Section 8.8.

Changes of Orbit

Read Section 8.8.

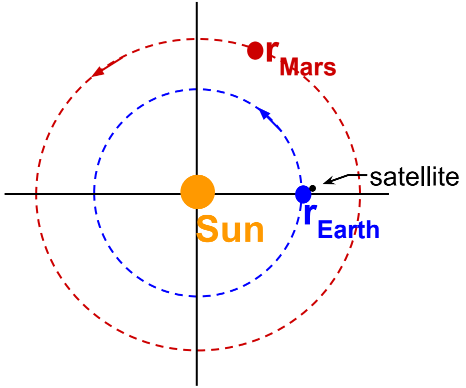
Here's a problem for NASA:
how to send a satellite from Earth to another planet, say Mars.

The satellite is too small to carry much fuel, so most of the trip is done in "free fall", i.e., no rocket power.

The satellite moves along a Keplerian orbit, starting at Earth and ending at the other planet.

The initial conditions for the satellite are $r(0) = r_{Earth}$ and $v(0) = v_{Earth}$.

As a first approximation we can approximate the orbits of *Earth and Mars* as circular.



Now send the satellite to Mars, and ...

- hit the moving target
- use the minimum amount of rocket fuel

Hohmann transfer orbit

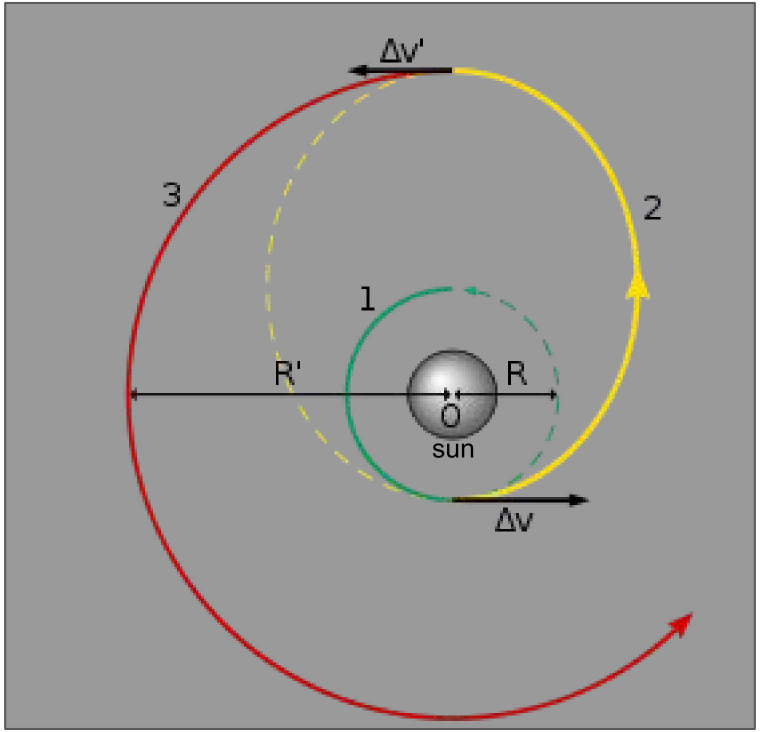
From Wikipedia, the free encyclopedia

In orbital mechanics, the **Hohmann transfer orbit** is an elliptical orbit used to transfer between two circular orbits of different radii in the same plane.

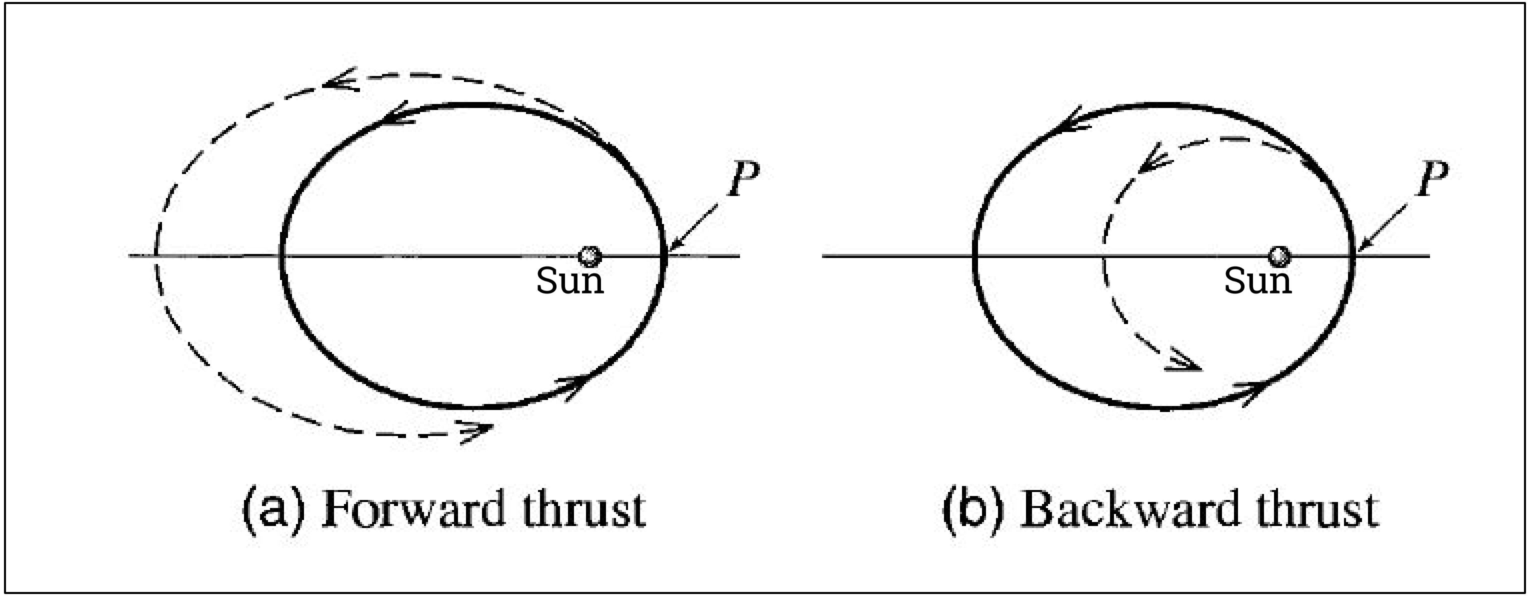
The orbital maneuver to perform the Hohmann transfer uses two engine impulses, one to move a spacecraft onto the transfer orbit and a second to move off it.

This maneuver was named after Walter Hohmann, the German scientist who published a description of it in his 1925 book *Die Erreichbarkeit der Himmelskörper* ("The Accessibility of Celestial Bodies")

Hohmann was influenced in part by the German science fiction author Kurd Lasswitz and his 1897 book *Two Planets*.



Hohmann transfer orbit, labelled 2, from a low orbit (1) to a higher orbit (3).

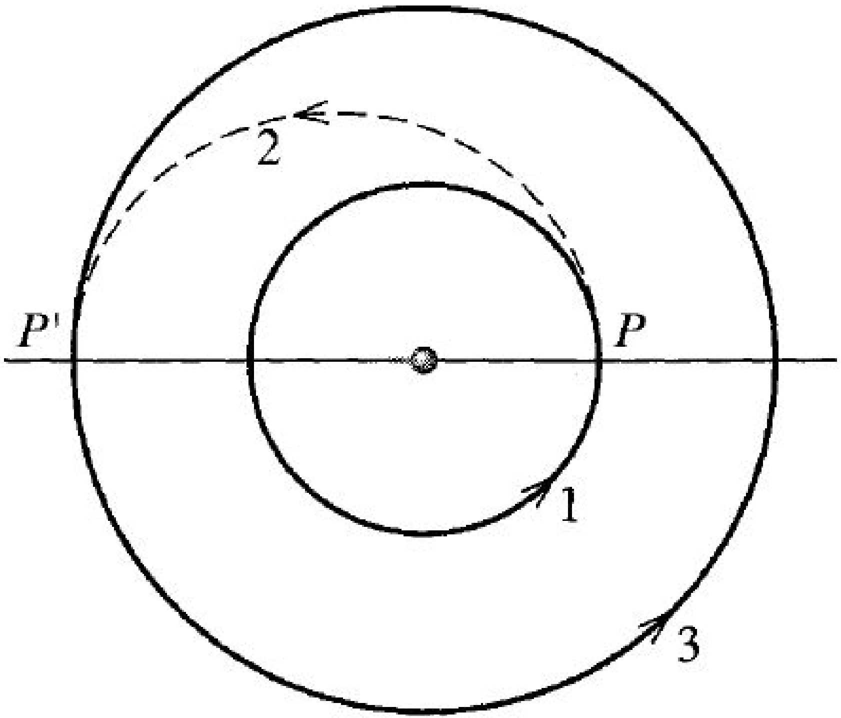


Forward thrust puts the satellite into a higher orbit, i.e., with larger orbital energy.

Backward thrust puts the satellite into a lower orbit, i.e., with smaller orbital energy.

Sending a satellite to Mars.

- The transfer orbit is ½ of an ellipse.
- Perihelion (P) is at $r = r_1 = r_{Earth}$.
- Aphelion (P') is at $r = r_3 = r_{Mars}$.
- Forward thrust at P puts the satellite into a higher orbit (2).
- Forward thrust at P' puts the satellite into an even higher orbit (3).
- At P the speed of the satellite is the same (approximately) as the speed of the Earth and $r = r_1 = r_{Earth}$.
- At P' the speed of the satellite must be raised to the speed of Mars at $r = r_3 = r_{Mars}$.



Do you remember the virial theorem for circular orbits?
$T = -U/2$ and $U = -\gamma/r$ and $E = T + U = U/2 = -\gamma/(2r)$

Summary of Kepler Orbits

Our results for the Kepler orbits can be summarized as follows: All of the possible orbits are given by Equation (8.59),

$$r(\phi) = \frac{c}{1 + \epsilon \cos \phi}, \tag{8.62}$$

and are characterized by the two constants of integration[5] ϵ and c. The dimensionless constant ϵ is related to the comet's energy by (8.58),

$$E = \frac{\gamma^2 \mu}{2\ell^2}(\epsilon^2 - 1). \tag{8.63}$$

It is, as we have seen, the eccentricity of the orbit that determines the orbit's shape as follows:

eccentricity	energy	orbit
$\epsilon = 0$	$E < 0$	circle
$0 < \epsilon < 1$	$E < 0$	ellipse
$\epsilon = 1$	$E = 0$	parabola
$\epsilon > 1$	$E > 0$	hyperbola

You can see from (8.62) that the constant c is a scale factor that determines the size of the orbit. It has the dimensions of length and is the distance from sun to comet when $\phi = \pi/2$. It is equal to $\ell^2/\gamma\mu$ or, since γ is the force constant Gm_1m_2,

$$c = \frac{\ell^2}{Gm_1m_2\mu}, \tag{8.64}$$

where m_1 is the mass of the comet, m_2 that of the sun, and μ is the reduced mass $\mu = m_1m_2/(m_1 + m_2)$, which is exceedingly close to m_1 since m_2 is so large.

If you need another equation,
you must derive it from these.

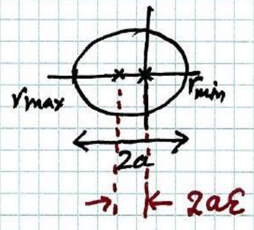
$$r_{min} = \frac{c}{1+\varepsilon} \qquad c = r_{min}(1+\varepsilon)$$

$$\ell^2 = G m_1 m_2 \mu c = \gamma \mu r_{min}(1+\varepsilon)$$

$$E = \frac{\gamma^2\mu}{2\ell^2}(\varepsilon^2 - 1) = \frac{\gamma^2\mu(\varepsilon^2-1)}{2\gamma\mu r_{min}(1+\varepsilon)}$$

$$E = \frac{\gamma}{2r_{min}}(\varepsilon - 1)$$

For $\varepsilon < 1$



$$2a = r_{max} + r_{min}$$

$$r_{max} = r_{min} + 2a\varepsilon$$

$$a = r_{min} + a\varepsilon$$

$$r_{min} = a(1-\varepsilon)$$

$$\Rightarrow E = \frac{-\gamma}{2a}$$

$$\frac{-\gamma}{r_{min} + r_{max}}$$

Tangential thrust at Perihelion

At perihelion the rocket fires for a short time; we can call it a *sharp impulse* at $r = r_1$.

Before the impulse:

$\gamma = GM_s m$

$$\frac{mv^2}{r} = \frac{GM_s m}{r^2} \quad \mapsto \quad E = E_1 = \frac{-\gamma}{2r_1}$$

After the impulse: we need

$$E = \gamma^2 m (2\ell^2)^{-1} (\varepsilon^2 - 1)$$

where

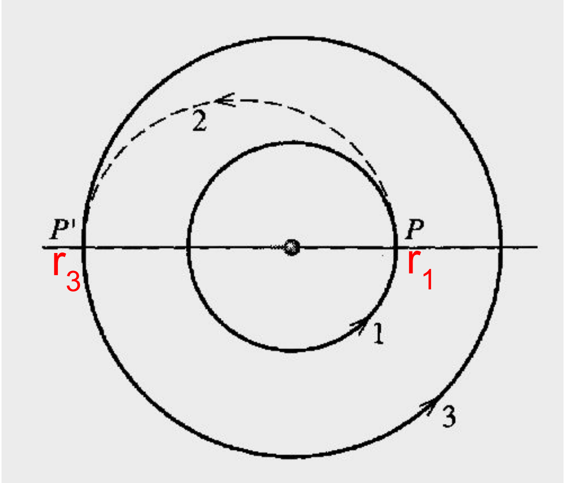
$$\ell^2 = \gamma m a (1 - \varepsilon^2)$$

(Kepler ellipse)

I.e., $E = E_2 = -\gamma/(2a) = -\gamma/(r_1+r_3)$

Note: $\boxed{2a = r_1+r_3}$

= E_2 = energy on the transfer orbit



∴ The impulse at P must provide energy equal to

$$\Delta E(P) = \frac{-\gamma}{r_1+r_3} - \frac{-\gamma}{2r_1} = \frac{\gamma(r_3-r_1)}{2r_1(r_1+r_3)} \quad \text{(positive)}$$

Tangential thrust at Aphelion

Similarly, the impulse at P′ must provide energy

$$\Delta E(P') = \frac{-\gamma}{2r_3} - \frac{-\gamma}{r_1+r_3} = \frac{\gamma(r_3-r_1)}{2r_3(r_1+r_3)} \quad \text{(positive)}$$

Taylor, Example 8.6.

Taylor does the calculation in terms of
/1/ the change of speed at P, and then
/2/ the change of speed at P'.

Because the time of the impulse is small, we can say that the kinetic energy increases but the potential energy (-γ/r) is constant during the impulse.

At P, before the impulse

$$r = r_1 \quad \text{and} \quad \tfrac{1}{2} m v_1^2 = \gamma/(2r_1) ;$$

after the impulse

$$r = r_1 \quad \text{and} \quad \tfrac{1}{2} m (v_1+\Delta v_1)^2 - \gamma/r_1 = E_2$$

$$\therefore \tfrac{1}{2} m (v_1+\Delta v_1)^2 = \gamma/r_1 - \gamma/(r_1+r_3) \;\; = \gamma\, r_3/r_1/(r_1+r_3)$$

Thus

$$\frac{(v_1+\Delta v_1)^2}{v_1^2} = \frac{2r_3}{r_1+r_3} \qquad \textbf{(8.72)}$$

Similarly at P', $r = r_3$; we find

$$\frac{v_3^2}{(v_3 - \Delta v_3)^2} = \frac{r_1 + r_3}{2r_1} \qquad \textbf{(8.73)}$$

Numerical example:

Sending a satellite from Earth to Mars

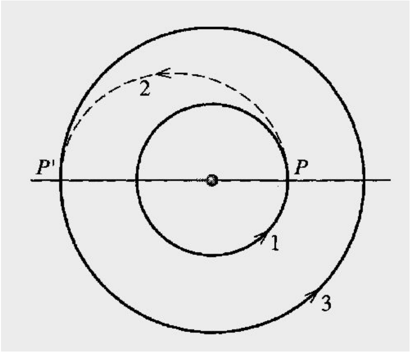
$r_1 = 149.5 \times 10^6$ km ; $r_3 = 227.9 \times 10^6$ km
(approximated as circular)

$v_1 = 29.8$ km/s ; $v_3 = 24.1$ km/s

$\Delta v_1 = 2.94$ km/s ; $\Delta v_3 = 2.65$ km/s

<u>Trick question:</u>
We increased the speed at P (+2.94 km/s) and again at P' (+2.65 km/s); but the final speed is less than the initial speed, $v_3 - v_1 = -5.65$ *km/s.*

Calculate the travel time to go from Earth to Mars on the Hohmann transfer orbit.



$r_1 = 149.5 \times 10^6$ km

$r_2 = 227.9 \times 10^6$ km

Recall Kepler's third law of planetary motion.

$$\tau^2 = \frac{4\pi^2 a^3}{GM} \quad \text{or} \quad \tau^2 \propto a^3$$

For the transfer orbit (2)

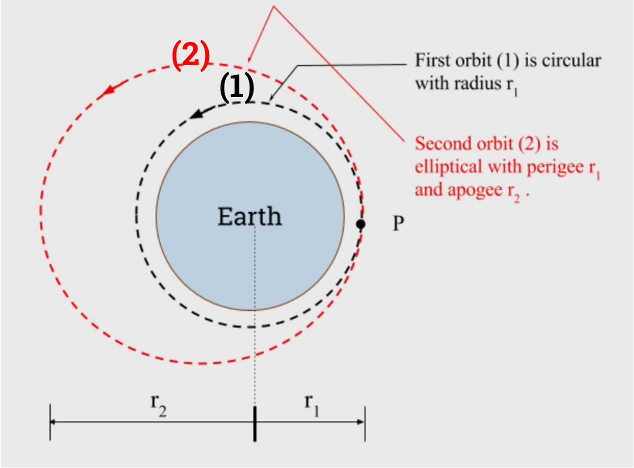
$$2a = r_1 + r_3 \quad \mapsto \quad a = 188.7 \times 10^6 \text{ km}$$

$\therefore \tau = 365.25$ days * $(188.7 / 149.5)^{3/2} = 517.7$ days

The travel time is $\frac{1}{2}\tau$ = 259 days **= 8½ months .**

Another example of change of orbit (Taylor p 317)

NASA wants to put a satellite in an elliptical orbit (2) *around the Earth.* First they arrange the satellite in a circular orbit (1). Then they apply a short impulsive tangential thrust at the point P to put the satellite into the elliptical orbit (2). Given r_1 and r_2 calculate the *increase of the velocity* that must be supplied by the impulsive thrust.



Study Section 8.8 for the final exam.