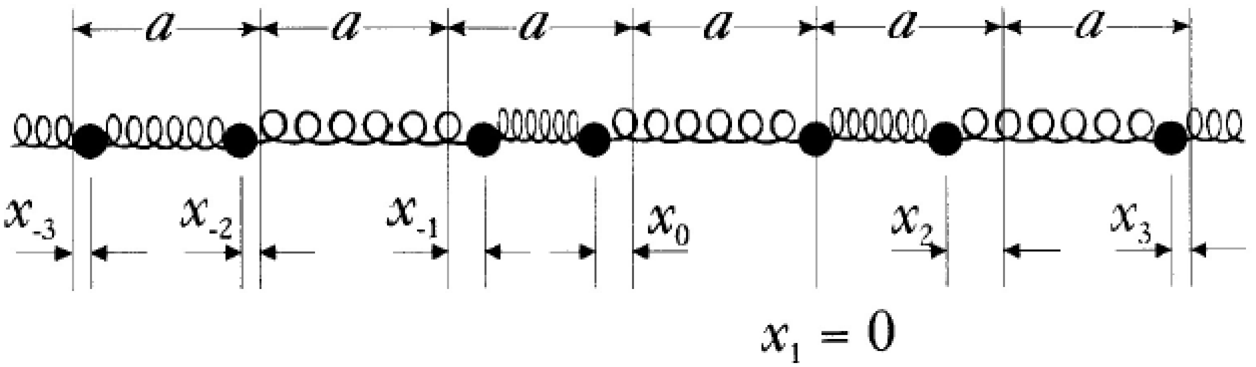
A chain of coupled oscillators

We now consider a long chain of oscillators of equal masses m and the same spring constant K. For simplicity we assume that the chain is infinitely long. (Infinity simplifies the problem!)

Let's suppose that, in equilibrium, the distance between particles is some constant a. The coordinate x_n of the mass i is its displacement from its equilibrium position



The Lagrangian of the system is given by

$$L = \frac{1}{2} m \sum_{n=-\infty}^{+\infty} \dot{x}_n^2 - \frac{1}{2} K \sum_{n=-\infty}^{+\infty} (x_{n+1} - x_n)^2$$

The Euler-Lagrange equations that follow from the Lagrangian are

$$\ddot{x}_n - \omega_0^2 \left(x_{n-1} - 2x_n + x_{n+1} \right) = 0$$

where $\omega_0 = \sqrt{\frac{K}{m}}$ as usual.

Writing this equation in a matrix form

$$\ddot{\vec{x}} + \Lambda \vec{x} = 0 \qquad \text{where } \vec{x} = (\ldots x_{-1}\ x_0\ x_1 \ldots)$$

The matrix Λ can be easily read off as

$$\Lambda = \omega_0^2 \begin{pmatrix} \ddots & & & & \\ & -2 & 1 & 0 & 0 \\ & 1 & -2 & 1 & 0 \\ & 0 & 1 & -2 & 1 \\ & 0 & 0 & 1 & -2 \\ & & & & & \ddots \end{pmatrix}$$

Most of the matrix elements are zero. Nonzero matrix elements are on or next to the diagonal of the matrix.

Instead of trying to diagonalize Λ analytically, we look directly for the solutions of the normal mode, but with a wavelike structure

$$x_n = C\, e^{i(kna - \omega t)}$$

While the solution x_n has to be real, let's not worry about it now.

Such a solution if it exists looks like a wave of amplitude C propagating down the chain of masses. To find such solutions, let's insert it into the equation of motion, obtaining

$$-\omega^2 x_n = \omega_0^2 \left[e^{-ika} - 2 + e^{ika} \right] x_n$$

This equation gives the dispersion relation

$$\omega(k) = 2\omega_0 \left| \sin \frac{ka}{2} \right|$$

This result is interpreted in the following way: A wavlike solution

$$x_n(t,k) = C(k)\, e^{i(kna - \omega(k)t)}$$

exists at every wave vector k whose frequency $\omega(k)$ satisfies the above equation.

A general solution is given by the superposition of waves of different wavevectors

$$x_n(t) = \int_{-\pi/a}^{\pi/a} dk\, C(k)\, e^{i(kna - \omega(k)t)}$$

and the constants C should be chosen from initial conditions.

Let's also expand the dispersion relation for small values of k. You may remember that long-distance features are encoded in small values of momentum. That means if we want to find out how things propagate over long distances, we have to look at small k.

$$\omega(k) \simeq \omega_0 a\, |k|$$

Therefore, the dispersion relation is linear. This sounds a lot like the propagation of light in vacuum where $\omega = c|\vec{k}|$ and c is the speed of light. In this case, however, the linear dispersion is the result of mechanical vibrations (as opposed to electromagnetic fluctuations) and the speed of light is replaced by the speed of sound!

$$\boxed{c_0 = \omega_0 a}$$

What happens in the presence of boundaries?

For a finite (but a long) chain, one should also consider boundary conditions at the endpoints. Suppose, for example, that the chain starts from the site 0. Also assume that the chain at its endpoint is "fixed" so that $x_0 = 0$. This endpoint condition can be satisfied by combining pairs of running waves so that they inteferе destructively and cancel out at the end points.

This can be achieved by the linear combination

$$x_n = C\, e^{-i\omega(k)t} \left(e^{ikna} - e^{-ikna}\right)/(2i)$$
$$= C\, e^{-i\omega(k)t} \sin(kna)$$

Remember that $\omega(k) = \omega(-k)$ so this combination is still a solution of the equation of motion. Also notice that the boundary condition is satisfied and $x_0 = 0$ at all times.