Chapter 2.
Projectiles and Charged Particles
Section 2.1. Air resistance
Section 2.2. Linear air resistance

Read Section 2.1.

Aerodynamic forces

When an object moves through air, it experiences a force.

The force is exerted by the air on the object; the reaction force is exerted by the object on the air.

The force on the object can be resolved into two components:

"drag" = component in the direction of $-\boldsymbol{v}$

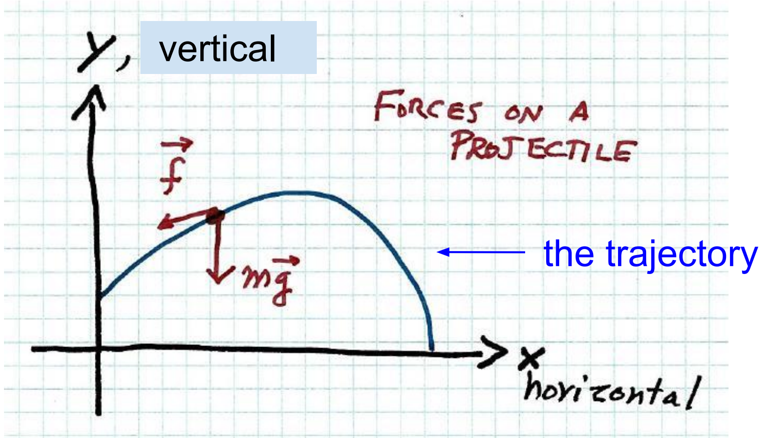
"lift" = component in the direction of $-\boldsymbol{g}$

Air resistance

Forget about lift.

Here we are concerned with the drag force; we'll denote it by $\boldsymbol{f}$.

Figure 2.1



The force of air resistance *f(v)*

The direction of ***f*** is parallel to $-\boldsymbol{v}$.

The magnitude of ***f*** depends on v (speed) and other properties of the object.

We'll write $\boldsymbol{f} = f(v)\ (-\boldsymbol{e}_v)$ *with magnitude*

$$f(v) = b\,v + c\,v^2 = f_{lin} + f_{quad}$$

- $f_{lin} = b\,v$ comes from viscosity; for a sphere,

  $b = \beta D$ (D = diameter)

  $\beta = 3\pi\,\eta$ (η = viscosity)

- $f_{quad} = c\,v^2$ comes from the inertia of the air; for a sphere,

  $c = 0.25\,\rho\,A = \gamma\,D^2$

  $\gamma \propto \rho$ (ρ = density)

Example 2.1 BASEBALLS AND LIQUID DROPS

↳↳ comparing the relative importance of f_{quad} and f_{lin}, consider 3 cases

For a sphere moving through air at STP,

$$f_{lin} = \beta D v \quad \& \quad \beta = 1.6 \times 10^{-4}\ Ns/m^2$$

$$f_{quad} = \gamma D^2 v^2 \quad \& \quad \gamma = 0.25\ Ns^2/m^4$$

in MKS units.

	D	v [m/s]	f_{quad} / f_{lin}	dominant
1. baseball	**7 cm**	**5**	**600**	cv^2
2. small raindrop	**1 mm**	**0.6**	**1**	**comparable**
3. tiny oil drop (Millikan expt)	**1.5 μm**	5×10^{-5}	10^{-7}	**bv**

2.2 Linear air resistance

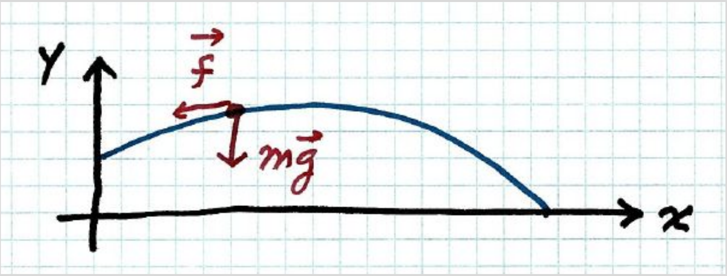
Now we'll specialize to $c = 0$.

Assume the force of air resistance on a projectile is

$$\vec{f} = -b\vec{v}$$

Then the equation of motion for the projectile moving through air is

$$m\dot{\vec{v}} = m\vec{g} - b\vec{v}$$



Cartesian components

x = horizontal coordinate;
y = vertical coordinate (*positive upward*)

$$m\dot{v}_x = -b\,v_x$$
$$m\dot{v}_y = -mg - b v_y$$

This is very nice, because the x and y coordinates *separate*;
so we can solve their equations separately.

⇨ Recall from PHY 183, we do the same thing if we *neglect* air resistance:

$x'' = 0$ so $x(t) = x_0 + v_{0x}\, t$

$y'' = -g$ so $y(t) = y_0 + v_{0y}\, t - \frac{1}{2} g\, t^2$

But today we are introducing frictional force components; so $x(t) \neq x_0+v_{0x}t$ and $y(t) \neq y_0+v_{0y}t-\frac{1}{2} g\, t^2$.

Horizontal motion with linear drag

Figure 2.3

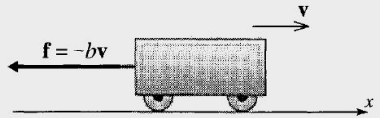


Figure 2.3 A cart moves on a horizontal frictionless track in a medium that produces a linear drag force.

$$m \dot{v}_x = -b v_x$$

The solution is obvious,

$$v_x(t) = C e^{-bt/m}$$

where C is a constant.

Determine C from the initial conditions

$$v_x(t) = v_{0x} e^{-bt/m}$$

or some other information.

Figure 2.4

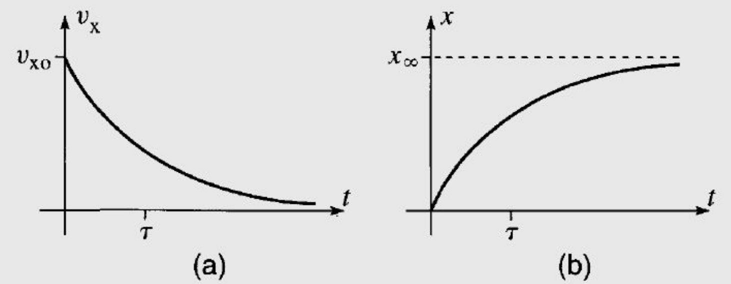


Figure 2.4 **(a)** The velocity v_x as a function of time, t, for a cart moving horizontally with a linear resistive force. As $t \to \infty$, v_x approaches zero exponentially. **(b)** The position x as a function of t for the same cart. As $t \to \infty$, $x \to x_\infty = v_{x0}\tau$.

$$dx = v_x \, dt$$

$$\int_{x_0}^{x} dx' = x - x_0 = \int_0^t v_x(t')\,dt'$$

$$= v_{0x}\left(-\frac{m}{b}\right) e^{-bt'/m} \Big]_0^t$$

$$= \frac{m v_{0x}}{b}\left[1 - e^{-bt/m}\right]$$

Vertical motion with linear drag

We want to solve this equation (*):

$$m\dot{v}_y = -mg - bv_y$$

The solution may be obtained in several ways ...

- *trial and error; see page ??*
- *separation of variables* ; see problems 2.x
- *particular + homogeneous; MTH 234*
  the third method only works for linear equations.

(*) I'm letting the y axis point upward; so $F_g = -mg$. See Taylor for y axis pointing downward.

Solution of differential equations by separation of variables

Suppose we have an equation of this form,

$$\frac{df}{dx} = K(f(x)) \qquad (1)$$

Separate the variables x and f,

$$\frac{df}{K(f)} = dx \qquad (2)$$

Now integrate both sides of the equation,

$$\int_{f_0}^{f} \frac{df'}{K(f')} = \int_{x_0}^{x} dx' = x - x_0 \qquad (3)$$

Eq.(3) gives x as a function of f. But what we want is f as a function of x. So finally use algebra to solve (3) for f. ⇒ f(x) = the solution (4)

Vertical motion with linear drag

We want to solve this equation:

$$m\frac{dv}{dt} = -mg - bv$$

Separation of variables:

$$m\,dv = (-mg - bv)\,dt$$

$$\frac{dv}{mg+bv} = -\frac{dt}{m}$$

Integrate:

$$\int_{v_0}^{v} \frac{dv'}{mg+bv'} = \frac{1}{b}\ln(mg+bv')\Big|_{v_0}^{v}$$

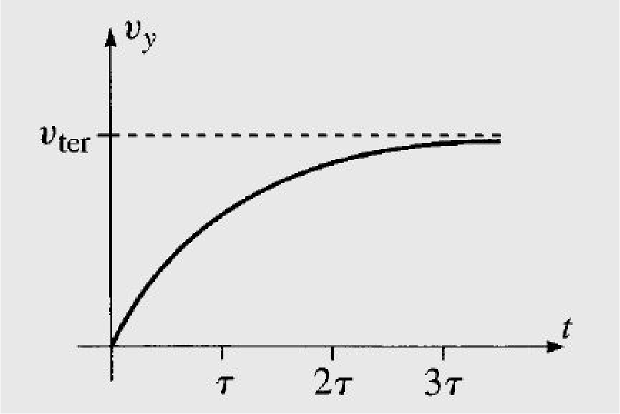
$$= \frac{1}{b}\ln\frac{mg+bv}{mg+bv_0} = \frac{-1}{m}\int_0^t dt = \frac{-t}{m}$$

Solve:

$$\frac{mg+bv}{mg+bv_0} = e^{-bt/m}$$

$$v = \left(v_0 + \frac{mg}{b}\right)e^{-bt/m} - \frac{mg}{b}$$

Figure 2.6



"Terminal velocity" and "time constant"

$$v_{ter} = -\frac{mg}{b}$$

$$\tau = m/b$$

$$e^{-bt/m} = e^{-t/\tau}$$

Determine y(t) by integration.

Example 2.2

TERMINAL SPEEDS OF

SMALL LIQUID DROPS

Related : this homework problem [16] ...

The terminal velocity of a drop of water (diameter = D) is the velocity at which

$$F = mg - bv - cv^2 = 0.$$

The parameter values for air at STP are

$b = (1.6 \times 10^{-4})D$ and $c = (0.25)D^2$, in MKS units;

also, $m = (0.52 \times 10^6)\ D^3$ in MKS units.

Determine v_{ter} as a function of D. Plot an accurate graph of v_{ter} versus D, from D = 0.1 mm to 3 mm. (Use a computer to make the plot.) [The result shows why water droplets in a cloud do not fall as rain.]

Test yourself:

For a sphere moving in a fluid w/ density ρ,

$$f_{quad} = 0.25\ \rho\ A\ v^2 . \qquad \text{(A = Area)}$$

Show that ***in air,***

$$f_{quad} = [\ 0.25\ Ns^2m^{-4}]\ D^2\ v^2 \qquad \text{(D = Diameter)}$$

Homework Assignment #3
due in class Friday, September 23

[11] Problem 2.2
[12] Problem 2.3
[13] Problem 2.10
[14] Problem 2.18
[15] Problem 2.26
[16] Assigned problem

Use the cover sheet.