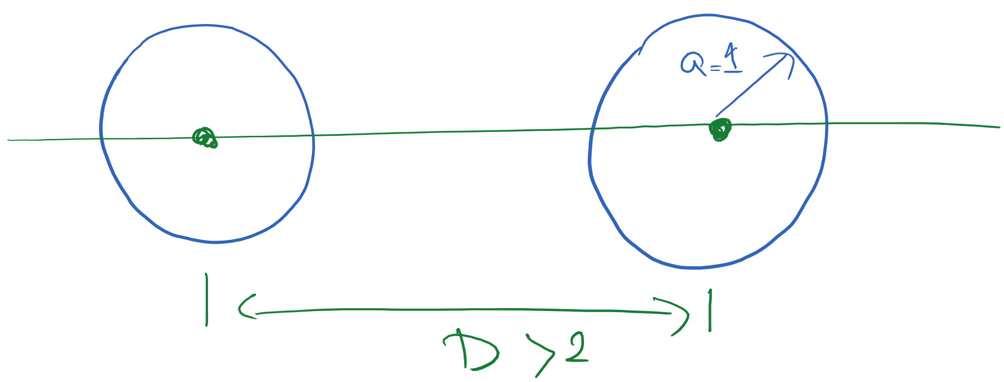
Chaotic scattering

I. Some scattering problems are quite simple. E.g., the Coulomb potential leads to Rutherford scattering, in which all parameters can be calculated exactly & $\Theta(b)$ is monotonic.

But some scattering problems are much more complicated. An illustrative example is 2-D scattering of point particles off of 2 hard disks.



For simplicity, the radius is set to $a=1$.

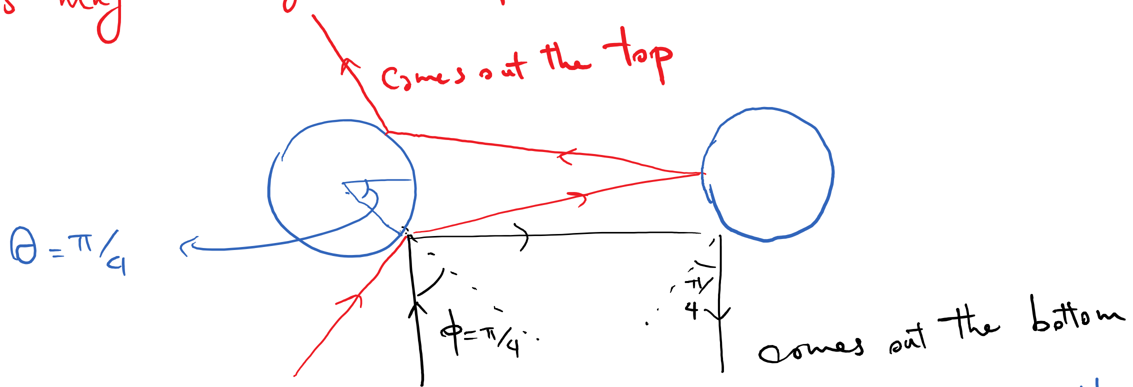
Note that particles may experience $0, 1, 2, \ldots$ collisions during the scattering. To prove that arbitrary many collisions can take place, note that there is a periodic orbit.

particle bounces back and forth between the disks forever

Trajectories very close to this one experience arbitrarily many collisions but eventually exit to one side.

Interestingly, there are also "non-periodic exceptional orbits" which approach the periodic one asymptotically. I.e. one can fine a particle in from ∞ in a such a way that (w/ ∞ fine tuning) it never comes out.

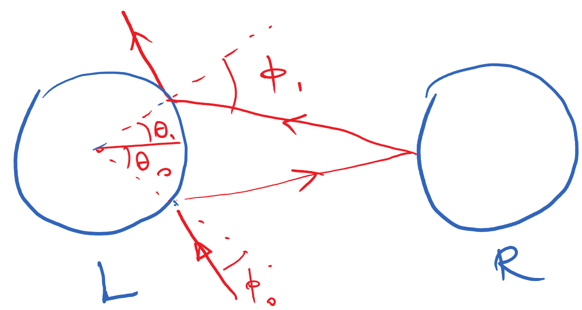
One can prove the existence of such trajectories by noting that particles incident on the same point but at different angles may emerge on opposite sides. E.g.



A small enough change of the angle ϕ will not alter on which side the particle emerges.

Now let the incident angle vary between the two angles shown in the above picture. There is an (open) set of incident angles for which the particle leave above, and an (open) set of incident angles for which it will leave below. $\Longrightarrow$ Between these two sets, there is angle whose trajectory ends up neither above nor below.

It turns out that this problem, in a sense, can be computed analytically. Assume the particle hits the Disk L (on the left) initially at (θ_0, ϕ_0), then Disk R (on the right), and then disk L again at (θ_1, ϕ_1) and so on



The key is to realize that if

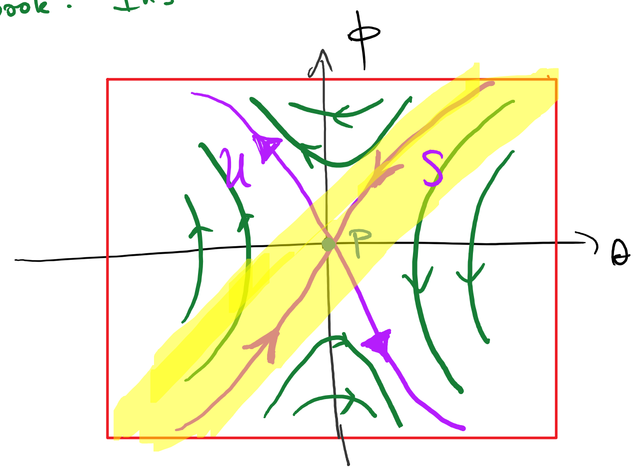
$$(\theta_1, \phi_1) = \mathcal{M}(\theta_0, \phi_0)$$

then in fact

$$(\theta_{n+1}, \phi_{n+1}) = \mathcal{M}(\theta_n, \phi_n)$$

Now it is just algebra to find the map $\mathcal{M}$

We won't show the equations. But you can see them in the text book. Instead the solution can be shown pictorially.



P = periodic orbit

S = stable exceptional orbits (attracted to P)

U = Unstable exceptional orbits (repelled from P)

$\Updownarrow$

stable if run time backwards

$\Updownarrow$

stable orbits = # unstable orbits

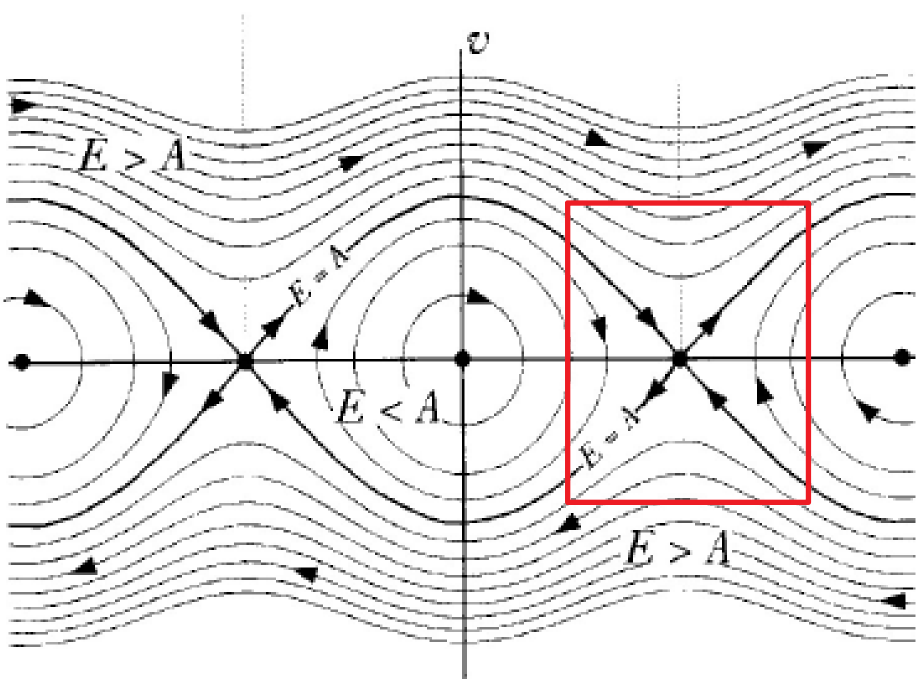
$\mathcal{M}$ moves (θ, ϕ) along curves shown

shaded area : region where particle hits another disk

Notice how similar the structure in this figure is to an unstable equilibrium point in the phase portrait of a physical dynamical system for example to the hyperbolic point of the figure below describing a plane pendulum.

	Two disks	Plane pendulum near unstable equilibrium point
Dynamical variables	(θ, ϕ)	(x, v) position & velocity
Map	Discrete: $(\theta_n, \phi_n) \to (\theta_{n+1}, \phi_{n+1})$	Continuous $(x(t), v(t))$

In the discrete map of dynamics, $(\theta, \phi) = (0, 0)$ is an equilibrium point. Its stable manifold corresponds to the line of initial conditions of the plane pendulum for which the pendulum approaches the state in which it stands upright.



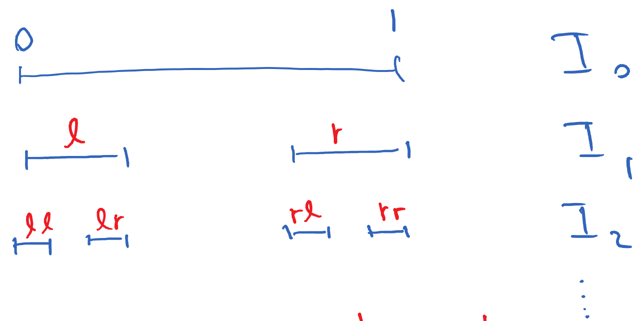
II. As you might imagine things become more complicated with three disks. What is interesting is that the complications reach an entirely new qualitative level.

For example, with three disks there are an infinite number of unstable periodic orbits. Even worse, the exceptional orbits form a "Cantor set", a fractal.

It is useful to talk about Cantor sets before deriving this fact. The simplest Cantor set is the "middle thirds" set first introduced by Cantor himself. Some of you may have seen this before.

Start with the closed interval $I_0 = [0,1]$ of the real line.
Remove the middle $1/3$ to form $I_1 = [0, 1/3] \cup [2/3, 1]$.
Then remove the middle $1/3$ of each interval in I_1 to form I_2

$$I_2 = \left[0, \tfrac{1}{9}\right] \cup \left[\tfrac{2}{9}, \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3}, \tfrac{7}{9}\right] \cup \left[\tfrac{8}{9}, 1\right], \quad \text{etc.}$$



Note that each interval can be labeled by a string of l's and r's representing the fact that at each split it is either the left or the right interval.

The middle thirds Cantor set is

$$C = \bigcap_{n=0}^{\infty} I_n$$

It is clear that $C \neq \emptyset$. Indeed, $0 \in C$, $1 \in C$, $\frac{1}{3} \in C$, etc.
Any endpoint of an interval in I_n lies in C.

In the sense that mathematicians say that the set C is "small" it has "measure zero." This means that

$$\int_C 1 \, dx = 0$$

We can see this by computing $\displaystyle\int_{I_n} 1 \, dx =$ "total length of I_n" $= \left(\frac{2}{3}\right)^n \xrightarrow[n\to\infty]{} 0$

On the other hand, C is "large" enough as it has the same number of points as the entire real line. We can see this by placing the pts in C in one-to-one correspondence with the points in $(0,1)$.

[The interval $(0,1)$ can be then put into one-to-one correspondence with $\mathbb{R}$ via functions like $\tanh : \mathbb{R} \to (-1,1)$.]

To do so, note that each infinite sequence of r's and l's identifies a point in $C = I_\infty$. Then replace l, r by $0, 1$ & regard the result as a binary one as a binary representation of a number in $(0,1)$

E.g. $\quad 1/3 \quad lrrrr\cdots \longrightarrow .0111\cdots = .1 \to 1/2$

$\quad 1/9 \quad llrrr\cdots \longrightarrow .00111\cdots = .01 \to 1/4$

$\underbrace{\qquad\qquad}_{\text{pt in } C} \qquad \underbrace{\qquad\qquad}_{\text{pt in } (0,1)}$

This shows that for every point in point in $(0,1)$, there is a point in C that maps to it. This means "$|C| \geq |(0,1)|$".

But on the other hand, $C \subset (0,1) \rightarrow$ "$|C| \leq |(0,1)|$"

$$\Longrightarrow \quad |C| = |(0,1)|$$

In fact, the Cantor set can be said to have the dimension d where $0<d<1$. Specifically, $d = \frac{\ln 2}{\ln 3}$. What does this mean?

$\longrightarrow$ Notice that in dimension d, it takes $N \sim \frac{V}{V(\text{unit ball})\,\epsilon^d}$ Balls of radius ϵ to "cover" a volume V.

Roughly speaking, $d = -\frac{\log N}{\log \epsilon}$.

Now consider covering the set I_n by intervals of length $\epsilon_n = \frac{1}{3^n}$.

It takes 2^n intervals of this size to cover I_n.

$$\rightsquigarrow \quad d_f = -\frac{\log N}{\log \epsilon_n} = \frac{n \log 2}{n \log 3} = \frac{\log 2}{\log 3} \approx 0.631$$

Now how all of this is related to periodic orbits of the 3-Disk problem?

Let I_n be the set of all orbits that bounce at least n times.

L

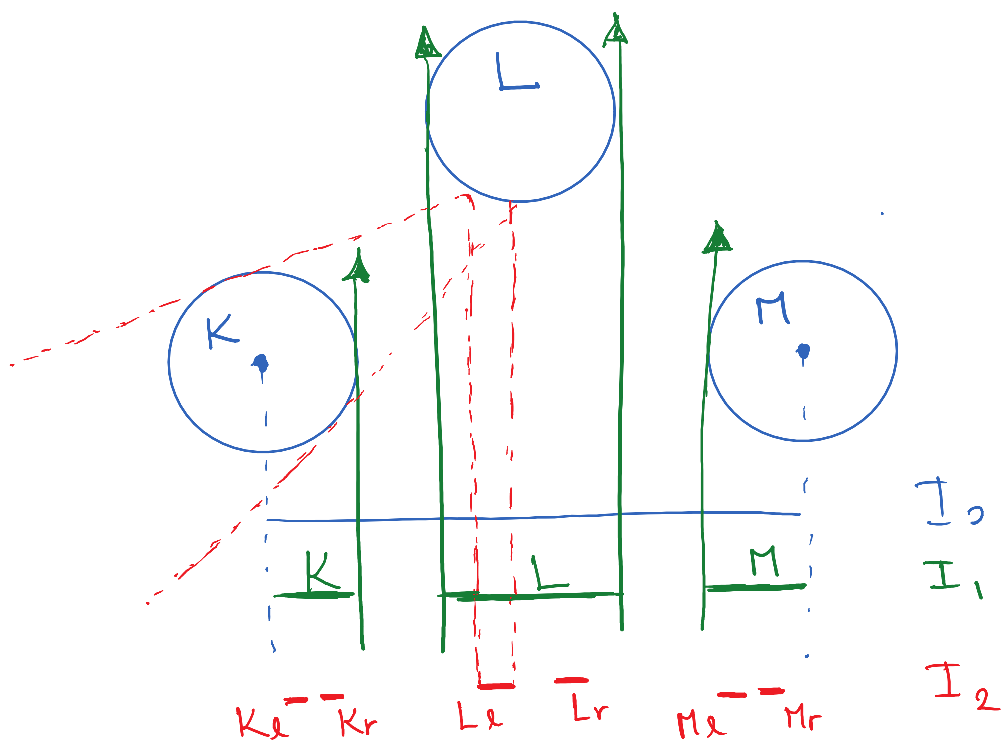
Write $I_n = I_n^K \cup I_n^L \cup I_n^M$

where $I_n^{K,L,M}$ is the subset in which the bounce n is off of the disk K, L, M, respectively.

K M

E.g. consider I_n^L. Some orbits hit M on bounce n+1 while some hit K. However, these set of orbits are separated by orbits that leave the system after n bounces & do not have an (n+1) st bounce at all. I.e., I_{n+1} formed from I_n by "removing the middle". Note also that the same fraction is removed at each stage $\Longrightarrow$ I_∞ is self similar just like the Cantor set! This I_∞ has a fractal dimension which depends on the distance D separating the disks & and which should be investigated numerically.

Pictorially, we can see this as follows.



Now, to each point in I_∞, we may assign a string that begins with K, L, M (first disk hit) followed by a string of $\{l, r\}$ s corresponding to whether disk hit on bounce hit n+1 is to left or right of disk hit on bounce n

Any orbit in I_∞ is a non-periodic exceptional orbit which is forever trapped in the scattering region. These orbits are labeled as, for example

$$L\,r\,l\,r\,r\,l\,\ldots$$

These orbits approach periodic orbits just like the two-disk example. In this case, however, there are infinitely many periodic orbits.

Because nearby orbits can have very different properties (E.g. The time delay function has infinitely many discontinuities !!!) This scattering process is called "irregular" or "chaotic".

Here is a plot of the time delay function taken from Ch. 4 of Jose and Saletan (Fig. 4.11)

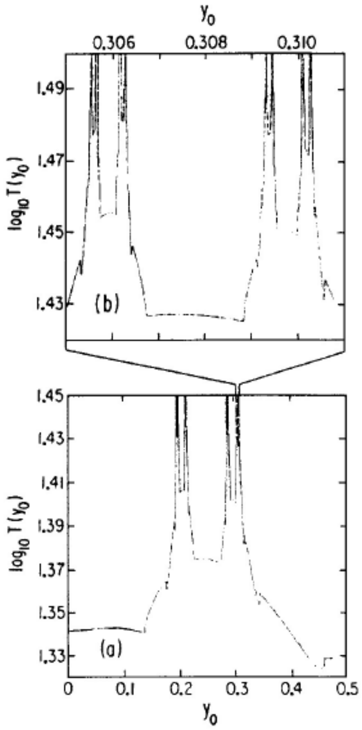


FIGURE 4.11
The logarithm of the time-delay function $T(y_0)$ in three-disk scattering with $D = 2.5$ (from Gaspard and Rice, 1989). y_0 is what we call the impact parameter b, and $T(y_0)$ is essentially what we call the dwell time (the authors define it more carefully, as the time the projectile spends inside a certain region that encloses the three disks). In Fig. (a) y_0 varies from 0 to 0.5. Figure (b) is a blow up of the y_0 interval from 0.305 to 0.311. The similarity of the two graphs reflects the self-similarity of the Cantor-like structure of the set of trapped orbits.