Coordinate independence of Lagrangians

Lagrangian equations were obtained without specifying in any way the particular generalized coordinates. w/ n coordinates. To observe this in detail, let $q'^{\alpha}(q,t)$ be a new set of generalized coordinates, in which case the functions must be invertible: $q^{\alpha}(q',t)$ exist. When the new coordinates are known so are their velocities:

$$\dot{q}^{\alpha} = \frac{\partial q^{\alpha}(q',t)}{\partial q'^{\beta}} \dot{q}^{\beta} + \frac{\partial q^{\alpha}(q',t)}{\partial t}$$

The Lagrangian is a function of $2n$ variables:

- n generalized coord.
- n time derivatives of coordinates

$\Big\}$ at a given time t

Lagrangian assigns a real number to each set of $2n$ numbers at a given time. But these $2n$ numbers describe the physical state of the system. When coordinates change, the state of a system is described by a different set of $2n$ coordinates (at the same time)

But the Lagrangian assigns the <u>same</u> real value to the transformed set of $2n+1$ numbers.

How do we know this?

Consider the problem in Cartesian coordinates

$$L_X(x_i, \dot{x}_i) = T - V = \frac{1}{2}\sum m_i \dot{\vec{x}}_i^{\,2} - V(\vec{x}_i)$$

We showed that the equations of motion in any generalized coordinates can be obtained by simply writing

$$\vec{x}_i = \vec{x}_i(q^\alpha, t)$$

and similarly for $\dot{\vec{x}}_i$ in terms of $q^\alpha, \dot{q}^\alpha$ and t

$$L_X(x_i, \dot{\vec{x}}_i) = L_X\left(\vec{x}_i(q^\alpha, t), \dot{\vec{x}}_i(q^\alpha, \dot{q}^\alpha, t)\right)$$

$$= L(q^\alpha, \dot{q}^\alpha, t)$$

But we could have chosen to go write things in a different coordinate system

$$L_X(x_i, \dot{\vec{x}}_i) = L'(q^{\alpha'}, \dot{q}^{\alpha'}, t)$$

What this means is that

$$L(q, \dot{q}, t) = L'(q', \dot{q}', t)$$

And the form of Euler-Lagrange equation should be the same too

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha} = 0 \longrightarrow \frac{d}{dt}\frac{\partial L'}{\partial \dot{q}'^\alpha} - \frac{\partial L'}{\partial q'^\alpha} = 0$$

[You will show this in Problem set 2]

Because of these properties the Lagrangian is called coordinate independent.

But, then, these equations are somehow making a statement that do not depend on coordinates, and the specific q^α and $\dot{q}^\alpha$ that appear in them are nonessential. Then there should be a way to write these equations in such a way that coordinates do not appear.

This is the geometrical approach to classical mechanics that is advocated in your text book.

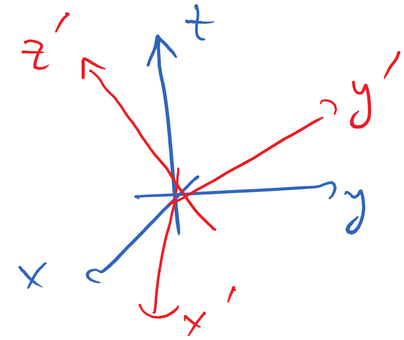
In conclusion, Lagrangian is more than just a scalar in the sense that it is not a vector.

Notice that energy is scalar too, but it is not invariant under general coordinate transformation.

For example: Free particle

$$E = T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

If we rotate coordinates (by a fixed angle)

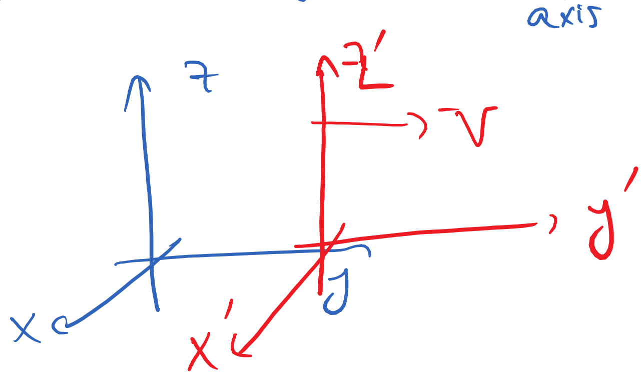


$$E' = \frac{1}{2}m(\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2)$$
$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = E$$

i.e., energy is invariant under rotation (unlike a vector)

But now let's consider a different transformation. For example

let's consider a Galilean transformation: Go to a reference frame moving at velocity V along y axis



$$E' = \frac{1}{2}m(\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2)$$
$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + m\dot{y}V$$
$$= E + m\dot{y}V \neq E$$

Energy is certainly not invariant under general coordinate transf.

On the other hand, Lagrangian is invariant under a general (curved, time-dependent, nonlinear) transformation.