Conservation laws and Poisson Brackets

Noether theorem gives us a lovely understanding of symmetries and conservation laws in Lagrangian Mechanics. Let's see what can be said in the Hamiltonian context, beginning with simplest cases.

- **Momentum conservation:**

$$\dot{p}_\alpha = 0 \quad \text{when} \quad 0 = \frac{\partial}{\partial q^\alpha}\bigg|_{\dot{q},t} L = \frac{\partial}{\partial q^\alpha}\bigg|_{p,t} H$$

i.e., when the Hamiltonian is independent of the coordinate q^α

- **Energy Conservation:**

$$\frac{d}{dt} H = 0 \quad (\text{energy is constant}) \quad \text{when} \quad \frac{\partial}{\partial t}\bigg|_{q,\dot{q}} L = 0 .$$

As we saw before, H is constant if it doesn't explicitly depend on time, i.e. $\frac{\partial}{\partial t} H = 0$

This can be seen from Hamilton's equations:

$$\frac{d}{dt} H = \frac{\partial H}{\partial q^\alpha} \dot{q}^\alpha + \frac{\partial H}{\partial p^\alpha} \dot{p}^\alpha + \frac{\partial H}{\partial t}$$

$$= \cancel{\frac{\partial H}{\partial q^\alpha} \frac{\partial H}{\partial p_\alpha}} - \cancel{\frac{\partial H}{\partial p_\alpha} \frac{\partial H}{\partial q^\alpha}} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

In general, a quantity $A(q,p)$ that is a function q,p but is not explicitly a function of time $\left(\frac{\partial A}{\partial t} = 0\right)$ is conserved when

$$0 = \frac{dA}{dt} = \frac{\partial A}{\partial q^\alpha} \dot{q}^\alpha + \frac{\partial A}{\partial p_\alpha} \dot{p}_\alpha$$

$$= \frac{\partial A}{\partial q^\alpha} \frac{\partial H}{\partial p_\alpha} - \frac{\partial A}{\partial p_\alpha} \frac{\partial H}{\partial q^\alpha} = 0$$

More generally,

$$\frac{d}{dt}A = \frac{\partial A}{\partial q^\alpha}\dot{q}^\alpha + \frac{\partial A}{\partial p_\alpha}\dot{p}_\alpha + \frac{\partial A}{\partial t}$$

$$= \underbrace{\frac{\partial A}{\partial q^\alpha}\frac{\partial H}{\partial p_\alpha} - \frac{\partial A}{\partial p_\alpha}\frac{\partial H}{\partial q^\alpha}}_{\substack{\text{this combination of derivatives} \\ \text{is central to Hamiltonian Mechanics}}} + \frac{\partial A}{\partial t}$$

For this reason, we give it a special name. Note that it is a bilinear operation on A, H. It is called the "Poisson Bracket". For any two functions $A(q,p;t)$ and $B(q,p;t)$ it is defined as

$$\{A,B\} \equiv \frac{\partial A}{\partial q^\alpha}\frac{\partial B}{\partial p_\alpha} - \frac{\partial A}{\partial p_\alpha}\frac{\partial B}{\partial q^\alpha}$$

so that, in particular, we have

$$\dot{A} = \{A,H\} + \left.\frac{\partial A}{\partial t}\right|_{p,q}$$

Note the following $\{q^\alpha, q^\beta\} = 0$, $\{p_\alpha, p_\beta\} = 0$

and $\{q^\alpha, p_\beta\} = \delta^\alpha{}_\beta$

(Similar to canonical commutation relations in QM.
In fact $\{A,B\} = \lim_{\hbar\to 0} \frac{-i}{\hbar}[A,B]$)

Some useful properties of Poisson brackets

1) Anti-symmetry $\{A,B\} = -\{B,A\}$

2) & 3) next page

2) Leibnitz rule $\{A, BC\} = \{A,B\}C + B\{A,C\}$

3) Jacobi identity

$$\{A, \{B,C\}\} + \{B, \{A,C\}\} + \{C, \{A,B\}\} = 0$$

As a result, the Poisson Brackets makes the space of functions $f(q,p)$ into a "Lie algebra".

Note that (3) implies that if $\{A,H\}=0$ & $\{B,H\}=0$, then $C = \{A,B\}$ also satisfies $\{C,H\}=0$.

$\Rightarrow$ conserved quantities "form an algebra".

$\Rightarrow$ New conservation laws from old!

Finally, a way to find at least some conservation laws other than just inspection.

In fact, it makes a similar statement about general time derivatives

$$\frac{d}{dt}\{A,B\} = \{\{A,B\},H\} + \frac{\partial}{\partial t}\{A,B\}$$

$$= \{\{A,H\},B\} + \{\{H,B\},A\} \qquad \text{(Jacobi)}$$

$$+ \left\{\frac{\partial A}{\partial t}\Big|_{p,q}, B\right\} + \left\{A, \frac{\partial B}{\partial t}\Big|_{p,q}\right\} \qquad \partial_t|_{p,q} \text{ commutes with } \frac{\partial}{\partial q} \ \&\ \frac{\partial}{\partial p}$$

$$= \{\dot{A}, B\} + \{A, \dot{B}\}$$

$\Rightarrow$ P.B.s respect $\frac{d}{dt}$ (even for t-dependent A & B)

Almost everything we said in this section translates to QM. Quantities that commute with the Hamiltonian (and do not explicitly depend on time) are constants of motion.

An example: Angular momentum

Consider the angular momentum of a particle $\vec{L} = \vec{r} \times \vec{p}$ or in terms of its components:

$$\begin{cases} L_x = y p_z - z p_y \\ L_y = z p_x - x p_z \\ L_z = x p_y - y p_x \end{cases}$$

Using properties (1), (2), (3) together with

$$\{x, p_x\} = 1, \qquad \{y, p_y\} = 1, \qquad \{z, p_z\} = 1$$

and $\{x, p_y\} = 0$, etc.

$\{x, y\} = 0$, etc.

$\{p_x, p_y\} = 0$, etc.

We can easily show that Poisson Brackets of angular momenta satisfy

$$\{L_x, L_y\} = L_z$$

$$\{L_y, L_z\} = L_x$$

$$\{L_z, L_x\} = L_y$$

If the Hamiltonian is rotationally symmetric (e.g. $H = \frac{\vec{p}^2}{2m} + V(r)$) we can also show that $\{H, L_x\} = 0$

as well as $\{H, L_y\} = 0$

and $\{H, L_z\} = 0$

(Show this explicitly) which simply means that different components of angular momentum vector are conserved.

But, if we just know that L_x and L_y are conserved, we immediately know that L_z is also conserved. This is because $\{L_x, L_y\} = L_z$, and using Jacobi identity, L_z should be conserved too.