Hamilton's equations

You must be wondering why another formalism?
There are mainly two reasons
1) Close ties to the operator formalism of Quantum Mechanics
2) Related to conservation laws

In particular, let's look at a mistake that is easy to make in the Lagrangian formalism. Consider a free particle in 2D

$$= \frac{1}{2} m \dot{\vec{x}}^2$$

In cylindrical coordinates, $L = \frac{1}{2} m \dot{r}^2 + \underbrace{\frac{1}{2} m r^2 \dot{\theta}^2}_{\frac{1}{2}\frac{\ell^2}{mr^2}}$

where the last two terms are written in terms of the angular momentum ℓ that is conserved.

Writing the Euler-Lagrange equation, we have

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r}$$

$$= m\ddot{r} - \frac{d}{dr}\left(\frac{1}{2}\frac{\ell^2}{mr^2}\right) = m\ddot{r} + \frac{\ell^2}{mr^3}$$

$\longrightarrow \quad m\ddot{r} = -\frac{\ell^2}{mr^3} \Rightarrow$ particle is driven to $r=0$, right?

No. What did we do wrong?

We are not allowed to set $\ell =$ const before computing EL equation, why

$$0 = (EL)_r = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\bigg|_{\theta,\dot{\theta}}\right) - \frac{\partial L}{\partial r}\bigg|_{\theta,\dot{\theta}}$$

But holding ℓ fixed does not fix $\dot{\theta}$ when r varies!!!

This suggests that it might be useful based on momenta instead of velocities. Hamiltonian formalism does precisely this.

How does it work?

Let $p_\alpha = \left.\dfrac{\partial L}{\partial \dot{q}^\alpha}\right|_q$ (generalized momentum)

We'd like to write the equations of motion in terms of momenta rather than velocities.

The Euler-Lagrange equations are $\dfrac{dp_\alpha}{dt} = \left.\dfrac{\partial L}{\partial q^\alpha}\right|_{\dot{q}}$ $(*)$

As we have seen, $\left.\dfrac{\partial L}{\partial q^\alpha}\right|_{\dot{q}} \neq \left.\dfrac{\partial L}{\partial q^\alpha}\right|_p$. But how different are they?

Note that $dL = \left.\dfrac{\partial L}{\partial q^\alpha}\right|_{\dot{q}} dq^\alpha + \left.\dfrac{\partial L}{\partial \dot{q}^\alpha}\right|_q d\dot{q}^\alpha$

But also $dL = \left.\dfrac{\partial L}{\partial q^\alpha}\right|_p dq^\alpha + \left.\dfrac{\partial L}{\partial p^\alpha}\right|_q dp^\alpha$ (fixed t implicit)

$$\Rightarrow \left.\frac{\partial L}{\partial q^\alpha}\right|_p = \left.\frac{\partial L}{\partial q^\alpha}\right|_{\dot{q}} + \frac{\partial L}{\partial \dot{q}^\beta}\left.\frac{\partial \dot{q}^\beta}{\partial q^\alpha}\right|_p$$

$$= \underbrace{\left.\frac{\partial L}{\partial q^\alpha}\right|_{\dot{q}}}_{\text{Need this term in } (*)} + p_\beta \left.\frac{\partial \dot{q}^\beta}{\partial q^\alpha}\right|_p$$

$$\Rightarrow \left.\frac{\partial L}{\partial q^\alpha}\right|_{\dot{q}} = \left.\frac{\partial}{\partial q^\alpha}\right|_p \left[L - p_\beta \dot{q}^\beta\right]$$

I.e., $(*)$ can be written as

$$\dot{p}^{\alpha} = \frac{\partial L}{\partial q^{\alpha}}\Big|_{\dot{q}} = \frac{\partial}{\partial q^{\alpha}}\Big|_{p_\beta} \Big[L - \underbrace{p_\beta \dot{q}^{\beta}}_{\text{Regard this as a function of } q, p} \Big]$$

But we can do even more!

→ Consider $\frac{\partial}{\partial p_\alpha}\Big|_{q} \Big[L - p_\beta \dot{q}^{\beta} \Big]$

Note that $\frac{\partial L}{\partial p_\alpha}\Big|_q = \frac{\partial L}{\partial \dot{q}^\beta}\Big|_q \frac{\partial \dot{q}^\beta}{\partial p_\alpha}\Big|_q = p_\beta \frac{\partial \dot{q}^\beta}{\partial p_\alpha}$

Thus $$\frac{\partial}{\partial p_\alpha}\Big|_q \Big[L - p_\beta \dot{q}^\beta \Big] = p_\beta \frac{\partial \dot{q}^\beta}{\partial p_\alpha} - p_\beta \frac{\partial \dot{q}^\beta}{\partial p_\alpha} - \dot{q}^\alpha$$

I.e. Define $H = p_\beta \dot{q}^\beta - L$ (Think of this as a function of q & p)

and find
$$\begin{cases} \dot{q}^\beta = \dfrac{\partial H}{\partial p_\beta}\Big|_{q,\ \text{other } p,\ t} \\ \dot{p}_\beta = -\dfrac{\partial H}{\partial q^\beta}\Big|_{p,\ \text{other } q,\ t} \end{cases}$$

Note that H is precisely the "energy" (which may or may not be $T \mp V$). As a result, $\frac{dH}{dt} = -\frac{\partial L}{\partial t}\Big|_{q,\dot{q}}$

Interestingly, we also have $\frac{dH}{dt} = \frac{\partial H}{\partial t}\Big|_{p,q}$

Proof: $$\frac{d}{dt} H = \underbrace{\frac{\partial H}{\partial q^\alpha}}_{} \underbrace{\dot{q}^\alpha}_{\partial H/\partial p_\alpha} + \frac{\partial H}{\partial p^\alpha} \underbrace{\dot{p}^\alpha}_{-\partial H/\partial q^\alpha} + \frac{\partial H}{\partial t}\Big|_{p,q} = \frac{\partial H}{\partial t}\Big|_{p,q}$$

⟹ If H not explicitly time dependent ⟶ "energy" is conserved