So, for conserved p_β, can we just replace this by a constant in H?

Almost. OK to compute all equations of motion except $\dot{q}^\beta = \frac{\partial H}{\partial p_\beta}$ for that particular value of β.

Typically, this equation is easy to solve since p_β is constant & can be handled separately

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Example: Hamiltonian and the canonical equations for a particle in a central force field.

$$L = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\,\dot{\phi}^2\right] - V(r)$$

in spherical coordinates (r, θ, ϕ). But we can focus on the 2D problem (r, θ) by ignoring all expressions in green.

The defining equations for conjugate momentum are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin^2\theta\,\dot{\phi}$$

Inverting these equations, we have

$$\dot{r} = \frac{p_r}{m}, \qquad \dot{\theta} = \frac{p_\theta}{mr^2}, \qquad \dot{\phi} = \frac{p_\phi}{mr^2\sin^2\theta}$$

The Hamiltonian is

$$H = p_\alpha \dot{q}^\alpha - L = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\theta} + V(r)$$

$$= \underbrace{\qquad\qquad\qquad}_{T} + V(r)$$

Hamiltonian's canonical equations of motion are

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \qquad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{1}{mr^3}\left[p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right] - V'(r)$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \qquad \dot{p}_\theta = \frac{p_\phi^2 \cos\theta}{mr^2 \sin^3\theta}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2 \sin^2\theta} \qquad \dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$$

In the 2D problem (or by setting $p_\phi = 0$), we find

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \qquad \dot{p}_r = -\frac{\partial H}{\partial r} = \frac{1}{mr^3} p_\theta^2 - V'(r)$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \qquad \dot{p}_\theta = 0$$

The last equation implies $p_\theta = \text{const.}$ and we can then solve the first two equations treating p_θ as a constant.

Also note that the first two equations are directly derived from the Hamiltonian by setting $p_\theta = \text{const.}$ (Show this!)

Another example:

Consider a relativistic particle of mass m

How to write the energy and momentum?

$$E = \frac{mc^2}{\sqrt{1-v^2/c^2}} = \frac{mc^2}{\sqrt{1-\dot{x}^2/c^2}}$$

Recall the gamma factor

$$\gamma(v) = \frac{1}{\sqrt{1-v^2/c^2}}$$

$$p = \frac{mv}{\sqrt{1-v^2/c^2}} = \frac{m\dot{x}}{\sqrt{1-\dot{x}^2/c^2}}$$

Now the energy can be written as

$$E^2 = c^2p^2 + m^2c^4$$

So
$$H = E = \sqrt{p^2c^2 + m^2c^4}$$

What is the Lagrangian?

$$L = P_\alpha \dot{x}^\alpha - H = \frac{m\dot{x}^2}{\sqrt{1-\dot{x}^2/c^2}} - \frac{mc^2}{\sqrt{1-\dot{x}^2/c^2}} = -m\sqrt{1-\dot{x}^2/c^2}$$

So the action reads

$$S = -m\int_{t_0}^{t_1} dt\sqrt{1-\dot{x}^2/c^2}$$

$$\Delta t = \frac{\Delta\tau}{\sqrt{1-v^2/c^2}}$$

Dilation of time

$$= -m\int_{\tau_0}^{\tau_1} d\tau$$

invariant under Lorentz transformation

$$= -m \underbrace{\Delta\tau}_{} \Big|_{t_0}^{t_1}$$

Proper time, or time elapsed on clock that travels along path

Variational Principle for the Hamiltonian

We showed that the Lagrangian follows from a variational principle.

$$\delta \int_{t_0}^{t_1} L\, dt = 0$$

where the coordinates at the endpoints were fixed $\delta q(t_0) = \delta q(t_1) = 0$

Can we formulate a similar variational principle in terms of the Hamiltonian? Yes!

To construct the variational principle, let's try something

$$S = \int_{t_0}^{t_1} L\, dt = \int_{t_0}^{t_1} \underbrace{\left[p_\alpha \dot{q}^\alpha - H(q,p) \right]}_{\text{we have inverted the equation}} dt$$

$$H = p_\alpha \dot{q}^\alpha - L$$

We know that $\delta S = 0 \implies EL_\alpha = 0$ if $\delta q^\alpha(t_0) = \delta q^\alpha(t_1) = 0$

and variations of $\dot{q}$ is not independent of variations of q because

$$\delta \dot{q} = \frac{d}{dt}(\delta q)$$

i.e., the variation $\delta q(t)$ also determines $\delta \dot{q}(t)$.

This means that the variation of $p(\dot{q}, q)$ is also determined from that of q. But what if we treat their variation independently?

Compute
$$\delta S = \int_{t_0}^{t_1} dt \left[\delta p_\alpha \dot{q}^\alpha + p_\alpha \delta \dot{q}^\alpha - \frac{\partial H}{\partial p_\alpha} \delta p_\alpha - \frac{\partial H}{\partial q^\alpha} \delta q^\alpha \right]$$

$$= \int_{t_0}^{t_1} dt \left[\delta p_\alpha \left(\dot{q}^\alpha - \frac{\partial H}{\partial p^\alpha} \right) + \underbrace{p_\alpha \delta \dot{q}^\alpha}_{\substack{\text{integration by parts} \\ \text{to remove dot from } \delta q}} - \frac{\partial H}{\partial q^\alpha} \delta q^\alpha \right]$$

$$= p_\alpha \delta q^\alpha \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} dt \left[\delta p_\alpha \left(\dot{q}^\alpha - \frac{\partial H}{\partial q^\alpha} \right) - \delta q^\alpha \left(\dot{p}_\alpha - \frac{\partial H}{\partial q^\alpha} \right) \right]$$

Now impose $\delta q^\alpha(t_0) = \delta q^\alpha(t_1) = 0$ & Find $\delta S = 0$

$\Longrightarrow$ Hamilton's equations

Interestingly, there is no need to impose any constraints on $p_\alpha(t_0)$ and $p_\alpha(t_1)$

This form of the variational principle is called the Hamilton principle.

A side note on QM

Remember
$$\langle q_1, t_1 \mid q_0, t_0 \rangle = \int_{\substack{\text{All paths from} \\ (q_0, t_0) \text{ to } (q_1, t_1)}} e^{i \int_{t_0}^{t_1} dt \, L}$$

In the Hamiltonian language, the equivalent description becomes
$$\langle q_1, t_1 \mid q_0, t_0 \rangle = \int_{\substack{\text{All trajectories in} \\ \text{"phase space" } (q,p) \\ \text{with fixed } q\text{'s at} \\ \text{and all values of } p\text{'s}}} e^{i \int_{t_0}^{t_1} dt \, [p \dot{q} - H]}$$

wavefunctions only depend on q's and not on both q's and p's

A sketch of the proof:

First: $e^{i\, dt\, p \dot{q}} = e^{i p_t (q_{t+dt} - q_t)} = \langle q_{t+dt} | p_t \rangle \langle p_t | q_t \rangle$

Second: $e^{-i dt H(p,q)} \approx 1 - i dt\, H(p_t, q_t) = 1 - i dt \dfrac{\langle q_t | H | p_t \rangle}{\langle q_t | p_t \rangle}$