Constraints in the Lagrangian Formulation

Now, it turns out that variational principles are well suited to the study of constrained systems

But, you say, we have already discussed constrained systems

Yes, but recall that we assumed constraints $f_I(\vec{X}_i) = 0$ could be solved by introducing $\vec{X}_i = \vec{X}_i(q^\alpha, t)$ where the q^α are unconstrained.

Sometimes this is so messy as to be a bad idea. & it is better to solve only <u>some</u> constraints in this way.

A variational approach is still useful.

So suppose $L = L(q^\alpha, \dot{q}^\alpha, t)$ & $f_I(q^\alpha, t)$ (for N q^α's)

again restrict to holonomic constraints

[Non holonomic constraints essentially never arise in modern physics & don't have a nice story]

We know that correct equation of motion would follow if one solve $f_I(q^\alpha, t)$ via $q^\alpha = q^\alpha(\underbrace{\tilde{q}}_{N-K \text{ of these}}, t)$

K = total number of constraints

and require $\delta S = 0$ for $S[q^\alpha(\tilde{q}, t)]$ with arbitrary $\delta \tilde{q}$.

However, it turns out we can do this calculation without exactly solving the constraints. We just note that the above is equivalent to requiring $\delta S = 0$ for all δq's which preserve the constraints; i.e., for which $\delta f_I = 0$

As usual,
$$\delta S = \delta \int L\,dt = \int \frac{\partial L}{\partial q^\alpha}\delta q^\alpha + \frac{\partial L}{\partial \dot q^\alpha}\delta \dot q^\alpha$$
$$= \int \left[\frac{\partial L}{\partial q^\alpha} - \frac{d}{dt}\frac{\partial L}{\partial \dot q^\alpha}\right]\delta q^\alpha$$
since $\delta q^\alpha = 0$ at the endpoints
$$= \langle EL \,|\, \delta q \rangle = 0$$

However, $|\delta q)$ is not completely arbitrary because it has to satisfy the condition

$$\delta f_I = \frac{\partial f_I}{\partial q^\alpha}\delta q^\alpha = 0$$
at each time

Let's write this condition in our vector notation too

$$\delta f_I = \left\langle \frac{\partial f}{\partial q} \,\middle|\, \delta q \right\rangle = 0$$

this means that $|\delta q\rangle \perp \left|\frac{\partial f}{\partial q}\right\rangle$

That satisfies the physical constraint

Now let's summarize what we have

$$\langle EL \mid \delta q \rangle = 0$$ for <u>all</u> vector $|\delta q\rangle$ that are perpendicular to $\left|\frac{\partial f_I}{\partial q}\right\rangle$

A little thought shows that the vector $|EL\rangle$ can have nonzero components only in the vector space spanned by $\left|\frac{\partial f_I}{\partial q}\right\rangle$. I.e.,

$$|EL\rangle = \sum_I \lambda_I \left|\frac{\partial f_I}{\partial q}\right\rangle$$

λ_I can be time dependent

Going back from the vector space to coordinates

$$EL_\alpha(t) \equiv \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\alpha} - \frac{\partial L}{\partial q} = \sum_I \lambda_I \frac{\partial f_I}{\partial q^\alpha} \quad (*)$$

forces of constraint

Solve these equations together with

$$f_I = 0$$ constraints.

Now why did I say that variational principle give a nice way to treat such problems? Because the constraints can be encoded directly into the variational problem.

E.g. note that (since λ_I does not depend on q & $\dot{q}$) the term q and $\dot{q}$ are much like potential terms.

In fact, consider $\tilde{L} = L - \sum_I \lambda_I f_I$

with the generalized coordinates q^α & $\underline{\underline{\lambda_I}}$

then $\tilde{S} = \int \tilde{L}\, dt$ & $\delta \tilde{S} = 0 \implies$

$$\delta \tilde{S} = \int dt \left[\frac{\partial \tilde{L}}{\partial q^\alpha} \delta q^\alpha + \frac{\partial \tilde{L}}{\partial \dot{q}^\alpha} \delta \dot{q}^\alpha + \frac{\partial \tilde{L}}{\partial \lambda_I} \delta \lambda_I \right]$$

$$= \int dt \left[\left(\frac{\partial \tilde{L}}{\partial q^\alpha} - \frac{d}{dt} \frac{\partial \tilde{L}}{\partial \dot{q}^\alpha} \right) \delta q^\alpha + \frac{\partial \tilde{L}}{\partial \lambda_I} \delta \lambda_I \right]$$

(again used $\delta q(t_0) = \delta q(t_1) = 0$)

$= 0$

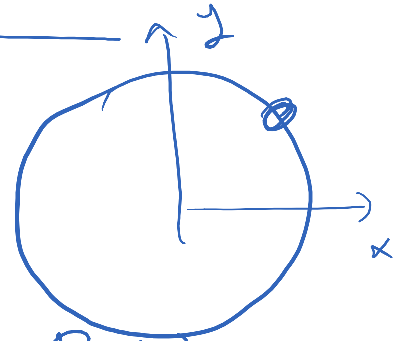
For $\delta S = 0$ for arbitrary δq^α and $\delta \lambda_I$, we should have

(1) $\dfrac{\partial \tilde{L}}{\partial \lambda_I(t)} = 0 \rightsquigarrow f_I = 0$ (our constraints!!)

(2) $\dfrac{d}{dt} \dfrac{\partial \tilde{L}}{\partial \dot{q}^\alpha} - \dfrac{\partial \tilde{L}}{\partial q^\alpha} = 0 \rightsquigarrow$ Equation (*)

Example: Motion on an expanding wire

Consider a bead that is constrained to a wire whose shape changes in time. For example, this could be a circular wire whose radius $R(t)$ expands linearly with time $R(t) = ct$



We can explicitly solve the constraint by defining an angle θ. But let's not do that; in general, for a complicated shape, it may not even be possible to do this.

Reminder: Energy is not conserved again. So we can't use that.

The constraint is then

$$f(x,y,t) = x^2 + y^2 - c^2t^2 = 0$$

In the absence of an external potential,

$$L = T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

Therefore, we can write the Lagrangian $\mathcal{L}$ that includes the constraint as

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \lambda(t)\left(x^2 + y^2 - c^2t^2\right)$$

equations of motion follows from (*)

$$\rightsquigarrow \begin{cases} m\ddot{x} - 2\lambda x = 0 & (1) \\ m\ddot{y} - 2\lambda y = 0 & (2) \end{cases}$$

and $f = x^2+y^2-c^2t^2 = 0 \rightsquigarrow$ constraint

Let's use the constraint to derive a relation between $\dot{x}, \dot{y}, \ddot{x}, \ddot{y}$

Since $f = 0 \rightsquigarrow \dot{f} = 0 \longrightarrow x\dot{x} + y\dot{y} - c^2 t = 0$

$$\rightsquigarrow \ddot{f} = 0 \longrightarrow \dot{x}^2 + x\ddot{x} + \dot{y}^2 + y\ddot{y} - c^2 = 0$$

On the other hand, from (1) and (2) together with $\ddot{f} = 0$, we find

$$\dot{x}^2 + \frac{2\lambda}{m}x^2 + \dot{y}^2 + \frac{2\lambda}{m}y^2 - c^2 = 0$$

use to solve λ

$$\lambda = m\,\frac{c^2 - \dot{x}^2 - \dot{y}^2}{2\,(x^2+y^2)} = m\,\frac{c^2 - \dot{x}^2 - \dot{y}^2}{2c^2t^2}$$

Now let's plug λ in Eqs. (1) and (2)

$$\begin{cases} m\ddot{x} = m\,\dfrac{c^2 - \dot{x}^2 - \dot{y}^2}{c^2t^2}\,x = C_x \\[2ex] m\ddot{y} = m\,\dfrac{c^2 - \dot{x}^2 - \dot{y}^2}{c^2t^2}\,y = C_y \end{cases}$$

components of constraint forces

Therefore, we actually managed to find the components of the constraint force (solving one of the unknowns in Newton's equation in terms of other variables)

$\Longrightarrow$ Two equations for two variables (x, y)

Well, at this point, we still have a complicated set of equations, and have some algebra to do. Nevertheless, the method of Lagrange multipliers can help to organize physics in a useful way.

We can understand these equations better in some limit. E.g. assume velocity $v = \sqrt{\dot{x}^2 + \dot{y}^2}$ is small compared to c. In this situation

$$\begin{cases} \ddot{x} \simeq \dfrac{1}{t^2} x \\ \ddot{y} \simeq \dfrac{1}{t^2} y \end{cases}$$

You can solve the differential equations together with the knowledge of initial conditions