(1)

Kepler problem

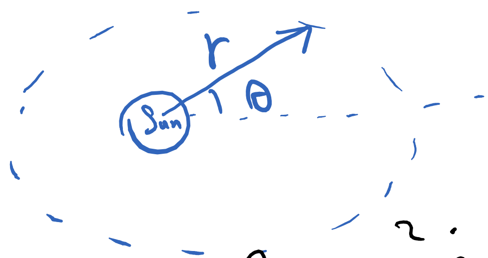
The analysis based on phase portrait is possible for one degree of freedom.

A larger system $\longrightarrow$ If a degree of freedom can be separated out.

Kepler problem

Two degrees of freedom:

In polar coordinates (r, θ)



$\rightarrow$ Conservation of angular momentum $\quad \underbrace{l = mr^2 \dot{\theta}}_{\theta \text{ can be eliminated in terms of } r} = \text{const}$

$\rightarrow$ Together with the conservation of energy, we have

Kinetic energy $T = \frac{1}{2} m \dot{\vec{x}}^2 = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2$

Potential energy $V = -\frac{GMm}{r} \equiv -\frac{k}{r}$

G: gravitational const

M: Mass of Sun

$$\underbrace{\frac{1}{2} m \dot{r}^2}_{\text{"Kinetic" energy}} + \underbrace{\frac{l^2}{2mr^2} - \frac{k}{r}}_{\text{looks like a potential}} = E$$

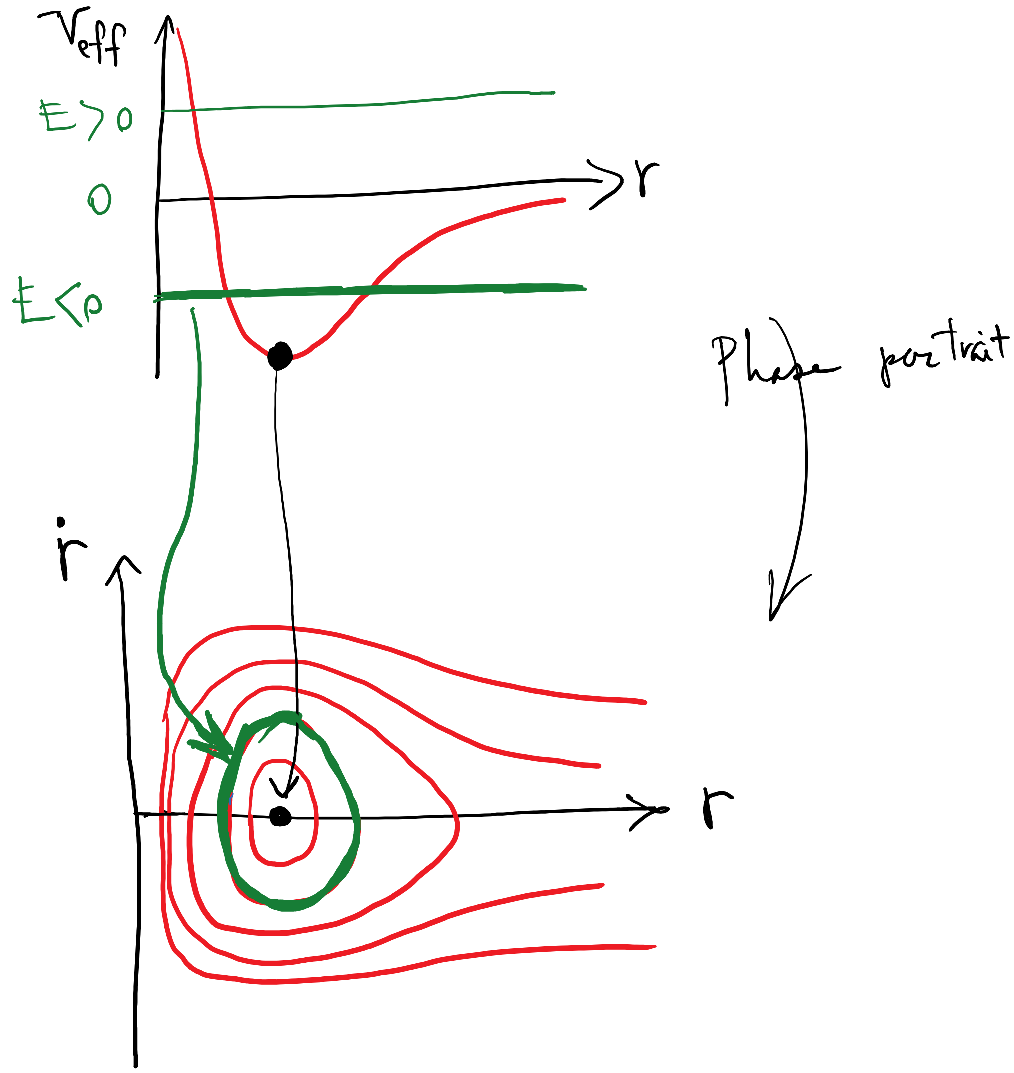
$\Longrightarrow$ "effective potential" $\equiv V_{eff}(r)$

② Near $r = 0$, $V_{eff} \to +\infty$ due to ℓ^2 Term

Near $r = \infty$, $V_{eff} \to 0$. But note that $\frac{1}{r} \gg \frac{1}{r^2}$

$\Rightarrow V_{eff} < 0$

$\rightsquigarrow$ So effective potential looks like



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Tricks to make the problem easier and allow an explicit solution.

1) Find $r(\theta)$ instead of $r(t)$

2) use a new coordinate $u = 1/r$

Note: $\frac{d}{dt} r = \frac{d\theta}{dt}\frac{d}{d\theta} r = \frac{\ell}{mr^2} \underbrace{\frac{d}{d\theta} r}_{\text{define } r'}$ [chain rule]

Rewrite the energy

$$E = \frac{\ell^2}{2mr^4}(r')^2 + \frac{\ell^2}{2mr^2} - \frac{k}{r}$$

But $u = 1/r$ simplifies kinetic term since

$$u' = -\frac{r'}{r^2}$$

$$\Rightarrow E = \frac{\ell^2}{2m} u'^2 + \frac{\ell^2}{2m} u^2 - k u$$

$$= \frac{\ell^2}{2m} u'^2 + \frac{\ell^2}{2m}\Big(u - \underbrace{\frac{km}{\ell^2}}_{\text{offset to } u}\Big)^2 - \underbrace{\frac{k^2 m}{2\ell^2}}_{\text{const}}$$

$\Rightarrow$ offset to E

can be written $= \frac{1}{2} M x'^2 + \frac{1}{2} K x^2 + E_0$

Note: Dynamics now follows from energy conservation

But we recognize above as energy of a harmonic oscillator (with $\underset{\text{time}}{t}$ replaced by θ)

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$\Rightarrow$ Solutions must be the same.

I.e., we can do the energy integral to find

$$u - \frac{km}{l^2} = A \cos(\theta - \theta_0)$$

with $\frac{l^2}{2m} A^2 = E + \frac{k^2 m}{2 l^2}$

Note: $u = \frac{1}{r}$ is singular at $u = 0$ $\longrightarrow$ Special point

Also $u < 0$ does not exist

$\Rightarrow$ 3 kinds of solutions $\left(A < \frac{km}{l^2},\ A = \frac{km}{l^2},\ A > \frac{km}{l^2}\right)$

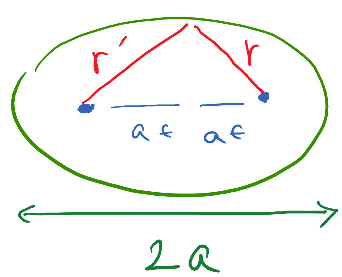
1) $A < \frac{km}{l^2} \Rightarrow u \neq 0$

$\rightsquigarrow$ Ellipse with $r = 0$ @ focus (not center)

We can see this as follows:

An ellipse has 2 foci. Let the distance between the two be $2a\epsilon$

$$r + r' = 2a$$



Now $\vec{r}\,' = \vec{r} - 2a\epsilon\,\hat{x}$

$\Rightarrow r'^2 = r^2 - 4a\epsilon r \cos\theta + 4a^2\epsilon^2$ (1)

But $r' = 2a - r \Rightarrow r'^2 = r^2 - 4ar + 4a^2$ (2)

$\xRightarrow{(1)\,\&\,(2)}$ $0 = -4ar(1 - \epsilon\cos\theta) + 4a^2(1 - \epsilon^2)$

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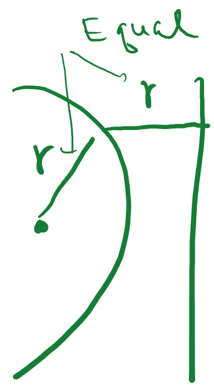
$$\Rightarrow \frac{1}{r} = \frac{1 - \epsilon \cos\theta}{a(1-\epsilon^2)}$$

$\Rightarrow$ same form as above
($=$ const $+$ term $\propto \cos\theta$)

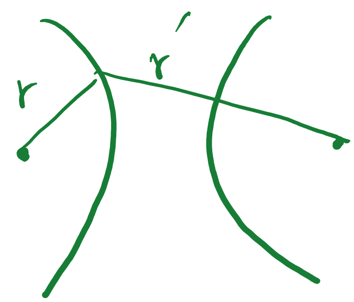
Similarly,

2) Parabola for $E = A$



3) Hyperbola for $E > A$

$$r' - r = 2a$$



Moral: It is often useful to change coordinates in novel ways.