Block Matrices

can think of Matrices in blocks

$[A \mid \vec{b}]$ augmented matrix

block 1 block 2 different sizes

Block Multiplication

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & \cdots \\ B_{21} & B_{22} & \cdots \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21}, & A_{11}B_{12} + A_{12}B_{22} & \cdots \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} & \cdots \end{bmatrix}$$

Blocks can be any size, can be columns + rows

Ex

$$\underset{A}{\left[\begin{array}{c|c} 1 & 2 \\ 3 & 4 \end{array}\right]} \underset{B}{\left[\begin{array}{cc} 1 & -1 \\ \hline -2 & 3 \end{array}\right]} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} + \begin{bmatrix} -4 & 6 \\ -8 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 5 \\ -5 & 9 \end{bmatrix}$$

Block Elimination

$$\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

we are used to Row operations

Now see in terms of block operations

$$\left[\begin{array}{c|c} I & \overset{\text{zero}}{0} \\ \hline E & I \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} IA + 0C & IB + 0D \\ \hline EA + IC & EB + ID \end{array}\right]$$

order of multiplication DOES matter

↓This is just $E_{21}E_{31}$ split

$$\left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -2 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right] \left[\begin{array}{c|cc} 2 & 4 & -2 \\ \hline 4 & 9 & -3 \\ -2 & -3 & 7 \end{array}\right] = \text{see next page}$$

$\Downarrow$

$$= \left[\begin{array}{c|c} \underbrace{(1)(2)}_{IA} + \underbrace{\begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix}}_{0C} & \underbrace{1\begin{bmatrix} 4 & -2 \end{bmatrix}}_{IB} + \underbrace{\begin{bmatrix} 0 & 0 \end{bmatrix}\begin{bmatrix} 9 & -3 \\ -3 & 7 \end{bmatrix}}_{0D} \\ \hline \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix} 2}_{EA} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix}}_{IC} & \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\begin{bmatrix} 4 & -2 \end{bmatrix}}_{EB} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 9 & -3 \\ -3 & 7 \end{bmatrix}}_{ID} \end{array}\right]$$

$$= \left[\begin{array}{c|c} 2 & \begin{bmatrix} 4 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \\ \hline \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \end{bmatrix} & \begin{bmatrix} -8 & 4 \\ 4 & -2 \end{bmatrix} + \begin{bmatrix} 9 & -3 \\ -3 & 7 \end{bmatrix} \end{array}\right] = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

Rules For Matrix Operations

Adding: we can add two matrices as long as they are the same size

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 9 \\ 2 & 9 \end{bmatrix}$$

$A + B = B + A$

$c(A+B) = cA + cB$

$A + (B+C) = (A+B) + C$

Scalar Multiplication

$$cA = \begin{bmatrix} | & | & & | \\ c\vec{a}_1 & c\vec{a}_2 & \cdots & c\vec{a}_n \\ | & | & & | \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$4A = \begin{bmatrix} 4 & 8 \\ 12 & 10 \end{bmatrix}$$

Matrix Multiplication

$$\underset{m \times n}{\begin{bmatrix} \quad \end{bmatrix}} \times \underset{n \times p}{\begin{bmatrix} \quad \end{bmatrix}} = \underset{m \times p}{\begin{bmatrix} \quad \end{bmatrix}}$$

Think about: AB

① dot product of each row of A w/ each col. of B

$$\underset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}} = \begin{bmatrix} (1,2)\cdot(1,-2) & (1,2)\cdot(-1,3) \\ 1-4=-3 & -1+6=5 \\ (3,4)\cdot(1,-2) & (3,4)\cdot(-1,3) \\ 3-8=-5 & -3+12=9 \end{bmatrix}$$

② Matrix A times the columns of B

$$\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}1 & -1\\ -2 & +3\end{bmatrix}$$

$$= \left[\begin{matrix} \underbrace{\underset{2\times 2}{\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}}\underset{2\times 1}{\begin{bmatrix}1\\ -2\end{bmatrix}}}_{2\times 1} & \quad & \underbrace{\underset{2\times 2}{\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}}\underset{2\times 1}{\begin{bmatrix}-1\\ +3\end{bmatrix}}}_{2\times 1}\end{matrix}\right]$$

③ Rows of A times matrix B

$$\left[\begin{matrix} \begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}1 & -1\\ -2 & -3\end{bmatrix} & \quad & \begin{bmatrix}3 & 4\end{bmatrix}\begin{bmatrix}1 & -1\\ -2 & -3\end{bmatrix} \\ & & \end{matrix}\right]$$

Examples

$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ $\quad 1 \times 3$

$B = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ $\quad 3 \times 1$

Ex)

$$\underset{1\times3 \cdot 3\times1}{AB} = \underset{1\times1}{\begin{bmatrix} \quad \end{bmatrix}} = (1,2,3)\cdot(2,4,6)$$

$$= 2 + 8 + 18 = 28$$

Ex)

$$\underset{3\times1}{B}\ \underset{1\times3}{A} = \underset{3\times3}{\begin{bmatrix} \quad \end{bmatrix}}$$

$$= \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \\ R_3C_1 & R_3C_2 & R_3C_3 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 8 & 12 \\ 6 & 12 & 18 \end{bmatrix}$$

$AB \neq BA$

Laws

$AB \neq BA$ $\qquad$ A, B, C Matrices

$c(A+B) = cA + cB$

$(A+B)C = AC + BC$

$(AB)C = A(BC)$