

Linear oscillations

If a dynamical system oscillates, it does so around a stable equilibrium point. Suppose that q_0 is the stable equilibrium point of a system with one degree of freedom. Let's consider the Lagrangian

$$L = T - V = \frac{1}{2} m \dot{q}^2 - V(q)$$

But the potential can be Taylor-expanded around equilibrium. Defining $q = q_0 + x$ and expanding to the second order in x we have

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} kx^2 - V(q_0) + \begin{matrix} \text{terms like} \\ x^3 \text{ or higher} \end{matrix}$$

the constant $k = -V''(q_0) > 0$

The second-order equation that follows from this Lagrangian is that of an oscillator

$$\ddot{x} + \omega^2 x = 0 \quad \text{with frequency } \omega = \sqrt{\frac{k}{m}}$$

This line of argument easily extends to a system with n degrees of freedom. Let the Lagrangian be

$$L = \frac{1}{2} m_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta - V(q)$$

Reminder on notation:

- Repeated indices indicate summation

- $q = \{q^\alpha\}$ is the collective set

of coordinates

Now assume that $-V(q)$ has a minimum @

some point $q = q_0$, so that $\frac{\partial V}{\partial q^\beta}|_{q_0} = 0$ for all β

q_0 is a stable equilibrium because it corresponds to the minimum of the potential

Now let's Taylor expand around \vec{q}_0

$$V(\vec{q}) = V(\vec{q}_0) + \frac{1}{2} \vec{x}^\alpha \vec{x}^\beta \left. \frac{\partial^2 V}{\partial \vec{x}^\alpha \partial \vec{x}^\beta} \right|_{\vec{x}=0} + \text{terms that are higher order in } \vec{x}$$

Here again, we have defined $\vec{q} = \vec{q}_0 + \vec{x}$.

The constant $V(\vec{q}_0)$ doesn't play any role, and can be set to zero. Then to the lowest nonzero order the potential is

$$V(\vec{x}) \approx \frac{1}{2} K_{\alpha\beta} \vec{x}^\alpha \vec{x}^\beta$$

$$\text{the symmetric matrix } K_{\alpha\beta} = \underbrace{\frac{\partial^2 V}{\partial \vec{x}^\alpha \partial \vec{x}^\beta}}$$

The Taylor series for the kinetic energy involves

$$m_{\alpha\beta}(\vec{q}) = M_{\alpha\beta} + (\text{linear or higher order in } \vec{x})$$

The matrices $m_{\alpha\beta}$ and $M_{\alpha\beta}$ may be assumed symmetric (why?)

Since the kinetic energy is always positive, the matrix $M_{\alpha\beta}$ forms a positive definite matrix: $M_{\alpha\beta} \vec{x}^\alpha \vec{x}^\beta > 0$ for any nonzero \vec{x} .

Putting kinetic and potential terms together to the lowest order in \vec{x} we have

$$L = \frac{1}{2} M_{\alpha\beta} \dot{\vec{x}}^\alpha \dot{\vec{x}}^\beta - \frac{1}{2} K_{\alpha\beta} \vec{x}^\alpha \vec{x}^\beta$$

(Question: Can we have cross terms such as $\vec{x}^\alpha \vec{x}^\beta$? When would they arise?)

The resulting equations of motion are

$$\ddot{\vec{x}}^\alpha + \Lambda_\beta^\alpha \vec{x}^\beta = 0$$

where the matrix $\Lambda_\beta^\alpha = (M^{-1})_{\alpha\gamma} K_{\gamma\beta}$

Normal modes

In general, the linear equations of motion are equations of n coupled oscillators. The rest of the treatment involves uncoupling them.

In a vector notation

$$\vec{x} = (x_1, x_2, \dots, x_n)$$

the equation of motion reads

$$\ddot{\vec{x}} + \Lambda \vec{x} = 0$$

where $\Lambda = M^{-1} K$ is a matrix ($s.$ are M and K).

Finding the normal modes requires finding the eigenvectors of Λ :

If \vec{x}_λ is an eigenvector belonging to eigenvalue λ , that is, if

$$\Lambda \vec{x}_\lambda = \lambda \vec{x}_\lambda,$$

equation of motion becomes

$$\ddot{\vec{x}}_\lambda = -\lambda \vec{x}_\lambda = -\omega^2 \vec{x}_\lambda$$

defining normal frequencies ω .

Each ω must be real, so λ must be nonnegative. This can be proven by using the properties (symmetry and positiveness) of M and K .

Let \vec{q} be an eigenvector: $\Lambda \vec{q} = \omega^2 \vec{q}$. Then

$$\vec{x}(t) = \vec{q} (\alpha e^{i\omega t} + \bar{\alpha} e^{-i\omega t}) = \vec{q} \underbrace{\left(\text{Re}(\alpha) \cos \omega t + \text{Im}(\alpha) \sin \omega t \right)}_{\text{To make it Real}}$$

The general solution is a linear combination of normal modes

$$\vec{x}(t) = \sum_{Y=1}^n \vec{x}_Y(t) = \sum_{Y=1}^n \vec{q}_Y \left[\alpha_Y e^{i\omega_Y t} + \bar{\alpha}_Y e^{-i\omega_Y t} \right]$$

The constants α_Y are determined from initial conditions

2n
initial conditions

→ 2n constants
(real and imaginary parts of α)

Because K is positive, but not necessarily positive definite, Λ may have a zero eigenvalue. In this case $\underbrace{\vec{x}}_{\lambda} = 0$ with the solution $\vec{x} = \vec{x}_0 + \vec{v}_0 t$ with constant vectors \vec{x}_0 and \vec{v}_0 . These are zero modes and the corresponding solution is the one-dimensional free particle.

for the corresponding λ

Next we consider the example of a water molecule consisting of three particles in a line connected by equal springs.

See the textbook

is the general solution of Eq. (4.47), where α is a complex number and $\Re(\alpha)$ and $\Im(\alpha)$ are its real and imaginary parts. The initial conditions determine α .

The general solution is a linear combination of normal modes:

$$\mathbf{x}(t) = \sum_{\gamma=1}^n \mathbf{x}_\gamma(t) \equiv \sum_{\gamma=1}^n \mathbf{a}_\gamma [\alpha_\gamma \exp(i\omega_\gamma t) + \alpha_\gamma^* \exp(-i\omega_\gamma t)]. \quad (4.48)$$

Here the \mathbf{a}_γ , $\gamma = \{1, \dots, n\}$, are the n normal modes, which are linearly independent and hence form a basis for $\mathbf{T}_{q_0}\mathbb{Q}$. Diagonalizing Λ is equivalent to transforming to the \mathbf{a}_γ basis (see the appendix). The α_γ are constants determined by the initial conditions. The normal modes thus yield $2n$ linearly independent solutions of the n second-order linear differential equations of motion, and their linear combinations are the general solutions. See Problem 9.

A general solution oscillates in several different normal modes simultaneously, each at its own frequency and with its own phase. This may make the motion appear quite complicated. Sometimes the initial conditions (amplitudes and phases) happen to be such that the system moves periodically into one of the x^γ directions and stays close to it for a while, then to another, etc. To an observer it may look as though it is oscillating along each of these x^γ for a while.

Because \mathbf{K} is positive, but not necessarily positive definite, Λ may have zero eigenvalues. In that case Eq. (4.45) becomes $\ddot{\mathbf{x}} = \mathbf{0}$. Thus there is no force and the potential does not change in this direction. The solution is then

$$\mathbf{x} = \mathbf{a} + \mathbf{v}t, \quad (4.49)$$

where \mathbf{a} and \mathbf{v} are constant vectors. These are *zero modes* in $\mathbf{T}_{q_0}\mathbb{Q}$, and the corresponding solution is the one-dimensional free particle. The displacement does not stay small and the approximation breaks down unless the force continues to vanish: the treatment works only for those zero directions in which all of the terms vanish in the Taylor expansion of V . The last sum of Eq. (4.48) changes correspondingly. This is illustrated in the following example.

WORKED EXAMPLE 4.2

An idealized *linear classical water molecule* consists of three particles in a line connected by equal springs and constrained to move along the line joining them (Fig. 4.17). The outer two particles 1 and 3 have equal masses μ and the central one has mass ν , and the spring constant is k . (a) Find the normal modes (describe them) and normal frequencies. (b) Write down the general solution. (c) Write down the solution with initial conditions $x_1(0) = -A$, $x_2(0) = A\nu/\mu$, $x_3 = 0$, and $\dot{x}_\gamma(0) = 0 \forall \gamma$.

Solution. (a) This is a linear problem ab initio. Choose the coordinate x_2 of mass ν to be zero at some arbitrary point along the line. The coordinates x_1 and x_3 of the

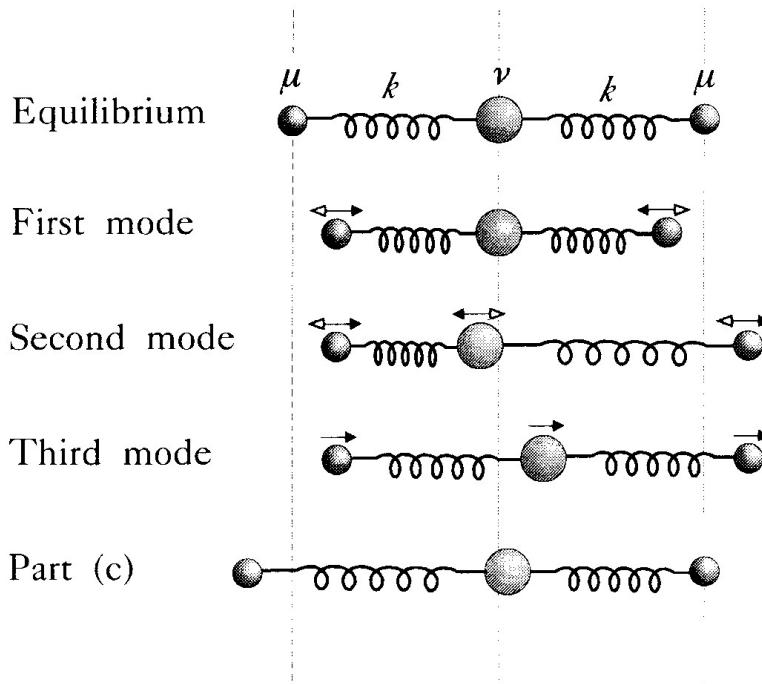


FIGURE 4.17

The linear classical water molecule. The top line shows all three particles in equilibrium. The next three lines show the three normal modes described in the text. In each mode the phases are indicated by the shade of the arrows. That is, in the first mode the phase difference between the two outer particles is π , and the central particle remains at rest. In the second mode the two outer particles are in phase, while the central particle is out of phase with them. In the third mode the molecule moves as a rigid body. The last line of the diagram illustrates the example that ends the discussion of this system.

other two masses will be their deviations from equilibrium when $x_2 = 0$. If $-l$ and l are their equilibrium positions when ν is at $x_2 = 0$, then x_1 and x_3 are the distances of the other two masses from $-l$ and l , respectively. (We use lower indices so as not to confuse squaring with the index 2.) The Lagrangian is

$$L = \frac{1}{2}\mu(\dot{x}_1^2 + \dot{x}_3^2) + \frac{1}{2}\nu\dot{x}_2^2 - \frac{1}{2}k[(x_1 - x_2)^2 + (x_2 - x_3)^2].$$

In the x coordinates, M and K are represented by the matrices

$$M = \begin{vmatrix} \mu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \mu \end{vmatrix} \quad \text{and} \quad K = k \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{vmatrix},$$

and Λ by the matrix

$$\Lambda = k \begin{vmatrix} 1/\mu & -1/\mu & 0 \\ -1/\nu & 2/\nu & -1/\nu \\ 0 & -1/\mu & 1/\mu \end{vmatrix}.$$

The solution to the eigenvector problem gives the following normal frequencies

ω_γ and column vectors a_γ of the normal modes \mathbf{a}_γ , $\gamma = 1, 2, 3$:

$$\omega_1 = \sqrt{\frac{k}{\mu}}, a_1 = \begin{vmatrix} 1 \\ 0 \\ -1 \end{vmatrix}; \quad \omega_2 = \sqrt{k(1/\mu + 2/\nu)}, a_2 = \begin{vmatrix} 1 \\ -2\mu/\nu \\ 1 \end{vmatrix}; \quad \omega_3 = 0, a_3 = \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix}.$$

In the first normal mode, $x_1 = -x_3$, so the two outer masses vibrate out of phase by 180° at frequency ω_1 and with equal amplitudes. The central mass remains fixed (its amplitude is zero).

In the second normal mode, $x_1 = x_3$, so the two outer masses vibrate in phase at frequency ω_2 and with equal amplitudes. Because $x_2 = -2x_1\mu/\nu$, the central mass vibrates out of phase by 180° at the same frequency and with $2\mu/\nu$ times the amplitude. In the first and second modes the center of mass remains stationary.

In the third normal mode, $x_1 = x_2 = x_3$, so the system moves as a whole. The center of mass moves at some fixed velocity v . Clearly there is no force. This is the zero mode. See Fig. 4.17.

(b) The general solution is

$$\begin{vmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \\ -1 \end{vmatrix} \{ \alpha_1 e^{i\omega_1 t} + \alpha_1^* e^{-i\omega_1 t} \} + \begin{vmatrix} 1 \\ -2\mu/\nu \\ 1 \end{vmatrix} \{ \alpha_2 e^{i\omega_2 t} + \alpha_2^* e^{-i\omega_2 t} \} + \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix} \{ a + vt \}.$$

The α_γ , a , and v are determined by the initial conditions, and in general the motion will not be in one of the normal modes.

(c) Some algebra leads to

$$\begin{vmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{vmatrix} = \frac{1}{2} A \begin{vmatrix} \cos \omega_1 t + \cos \omega_2 t \\ -2(\mu/\nu) \cos \omega_2 t \\ -\cos \omega_1 t + \cos \omega_2 t \end{vmatrix}.$$

The center of mass remains fixed. The central mass performs simple harmonic motion at frequency ω_2 , while the two outer masses perform more complicated motion at a combination of frequencies ω_1 and ω_2 . If $\omega_1 \approx \omega_2$ (i.e., if $v \gg \mu$), the phenomenon of beats occurs (Halliday et al., 1993), but the two outer masses beat out of phase. That is, when the amplitude of one is large, the amplitude of the other is small, and it appears that only one of the outer masses is oscillating, and at other times only the other one. Roughly speaking, energy flows back and forth between the two of them at the beat frequency $\frac{1}{2}(\omega_1 - \omega_2)$.

4.2.2 COMMENSURATE AND INCOMMENSURATE FREQUENCIES

THE INVARIANT TORUS \mathbb{T}

We return in this subsection to the two-freedom harmonic oscillator of Eq. (3.38), whose vector field Δ lies on the toroidal invariant submanifold \mathbb{T} described after Eq. (3.39). It