PHY820/422 HW #8 — Due Monday 11/06/17 @ 5pm
Hamiltonians & Poisson Brackets

1. **Conserved quantities from Hamiltonian.** Given the Hamiltonian

$$H = xp_x - yp_y - ax^2 + by^2 \tag{1}$$

where a and b are constants show that the three functions

$$f_1 = \frac{p_y - by}{x}, \quad f_2 = xy, \quad f_3 = xe^{-t} \tag{2}$$

are constants of motions. Are they functionally independent? Do there exist other independent constants of the motion? Find them if they do exist. Show explicitly that the Poisson Bracket of any two conserved quantities itself is conserved.

2. **Conserved quantities from Hamiltonian.** For a one-dimensional system with the Hamiltonian

$$H = \frac{p^2}{2} - \frac{1}{2q^2} \tag{3}$$

   (a) Show that there is a constat of the motion

$$D = \frac{pq}{2} - Ht \tag{4}$$

   (b) As a generalization of part (a), for motion in a plane with the Hamiltonian

$$H = |\mathbf{p}|^n - ar^{-n} \tag{5}$$

   where $\mathbf{p}$ is the vector of the momenta conjugate to the Cartesian coordinates, show that there is constant of the motion

$$D = \frac{\mathbf{p} \cdot \mathbf{r}}{n} - Ht \tag{6}$$

   (c) The transformation $Q = \lambda q$, $P = p/\lambda$ is obviously canonical. However, the same transformation with t *time dilation*, $Q = \lambda q$, $P = p/\lambda$, $t' = \lambda^2 t$, is not. Show that, however, the equations of motion for p and q in the Hamiltonian in part (a) are invariant under the transformation. The constant of the motion is said to be associated with this invariance.

3. **Poisson Brackets.**

   (a) Prove that the Poisson bracket satisfies the Jacobi idenitity, i.e., that $\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0$ for any three dynamical variables A, B, and C.

(b) Prove that the Poisson bracket satisfies the Leibnitz rule, i.e., that $\{A, BC\} = B\{A, C\} + \{A, B\}C$.

4. **Poisson Brackets of angular and linear momentum.** Using the definition of angular momenta (see lecture notes), show the following relations

$$\{L_x, L_y\} = L_z, \quad \{L_y, L_z\} = L_x, \quad \{L_z, L_x\} = L_y,$$
$$\{p_x, L_z\} = -p_y, \quad \{p_y, L_z\} = p_x, \quad \{p_z, L_x\} = p_y, \quad \{p_z, L_y\} = -p_x$$

5. *** Bonus * Laplace-Rung-Lenz vector**. For the Kepler problem, there exists in addition to the angular momentum, $\mathbf{L}$, another conserved vector quantity, $\mathbf{A}$, the Laplace-Rung-Lenz vector defined as

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \frac{mk\mathbf{r}}{r} \tag{7}$$

   (a) Using your favorite method, show that this vector is conserved. (My favorite method is using Poisson Brackets.)

   (b) Use this fact to derive the orbit equation of the Kepler problem.

   (c) Find the Poisson bracket of the components of this vector (A_x, A_y, A_z) with each other and with those of angular momentum, (L_x, L_y, L_z). Show explicitly that the Poisson Bracket of all conserved quantities are themselves conserved.

6. **Examples of canonical transformations.** Show that

   (a) the following transformation is canonical

$$Q = \log\left(\frac{1}{q}\sin p\right), \quad P = q\cot p \tag{8}$$

   (b) the following transformation is canonical for any choice of the constant α

$$Q = \arctan\frac{\alpha q}{p}, \quad P = \frac{\alpha q^2}{2}\left(1 + \frac{p^2}{\alpha^2 q^2}\right) \tag{9}$$

   (c) the following transformation is canonical for any choice of α

$$Q = q\cos\alpha - p\sin\alpha, \quad P = q\sin\alpha + p\cos\alpha \tag{10}$$

   What canonical transformations do $\alpha = 0, \pi/2$ represent?

7. **Gauge canonical transformation.** We saw that the equations of motion are unchanged when a total derivative $d\psi(q,t)/dt$ is added the Lagrangian. Such a transformation is also called a gauge transformation. How does the Hamiltonian change under such a gauge transformation? [Remark: The momenta p_α change to new momenta p'_α, and the transformation from (q, p) to (q, p') is a particular kind of canonical transformation called a gauge canonical transformation.]

Solutions

1. To determine whether the functions f_1,f_2, and f_3 we need to calculate their total time derivative which can be expressed as

$$\frac{df_i}{dt} = \{f_i, H\} + \frac{\partial f_i}{\partial t} \tag{11}$$

where $\{f_i, H\} = \sum_k (\frac{\partial f_i}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial f_i}{\partial p_k}\frac{\partial H}{\partial q_k})$ is the Poisson bracket of the function f_i and the Hamiltonian H. Thus we have that

$$\frac{df_1}{dt} = \frac{\partial f_1}{\partial x}\frac{\partial H}{\partial p_x} - \frac{\partial H}{\partial x}\frac{\partial f_1}{\partial p_x} + \frac{\partial f_1}{\partial y}\frac{\partial H}{\partial p_y} - \frac{\partial H}{\partial y}\frac{\partial f_1}{\partial p_y} + \frac{\partial f_1}{\partial t} \tag{12}$$

$$\frac{df_1}{dt} = -(\frac{p_y - by}{x^2})x + \frac{by}{x} - \frac{1}{x}(-p_y + 2by) = 0 \tag{13}$$

In a similar way we have that

$$\frac{df_2}{dt} = xy - xy = 0 \tag{14}$$

$$\frac{df_3}{dt} = xe^{-t} - xe^{-t} = 0 \tag{15}$$

Another constant of motion is the Hamiltonian since it doesn't depend explicitly on time hence

$$\frac{dH}{dt} = \{H, H\} + \frac{\partial H}{\partial t} = \{H, H\} = 0 \tag{16}$$

where the last equality used the trivial property that the Poisson bracket of a function with itself is zero. Additional constants of motion can be obtained from the previous functions f_1,f_2, and f_3 by taking the Poisson brackets between themselves. This follows from Poisson's Theorem which states that the Poisson bracket of two quantities that are constants of motion is itself a constant of motion. To show this explicitly, let's compute the Poisson Brackets explicitly

$$\{f_1, f_2\} = -1, \quad \{f_2, f_3\} = 0, \quad \{f_3, f_1\} = 0, \tag{17}$$

$$\{H, f_1\} = f_1, \quad \{H, f_2\} = 0, \quad \{H, f_3\} = 0, \tag{18}$$

As one can seem the right hand side of these equations are either a constant (or, even zero), or are themselves conserved quantities.

2. (a) Using the Poisson bracket formalism we have that

$$\frac{dD}{dt} = \{D, H\} + \frac{\partial D}{\partial t} = \{\frac{pq}{2} - Ht, H\} + \frac{\partial}{\partial t}(\frac{pq}{2} - Ht) = \{\frac{pq}{2}, H\} - H$$
$$= \frac{p}{2}p - \frac{q}{2}\frac{1}{q^3} - \frac{p^2}{2} + \frac{1}{2q^2} = 0 \tag{19}$$

Therefore we conclude that D is a constant of the motion

(b) For this case we have that

$$\frac{dD}{dt} = \{\frac{\vec{p}\cdot\vec{r}}{n} - Ht, H\} + \frac{\partial D}{\partial t} = \{\frac{p_x x + p_y y + p_z z}{n}, H\} - H \tag{20}$$

To make the calculations easier, we will exploit the following property of Poisson brackets

$$\{A, BC\} = \{A, B\}C + B\{A, C\} \equiv \{BC, A\} = \{B, A\}C + B\{C, A\} \tag{21}$$

Now consider the first term of $\frac{dD}{dt}$

$$\{\frac{p_x x}{n}, H\} = \{p_x, H\}\frac{x}{n} + \frac{p_x}{n}\{x, H\} \tag{22}$$

Next we will use the following property of Poisson brackets

$$\{q_i, f(\vec{q}, \vec{p}, t)\} = \frac{\partial f}{\partial p_i} \quad \{p_i, f(\vec{q}, \vec{p}, t)\} = -\frac{\partial f}{\partial q_i} \tag{23}$$

Thus the first term of the time derivative of D can be written now as

$$\{\frac{p_x x}{n}, H\} = -\frac{\partial H}{\partial x}\frac{x}{n} + \frac{p_x}{n}\frac{\partial H}{\partial x} \tag{24}$$

It is easy to see that

$$\frac{\partial H}{\partial r_i} = an\frac{r^{-n}}{r^2}r_i \quad \frac{\partial H}{\partial p_i} = n\frac{p^n}{p^2}p_i \quad (r_1, r_2, r_3) = (x, y, z), (p_1, p_2, p_3) = (p_x, p_y, p_z) \tag{25}$$

Thus we have that

$$\{\frac{p_x x}{n}, H\} = -a\frac{r^{-n}}{r^2}x^2 + \frac{p^n}{p^2}p_x^2 \tag{26}$$

Similarly one finds that

$$\{\frac{p_y y}{n}, H\} = -a\frac{r^{-n}}{r^2}y^2 + \frac{p^n}{p^2}p_y^2 \quad \{\frac{p_z z}{n}, H\} = -a\frac{r^{-n}}{r^2}z^2 + \frac{p^n}{p^2}p_z^2 \tag{27}$$

Thus, the time derivative of D can now be expressed as

$$\frac{dD}{dt} = -a\frac{r^{-n}}{r^2}(x^2 + y^2 + z^2) + \frac{p^n}{p^2}(p_x^2 + p_y^2 + p_z^2) - H \tag{28}$$

$$\frac{dD}{dt} = -ar^{-n} + p^n - H = H - H = 0 \tag{29}$$

Therefore we can conclude that D is a constant of motion.

(c) The equations of motion for this Hamiltonian are

$$\dot{q} = \frac{\partial H}{\partial p} = p \quad \dot{p} = -\frac{\partial H}{\partial q} = -\frac{1}{q^3} \tag{30}$$

hence we obtain that

$$\ddot{q} = -\frac{1}{q^3} \tag{31}$$

By applying the transformation $Q = \lambda q$, $P = \frac{p}{\lambda}$, $t' = \lambda^2 t$ we have that the new Hamiltonian is given by

$$H'(Q, P, t) = \frac{\lambda^2 P^2}{2} - \frac{\lambda^2}{2Q^2} \tag{32}$$

The equations of motion are then given by

$$\dot{Q} = \frac{\partial H'}{\partial P} = \lambda^2 P \quad \dot{P} = -\frac{\partial H'}{\partial Q} = -\frac{\lambda^2}{Q^3} \tag{33}$$

thus we obtain that

$$\ddot{Q} = -\frac{\lambda^4}{Q^3} \Rightarrow \ddot{q} = -\frac{1}{q^3} \tag{34}$$

3. (a) Proving the Jacobi identity for Poisson brackets requires going through lengthy, but straightforward algebra. If done correctly one should arrive at the right answer.

(b) By writing the Poisson bracket explicitly we have that

$$\{f, gh\} = \sum_i (\frac{\partial f}{\partial q_i}\frac{\partial (gh)}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial (gh)}{\partial q_i}) \tag{35}$$

by applying the product rule we obtain

$$\{f, gh\} = \sum_i (\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}h - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}h + g\frac{\partial f}{\partial q_i}\frac{\partial h}{\partial p_i} - g\frac{\partial f}{\partial p_i}\frac{\partial h}{\partial q_i}) \tag{36}$$

$$\therefore \{f, gh\} = \{f, g\}h + g\{f, h\} \tag{37}$$

4. Here we will only show that $\{L_x, L_y\} = L_z$ and $\{p_x, L_z\} = -p_y$. The other relations can be proved by following a similar procedure. To prove $\{L_x, L_y\} = L_z$ we will first write explicitly L_x and L_y as

$$L_x = yp_z - zp_y \quad L_y = zp_x - xp_z \tag{38}$$

thus we have that

$$\{L_x, L_y\} = \{yp_z - zp_y, zp_x - xp_z\} = \{yp_z, zp_x - xp_z\} - \{zp_y, zp_x - xp_z\} \tag{39}$$

By applying Leibnitz's rule we obtain

$$\{L_x, L_y\} = \{y, zp_x - xp_z\}p_z + y\{p_z, zp_x - xp_z\} - \{z, zp_x - xp_z\}p_y - z\{p_y, zp_x - xp_z\} \tag{40}$$

Next by applying the following property of Poisson brackets

$$\{q_i, f(\vec{q}, \vec{p}, t)\} = \frac{\partial f}{\partial p_i} \quad \{p_i, f(\vec{q}, \vec{p}, t)\} = -\frac{\partial f}{\partial q_i} \tag{41}$$

we can see that the first and fourth terms of $\{L_x, L_y\}$ are zero. Thus we have that

$$\{L_x, L_y\} = -y\frac{\partial}{\partial z}(zp_x - xp_z) - p_y\frac{\partial}{\partial p_z}(zp_x - xp_z) = -yp_x + xp_y = L_z \tag{42}$$

To prove that $\{p_x, L_z\} = -p_y$ we simply apply the property $\{p_i, f(\vec{q}, \vec{p}, t)\} = -\frac{\partial f}{\partial q_i}$ to find that

$$\{p_x, L_z\} = \{p_x, xp_y - yp_x\} = -\frac{\partial}{\partial x}(xp_y - yp_x) = -p_y \tag{43}$$

5. (a) For the Kepler problem we can write the Hamiltonian as

$$H = \frac{\vec{p}^2}{2m} - \frac{k}{r} \tag{44}$$

Let us consider the time derivative of the components the Laplace-Rung-Lenz vector

$$\frac{dA_i}{dt} = \{A_i, H\} \tag{45}$$

Let us consider the x component first,

$$\frac{dA_x}{dt} = \{p_yL_z - p_zL_y - \frac{mkx}{r}, \frac{p_x^2 + p_y^2 + p_z^2}{2m} - \frac{k}{r}\} \tag{46}$$

To simplify this, first note that

$$\{L_i, \vec{p}^2\} = 2p_j\{L_i, p_j\} = 2p_i p_k \epsilon_{ijk} = 2(\vec{p}\times\vec{p})_i = 0 \tag{47}$$

This relation also holds if $\vec{p}^2$ is replaced by any function of $\vec{p}^2$. Similarly,

$$\{L_i, \vec{r}^2\} = 0 \tag{48}$$

or, more generally,

$$\{L_i, f(r)\} = 0 \tag{49}$$

for an arbitrary function of $r = |\vec{r}|$. These relation follows from the fact that L_is serve as the generator of rotation, which commute with *scalar* functions that are invariant under rotation.

We also use the fact that

$$\{p_i, g(x, y, z)\} = -\partial_i f \tag{50}$$

and

$$\{p_i, f(r)\} = -(\partial_i r)f'(r) = \frac{r_i}{r^2}f'(r) \tag{51}$$

We thus find

$$\begin{aligned}
\frac{dA_x}{dt} &= \left\{p_y L_z - p_z L_y\,,\, -\frac{k}{r}\right\} + \left\{-\frac{mkx}{r}\,,\, \frac{p_x^2+p_y^2+p_z^2}{2m}\right\} && (52)\\
&= -\frac{k}{r^3}(yL_z - zL_y) - k(p_x\partial_x + p_y\partial_y + p_z\partial_z)\frac{x}{r} && (53)\\
&= -\frac{k}{r^3}\left(\vec{r}\times(\vec{r}\times\vec{p})\right)_x + \frac{k}{r^3}(x\vec{p}\cdot\vec{r} - p_x r^2) && (54)\\
&= 0 && (55)
\end{aligned}$$

(b) From the definition of $\vec{A}$, one can easily see that

$$\vec{A}\cdot L = 0 \tag{56}$$

since the angular momentum $\vec{L}$ is perpendicular to $\vec{p}\times\vec{L}$ and r. It follows from this orthogonality of $\vec{A}$ to $\vec{L}$ that $\vec{A}$ must become fixed vector in the plane of the orbit. If θ is used to denote the angle between $\vec{r}$ and the fixed direction of $\vec{A}$, then the dot product of $\vec{r}$ and $\vec{A}$ is given by

$$\vec{A}\cdot\vec{r} = Ar\cos\theta = \vec{r}\cdot(\vec{p}\times\vec{L}) - mkr \tag{57}$$

Now, by permutation of the terms in the triple dot product, we have

$$\vec{r}\cdot(\vec{p}\times\vec{L}) = \vec{L}\cdot(\vec{r}\times\vec{p}) = l^2 \tag{58}$$

so that Eq. (57) becomes

$$Ar\cos\theta = l^2 - mkr \tag{59}$$

or

$$\frac{1}{r} = \frac{mk}{l^2}\left(1 + \frac{A}{mk}\cos\theta\right) \tag{60}$$

which is the orbit equation for the Kepler problem.

(c) The Poisson bracket between the Laplace-Rung-Lenz vector components with the components of angular momenta is given by

$$\{A_i, L_j\} = \epsilon_{ijk} A_k \tag{61}$$

similar to the Poisson Bracket of the angular momentum with any vector quantity (such as $\vec{r}$ and $\vec{p}$).

The Poisson bracket between components of the Laplace-Rung-Lenz vector require more significant amount of work to obtain. Some reduction in the length of the derivation is obtained by identifying $\vec{C} = \vec{p} \times \vec{L}$, and first evaluating Poisson Brackets between different components of this vector and those with $\vec{r}/r$. After some tedious algebra, it is found that, for example, the first and second component of this vector is given by

$$\{A_x, A_y\} = -(\vec{p}^2 - \frac{2mk}{r})L_z \tag{62}$$

6. (a) To verify if $Q = \log(\frac{\sin(p)}{q})$ and $P = q\cot(p)$ form a canonical transformation, their Poisson bracket must be equal to 1, thus

$$\{Q, P\} = \{\log(\frac{\sin(p)}{q}), q\cot(p)\} = \frac{\partial}{\partial q}(\log(\frac{\sin(p)}{q}))\frac{\partial}{\partial p}(q\cot(p)) - \frac{\partial}{\partial p}(\log(\frac{\sin(p)}{q}))\frac{\partial}{\partial q}(q\cot(p)) \tag{63}$$

$$\{Q, P\} = \frac{1}{q}(q\csc^2(p)) - \frac{\cos(p)}{\sin(p)}\cot(q) = \csc^2(p) - \cot^2(p) = 1 \tag{64}$$

hence Q and P form a canonical transformation

(b) Following the same procedure as part (a) we have that

$$\{Q.P\} = \frac{\partial}{\partial q}(\arctan(\frac{\alpha q}{p}))\frac{\partial}{\partial p}(\frac{\alpha q^2}{2}(1+\frac{p^2}{\alpha^2 q^2})) - \frac{\partial}{\partial p}(\arctan(\frac{\alpha q}{p}))\frac{\partial}{\partial q}(\frac{\alpha q^2}{2}(1+\frac{p^2}{\alpha^2 q^2})) \tag{65}$$

$$\{Q, P\} = \frac{\alpha p}{\alpha^2 q^2 + p^2}\frac{p}{\alpha} + \frac{\alpha q}{\alpha^2 q^2 + p^2}\alpha q = 1 \tag{66}$$

therefore Q and P form also a canonical transformation

(c) Just like the previous parts we have that

$$\{Q, P\} = \cos^2(\alpha) + \sin^2(\alpha) = 1 \tag{67}$$

If $\alpha = 0$ this corresponds to the identity transformation $Q = q$ and $P = p$. If $\alpha = \frac{\pi}{2}$ this corresponds to switching the generalized coordinate with the generalized momentum, that is $Q = -p$ and $P = q$

7. Let $L(\vec{q}, \dot{\vec{q}}, t)$ be the Lagrangian associated with the Hamiltonian $H(\vec{q}, \vec{p}, t)$. Now we define a new Lagrangian in terms of the old one using the total time derivative of the function $\psi(\vec{q}), t$ as

$$L'(\vec{q}, \dot{\vec{q}}, t) = L(\vec{q}, \dot{\vec{q}}, t) + \frac{d\psi(\vec{q}, t)}{dt} = L(\vec{q}, \dot{\vec{q}}, t) + \frac{\partial \psi}{\partial t} + \sum_i \frac{\partial \psi}{\partial q_i} \dot{q}_i \tag{68}$$

We now determine the canonical momenta for the new Lagrangian

$$p'_j = \frac{\partial L'}{\partial \dot{q}_j} = \frac{\partial L}{\partial \dot{q}_j} + \frac{\partial}{\partial \dot{q}_j} (\sum_i \frac{\partial \psi}{\partial q_i} \dot{q}_i) = p_j + \frac{\partial \psi}{\partial q_j} \tag{69}$$

Now the new Hamiltonian is given by

$$H'(\vec{q}, \vec{p}', t) = \sum_i \dot{q}_i(\vec{q}, \vec{p}' - \nabla\psi, t) \cdot p'_i - L' = \sum_i \dot{q}_i(\vec{q}, \vec{p}' - \nabla\psi, t) \cdot p'_i - (L(\vec{q}, \dot{\vec{q}}, t) + \frac{\partial \psi}{\partial t} + \sum_i \frac{\partial \psi}{\partial q_i} \dot{q}_i) \tag{70}$$

hence we have that

$$H'(\vec{q}, \vec{p}', t) = \sum_i \dot{q}_i(\vec{q}, \vec{p}' - \nabla\psi, t)(p'_i - \frac{\partial \psi}{\partial q_i}) - L(\vec{q}, \dot{\vec{q}}, t) - \frac{\partial \psi}{\partial t} \tag{71}$$

By identifying the first two terms in the previous equation as the old Hamiltonian

$$H(\vec{q}, \vec{p}' - \nabla\psi, t) = \sum_i \dot{q}_i(\vec{q}, \vec{p}' - \nabla\psi, t)(p'_i - \frac{\partial \psi}{\partial q_i}) - L(\vec{q}, \dot{\vec{q}}(\vec{q}, \vec{p}' - \nabla\psi, t), t) \tag{72}$$

we have that

$$H'(\vec{q}, \vec{p}', t) = H(\vec{q}, \vec{p}' - \nabla\psi, t) - \frac{\partial \psi}{\partial t} \tag{73}$$

Thus the old Hamiltonian differs from the new one since the canonical momenta p_i are replaced by $p_i - \frac{\partial \psi}{\partial q_i}$ and the additional factor of the partial derivative with respect to time of ψ.