(t) = q^\alpha(t) - \underline{q}^\alpha(t)$$

Later on, we also have to worry about

$$\delta \dot{q}^\alpha \equiv \dot{q}(t) - \underline{\dot{q}}^\alpha(t) = \frac{d}{dt}\left(q^\alpha(t) - \underline{q}^\alpha(t)\right) \equiv \frac{d}{dt}\delta q^\alpha(t)$$

therefore, it follows from our definition that

"variation of time derivative" = "time derivative of variation"

$$\delta \dot{q}^\alpha = \delta \frac{d}{dt} q^\alpha = \frac{d}{dt} \delta q^\alpha$$

that is, δ and $\frac{d}{dt}$ commute.

[This should remind you of the fact that $\frac{d}{dt}$ & $\frac{\partial}{\partial q^\alpha}$ commute]

Now → the variation of the action comes from that of the Lagrangian

$$\delta S = \int_{t_0}^{t_1} \delta L \, dt$$

Now $L(q^\alpha, \dot{q}^\alpha, t)$ changes through δq^α & $\delta \dot{q}^\alpha$.

[Note though that $\delta t = 0$; we are varying the <u>path</u> $\delta q^\alpha(t)$
that is, at each $t_0 < t < t_1$, we just change $q^\alpha(t) \to q^\alpha(t) + \delta q^\alpha(t)$
↓ fixed
and $\dot{q}^\alpha(t) \to \dot{q}^\alpha(t) + \delta \dot{q}^\alpha(t)$]

So, we have

$$\delta S = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^\alpha} \delta q^\alpha + \frac{\partial L}{\partial \dot{q}^\alpha} \delta \dot{q}^\alpha \right) dt$$

Using the fact that $\delta \dot{q}^\alpha = \frac{d}{dt}\delta q^\alpha$, we have

$$\delta S = \int_{t_0}^{t_1} dt \left(\frac{\partial L}{\partial q^\alpha}\delta q^\alpha + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\delta q^\alpha\right) - \delta q^\alpha \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha}\right)$$

$$= \frac{\partial L}{\partial \dot{q}^\alpha}\delta q^\alpha \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \underbrace{\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha}\right)}_{} \delta q^\alpha \, dt$$

Now, suppose that we impose $\delta q^\alpha(t_0) = \delta q^\alpha(t_1) = 0$

i.e., require variations to vanish at the endpoints.

Then, note that when Euler-Lagrange equation satisfied ($EL_\alpha = 0$)

$\Longrightarrow \delta S = 0$; i.e. S is stationary when varied around a solution of the equation of motion keeping the end points fixed.

Note: we expand $q^\alpha(t) = \underline{q}^\alpha(t) + \delta q^\alpha(t)$

with $\underline{q}^\alpha(t)$ satisfying EOM

but $\underline{\underline{\text{not}}}$ $\delta q^\alpha(t)$

In fact, $\delta S = 0 \Longrightarrow EL_\alpha^{(t)} = 0$

Roughly speaking, this is because

$$0 = \delta S = \int dt \; \delta q^\alpha(t) \, EL_\alpha(t)$$

should be valid for $\underline{\text{any}}$ choice of $\delta q^\alpha(t)$.

For an elegant proof, we follow the textbook.

Note that $q^\alpha : [t_0, t_1] \longrightarrow \mathbb{R}^N$ can be thought of

as a vector (in a space of $L^2(t_0, t_1)$ functions)

→ Same for δq^α

→ Same for $EL_\alpha(t)$!

We can define the inner product

$$(f, h) = \int f_\alpha(t)\, L^\alpha(t)\, dt$$

In this notation

$$\delta S = \langle EL \mid \delta q \rangle = 0 \Rightarrow |EL\rangle \perp |\delta q\rangle$$

for all possible $|\delta q\rangle$ (such that $\delta q(t_0) = \delta q(t_1) = 0$)

But all possible $|\delta q\rangle$ spans the whole vector space

And if $|EL\rangle \perp$ all vectors $\Rightarrow |EL\rangle = 0$ ✓

$\boxed{\text{Aside}}$

In Newtonian mechanics $\delta S = 0$ usually picks out a minimum of S ⇒ "principle of <u>least</u> action"

But physically this is not important which is good, since there are exceptions & we will not belabor this point.