Section 8.6.

The Kepler orbits

Read Section 8.6.

Recall equations from Section 8.5

$$\ell = \mu r^2 \dot{\phi} \qquad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r)$$

$$r(\phi) = \frac{1}{u(\phi)} \quad \text{and} \quad \frac{d^2u}{d\phi^2} = -u - \frac{\mu}{\ell^2}\frac{F_r}{u^2}$$

$$F_r = -\frac{dU}{dr}$$

8.6. The Kepler orbits

From Newton's theory of gravitation,

$$F_r(r) = - G\, m_1\, m_2 \,/\, r^2 \,.$$

Write this as

$$F_r(r) = - \gamma \,/r^2 \,. \quad \text{where } \gamma = Gm_1m_2 = GM\mu$$

The potential energy is

$$U(r) = - \ \gamma \,/r \,.$$

The orbit equation, in terms of u (= 1/r) is

$$u''(\varphi) = - u(\varphi) + \gamma\, \mu \,/l^2 \ .$$

prime (') means d /dφ

We know this equation: it's an inhomogeneous linear differential equation.

$$u''(\varphi) = -\,u(\varphi) + \gamma\,\mu\,/l^{\,2}$$

inhomogeneous linear diff. eq.

The solution is any particular solution plus a general solution of the homogeneous equation.

<u>Proof.</u>

Write $u(\varphi) = u_p(\varphi) + w(\varphi)$

Then

$$u'' = u_p'' + w''$$

$$= -u_p + \frac{\gamma\mu}{\ell^2} + (-w)$$

$$= -u + \frac{\gamma\mu}{\ell^3} \quad \text{as required}$$

<u>The particular solution</u>

Really easy,

$$u_p(\varphi) = \gamma\,\mu\,/l^{\,2}$$

because this is a constant.

<u>The general solution of the homogeneous equation</u>

$$w''(\varphi) = -\,w(\varphi)$$

This is also easy,

$$w(\varphi) = A\cos(\varphi - \delta)$$

where A and δ are two constants which must be determined from the initial conditions or some other information.

$$u(\varphi) = A \cos(\varphi - \delta) + \gamma \mu / l^2 \quad (☆)$$

We'll eventually see that this describes an ellipse. [Problem 8.16]

Remember $u(\varphi) = 1/r(\varphi)$.

Theorem 1. Equation (☆) describes a closed curve.

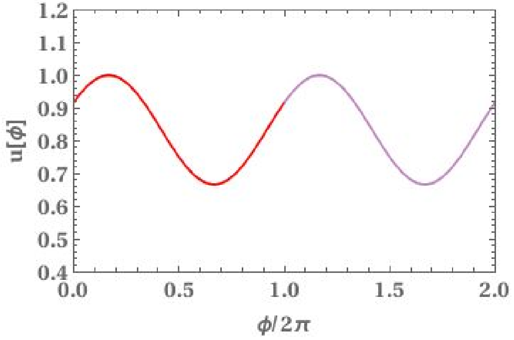
Proof. As the angle φ goes from 0 to 2π, i.e., one angular cycle, the radial coordinate r goes through one radial cycle because cos(φ − δ) goes through one cycle.

Or, more simply,

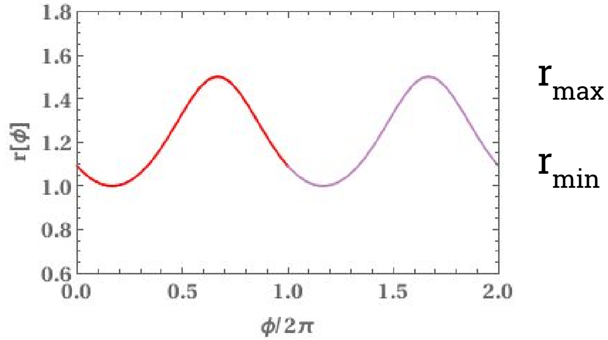
$$r(\varphi + 2\pi) = r(\varphi)$$

Graphical analysis

※ Plot u(φ)



※ Plot r(φ)



※ Plot the orbit, i.e., curve {r,φ}

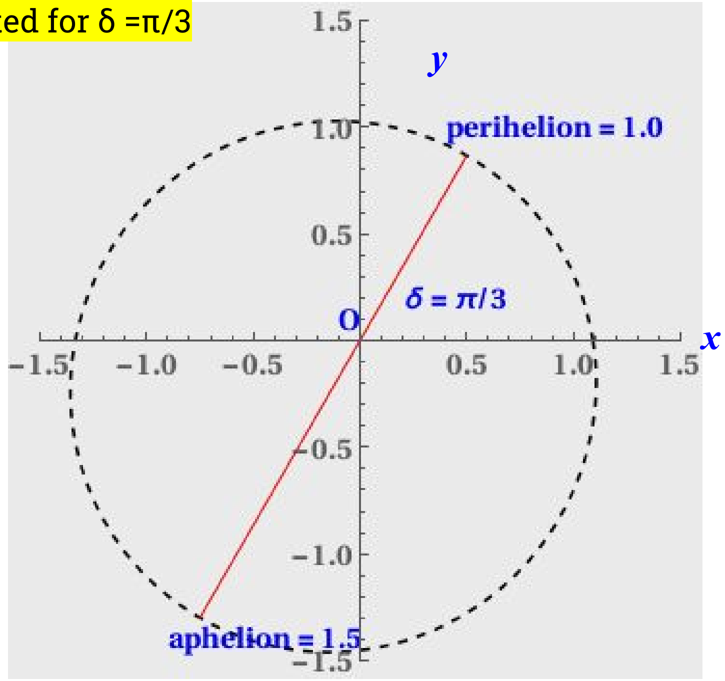
$$u(\varphi) = A \cos(\varphi - \delta) + \gamma\, \mu / l^2 \qquad (☆)$$

※ ParametricPlot[

{ Cos[phi] /u[phi] , Sin[phi] /u[phi] },

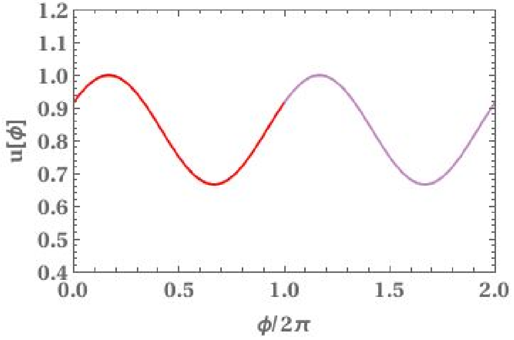
{phi, 0, 2 Pi}]

plotted for δ =π/3

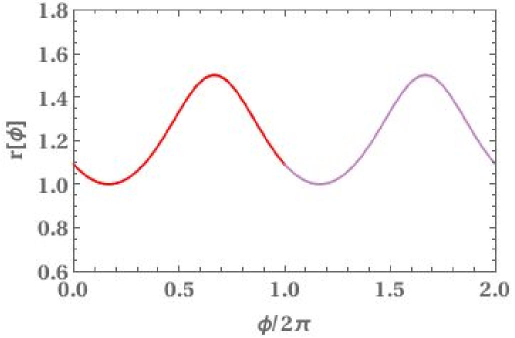


Graphical analysis

※ Plot u(φ)



※ Plot r(φ)



r_{max}

r_{min}

To make it simple, ***let δ = 0***.
I.e., set up the xy-coordinate system such that the perihelion is on the positive x axis.

$$u(\varphi) = A \cos(\varphi) + \gamma \mu / l^2$$

Also, define some new parameters.

Write $r(\phi) = \dfrac{c}{1+\varepsilon \cos\phi}$ $\begin{cases} r_{min} = c/(1+\varepsilon) \\ r_{max} = c/(1-\varepsilon) \end{cases}$

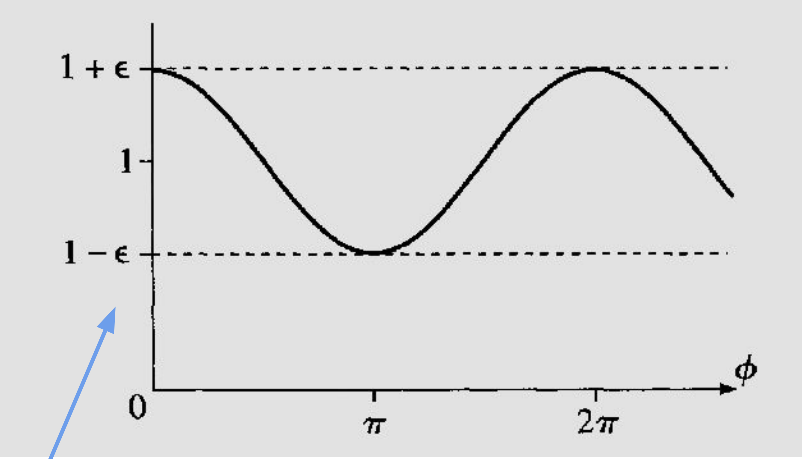
$$u(\phi) = \frac{1}{c} + \frac{\varepsilon}{c}\cos\phi \Rightarrow c = \frac{l^2}{\gamma\mu} = \frac{l^2}{\mu^2 GM}$$

and $\dfrac{\varepsilon}{c} = A$; $\varepsilon = \dfrac{l^2 A}{\mu^2 GM}$

The orbit equation in polar coordinates is

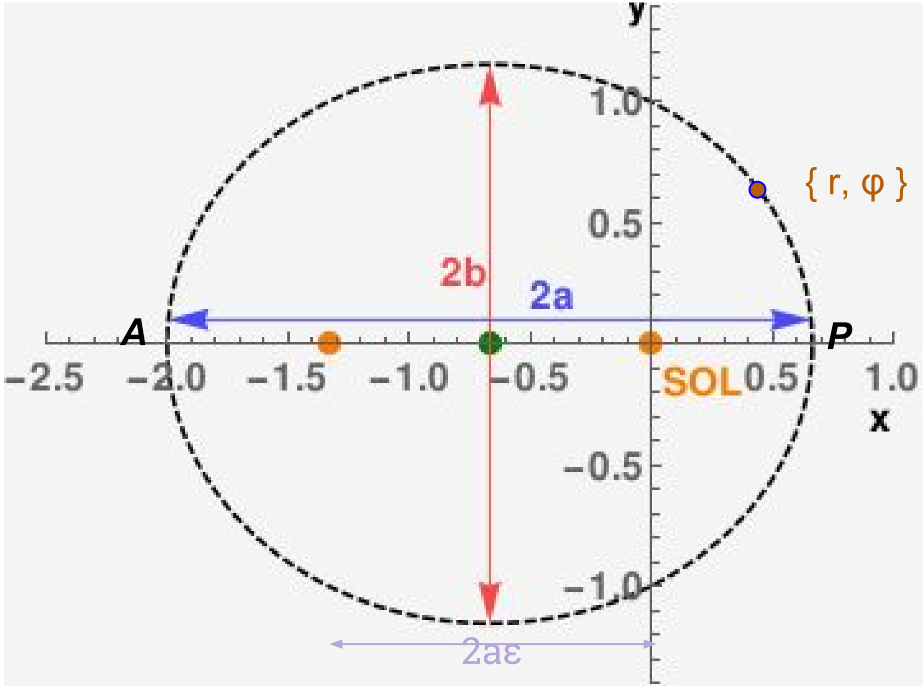
$$r(\varphi) = \frac{c}{1 + \varepsilon \cos\varphi} .$$

FIGURE 8.9



The vertical axis is $1 + \varepsilon \cos\varphi$, which is equal to $c / r(\varphi)$.

FIGURE 8.10
ELLIPSE GEOMETRY



Problem 8.16: Prove $\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1$ where $d = \varepsilon a$.

- Dashed curve = the ellipse
- In polar coordinates,
  $r(\varphi) = c \,/\, (1 + \varepsilon \cos \varphi)$
  (This ellipse has $c = 1$ and $\varepsilon = 0.5$.)
- Orange dots = the focal points. The sun is at one of the focal points. dist.=$2a\varepsilon$
- Green dot = the center of the ellipse
- Perihelion = minimum distance r_{min} $= c/(1+\varepsilon)$; $x_p = c/(1+\varepsilon)$ & $y_p = 0$
  Aphelion = maximum distance r_{max} $= c/(1-\varepsilon)$; $x_a = -c/(1-\varepsilon)$ & $y_a = 0$
- Blue distance = major axis = $2a = r_{min} + r_{max}$; the semimajor axis is a;
  $a = c/(1-\varepsilon^2) = ½\,(x_p - x_a)$
- Red distance = minor axis = 2b; the semiminor axis is b;
  $b = c / \mathrm{sqrt}(1-\varepsilon^2)$ ***exercise!***

Example 8.4 : Halley's Comet

Edmund Halley (1656 – 1742) was a famous astronomer, and a friend of Isaac Newton.
Pronunciation of his name : rhymes with valley.

Interesting history.

One day (1687) Halley visited Newton, who was at that time an obscure math professor at Cambridge University. Halley asked Newton if he could explain Kepler's 3 laws of planetary motion. Newton said he figured it out a few years earlier. But he couldn't find the paper with his calculations, and he promised to send Halley a copy.

When Halley saw it, he was amazed that Newton had solved the most important problem in astronomy, but had never told anyone about it. So Halley arranged to get Newton's work published.

That is how Newton became famous.

Then Halley applied Newton's theory to observations of comets; and he proved that comets revolve around the sun and return to the sun after each period of revolution.

Observations of the comet

1531 (known to Halley)
~76 years
1607 (")
~75 years
1682 (" ; did Halley & Newton see it?)
~76 years
so he predicted it would return in 1758
most recent 1986 (I saw it in a telescope)
~75 years
next time 2061

Before Newton and Halley, people thought these were all different comets. But it's just the same one, seen near perihelion.

Wikipeida: ***Halley's Comet*** *or* ***Comet Halley****, officially designated* ***1P/Halley****, is a short-period comet visible from Earth every 75–76 years.*

Example 8.4 : Halley's Comet

$$\varepsilon = 0.967$$

$$\text{perihelion} = 0.59 \text{ AU}$$

Compare: Mercury has $0.31 < r < 0.47$ AU
Earth has $0.983 < r < 1.017$ AU
Neptune has $29.8 < r < 30.3$ AU

Calculate the aphelion distance for the orbit of Halley's comet.

$$r_{min} = \frac{C}{1+\varepsilon} \quad \text{and} \quad r_{max} = \frac{C}{1-\varepsilon}$$

$$r_{max} = \frac{1+\varepsilon}{1-\varepsilon}\, r_{min} = \frac{1.967}{0.033}\, 0.59 \text{ AU}$$

$$r_{max} = 35.2 \text{ AU}$$

The period of revolution of a planet, comet, satellite, etc.

This was known to Kepler.
By analyzing observations, he discovered ...

Kepler's third law (published 1619)
Kepler claimed that $\tau^2 \propto a^3$ for all the planets;

τ = period of revolution
a = semimajor axis

In other words, τ^2 / a^3 is the same for all the planets.

It's not precisely true, but it's close. Given the observations available to Kepler, it is remarkable that he came so close.

(next lecture)

Calculate the period of Halley's Comet.

By Kepler's 3rd law,

$$\frac{\tau_H^2}{a_H^3} = \frac{\tau_E^2}{a_E^3} = \frac{(1\ \text{year})^2}{(1\ \text{AU})^3}$$

$$a_H = \frac{1}{2}(r_{min} + r_{max}) = \frac{1}{2}(0.59 + 35.2)\,\text{AU}$$

$$= 17.9\ \text{AU}$$

$$\therefore \tau_H = 1\,\text{year}\left[\frac{17.9}{1}\right]^{3/2} = 75.7\ \text{year}$$

Homework Assignment 14

due in class Friday December 9

[76] Problem 8.19 ★★

[77] Problem 8.25 ★★★

[78] Problem 8.27 ★★★

[79] Problem 8.28 ★

[80] Problem 8.34 ★★

USE THE COVER SHEET.