Review of Classical Mechanics:

Newton's law $\quad m\dfrac{d^2\vec{x}}{dt^2} = \vec{F}(\vec{x})$

i.e., $\vec{F} = -\vec{\nabla} V$

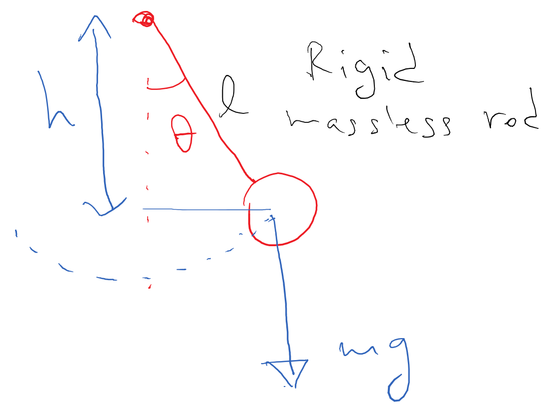
$\longrightarrow$ Conservation of energy (if the force is conservative)

$$\underbrace{\frac{1}{2} m \dot{\vec{x}}^2}_{\text{Kinetic energy}} + \underbrace{V(\vec{x})}_{\text{potential energy}} = E$$

= Total energy is conserved.

A) One-dimensional motion: simple pendulum

$$V = mg\,h = -\underbrace{mg\ell}_{\text{Call this } A} \cos\theta$$



$$\frac{1}{2}\underbrace{m\ell^2}_{\text{Call this } I}\dot{\theta}^2 - A\cos\theta = E$$

Try to Solve this equation

$$\dot{\theta} = \sqrt{\frac{2}{I}}\sqrt{E + A\cos\theta}$$

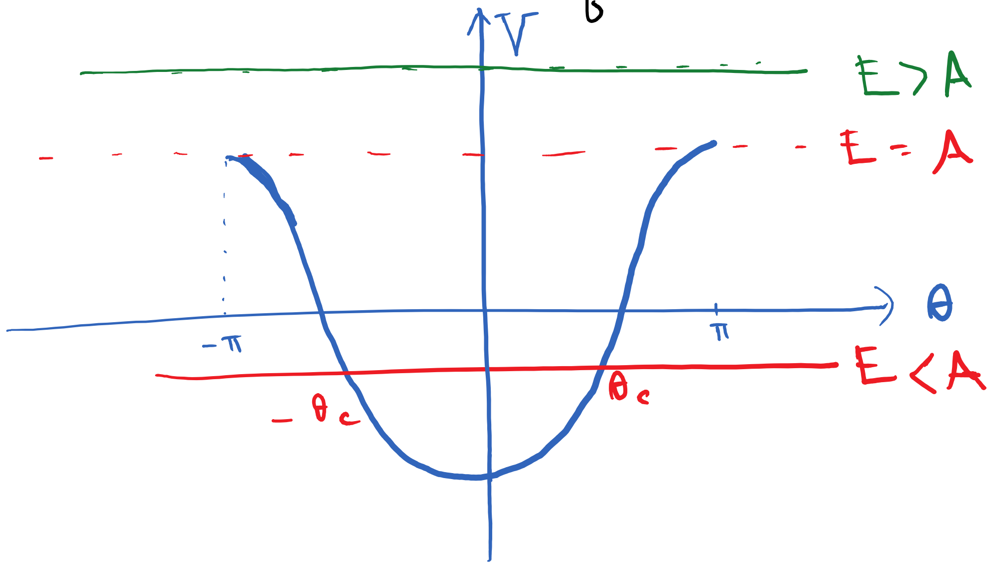
$$\longrightarrow \quad \boxed{t - t_0 = \sqrt{\frac{I}{2}} \int_{\theta_0}^{\theta} \frac{d\theta}{\sqrt{E + A\cos\theta}}}$$

$(*)$

As you probably know, this can't be evaluated in terms of elementary functions & is instead related to complete elliptic integrals.

②

But how can we understand qualitative features of this equation?



E < A : Oscillates

θ_c is the turning point $\quad E = V(\theta_c)$

Near θ_c, $\quad V(\theta) = V(\theta_c) + V'(\theta_c)(\theta-\theta_c) + V''(\theta_c)(\theta-\theta_c)^2/2 + \cdots$

Suppose $V'(\theta_c) \neq 0 \iff$ Non-zero force at θ_c

For θ near θ_c, we can expand

$$\int_\theta^{\theta_c} \frac{d\theta}{\sqrt{E-V}} \approx \int_\theta^{\theta_c} \frac{d\theta}{\sqrt{\cancel{V(\theta_c)} - \cancel{V(\theta_c)} + V'(\theta_c - \theta) + \cdots}}$$

$$\approx \underbrace{\sqrt{\frac{\theta_c - \theta}{V'}}}_{\text{Finite.}} + \cdots$$

so reaches θ_c, turns around & goes back

③

Of course, for $\theta_c \ll 1$, we can use

$$E = \frac{1}{2} I \dot{\theta}^2 - A \cos\theta = \frac{I \dot{\theta}^2}{2} - A + \frac{A}{2}\theta^2 + \cdots$$

= Harmonic oscillator w/ freq $\omega = \sqrt{\frac{A}{I}}$

More generally we can use (*) above to compute the period numerically as a function of θ_c.

$E = A$: $V'(\theta_c) = 0$ & find

$$\int_{\theta_c}^{\theta} \frac{d\theta}{\sqrt{E - V}} \approx \int_{\theta_c}^{\theta} \frac{d\theta}{\sqrt{-V''(\theta-\theta_c)^2/2}} \propto \log(\theta_c - \theta)$$

$\Longrightarrow$ $\theta = \theta_c$ at $t = \infty$

For $E > A$: The pendulum just goes over the potential

④

Phase portrait

Velocity: $v = \pm \sqrt{\frac{2}{m}\left(E + A\cos\theta\right)}$

- E close to A: $\cos\theta \simeq 1 - \frac{\theta^2}{2}$

the $v - \theta$ relation becomes

$$\frac{m v^2}{2(E+A)} + \frac{A\theta^2}{2(E+A)} = 1$$

→ Ellipse centered at $\theta = 0$