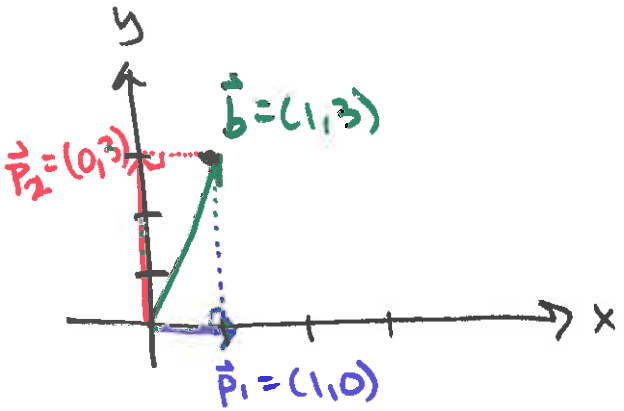
Projections

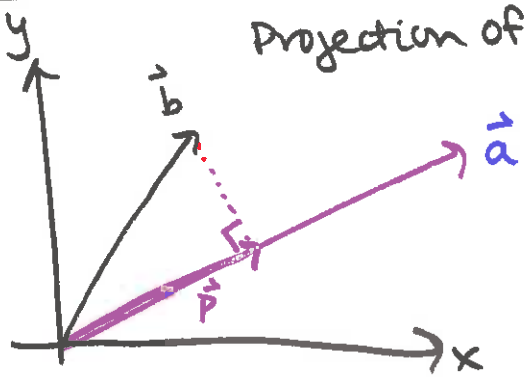
- when a vector $\vec{b}$ is projected onto a line, its projection $\vec{p}$ is the part of $\vec{b}$ along that line

Ex) $\vec{b} = (1,3)$ projection of $\vec{b}$ onto the x-axis is $(1,0)$

projection of $\vec{b}$ onto the y-axis is $(0,3)$



Projection of a vector onto a line



projection of $\vec{b}$ onto $\vec{a}$ (part of $\vec{b}$ in direction of $\vec{a}$)

$$\vec{p} = \frac{\vec{a}^T \vec{b}}{\vec{a}^T \vec{a}} \vec{a}$$

$$\text{proj}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{\underbrace{\|\vec{a}\|^2}_{\text{scalar}}} \vec{a}$$

#8 Project $\vec{b} = (1,1)^T$ onto the lines through $\vec{a}_1 = (1,0)^T$ and $\vec{a}_2 = (1,2)^T$.

Draw the projections $\vec{p}_1$ & $\vec{p}_2$.

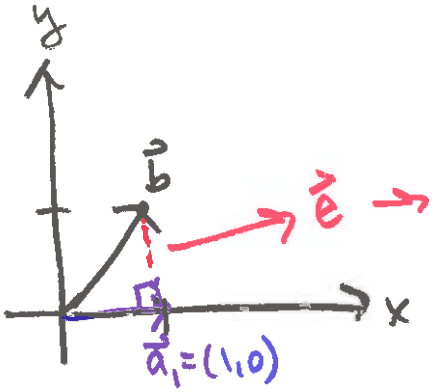
$\vec{p}_1 \rightarrow$ projection of $\vec{b}$ onto $\vec{a}_1$

(finding part of $\vec{b}$ in direction of $\vec{a}_1$)

$$\text{proj}_{\vec{a}_1} \vec{b} = \vec{p}_1 = \frac{\vec{a}_1^T \vec{b}}{\vec{a}_1^T \vec{a}_1} \vec{a}_1 = \frac{1}{1} \vec{a}_1 = (1,0)$$

$$\underset{1\times 2}{\vec{a}_1^T} \underset{2\times 1}{\vec{b}} = \underset{1\times 2}{(1,0)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \underset{1\times 1}{1} = \vec{a}_1 \cdot \vec{b}$$

$$\vec{a}_1^T \vec{a}_1 = \|\vec{a}_1\|^2 = \underset{1\times 2}{(1,0)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{2\times 1} = 1$$



$\vec{e} \rightarrow$ part of $\vec{b}$ perpindicular to $\vec{a}_1$

Find $\vec{p}_2 = \text{proj}_{\vec{a}_2} \vec{b}$ & draw it.

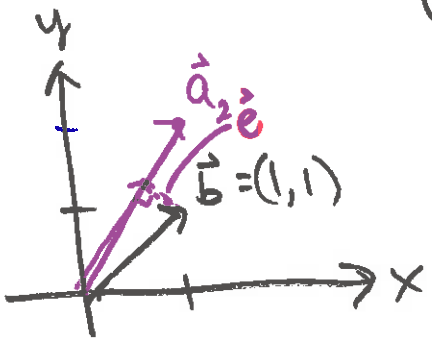
$$\vec{p}_2 = \frac{\vec{a}_2^T \vec{b}}{\vec{a}_2^T \vec{a}_2} \vec{a}_2$$

$$= \frac{3}{5} \vec{a}_2$$

$$= \frac{3}{5}(1,2)$$

$$\vec{a}_2^T \vec{b} = (1\ 2)\begin{pmatrix}1\\1\end{pmatrix} = 3$$

$$\vec{a}_2^T \vec{a}_2 = (1\ 2)\begin{pmatrix}1\\2\end{pmatrix} = 5$$



Ex] Project $\vec{b} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ onto $\vec{a} = \begin{bmatrix}1\\2\\2\end{bmatrix}$

and find $\vec{p} = \hat{x}\,\vec{a}$

$$\hat{x} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|^2} = \frac{\vec{a}^T \vec{b}}{\vec{a}^T \vec{a}}$$

$$\hat{x} = \frac{5}{9} \qquad \vec{p} = \frac{5}{9}(1,2,2)$$

$$\vec{p} = \frac{5}{9}\begin{bmatrix}1\\2\\2\end{bmatrix}$$

error: $\vec{e} = \vec{b} - \vec{p}$

$$\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{5}{9}\begin{bmatrix}1\\2\\2\end{bmatrix} = \begin{bmatrix}4/9\\-1/9\\-1/9\end{bmatrix}$$

note: $\vec{e} \perp \vec{a}$ $\quad \vec{e} \cdot \vec{a} = 0$

Part 2 Find projection matrix

(matrix that projects any vector $\vec{b}$ onto a line through $\vec{a}$)

$\vec{p} = P\vec{b}$

↑ find this matrix

$$P = \frac{\vec{a}\,\vec{a}^T}{\vec{a}^T\vec{a}}$$

$\vec{a}\,\vec{a}^T$ ← matrix

$\vec{a}^T\vec{a}$ ↑ constant

$\vec{a} = (1\ 2\ 2)^T$

$$\vec{a}\,\vec{a}^T = \underset{3\times1}{\begin{pmatrix}1\\2\\2\end{pmatrix}}\underset{1\times3}{\begin{pmatrix}1 & 2 & 2\end{pmatrix}} = \underset{2\times2}{\begin{pmatrix}1 & 2 & 2\\2 & 4 & 4\\2 & 4 & 4\end{pmatrix}}$$

$$\vec{a}^T\vec{a} = \begin{pmatrix}1 & 2 & 2\end{pmatrix}\begin{pmatrix}1\\2\\2\end{pmatrix} = 1+4+4=9$$

$$P = \frac{1}{9}\begin{bmatrix}1 & 2 & 2\\ 2 & 4 & 4\\ 2 & 4 & 4\end{bmatrix}$$

$\frac{1}{9}$: $\vec{a}^T\vec{a}$

$\begin{bmatrix}1 & 2 & 2\\ 2 & 4 & 4\\ 2 & 4 & 4\end{bmatrix}$: $\vec{a}\vec{a}^T$

Take any vector $\vec{b}$ & find its projection onto $\vec{a}$

$P\vec{b}$

$$\underset{3\times 3}{\begin{bmatrix} & P & \end{bmatrix}}\underset{3\times 1}{\begin{bmatrix} & b & \end{bmatrix}} = \underset{3\times 1}{\begin{bmatrix} & \text{proj}_{\vec{a}}\vec{b} & \end{bmatrix}}$$

Property: $P^2 = P$ projection matrices

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