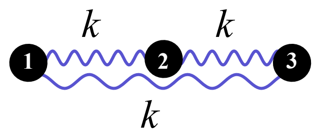
PHY820/422 MidTerm Exam 2

1. Consider three particles with the same mass m that are all connected to each other via springs as shown in the figure below. The two springs that are connected to particle 2 have the unstretched length equal to 0, while the spring connecting particle 1 to 3 has an unstreched length equal to a. All force constants are k.



   (a) Using your favorite method, find the equilibrium positions of the three particles.

   (b) Do you expect a zero mode of small oscillations around equilibrium? Explain physically why or why not?

   (c) Determine the eigenfrequencies of small oscillations around equilibrium. (You do not need to determine the normal modes.)

   (d) Does your answer to part (b) change if the force constant of the spring connecting particle 1 to particle 3 would be different from k?

2. (a) What is the total cross section for the potential

$$V(r) = \frac{k}{r} \tag{1}$$

   (b) What is the total cross section for the potential

$$V(r) = \begin{cases} \frac{k}{r} - \frac{k}{a}, & r < a \\ 0, & r > a \end{cases} \tag{2}$$

3. Consider a particle moving in two dimensions (x, y) with the Hamiltonian

$$H = \frac{1}{2m}p_x^2 + \frac{1}{2m}(p_y + bx)^2 \tag{3}$$

   where m is the mass of the particle, and b is a constant. p_x and p_y represent the momenta conjugate to x and y, respectively.

   (a) Identify conserved quantities. Explain why they are conserved.

   (b) Use Hamilton's equations to derive the equations of motion. Write these equations solely in terms of x, y, and their time derivatives as well as possible constants of motion. [You are not expected to solve these equations.]

   (c) * **Bonus** * Explain what the equations of motion physically describe? (Hint: What is the physical significance of b?)

Solutions

1. (a) The Lagrangian for this system is

$$L = \frac{1}{2}m\left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2\right) - \frac{1}{2}k\left[(q_2 - q_1)^2 + (q_3 - q_2) + (q_3 - q_1 - a)^2\right] \tag{4}$$

Using the Euler-Lagrange equations, we find the equations of motion,

$$m\ddot{q}_1 = -k(2q_1 - q_2 - q_3 + a), \tag{5}$$
$$m\ddot{q}_2 = -k(2q_2 - q_1 - q_3), \tag{6}$$
$$m\ddot{q}_3 = -k(2q_3 - q_1 - q_2 - a) \tag{7}$$

To find the equilibrium position, we must set the right-hand-side of these equations to zero. This will nevertheless not completely determine the equilibrium position because of translation invariance: Changing the positions of all particles by a constant does not affect the physics. To find *an* equilibrium, we set $q_2 = 0$ which then gives us the equilibrium positions

$$\bar{q}_1 = -\frac{a}{3}, \quad \bar{q}_2 = 0, \quad \bar{q}_3 = \frac{a}{3} \tag{8}$$

(b) A zero mode must exist because the center of mass coordinate can move at a constant velocity.

(c) Expanding around equilibrium, $x_i = q_i - \bar{q}_i$, we then find

$$\ddot{\mathbf{x}} + \mathbf{\Lambda}\,\mathbf{x} = 0 \tag{9}$$

where $\mathbf{x}$ represents the matrix (x_1, x_2, x_3), and the matrix $\mathbf{\Lambda}$ is given by

$$\mathbf{\Lambda} = \frac{k}{m}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \tag{10}$$

The eigenfrequencies can be obtained as the square roots of the eigenvalues of this matrix,

$$\omega_1 = 0, \quad \omega_2 = \omega_3 = \sqrt{\frac{3k}{m}} \tag{11}$$

(d) No. The zero exists due to translation invariance.

2. (a) The cross section is infinite since the interaction is long-ranged (decays less slowly than $1/r^2$ as $r \to \infty$).

(b) The cross section is πa^2. Only the particles in the circular cross section of radius a get scattered.

3. (a) Some conserved quantities are the Hamiltonian, since it does not explicitly depend on time ($\partial H/\partial t = 0$), and the momentum along the y direction, p_y, since the Hamiltonian does not explicitly depend on the corresponding coordinate ($\partial H/\partial y = 0$).

(b) Using Hamilton's equations, we find

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p_x}{m}, \qquad \dot{p}_x = -\frac{\partial H}{\partial x} = -\frac{b}{m}\left(p_y + bx\right) \tag{12}$$

$$\dot{y} = \frac{\partial H}{\partial p} = \frac{p_y + bx}{m}, \qquad \dot{p}_y = 0 \tag{13}$$

from which it follows explicitly that $p_y = c$ is a constant of motion. The equation of motion follows by eliminating the momenta from the Hamilton's equations to find

$$\ddot{x} = -\frac{b}{m}(c + bx), \qquad \ddot{y} = \frac{b}{m}\dot{x} \tag{14}$$

(c) One may notice that the equations of motion (using the Hamilton's equations) can also be written as

$$m\ddot{x} = -b\dot{y}, \quad m\dot{y} = b\dot{x} \tag{15}$$

In a vector notation, these equations can be cast as

$$m\ddot{\mathbf{x}} = q\mathbf{v} \times \mathbf{B} \tag{16}$$

where $\mathbf{v} = \dot{\mathbf{x}}$ is the velocity, and $\mathbf{B} = -(b/q)\hat{\mathbf{z}}$ is a magnetic field along the z direction (q is assumed to be the particle's charge).