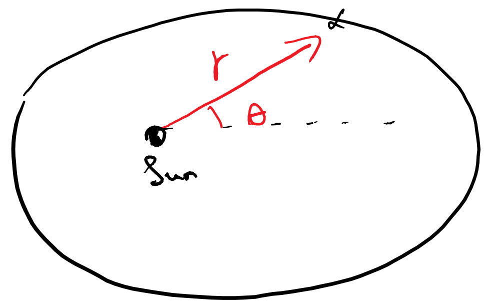
# A familiar example: Now in the Lagrangian formulation.

- Central force




Lagrangian $L = \frac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)$

There are now two equations of motio

one for $r$ : $\frac{d}{dt}\left(m r^2 \dot{\theta}\right) = 0$ (1)

one for $\theta$ : $m\ddot{r} - m r \dot{\theta}^2 + \frac{dV}{dr} = 0$ (2)

(1) is immediately integrated $\longrightarrow$ $m r^2 \dot{\theta}$ = Const $= \ell$ = angular momentum

In the above example, The Lagrangian is indep. of the angular coordinate $\theta$ due to rotational symmetry. More generally, The Lagrangian could be indep. of some generalized coordinate $q^\alpha$.

We say that the coordinate $q^\alpha$ is "cyclic". And The associated Euler-Lagrange equation becomes

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right) - \cancelto{0}{\frac{\partial L}{\partial q^\alpha}} \rightsquigarrow \text{Conservation law for}$$

$$P_\alpha \equiv \frac{\partial L}{\partial \dot{q}^\alpha}$$

So cyclic coordinates lead to conservation laws (of generalized momentum)

There is another conservation:

– Conservation of "energy" (I put "energy" in quotation marks because it might not coincide w/ mechanical energy $L = T + V$ if the constraints or coordinate transformation are time dependent. See previous lectures.)

If the Lagrangian is not explicitly time dependent,

i.e., $\dfrac{\partial L(q, \dot{q})}{\partial t} = 0 \implies W = \dfrac{\partial L}{\partial \dot{q}^\alpha} \dot{q}^\alpha - L$

$$\equiv p_\alpha \dot{q}^\alpha - L$$

is conserved, i.e. $\dfrac{dW}{dt} = 0$.

Our descriptions of (generalized) momentum conservation and energy conservation were quite different. They also lead to rather different looking formulas:

$$p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha} \quad \text{vs.} \quad E = p_\alpha \dot{q}^\alpha - L$$

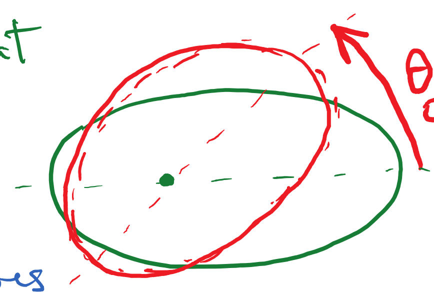
Yet they are both conserved quantities (in the right contexts)

Do these conservation rules have anything in common?

Yes! Both are associated w/ symmetries of the system.

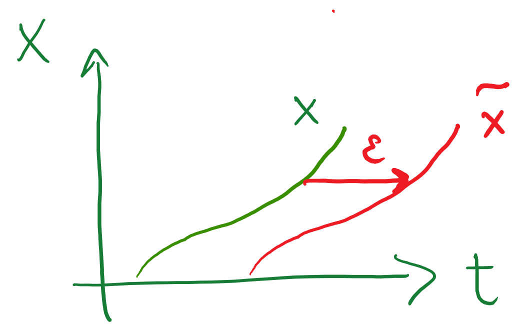
$p_\alpha$ is conserved when $L$ is independent of $q^\alpha$, i.e. when $q^\alpha \longrightarrow q^\alpha + \text{const.}$ is a symmetry, meaning that shifting any solution in space gives another solution

Example: Central force. If there is a solution (such as an ellipse), a rotation of that is also a solution ($\theta \longrightarrow \theta + \theta_0$).




Similarly, $E$ is conserved when $L$ does not explicitly depend on time. I.e., when $t \longrightarrow t + \text{const}$ is a symmetry meaning that shifting a solution in time is also solution

Example: If $x = f(t)$ is a solution, so is $\tilde{x} = f(t - \varepsilon)$




There is a unifying principle behind both statements.

$\Longrightarrow$ Noether's theorem shows that any continuous symmetry yields a conservation law.

Emmy Noether was a German mathematician in the early 1900's. Theorem proved in 1915, published in 1918.
Like Hilbert, she mostly worked in mathematics but made important contributions to physics. She was invited by Hilbert to teach at University of Göttingen; however, that was blocked because she was a woman. [See wikipedia]

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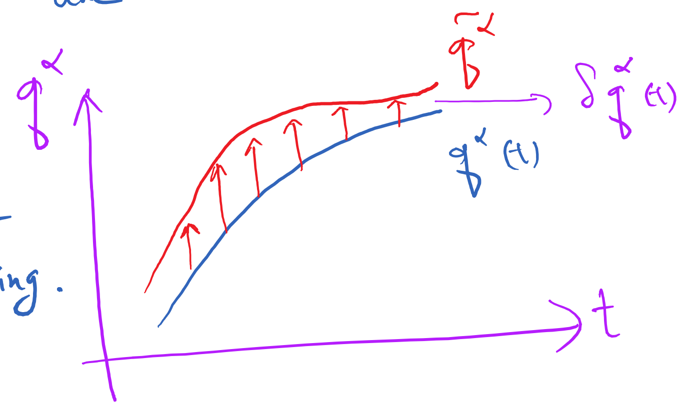
Now let's define things more precisely.

First, by a transformation, we mean an operation that takes one trajectory $q^\alpha(t)$ and makes a new one

$$\tilde{q}^\alpha(t) = q^\alpha(t) + \delta q^\alpha(t).$$

We will consider infinitesimal transformations in the following.




Let's also suppose that $\delta q^\alpha(t) = \delta q^\alpha(q, \dot{q}, t)$

Note: this also determines $\delta \dot{q}^\alpha(t) = \frac{d}{dt} \delta q^\alpha(q, \dot{q}, t)$

As we discussed, symmetry implies that if we have a solution to the equation of motion, we can construct new solutions that are related to the original one by the symmetry. Let's make this statement more precise.

Consider a solution $q^\alpha(t)$ to the Euler-Lagrange equation

i.e. $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} - \frac{\partial L}{\partial q^\alpha} = 0$$ is satisfied for $q^\alpha = q^\alpha(t)$

and, as usual $L(q, \dot{q}, t)$ is the Lagrangian.

Now if there is another function $\tilde{q}^\alpha(t)$ that is related to $q^\alpha(t)$ by symmetry, it should also satisfy the equation of motion

i.e. $$\frac{d}{dt}\frac{\partial L(\tilde{q}^\alpha, \dot{\tilde{q}}^\alpha, t)}{\partial \dot{\tilde{q}}^\alpha} - \frac{\partial L(\tilde{q}^\alpha, \dot{\tilde{q}}^\alpha, t)}{\partial \tilde{q}^\alpha} = 0$$

Notice that we only replaced the new function $\tilde{q}^\alpha(t)$ in the previous equation.

On the other hand, we know that Euler-Lagrange equations are "covariant" under a general coordinate transformation. I.e., they take the same form in all coordinate systems. Therefore for arbitrary generalized coordinates, we have

$$\frac{d}{dt}\frac{\partial \tilde{L}(\tilde{q}^\alpha, \dot{\tilde{q}}^\alpha, t)}{\partial \dot{\tilde{q}}^\alpha} - \frac{\partial \tilde{L}(\tilde{q}^\alpha, \dot{\tilde{q}}^\alpha, t)}{\partial \tilde{q}^\alpha} = 0$$

Notice that in the new coordinate system, the Lagrangian can be a different function $\tilde{L}$.

The comparison of the two equations suggest that if we have a symmetry, then the Lagrangian satisfies

$$L(\bar{q}, \dot{\bar{q}}, t) = \tilde{L}(\bar{q}, \dot{\bar{q}}, t)$$

$\Rightarrow$ Lagrangian is invariant under a symmetry transformation

In particular, this implies the action has to be invariant because it is just the (time) integral of the Lagrangian.

Important: We shall see that there is a slightly more general possibility that the action is not invariant under the symmetry transformation but can be different by boundary terms. But, for the start, let's assume that the action is invariant under symmetry transformation.

In summary, under a symmetry transformation, the action remains invariant. More precisely, an infinitesimal transformation $q^\alpha(t) \to q^\alpha(t) + \delta q^\alpha$.

I.e., $\delta S = 0$ for any $q^\alpha(t)$ and not just those that solve EL equations.

Now let us calculate $\delta S$ as usual

$$0 = \delta S = \delta \int_{t_0}^{t_1} L\,dt$$

$$(\text{as usual}) = \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial q^\alpha} \delta q^\alpha + \frac{\partial L}{\partial \dot{q}^\alpha} \delta \dot{q}^\alpha\right] dt$$

$$= \int_{t_0}^{t_1} \left[\left(\frac{\partial L}{\partial q^\alpha} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha}\right)\delta q^\alpha + \underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\delta q^\alpha\right)}_{\text{we must keep this}}\right]$$

$\delta q^\alpha$ does not necessarily satisfy the boundary condition of our variational principle. $\Longleftarrow$ boundary term since in general $\delta q^\alpha \neq 0$ at the endpoints.

We have not yet used Euler-Lagrange equations. Above is valid for any history. But now suppose we apply our transformation to a solution; i.e. suppose $q^\alpha(t)$ satisfies

$$\frac{\partial L}{\partial q^\alpha} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} = 0 \implies 0 = \delta S = \left.\frac{\partial L}{\partial \dot{q}^\alpha}\delta q^\alpha\right|_{t_0}^{t_1}$$

$$\implies \frac{\partial L}{\partial \dot{q}^\alpha}\delta q^\alpha \text{ must be the same at } t_0 \text{ \& } t_1$$

But these times were arbitrary

$$\boxed{\implies \frac{\partial L}{\partial \dot{q}^\alpha}\delta q^\alpha \text{ conserved along solutions to the EOM}}$$

In particular, if $q^1 \to q^1 + \lambda$ is a symmetry, then

$\delta q^\alpha = \lambda \delta^\alpha_1$ (Kronecker delta) $\Rightarrow$

$$\frac{\partial L}{\partial \dot{q}^\alpha} \delta q^\alpha = \frac{\partial L}{\partial \dot{q}^1} \equiv P_1 = \text{Conserved}$$

We get conservation of generalized momentum.

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As a more general example, consider a collection of particles that can interact with each other with a potential that only depends on their relative position

$$L = \sum_i \frac{1}{2} m_i \dot{\vec{x}}_i^2 - \sum_{i,j} V(\vec{x}_i - \vec{x}_j)$$

There is a symmetry under $\vec{x}_i \to \vec{x}_i + \underbrace{\vec{\varepsilon}}_{\text{uniform translation}}$

Therefore $\delta \vec{x}_i = \vec{\varepsilon}$ and the conserved quantity is

$$\sum_i \frac{\partial L}{\partial \dot{\vec{x}}_i} \delta \vec{x}_i = \sum_i m_i \dot{\vec{x}}_i \cdot \vec{\varepsilon}$$

By choosing $\vec{\varepsilon}$ in different directions ($\hat{x}$, $\hat{y}$, or $\hat{z}$) it follows that the total momentum is conserved

$$\vec{P} = \sum_i m_i \dot{\vec{x}}_i = \text{const}$$

Another example: this time on rotation

A simple way to see this is to go to polar coordinates where the conservation of angular momentum follows from the conservation of generalized momentum corresponding to the angular variable.

But here we want to show this in Cartesian coordinates. Let's start from a Lagrangian that is invariant under rotation

$$L = \frac{1}{2} m \left( \dot{x}^2 + \dot{y}^2 \right) - V\left( \sqrt{x^2 + y^2} \right)$$

A rotation by an angle $\theta$ takes us to

$$\begin{cases} \tilde{x} = x \cos\theta + y \sin\theta \approx x + y\theta \\ \tilde{y} = -x \sin\theta + y \cos\theta \approx -x\theta + y \end{cases}$$

For an infinitesimal transformation,

Therefore,

$$\begin{cases} \delta x = y\,\theta \\ \delta y = -x\,\theta \end{cases}$$

(Notice that $\delta x$ and $\delta y$ are not just constants)

Now Noether's theorem tells us that

$$P_\alpha \, \delta q^\alpha = P_x (y\,\theta) - P_y (x\,\theta) = -\theta \left[ -y P_x + x P_y \right]$$

is conserved. But this is nothing but the angular momentum in Cartesian coordinates.

It seems that we only get conservation laws of the sort of momentum conservation. What about energy conservation?

It turns out that Noether's theorem can be generalized.
→ Suppose the action is not invariant under the symmetry but can change up to a boundary term as

$$\delta S = \text{pure boundary term}$$
$$= \Phi(q, \dot{q}, t)\Big|_{t_0}^{t_1}$$

This is because in our previous argument the fact that $L(\tilde{q}, \dot{\tilde{q}}, t)$ and $L(\tilde{q}, \dot{\tilde{q}}, t)$ give the same equation of motion can also be satisfied if they just differ by a total derivative.

In this case, $\frac{\partial L}{\partial \dot{q}^\alpha} \delta q^\alpha - \Phi$ is conserved

This will yield energy conservation when the symmetry is a time translation. Let's check this.

A time translation $t \to \tilde{t} = t + \varepsilon$ means that we consider a new trajectory $\tilde{q}^\alpha(\tilde{t})$ defined by

$$\tilde{q}^\alpha(\tilde{t}) = q^\alpha(t)$$

This means

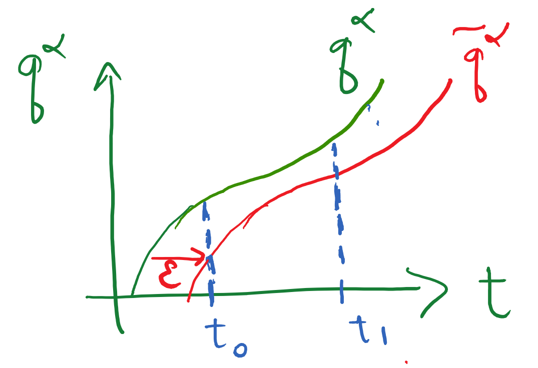
$$\tilde{q}^\alpha(t+\varepsilon) = q^\alpha(t)$$

or

$$\tilde{q}^\alpha(t) = q^\alpha(t-\varepsilon) = q^\alpha(t) - \varepsilon\,\dot{q}^\alpha + \cdots \equiv q^\alpha(t) + \delta q^\alpha$$

Taylor expand for infinitesimal $\varepsilon$

The change of the action is




$$\delta S = \int_{t_0}^{t_1} L(\tilde{q},\dot{\tilde{q}})\,dt - \int_{t_0}^{t_1} L(q,\dot{q})\,dt$$

$$= \int_{t_0-\varepsilon}^{t_1-\varepsilon} L(q,\dot{q})\,dt - \int_{t_0}^{t_1} L(q,\dot{q})\,dt$$

$$= -\varepsilon\, L(q,\dot{q})\Big|_{t_0}^{t_1} \equiv \Phi\Big|_{t_0}^{t_1}$$

We can also show this explicitly. First notice $\tilde{\dot{q}}^\alpha = \dot{q}^\alpha - \varepsilon\,\ddot{q}^\alpha + \cdots$ that just follows from symmetry transformation of $q^\alpha$. Next

$$\delta S = \int_{t_0}^{t_1} \delta L(q,\dot{q})\,dt = \int_{t_0}^{t_1}\left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}\right]dt$$

$$= -\varepsilon\int_{t_0}^{t_1}\left(\frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}\right)dt = -\varepsilon\int_{t_0}^{t_1}\frac{dL}{dt} = -\varepsilon L\Big|_{t_0}^{t_1}$$

$\Longrightarrow$ Conservation of energy follows from

$$\frac{\partial L}{\partial \dot{q}^\alpha}\delta q^\alpha - \Phi = p_\alpha(-\varepsilon\,\dot{q}^\alpha) + \varepsilon L$$

$$= -\varepsilon\,\underbrace{\left[p_\alpha\dot{q}^\alpha - L\right]}_{\text{"Energy"}} = \text{Conserved}$$

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