PHY820/422 HW #7 — Due Monday 10/30/17 @ 5pm
Linear oscillations in long chains & Hamiltonians

1. **Finite chain with fixed boundary conditions.** We saw that for an infinite chain there is a wave-like solution for every wavevector k oscillating at a frequency $\omega(k)$ that is a particular function of k. In this problem, we want to study a finite chain of $N+1$ particles with fixed boundary conditions at the two endpoints $x_0 = x_{N+1} = 0$.

   (a) Using a superposition of running waves propagating to the left or to the right, construct a solution that satisfies the boundary condition $x_0 = 0$.

   (b) We still need to satisfy the boundary condition at the other end of the chain, $x_{N+1} = 0$, but there is no more freedom in choosing arbitrary superposition of running waves (convince yourself). Therefore, in this case, a solution will not exist for every wavevector k. Find "allowed" values of k for which both boundary conditions are satisfied.

2. **Finite chain with periodic boundary conditions.** A chain of $N+1$ particles interacting through springs (as in Section 4.2.3 of the textbook) is subject to the condition that $x_0 = x_{N+1}$; that is, we impose the periodic boundary conditions, or the chain is looped on itself (similar to Problem 5 of the previous homework). This means that there are just N independent particles.

   (a) Find the normal modes. (The normal modes are running waves.)

   (b) Consider the limit of a small wavevector ($k \to 0$), and expand the dispersion relation to find the speed of sound.

3. **Hamiltonian and Hamilton's equations.** The Lagrangian for a particle moving in three dimensions (in terms of x, y, and z coordinates) can be written as

$$L = a\dot{x}^2 + b\frac{\dot{y}}{x} + c\dot{x}\dot{y} + fy^2\dot{x}\dot{z} + g\dot{y} - k\sqrt{x^2+y^2} \tag{1}$$

   where a, b, c, f, g, and k are constants. What is the Hamiltonian? What quantities are conserved?

4. **Hamiltonian and Hamilton's equations.** The Hamiltonian of a particle moving in one dimension is described by

$$L(q,\dot{q}) = \frac{1}{2}m\dot{q}^2 + mu\dot{q} - V(q) \tag{2}$$

   where u is a constant.

   (a) Find the equation of motion from the Euler-Lagrange equation.

(b) Construct the Hamiltonian $H(q, p)$ and obtain the Hamilton's equations. Show that the two set of (Hamiltonian and Euler-Lagrange) equations are consistent.

5. **Hamiltonian and Hamilton's equations.** The Hamiltonian of a particle moving in one dimension is described by

$$H(p, q) = \frac{p^2}{2m} - up + V(q) \tag{3}$$

where u is a constant.

(a) Find the equation of motion from Hamilton's equations. (Write an equation that involves only q and its time derivatives.)

(b) Construct the Lagrangian $L(q, \dot{q})$ and derive the Euler-Lagrange equations. Show that the two set of (Hamiltonian and Euler-Lagrange) equations are consistent with each other.

6. **Conserved quantities in Lagrangian and Hamiltonian formalisms.** Consider a system with two degrees of freedom x and y

$$L = \dot{x}^2 + \frac{1}{x}\dot{y}^2 \tag{4}$$

(a) Derive the equations of motion for both degrees of freedom. Identify the cyclic coordinate, and deduce the constant of motion.

(b) Define a new Lagrangian $\tilde{L}$ that is obtained from the above Lagrangian by eliminating the cyclic coordinate and its velocity using the constant of motion. Derive the Euler-Lagrange equation for the remaining coordinate. Is this equation consistent with your result in the previous part. Why or why not?

(c) Next construct the Hamiltonian $H(x, y, p_x, p_y)$ for the Lagrangian L.

(d) Derive the equations of motion from the Hamilton's equations, and show that they are consistent with the Euler-Lagrange equations

(e) This time, first identify the conserved momentum, and replace it by a constant in the Hamiltonian (if p_α is conserved for a certain α, replace it by $p_\alpha = a = \text{const}$). Now derive the Hamilton's equations of motion (treat the conserved momentum as a constant). Do you find the same equations of motion. Why or why not?

7. **Another exercise in Hamiltonian dynamics.** Formulate the double-pendulum problem (See Problem 4 of Homework 2) in terms of the Hamiltonian and the Hamilton's equations of motion.

8. ***Bonus* Lagrangian from Hamiltonian.** Given a Hamiltonian function $H(q, p, t)$, how does one obtain the corresponding Lagrangian? That is, (a) describe the inverse of the procedure that leads to the Hamiltonian. (b) Show that the Euler-Lagrange equations are derivable from Hamilton's equations.