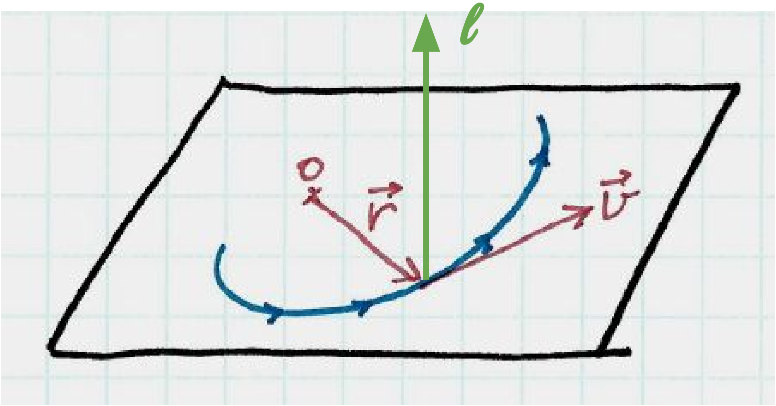
Section 3.4.

Angular Momentum for a Single Particle

The definition

Consider a particle with position vector $\boldsymbol{r}$ (w.r.t. the chosen origin O) and momentum $\boldsymbol{p}$ = m $\boldsymbol{v}$.



Define the angular momentum, of the particle about the origin O; notation = $\boldsymbol{\ell}$;

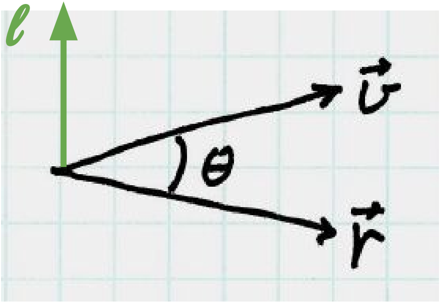
$$\boldsymbol{\ell} = \boldsymbol{r} \times \boldsymbol{p}$$

$\boldsymbol{\ell}$ is a vector.

(Review cross product; page 7 and right hand rule)

★ The direction is perpendicular to the plane spanned by $\boldsymbol{r}$ and $\boldsymbol{p}$.

★ The magnitude is $r\,p\,\sin\theta$.



Theorem.

$d\boldsymbol{\ell}/dt$ is equal to the torque.

Proof.

$$\frac{d\vec{\ell}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) \quad \text{where } \vec{p} = m\vec{v}$$

$$= m\left[\underbrace{\frac{d\vec{r}}{dt} \times \vec{v}}_{\vec{v} \times \vec{v} = 0} + \vec{r} \times \underbrace{\frac{d\vec{v}}{dt}}_{\vec{a} = \vec{F}/m}\right]$$

$$= \vec{r} \times \vec{F}$$

$$= \text{definition of torque}$$

Figure 3.5

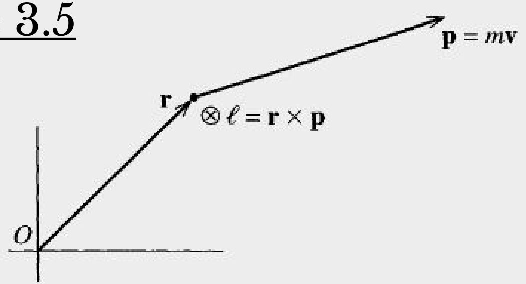


Figure 3.5 For any particle with position **r** relative to the origin O and momentum **p**, the angular momentum about O is defined as the vector $\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p}$. For the case shown, $\boldsymbol{\ell}$ points into the page.

Figure 3.6

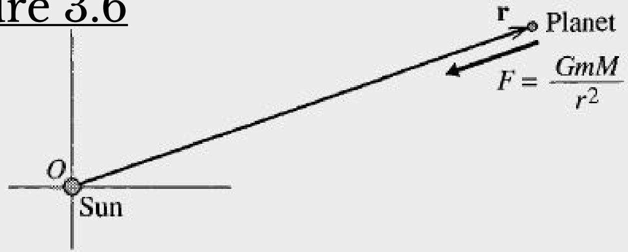


Figure 3.6 A planet (mass m) is subject to the central force of the sun (mass M). If we choose the origin at the sun, then $\mathbf{r} \times \mathbf{F} = 0$, and the planet's angular momentum about O is constant.

<u>Example</u>

Kepler's second law

<u>Figure 3.7</u>

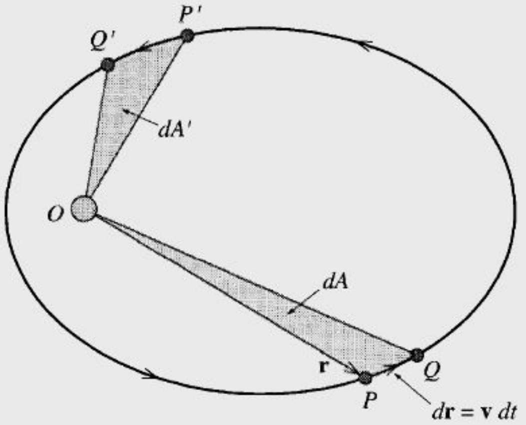


Figure 3.7 The orbit of a planet with the sun fixed at O. Kepler's second law asserts that if the two pairs of points P, Q and P', Q' are separated by equal time intervals, $dt = dt'$, then the two areas dA and dA' are equal.

Kepler published three laws of planetary orbits, in 1609 and 1619. He determined these laws from a mathematical analysis of planetary observations —– very difficult in the 17th century but Kepler was a mathematical genius.

Kepler's second law:

> *The radial vector sweeps out equal areas in equal times.*

We'll prove that this follows from conservation of angular momentum.

(not known at the time of Kepler; Newton)

Comment:

"Kepler's second law", or, conservation of angular momentum, applies to ***all*** central forces.

I.e., it's not just for planetary orbits.

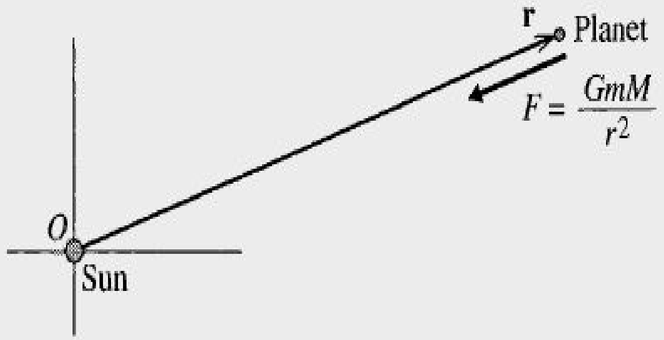
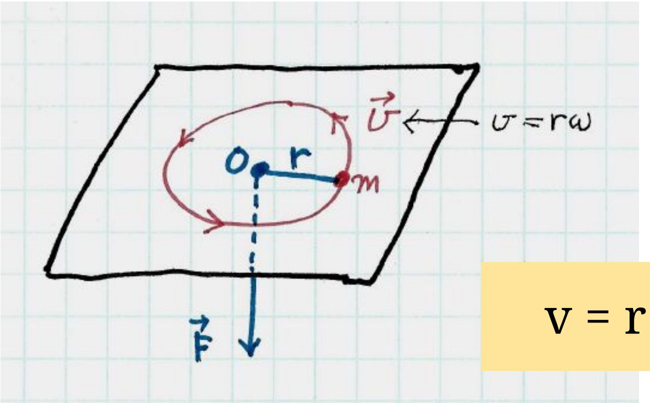


Figure 3.6 A planet (mass m) is subject to the central force of the sun (mass M). If we choose the origin at the sun, then $\mathbf{r} \times \mathbf{F} = 0$, and the planet's angular momentum about O is constant.

<u>TAYLOR PROBLEM 3.25.</u>



$v = r\,\omega$

The mass *m* slides without friction on a horizontal surface. It is attached to a string as shown. The string goes through a hole in the surface, *O*; and it can be pulled down beneath the surface to change the distance *r* from *m* to *O*.

A. Initially, $r = r_0$ and $\omega = \omega_0$. Calculate F required to keep r constant.

(A) $F_r = m a_r \qquad a_r = -r\dot{\phi}^2 = -r\omega^2 = -v^2/r$

$F = m r \omega^2 = m \frac{v^2}{r}$

B. Then the string is pulled down by distance $r_0/2$. Determine the final angular velocity.

(B) $\ell = m r_0^2 \omega_0^2 = m r^2 \omega^2$

$r = r_0/2 \Rightarrow \omega = 4\omega_0$

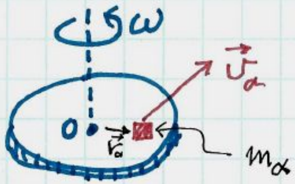
C. Calculate the work done pulling the string.

(C) $W = \Delta\, K.E.$

$= \frac{1}{2} m (r\omega)^2 - \frac{1}{2} m (r_0 \omega_0)^2$

$= \frac{3}{2} m r_0^2 \omega_0^2$

Angular momentum of a rotating disk (ω) about the symmetry axis of the disk



$m_\alpha \to dm$ in continuum limit.

$$\vec{L} = \sum_\alpha \vec{\ell}_\alpha = \sum_\alpha m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha$$

$$= \sum_\alpha m_\alpha \, r_\alpha \hat{e}_r \times r_\alpha \omega \hat{e}_\phi$$

$$= \sum_\alpha m_\alpha r_\alpha^2 \, \omega \, \hat{e}_z = I\omega \, \hat{e}_z$$

Moment of Inertia

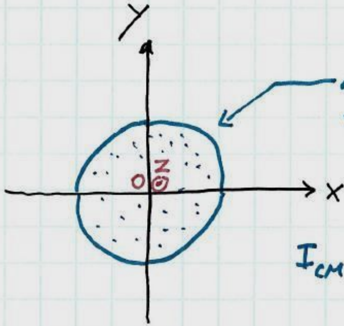
$$I = \sum_\alpha m_\alpha r_\alpha^2 \longrightarrow \int r_\perp^2 \, dm$$

For a uniform disk w/ $\rho = \frac{M}{\pi R^2} \leftarrow \frac{dm}{dA}$

$$I = \iint r^2 \frac{M}{\pi R^2} \, r \, dr \, d\phi$$

$$= \frac{M}{\pi R^2} \cdot \frac{R^4}{4} \cdot 2\pi = \frac{1}{2} M R^2$$

THE PARALLEL AXIS THEOREM – an example

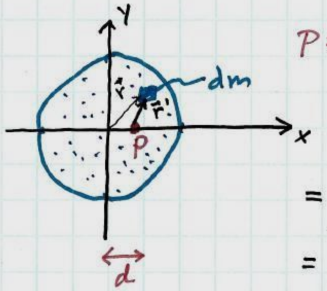


disk; symmetry about z axis

CoM = 0

$$I_{CM} = \int r_\perp^2 \, dm = \int (x^2+y^2) \, dm$$

$$= \frac{1}{2} M R^2$$



P = parallel axis at distance d

$$I_P = \int (x'^2 + y'^2) \, dm$$

$$= \int \left[(d-x)^2 + y^2 \right] dm$$

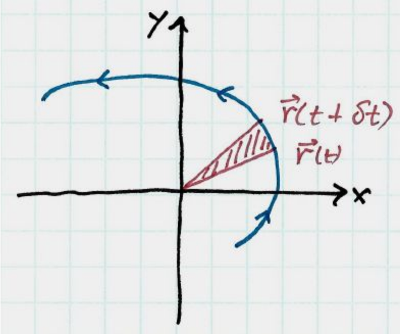
$$= \int \left[d^2 - 2xd + x^2 + y^2 \right] dm$$

$$= M d^2 + I_{CM}$$ because $\langle x \rangle = 0$, $\int x \, dm = 0$

$$I_P = I_{CM} + M d^2$$

KEPLER'S SECOND LAW …

First, the orbit lies in a plane because the vector $\boldsymbol{\ell}$ is a constant of the motion.



The area swept out by the radial vector from time t to t+δt is

$$\delta A = \frac{1}{2}\, r\, r\, \delta\phi$$

$$\frac{\delta A}{\delta t} = \frac{1}{2} r^2 \frac{\delta \phi}{\delta t}$$

The angular momentum is

$$\vec{\ell} = \vec{r} \times m\vec{v}$$

$$\vec{r} = r(t)\, \hat{e}_r(t)$$

$$\vec{v} = \dot{r}\, \hat{e}_r + r\, \dot{\hat{e}}_r$$

$$= \dot{r}\, \hat{e}_r + r\dot{\phi}\, \hat{e}_\phi$$

$$\vec{\ell} = m\, r\hat{e}_r \times (\dot{r}\hat{e}_r + r\dot{\phi}\,\hat{e}_\phi)$$

$$= m r^2 \dot{\phi}\, \hat{e}_z$$

$$\therefore \quad \frac{dA}{dt} = \frac{\ell}{2m} \quad \text{is constant}$$

Thus the area rate is constant because angular momentum is constant.

Homework Assignment #6

due in class Friday, October 14

[27] Problem 3.16 *

[28] Problem 3.20 **

[29] Problem 3.22 **

[30] Problem 3.27 **

[30x] Problem 3.32 **

[30xx] Problem 3.35 **

Use the cover sheet.

The first midterm exam is this Friday, October 7.