Phy 321 - Spring 2017

HW 11.1

1.(a) $T = \text{"}\frac{1}{2} M v^2\text{"}$

$$= \frac{1}{2} M\left[(R\dot\theta)^2 + (R\sin\theta\,\omega)^2\right]$$

$$V = \text{"}mgh\text{"} = -MgR\cos\theta$$

$$\boxed{\mathcal{L} = T - V = \frac{1}{2}MR^2\left(\dot\theta^2 + \omega^2\sin^2\theta\right) + MgR\cos\theta}$$

(b) $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial\dot\theta}\right) = \frac{\partial \mathcal{L}}{\partial\theta}$$

$$\frac{d}{dt}\left[MR^2\dot\theta\right] = \frac{1}{2}MR^2\left(2\omega^2\sin\theta\cos\theta\right) - MgR\sin\theta$$

$$MR^2\ddot\theta = MR^2\omega^2\sin\theta\cos\theta \cancel{-MgR\sin\theta} - MgR\sin\theta$$

$$\boxed{\ddot\theta = \omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta}$$

(c) T+V is not constant for this problem, because the torque that makes ω be constant depends on θ and $\dot{\theta}$.

Specifically,

$$T+V = \frac{1}{2}MR^2(\dot{\theta}^2 + \omega^2 \sin^2\theta) - MgR\cos\theta$$

$$\frac{d}{dt}(T+V) = \frac{1}{2}MR^2(2\dot{\theta}\ddot{\theta} + 2\omega^2 \sin\theta\cos\theta\,\dot{\theta}) + MgR\sin\theta\,\dot{\theta}$$

$$= \dot{\theta}\left[MR^2\ddot{\theta} + MR^2\omega^2\sin\theta\cos\theta + MgR\sin\theta\right]$$

$$= \dot{\theta}\left[MR^2\right]\left[\omega^2\sin\theta\cos\theta - \frac{g}{\cancel{R}}\sin\theta + \omega^2\sin\theta\cos\theta + \frac{g}{\cancel{R}}\sin\theta\right]$$

$$\frac{d}{dt}(T+V) = 2MR^2\omega^2 \sin\theta \cos\theta \dot{\theta}$$

this is not 0 (unless $\omega = 0$ or $\theta \equiv 0$)

2.(a) $T = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$$= \frac{1}{2}M[(\dot{\rho}\cos\phi - \rho\sin\phi\,\dot{\phi})^2 + (\dot{\rho}\sin\phi + \rho\cos\phi\,\dot{\phi})^2 + (\dot{\rho}/\tan\alpha)^2]$$

$$= \frac{1}{2}M[\dot{\rho}^2 + (\rho\dot{\phi})^2 + \dot{\rho}^2\cos^2\alpha/\sin^2\alpha]$$

$$= \frac{1}{2}M[\dot{\rho}^2/\sin^2\alpha + (\rho\dot{\phi})^2]$$

$$V = Mgz = Mg\rho/\tan\alpha$$

$$\boxed{\mathcal{L} = T - V = \frac{1}{2}M[\dot{\rho}^2/\sin^2\alpha + (\rho\dot{\phi})^2] - Mg\rho/\tan\alpha}$$

(b) $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\rho}}\right) = \frac{\partial \mathcal{L}}{\partial \rho}$

$$\Rightarrow \frac{d}{dt}(M\dot{\rho}/\sin^2\alpha) = M\rho\dot{\phi}^2 - Mg/\tan\alpha$$

$$\boxed{\ddot{\rho}/\sin^2\alpha = \rho\dot{\phi}^2 - g/\tan\alpha}$$

and
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi}$$

$$\Rightarrow \quad \frac{d}{dt}\left[M\rho^2\dot{\phi}\right] = 0$$

$$\Rightarrow \quad \boxed{\left(\rho^2\dot{\phi}\right) = \text{constant}}$$

Remarks — not required —

(1) $\rho^2\dot{\phi}$ = constant is angular momentum constant (no torque)

(2) $T + V$ = constant, so

$$E = \frac{1}{2}M\left[\dot{\rho}^2/\sin^2\alpha + \rho^2\dot{\phi}^2\right] + Mg\rho/\tan\alpha$$

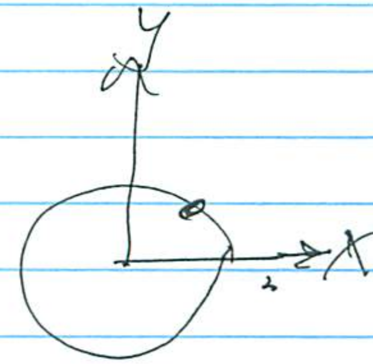
$$= \frac{1}{2}M\left[\dot{\rho}^2/\sin^2\alpha + \frac{1}{\rho^2}\underbrace{\left(\rho^2\dot{\phi}\right)^2}_{\text{const.}}\right] + Mg\rho/\tan\alpha$$

HW 11.5

3. Point P is at

$$x = R\cos(\omega t)$$

$$y = R\sin(\omega t)$$



Center of rod is at

$$\begin{pmatrix} x_{cm} \\ y_{cm} \end{pmatrix} = \begin{pmatrix} R\cos(\omega t) \\ R\sin(\omega t) \end{pmatrix} + \begin{pmatrix} \frac{l}{2}\sin\phi \\ -\frac{l}{2}\cos\phi \end{pmatrix},$$

$$T_{cm} = \frac{1}{2}M\left(\dot{x}_{cm}^2 + \dot{y}_{cm}^2\right)$$

$$= \frac{1}{2}M\left\{\left(-R\omega\sin\omega t + \frac{l}{2}\cos\phi\,\dot{\phi}\right)^2 + \left(R\omega\cos\omega t + \frac{l}{2}\sin\phi\,\dot{\phi}\right)^2\right\}$$

$$= \frac{1}{2}M\left[(R\omega)^2 + \left(\frac{l}{2}\dot{\phi}\right)^2 + R\omega l\dot{\phi}\left(-\cos\phi\sin\omega t + \sin\phi\cos\omega t\right)\right]$$

Also have

$$T_{rot} = \frac{1}{2}\left(\frac{Ml^2}{12}\right)\dot{\phi}^2$$

HW11.6

$$T = T_{cm} + T_{rot}$$

$$= \frac{1}{2} M \left\{ (R\omega)^2 + \left(\frac{l}{2}\dot{\phi}\right)^2 + R\omega l \dot{\phi} (-\cos\phi \sin\omega t + \sin\phi \cos\omega t) + \frac{l^2}{12}\dot{\phi}^2 \right\}$$

$$= \frac{1}{2} M \left\{ (R\omega)^2 + \frac{l^2 \dot{\phi}^2}{3} + R\omega l \dot{\phi} (-\cos\phi \sin\omega t + \sin\phi \cos\omega t) \right\}$$

$$V = Mg\, y_{cm} = Mg \left(R \sin\omega t - \frac{l}{2} \cos\phi\right)$$

$$\mathcal{L} = T - V$$

$$\boxed{\mathcal{L} = \frac{1}{2} M \left\{ (R\omega)^2 + \frac{l^2 \dot{\phi}^2}{3} + R\omega l \dot{\phi} (\sin\phi \cos\omega t - \cos\phi \sin\omega t) \right\} - Mg \left(R \sin\omega t - \frac{l}{2} \cos\phi\right)}$$

HW 11.7

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\phi} = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\frac{d}{dt}\left\{ \frac{M\ell^2\dot\phi}{3} + \frac{1}{2}MR\omega\ell\left(\sin\phi\cos\omega t - \cos\phi\sin\omega t\right) \right\}$$

$$= \frac{1}{2}M R\omega\ell\dot\phi\left(\cos\phi\cos\omega t + \sin\phi\sin\omega t\right)$$

$$+ \frac{Mg\ell}{2}\left(-\sin\phi\right)$$

$$\frac{M\ell^2}{3}\ddot\phi + \frac{1}{2}MR\omega\ell\Big(-\sin\phi\sin\omega t\,\omega$$

$$\underline{+\cos\phi\,\dot\phi\cos\omega t}$$

$$-\cos\phi\cos(\omega t)\,\omega$$

$$\underline{+\sin\phi\,\dot\phi\sin\omega t}\Big)$$

$$= \frac{1}{2}MR\omega\ell\dot\phi\left(\underline{\cos\phi\cos\omega t + \sin\phi\sin\omega t}\right)$$

$$-\frac{Mg\ell}{2}\sin\phi$$

divide by M

$$\frac{\ell^2}{3}\ddot{\phi} - \frac{1}{2}R\omega^2\ell\left(\sin\phi\sin\omega t + \cos\phi\cos\omega t\right)$$

$$+ \frac{g\ell}{2}\sin\phi = 0$$

$$\boxed{\ddot{\phi} - \frac{3}{2}\frac{R\omega^2}{\ell}\left(\sin\phi\sin\omega t + \cos\phi\cos\omega t\right) + \frac{3}{2}\frac{g}{\ell}\sin\phi = 0}$$

or

$$\boxed{\ddot{\phi} - \frac{3}{2\ell}R\omega^2\cos(\phi - \omega t) + \frac{3}{2}\frac{g}{\ell}\sin\phi = 0}$$

4. (a)

$$\begin{cases} x = R\sin\theta\cos\phi \\ y = R\sin\theta\sin\phi \\ z = R\cos\theta \end{cases}$$

$$\dot{x} = R\sin\theta(-\sin\phi\,\dot{\phi}) + R\cos\theta\cos\phi\,\dot{\theta}$$

$$\dot{y} = R\sin\theta(\cos\phi\,\dot{\phi}) + R\cos\theta\sin\phi\,\dot{\theta}$$

$$\dot{z} = -R\sin\theta\,\dot{\theta}$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$= \frac{1}{2}mR^2\Big[(-\sin\theta\sin\phi\,\dot{\phi} + \cos\theta\cos\phi\,\dot{\theta})^2 + (\sin\theta\cos\phi\,\dot{\phi} + \cos\theta\sin\phi\,\dot{\theta})^2 + (\sin\theta\,\dot{\theta})^2\Big]$$

$$= \frac{1}{2}mR^2\left(\sin^2\theta\,\dot{\phi}^2 + \cos^2\theta\,\dot{\theta}^2 + \sin^2\,\dot{\theta}^2\right)$$

$$= \frac{1}{2}mR^2\left[\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right]$$

HW 11-10

$$V = -Mgz = -MgR\cos\theta$$

Because z axis points down

$$\boxed{\mathcal{L} = T - V = \frac{1}{2}MR^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) + MgR\cos\theta}$$

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = MR^2\sin^2\theta\,\dot{\phi}$$

$$F_\phi = \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\dot{p}_\phi = F_\phi \Rightarrow \boxed{MR^2\sin^2\theta\,\dot{\phi} = \text{const}}$$

or $\sin^2\theta\,\ddot{\phi} + 2\sin\theta\cos\theta\,\dot{\theta}\dot{\phi} = 0$

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = MR^2\dot{\theta}$$

$$F_\theta = \frac{\partial \mathcal{L}}{\partial \theta} = MR^2\sin\theta\cos\theta\,\dot{\phi}^2 - MgR\sin\theta$$

$\dot{p}_\theta = F_\theta \implies$

$$\boxed{MR^2 \ddot{\theta} = MR^2 \sin\theta \cos\theta \, \dot{\phi}^2 - MgR \sin\theta}$$

4(b) No ϕ in L, so

p_ϕ = z-comp. of angular momentum

= constant