PHY-820 - CLASSICAL MECHANICS

Final Exam

Problem 2 - 10 Points

Consider a roller coaster in which a car of mass m is attached to a frictionless three-dimensional track. The track has a circular footprint with radius R, i.e. the x and y coordinates defining the horizontal plane satisfy $x^2 + y^2 = R^2$. The vertical z coordinate of the track is made to depend on the azimuthal angle θ, measured in radians, as $z = h \cdot (1 + \sin\theta)$.

a) [2pts] Determine the Lagrangian L in terms of θ and $\dot{\theta}$.

b) [2pts] Derive the equations of motion.

c) [2pts] Determine the generalized momentum p_θ, and express $\dot{\theta}$ by p_θ.

d) [2pts] Determine the Hamiltonian H of the system.

e) [2pts] Are p_θ and/or the Hamiltonian H conserved, why or why not?

Solution -

a) $L = \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} m \dot{z}^2 - mgz \quad , \quad \dot{z} = h \cos\theta \, \dot{\theta}$

$= \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} m h^2 \cos^2\theta \, \dot{\theta}^2 - mgh(1+\sin\theta)$

b) $\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$

$\rightarrow \frac{d}{dt} \left[m (R^2 + h^2 \cos^2\theta) \, \dot{\theta} \right] + \sin\theta \cos\theta \, m h^2 \dot{\theta}^2$

$+ mgh \cos\theta = 0$

c) $P_\theta = \frac{\partial L}{\partial \dot{\theta}} = m R^2 \dot{\theta} + m h^2 \cos^2\theta \, \dot{\theta} = m(R^2 + h^2 \cos^2\theta) \dot{\theta}$

$\rightarrow \dot{\theta} = \frac{P_\theta}{m(R^2 + h^2 \cos^2\theta)}$

$\rightarrow H = \dot{\theta} P_\theta - L = \dot{\theta} P_\theta - \frac{1}{2} P_\theta \dot{\theta} + mgh(1+\sin\theta)$

$= \frac{1}{2} P_\theta \dot{\theta} + mgh(1+\sin\theta)$

$= \frac{P_\theta^2}{2m(R^2 + h^2 \cos^2\theta)} + mgh(1+\sin\theta)$

d) Using hamilton's equations, one can show that the equation of motion is consistent with the one in part b.

e) H is conserved because hamiltonian is not explicitly time dependant, but P_θ is not conserved due to the explicit dependence of H on θ.

- Problem:

Consider the Hamiltonian

$$H = \frac{(P_x + at)^2}{2m} + V(x)$$

find the equation of motion

Solution -

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{P_x + at}{m} \rightarrow P_x = m\dot{x} - at \quad (I)$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x} \quad (II)$$

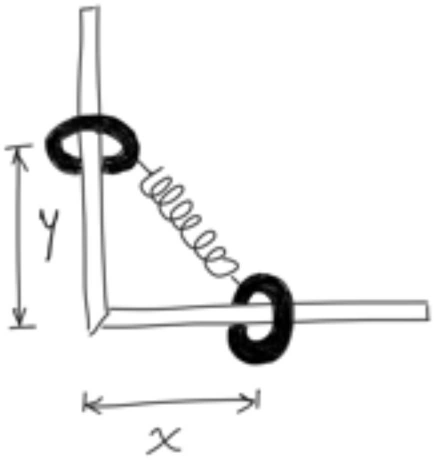
$$(I),(II) \Rightarrow m\ddot{x} - a = -\frac{\partial V}{\partial x}$$

$$\rightarrow m\ddot{x} = -\frac{\partial V}{\partial x} - a$$

Therefore the linear term added to momentum in H acts as a constant force.

- Problem:

Consider two rods joined at the right angle. Two beads are free to move on the rods & are connected to each other via a spring with force constant k & unstretched length l. Find the normal modes.



Solution-

The Lagrangian of the system is

$$\mathcal{L} = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - \frac{k}{2}\left(\sqrt{x^2+y^2} - \ell\right)^2$$

We make a change of coordinates:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

The Lagrangian in the new coordinates takes the form

$$\mathcal{L} = \frac{1}{2} m\dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - \frac{k}{2}(r-\ell)^2$$

The system has two normal modes:

1) $r = \ell$ & $\ddot{\theta} = 0$ which is a zero mode of the system.

2) $\theta = \text{constant}$ & $\ddot{r} + \frac{k}{m}(r-\ell) = 0$ which gives rise to an eigen-frequency $\omega = \sqrt{\frac{k}{m}}$.