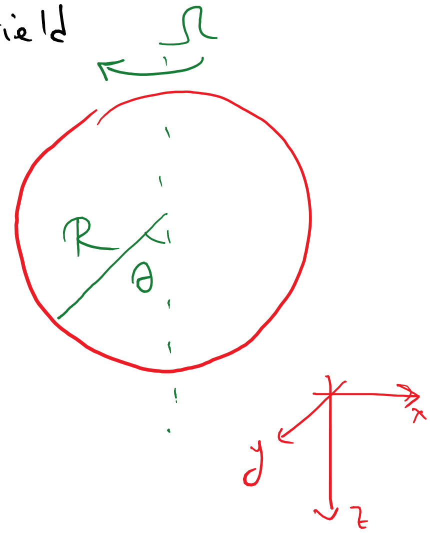
Next we solve a few examples. We'll start with a standard example of a problem that is easy to solve with Lagrangians: A bead on a frictionless rotating loop of wire in a gravitational field



Loop makes angle $\phi = \omega t$ w.r.t. XZ plane

$\downarrow g$

potential energy $V = -mgR\cos\theta$

let's solve constraints as follows

$$\begin{cases} x = R\sin\theta\cos\Omega t \\ y = R\sin\theta\sin\Omega t \\ z = R\cos\theta \end{cases}$$

$$\Longrightarrow \begin{cases} \dot{x} = R\cos\theta\cos\Omega t\,\dot{\theta} - \omega R\sin\theta\cos\Omega t \\ \dot{y} = R\cos\theta\sin\Omega t\,\dot{\theta} + \omega R\sin\theta\sin\Omega t \\ \dot{z} = -R\sin\theta\,\dot{\theta} \end{cases}$$

Now let's form the Lagrangian

Cross terms cancel

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \underbrace{\frac{1}{2}mR^2\left(\dot{\theta}^2 + \Omega^2\sin^2\theta\right)}_{\text{Kinetic energy in spherical coordinates when } \dot{R}=0}$$

Can we use conservation of energy? No! The wire dose work on the bead

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$$\Rightarrow \quad L = T - V = \frac{1}{2} m R^2 \dot{\theta}^2 + \underbrace{\frac{1}{2} m R^2 \Omega^2 \sin^2\theta + mgR\cos\theta}$$

$$= \frac{1}{2} m R^2 \dot{\theta}^2 - \tilde{V}$$ Can solve by energy methods

But let's instead derive the Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(m R^2 \dot{\theta}\right) + \frac{\partial \tilde{V}}{\partial \theta} = 0$$

$$\Rightarrow \quad \ddot{\theta} = \Omega^2 \sin\theta \cos\theta - \frac{g}{R}\sin\theta \equiv F(\theta)$$

divided by mR^2

Equilibrium occurs where $\ddot{\theta} = 0$ at

$$\begin{cases} \theta = 0 \\ \theta = \pi \\ \theta = \theta_0 \quad \text{where} \quad \cos\theta_0 = \dfrac{g}{R\Omega^2} \end{cases}$$

$\rightarrow$ this makes sense only if $\Omega \geq \sqrt{\frac{g}{R}}$

- When Ω is very large $\cos\theta_0 = 1$ $\rightsquigarrow$ $\theta_0 = \frac{\pi}{2}$
- As Ω approaches $\sqrt{\frac{g}{R}}$ from above, θ_0 approaches zero and merges with the equilibrium point at $\theta = 0$.

But, which one of the solutions is chosen by the system?

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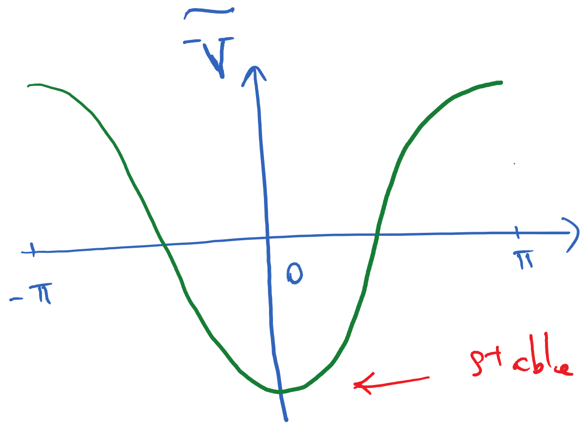
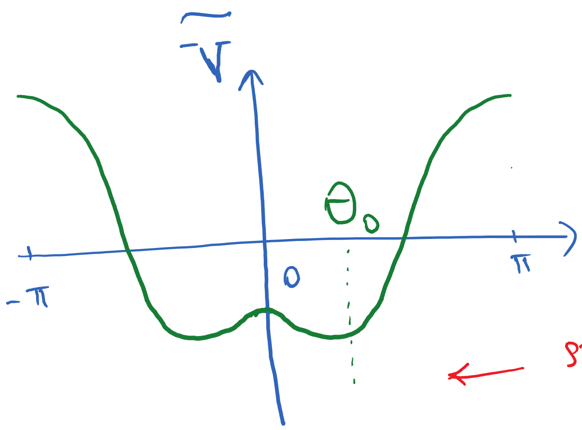
Stability: useful to go back to the interpretation in terms of potential energy $\tilde{V}$. Note that

$F(\theta) = -\frac{d\tilde{V}}{d\theta}$



$\boxed{\frac{R\Omega^2}{g} < 1}$



$\boxed{\frac{R\Omega^2}{g} > 1}$