3.5/3.6 Dimensions of the 4 subspaces

4 Fundamental Subspaces of $A_{m \times n}$

(1) The row space is $C(A^T)$ is a subspace of $\mathbb{R}^n$

(2) The column space is $C(A)$ is a subspace of $\mathbb{R}^m$

(3) The nullspace is $N(A)$ is a subspace of $\mathbb{R}^n$

(4) The left nullspace $N(A^T)$ is a subspace of $\mathbb{R}^m$

Find bases and dimensions for the 4 subspaces associated with A and B.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}$$

A:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} R_2 \to R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

pivot: x_1 column 1 + row 1

free: x_2, x_3

Row space: basis for row space row1 of A

$(1, 2, 4)$ (dimension is 1) in $\mathbb{R}^3$

Ex) Is $(1,3,4)$ in Row space? NO

not a scalar multiple of $(1,2,4)$

column space: has the same dimension as row space, x_1 is our pivot, take column from A

basis of column space is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\mathbb{R}^2$

Null Space: free variables are $x_2 + x_3$ here

dimension of 2

to get the basis... find special solutions

$x_2 = 1, x_3 = 0$

$x_1 + 2x_2 + 4x_3 = 0$

$x_1 + 2 = 0$ $x_1 = -2$

$x_2 = 0, x_3 = 1$

$x_1 + 2x_2 + 4x_3 = 0$

$x_1 = -4$

basis for the null space is

$$\vec{s}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \quad \vec{s}_2 = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$$

$$\alpha \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$$ $\rightarrow$ any vector that is a linear combination of $\vec{s}_1 + \vec{s}_2$ is in the $N(A)$

Left Null Space $N(A^T)$

$$A^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 8 \end{bmatrix} \xrightarrow{\text{Row Operation}} \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

3×2

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 4R_1$

pivot "x_1"

free "x_2"

$x_2 = 1$

$1x_1 + 2x_2 = 0$

$x_1 = -2$

basis vector is $(-2, 1)$

dimension of left Null space is 1, rows of A are dependent

B

$$B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \end{bmatrix}$$ 2 pivots: x_1 + x_2

1 free variable: x_3

row space: dimension 2 because 2 pivots

basis: $(1, 2, 4)$

$(2, 5, 8)$

column space: dimension 2, columns 1 + 2 of B form the basis

$(1,2)$ and $(2,5)$

Note: $\alpha(1,2) + \beta(2,5) = 0$ only when $\alpha + \beta$ are 0

<u>Nullspace:</u> dimension 1: x_3 is free

$x_3 = 1$

$x_1 + 2x_2 + 4x_3 = 0$

$x_2 = 0$

$x_1 + 4 = 0$

$x_1 = -4$

basis is $(-4, 0, 1)$

Left Null Space

$B^T = \begin{bmatrix} 1 & 2 \\ 2 & 5 \\ 4 & 8 \end{bmatrix} \rightarrow$ Row ops $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 4R_1$

no free variables

$N(B^T)$ is $\vec{0}$, no special solution

Rows of B are linearly independent

Hmwk check 6

3.2 #18 5th Ed

column space $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\\1\end{pmatrix}$

and null space $\begin{pmatrix}1\\0\\1\end{pmatrix}$ & $\begin{pmatrix}0\\0\\1\end{pmatrix}$

Construct a matrix . . . impossible

dimension of column space: 2 (2 pivot variables)

dimension of null space is: 2 (2 free variables)

$A\vec{x}=\vec{b}$

need 3x3 matrix

not possible to have 2 pivots + 2 free variables in 3x3

3.2

#21 [scribbled out] Why does no $\boxed{3\times3}$ $n=3$ matrix have a nullspace equal to its column space?

With r pivots this would mean

$$\underbrace{r}_{\text{rank}} = \underbrace{n-r}_{\text{dim. of null space}}$$

$$r = 3 - r$$

$$2r = 3$$

$r = \frac{3}{2}$?? Not possible

#15 Construct a matrix for which $N(A)$ = all ~~multiples~~ combinations of $(2,2,1,0)$ and $(3,1,0,1)$

Know: $\vec{s}_1 = (2,2,1,0)$ & $s_2 = (3,1,0,1)$

2 free variables, x_3 & x_4

x_1 & x_2 are pivots

#15 cont:

$\vec{s}_1 = (2,2,1,0)$ $\vec{s}_2 = (3,1,0,1)$

The Easiest matrix to start w/ is the one with identity in x_1 & x_2 columns

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$
$$\left[\begin{array}{cc|cc} 1 & 0 & \boxed{-2} & -3 \\ 0 & 1 & \boxed{-2} & -1 \end{array}\right]$$

↑ fill this in

Equations

$x_1 + 0x_2 + \square x_3 + \triangle x_4 = 0$

know from $\vec{s}_1$ that $x_1 = 2$ & $x_2 = 2$ & $x_3 = 1, x_4 = 0$

$2 + 0 + \square \cdot 1 + 0 = 0$ so coefficient for x_3 must be -2

$0x_1 + x_2 + \square x_3 + \triangle x_4 = 0$ now, $x_2 = 2, x_3 = 1$

$2 + \square \cdot 1 = 0$ $\square = -2$

We could start with a different matrix

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & \end{matrix} \\ \left[\begin{matrix} 2 & 1 & \cdots \\ 0 & 1 & \end{matrix}\right. \end{array}$$

but this just makes things harder to solve.