Journal of the Association of Arab Universities for Basic and Applied Sciences (2013) **14**, 28–31

University of Bahrain
Journal of the Association of Arab Universities for Basic and Applied Sciences
www.elsevier.com/locate/jaaubas
www.sciencedirect.com

ORIGINAL ARTICLE

Constrained motion of a particle on an elliptical path

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Received 24 June 2012; revised 18 October 2012; accepted 16 November 2012
Available online 20 December 2012

KEYWORDS
Constrained motion;
Dirac brackets

Abstract In this work we investigate the constrained motion of a particle on an ellipse in the framework of two approaches: the standard approach and Dirac's approach. The Poisson brackets and the Dirac brackets of the generalized variables are calculated by using Scardicchio's technique, and it is observed that there exist nonconserved quantities in the system. On the quantum level, the nonvanishing commutators between the generalized variables lead to the principle of uncertainty. However, in the limit case of motion on a circle the generalized momentum is found to be a constant of motion.

1. Introduction

The classical dynamics of a particle constrained to move on an ellipse is an interesting example of analysing the classical dynamics of constrained particles. The analysis of such a constrained system includes the construction of Dirac brackets between observable quantities. Moreover, Dirac gave very general rules to construct the Hamiltonian and calculate sensible brackets that can be used to describe the classical and, by the canonical quantization procedure, the quantum dynamics.

In fact, Dirac bracket is a generalization of the Poisson bracket to correctly treat systems with second class constraints in Hamiltonian mechanics. The first systematic attempt to provide mathematically consistent quantization procedure for constrained systems was given by Dirac (1950, 1964), who derived a formal "replacement" for the canonical Poisson brackets which play today a fundamental role in the canonical formalism for constrained Hamiltonian systems on both classical and quantum levels.

A unified geometric description picture of various Dirac brackets for regular and singular Lagrangians with holonomic or non-holonomic constraints has been investigated in reference (Ibort et al., 1999). A classification of constraints into first and second class as envisaged by Dirac also emerges naturally from this picture.

In spite of the considerable attention paid to that formula in the mathematical literature (Marsden and Ratiu, 1994; Bhaskara and Viswanath, 1988; Sudarshan and Mukunda, 1974), and several attempts to use the Dirac brackets in the quantization of gauge invariant systems (Deriglazov et al., 1996; Ferrari and Lazzizzera, 1997), until recently there were few attempts to actually use this formalism in more conventional applications.

Nguyen and Turski (2001a,b), recently provided few examples of these applications in classical and continuum mechanics, and they proposed a canonical description for constrained dissipative systems through an extension of the concept of Dirac brackets developed originally for the conservative constrained Hamiltonian dynamics, to the non-Hamiltonian, namely metric and mixed metriplectic, constrained dynamics (Nguyen and Turski, 2001a,b). It turns out that this generalized unified formula for the Dirac brackets is very useful in the description and analysis of a wider class of dynamical systems.

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Peer review under responsibility of University of Bahrain.

Production and hosting by Elsevier

1815-3852
http://dx.doi.org/10.1016/j.jaubas.2012.11.002

Scardicchio (2002) investigated the Dirac approach to analyse the classical and quantum dynamics of a particle constrained on a circle. The method of Lagrange multipliers is used for quantizing the system.

The paper is organized into four sections. In Section 2 we discuss the motion of a constrained particle on an ellipse within the framework of the standard approach and the Dirac approach. In Section 3 we discuss as a special case, the motion on a circle. In Section 4 we present a conclusion.

2. Particle on an ellipse

Let us treat a particle moving on an ellipse as a constrained system. We will investigate the motion in the framework of two approaches: the standard- and the Dirac approach.

2.1. Standard approach

Considering the motion in the horizontal xy-plane, the Lagrangian that describes the motion of a particle on an ellipse is expressed as

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \tag{1}$$

subject to the constraint

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{2}$$

Upon using the parametric equations:

$$x = a\cos\theta; \quad y = b\sin\theta \tag{3}$$

the Lagrangian (1) then reduced to

$$L = \frac{1}{2}m(a^2\sin^2\theta + b^2\cos^2\theta)\dot{\theta}^2. \tag{4}$$

In accordance to the standard procedure for transforming a Lagrangian system into a Hamiltonian one, we introduce the generalized momentum

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = m(a^2\sin^2\theta + b^2\cos^2\theta)\dot{\theta}. \tag{5}$$

Therefore the Hamiltonian function, formed in accordance with

$$H = p_\theta\dot{\theta} - L$$

becomes

$$H = \frac{p_\theta^2}{2m(a^2\sin^2\theta + b^2\cos^2\theta)}. \tag{6}$$

The Poisson bracket of two dynamical variables u and v is defined as

$$\{u, v\} = \sum_{i=1}^{N} \frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}$$

where u and v are regarded as functions of the coordinates and momenta q_i and p_i. The time evolution of a dynamical variable can also be written in terms of a Poisson bracket by noting that

$$\frac{du}{dt} = \sum_{i=1}^{N}\left(\frac{\partial u}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial u}{\partial p_i}\frac{dp_i}{dt}\right) = \sum_{i=1}^{N}\left(\frac{\partial u}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial H}{\partial q_i}\right)$$
$$= \{u, H\}.$$

It is easily demonstrated that

$$\dot{p}_\theta = \{p_\theta, H\} = \frac{1}{2}(a^2 - b^2)\dot{\theta}^2\sin 2\theta. \tag{7}$$

However, this result shows that p_θ is not conserved, i.e., it is not a constant of motion.

2.2. Dirac's approach

Using the method of Lagrange multipliers (Goldstein, 2002), the Lagrangian (1) can be written as

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \lambda(b^2x^2 + a^2y^2 - a^2b^2). \tag{8}$$

where λ is a Lagrange multiplier and treated as an independent variable.

The conjugate momenta are

$$p_x = \frac{\partial L}{\partial\dot{x}} = m\dot{x}; \tag{9}$$

$$p_y = \frac{\partial L}{\partial\dot{y}} = m\dot{y}; \tag{10}$$

$$p_\lambda = \frac{\partial L}{\partial\dot{\lambda}} = \phi_1 \approx 0. \tag{11}$$

where "$\approx$" means weak equality in Dirac's sense. Eq. (11) is the primary constraint of this system. From Eqs. (8)–(11), the total Hamiltonian is given by

$$H_T = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \lambda(b^2x^2 + a^2y^2 - a^2b^2) + u_1p_\lambda. \tag{12}$$

where $u_1 = \dot{\lambda}$ is a Lagrange multiplier. To keep consistency for the system, all constraints are to be imposed after working out all Poisson brackets. In order that the system be compatible with the dynamical evolution, we require all constraints to be conserved throughout all the time. This requirement is called a consistency condition. In our case, it is necessary that $\phi_2 \equiv \dot{\phi}_1 \approx 0$. We have thus a new constraint (secondary constraint) $\phi_2 \approx 0$ on the system. Furthermore, we impose the consistency condition on $\phi_2 \approx 0$. In general, the above arguments continue till either no new constraint turns up further or a condition on a Lagrange multiplier in the Hamiltonian is obtained.

In doing so, the consistency condition for the primary constraint ϕ_1

$$\dot{\phi}_1 = \{\phi_1, H_T\} = -(b^2x^2 + a^2y^2 - a^2b^2) \tag{13}$$

leads to a new secondary constraint, ϕ_2

$$\phi_2 = b^2x^2 + a^2y^2 - a^2b^2 \approx 0. \tag{14}$$

The consistency condition for ϕ_2

$$\dot{\phi}_2 = \{\phi_2, H_T\} = \frac{b^2}{m}(2xp_x - 1) + \frac{a^2}{m}(2yp_y - 1) \tag{15}$$

gives a new constraint, ϕ_3

$$\phi_3 = \frac{2b^2}{m}xp_x + \frac{2a^2}{m}yp_y - \frac{(a^2 + b^2)}{m} \approx 0. \tag{16}$$

And by imposing the consistency condition for the constraint ϕ_3

$$\dot{\phi}_3 = \{\phi_3, H_T\} = \frac{4b^2}{m}p_x^2 + \frac{4a^2}{m}p_y^2 - \frac{4b^4}{m}\lambda x^2 - \frac{4a^4}{m}\lambda y^2, \tag{17}$$

we arrive at a new constraint, ϕ_4

$$\phi_4 = \frac{4}{m}(b^2p_x^2 + a^2p_y^2) - \frac{4\lambda}{m}(b^4x^2 + a^4y^2) \approx 0. \tag{18}$$

In making use of the consistency condition to this constraint

$$\dot{\phi}_4 = \{\phi_4, H_T\} = 8\lambda(1-2xp_x)\left[\tfrac{b^4}{m}+\tfrac{b^4}{2m^2}\right] + 8\lambda(1-2yp_y)\left[\tfrac{a^4}{m}+\tfrac{a^4}{2m^2}\right] - u_1\tfrac{4}{m}(b^2x^2+a^2y^2) \tag{19}$$

we get an equation for u_1

$$u_1 = 2\lambda\frac{(1-2xp_x)\left[b^4+\frac{b^4}{2m}\right]+(1-2yp_y)\left[a^4+\frac{a^4}{2m}\right]}{b^2x^2+a^2y^2}. \tag{20}$$

Since we regard the constraint equations as strong equations, one can ignore the term $u_1\phi_1$ from the total Hamiltonian. We can also ignore the Lagrangian multiplier term because of ϕ_2. So the Hamiltonian becomes the free one

$$H_T = \frac{p_x^2}{2m} + \frac{p_y^2}{2m}. \tag{21}$$

The above Hamiltonian seems exactly that of unconstrained one, and the equations of motion would be the same. In fact, this is not correct because we will change the canonical brackets, so that the information of the constrained Hamiltonian is included in these new canonical brackets.

The Poisson brackets between the four constraints can be written in antisymmetric matrix, $M_{ij} = \{\phi_i,\phi_j\}$, as:

$$M = \begin{pmatrix} 0 & 0 & 0 & M_{14} \\ 0 & 0 & M_{23} & M_{24} \\ 0 & M_{32} & 0 & M_{34} \\ M_{41} & M_{42} & M_{43} & 0 \end{pmatrix}, \tag{22}$$

where

$$\begin{aligned} M_{14} &= \frac{4a^2b^2}{m}; \\ M_{23} &= \frac{4}{m}(b^4x^2 + a^4y^2); \\ M_{24} &= \frac{16}{m}(b^4xp_x + a^4yp_y) - \frac{8}{m}(b^4 + a^4); \\ M_{34} &= \frac{16}{m^2}(b^4p_x^2 + a^4p_y^2) + \frac{16\lambda}{m^2}(b^4x^2 + a^4y^2). \end{aligned} \tag{23}$$

This matrix is inverted to give another antisymmetric one, G:

$$G = M^{-1} = \frac{1}{D}\begin{pmatrix} 0 & G_{12} & G_{13} & G_{14} \\ G_{21} & 0 & G_{23} & 0 \\ G_{31} & G_{32} & 0 & 0 \\ G_{41} & 0 & 0 & 0 \end{pmatrix}, \tag{24}$$

where

$$D = M_{14}M_{23}M_{32}M_{41} \tag{25}$$

is the determinant of M, and the elements of the antisymmetric matrix G are

$$\begin{aligned} G_{12} &= M_{41}M_{23}M_{34}; \\ G_{13} &= M_{41}M_{24}M_{32}; \\ G_{14} &= -M_{23}M_{32}M_{41}; \\ G_{23} &= -M_{14}M_{32}M_{41}. \end{aligned} \tag{26}$$

It is an important part of Dirac's development of Hamiltonian mechanics to handle more general Lagrangians. More abstractly the two forms implied from the Dirac bracket is the restriction of the symplectic form to the constraint surface in the phase space. Suppose we have a constrained system represented by a Lagrangian L and a set of constraints $\phi_m \approx 0$, while computing the Hamiltonian and the Poisson brackets of the system, the constraints must be taken care of. This is done by the Dirac procedure, where instead of Poisson brackets, the brackets are the Dirac brackets, defined by

$$[q, p_q]_D \equiv \{q, p_q\} - \sum_{ij}\{q, \phi_i\}G_{ij}\{\phi_j, p_q\}. \tag{27}$$

Accordingly, the Dirac brackets of canonical variables are given as follows:

$$\begin{aligned} &[\lambda, p_\lambda]_D = 0; \\ &[x, p_x]_D = 1 - \frac{b^4x^2}{b^4x^2 + a^4y^2}; \\ &[y, p_y]_D = 1 - \frac{a^4y^2}{b^4x^2 + a^4y^2}; \\ &[x, p_y]_D = -\frac{a^2b^2xy}{b^2x^2 + a^2y^2} = -xy; \\ &[y, p_x]_D = -\frac{a^2b^2xy}{b^2x^2 + a^2y^2} = -xy; \\ &[x, y]_D = 0; \\ &[p_x, p_y]_D = -a^2b^2\left(\frac{x}{b^4x^2 + a^4y^2}p_y + p_x\frac{y}{b^4x^2 + a^4y^2}\right). \end{aligned} \tag{28}$$

From these results some quantities cannot be measured simultaneously, this leads to the principle of uncertainty between these quantities.

We are in a level now to quantize the system. Introduce the constraint ϕ_3

$$\phi_3 = 2b^2xp_x + 2a^2yp_y - (a^2 + b^2) \approx 0 \tag{29}$$

and using Eq. (3), one can write

$$p_y = \frac{-b}{a}\frac{\cos\theta}{\sin\theta}p_x + \frac{a^2 + b^2}{2a^2b\sin\theta}. \tag{30}$$

The z-component of angular momentum

$$L_z = xp_y - yp_x \tag{31}$$

can be written in terms of a generalized coordinate θ as

$$L_z = a\cos\theta\left[\frac{-b}{a}\frac{\cos\theta}{\sin\theta}p_x + \frac{a^2 + b^2}{2a^2b\sin\theta}\right] - b\sin\theta p_x \tag{32}$$

or

$$L_z = \frac{-b}{\sin\theta}p_x + \frac{a^2 + b^2}{2ab\sin\theta}\cos\theta \tag{33}$$

Thus, the x-component of linear momentum reads

$$\begin{aligned} p_x &= \frac{-\sin\theta}{b}L_z + \frac{\cos\theta}{2ab^2}(a^2 + b^2) \\ &= \frac{-a\sin\theta}{a^2\sin^2\theta + b^2\cos^2\theta}p_\theta. \end{aligned} \tag{34}$$

When using canonical quantization with a constrained Hamiltonian system, the commutator of the operators is set to $i\hbar$times their classical Poisson bracket i.e., quantization proceeds by the replacement $\{q, p_q\} = \frac{1}{i\hbar}[q, p_q]$. Therefore the commutation relations between the quantum operators x, y and p_x give

$$[x, p_x] = i\hbar \frac{a}{b} \sin^2\theta = i\hbar \frac{a}{b}\left(1 - \frac{x^2}{a^2}\right). \tag{35}$$

$$[y, p_x] = -i\hbar \sin\theta\cos\theta = -i\hbar \frac{xy}{ab}. \tag{36}$$

Similarly, the y-component of linear momentum reads

$$p_y = \frac{-\cos\theta}{a} L_z + \frac{\sin\theta}{2a^2 b}(a^2 + b^2) = \frac{b\cos\theta}{a^2\sin^2\theta + b^2\cos^2\theta} p_\theta \tag{37}$$

and the commutation relations between the quantum operators x, y and p_y give

$$[x, p_y] = i\hbar \frac{xy}{ab}; \tag{38}$$

$$[y, p_y] = i\hbar \frac{b}{a}\frac{x^2}{a^2} = i\hbar \frac{b}{a}\left(1 - \frac{y^2}{b^2}\right). \tag{39}$$

Finally, the commutation relation between p_x, and p_y gives

$$\begin{aligned}[p_x, p_y] &= -\frac{i\hbar}{ab} L_z = -\frac{i\hbar}{ab}(xp_y - yp_x) \\ &= \frac{-i\hbar}{a^2\cos^2\theta + b^2\sin^2\theta} p_\theta.\end{aligned} \tag{40}$$

3. Motion on a circle

If $a = b$, the equation of constraint, Eq. (2), is reduced to an equation of a circle of radius a

$$x^2 + y^2 = a^2. \tag{41}$$

Making use of Eq. (40), the z-component of angular momentum can be written as

$$L_z = \frac{ab}{a^2\cos^2\theta + b^2\sin^2\theta} p_\theta \tag{42}$$

and in the limit $a = b$, it reduces to

$$L_z = p_\theta. \tag{43}$$

This result shows that the conjugate momentum, p_θ, represents the z-component of the angular momentum, and by using Eq. (7), it becomes a constant of motion in the classical limit.

Moreover substituting Eqs. (34) and (37) in Eq. (21), we get

$$H = \frac{p_\theta^2}{2m(a^2\cos^2\theta + b^2\sin^2\theta)}. \tag{44}$$

In the limit $a = b$, this Hamiltonian reduces to

$$H = \frac{p_\theta^2}{2ma^2} \tag{45}$$

which commutes with p_θ i.e., it is also a constant of motion in the quantum level.

Finally, one can show that the Dirac brackets Eq. (28) concide exactly with those obtained by Scardicchio (2002) in the limit $a = b$.

4. Conclusion

We consider a constrained motion of a particle on an ellipse using the standard approach and Dirac's approach. In the standard approach we calculate the time evolution of the dynamical variable p_θ, and the result shows that this quantity is not a constant of motion. However, in Dirac's approach we evaluate the Dirac brackets between the phase space variables, and we get results that differ from the corresponding Poisson brackets between the same generalized variables.

At the quantum level, the nonvanishing commutators between the generalized variables, lead to the uncertainty principle for such variables. Hence, there exist nonconserved quantities in the system. In the limit $a = b$ which is the case of a constrained motion on a circle, the conjugate momentum, p_θ, becomes a conserved quantity.

In the future we hope to study the constraint motion in three dimensions such as the motion of a particle on an ellipsoid besides the motion on a sphere.

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