PHY820/422 HW #4 — Due Monday 10/2/17 @ 5pm
Lagrangians: More practice

1. Using the variational principle, the goal of this problem is to show something really simple: A free particle undergoes a uniform linear motion, or, more precisely, it moves from a position x_1 at time t_1 to a position x_2 at time t_2 at a constant speed $v = (x_2 - x_1)/(t_2 - t_1)$. The equation of motion is determined from the minimum of the action over all possible trajectories $x(t)$ such that $x(t_1) = x_1$ and $x(t_2) = x_2$. To treat the boundary conditions, let's consider a general trajectory of the form

$$x(t) = x_1 + \frac{x_2 - x_1}{t_2 - t_1}(t - t_1) + f(t) \tag{1}$$

where the function $f(t)$ has the property that $f(t_1) = f(t_2) = 0$ but is otherwise completely arbitrary.

   (a) We first need to define the space of all possible trajectories. To do so, let's define a complete set of functions as

$$f_n(t) = \sqrt{\frac{2}{t_2 - t_1}} \sin\left(\frac{n\pi(t - t_1)}{t_2 - t_1}\right), \qquad n = 1, 2, 3, \cdots \tag{2}$$

   Convince yourself that this is a complete basis. (An analogy with the quantum mechanical problem of a particle in a box may be useful.) Furthermore, show that it defines an orthonormal basis, i.e.,

$$\int_{t_1}^{t_2} f_n(t) f_m(t) dt = \delta_{nm} \tag{3}$$

   where the Kronecker delta function δ_{nm} is unity if $n = m$ and zero otherwise.

   (b) Consider a general trajectory $f(t) = \sum_{n=1,2,\cdots} a_n f_n(t)$ with arbitrary coefficients $a_1, a_2, \cdots$. Using the orthornomal property of these functions[1], construct the action $S = (m/2) \int_{t_1}^{t_2} \dot{x}^2 dt$ in terms of the coefficients a_n. Show that the minimum of the action is given by all $a_n = 0$, hence $f(t) = 0$ and particle moves at a constant speed.

2. A particle is free to move on a circle under the influence of no forces other than those than constrain it to the circle. It starts at an angle θ_1 at time t_1 and ends at another angle θ_2 at time t_2.

   (a) Show that there are a (countably) infinite number of physical paths $\theta(t)$ the particle can take in doing so; label different trajectories by n. Therefore, the variational principle in this case does not deterministically specify the motion.

[1] You also need the fact that $\int_{t_1}^{t_2} f_n(t) = 0$, that is, the function f_n is also orthogonal to a constant function.

(b) **Bonus:** In quantum mechanics, on the other hand, there is no unique trajectory, but instead one should sum over all possible trajectories where each trajectory contributes a phase e^{iS_n} with S_n the action corresponding to the trajectory labeled by n. What quantum number does n represent?

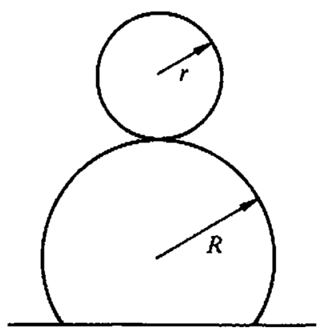
3. José and Saletan, Chapter 3, problem 22

   Use the Lagrange multiplier method to solve the following problem. A particle in a uniform gravitational field is free to move without friction on a paraboloid of revolution whose symmetry axis is vertical (opening upward). Obtain the force of constraint. Prove that for given energy and angular momentum about the symmetry axis there are a minimum and maximum height to which the particle will go.

   Note: Use parabolic coordinates (z, ρ, ϕ) with z along the vertical axis, and ρ and ϕ the radial and (polar) angular coordinates. The paraboloid is defined by the equation $z = a\rho^2$.

4. Goldstein (Ed. 2), Chapter 1, Problem 12

   A uniform hoop of mass m and radius r rolls without slipping on a fixed cylinder of radius R as shown in the figure. The only external force is that of gravity. If the smaller cylinder starts rolling from rest on top of the bigger cylinder, find by method of Lagrange multipliers the point at which the hoop falls off the cylinder.



5. Follow up on the problem covered during class.

   In class, we discussed the problem of a bead that is constrained to a loop whose radius is changing linearly in time, $R(t) = ct$. Recall that the equations of motion are given by

$$\ddot{x} = \frac{c^2 - \dot{x}^2 - \dot{y}^2}{c^2 t^2} x$$
$$\ddot{y} = \frac{c^2 - \dot{x}^2 - \dot{y}^2}{c^2 t^2} y$$

   (a) Show that the constraint force is in the radial direction, and find its magnitude. Is the constraint force pointing towards or away from the center? Does your answer change if, instead of an expanding wire, the radius is decreasing in time at a constant rate? Explain.

(b) Repeat this problem, using the method of Lagrange multipliers, for a loop whose radius is a general function of time $R = R(t)$. Derive the new equations of motion, determine the constraint forces, and discuss how your answers to part (a) may be different. Show that the constraint force is precisely the force needed to generate a radial acceleration $\ddot{R}(t) - v_\theta^2/R(t)$ with v_θ the angular velocity tangent to the loop.