Linear Algebra Problems

Math 504 – 505 Jerry L. Kazdan

Topics

The level of difficulty of these problems varies wildly. Some are entirely appropriate for a high school course. Others definitely inappropriate.

Although problems are categorized by topics, this should not be taken very seriously. Many problems fit equally well in several different topics.

NOTE: To make this collection more stable *no* new problems will be added in the future. Of course corrections and clarifications will be inserted.

I have never formally written solutions to these problems. However, I have frequently used some in Homework and Exams in my own linear algebra courses – in which I often have written solutions. See my web page: `https://www.math.upenn.edu/~kazdan/`

NOTATION: We occasionally write $M(n, \mathbb{F})$ for the ring of all $n \times n$ matrices over the field $\mathbb{F}$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$.

1 Basics

1. At noon the minute and hour hands of a clock coincide.
   a) What in the first time, T_1, when they are perpendicular?
   b) What is the next time, T_2, when they again coincide?

2. Which of the following sets are linear spaces?

a) $\{X = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ with the property $x_1 - 2x_3 = 0\}$

b) The set of solutions $\vec{x}$ of $A\vec{x} = 0$, where A is an $m \times n$ matrix.

c) The set of 2×2 matrices A with $\det(A) = 0$.

d) The set of polynomials $p(x)$ with $\int_{-1}^{1} p(x)\,dx = 0$.

e) The set of solutions $y = y(t)$ of $y'' + 4y' + y = 0$.

f) The set of solutions $y = y(t)$ of $y'' + 4y' + y = 7e^{2t}$.

g) Let S_f be the set of solutions $u(t)$ of the differential equation $u'' - xu = f(x)$. For which continuous functions f is S_f a linear space? Why? [NOTE: You are not being asked to actually solve this differential equation.]

3. Which of the following sets of vectors are bases for $\mathbb{R}^2$?

a). $\{(0,\ 1),\ (1,\ 1)\}$

b). $\{(1,\ 0),\ (0,\ 1),\ (1,\ 1)\}$

c). $\{(1,\ 0),\ (-1,\ 0\}$

d). $\{(1,\ 1),\ (1,\ -1)\}$

e). $\{((1,\ 1),\ (2,\ 2)\}$

f). $\{(1,\ 2)\}$

4. For which real numbers x do the vectors: $(x, 1, 1, 1)$, $(1, x, 1, 1)$, $(1, 1, x, 1)$, $(1, 1, 1, x)$ *not* form a basis of $\mathbb{R}^4$? For each of the values of x that you find, what is the dimension of the subspace of $\mathbb{R}^4$ that they span?

5. Let $C(\mathbb{R})$ be the linear space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

a) Let S_c be the set of differentiable functions $u(x)$ that satisfy the differential equation

$$u' = 2xu + c$$

for all real x. For which value(s) of the real constant c is this set a linear subspace of $C(\mathbb{R})$?

b) Let $C^2(\mathbb{R})$ be the linear space of all functions from $\mathbb{R}$ to $\mathbb{R}$ that have two continuous derivatives and let S_f be the set of solutions $u(x) \in C^2(\mathbb{R})$ of the differential equation

$$u'' + u = f(x)$$

for all real x. For which polynomials $f(x)$ is the set S_f a linear subspace of $C(\mathbb{R})$?

c) Let $\mathcal{A}$ and $\mathcal{B}$ be linear spaces and $L : \mathcal{A} \to \mathcal{B}$ be a linear map. For which vectors $y \in \mathcal{B}$ is the set

$$\mathcal{S}_y := \{x \in \mathcal{A} \mid Lx = y\}$$

a linear space?

6. Let $\mathcal{P}_k$ be the space of polynomials of degree at most k and define the linear map $L : \mathcal{P}_k \to \mathcal{P}_{k+1}$ by $Lp := p''(x) + xp(x)$.

a) Show that the polynomial $q(x) = 1$ is not in the image of L. [SUGGESTION: Try the case $k = 2$ first.]

b) Let $V = \{q(x) \in \mathcal{P}_{k+1} \,|\, q(0) = 0\}$. Show that the map $L : \mathcal{P}_k \to V$ is invertible. [Again,try $k = 2$ first.]

7. Compute the dimension and find bases for the following linear spaces.

   a) Real anti-symmetric 4×4 matrices.

   b) Quartic polynomials p with the property that $p(2) = 0$ and $p(3) = 0$.

   c) Cubic polynomials $p(x, y)$ in two real variables with the properties: $p(0, 0) = 0$, $p(1, 0) = 0$ and $p(0, 1) = 0$.

   d) The space of linear maps $L : \mathbb{R}^5 \to \mathbb{R}^3$ whose kernels contain $(0, 2, -3, 0, 1)$.

8. a) Compute the dimension of the intersection of the following two planes in $\mathbb{R}^3$

$$x + 2y - z = 0, \qquad 3x - 3y + z = 0.$$

   b) A map $L : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by the matrix $L := \begin{pmatrix} 1 & 2 & -1 \\ 3 & -3 & 1 \end{pmatrix}$. Find the nullspace (kernel) of L.

9. If A is a 5×5 matrix with $\det A = -1$, compute $\det(-2A)$.

10. Does an 8-dimensional vector space contain linear subspaces V_1, V_2, V_3 with no common non-zero element, such that

    a). $\dim(V_i) = 5$, $i = 1, 2, 3$? b). $\dim(V_i) = 6$, $i = 1, 2, 3$?

11. Let U and V both be two-dimensional subspaces of $\mathbb{R}^5$, and let $W = U \cap V$. Find all possible values for the dimension of W.

12. Let U and V both be two-dimensional subspaces of $\mathbb{R}^5$, and define the set $W := U + V$ as the set of all vectors $w = u + v$ where $u \in U$ and $v \in V$ can be any vectors.

    a) Show that W is a linear space.

    b) Find all possible values for the dimension of W.

13. Let A be an $n \times n$ matrix of real or complex numbers. Which of the following statements are *equivalent* to: "the matrix A is invertible"?

    a) The columns of A are linearly independent.

b) The columns of A span $\mathbb{R}^n$.

c) The rows of A are linearly independent.

d) The kernel of A is 0.

e) The only solution of the homogeneous equations $Ax = 0$ is $x = 0$.

f) The linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^n$ defined by A is 1-1.

g) The linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^n$ defined by A is onto.

h) The rank of A is n.

i) The adjoint, A^*, is invertible.

j) $\det A \neq 0$.

14. Call a subset S of a vector space V a *spanning set* if $\mathrm{Span}(S) = V$. Suppose that $T : V \to W$ is a linear map of vector spaces.

    a) Prove that a linear map T is 1-1 if and only if T sends linearly independent sets to linearly independent sets.

    b) Prove that T is onto if and only if T sends spanning sets to spanning sets.

2 Linear Equations

15. Solve the given system – or show that no solution exists:

$$\begin{aligned} x + 2y \phantom{{}+ 4z} &= 1 \\ 3x + 2y + 4z &= 7 \\ -2x + y - 2z &= -1 \end{aligned}$$

16. Say you have k linear algebraic equations in n variables; in matrix form we write $AX = Y$. Give a proof or counterexample for each of the following.

    a) If $n = k$ there is always *at most one* solution.

    b) If $n > k$ you can *always* solve $AX = Y$.

    c) If $n > k$ the nullspace of A has dimension greater than zero.

    d) If $n < k$ then for *some* Y there is *no* solution of $AX = Y$.

    e) If $n < k$ the *only* solution of $AX = 0$ is $X = 0$.

17. Let $A : \mathbb{R}^n \to \mathbb{R}^k$ be a linear map. Show that the following are equivalent.

a) A is injective (hence $n \leq k$). [*injective* means one-to-one]

b) $\dim \ker(A) = 0$.

c) A has a *left inverse* B, so $BA = I$.

d) The columns of A are linearly independent.

18. Let $A : \mathbb{R}^n \to \mathbb{R}^k$ be a linear map. Show that the following are equivalent.

a) A is surjective (hence $n \geq k$).

b) $\dim \operatorname{im}(A) = k$.

c) A has a *right inverse* B, so $AB = I$.

d) The columns of A span $\mathbb{R}^k$.

19. Let A be a 4×4 matrix with determinant 7. Give a proof or counterexample for each of the following.

a) For some vector $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution.

b) For some vector $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

c) For some vector $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has no solution.

d) For all vectors $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution.

20. Let $A : \mathbb{R}^n \to \mathbb{R}^k$ be a real matrix, not necessarily square.

a) If two rows of A are the same, show that A is not onto by finding a vector $y = (y_1, \ldots, y_k)$ that is not in the image of A. [Hint: This is a mental computation if you write out the equations $Ax = y$ explicitly.]

b) What if $A : \mathbb{C}^n \to \mathbb{C}^k$ is a complex matrix?

c) More generally, if the rows of A are linearly dependent, show that it is not onto.

21. Let $A : \mathbb{R}^n \to \mathbb{R}^k$ be a real matrix, not necessarily square.

a) If two columns of A are the same, show that A is not one-to-one by finding a vector $x = (x_1, \ldots, x_n)$ that is in the nullspace of A.

b) More generally, if the columns of A are linearly dependent, show that A is not one-to-one.

22. Let A and B be $n \times n$ matrices with $AB = 0$. Give a proof or counterexample for each of the following.

a) Either $A = 0$ or $B = 0$ (or both).

b) $BA = 0$

c) If $\det A = -3$, then $B = 0$.

d) If B is invertible then $A = 0$.

e) There is a vector $V \neq 0$ such that $BAV = 0$.

23. Consider the system of equations

$$\begin{aligned} x + y - \ \ z &= a \\ x - y + 2z &= b. \end{aligned}$$

a) Find the general solution of the homogeneous equation.

b) A particular solution of the inhomogeneous equations when $a = 1$ and $b = 2$ is $x = 1$, $y = 1$, $z = 1$. Find the most general solution of the inhomogeneous equations.

c) Find some particular solution of the inhomogeneous equations when $a = -1$ and $b = -2$.

d) Find some particular solution of the inhomogeneous equations when $a = 3$ and $b = 6$.

[Remark: After you have done part a), it is possible immediately to write the solutions to the remaining parts.]

24. Solve the equations $\begin{aligned} 2x + 3y + 2z &= 1 \\ x + 0y + 3z &= 2 \\ 2x + 2y + 3z &= 3 \end{aligned}$ for x, y, and z.

HINT: If $A = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}$, then $A^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}$.

25. Consider the system of linear equations $\begin{aligned} kx + \ y + \ z &= 1 \\ x + ky + \ z &= 1 \\ x + \ y + kz &= 1 \end{aligned}$.

For what value(s) of k does this have (i) a unique solution? (ii), no solution? (iii) infinitely many solutions? (Justify your assertions).

26. Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix}$.

a) Find the general solution $\mathbf{Z}$ of the homogeneous equation $A\mathbf{Z} = 0$.

b) Find some solution of $A\mathbf{X} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

c) Find the general solution of the equation in part b).

d) Find some solution of $A\mathbf{X} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$ and of $A\mathbf{X} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

e) Find some solution of $A\mathbf{X} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

f) Find some solution of $A\mathbf{X} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$. [Note: $\left(\begin{smallmatrix} 7 \\ 2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right) + 2\left(\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right)$].

[Remark: After you have done parts a), b) and e), it is possible immediately to write the solutions to the remaining parts.]

27. Consider the system of equations

$$\begin{aligned} x + y - z &= a \\ x - y + 2z &= b \\ 3x + y \phantom{{}+2z} &= c \end{aligned}$$

a) Find the general solution of the homogeneous equation.

b) If $a = 1$, $b = 2$, and $c = 4$, then a particular solution of the inhomogeneous equations is $x = 1$, $y = 1$, $z = 1$. Find the most general solution of these inhomogeneous equations.

c) If $a = 1$, $b = 2$, and $c = 3$, show these equations have *no* solution.

d) If $a = 0$, $b = 0$, $c = 1$, show the equations have *no* solution. [Note: $\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 2 \\ 4 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right)$].

e) Let $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix}$. Find a basis for $\ker(A)$ and $\operatorname{image}(A)$.

28. Let A be a square matrix with integer elements. For each of the following give a proof or counterexample.

a) If $\det(A) = \pm 1$, then for any vector y with integer elements there is a vector x with integer elements that solves $Ax = y$.

b) If $\det(A) = 2$, then for any vector y with even integer elements there is a vector x with integer elements that solves $Ax = y$.

c) If all of the elements of A are positive integers and $\det(A) = +1$, then given any vector y with non-negative integer elements there is a vector x with non-negative integer elements that solves $Ax = y$.

d) If the elements of A are rational numbers and $\det(A) \neq 0$, then for any vector y with rational elements there is a vector x with rational elements that solves $Ax = y$.

3 Linear Maps

29. a) Find a 2×2 matrix that rotates the plane by $+45$ degrees ($+45$ degrees means 45 degrees *counterclockwise*).

    b) Find a 2×2 matrix that rotates the plane by $+45$ degrees followed by a reflection across the horizontal axis.

    c) Find a 2×2 matrix that reflects across the horizontal axis followed by a rotation the plane by $+45$ degrees.

    d) Find a matrix that rotates the plane through $+60$ degrees, keeping the origin fixed.

    e) Find the inverse of each of these maps.

30. a) Find a 3×3 matrix that acts on $\mathbb{R}^3$ as follows: it keeps the x_1 axis fixed but rotates the x_2 x_3 plane by 60 degrees.

    b) Find a 3×3 matrix A mapping $\mathbb{R}^3 \to \mathbb{R}^3$ that rotates the x_1 x_3 plane by 60 degrees and leaves the x_2 axis fixed.

31. Consider the homogeneous linear system $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 3 & 0 & 1 \\ 1 & 3 & -2 & -2 \\ 0 & 0 & 2 & 3 \end{pmatrix}.$$

    Identify which of the following statements are correct?

    a) $Ax = 0$ has no solution.

    b) $\dim \ker A = 2$

    c) $Ax = 0$ has a unique solution.

    d) For any vector $b \in \mathbb{R}^3$ the equation $Ax = b$ has at least one solution.

32. Find a real 2×2 matrix A (other than $A = I$) such that $A^5 = I$.

33. Proof or counterexample. In these L is a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$, so its representation will be as a 2×2 matrix.

a) If L is invertible, then L^{-1} is also invertible.

b) If $LV = 5V$ for all vectors V, then $L^{-1}W = (1/5)W$ for all vectors W.

c) If L is a rotation of the plane by 45 degrees *counterclockwise*, then L^{-1} is a rotation by 45 degrees *clockwise*.

d) If L is a rotation of the plane by 45 degrees counterclockwise, then L^{-1} is a rotation by 315 degrees counterclockwise.

e) The zero map ($0\mathbf{V} = 0$ for all vectors $\mathbf{V}$) is invertible.

f) The identity map ($I\mathbf{V} = \mathbf{V}$ for all vectors $\mathbf{V}$) is invertible.

g) If L is invertible, then $L^{-1}0 = 0$.

h) If $L\mathbf{V} = 0$ for some non-zero vector $\mathbf{V}$, then L is not invertible.

i) The identity map (say from the plane to the plane) is the only linear map that is its own inverse: $L = L^{-1}$.

34. Let R, M, and N be linear maps from the (two dimensional) plane to the plane given in terms of the standard $\mathbf{i}, \mathbf{j}$ basis vectors by:

$$R\mathbf{i} = \mathbf{j}, \quad R\mathbf{j} = -\mathbf{i} \qquad M\mathbf{i} = -\mathbf{i}, \quad M\mathbf{j} = \mathbf{j} \qquad N\mathbf{v} = -\mathbf{v} \text{ for } \textit{all} \text{ vectors } \mathbf{v}$$

a) Describe (pictures?) the actions of the maps R, R^2, R^{-1}, M, M^2, M^{-1} and N [compare Problem 48]

b) Describe the actions of the maps RM, MR, RN, NR, MN, and NM [here we use the standard convention that the map RM means *first* use M *then* R]. Which pairs of these maps commute?

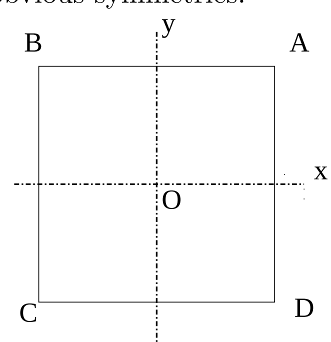
c) Which of the following identities are correct—and why?

1)	$R^2 = N$	2)	$N^2 = I$	3)	$R^4 = I$	4)	$R^5 = R$
5)	$M^2 = I$	6)	$M^3 = M$	7)	$MNM = N$	8)	$NMN = R$

d) Find matrices representing each of the maps R, R^2, R^{-1}, M, and N.

e) [Symmetries of a Square] Consider a square centered at the origin in the plane $\mathbb{R}^2$ with its vertices at A, B, C, D. It has the following obvious symmetries:

Rotation I by 0 degrees (identity map)
Rotation R by 90 degrees counterclockwise
Rotation R^2 by 180 degrees counterclockwise
Rotation R^3 by 270 degrees counterclockwise
Reflection G across the horizontal (x) axis
Reflection M across the vertical (y) axis
Reflection S across the diagonal AC
Reflection T across the diagonal BD



Show that the square has no other symmetries.
Also, show that $SR = G$, $SR^2 = T$, and $SR^3 = M$.

f) Investigate the symmetries of an equilateral triangle in the plane.

[See `https://en.wikipedia.org/wiki/Dihedral_group` for more on the symmetries of regular polygons by the valuable device of representing the symmetries as matrices.]

35. Give a proof or counterexample the following. In each case your answers should be brief.

a) Suppose that u, v and w are vectors in a vector space V and $T : V \to W$ is a linear map. If u, v and w are linearly dependent, is it true that $T(u)$, $T(v)$ and $T(w)$ are linearly dependent? Why?

b) If $T : \mathbb{R}^6 \to \mathbb{R}^4$ is a linear map, is it possible that the nullspace of T is one dimensional?

36. Identify which of the following collections of matrices form a *linear subspace* in the linear space $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ of all 2×2 real matrices?

a) All invertible matrices.

b) All matrices that satisfy $A^2 = 0$.

c) All anti-symmetric matrices, that is, $A^T = -A$.

d) Let B be a fixed matrix and $\mathcal{B}$ the set of matrices with the property that $A^T B = BA^T$.

37. Identify which of the following collections of matrices form a *linear subspace* in the linear space $\mathrm{Mat}_{3\times 3}(\mathbb{R})$ of all 3×3 real matrices?

a) All matrices of rank 1.

b) All matrices satisfying $2A - A^T = 0$.

c) All matrices that satisfy $A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

38. Let V be a vector space and $\ell : V \to \mathbb{R}$ be a linear map. If $z \in V$ is not in the nullspace of ℓ, show that every $x \in V$ can be decomposed uniquely as $x = v + cz$, where v is in the nullspace of ℓ and c is a scalar. [MORAL: The nullspace of a linear functional has codimension one.]

39. For each of the following, answer TRUE or FALSE. If the statement is false in even a single instance, then the answer is FALSE. There is no need to justify your answers to this problem – but you should know either a proof or a counterexample.

a) If A is an invertible 4×4 matrix, then $(A^T)^{-1} = (A^{-1})^T$, where A^T denotes the transpose of A.

b) If A and B are 3×3 matrices, with $\operatorname{rank}(A) = \operatorname{rank}(B) = 2$, then $\operatorname{rank}(AB) = 2$.

c) If A and B are invertible 3×3 matrices, then $A + B$ is invertible.

d) If A is an $n \times n$ matrix with rank less than n, then for any vector b the equation $Ax = b$ has an infinite number of solutions.

e)) If A is an invertible 3×3 matrix and λ is an eigenvalue of A, then $1/\lambda$ is an eigenvalue of A^{-1} ,

40. For each of the following, answer TRUE or FALSE. If the statement is false in even a single instance, then the answer is FALSE. There is no need to justify your answers to this problem – but you should know either a proof or a counterexample.

a) If A and B are 4×4 matrices such that $\operatorname{rank}(AB) = 3$, then $\operatorname{rank}(BA) < 4$.

b) If A is a 5×3 matrix with $\operatorname{rank}(A) = 2$, then for every vector $b \in \mathbb{R}^5$ the equation $Ax = b$ will have at least one solution.

c) If A is a 4×7 matrix, then A and A^T have the same rank.

d) Let A and $B \neq 0$ be 2×2 matrices. If $AB = 0$, then A must be the zero matrix.

41. Let $A : \mathbb{R}^3 \to \mathbb{R}^2$ and $B : \mathbb{R}^2 \to \mathbb{R}^3$, so $BA : \mathbb{R}^3 \to \mathbb{R}^3$ and $AB : \mathbb{R}^2 \to \mathbb{R}^2$.

a) Show that BA can *not* be invertible.

b) Give an example showing that AB might be invertible (in this case it usually is).

42. Let A, B, and C be $n \times n$ matrices.

a) If A^2 is invertible, show that A is invertible.

[NOTE: You cannot naively use the formula $(AB)^{-1} = B^{-1}A^{-1}$ because it presumes you *already* know that both A and B are invertible. For non-square matrices, it is possible for AB to be invertible while neither A nor B are (see the last part of the previous Problem 41).]

b) Generalization. If AB is invertible, show that both A and B are invertible.

If ABC is invertible, show that A, B, and C are also invertible.

43. Let A be a real square matrix satisfying $A^{17} = 0$.

a) Show that the matrix $I - A$ is invertible.

b) If B is an invertible matrix, is $B - A$ also invertible? Proof or counterexample.

44. Suppose that A is an $n \times n$ matrix and there exists a matrix B so that

$$AB = I.$$

Prove that A is invertible and $BA = I$ as well.

45. Let A be a square real (or complex) matrix. Then A is invertible if and only if zero is *not* an eigenvalue. Proof or counterexample.

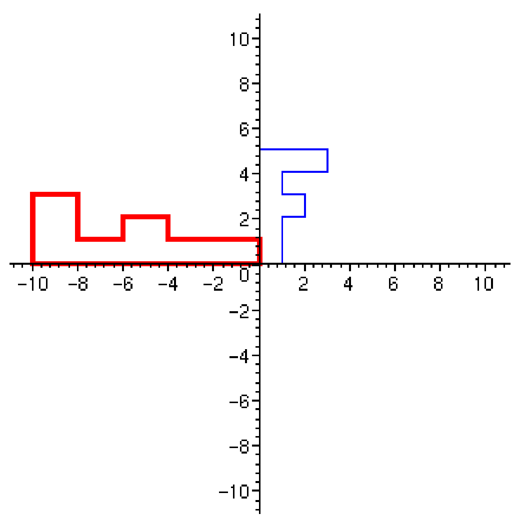
46. Let $\mathcal{M}_{(3,2)}$ be the linear space of all 3×2 real matrices and let the linear map $L : \mathcal{M}_{(3,2)} \to \mathbb{R}^5$ be *onto*. Compute the dimension of the nullspace of L.

47. Think of the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as mapping one plane to another.

   a) If two lines in the first plane are parallel, show that after being mapped by A they are also parallel – although they might coincide.

   b) Let Q be the unit square: $0 < x < 1, 0 < y < 1$ and let Q' be its image under this map A. Show that the area$(Q') = |ad - bc|$. [More generally, the area of any region is magnified by $|ad - bc|$ ($ad - bc$ is called the *determinant* of a 2×2 matrix]

48. a). Find a linear map of the plane, $A : \mathbb{R}^2 \to \mathbb{R}^2$ that does the following transformation of the letter **F** (here the smaller **F** is transformed to the larger one):



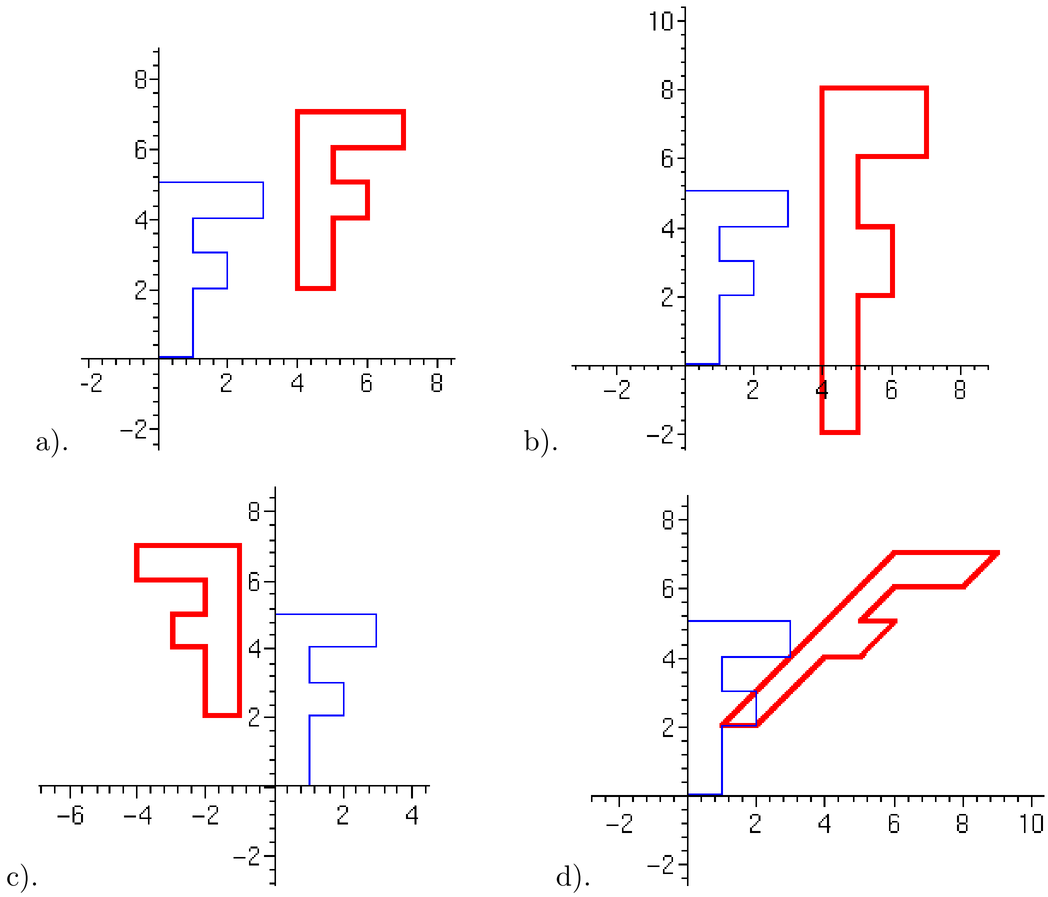
   b). Find a linear map of the plane that inverts this map, that is, it maps the larger **F** to the smaller.

49. Linear maps $F(X) = AX$, where A is a matrix, have the property that $F(0) = A0 = 0$, so they necessarily leave the origin fixed. It is simple to extend this to include a translation,

$$F(X) = V + AX,$$

where V is a vector. Note that $F(0) = V$.

Find the vector V and the matrix A that describe each of the following mappings [here the light blue F is mapped to the dark red F].



50. Find all linear maps $L\colon \mathbb{R}^3 \to \mathbb{R}^3$ whose kernel is exactly the plane $\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + 2x_2 - x_3 = 0 \}$.

51. Let A be a matrix, not necessarily square. Say $\mathbf{V}$ and $\mathbf{W}$ are particular solutions of the equations $A\mathbf{V} = \mathbf{Y}_1$ and $A\mathbf{W} = \mathbf{Y}_2$, respectively, while $\mathbf{Z} \neq 0$ is a solution of the homogeneous equation $A\mathbf{Z} = 0$. Answer the following in terms of $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{Z}$.

a) Find some solution of $A\mathbf{X} = 3\mathbf{Y}_1$.

b) Find some solution of $A\mathbf{X} = -5\mathbf{Y}_2$.

c) Find some solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$.

d) Find another solution (other than $\mathbf{Z}$ and 0) of the homogeneous equation $A\mathbf{X} = 0$.

e) Find *two* solutions of $A\mathbf{X} = \mathbf{Y}_1$.

f) Find another solution of $A\mathbf{X} = 3\mathbf{Y}_1 - 5\mathbf{Y}_2$.

g) If A is a square matrix, then $\det A = ?$

h) If A is a square matrix, for any given vector $\mathbf{W}$ can one always find at least one solution of $A\mathbf{X} = \mathbf{W}$? Why?

52. Let V be an n-dimensional vector space and $T : V \to V$ a linear transformation such that the image and kernel of T are identical.

    a) Prove that n is even.

    b) Give an example of such a linear transformation T.

53. Let V, W be two-dimensional real vector spaces, and let $f_1, \ldots, f_5$ be linear transformations from V to W. Show that there exist real numbers $a_1, \ldots, a_5$, not all zero, such that $a_1 f_1 + \cdots + a_5 f_5$ is the zero transformation.

54. Let $V \subset \mathbb{R}^{11}$ be a linear subspace of dimension 4 and consider the family $\mathcal{A}$ of all linear maps $L : \mathbb{R}^{11} - > \mathbb{R}^9$ each of whose nullspace contain V.

    Show that $\mathcal{A}$ is a linear space and compute its dimension.

55. Let L be a 2×2 matrix. For each of the following give a proof or counterexample.

    a) If $L^2 = 0$ then $L = 0$.

    b) If $L^2 = L$ then either $L = 0$ or $L = I$.

    c) If $L^2 = I$ then either $L = I$ or $L = -I$.

56. Find all four 2×2 diagonal matrices A that have the property $A^2 = I$.

    Geometrically interpret each of these examples as linear maps.

57. Find an example of 2×2 matrices A and B so that $AB = 0$ but $BA \neq 0$.

58. Let A and B be $n \times n$ matrices with the property that $AB = 0$. For each of the following give a proof or counterexample.

    a) Every eigenvector of B is also an eigenvector of A.

    b) At least one eigenvector of B is also an eigenvector of A.

59. a) Give an example of a square real matrix that has rank 2 and all of whose eigenvalues are zero.

    b) Let A be a square real matrix all of whose eigenvalues are zero. Show that A is diagonalizable if and only if $A = 0$.

60. Let $\mathcal{P}_3$ be the linear space of polynomials $p(x)$ of degree at most 3. Give a non-trivial example of a linear map $L : \mathcal{P}_3 \to \mathcal{P}_3$ that is *nilpotent*, that is, $L^k = 0$ for some integer k. [A trivial example is the zero map: $L = 0$.]

61. Say $A \in M(n, \mathbb{F})$ has rank k. Define

$$\mathcal{L} := \{\, B \in M(n, \mathbb{F}) \mid BA = 0 \,\} \qquad \text{and} \qquad \mathcal{R} := \{\, C \in M(n, \mathbb{F}) \mid AC = 0 \,\}.$$

    Show that $\mathcal{L}$ and $\mathcal{R}$ are linear spaces and compute their dimensions.

62. Let A and B be $n \times n$ matrices.

    a) Show that the $\operatorname{rank}(AB) \le \operatorname{rank}(A)$. Give an example where strict inequality can occur.

    b) Show that $\dim(\ker AB) \ge \dim(\ker A)$. Give an example where strict inequality can occur.

63. Let $\mathcal{P}_1$ be the linear space of real polynomials of degree at most one, so a typical element is $p(x) := a + bx$, where a and b are real numbers. The derivative, $D : \mathcal{P}_1 \to \mathcal{P}_1$ is, as you should expect, the map $DP(x) = b = b + 0x$. Using the basis $e_1(x) := 1$, $e_2(x) := x$ for $\mathcal{P}_1$, we have $p(x) = ae_1(x) + be_2(x)$ so $Dp = be_1$.

    Using this basis, find the 2×2 matrix M for D. Note the obvious property $D^2 p = 0$ for any polynomial p of degree at most 1. Does M also satisfy $M^2 = 0$? Why should you have expected this?

64. Let $\mathcal{P}_2$ be the space of polynomials of degree at most 2.

    a) Find a basis for this space.

    b) Let $D : \mathcal{P}_2 \to \mathcal{P}_2$ be the derivative operator $D = d/dx$. Using the basis you picked in the previous part, write D as a matrix. Compute D^3 in this situation. Why should you have predicted this without computation?

65. Let $\mathcal{P}_3$ be the space of polynomials of degree at most 3 anD let $D : \mathcal{P}_3 \to \mathcal{P}_3$ be the derivative operator.

    a) Using the basis $e_1 = 1$, $e_2 = x$, $e_3 = x^2$, $\epsilon_4 = x^3$ find the matrix D_e representing D.

b) Using the basis $\epsilon_1 = x^3$, $\epsilon_2 = x^2$, $\epsilon_3 = x$, $\epsilon_4 = 1$ find the matrix D_ϵ representing D.

c) Show that the matrices D_e and D_ϵ are similar by finding an invertible map $S : \mathcal{P}_3 \to \mathcal{P}_3$ with the property that $D_\epsilon = S D_e S^{-1}$.

66. a) Let $\{e_1, e_2, \dots, e_n\}$ be the standard basis in $\mathbb{R}^n$ and let $\{v_1, v_2, \dots, v_n\}$ be another basis in $\mathbb{R}^n$. Find a matrix A that maps the standard basis to this other basis.

    b) Let $\{w_1, w_2, \dots, w_n\}$ be yet another basis for $\mathbb{R}^n$. Find a matrix that maps the $\{v_j\}$ basis to the $\{w_j\}$ basis. Write this matrix explicitly if both bases are orthonormal.

67. Consider the two linear transformations on the vector space $V = \mathbf{R}^n$:

    $R =$ right shift: $(x_1, \dots, x_n) \to (0, x_1, \dots, x_{n-1})$

    $L =$ left shift: $(x_1, \dots, x_n) \to (x_2, \dots, x_n, 0)$.

    Let $A \subset \operatorname{End}(V)$ be the real algebra generated by R and L. Find the dimension of A considered as a real vector space.

68. Let $\mathcal{S} \subset \mathbb{R}^3$ be the subspace spanned by the two vectors $v_1 = (1, -1, 0)$ and $v_2 = (1, -1, 1)$ and let $\mathcal{T}$ be the *orthogonal complement* of $\mathcal{S}$ (so $\mathcal{T}$ consists of all the vectors orthogonal to $\mathcal{S}$).

    a) Find an orthogonal basis for $\mathcal{S}$ and use it to find the 3×3 matrix P that projects vectors orthogonally into $\mathcal{S}$.

    b) Find an orthogonal basis for $\mathcal{T}$ and use it to find the 3×3 matrix Q that projects vectors orthogonally into $\mathcal{T}$.

    c) Verify that $P = I - Q$. How could you have seen this in advance?

69. Given a *unit* vector $\mathbf{w} \in \mathbb{R}^n$, let $W = \operatorname{span}\{\mathbf{w}\}$ and consider the linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$T(\mathbf{x}) = 2\operatorname{Proj}_W(\mathbf{x}) - \mathbf{x},$$

    where $\operatorname{Proj}_W(\mathbf{x})$ is the orthogonal projection onto W. Show that T is one-to-one.

70. [The Cross Product as a Matrix]

    a) Let $\mathbf{v} := (a, b, c)$ and $\mathbf{x} := (x, y, z)$ be any vectors in $\mathbb{R}^3$. Viewed as column vectors, find a 3×3 matrix $A_\mathbf{v}$ so that the *cross product* $\mathbf{v} \times \mathbf{x} = A_\mathbf{v}\mathbf{x}$.

    Answer:

$$\mathbf{v} \times \mathbf{x} = A_\mathbf{v}\mathbf{x} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

    where the anti-symmetric matrix $A_\mathbf{v}$ is defined by the above formula.

b) From this, one has $\mathbf{v} \times (\mathbf{v} \times \mathbf{x}) = A_{\mathbf{v}}(\mathbf{v} \times \mathbf{x}) = A_{\mathbf{v}}^2\mathbf{x}$ (why?). Combined with the cross product identity $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \langle \mathbf{u},\, \mathbf{w} \rangle \mathbf{v} - \langle \mathbf{u},\, \mathbf{v} \rangle \mathbf{w}$, show that

$$A_{\mathbf{v}}^2\mathbf{x} = \langle \mathbf{v},\, \mathbf{x} \rangle \mathbf{v} - \|\mathbf{v}\|^2\mathbf{x}.$$

c) If $\mathbf{n} = (a, b, c)$ is a *unit* vector, use this formula to show that (perhaps surprisingly) the orthogonal projection of $\mathbf{x}$ into the plane perpendicular to $\mathbf{n}$ is given by

$$\mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n} = -A_{\mathbf{n}}^2\mathbf{x} = -\begin{pmatrix} -b^2 - c^2 & ab & ac \\ ab & -a^2 - c^2 & bc \\ ac & bc & -a^2 - b^2 \end{pmatrix} \mathbf{x}$$

(See also Problems 193, 233, 234, 235, 273).

71. Let V be a vector space with $\dim V = 10$ and let $L : V \to V$ be a linear transformation. Consider $L^k : V \to V$, $k = 1, 2, 3, \ldots$. Let $r_k = \dim(\operatorname{Im} L^k)$, that is, r_k is the dimension of the image of L^k, $k = 1, 2, \ldots$.

Give an example of a linear transformation $L : V \to V$ (or show that there is no such transformation) for which:

a) $(r_1, r_2, \ldots) = (10, 9, \ldots)$; b) $(r_1, r_2, \ldots) = (8, 5, \ldots)$; c) $(r_1, r_2, \ldots) = (8, 6, 4, 4, \ldots)$.

72. Let $\mathcal{S}$ be the linear space of infinite sequences of real numbers $x := (x_1,\, x_2,\, \ldots)$. Define the linear map $L : \mathcal{S} \to \mathcal{S}$ by

$$Lx := (x_1 + x_2,\, x_2 + x_3,\; x_3 + x_4, \ldots).$$

a) Find a basis for the nullspace of L. What is its dimension?

b) What is the image of L? Justify your assertion.

c) Compute the eigenvalues of L and an eigenvector corresponding to each eigenvalue.

73. Let A be a real matrix, not necessarily square.

a) If A is onto, show that A^* is one-to-one.

b) If A is one-to-one, show that A^* is onto.

74. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a self-adjoint map (so A is represented by a symmetric matrix). Show that $\operatorname{image}(A)^\perp = \ker(A)$.

75. Let A be a real matrix, not necessarily square.

a) Show that both A^*A and AA^* are self-adjoint.

b) Show that $\ker A = \ker A^*A$. [HINT: Show separately that $\ker A \subset \ker A^*A$ and $\ker A \supset \ker A^*A$. The identity $\langle \vec{x},\, A^*A\vec{x}\rangle = \langle A\vec{x},\, A\vec{x}\rangle$ is useful.]

c) If A is one-to-one, show that A^*A is invertible

d) If A is onto, show that AA^* is invertible.

e) Show that the non-zero eigenvalues of A^*A and AA^* agree. Generalize. [See Problem 124].

76. Let $L : \mathbb{R}^n \to \mathbb{R}^k$ be a linear map. Show that

$$\dim\ker(L) - \dim(\ker L^*) = n - k.$$

Consequently, for a square matrix, $\dim\ker A = \dim\ker A^*$. [In a more general setting, $\operatorname{ind}(L) := \dim\ker(L) - \dim(\ker L^*)$ is called the *index* of a linear map L. It was studied by Atiyah and Singer for elliptic differential operators.]

77. Let $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^n$. If $\|\vec{v}\| = \|\vec{w}\|$, show there is an orthogonal matrix R with $R\vec{v} = \vec{w}$ and $R\vec{w} = \vec{v}$.

4 Rank One Matrices

78. Let $A = (a_{ij})$ be an $n \times n$ matrix whose rank is 1. Let $v := (v_1, \ldots, v_n) \neq 0$ be a basis for the image of A.

a) Show that $a_{ij} = v_i w_j$ for some vector $w := (w_1, \ldots, w_n) \neq 0$.

b) If A has a non-zero eigenvalue λ_1, show that

c) If the vector $z = (z_1, \ldots, z_n)$ satisfies $\langle z,\, w\rangle = 0$, show that z is an eigenvector with eigenvalue $\lambda = 0$.

d) If $\operatorname{trace}(A) \neq 0$, show that $\lambda = \operatorname{trace}(A)$ is an eigenvalue of A. What is the corresponding eigenvector?

e) If $\operatorname{trace}(A) \neq 0$, prove that A is similar to the $n \times n$ matrix

$$\begin{pmatrix} c & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \end{pmatrix},$$

where $c = \operatorname{trace}(A)$

f) If $\operatorname{trace}(A) = 1$, show that A is a projection, that is, $A^2 = A$.

g) What can you say if $\operatorname{trace}(A) = 0$?

h) Show that $\det(A + I) = 1 + \det A$.

79. Let A be the rank one $n \times n$ matrix $A = (v_i v_j)$, where $\vec{v} := (v_1, \ldots, v_n)$ is a non-zero real vector.

a) Find its eigenvalues and eigenvectors.

b) Find the eigenvalues and eigenvectors for $A + cI$, where $c \in \mathbb{R}$.

c) Find a formula for $(I + A)^{-1}$. [Answer: $(I + A)^{-1} = I - \frac{1}{1 + \|\vec{v}\|^2} A$.]

80. [Generalization of Problem 79(b)] Let W be a linear space with an inner product and $A : W \to W$ be a linear map whose image is one dimensional (so in the case of matrices, it has rank one). Let $\vec{v} \neq 0$ be in the image of A, so it is a basis for the image. If $\langle \vec{v}, (I + A)\vec{v} \rangle \neq 0$, show that $I + A$ is invertible by finding a formula for the inverse.

Answer: The solution of $(I + A)\vec{x} = \vec{y}$ is $\vec{x} = \vec{y} - \dfrac{\|\vec{v}\|^2}{\|\vec{v}\|^2 + \langle \vec{v}, A\vec{v} \rangle} A\vec{y}$ so

$$(I + A)^{-1} = I - \frac{\|\vec{v}\|^2}{\|\vec{v}\|^2 + \langle \vec{v}, A\vec{v} \rangle} A.$$

5 Algebra of Matrices

81. Which of the following are *not* a basis for the vector space of all symmetric 2×2 matrices? Why?

a) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -2 & -2 \\ -2 & 1 \end{pmatrix}$

e) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

f) $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, $\begin{pmatrix}-1 & 2\\ 2 & -1\end{pmatrix}$, $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

82. For each of the sets $\mathcal{S}$ below, determine if it is a linear *subspace* of the given real vector space V. If it is a subspace, write down a basis for it.

a) $V = \mathrm{Mat}_{3\times 3}(\mathbb{R})$, $\mathcal{S} = \{A \in V \mid \mathrm{rank}(A) = 3\}$.

b) $V = \mathrm{Mat}_{2\times 2}(\mathbb{R})$, $\mathcal{S} = \{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in V \mid a + d = 0\}$.

83. Every real upper triangular $n\times n$ matrix (a_{ij}) with $a_{ii} = 1$, $i = 1, 2, \ldots, n$ is invertible. Proof or counterexample.

84. Let $L\colon V \to V$ be a linear map on a vector space V.

a) Show that $\ker L \subset \ker L^2$ and, more generally, $\ker L^k \subset \ker L^{k+1}$ for all $k \ge 1$.

b) If $\ker L^j = \ker L^{j+1}$ for some integer j, show that $\ker L^k = \ker L^{k+1}$ for all $k \ge j$. Does your proof require that V is finite dimensional?

c) Let A be an $n \times n$ matrix. If $A^j = 0$ for some integer j (perhaps $j > n$), show that $A^n = 0$.

85. Let $L\colon V \to V$ be a linear map on a vector space V and $z \in V$ a vector with the property that $L^{k-1}z \neq 0$ but $L^k z = 0$. Show that z, Lz, $\ldots$ $L^{k-1}z$ are linearly independent.

86. Let A, B, and C be any $n \times n$ matrices.

a) Show that $\mathrm{trace}(AB) = \mathrm{trace}(BA)$.

b) Show that $\mathrm{trace}(ABC) = \mathrm{trace}(CAB) = \mathrm{trace}(BCA)$.

c) $\mathrm{trace}(ABC) \stackrel{?}{=} \mathrm{trace}(BAC)$. Proof or counterexample.

87. There are no square matrices A, B with the property that $AB - BA = I$. Proof or counterexample.

Remark: In quantum physics, the operators $Au = du/dx$ and $Bv(x) = xv(x)$ do satisfy $(AB - BA)w = w$.

88. Let A and B be $n \times n$ matrices. If $A + B$ is invertible, show that $A(A + B)^{-1}B = B(A + B)^{-1}A$. [Don't assume that $AB = BA$].

89. Let A be an $n \times n$ matrix. If $AB = BA$ for *all* invertible matrices B, show that $A = cI$ for some scalar c.

90. a) For non-zero real numbers one uses $\dfrac{1}{a} - \dfrac{1}{b} = \dfrac{b-a}{ab}$. Verify the following analog for invertible matrices A, B:

$$A^{-1} - B^{-1} = A^{-1}(B-A)B^{-1}.$$

The following version is also correct

$$A^{-1} - B^{-1} = B^{-1}(B-A)A^{-1}.$$

b) Let $A(t)$ be a family of invertible real matrices depending smoothly on the real parameter t and assume they are invertible. Show that the inverse matrix $A^{-1}(t)$ is invertible and give a formula for the derivative of $A^{-1}(t)$ in terms of $A'(t)$ and $A^{-1}(t)$. Thus one needs to investigate

$$\lim_{h \to 0} \frac{A^{-1}(t+h) - A^{-1}(t)}{h}.$$

91. Let $A(t)$ be a family of real square matrices depending smoothly on the real parameter t.

a) Find a formula for the derivative of $A^2(t)$.

b) Find an example where $A^2(t)' \neq 2AA'$.

92. Let $A : \mathbb{R}^\ell \to \mathbb{R}^n$ and $B : \mathbb{R}^k \to \mathbb{R}^\ell$. Prove that

$$\operatorname{rank} A + \operatorname{rank} B - \ell \leq \operatorname{rank} AB \leq \min\{\operatorname{rank} A, \operatorname{rank} B\}.$$

[Hint: Observe that $\operatorname{rank}(AB) = \operatorname{rank} A|_{\mathrm{Image}\,(B)}$.]

93. a) Let $U \subset V$ and W be finite dimensional linear spaces and $L : V \to W$ a linear map. Show that

$$\dim(\ker L|_U) \leq \dim \ker L = \dim V - \dim \operatorname{Im}(L)$$

b) [Frobenius] Let A, B, and C be matrices so that the products AB and BC are defined. Use the obvious

$$\dim(\ker A|_{\operatorname{Im} BC}) = \dim \operatorname{Im} BC - \dim \operatorname{Im} ABC$$

and the previous part to show that

$$\operatorname{rank}(BC) + \operatorname{rank}(AB) \leq \operatorname{rank}(ABC) + \operatorname{rank}(B).$$

6 Eigenvalues and Eigenvectors

94. a) Find a 2×2 real matrix A that has an eigenvalue $\lambda_1 = 1$ with eigenvector $E_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and an eigenvalue $\lambda_2 = -1$ with eigenvector $E_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

    b) Compute the determinant of $A^{10} + A$.

95. Give an example of a matrix A with the following three properties:

    i). A has eigenvalues -1 and 2.

    ii). The eigenvalue -1 has eigenvector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

    iii). The eigenvalue 2 has eigenvectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

96. Let A be a square matrix. If the eigenvectors $v_1, \ldots v_k$ have distinct eigenvalues, show that these vectors are linearly independent.

97. Let A be an invertible matrix with eigenvalues λ_1, λ_2, $\ldots, \lambda_k$ and corresponding eigenvectors $\vec{v}_1$, $\vec{v}_2$, $\ldots, \vec{v}_k$. What can you say about the eigenvalues and eigenvectors of A^{-1}? Justify your response.

98. Two matrices A, B can be *simultaneously diagonalized* if there is an invertible matrix that diagonalizes both of them. In other words, if there is a basis in which both matrices are diagonalized.

    a) If A and B can be simultaneously diagonalized, show that $AB = BA$.

    b) Conversely, if $AB = BA$, and if one of these matrices, say A, has *distinct* eigenvalues (so the eigenspaces all have dimension one), show they can be simultaneously diagonalized.

    SUGGESTION: Say λ is an eigenvalue of A and $\vec{v}$ a corresponding eigenvector: $A\vec{v} = \lambda\vec{v}$. Show that $B\vec{v}$ satisfies $A(B\vec{v}) = \lambda B\vec{v}$ and deduce that $B\vec{v} = c\vec{v}$ for some constant c (possibly zero). Thus, the eigenvectors of A are also eigenvectors of B. Why does this imply that (in this case where A has distinct eigenvalues) in a basis where A is diagonal, so is B?

    REMARK: This result extends to any two commuting $n \times n$ matrices A and B, assuming that A and B can each be diagonalized.

99. Let A be an $n \times n$ real self-adjoint matrix and $\mathbf{v}$ an eigenvector with eigenvalue λ. Let $W = \operatorname{span}\{\mathbf{v}\}$.

a) If $\mathbf{w} \in W$, show that $A\mathbf{w} \in W$

b) If $\mathbf{z} \in W^{\perp}$, show that $A\mathbf{z} \in W^{\perp}$.

100. Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$.

a) What is the dimension of the image of A? Why?

b) What is the dimension of the kernel of A? Why?

c) What are the eigenvalues of A? Why?

d) What are the eigenvalues of $B := \begin{pmatrix} 4 & 1 & 2 \\ 1 & 4 & 2 \\ 1 & 1 & 5 \end{pmatrix}$? Why? [Hint: $B = A + 3I$].

101. Diagonalize the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

by finding the eigenvalues of A listed in increasing order, the corresponding eigenvectors, a diagonal matrix D, and a matrix P such that $A = PDP^{-1}$.

102. If a matrix A is diagonalizable, show that for any scalar c so is the matrix $A + cI$.

103. Let $A = \begin{pmatrix} a & b-a \\ 0 & b \end{pmatrix}$

a) Diagonalize A.

b) Use this to compute A^k for any integer $k \geq 0$.

104. An $n \times n$ matrix is called *nilpotent* if A^k equals the zero matrix for some positive integer k. (For instance, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is nilpotent.)

a) If λ is an eigenvalue of a nilpotent matrix A, show that $\lambda = 0$. (Hint: start with the equation $A\vec{x} = \lambda\vec{x}$.)

b) Show that if A is both nilpotent and diagonalizable, then A is the zero matrix. [Hint: use Part a).]

c) Let A be the matrix that represents $T : \mathcal{P}_5 \to \mathcal{P}_5$ (polynomials of degree at most 5) given by differentiation: $Tp = dp/dx$. Without doing any computations, explain why A must be nilpotent.

105. Identify which of the following matrices have two linearly independent eigenvectors.

$$A=\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad B=\begin{pmatrix}1&1\\0&1\end{pmatrix},\qquad C=\begin{pmatrix}1&1\\0&2\end{pmatrix},\qquad D=\begin{pmatrix}1&1\\2&2\end{pmatrix},$$

$$E=\begin{pmatrix}0&0\\1&0\end{pmatrix},\qquad F=\begin{pmatrix}0&0\\0&0\end{pmatrix},\qquad G=\begin{pmatrix}1&2\\2&0\end{pmatrix},\qquad H=\begin{pmatrix}3&0\\1&-3\end{pmatrix}.$$

106. Find an orthogonal matrix R that diagonalizes $A:=\begin{pmatrix}1&-1&0\\-1&1&0\\0&0&2\end{pmatrix}$

107. This problem is a rich source of classroom examples that are computationally simple.

Let a, b, c, d, and e be real numbers. For each of the following matrices, find their eigenvalues, corresponding eigenvectors, and orthogonal matrices that diagonalize them.

$$A=\begin{pmatrix}a&b\\b&a\end{pmatrix},\qquad B=\begin{pmatrix}a&b&0\\b&a&0\\0&0&c\end{pmatrix},\qquad C=\begin{pmatrix}a&b&0&0&0\\b&a&0&0&0\\0&0&c&d&0\\0&0&d&c&0\\0&0&0&0&e\end{pmatrix}.$$

108. Let A be a square matrix. Proof or Counterexample.

a) If A is diagonalizable, then so is A^2.

b) If A^2 is diagonalizable, then so is A.

109. Let A be an $m\times n$ matrix, and suppose $\vec{v}$ and $\vec{w}$ are orthogonal eigenvectors of A^TA. Show that $A\vec{v}$ and $A\vec{w}$ are orthogonal.

110. Let A be an invertible matrix. If $\mathbf{V}$ is an eigenvector of A, show it is also an eigenvector of both A^2 and A^{-2}. What are the corresponding eigenvalues?

111. True or False – and Why?.

a) A 3×3 real matrix need not have any real eigenvalues.

b) If an $n\times n$matrix A is invertible, then it is diagonalizable.

c) If A is a 2×2 matrix both of whose eigenvalues are 1, then A is the identity matrix.

d) If $\vec{v}$ is an eigenvector of the matrix A, then it is also an eigenvector of the matrix $B:=A+7I$.

112. Let L be an $n \times n$ matrix with real entries and let λ be an eigenvalue of L. In the following list, identify all the assertions that are correct.

a) $a\lambda$ is an eigenvalue of aL for any scalar a.

b) λ^2 is an eigenvalue of L^2.

c) $\lambda^2 + a\lambda + b$ is an eigenvalue of $L^2 + aL + bI_n$ for all real scalars a and b.

d) If $\lambda = a + ib$, with a, $b \neq 0$ some real numbers, is an eigenvalue of L, then $\bar{\lambda} = a - ib$ is also an eigenvalue of L.

113. Let C be a 2×2 matrix of real numbers. Give a proof or counterexample to each of the following assertions:

a) $\det(C^2)$ is non-negative.

b) $\operatorname{trace}(C^2)$ is non-negative.

c) All of the elements of C^2 are non-negative.

d) All the eigenvalues of C^2 are non-negative.

e) If C has two distinct eigenvalues, then so does C^2.

114. Let $A \in M(n, \mathbb{F})$ have an eigenvalue λ with corresponding eigenvector v.

True or False

a) $-v$ is an eigenvector of $-A$ with eigenvalue $-\lambda$.

b) If v is also an eigenvector of $B \in M(n, \mathbb{F})$ with eigenvalue μ, then $\lambda\mu$ is an eigenvalue of AB.

c) Let $c \in \mathbb{F}$. Then $(\lambda + c)^2$ is an eigenvalue of $A^2 + 2cA + c^2 I$.

d) Let μ be an eigenvalue of $B \in M(n, \mathbb{F})$, Then $\lambda + \mu$ is an eigenvalue of $A + B$.

e) Let $c \in \mathbb{F}$. Then $c\lambda$ is an eigenvalue of cA.

115. Suppose that A is a 3×3 matrix with eigenvalues $\lambda_1 = -1$, $\lambda_2 = 0$ and $\lambda_3 = 1$, and corresponding eigenvectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \qquad \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

a) Find the matrix A .

b) Compute the matrix A^{20}.

116. Let A be a square matrix and $p(\lambda)$ any polynomial. If $\lambda)$ is an eigenvalue of A, show that $p(\lambda)$ is an eigenvalue of the matrix $p(A)$ with the same eigenvector.

117. Let $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ be the standard basis for $\mathbb{R}^3$ and let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation with the properties

$$L(\vec{e}_1) = \vec{e}_2, \qquad L(\vec{e}_2) = 2\vec{e}_1 + \vec{e}_2, \qquad L(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) = \vec{e}_3.$$

Find a vector $\vec{v}$ such that $L(\vec{v}) = k\vec{v}$ for some real number k.

118. Let M be a 2×2 matrix with the property that the sum of each of the rows and also the sum of each of the columns is the *same* constant c. Which (if any) any of the vectors

$$U := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad V := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad W := \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

must be an eigenvector of M?

119. Let A and B be $n \times n$ complex matrices that commute: $AB = BA$. If λ is an eigenvalue of A, let $\mathcal{V}_\lambda$ be the subspace of all eigenvectors having this eigenvalue.

a) Show there is an vector $v \in \mathcal{V}_\lambda$ that is also an eigenvector of B, possibly with a different eigenvalue.

b) Give an example showing that some vectors in $\mathcal{V}_\lambda$ may not be an eigenvectors of B.

c) If all the eigenvalues of A are distinct (so each has algebraic multiplicity one), show that there is a basis in which both A and B are diagonal. Also, give an example showing this may be false if some eigenvalue of A has multiplicity greater than one.

120. Let A be a 3×3 matrix with eigenvalues λ_1, λ_2, λ_3 and corresponding linearly independent eigenvectors V_1, V_2, V_3 which we can therefore use as a basis.

a) If $X = aV_1 + bV_2 + cV_3$, compute AX, A^2X, and $A^{35}X$ in terms of λ_1, λ_2, λ_3, V_1, V_2, V_3, a, b and c (only).

b) If $\lambda_1 = 1$, $|\lambda_2| < 1$, and $|\lambda_3| < 1$, compute $\lim_{k\to\infty} A^k X$. Explain your reasoning clearly.

121. Let Z be a complex square matrix whose self-adjoint part is positive definite, so $Z + Z^*$ is positive definite.

a) Show that the eigenvalues of Z have positive real part.

b) Is the converse true? Proof or counterexample.

122. The *characteristic polynomial* of a square matrix is the polynomial $p(\lambda) = \det(\lambda I - A)$.

a) If two square matrices are similar, show that they have the same characteristic polynomials.

b) Conversely, if two matrices have the same characteristic polynomials, are they similar? Proof or counterexample.

123. Show that $p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$ is the *characteristic polynomial* of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix},$$

that is, $\det(\lambda I - A) = p(\lambda)$. In particular, every polynomial is the characteristic polynomial of a matrix.

If λ is an eigenvalue of A show that $(1, \lambda, \lambda^2, \ldots, \lambda^{n-1})$ is a corresponding eigenvector.

The above matrix is called the *matrix of* $p(\lambda)$. It arises when writing a linear nth order ordinary differential equation as a system of first order equations. Gershgorin's Theorem (Problem 134) can be applied to the matrix to find simple, but useful, estimates of the roota of any polynomial.

124. Let A be an $n \times k$ matrix and B a $k \times n$ matrix. Then both AB and BA are square matrices.

a) If $\lambda \neq 0$ is an eigenvalue of AB, show it is also an eigenvalue of BA. In particular, the non-zero eigenvalues of A^*A and AA^* agree.

b) If $v_1, \ldots, v_k$ are linearly independent eigenvectors of BA corresponding to the same eigenvalue, $\lambda \neq 0$, show that $Av_1, \ldots, Av_k$ are linearly independent eigenvectors of AB corresponding to λ. Thus the eigenspaces of AB and BA corresponding to a non-zero eigenvalue have the same geometric multiplicity.

c) (This gives a sharper result to the first part) We seek a formula relating the characteristic polynomials $p_{AB}(\lambda)$ of AB and $p_{BA}(\lambda)$ of BA, respectively. Show that

$$\lambda^k p_{AB}(\lambda) = \lambda^n p_{BA}(\lambda).$$

In particular if A and B are square, then AB and BA have the same characteristic polynomial. [Suggestion: One approach uses *block matrices*: let $P = \begin{pmatrix} \lambda I_n & A \\ B & I_k \end{pmatrix}$ and $Q = \begin{pmatrix} I_n & 0 \\ -B & \lambda I_k \end{pmatrix}$, where I_m is the $m \times m$ identity matrix. Then use $\det(PQ) = \det(QP)$.]

125. Let $\alpha_1, \ldots, \alpha_n$ be positive real numbers and let $A = (a_{ij})$ where $a_{ij} = \alpha_i/\alpha_j$. Say as much as you can about the eigenvalues and eigenvectors of A.

126. Compute the value of the determinant of the 3×3 complex matrix X, provided that $\operatorname{tr}(X) = 1$, $\operatorname{tr}(X^2) = -3$, $\operatorname{tr}(X^3) = 4$. [Here $\operatorname{tr}(A)$ denotes the the *trace*, that is, the sum of the diagonal entries of the matrix A.]

127. Let $A := \begin{pmatrix} 4 & 4 & 4 \\ -2 & -3 & -6 \\ 1 & 3 & 6 \end{pmatrix}$. Compute

a) the characteristic polynomial,

b) the eigenvalues,

c) *one* of the corresponding eigenvectors.

128. Let A be a square matrix. In the following, a sequence of matrices C_j *converges* if all of its elements converge.
Prove that the following are equivalent:
(i) $A^k \to 0$ as $k \to \infty$ [each of the elements of A^k converge to zero].
(ii) All the eigenvalues λ_j of A have $|\lambda_j| < 1$.
(iii) The matrix geometric series $\sum_0^\infty A^k$ converges to $(I - A)^{-1}$.

129. Let A be a square matrix and let $\|B\|$ be any norm on matrices [one example is $\|B\| = \max_{i,j} |b_{ij}|$]. To what extent are the conditions in the previous problem also equivalent to the condition that $\|A^k\| \to 0$?

130. a) Prove that the set of invertible real 2×2 matrices is dense in the set of all real 2×2 matrices.

b) The set of diagonalizable 2×2 matrices dense in the set of all real 2×2 matrices. Proof or counterexample?

131. a) Identify all possible eigenvalues of an $n \times n$ matrix A that satisfies the matrix equation: $A - 2I = -A^2$. Justify your answer.

b) Must A be invertible?

132. [Spectral Mapping Theorem] Let A be a square matrix.

a) If $A(A - I)(A - 2I) = 0$, show that the only possible eigenvalues of A are $\lambda = 0$, $\lambda = 1$, and $\lambda = 2$.

b) Let p any polynomial. Show that the eigenvalues of the matrix $p(A)$ are precisely the numbers $p(\lambda_j)$, where the λ_j are the eigenvalues of A.

133. Let $A = (a_{ij})$ be an $n \times n$ matrix with the property that its absolute row sums are at most 1, that is, $|a_{i1}| + \cdots + |a_{in}| \le 1$ for all $i = 1, \ldots, n$. Show that all of its (possibly complex) eigenvalues are in the unit disk: $|\lambda| \le 1$.

[Suggestion: Let $v = (v_1, \ldots, v_n) \neq 0$ be an eigenvector and say v_k is the largest component, that is, $|v_k| = \max_{j=1,\ldots,n} |v_j|$. Then use the k^{th} row of $\lambda v = Av$, that is, $\lambda v_k = a_{k1}v_1 + \cdots + a_{kn}v_n$].

Remark: This is a special case of: "for any matrix norm, if $\|A\| < 1$ then $I - A$ is invertible." However, the proof of this special case can be adapted to give a deeper estimate for the eigenvalues of a matrix. See Problem 134

134. [Gershgorin] Let $A = (a_{ij})$ be an $n \times n$ matrix with eigenvalue λ (possible complex) and corresponding eigenvector $v = (v_1, \ldots, v_n) \neq 0$ and say v_k is the largest component of v, that is, $|v_k| = \max_{j=1,\ldots,n}|v_j|$. Then the k^{th} row of $Av = \lambda v$ can be written as

$$(\lambda - a_{kk})v_k = \sum_{j \neq k} a_{kj} v_j .$$

a) Show that $\quad |\lambda - a_{kk}| \le \sum_{j \neq k} |a_{kj}|$, That is, λ lies in the disk D_k in thw complex plane centered at a_{kk} with radius $R_k := \sum_{j \neq k} |a_{kj}|$. These are called *Gershgorin disks*. Although k is unknown, every eigenvalue of A must lie in at least one of these disks.

b) The same eigenvalue estimate is true if we use the absolute column sums. Why?

c) Use Gershgorin disks to crudely estimate the eigenvalues of $\begin{pmatrix} 10+2i & 1 & 1 \\ 2 & 4 & 1 \\ 0 & 0.1 & 7i \end{pmatrix}$.

d) For the tridiagonal matrix in Problem 285 with $\alpha = 2$ and $\beta = 1$ compare the exact eigenvalues with the estimate found using Gershgorin disks.

e) Although each eigenvalue is in some Gershgorin disk, some disks may not contain an eigenvalue. Show how this is illustrated by the matrices $\left(\begin{smallmatrix} 0 & 1 \\ 4 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & -2 \\ 1 & -1 \end{smallmatrix}\right)$ [1] Another instructive example is $\left(\begin{smallmatrix} 1 & -1 \\ 1 & -1 \end{smallmatrix}\right)$.

f) Say one of the Gershgorin disks does not intersect any of of the others. Show that this disk has exactly one eigenvalue. [Suggestion: Let C be the diagonal matrix consisting of the diagonal elements of A and let

$$A(t) = (1-t)C + tA = C + t(A - C), \qquad 0 \le t \le 1,$$

[1] https://en.wikipedia.org/wiki/Gershgorin_circle_theorem

so $A(0) = C$ and $A(1) = A$. Use that the eigenvalues of $A(t)$ depend continuously on t (Problem 298).]

g) For any polynomial $p(z)$ apply Gershgorin's theorem to the companion matrix (Problem 123) to estimate the roots of the polynomial.

Remark: In practice, a valuable numerical method for estimating the eigenvalues of a matrix does *not* attempt to find the roots of the charachteristic polynomial, but instead finds a sequence of similar matrices (so they have the same eigenvalues as A) whose off-diagonal elements get very small and use Gershgorin's estimate.

135. Say the matrix A has a simple eigenvalue λ_0 and corresponding eigenvector $\vec{v}_0$. Show there is no vector $\vec{z}$ linearly independent of $\vec{v}_0$ with the property $A\vec{z} = \lambda_0\vec{z} + c\vec{v}_0$ for any value of the scalar c.

7 Inner Products and Quadratic Forms

136. Let V, W be vectors in the plane $\mathbb{R}^2$ with lengths $\|V\| = 3$ and $\|W\| = 5$. What are the maxima and minima of $\|V + W\|$? When do these occur?

137. Let V, W be vectors in $\mathbb{R}^n$.

a) Show that the Pythagorean relation $\|V + W\|^2 = \|V\|^2 + \|W\|^2$ holds if and only if V and W are orthogonal.

b) Prove the *parallelogram identity* $\|V + W\|^2 + \|V - W\|^2 = 2\|V\|^2 + 2\|W\|^2$ and interpret it geometrically. [This is true in any inner product space].

138. Prove *Thales' Theorem*: an angle inscribed in a semicircle is a right angle. Prove the converse: given a right triangle whose vertices lie on a circle, then the hypotenuse is a diameter of the circle.

[Remark: Both Thales' theorem and its converse are valid in any inner product space].

139. Let $A = (-6, 3)$, $B = (2, 7)$, and C be the vertices of a triangle. Say the altitudes through the vertices A and B intersect at $Q = (2, -1)$. Find the coordinates of C.

[The *altitude* through a vertex of a triangle is a straight line through the vertex that is perpendicular to the opposite side — or an extension of the opposite side. Although not needed here, the three altitudes always intersect in a single point, sometimes called the orthocenter of the triangle.]

140. Find all vectors in the plane (through the origin) spanned by $\mathbf{V} = (1, 1-2)$ and $\mathbf{W} = (-1, 1, 1)$ that are perpendicular to the vector $\mathbf{Z} = (2, 1, 2)$.

141. For real $c > 0$, $c \neq 1$, and distinct points $\vec{p}$ and $\vec{q}$ in $\mathbb{R}^k$, consider the points $\vec{x} \in \mathbb{R}^k$ that satisfy

$$\|\vec{x} - \vec{p}\| = c\|\vec{x} - \vec{q}\|.$$

Show that these points lie on a sphere, say $\|\vec{x} - \vec{x}_0\| = r$, so the center is at $\vec{x}_0$ and the radius is r. Thus, find center and radius of this sphere in terms of $\vec{p}$, $\vec{q}$ and c.

What if $c = 1$?

142. In $\mathbb{R}^3$, let N be a non-zero vector and X_0 and P points.

a) Find the equation of the plane through the origin that is orthogonal to N, so N is a normal vector to this plane.

b) Compute the distance from the point P to the origin.

c) Find the equation of the plane parallel to the above plane that passes through the point X_0.

d) Find the distance between the parallel planes in parts a) and c).

e) Let S be the sphere centered at P with radius r. For which value(s) of r is this sphere tangent to the plane in part c)??

143. Let U, V, W be orthogonal vectors and let $Z = aU + bV + cW$, where a, b, c are scalars.

a) (Pythagoras) Show that $\|Z\|^2 = a^2\|U\|^2 + b^2\|V\|^2 + c^2\|W\|^2$.

b) Find a formula for the coefficient a in terms of U and Z only. Then find similar formulas for b and c. [Suggestion: take the inner product of $Z = aU + bV + cW$ with U].

Remark The resulting simple formulas are one reason that orthogonal vectors are easier to use than more general vectors. This is vital for Fourier series.

c) Solve the following equations:

$$\begin{aligned} x + y + z + w &= 2 \\ x + y - z - w &= 3 \\ x - y + z - w &= 0 \\ x - y - z + w &= -5 \end{aligned}$$

[Suggestion: Observe that the columns vectors in the coefficient matrix are orthogonal.]

144. For certain polynomials $\mathbf{p}(t)$, $\mathbf{q}(t)$, and $\mathbf{r}(t)$, say we are given the following table of inner products:

$\langle\ ,\ \rangle$	$\mathbf{p}$	$\mathbf{q}$	$\mathbf{r}$
$\mathbf{p}$	4	0	8
$\mathbf{q}$	0	1	0
$\mathbf{r}$	8	0	50

For example, $\langle \mathbf{q}, \mathbf{r}\rangle = \langle \mathbf{r}, \mathbf{q}\rangle = 0$. Let E be the span of $\mathbf{p}$ and $\mathbf{q}$.

a) Compute $\langle \mathbf{p}, \mathbf{q}+\mathbf{r}\rangle$.

b) Compute $\|\mathbf{q}+\mathbf{r}\|$.

c) Find the orthogonal projection $\mathrm{Proj}_E\mathbf{r}$. [Express your solution as linear combinations of $\mathbf{p}$ and $\mathbf{q}$.]

d) Find an orthonormal basis of the span of $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$. [Express your results as linear combinations of $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$.]

145. Let V be the real vector space of continuous real-valued functions on the closed interval $[0,1]$, and let $w \in V$. For $p, q \in V$, define $\langle p, q\rangle = \int_0^1 p(x)q(x)w(x)\,dx$.

a) Suppose that $w(a) > 0$ for all $a \in [0,1]$. Does it follow that the above defines an inner product on V? Justify your assertion.

b) Does there exist a choice of w such that $w(1/2) < 0$ and such that the above defines an inner product on V? Justify your assertion.

146. Let $w(x)$ be a positive continuous function on the interval $0 \le x \le 1$, n a positive integer, and $\mathcal{P}_n$ the vector space of polynomials $p(x)$ whose degrees are at most n equipped with the inner product

$$\langle p, q\rangle = \int_0^1 p(x)q(x)w(x)\,dx.$$

a) Prove that $\mathcal{P}_n$ has an orthonormal basis p_0, $p_1, \ldots, p_n$ with the degree of p_k is k for each k.

b) Prove that $\langle p_k, p_k'\rangle = 0$ for each k.

147. [Linear functionals] In R^n with the usual inner product, a *linear functional* $\ell : \mathbb{R}^n \to \mathbb{R}$ is just a linear map into the reals (in a complex vector space, it maps into the complex numbers $\mathbb{C}$). Define the norm, $\|\ell\|$, as

$$\|\ell\| := \max_{\|x\|=1} |\ell(x)|.$$

a) Show that the set of linear functionals with this norm is a normed linear space.

b) If $v \in \mathbb{R}^n$ is a given vector, define $\ell(x) = \langle x, v \rangle$. Show that ℓ is a linear functional and that $\|\ell\| = \|v\|$.

c) [REPRESENTATION OF A LINEAR FUNCTIONAL] Let ℓ be any linear functional. Show there is a unique vector $v \in \mathbb{R}^n$ so that $\ell(x) := \langle x, v \rangle$.

d) [EXTENSION OF A LINEAR FUNCTIONAL] Let $U \subset \mathbb{R}^n$ be a subspace of $\mathbb{R}^n$ and ℓ a linear functional defined on U with norm $\|\ell\|_U$. Show there is a unique extension of ℓ to $\mathbb{R}^n$ with the property that $\|\ell\|_{\mathbb{R}^n} = \|\ell\|_U$.

[In other words define ℓ on all of $\mathbb{R}^n$ so that on U this extended definition agrees with the original definition and so that its norm is unchanged].

148. a) Let A be a positive definite $n \times n$ real matrix, $b \in \mathbb{R}^n$, and consider the quadratic polynomial

$$Q(x) := \tfrac{1}{2}\langle x, Ax \rangle - \langle b, x \rangle.$$

Show that Q is bounded below, that is, there is a constant m so that $Q(x) \geq m$ for all $x \in \mathbb{R}^n$.

b) Show that Q blows up at infinity by showing that there are positive constants R and c so that if $\|x\| \geq R$, then $Q(x) \geq c\|x\|^2$.

c) If $x_0 \in \mathbb{R}^n$ minimizes Q, show that $Ax_0 = b$. [Moral: One way to solve $Ax = b$ is to minimize Q.]

d) Give an example showing that if A is only positive semi-definite, then $Q(x)$ may not be bounded below.

149. Let A be a square matrix of real numbers whose columns are (non-zero) orthogonal vectors.

a) Show that $A^T A$ is a diagonal matrix — whose inverse is thus obvious to compute.

b) Use this observation (or any other method) to discover a simple general formula for the inverse, A^{-1} involving only its transpose, A^T, and $(A^T A)^{-1}$. In the special case where the columns of A are orthonormal, your formula should reduce to $A^{-1} = A^T$.

c) Apply this to again solve the equations in Problem (143c).

150. [Gram-Schmidt Orthogonalization]

a) Let $A := \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Briefly show that the bilinear map $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ defined by $(x, y) \mapsto x^T A y$ gives a scalar product.

b) Let $\alpha : \mathbb{R}^3 \to \mathbb{R}$ be the linear functional $\alpha : (x_1, x_2, x_3) \mapsto x_1 + x_2$ and let $v_1 := (-1, 1, 1)$, $v_2 := (2, -2, 0)$ and $v_3 := (1, 0, 0)$ be a basis of $\mathbb{R}^3$. Using the scalar product of the previous part, find an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ with $e_1 \in \operatorname{span}\{v_1\}$ and $e_2 \in \ker \alpha$.

151. Let $A : \mathbb{R}^n \to \mathbb{R}^k$ be a linear map defined by the matrix A. If the matrix B satisfies the relation $\langle AX, Y\rangle = \langle X, BY\rangle$ for all vectors $X \in \mathbb{R}^n$, $Y \in \mathbb{R}^k$, show that B is the transpose of A, so $B = A^T$. [This basic property of the transpose,

$$\langle AX, Y\rangle = \langle X, AY\rangle,$$

is the *only* reason the transpose is important.]

152. Let V be the linear space of $n \times n$ matrices with real entries. Define a linear transformation $T : V \to V$ by the rule $T(A) = \frac{1}{2}(A + A^T)$. [Here A^T is the matrix transpose of A.]

a) Verify that T is linear.

b) Describe the image of T and find it's dimension. [Try the cases $n = 2$ and $n = 3$ first.]

c) Describe the image of T and find it's dimension.

d) Verify that the rank and nullity add up to what you would expect. [Note: This map T is called the *symmetrization operator*.]

153. Proof or counterexample. Here v, w, z are vectors in a real inner product space H.

a) Let v, w, z be vectors in a real inner product space. If $\langle v, w\rangle = 0$ and $\langle v, z\rangle = 0$, then $\langle w, z\rangle = 0$.

b) If $\langle v, z\rangle = \langle w, z\rangle$ for all $z \in H$, then $v = w$.

c) If A is an $n \times n$ symmetric matrix then A is invertible.

154. In $\mathbb{R}^4$, compute the distance from the point $(1, -2, 0, 3)$ to the hyperplane $x_1 + 3x_2 - x_3 + x_4 = 3$.

155. Find the (orthogonal) projection of $\mathbf{x} := (1, 2, 0)$ into the following subspaces:

a) The line spanned by $\mathbf{u} := (1, 1, -1)$.

b) The plane spanned by $\mathbf{u} := (0, 1, 0)$ and $\mathbf{v} := (0, 0, -2)$

c) The plane spanned by $\mathbf{u} := (0, 1, 1)$ and $\mathbf{v} := (0, 1, -2)$

d) The plane spanned by $\mathbf{u} := (1, 0, 1)$ and $\mathbf{v} := (1, 1, -1)$

e) The plane spanned by $\mathbf{u} := (1,0,1)$ and $\mathbf{v} := (2,1,0)$.

f) The subspace spanned by $\mathbf{u} := (1,0,1)$, $\mathbf{v} := (2,1,0)$ and $\mathbf{w} := (1,1,0)$.

156. Let $\mathcal{S} \subset \mathbb{R}^4$ be the vectors $X = (x_1, x_2, x_3, x_4)$ that satisfy $x_1 + x_2 - x_3 + x_4 = 0$.

a) What is the dimension of $\mathcal{S}$?

b) Find a basis for the orthogonal complement of $\mathcal{S}$.

157. Let $\mathcal{S} \subset \mathbb{R}^4$ be the subspace spanned by the two vectors $v_1 = (1,-1,0,1)$ and $v_2 = (0,0,1,0)$ and let $\mathcal{T}$ be the orthogonal complement of $\mathcal{S}$.

a) Find an orthogonal basis for $\mathcal{T}$.

b) Compute the orthogonal projection of $(1,1,1,1)$ into $\mathcal{S}$.

158. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map with the property that $L\mathbf{v} \perp \mathbf{v}$ for every $\mathbf{v} \in \mathbb{R}^3$. Prove that L cannot be invertible.
Is a similar assertion true for a linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$?

159. In a complex vector space (with a hermitian inner product), if a matrix A satisfies $\langle X,\, AX\rangle = 0$ for all vectors X, show that $A = 0$. [The previous problem shows that this is false in a real vector space].

160. Using the inner product $\langle f,\, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$, for which values of the real constants α, β, γ are the quadratic polynomials $p_1(x) = 1$, $p_2(x) = \alpha + x$ $p_3(x) = \beta + \gamma x + x^2$ orthogonal? [ANSWER: $p_2(x) = x$, $p_3(x) = x^2 - 1/3$.]

161. Using the inner product of the previous problem, let $\mathcal{B} = \{1,\, x,\, 3x^2 - 1\}$ be an orthogonal basis for the space $\mathcal{P}_2$ of quadratic polynomials and let $\mathcal{S} = \operatorname{span}(x,\, x^2) \subset \mathcal{P}_2$. Using the basis $\mathcal{B}$, find the linear map $P : \mathcal{P}_2 \to \mathcal{P}_2$ that is the orthogonal projection from $\mathcal{P}_2$ onto $\mathcal{S}$.

162. Let $\mathcal{P}_2$ be the space of quadratic polynomials.

a) Show that $\langle f,\, g\rangle = f(-1)g(-1) + f(0)g(0) + f(1)g(1)$ is an inner product for this space.

b) Using this inner product, find an orthonormal basis for $\mathcal{P}_2$.

c) Is this also an inner product for the space $\mathcal{P}_3$ of polynomials of degree at most three? Why?

163. Let $\mathcal{P}_2$ be the space of polynomials $p(x) = a + bx + cx^2$ of degree at most 2 with the inner product $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$. Let ℓ be the functional $\ell(p) := p(0)$. Find $h \in \mathcal{P}_2$ so that $\ell(p) = \langle h, p\rangle$ for all $p \in \mathcal{P}_2$.

164. Let $C[-1,1]$ be the real inner product space consisting of all continuous functions $f : [-1,1] \to \mathbb{R}$, with the inner product $\langle f, g\rangle := \int_{-1}^{1} f(x)g(x)\,dx$. Let W be the subspace of odd functions, i.e. functions satisfying $f(-x) = -f(x)$. Find (with proof) the orthogonal complement of W.

165. Find the function $f \in \operatorname{span}\{1 \sin x, \cos x\}$ that minimizes $\|\sin 2x - f(x)\|$, where the norm comes from the inner product

$$\langle f, g\rangle := \int_{-\pi}^{\pi} f(x)g(x)\,dx \quad \text{on} \quad C[-\pi, \pi].$$

166. a) Let $V \subset \mathbb{R}^n$ be a subspace and $Z \in \mathbb{R}^n$ a given vector. Find a unit vector X that is perpendicular to V with $\langle X, Z\rangle$ as large as possible.

b) Compute $\max \displaystyle\int_{-1}^{1} x^3 h(x)\,dx$ where $h(x)$ is any continuous function on the interval $-1 \le x \le 1$ subject to the restrictions

$$\int_{-1}^{1} h(x)\,dx = \int_{-1}^{1} xh(x)\,dx = \int_{-1}^{1} x^2 h(x)\,dx = 0; \qquad \int_{-1}^{1} |h(x)|^2\,dx = 1.$$

c) Compute $\displaystyle\min_{a,b,c} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2\,dx$.

167. [Dual variational problems] Let $V \subset \mathbb{R}^n$ be a linear space, $Q : R^n \to V^\perp$ the orthogonal projection into $V^\perp$, and $x \in \mathbb{R}^n$ a given vector. Note that $Q = I - P$, where P in the orthogonal projection into V

a) Show that $\displaystyle\max_{\{z \perp V,\, \|z\|=1\}} \langle x, z\rangle = \|Qx\|$.

b) Show that $\displaystyle\min_{v \in V} \|x - v\| = \|Qx\|$.

[Remark: *dual variational problems* are a pair of maximum and minimum problems whose extremal values are equal.]

168. [Completing the Square]

a) Let $\vec{x}$ and $\vec{p}$ be points in $\mathbb{R}^n$. Under what conditions on the scalar c is the set

$$\|\vec{x}\|^2 + 2\langle \vec{p}, \vec{x}\rangle + c = 0$$

a sphere $\|\vec{x} - \vec{x}_0\| = R \ge 0$? Compute the center, $\vec{x}_0$, and radius, R, in terms of $\vec{p}$ and c.

b) Let

$$Q(\vec{x}) = \sum a_{ij}x_ix_j + 2\sum b_ix_i + c$$
$$= \langle \vec{x},\, A\vec{x}\rangle + 2\langle \vec{b},\, \vec{x}\rangle + c$$

be a real quadratic polynomial so $\vec{x} = (x_1, \ldots, x_n)$, $\vec{b} = (b_1, \ldots, b_n)$ are real vectors and $A = (a_{ij})$ is a real symmetric $n \times n$ matrix. Just as in the case $n = 1$ (which you should do first), if A is invertible find a vector $\vec{v}$ (depending on A and $\vec{b}$) so that the change of variables $\vec{y} == \vec{x} - \vec{v}$ (this is a translation by the vector $\vec{v}$) so that in the new $\vec{y}$ variables Q has the simpler form

$$Q = \langle \vec{y},\, A\vec{y}\rangle + \gamma \quad \text{that is,} \quad Q = \sum a_{ij}y_iy_j + \gamma,$$

where $\gamma = c - \langle \vec{b},\, A^{-1}\vec{b}\rangle$.
As an example, apply this to $Q(\vec{x}) = 2x_1^2 + 2x_1x_2 + 3x_2 - 4$.

169. Let A be a positive definite $n \times n$ real matrix, $\vec{b}$ a real vector, and $\vec{N}$ a real unit vector.

a) For which value(s) of the real scalar c is the set

$$E := \{\, \vec{x} \in \mathbb{R}^3 \;\big|\; \langle \vec{x},\, A\vec{x}\rangle + 2\langle \vec{b},\, \vec{x}\rangle + c = 0 \,\}$$

(an ellipsoid) non-empty? [Answer: $c \le \langle \vec{b},\, A^{-1}\vec{b}\rangle$. If $n = 1$, this of course reduces to a familiar condition.]

b) For what value(s) of the scalar d is the plane $P := \{\, \vec{x} \in \mathbb{R}^3 \;\big|\; \langle \vec{N},\, \vec{x}\rangle = d \,\}$ tangent to the above ellipsoid E (assumed non-empty)?
[Suggestion: First discuss the case where $A = I$ and $\vec{b} = 0$. Then show how by a change of variables, the general case can be reduced to this special case. See also Problem 142].]
[Answer:

$$d = -\langle \vec{N},\, A^{-1}\vec{b}\rangle \pm \sqrt{\langle \vec{N},\, A^{-1}\vec{N}\rangle}\sqrt{\langle \vec{b},\, A^{-1}\vec{b}\rangle - c}.$$

For $n = 1$ this is just the solution $d = \frac{-b \pm \sqrt{b^2 - ac}}{a}$ of the quadratic equation $ax^2 + 2bx + c = 0$.]

170. More completing the square. Let $P_1, \ldots, P_k$ be distinct points in $\mathbb{R}^n$. Find a unique point X_0 in $\mathbb{R}^n$ at which the function

$$Q(X) = \|X - P_1\|^2 + \cdots + \|X - P_k\|^2$$

achieves its minimum value by "completing the square" to obtain the identity

$$Q(X) = k\left\|X - \frac{1}{k}\sum_{n=1}^{k} P_n\right\|^2 + \sum_{j=1}^{k} \|P_j|^2 - \frac{1}{k}\|\sum_{j=1}^{k} P_j\|^2.$$

[Of course one can also solve this using calculus.]

171. Let $v_1 \ldots v_k$ be vectors in a linear space with an inner product $\langle\ ,\ \rangle$. Define the *Gram determinant* by $G(v_1, \ldots, v_k) = \det(\langle v_i,\, v_j\rangle)$.

a) If the $v_1 \ldots v_k$ are orthogonal, compute their Gram determinant.

b) Show that the $v_1 \ldots v_k$ are linearly independent if and only if their Gram determinant is not zero.

c) Better yet, if the $v_1 \ldots v_k$ are linearly independent, show that the symmetric matrix $(\langle v_i,\, v_j\rangle)$ is positive definite. In particular, the inequality $G(v_1, v_2) \geq 0$ is the *Schwarz inequality*.

d) Conversely, if A is any $n \times n$ positive definite matrix, show that there are vectors $v_1, \ldots,\ v_n$ so that $A = (\langle v_i,\, v_j\rangle)$.

e) Let $\mathcal{S}$ denote the subspace spanned by the linearly independent vectors $w_1 \ldots w_k$. If X is any vector, let $P_{\mathcal{S}}X$ be the orthogonal projection of X into $\mathcal{S}$, prove that the distance $\|X - P_{\mathcal{S}}X\|$ from X to $\mathcal{S}$ is given by the formula

$$\|X - P_{\mathcal{S}}X\|^2 = \frac{G(X, w_1, \ldots, w_k)}{G(w_1, \ldots, w_k)}.$$

172. (continuation) Consider the space of continuous real functions on $[0, 1]$ with the inner product, $\langle f,\, g\rangle := \int_0^1 f(x)g(x)\, dx$ and related norm $\|f\|^2 = \langle f,\, f\rangle$. Let $\mathcal{S}_k := \operatorname{span}\{x^{n_1}, x^{n_2}, \ldots, x^{n_k}\}$, where $\{n_1, n_2, \ldots, n_k\}$ are distinct positive integers. Let $h(x) := x^\ell$ where $\ell > 0$ is a positive integer – but *not* one of the n_j's. Prove that

$$\lim_{k\to\infty} \|h - P_{\mathcal{S}_k} h\| = 0 \qquad \text{if and only if} \qquad \sum \frac{1}{n_j} \text{ diverges.}$$

This, combined with the Weierstrass Approximation theorem, proves **Muntz's Theorem:** *Linear combinations of $x^{n_1}, x^{n_2}, \ldots, x^{n_k}$ are dense in $L_2(0, 1)$ if and only if $\sum \frac{1}{n_j}$ diverges.*

173. Let $L : V \to W$ be a linear map between the finite dimensional linear spaces V and W, both having inner products.

a) Show that $(\mathrm{im} L)^\perp = \ker L^*$, where L^* is the adjoint of L.

b) Show that $\dim \mathrm{im} L = \dim \mathrm{im} L^*$. [Don't use determinants.]

c) [Fredholm Alternative] Show that the equation $Lx = y$ has a solution if and only if y is orthogonal to the kernel of L^*.

174. Let A and B be $n \times n$ matrices. If $B^*A = 0$, show that

a) $\operatorname{Im} A$ and $\operatorname{Im} B$ are orthogonal.

b) $\operatorname{rank}(A+B) = \operatorname{rank} A + \operatorname{rank} B$.

175. Let $L : \mathbb{R}^n \to \mathbb{R}^k$ be a linear map. Show that

$$\dim \ker(L) - \dim \ker(L^*) = n - k.$$

($\ker(L^*)$ is often called the *cokernel* of L).

176. Let U, V, and W be finite dimensional vector spaces with inner products. If $A : U \to V$ and $B : V \to W$ are linear maps with adjoints A^* and B^*, define the linear map $C : V \to V$ by

$$C = AA^* + B^*B.$$

If $U \xrightarrow{A} V \xrightarrow{B} W$ is *exact* [that is, image $(A) = \ker(B)$], show that $C : V \to V$ is invertible.

177. [Bilinear and Quadratic Forms] Let ϕ be a bilinear form over the finite dimensional real vector space V. ϕ is called *non-degenerate* if $\phi(x, y) = 0$ for all $y \in V$ implies $x = 0$.

True or False

a) If ϕ is non-degenerate, then $\psi(x, y) := \frac{1}{2}[\phi(x, y) + \phi(y, x)]$ is a scalar product.

b) If $\phi(x, y) = -\phi(y, x)$ for all x, $y \in V$, then $\phi(z, z) = 0$ for all $z \in V$.

c) If ϕ is symmetric and $\phi(x, x) = 0$ for all $x \in V$, then $\phi = 0$.

d) Assume the bilinear forms ϕ and ψ are both symmetric and positive definite. Then $\{z \in V \mid \phi(x, z)^3 + \psi(y, z)^3 = 0\}$ is a subspace of V.

e) If ϕ and ψ are bilinear forms over V, then $\{z \in V \mid \phi(x, z)^2 + \psi(y, z)^2 = 0\}$ is a subspace of V.

8 Norms and Metrics

178. Let $\mathcal{P}_n$ be the space of real polynomials with degree at most n. Write $p(t) = \sum_{j=0}^{n} a_j t^j$ and $q(t) = \sum_{j=0}^{n} b_j t^j$.

True or False

a) Define $d : \mathcal{P}_n \times \mathcal{P}_n \to \mathbb{R}$ by $d(p,q) := \sum_{j=0}^{n} |a_j - b_j|$. Then $\|p\| = d(p, 0)$ is a norm on $\mathcal{P}_n$.

b) For $p \in \mathcal{P}_n$ let $\|p\| := 0$ when $p = 0$ and $\|p\| := \max(0, NP(p))$ for $p \neq 0$. Here $NP(p)$ is the set of all the real zeroes of p. Claim: $\|p\|$ is a norm on $\mathcal{P}_n$.

c) Define a norm $\|\cdot\|$ on $\mathcal{P}_n$ by $\|p\| := \max_{t\in[0,1]} |p(t)|$. Then there is a bilinear form ϕ on $\mathcal{P}_n$ with $\phi(p,p) = \|p\|^2$ for all $p \in \mathcal{P}_n$.

d) Let $\langle \cdot, \cdot \rangle$ be a scalar product on $\mathcal{P}_n$ and $\|\cdot\|$ the associated norm. If α is an endomorphism of $\mathcal{P}_n$ with the property that $\|\alpha(p)\| = \|p\|$ for all $p \in \mathcal{P}_n$, then α is orthogonal in this scalar product.

e) The real function $(p,q) \mapsto (pq)'(0)$, where f' is the derivative of f, defines a scalar product on the subspace $\{p \in \mathcal{P}_n \mid p(0) = 0\}$.

179. [Inner Products of Matrices] Let A and B be real $n \times n$ matrices and define their *inner product* by the rule

$$\langle A, B\rangle = \operatorname{trace}(AB^*) = \sum_{i,j} a_{ij} b_{ij}.$$

Show that this has all the properties of an abstract inner product, namely:

$\langle A, A\rangle \geq 0$ and $\langle A, A\rangle = 0$ if and only if $A = 0$ (positive definite).

$\langle A, B\rangle = \langle B, A\rangle$ (symmetry)

$\langle A+B, C\rangle = \langle A, C\rangle + \langle B, C\rangle$ and $\langle \alpha A, B\rangle = \alpha\langle A, B\rangle$ for any real α (linearity).

9 Projections and Reflections

180. Orthogonal Projections of Rank 1 and $n-1$.

a) Let $\vec{v} \in \mathbb{R}^n$ be a unit vector and Px the orthogonal projection of $x \in \mathbb{R}^n$ in the direction of $\vec{v}$, that is, if $x = \text{const.}\,\vec{v}$, then $Px = x$, while if $x \perp \vec{v}$, then $Px = 0$. Show that $P = \vec{v}\vec{v}^T$ (here $\vec{v}^T$ is the transpose of the column vector $\vec{v}$). In matrix notation, is v_i are the components of $\vec{v}$, then $(P)_{ij} = v_i v_j$.

b) Continuing, let Q be the orthogonal projection into the subspace perpendicular to $\vec{v}$. It has rank $n-1$ Show that $Q = I - P = I - \vec{v}\vec{v}^T$.

c) Let $\vec{u}$ and $\vec{v}$ be orthogonal unit vectors and let R be the orthogonal projection into the subspace perpendicular to both $\vec{u}$ and $\vec{v}$. Show that $R = I - \vec{u}\vec{u}^T - \vec{v}\vec{v}^T$.

d) Let $Q : \mathbb{R}^3 \to \mathbb{R}^3$ be a matrix representing an orthogonal projection. From the above formulas, it is a symmetric matrix. If its diagonal elements are $5/6$, $2/3$, and $1/2$, find Q (it is almost uniquely determined).

181. A linear map $P\colon X \to X$ acting on a vector space X is called a *projection* if $P^2 = P$ (this P is not necessarily an "orthogonal projection").

a) Show that the matrix $P = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ is a projection. Draw a sketch of $\mathbb{R}^2$ showing the vectors $(1,2)$, $(-1,0)$, and $(0,3)$ and their images under the map P. Also indicate both the image, V, and nullspace, W, of P.

b) Repeat this for $Q := I - P$.

c) If the image and nullspace of a projection P are orthogonal then P is called an *orthogonal projection*. Let $M = \left(\begin{smallmatrix} 0 & a \\ 0 & c \end{smallmatrix}\right)$. For which real value(s) of a and c is this a projection? An orthogonal projection?

182. More on general projections, so all one knows is that $P : X \to X$ is a linear map that satisfies $P^2 = P$. Let $V := \text{image}(P)$ and $W := \ker(P)$.

a) Show that V and W are *complementary subspaces*, that is, every vector $\vec{x} \in X$ can be written in the form $\vec{x} = \vec{v} + \vec{w}$, where $\vec{v} \in V$ and $\vec{w} \in W$ are *uniquely* determined. The usual notation is $X = V \oplus W$ with, in this case, $P\vec{x} = \vec{x}$ for all $\vec{x} \in V$, $P\vec{x} = 0$ for all $\vec{x} \in W$. Thus, P is the projection onto V. [Suggestion: You can write any $x \in X$ uniquely as $\vec{x} = (I-P)\vec{x} + P\vec{x}$. In other words, $X = \ker(P) \oplus \ker(I-P)$.]

b) Show that $Q := I - P$ is also a projection, but it projects onto W.

c) If P is written as a matrix, it is similar to the block matrix $M = \begin{pmatrix} I_V & 0 \\ 0 & 0_W \end{pmatrix}$, where I_V is the identity map on V and 0_W the zero map on W.

d) Show that $\dim \text{image}\,(P) = \text{trace}\,(P)$.

e) If two projections P and $\hat{P}$ on V have the same rank, show they are similar.

183. [Continuation of Problem 182] If X has an inner product, show that the subspaces V and W are orthogonal if and only if $P = P^*$. Moreover, if $P = P^*$, then $\|\vec{x}\|^2 = \|P\vec{x}\|^2 + \|Q\vec{x}\|^2$, where $Q := I - P$. P and Q are the *orthogonal projections* into V and W, respectively.

184. Let P be a projection, so $P^2 = P$. If $c \neq 1$, find a short simple formula for $(I - cP)^{-1}$. [HINT: the formula $1/(1-t) = 1 + t + t^2 + \cdots$ helped me guess the answer.]

185. [See Problem 182] A linear map $R\colon X \to X$ acting on a vector space X is called a *reflection* if $R^2 = I$. Two special cases are when $R = \pm I$.

a) Show that $R = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is a reflection. For the vector $\vec{x} = (2,\, 1)$ draw a sketch showing both $\vec{x}$ and $R\vec{x}$.

b) Show that the matrix $R = \left(\begin{smallmatrix} -1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ is a reflection. Draw a sketch of $\mathbb{R}^2$ showing the vectors $(1,2)$, $(-1,0)$, (and $(0,3)$ and their images under R. Also indicate both the subspaces V and W of vectors that are mapped to themselves: $R\vec{v} = \vec{v}$, and those that are mapped to their opposites: $R\vec{w} = -\vec{w}$. [From your sketch it is clear that in this example V and W are not orthogonal so this R is not an "orthogonal reflection".]

c) More generally, show that for any reflection one can write $X = V \oplus W$ so that $R\vec{v} = \vec{v}$ for all $\vec{v} \in V$ and $R\vec{w} = -\vec{w}$ for all $\vec{w} \in W$. Thus, R is the reflection across V.

d) Show that R is similar to the block matrix $M = \begin{pmatrix} I_V & 0 \\ 0 & -I_W \end{pmatrix}$, where I_V is the identity map on V.

e) X has an inner product and the above subspaces V and W are orthogonal, then R is called an *orthogonal reflection*. Let $S = \left(\begin{smallmatrix} -1 & c \\ 0 & 1 \end{smallmatrix}\right)$. For which value(s) of c is this an orthogonal reflection?

f) Let $M := \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$. For which value(a) of a, b, and c is M a reflection? An orthogonal reflection?

186. [Continuation] More generally, show that for a reflection R, the above subspaces V and W are orthogonal if and only if $R = R^*$. This property characterizes an *orthogonal reflection.*

187. If the matrix R is a reflection (that is, $R^2 = I$) and $c \neq \pm 1$ show that $I - cR$ is invertible by finding a simple explicit formula for the inverse. [HINT: See Problem 184.]

188. If a real square matrix R is both symmetric and an orthogonal matrix, show that it an reflection across some subspace.

189. Show that projections P and reflections R are related by the formula $R = 2P - I$. This makes obvious the relation between the above several problems.

190. Let X be a linear space and $A : X \to X$ a linear map with the property that

$$(A - \alpha I)(A - \beta I) = 0, \tag{1}$$

where α and β are scalars with $\alpha \neq \beta$.

This problem generalizes the above Problems 182 and 185 on projections, $P^2 - P = 0$, and reflections, $R^2 - I = 0$. [See Problem 192 for a related problem where $A = D$ is the first derivative operator and $Lu := (D - \alpha I)(D - \beta I)u = 0$ is a second order constant coefficient linear differential operator.]

a) If $\vec{v}$ is an eigenvalue of A, compute $A\vec{v}$ and use this to find the eigenvalues of A.

b) Show that $\ker(A - \alpha I) \cap \ker(A - \beta I) = \{0\}$.

c) Show that $X = \ker(A - \alpha I) \oplus \ker(A - \beta I)$.

[Suggestion: Several possible approaches. One is to observe that

$$\text{if} \quad P := \frac{A - \alpha I}{\beta - \alpha}, \quad \text{then} \quad P(P - 1) = \frac{(A - \alpha I)(A - \beta I)}{\beta - \alpha}.$$

This substitution changes equation (1) to $P(P - I) = 0$ treated in Problem 182.

A more direct approach (it is useful in Problem 191) is: if $\vec{x} \in X$, seek vectors $\vec{x}_1 \in \ker(A - \alpha I)$ and $\vec{x}_2 \in \ker(A - \beta I)$k so that $\vec{x} = \vec{x}_1 + \vec{x}_2$ by computing $(A - \alpha I)\vec{x}$ and $(A - \beta I)\vec{x}$].

d) If $X = \mathbb{R}^n$, show it has a basis in which the matrix representing A is the block diagonal matrix

$$A = \begin{pmatrix} \alpha I_k & 0 \\ 0 & \beta I_{n-k} \end{pmatrix},$$

where $k = \dim \ker(A - \alpha I)$.

e) If X has an inner product and $A = A^*$, show that $\ker(A - \alpha I)$ and $\ker(A - \beta I)$ are orthogonal. [See Problems 183 and 186].

191. [Generalization of Problem 190] Let X be a linear space and $A : X \to X$ a linear map with the property that

$$(A - \alpha_1 I)(A - \alpha_2 I) \cdots (A - \alpha_k I) = 0,$$

where the α_i are scalars with $\alpha_i \neq \alpha_j$ for $i \neq j$.

a) If $\vec{v}$ is an eigenvector of A compute $A\vec{v}$. Use this to find the eigenvalues of A.

b) Show that $\ker(A - \alpha_i I) \cap \ker(A - \alpha_j I) = \{0\}$ for $i \neq j$.

c) Show that $X = \ker(A - \alpha_1 I) \oplus \ker(A - \alpha_2 I) \oplus \cdots \oplus \ker(A - \alpha_k I)$.

]Suggestion: Seek $\vec{x} = \vec{x}_1 + \cdots + \vec{x}_k$, where $\vec{x}_i \in \ker(A - \alpha_i I)$, observing that

$$[(A - \alpha_2 I) \cdots (A - \alpha_k I)]\vec{x} = (\alpha_1 - \alpha_2) \cdots (\alpha_1 - \alpha_k)\vec{x}_1].$$

This gives $\vec{x}_1$. There are similar formulas for $\vec{x}_2$ etc.

d) If $X = \mathbb{R}^n$, show it has a basis in which the matrix representing A is the block diagonal matrix

$$A = \begin{pmatrix} \alpha_1 I_1 & 0 & 0 & 0 \\ 0 & \alpha_2 I_2 & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_k I_k \end{pmatrix},$$

where I_j is the identity matrix on the subspace $\ker(A - \alpha_j I)$.

e) If X has an inner product and $A = A^*$, show that $\ker(A - \alpha_i I)$ and $\ker(A - \alpha_j I)$ are orthogonal for $i \neq j$.

192. This problem applies the ides in Problem 190 to the linear constant coefficient ordinary differential operator

$$Lu := (D - \alpha I)(D - \beta I)u = 0, \quad \text{where} \quad \alpha \neq \beta.$$

The key observation is Problem 190 also applies immediately to the case where equation (1) holds only on a *subspace*. Let X be the linear space of twice differentiable functions $u(t)$ that satisfy $Lu = 0$, that is, $X = \ker(L)$.

a) Show that $\ker(D - \alpha I) \cap \ker(D - \beta I) = \{0\}$.

b) Show that $\ker(L) = \ker(D - \alpha I) \oplus \ker(D - \beta I)$.

c) If $u'' - 4u = 0$, deduce that $u(t) = c_1 e^{2t} + c_2 e^{-2t}$ for some constants c_1 and c_2. [Remark: To understand $\ker(D - \alpha I)$, see Problem 246]

d) Extend this idea to show that if $Mu := (D^2 u - \alpha I)(D^2 - \beta I)u$, where $\alpha \neq \beta$, then

$$\ker M = \ker(D^2 - \alpha I) \oplus \ker(D^2 - \beta I).$$

193. [Orthogonal Projections as Matrices. See also Problems 70, 233, 234 , 235, 273].

Let $\mathbf{n} := (a, b, c) \in \mathbb{R}^3$ be a *unit* vector and $\mathcal{S}$ the plane of vectors (through the origin) orthogonal to $\mathbf{n}$.

a) Show that the *orthogonal projection of* $\mathbf{x}$ *in the direction of* $\mathbf{n}$ can be written in the matrix form

$$\langle \mathbf{x}, \mathbf{n} \rangle \mathbf{n} = (\mathbf{n}\mathbf{n}^T)\mathbf{x} = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where $\langle \mathbf{x}, \mathbf{n} \rangle$ is the usual inner product, $\mathbf{n}^T$ is the transpose of the column vector $\mathbf{n}$, and $\mathbf{n}\mathbf{n}^T$ is matrix multiplication.

b) Show that the *orthogonal projection* P of a vector $\mathbf{x} \in \mathbb{R}^3$ into $\mathcal{S}$ is

$$P\mathbf{x} = \mathbf{x} - \langle \mathbf{x}, \mathbf{n} \rangle \mathbf{n} = (I - \mathbf{n}\mathbf{n}^T)\mathbf{x},$$

Apply this to compute the orthogonal projection of the vector $\mathbf{x} = (1, -2, 3)$ into the plane in $\mathbb{R}^3$ whose points satisfy $x - y + 2z = 0$.

c) Find a formula similar to the previous part for the orthogonal *reflection* R of a vector across $\mathcal{S}$. Then apply it to compute the orthogonal reflection of the vector $\mathbf{v} = (1, -2, 3)$ across the plane in $\mathbb{R}^3$ whose points satisfy $x - y + 2z = 0$.

d) Find a 3×3 matrix that projects a vector in $\mathbb{R}^3$ into the plane $x - y + 2z = 0$.

e) Find a 3×3 matrix that reflects a vector in $\mathbb{R}^3$ across the plane $x - y + 2z = 0$.

10 Similar Matrices

194. Let C and B be square matrices with C invertible. Show the following.

a) $(CBC^{-1})^2 = C(B^2)C^{-1}$

b) Similarly, show that $(CBC^{-1})^k = C(B^k)C^{-1}$ for any $k = 1, 2, \ldots$.

c) If B is also invertible, is it true that $(CBC^{-1})^{-2} = C(B^{-2})C^{-1}$? Why?

195. Let $A = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$.

a) Find an invertible matrix C such that $D := C^{-1}AC$ is a diagonal matrix. Thus, $A = CDC^{-1}$.

b) Compute A^{50}.

196. Determine whether any of the following three matrices are similar over $\mathbb{R}$:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

197. Let $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

a) Are A and B similar? Why?

b) Show that B is not similar to any diagonal matrix.

198. a) Show that the matrices $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ are similar. [This is a counterexample to the plausible suspicion "if the matrices A and $2A$ are similar, then $A = 0$."]

b) Let $A(s) = \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}$ and let $M = A(1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ If $s \neq 0$, show that $A(s)$ is similar to M.

REMARK: This is a simple and fundamental counterexample to the assertion: "If $A(s)$ depends smoothly on the parameter s and is similar to M for all $s \neq 0$, then $A(0)$ is also similar to M."

199. Say a matrix A is *similar* to the matrix $B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. Give a proof or counterexample for each of the following assertions.

a). $A^2 = A$
b). $\det A = 0$.
c). $\operatorname{trace} A = 1$.
d). $\lambda = 0$ is an eigenvalue of A.
e). $\lambda = 1$ is an eigenvalue of A.
f). $v = (1,\, 0)$ is an eigenvector of A.

200. Let V be a finite dimensional linear space. If $e = \{e_1, \dots, e_n\}$ and $\epsilon = \{\epsilon_1, \dots, \epsilon_n\}$ are bases for V, we let V_e and V_ϵ refer to vectors when written using these bases, respectively. Let $L : V \to V$ be a linear map. Using the bases we can represent L by the matrices L_e and L_ϵ (Example: see Problem 65).

Show that the matrices L_e and L_ϵ are similar by finding an invertible linear map $S : V \to V$ so that $L_\epsilon S = S L_e$.

REMARK: The following diagram may help understand this. What we call a "change of basis" is an example of a fundamental procedure known to everyone, yet often seems exotic when it arises in a mathematical setting. As an illustration, say you have a problem stated in a "language (Hungarian?)" that is difficult for you. To work with it, first "translate" it (find S) into a "language" that is simpler for you. Solve the translated version and then translate the solution back (S^{-1}). Symbolically, reading from *right* to *left*,

$$\mathsf{Q}S = S\mathsf{P} \quad \text{that is,} \quad P = S^{-1}QS$$

$$\begin{tikzcd} \textit{old} \arrow[r, "\mathsf{P}", "\text{(original version)}"'] \arrow[d, "S"'] & \textit{old} \\ \textit{new} \arrow[r, "\mathsf{Q}", "\text{(new version)}"'] & \textit{new} \arrow[u, "S^{-1}"'] \end{tikzcd}$$

The goal is to choose the new setting and S so the new problem Q is easier than P. Finding a new basis in which a matrix is diagonal is a standard example in linear algebra.

201. Let A be a square real matrix. For each of the following assertions, either give a proof or find a counterexample.

a) If A is similar to the identity matrix, then $A = I$.

b) If A is similar to the zero matrix, then $A = 0$.

c) If A is similar to $2A$, then $A = 0$.

d) If all the eigenvalues of A are zero, then $A = 0$.

e) If A is similar to a matrix B with the property $B^2 = 0$, then $A^2 = 0$.

f) If A is similar to a matrix B one of whose eigenvalues is 7, then one eigenvalue of A is 7.

g) If A is similar to a matrix B that can be diagonalized, then A can be diagonalized.

h) If A can be diagonalized and $A^2 = 0$, then $A = 0$.

i) If A is similar to a projection P (so $P^2 = P$), then A is a projection.

j) If A is similar to a real orthogonal matrix, then A is an orthogonal matrix.

k) If A is similar to a symmetric matrix, then A is a symmetric matrix.

202. Say the square matrix A is similar to B.

a) Is A^2 similar to B^2? Proof or counterexample.

b) Is $I + 3A - 7A^4$ similar to $I + 3B - 7B^4$? Proof or counterexample.

c) Generalize.

203. A square matrix M is *diagonalized* by an invertible matrix S if SMS^{-1} is a diagonal matrix. Of the following three matrices, one can be diagonalized by an orthogonal matrix, one can be diagonalized but not by any orthogonal matrix, and one cannot be diagonalized. State which is which — and why.

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 \\ 2 & -5 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & -2 \\ 2 & -5 \end{pmatrix}.$$

204. Repeat the previous problem for the matrices

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

205. Let A be the matrix

$$A = \begin{pmatrix} 1 & \lambda & 0 & 0 & \dots & 0 \\ 0 & 1 & \lambda & 0 & \dots & 0 \\ 0 & 0 & 1 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \lambda \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Show that there exists a matrix B with $BAB^{-1} = A^T$ (here A^T is the transpose of A).

206. Let A be an $n \times n$ matrix with coefficients in a field $\mathbb{F}$ and let S be an invertible matrix.

a) If $SAS^{-1} = \lambda A$ for some $\lambda \in \mathbb{F}$, show that either $\lambda^n = 1$ or A is nilpotent.

b) If n is odd and $SAS^{-1} = -A$, show that 0 is an eigenvalue of A.

c) If n is odd and $SAS^{-1} = A^{-1}$, show that 1 is an eigenvalue of A.

207. a) Find a 2×2 matrix with entries in the Boolean field F_2 that is diagonalizable over the field F_4 of order 4 but not over F_2.

b) Find a 2×2 matrix with entries in F_2 that is not diagonalizable over the field F_4.

208. In $\mathbb{R}^n$, say you are given a vector $\vec{v} \neq 0$ and let $\vec{e}_1 = (1, 0, \ldots, 0)$.

a) Find an invertible matrix A that maps $\vec{e}_1 \to \vec{v}$.

b) If $\vec{v}$ and $\vec{w}$ are linearly independent vectors in $\mathbb{R}^n$, describe how to find an invertible matrix A with both $A\vec{v} = \vec{w}$ and $A\vec{w} = \vec{v}$.

209. Let $A(t) = \begin{pmatrix} 1+t & 1 \\ -t^2 & 1-t \end{pmatrix}$.

a) Show that $A(t)$ is similar to $A(0)$ for all t.

b) Show that $B(t) := A(0) + A'(0)t$ is similar to $A(0)$ only for $t = 0$.

210. Let A, B, and C be $n \times n$ matrices.

a) Show that $\operatorname{trace}(AB) = \operatorname{trace}(BA)$

b) Is it true that $\operatorname{trace}(ABC) = \operatorname{trace}(CAB)$? Proof or counterexample.

c) Is it true that $\operatorname{trace}(ABC) = \operatorname{trace}(ACB)$? Proof or counterexample.

211. Let f be any function defined on $n \times n$ matrices with the property that $f(AB) = f(BA)$ [*Example*: $f(A) = \operatorname{trace}(A)$]. If A and C are similar, show that $f(A) = f(C)$.

212. Let $h(A)$ be a scalar-valued function defined on all square matrices A having the property that if A and B are similar, then $h(A) = h(B)$. If h is also linear, show that $h(A) = c \operatorname{trace}(A)$ where c is a constant.

213. Let $\{A, B, C, \ldots\}$ be linear maps over a finite dimensional vector space V. Assume these matrices all commute pairwise, so $AB = BA$, $AC = CA$, $BC = CB$, etc.

a) Show that there is some basis for V in which all of these are represented simultaneously by upper triangular matrices.

b) If each of these matrices can be diagonalized, show that there is some basis for V in which all of these are represented simultaneously by diagonal matrices.

11 Symmetric and Self-adjoint Maps

214. Proof or Counterexample. Here A is a real symmetric matrix.

a) Then A is invertible.-

b) A is invertible if and only if $\lambda = 0$ is *not* an eigenvalue of A.

c) The eigenvalues of A are all real.

d) If A has eigenvectors v, w corresponding to eigenvalues λ, μ with $\lambda \neq \mu$, then $\langle v, w \rangle = 0$.

e) If A has linearly independent eigenvectors v and w then $\langle v, w \rangle = 0$.

f) If B is any square real matrix, then $A := B^*B$ is positive semi-definite.

g) If B is any square matrix, then $A := B^*B$ is positive definite if and only if B is invertible.

h) If C is a real anti-symmetric matrix (so $C^* = -C$), then $\langle v, Cv \rangle = 0$ for all real vectors v.

215. Let a, b, c be real numbers, and consider the matrix $A = \begin{pmatrix} a & b & c \\ b & c & b \\ c & b & a \end{pmatrix}$.

a) Explain why all the eigenvalues of A must be real.

b) Show that some eigenvalue λ of A has the property that for every vector $v \in \mathbb{R}^3$, $v \cdot Av \leq \lambda \|v\|^2$. (Note: You are not being asked to compute the eigenvalues of A.)

216. True or False.

a) The vector space of all 4×4 matrices that are both symmetric and anti-symmetric (also called "skew-symmetric") has dimension one.

b) If T is a linear transformation between the linear spaces V and W, then the set $\{v \in V \mid T(v) = 0\}$ is a linear subspace of V.

c) The vectors $v_1, v_2, \ldots, v_n$ in $\mathbb{R}^n$ are linearly independent if, and only if, span $\{v_1, v_2, \ldots, v_n\}$ is all of $\mathbb{R}^n$.

d) If A is an $n \times n$ matrix such that nullity$(A) = 0$, then A is the identity matrix.

e) If A is an $k \times n$ matrix with rank k, then the columns of A are linearly independent.

217. Let A and B be symmetric matrices with A positive definite.

a) Show there is a change of variables $y = Sx$ (so S is an invertible matrix) so that $\langle x, Ax\rangle = \|y\|^2$ (equivalently, $S^T AS = I$). One often rephrases this by saying that a positive definite matrix is *congruent* to the identity matrix.

b) Show there is a linear change of variables $y = Px$ so that *both* $\langle x, Ax\rangle = \|y\|^2$ and $\langle x, Bx\rangle = \langle y, Dy\rangle$, where D is a diagonal matrix.

c) If A is a positive definite matrix and B is positive semi-definite, show that

$$\operatorname{trace}(AB) \geq 0$$

with equality if and only if $B = 0$.

218. [Congruence of Matrices] Two symmetric matrices A, B in $M(n,\mathbb{F})$ are called *congruent* if there is an invertible matrix $T \in M(n,\mathbb{F})$ with $A = T^*BT$ (here T^* is the hermitian adjoint of T); equivalently, if

$$\langle Tx, ATy\rangle = \langle x, By\rangle \qquad \text{for all vectors } x, y,$$

so T is just a change of coordinates.

True or False?

a) Over $\mathbb{R}$ the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is congruent to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

b) If A and B are congruent over $\mathbb{C}$, then A and B are similar over $\mathbb{C}$.

c) If A is real and all of its eigenvalues are positive, then over $\mathbb{R}$ A is congruent to the identity matrix.

d) Over $\mathbb{R}$ if A is congruent to the identity matrix, then all of its eigenvalues are positive.

219. [Rayleigh Quotient] Let A be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ and corresponding orthonormal eigenvectors $v_1, \ldots, v_n$.

a) Show that

$$\lambda_1 = \min_{x\neq 0} \frac{\langle x, Ax\rangle}{\|x\|^2} \qquad \text{and} \qquad \lambda_n = \max_{x\neq 0} \frac{\langle x, Ax\rangle}{\|x\|^2}.$$

Also, show directly that if $v \neq 0$ minimizes $\langle x, Ax\rangle/\|x\|^2$, then v is an eigenvalue of A corresponding the minimum eigenvalue of A.

b) Show that

$$\lambda_2 = \min_{x\perp v_1,\ x\neq 0} \frac{\langle x, Ax\rangle}{\|x\|^2}.$$

Remark: The Courant Fischer Min-max Theorem is a useful generalization. The same ideas can be applied to the eigenvalues of the Laplacian, including the Schrödinger equation in quantum mechanics.

220. Let $A = (a_{ij})$ be an $n \times n$ real symmetric matrix with eigenvalues $\lambda_1 \le \cdots \le \lambda_n$ and let $C = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{smallmatrix}\right)$ be the upper-left 2×2 block of A with eigenvalues $\mu_1 \le \mu_2$.

a) Show that $\lambda_1 \le \mu_1$ and $\lambda_n \ge \mu_2$.

b) Determine the information this gives about the eigenvalues of $\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}$.

221. Let A be a positive definite symmetric matrix.

a) Show that it has a unique positive definite square root, that is, a positive definite matrix P so that $P^2 = A$.

b) Find a linear change of variables so that the points on the ellipsoid $\langle x, Ax \rangle = 1$ become points on the sphere $\|y\|^2 = 1$.

c) If B is any symmetric matrix (with the same size as A) show there is a linear map that simultaneously diagonalizes both.

222. Let $M = (m_{ij})$ be a real symmetric $n \times n$ matrix and let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Further, let $Q(x$ be the quadratic polynomial

$$Q(x) = \sum_{i,j} m_{ij} x_i x_j.$$

In terms of the rank and signature of M, give a necessary and sufficient condition that the set $\{\, x \in \mathbb{R}^n \mid Q(x) = 1 \,\}$ is bounded and non-empty.

223. Suppose that A is a real $n \times n$ symmetric matrix with two equal eigenvalues. If v is any vector, show that the vectors v, $Av, \ldots,$ $A^{n-1}v$ are linearly dependent.

224. Let A be a positive definite $n \times n$ matrix with diagonal elements a_{11}, a_{22}, $\ldots$, a_{nn}. Show that

$$\det A \le \prod a_{ii}.$$

225. Let A be a positive definite $n \times n$ matrix. Show that $\det A \le \left(\dfrac{\operatorname{trace} A}{n} \right)^n$. When can equality occur?

226. Let Q and M be symmetric matrices with Q invertible. Show there is a matrix A such that $AQ + QA^* = M$.

227. Let the real matrix A be *anti-symmetric* (or *skew-symmetric*), that is, $A^* = -A$.

a) Give an example of a 2×2 anti-symmetric matrix.

b) Show that the diagonal elements of any $n \times n$ anti-symmetric matrix must all be zero.

c) Show that every square matrix can (uniquely?) be written as the sum of a symmetric and an anti-symmetric matrix.

d) Show that the eigenvalues of a real anti-symmetric matrix are purely imaginary.

e) Show that $\langle \mathbf{V}, A\mathbf{V} \rangle = 0$ for every vector $\mathbf{V}$.

f) If A is an $n \times n$ anti-symmetric matrix and n is odd, show that $\det A = 0$ — and hence deduce that A cannot be invertible.

g) If n is even, show that $\det A \geq 0$. Show by an example that A may be invertible.

h) If A is a real invertible $2n \times 2n$ anti-symmetric matrix, show there is a real invertible matrix S so that

$$A = SJS^*,$$

where $J := \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$; here I_k is the $k \times k$ identity matrix. [Note that $J^2 = -I$ so the matrix J is like the complex number $i = \sqrt{-1}$.

228. a) Compute $\displaystyle\iint_{\mathbb{R}^2} \frac{dx_1\, dx_2}{[1 + \langle x, Cx \rangle]^2}$, where C is a positive definite (symmetric) 2×2 matrix, and $x = (x_1, x_2) \in \mathbb{R}^2$.

SUGGESTION: If $C = I$, this is routine using polar coordinates. For the general case, prove (and use) that every positive definite matrix C has a square root, that is, there is a positive definite matrix P with $P^2 = C$.

b) Use part a) to compute

$$\iint_{\mathbb{R}^2} \frac{dx\, dy}{(1 + 4x^2 + 9y^2)^2}, \quad \iint_{\mathbb{R}^2} \frac{dx\, dy}{(1 + x^2 + 2xy + 5y^2)^2}, \quad \iint_{\mathbb{R}^2} \frac{dx\, dy}{(1 + 5x^2 - 4xy + 5y^2)^2}.$$

c) Generalizing part a), let $h(t)$ be a given function and say you know that $\int_0^\infty h(t)\, dt = \alpha$. If $x = (x_1, x_2) \in \mathbb{R}^2$ and C is a positive definite 2×2 matrix, show that

$$\iint_{\mathbb{R}^2} h(\, \langle x, Cx \rangle\,)\, dA = \frac{\pi \alpha}{\sqrt{\det C}}.$$

d) Compute $\displaystyle\iint_{\mathbb{R}^2} e^{-(5x^2 - 4xy + 5y^2)}\, dx\, dy$.

e) Compute $\displaystyle\iint_{\mathbb{R}^2} e^{-(5x^2 - 4xy + 5y^2 - 2x + 3)}\, dx\, dy$. [SUGGESTION: see Problem 168]

229. Let A be a positive definite $n \times n$ positive definite matrix and $b \in \mathbb{R}^n$.

a) Show that

$$\iint_{\mathbb{R}^n} e^{-[\langle x,\, Ax\rangle]}\, dx = \frac{\pi^{n/2}}{\sqrt{\det A}}.$$

[This assumes you can already do the special case $A = I$. For the general case, use that if A is positive definite it has a positive definite square root: $A = P^2$.]

b) Use Problem 168 to generalize this and obtain the formula

$$\iint_{\mathbb{R}^n} e^{-[\langle x,\, Ax\rangle + \langle b,\, x\rangle + c]}\, dx = \frac{\pi^{n/2}}{\sqrt{\det A}}\, e^{\langle b,\, A^{-1}b\rangle - c}.$$

230. Let S be any symmetric matrix and A a positive definite matrix.

a) Show that

$$\iint_{R^n} \langle x,\, Sx\rangle\, e^{-\|x\|^2}\, dx = \frac{1}{2}\pi^{n/2}\operatorname{trace}(S).$$

b) Show that

$$\iint_{R^n} \langle x,\, Sx\rangle\, e^{-\langle x,\, Ax\rangle}\, dx = \frac{\pi^{n/2}\operatorname{trace}(SA^{-1})}{2\sqrt{\det A}}.$$

12 Orthogonal and Unitary Maps

231. Let the real $n \times n$ matrix A be an *isometry*, that is, it preserves length:

$$\|Ax\| = \|x\| \quad \text{for all vectors} \quad x \in \mathbb{R}^n. \tag{2}$$

These are the *orthogonal transformations.*

a) Show that (2) is equivalent to $\langle Ax,\, Ay\rangle = \langle x,\, y\rangle$ for all vectors x, y, so A preserves inner products. HINT: use the *polarization* identity:

$$\langle x,\, y\rangle = \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2\big). \tag{3}$$

This shows how, in a real vector space, to recover a the inner product if you only know how to compute the (euclidean) length.

b) Show that (2) is equivalent to $A^{-1} = A^*$.

c) Show that (2) is equivalent to the columns of A being unit vectors that are mutually orthogonal.

d) Show that (2) implies $\det A = \pm 1$ and that all eigenvalues satisfy $|\lambda| = 1$.

e) If $n = 3$ and $\det A = +1$, show that $\lambda = 1$ is an eigenvalue.

f) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ have the property (2), namely $\|F(x)\| = \|x\|$ for all vectors $x \in \mathbb{R}^n$. Then F is an orthogonal transformation. Proof or counterexample.

g) Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a *rigid motion*, that is, it preserves the distance between any two points: $\|F(x) - F(y)\| = \|x - y\|$ for all vectors $x, y \in \mathbb{R}^n$. Show that $F(x) = F(0) + Ax$ for some orthogonal transformation A.

232. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible matrix that permutes the standard basis vectors. Then A is an orthogonal matrix. Proof or counterexample.

233. Recall (see Problem 193) that $\mathbf{u} := \mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n}$ is the projection of $\mathbf{x}$ into the plane perpendicular to the unit vector $\mathbf{n}$. Show that in $\mathbb{R}^3$ the vector

$$\mathbf{w} := \mathbf{n} \times \mathbf{u} = \mathbf{n} \times [\mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n}] = \mathbf{n} \times \mathbf{x}$$

is orthogonal to both $\mathbf{n}$ and $\mathbf{u}$, and that $\mathbf{w}$ has the same length as $\mathbf{u}$. Thus $\mathbf{n}$, $\mathbf{u}$, and $\mathbf{w}$ are orthogonal with $\mathbf{u}$, and $\mathbf{w}$ in the plane perpendicular to the axis of rotation $\mathbf{n}$. (See also Problems 70, 234, 235, 273).

234. [ROTATIONS IN $\mathbb{R}^3$] Let $\mathbf{n} \in \mathbb{R}^3$ be a unit vector. Find a formula for the 3×3 matrix that determines a rotation of $\mathbb{R}^3$ through an angle θ with $\mathbf{n}$ as axis of rotation (assuming the axis passes through the origin). Here we outline one approach to find this formula — but before reading further, try finding it on your own.

a) (Example) Find a matrix that rotates $\mathbb{R}^3$ through the angle θ using the vector $(1, 0, 0)$ as the axis of rotation.

b) More generally, let $\mathcal{S}$ be the plane through the origin that is orthogonal to $\mathbf{n}$. If $\mathbf{u} \in \mathcal{S}$ is a non-zero vector, let $\mathbf{w} \in \mathcal{S}$ be orthogonal to $\mathbf{u}$ with $\|\mathbf{w}\| = \|\mathbf{u}\|$ (this determines $\mathbf{w}$ except for a factor of ± 1). Explain why by varying θ the vectors

$$\mathbf{z}(\theta) := \cos\theta\,\mathbf{u} + \sin\theta\,\mathbf{w}$$

sweep out the rotations of $\mathbf{u}$ in the plane $\mathcal{S}$.

c) Given a vector $\mathbf{x}$ use this to show that the map

$$R_{\mathbf{n}} : \mathbf{x} \mapsto (\mathbf{x} \cdot \mathbf{n})\mathbf{n} + \cos\theta\,\mathbf{u} + \sin\theta\,\mathbf{w}$$

d) Using Problems 193 and 233 to write $\mathbf{u}$ and $\mathbf{w}$, in terms of $\mathbf{n}$ and $\mathbf{x}$, show that the following map rotates $\mathbf{x}$ through an angle θ with $\mathbf{n}$ as axis of rotation. [Note: as above one needs more information to be able to distinguish between θ and $-\theta$].

$$\begin{aligned} R_{\mathbf{n}}\mathbf{x} &= (\mathbf{x} \cdot \mathbf{n})\mathbf{n} + \cos\theta\,[\mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n}] + \sin\theta\,(\mathbf{n} \times \mathbf{x}) \\ &= \mathbf{x} + \sin\theta\,(\mathbf{n} \times \mathbf{x}) + (1 - \cos\theta)[(\mathbf{x} \cdot \mathbf{n})\mathbf{n} - \mathbf{x}]. \end{aligned}$$

Thus, using Problem 70, if $\mathbf{n}=(a,b,c)\in\mathbb{R}^3$ deduce that:

$$R_{\mathbf{n}}=I+\sin\theta\begin{pmatrix}0&-c&b\\c&0&-a\\-b&a&0\end{pmatrix}+(1-\cos\theta)\begin{pmatrix}-b^2-c^2&ab&ac\\ab&-a^2-c^2&bc\\ac&bc&-a^2-b^2\end{pmatrix}.$$

In the notation of Problem 70 (but using the unit vector $\mathbf{n}$ rather than $\mathbf{v}$), this is

$$R_{\mathbf{n}}=I+\sin\theta\,A_{\mathbf{n}}+(1-\cos\theta)\,A_{\mathbf{n}}^2 \tag{4}$$

(see more on this in Problems 235, 273).

e) Use this formula to find the matrix that rotates $\mathbb{R}^3$ through an angle of θ using as axis the line through the origin and the point $(1,1,1)$.

235. [THE AXIS OF A ROTATION IN $\mathbb{R}^3$]. Given a unit vector $\mathbf{n}$, Problem 234 equation (4) gives a formula for an orthogonal matrix of a rotation with axis of rotation $\mathbf{n}$.

a) Say you are just given a 3×3 orthogonal matrix R with $\det R=1$. How can you determine the axis of rotation?

The axis of rotation $\mathbf{n}$ is encoded in the matrix $A_{\mathbf{n}}$. But since $A_{\mathbf{n}}^2$ is a symmetric matrix, then $\sin\theta A_{\mathbf{n}}$ is the anti-symmetric part of the orthogonal matrix R in the decomposition (4). Give the details of this.

b) Apply this procedure to recover the axis of rotation for the orthogonal matrix you found in Problem 234 (d).

236. a) Let V be a complex vector space and $A:V\to V$ a unitary operator. Show that A is diagonalizable.

b) Does the same remain true if V is a real vector space, and A is orthogonal?

237. For a complex vector space with a hermitian inner product one can define a unitary matrix U just as in Problem 231 as one that preserves the length:

$$\|Uv\|=\|v\|$$

for all complex vectors v.

a) In this situation, for any complex vectors u, v prove the *polarization identity*

$$\langle u,\,v\rangle=\frac{1}{4}\left[\left(\|u+v\|^2-\|u-v\|^2\right)+i\left(\|u+iv\|^2-\|u-iv\|^2\right)\right].$$

b) Extend Problem 231 to unitary matrices.

238. Show that the only real matrix that is orthogonal, symmetric and positive definite is the identity.

239. Let V be a finite dimensional vector space over $\mathbb{R}$and W a finite dimensional vector space over $\mathbb{C}$.

True or False

a) Let α be an endomorphism of W. In a unitary basis for W say M is a diagonal matrix all of whose eigenvalues satisfy $|\lambda| = 1$. Then α is a unitary matrix.

b) The set of orthogonal endomorphisms of V forms a ring under the usual addition and multiplication.

c) Let $\alpha \neq I$ be an orthogonal endomorphism of V with determinant 1. Then there is no $v \in V$ (except $v = 0$) satisfying $\alpha(v) = v$.

d) Let α be an orthogonal endomorphism of V and $\{v_1, \ldots, v_k\}$ a linearly independent set of vectors in V. Then the vectors $\{\alpha(v_1), \ldots, \alpha(v_k)\}$ are linearly independent.

e) Using the standard scalar product for $\mathbb{R}^3$, let $v \in \mathbb{R}^3$ be a unit vector, $\|v\| = 1$, and define the endomorphism $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ using the cross product: $\alpha(x) := v \times x$. Then the subspace $v^{\perp}$ is an invariant subspace of α and α is an orthogonal map on this subspace.

240. Let $R(t)$ be a family of real orthogonal matrices that depend smoothly on the real parameter t.

a) If $R(0) = I$, show that the derivative, $A := R'(0)$ is anti-symmetric, that is, $A^* = -A$. [This shows that for the Lie group of orthogonal matrices the corresponding Lie algebra consists of anti-symmetric matrices.]

b) Let the vector $x(t)$ be a solution of the differential equation $x' = A(t)x$, where the matrix $A(t)$ is anti-symmetric (the Frenet-Serret equations in differential geometry have this form). Show that its (Euclidean) length is constant, $\|x(t)\| =$const.

If also $y' = A(t)y$, show that $\langle x(t),\, y(t)\rangle$ is a constant.

Using this $x(t)$, if we define the map $R(t)$ by $R(t)x(0) := x(t)$, show that $R(t)$ is an orthogonal transformation.

c) Let $A(t)$ be an anti-symmetric matrix and let the square matrix $R(t)$ satisfy the differential equation $R' = AR$ with $R(0)$ an orthogonal matrix. Show that $R(t)$ is an orthogonal matrix.

13 Normal Matrices

241. A square matrix M is called *normal* if it commutes with its adjoint: $AA^* = A^*A$. For instance all self-adjoint and all orthogonal matrices are normal.

a) Give an example of a normal matrix that is neither self-adjoint nor orthogonal.

b) Show that M is normal if and only if $\|MX\| = \|M^*X\|$ for all vectors X.

c) Let M be normal and V and eigenvector with eigenvalue λ. Show that V is also an eigenvalue of M^*, but with eigenvalue $\bar{\lambda}$. [SUGGESTION: Notice that $L := M - \lambda I$ is also normal.]

d) If M is normal, show that the eigenvectors corresponding to distinct eigenvalues are orthogonal.

242. Here A and B are $n \times n$ complex matrices.

True or False

a) If A is normal and $\det(A) = 1$, then A is unitary.

b) If A is unitary, then A is normal and $\det(A) = 1$.

c) If A is normal and has real eigenvalues, then A is hermitian (that is, self-adjoint).

d) If A and B are hermitian, then AB is normal.

e) If A is normal and B is unitary, then $\bar{B}^T AB$ is normal.

14 Symplectic Maps

243. Let B be a real $n \times n$ matrix with the property that $B^2 = -I$.

a) Show that n must be even, $n = 2k$.

b) Show that B is similar to the block matrix $J := \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$, where here I_k is the $k \times k$ identity matrix. [HINT: Write $x_1 := \frac{1}{2}(I - B)x$ and $x_2 := \frac{1}{2}(I + B)x$. Note that $x_1 + x_2 = x$. Compute $Bx =$?].

c) Let C be a real $n \times n$ matrix with the property that $(C - \lambda I)(C - \bar{\lambda} I) = 0$, where $\lambda = \alpha + i\beta$ with α and β real and $\beta \neq 0$. Show that C is similar to the matrix $K := \alpha I + \beta J$ with J as above.

244. Let $J := \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$, where I_k is the $k \times k$ identity matrix. Note that $J^2 = -I$. A real $2k \times 2k$ matrix S is *symplectic* if it preserves the bilinear form $B[x, y] := \langle x,\, Jy \rangle$; thus $B[Sx, Sy] = B[x, y]$ for all vectors x, y in $\mathbb{R}^{2k}$.

a) Is J itself symplectic?

b) Show that a symplectic matrix is invertible and that the inverse is also symplectic.

c) Show that the set $Sp(2k)$ of $2k \times 2k$ symplectic matrices forms a group. [In many ways this is analogous to the orthogonal group].

d) Show that a matrix S is symplectic if and only if $S^*JS = J$. Then deduce that $S^{-1} = -JS^*J$ and that S^* is also symplectic.

e) Show that if S is symplectic, then S^* is similar to S^{-1}. Thus, if λ is an eigenvalue of S, then so are $\bar{\lambda}$, $1/\lambda$, and $1/\bar{\lambda}$.

f) Write a symplectic matrix S have the block form $S := \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, B, C, and D are $k \times k$ real matrices. Show that S is symplectic if and only if

$$A^*C = C^*A, \quad B^*D = D^*B, \quad \text{and} \quad A^*D - C^*B = I.$$

Show that

$$S^{-1} = \begin{pmatrix} D^* & -B^* \\ -C^* & A^* \end{pmatrix}.$$

g) If S is symplectic, show that $\det S = +1$. One approach is to use the previous part, picking the block matrices X and Y so that

$$\begin{pmatrix} I & 0 \\ X & Y \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix}.$$

h) Let $S(t)$ be a family of real symplectic matrices that depend smoothly on the real parameter t with $S(0) = I$. Show that the derivative $T := S'(0)$ has the property that JT is self-adjoint.

i) Let the matrix $S(t)$ be a solution of the differential equation $S'(t) = TS$ with $S(0)$ a symplectic matrix, where T is a real square matrix with the property that JT is self-adjoint. Show that $S(t)$ is a symplectic matrix.

15 Differential Equations

245. Let $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ be a solution of the system of differential equations

$$\begin{aligned} x_1' &= \ \ cx_1 + x_2 \\ x_2' &= -x_1 + cx_2 \end{aligned}.$$

For which value(s) of the real constant c do *all* solutions $\vec{x}(t)$ converge to 0 as $t \to \infty$?

246. Let L be the ordinary differential operator

$$Lu := (D - a)u,$$

where $Du = du/dt$, a is a constant, and $u(t)$ is a differentiable function.

Show that the kernel of L is exactly the functions of the form $u(t) = ce^{at}$, where c is any constant. In particular, the dimension of the kernel of L is one.

[HINT: Let $v(t) := e^{-at}u(t)$ and show that $v(t) =$ constant.]

247. Let V be the linear space of smooth real-valued functions and $L : V \to V$ the linear map defined by $Lu := u'' + u$.

a) Compute $L(e^{2x})$ and $L(x)$.

b) Find particular solutions of the inhomogeneous equations

$$a).\ u'' + u = 7e^{2x}, \qquad b).\ w'' + w = 4x, \qquad c).\ z'' + z = 7e^{2x} - 3x$$

c) Find the kernel (=nullspace) of L. What is its dimension?

248. Let $\mathcal{P}_N$ be the linear space of polynomials of degree at most N and $L : \mathcal{P}_N \to \mathcal{P}_N$ the linear map defined by $Lu := au'' + bu' + cu$, where a, b, and c are constants. Assume $c \neq 0$.

a) Compute $L(x^k)$.

b) Show that nullspace (=kernel) of $L : \mathcal{P}_N \to \mathcal{P}_N$ is 0.

c) Show that for every polynomial $q(x) \in \mathcal{P}_N$ there is one and only one solution $p(x) \in \mathcal{P}_N$ of the ODE $Lp = q$.

d) Find some solution $v(x)$ of $v'' + v = x^2 - 1$.

249. Consider the differential equation $y^{(4)} - y = ce^{2x}$ where c is a real constant.

a) Let S_c be the set of solutions of this equation. For which c is this set a vector space? Why?.

b) For each such c, find this solution space explicitly, and find a basis for it.

250. a) If A is a constant matrix (so it does not depend on t), compute the derivative of e^{tA} with respect to the real parameter t.

b) If $M := \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$, find a constant matrix A so that $M = e^{tA}$ for all real t.

c) If $N := \begin{pmatrix} \cosh t & -\sinh t \\ \sinh t & \cosh t \end{pmatrix}$, show there is *no* constant matrix A so that $N = e^{tA}$.

251. Let A be a square constant matrix. Show that the (unique) solution $X(t)$ of the matrix differential equation

$$\frac{dX(t)}{dt} = AX(t), \qquad \text{with } X(0) = I$$

is $X(t) = e^{tA}$. [For e^A see problem 271].

252. Let $\vec{x}(t)$ be the solution of the initial value problem

$$\vec{x}\,'(t) = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \vec{x}(t) \qquad \text{with} \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Compute $x_3(1)$.

253. Consider the following system of differential equations subject to the initial conditions $y_1(0) = 1$, and $y_2(0) = 3$.

$$\begin{aligned} \frac{dy_1}{dx} &= 3y_1 - y_2 \\ \frac{dy_2}{dx} &= y_1 + y_2 \end{aligned}$$

a) Solve this system for $y_1(x)$ and $y_2(x)$.

b) What is $y_1(1)$?

254. Let $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ be the vector-valued function that solves the initial value problem

$$\vec{x}\,' = \begin{pmatrix} -1 & 0 \\ 4 & -1 \end{pmatrix} \vec{x}, \qquad \text{with} \qquad \vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Compute $x_2(2)$.

255. Solve the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 2x + y \\ \frac{dy}{dt} &= x + 2y \end{aligned}$$

for the unknown functions $x(t)$ and $y(t)$, subject to the initial conditions $x(0) = 1$ and $y(0) = 5$.

256. Determine the general (real-valued) solution $\vec{x}(t)$ to the system $\vec{x}\,' = A\vec{x}$, where

$$A = \begin{pmatrix} 7 & 1 \\ -4 & 3 \end{pmatrix}.$$

257. Determine the general (real-valued) solution $\vec{x}(t)$ to the system $\vec{x}\,' = A\vec{x}$, where

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -3 & -1 \\ 0 & 2 & -1 \end{pmatrix}.$$

258. Let A be a 2×2 matrix with real entries and we seek a solution $\vec{x}(t)$ of the vector differential equation $\vec{x}\,' = A\vec{x}$. Suppose we know that one solution of this equation is given by $e^t \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix}$. Find the matrix A and the solution to $\vec{x}\,' = A\vec{x}$ that satisfies $\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

259. Carefully determine whether or not the set $\{3,\, x - 3,\, 5x + e^{-x}\}$ forms a basis for the space of solutions of the differential equation $y''' + y'' = 0$.

260. Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and make the change of variable $y = Sx$ where $S = (s_{ki})$ is an invertible constant real matrix. Thus, $y_k = \sum_i s_{ki} x_i$.

a) Under this change of variables show that for a smooth function $u(x_1, \dots, x_n)$

$$\frac{\partial u}{\partial x_i} = \sum_k \frac{\partial u}{\partial y_k} s_{ki} \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{k\ell} \frac{\partial^2 u}{\partial y_k \partial y_\ell} s_{ki} s_{\ell j}.$$

b) If $A := (a_{ij})$ is a constant real symmetric matrix, show that

$$\sum_{ij} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{k\ell} b_{k\ell} \frac{\partial^2 u}{\partial y_k \partial y_\ell},$$

where

$$b_{k\ell} = \sum_{ij} s_{ki} a_{ij} s_{\ell j}, \quad \text{that is,} \quad B = SAS^T,$$

where S^T is the transpose of S. Because A is a symmetric matrix, we can always find an orthogonal matrix S so that B is a diagonal matrix.

c) For each of the differential operators

$$Lu := 3u_{x_1 x_1} + 2u_{x_1 x_2} + 3u_{x_2 x_2} \quad \text{and} \quad Mu := u_{x_1 x_1} + 4u_{x_1 x_2} + u_{x_2 x_2}$$

find changes of coordinates $y = Sx$ so that in the new coordinates these have th e simpler form $\alpha u_{y_1 y_1} + \beta u_{y_2 y_2}$.

261. a) Let $V = (v_1, \dots, v_n) \neq 0$ be a real vector and A be the $n \times n$ matrix $A = (v_i v_j)$, so the product $v_i v_j$ is the ij element of A. Find the eigenvalues of A. [Note: A has rank one since each column is a multiple of V. Thus $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0$. What is the trace of A? What is λ_n?]

Let $B = A + cI$, where $c \in \mathbb{R}$. Compute the eigenvalues and the determinant of B.

b) If $u(x_1, x_2, \dots, x_n)$ is a given smooth function, let $u'' := \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)$ be its second derivative (Hessian) matrix. Find all solutions of $\det(u'') = 1$ in the special case where $u = u(r)$ depends only on $r = \sqrt{x_1^2 + \cdots x_n^2}$, the distance to the origin.

c) Let $x = (x_1, x_2, \dots, x_n)$ and A be a square matrix with $\det A = 1$. If $u(x)$ satisfies $\det(u'') = 1$ (see above), and $v(x) := u(Ax)$, show that $\det(v'') = 1$ also.

d) Let $u(x_1, \dots, x_n) = f(a_1 x_1 + \cdots + a_n x_n)$, where the a_j are constants and $f(t)$, $t \in \mathbb{R}$, is a smooth function. Show that $\det u''(x) = 0$.

e) If $v(x_1, \dots, x_n) := c(x_1^2 + \cdots + x_n^2) + f(a_1 x_1 + \cdots + a_n x_n)$ where c is a constant, compute $\det u''(x)$.

REMARK: The differential operator $\det(u'')$ is interesting because its symmetry group is so large. Partial differential equations involving $\det(u'')$ are called *Monge-Ampère equations.*

16 Least Squares

262. Find the straight line $y = a + mx$ that is the best least squares fit to the points $(0, 0)$, $(1, 3)$, and $(2, 7)$.

Here, "best" is defined customarily as follows. Let $f(x) = a + mx$. Given the data points (x_1, y_1), ..., (x_n, y_n), best means picking a and b to minimize

$$E(a, m) := |f(x_1) - y_1|^2 + |f(x_2) - y_2|^2 + \cdots + |f(x_n) - y_n|^2.$$

But for many cases, such as when the data are $(\text{height}_j, \text{weight}_j)$ of the j^{th} person in a medical experiment, alternate definitions may be more appropriate. See problems 266 and 267 below.

263. The water level in the North Sea is mainly determined by the so-called M2 tide, whose period is about 12 hours see [`https://en.wikipedia.org/wiki/Tide`]. The height $H(t)$ thus roughly has the form

$$H(t) = c + a \sin(2\pi t/12) + b \cos(2\pi t/12),$$

where time t is measured in hours (note $\sin(2\pi t/12$ and $\cos(2\pi t/12)$ are periodic with period 12 hours). Say one has the following measurements:

t (hours)	0	2	4	6	8	10
$H(t)$ (meters)	1.0	1.6	1.4	0.6	0.2	0.8

Use the method of least squares as in the previous problem (but replace $f(x)$ by $H(t)$) to find the best constants a, b, and c in $H(t)$ for this data.

264. Let $L : \mathbb{R}^n \to \mathbb{R}^k$ be a linear map. If the equation $Lx = b$ has no solution, instead frequently one wants to pick x to minimize the error: $\|Lx - b\|$ (here we use the Euclidean distance). Assume that the nullspace of L is zero.

   a) Show that the desired x is a solution of the *normal equations* $L^*Lx = L^*b$ (here L^* is the adjoint of L.). Note that since the nullspace of L is zero, $L^*L : \mathbb{R}^n \to \mathbb{R}^n$ is invertible (why?).

   b) Apply this to find the optimal horizontal line that fits the three data points $(0, 1)$, $(1, 2)$, $(4, 3)$.

   c) Similarly, find the optimal straight line (not necessarily horizontal) that fits the same data points.

265. Let $A : \mathbb{R}^n \to \mathbb{R}^k$ be a linear map. If A is not one-to-one, but the equation $Ax = y$ has some solution, then it has many. Is there a "best" possible answer? What can one say? Think about this before reading the next paragraph.

   If there is some solution of $Ax = y$, show there is exactly one solution x_1 of the form $x_1 = A^*w$ for some w, so $AA^*w = y$. Moreover of all the solutions x of $Ax = y$, show that x_1 is closest to the origin (in the Euclidean distance). [REMARK: This situation is related to the case where where A is not onto, so there may not be a solution — but the method of least squares gives an "best" approximation to a solution.]

266. Let $P_j = (x_j, y_j)$, $j = 1, \ldots, , n$ be (data) points in the plane $\mathbb{R}^2$, say the height and weight of the j^{th} person in a medical test. How can we find the straight line $\mathcal{L} := \{(x, y) \in \mathbb{R}^2 \mid ax + by = c\}$ that best fits this data in the sense that it minimizes the function

$$Q(\mathcal{L}) := \sum_{j=1}^{n} [\text{distance}(P_j, \mathcal{L})]^2?$$

   a) Thus, the problem is to determine the parameters a, b, and c. As will be clear in your computation, it is simplest first to investigate the special case where $\sum_{j=1}^{n} P_j = 0$.

b) Apply this procedure to the data in problem 262.

REMARK: Also see the next problem.

267. Let P_1, P_2,..., P_k be k points (think of them as *data*) in $\mathbb{R}^3$ and let $\mathcal{S}$ be the plane

$$\mathcal{S} := \left\{X \in \mathbb{R}^3 \,:\, \langle X,\, N\rangle = c\right\},$$

where $N \neq 0$ is a unit vector normal to the plane and c is a real constant.

This problem outlines how to find the plane that *best approximates the data points* in the sense that it minimizes the function

$$Q(N,c) := \sum_{j=1}^{k}[\text{distance}\,(P_j,\mathcal{S})]^2.$$

Determining this plane means finding N and c.

a) Show that for a given point P, then

$$\text{distance}\,(P,\mathcal{S}) = |\langle P - X,\, N\rangle| = |\langle P,\, N\rangle - c\,|,$$

where X is any point in $\mathcal{S}$

b) First do the special case where the center of mass $\bar{P} := \frac{1}{k}\sum_{j=1}^{k} P_j$ is at the origin, so $\bar{P} = 0$. Show that for any P, then $\langle P,\, N\rangle^2 = \langle N,\, PP^*N\rangle$. Here view P as a column vector so PP^* is a $k \times k$ matrix.

Use this to observe that the desired plane $\mathcal{S}$ is determined by letting N be an eigenvector of the matrix

$$A := \sum_{j=1}^{k} P_j P_j^T$$

corresponding to it's lowest eigenvalue. What is c in this case?

c) Reduce the general case to the previous case by letting $V_j = P_j - \bar{P}$.

d) Find the equation of the line $ax + by = c$ that, in the above sense, best fits the data points $(-1,3)$, $(0,1)$, $(1,-1)$, $(2,-3)$.

e) Let $P_j := (p_{j1},\ldots,p_{j3})$, $j = 1,\ldots,k$ be the coordinates of the j^{th} data point and $Z_\ell := (p_{1\ell},\ldots,p_{k\ell})$, $\ell = 1,\ldots,3$ be the vector of ℓ^{th} coordinates. If a_{ij} is the ij element of A, show that $a_{ij} = \langle Z_i,\, Z_j\rangle$. Note that this exhibits A as a *Gram matrix* (see Problem 171).

f) Generalize to where P_1, P_2,..., P_k are k points in $\mathbb{R}^n$.

NOTE: In statistics this approach to understanding a data matrix $A = (a_{ij})$, which is usually not square, is called *principal component analysis* or *singular value decomposition*.

17 Markov Chains

268. In a large city, a car rental company has three locations: the Airport, the City, and the Suburbs. One has data on which location the cars are returned daily:

- RENTED AT AIRPORT: 2% are returned to the City and 25% to the Suburbs. The rest are returned to the Airport.
- RENTED IN CITY : 10% returned to Airport, 10% returned to Suburbs. The rest are returned to the City.
- RENTED IN SUBURBS: 25% are returned to the Airport and 2% to the city. The rest are returned to the Suburbs.

If initially there are 35 cars at the Airport, 150 in the city, and 35 in the suburbs, what is the long-term distriubtion of the cars?

SUGGESTION: Let $D_j := (a_j,\, c_j,\, s_j)$ be the number of cars at the Airport, City, and Suburbs at the beginning of the j^{th} day. Find a matrix M so that $D_{j+1} = MD_j$. If there is an "equilibrium" distribution $D = \lim_{j\to\infty} D_j$, show that $D = MD$.

269. Let $A = (a_{ij})$ be an $n \times n$ matrix with the property that $a_{ij} \geq 0$ for all i, j and that the sum of the elements in each of its columns is 1. These are called *transition matrices for Markov Chains*. An example is the matrix M in the previous problem.

a) Let $v = (v_1, \ldots, v_n)$ be any column vector with $v_j \geq 0$ and the sum of whose elements is 1. Show that the vector $w := Av$ has the same properties and hence that for any integer $k \geq 0$ the sum of each column of A^k is also 1. Thus A^k is also the transition matrix for a Markov Chain.

b) Show that $\lambda = 1$ is an eigenvalue of A. SUGGESTION: Let q be the column vector all of whose elements are 1. Compute $A^* q$.]

c) Show that all of the eigenvalues of A satisfy $|\lambda| \leq 1$.[See Problem 133.]

d) If in addition $0 < a_{ij}$ for all i, j, show that A^k, $k = 1, 2, \ldots$ converges to some matrix A_∞ and that all of its columns are identical.

This will take some thinking. For me the simplest proof uses the

AVERAGING INEQUALITY *If one takes a weighted average* $\overline{x} = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$ *of real numbers* $x_1, \ldots, x_n$, *xhere* $0 < \gamma \leq c_j$ *and* $c_1 + \cdots + c_n = 1$, *then the average lies strictly between the max and min of the* x_j *with the quantitative estimate*

$$x_{\min} + \gamma(x_{\max} - x_{\min}) \leq \overline{x} \leq x_{\max} - \gamma(x_{\max} - x_{\min}).$$

Apply this to where γ be the smallest element of A.

[Convergence of the A^k may fail if you only assume that $a_{ij} \geq 0$. Example: $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$].

e) Let p be one of the (identical) columns of A_∞. Show that p is a *fixed point* of A, that is, $Ap = p$ and that for any vector v with the properties of part (a) we have $A^k v \to p,\ k = 1, 2, \ldots$.

270. (Franz Pedit) Five mathematicians - Alex, Franz, Jenia, Paul and Rob - sit around a table, each with a huge plate of cheese. Instead of eating it, every minute each of them simultaneously passes half of the cheese in front of him to his neighbor on the left and the other half to his neighbor on the right. Is it true that the amount of cheese on Franz's plate will converge to some limit as time goes to infinity?

The next week they meet again, adding a sixth friend, Herman, and follow the same procedure. What can you say about the eventual distribution of the cheese?

18 The Exponential Map

271. For any square matrix A, define the *exponential* e^A by the usual power series

$$e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

a) Show that the series always converges.

b) If A is a 3×3 diagonal matrix, compute e^A.

c) If $A^2 = 0$, compute e^A.

d) If $A^2 = A$, compute e^A.

e) Show that $e^{(s+t)A} = e^{sA} e^{tA}$ for all real or complex $s,\ t$.

f) If $AB = BA$, show that $e^{A+B} = e^A e^B$. In particular, $e^{-A} e^A = I$ so e^A is always invertible.

g) If $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, verify that $e^A e^B \neq e^{A+B}$.

h) Compute e^A for the matrix $\quad A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

i) If P is a projection (so $P^2 = P$) and $t \in \mathbb{R}$, compute e^{tP}.

j) If R is a reflection (so $R^2 = I$) and $t \in \mathbb{R}$, compute e^{tR}.

k) For real t show that
$$e^{\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$
(The matrix on the right is a rotation of $\mathbb{R}^2$ through the angle t).

l) If A is a real anti-symmetric matrix, show that e^A is an orthogonal matrix.

m) If a (square) matrix A satisfies $A^2 = \alpha^2 I$, show that
$$e^A = \cosh \alpha \, I + \frac{\sinh \alpha}{\alpha} A.$$

n) If a square matrix A satisfies $A^3 = \alpha^2 A$ for some real or complex α, show that
$$e^A = I + \frac{\sinh \alpha}{\alpha} A + \frac{\cosh \alpha - 1}{\alpha^2} A^2.$$
(if A is invertible then $A^2 = \alpha^2 I$ so this formula reduces to the previous part). What is the corresponding formula if $A^3 = -\alpha^2 A$?

272. If A is a diagonal matrix, show that
$$\det(e^A) = e^{\operatorname{trace}(A)}.$$
Is this formula valid for any matrix, not just a diagonal matrix?

273. a) Let $\mathbf{v} = (a, b, c)$ be any vector. Using the matrix notation $A_{\mathbf{v}} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$ from Problem 70, show that
$$A_{\mathbf{v}}^3 = -|\mathbf{v}|^2 A_{\mathbf{v}}.$$

b) Use this (and the definition of $e^A = \sum_k A^k/k!$ from Problem 271) to verify that
$$e^{A_{\mathbf{v}}} = I + \frac{\sin |\mathbf{v}|}{|\mathbf{v}|} A_{\mathbf{v}} + \frac{1 - \cos |\mathbf{v}|}{|\mathbf{v}|^2} A_{\mathbf{v}}^2.$$
Note: this $e^{A_{\mathbf{v}}}$ is closely related to the formula for the rotation $R_{\mathbf{n}}$ of Problem 234. [See Duistermaat and Kolk, *Lie Groups*, Section 1.4 for an explanation. There the anti-symmetric matrix $A_{\mathbf{v}}$ is viewed as an element of the Lie algebra associated with the Lie group of 3×3 orthogonal matrices.] (See also Problems 70, 193, 233, 234, 235).

274. Let A be an $n \times n$ upper-triangular matrix all of whose diagonal elements are zero. Show that the matrix e^A is a polynomial in A of degree at most $n - 1$.

275. Say the square matrix A is similar to B. Is e^A similar to e^B? Proof or counterexample.

19 Jordan Form

276. Show that the following matrices are similar to upper triangular matrix –s but are not diagonalizable.

$$a).\ \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \qquad\qquad b).\ \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

277. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Suppose that there exist $v \neq 0$ and w in $\mathbb{R}^2$ such that $T(v) = v$ and $T(w) \neq w$. Show that T is diagonalizable if and only if it has an eigenvalue unequal to 1.

278. [Jordan Normal Form] Let

$$A := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 \end{pmatrix},$$

a) In the Jordan normal form for A, how often does the largest eigenvalue of A occur on the diagonal?

b) For A, find the dimension of the eigenspace corresponding to the eigenvalue 4.

279. Let $A \in M(4, \mathbb{F})$, where here $\mathbb{F}$ is any field. Let $\chi_A =$ the characteristic polynomial of A and $p(t) := t^4 + 1 \in \mathbb{F}[t]$.

True or False?

a) If $\chi_A = p$, then A is invertible.

b) If $\chi_A = p$, then A is diagonalizable over $\mathbb{F}$.

c) If $p(B) = 0$ for some matrix $B \in M(8, \mathbb{F})$, then P is the characteristic polynomial of B.

d) There is a unique monic polynomial $q \in \mathbb{F}[t]$ of degree 4 such that $q(A) = 0$.

e) A matrix $B \in M(n, \mathbb{F})$ is always nilpotent if its minimal polynomial is t^k for some integer $k \geq 1$.

280. Determine the Jordan normal form of
$$B := \begin{pmatrix} -3 & -2 & 0 \\ 4 & 3 & 0 \\ 2 & 1 & -1 \end{pmatrix}.$$

281. Let $L\colon V \to V$ be a linear map on a linear space V.

a) Show that $\ker L \subset \ker L^2$ and, more generally, $\ker L^j \subset \ker L^{j+1}$ for all $j \geq 1$, so the kernel of L^j can only get larger as j increases.

b) If $\ker L^j = \ker L^{j+1}$ for some integer j, show that $\ker L^k = \ker L^{k+1}$ for all $k \geq j$. [Hint: $L^{k+2} = L^{k+1}L$.]

Moral: If at some step the kernel of L^j does not get larger, then it never gets larger for any $k > j$.

Does your proof require that V is finite dimensional?

c) Let A be an $n \times n$ matrix. If $A^k = 0$ for some integer k we say that A is *nilpotent*. If $A : \mathbb{R}^7 \to \mathbb{R}^7$ is nilpotent, show that $A^7 = 0$. [Hint: What is the largest possible value of $\dim(\ker A)$?]

Generalize this to a nilpotent $A : \mathbb{R}^n \to \mathbb{R}^n$.

282. Let A be an $n \times n$ real matrix. If A is nilpotent, that is, $A^k = 0$ for some positive integer k, show that $A^n = 0$.

20 Derivatives of Matrices

283. Let $A(t) = (a_{ij}(t))$ be a square real matrix whose elements are smooth functions of the real variable t and write $A'(t) = (a'_{ij}(t))$ for the matrix of derivatives. [There is an obvious equivalent coordinate-free definition of the derivative of a matrix using $\lim_{h\to 0}[A(t+h) - A(t)]/h$].

a) Compute the derivative: $dA^3(t)/dt$.

b) If $A(t)$ is invertible, find the formula for the derivative of $A^{-1}(t)$. Of course it will resemble the 1×1 case $-A'(t)/A^2(t)$.

284. Let $A(t)$ be a square real matrix whose elements are smooth functions of the real variable t. Assume $\det A(t) > 0$.

a) Show that $\dfrac{d}{dt} \log \det A = \operatorname{trace}(A^{-1}A')$.

b) Conclude that for any invertible matrix $A(t)$

$$\frac{d \det A(t)}{dt} = \det A(t) \operatorname{trace}[A^{-1}(t)A'(t)].$$

c) If $\det A(t) = 1$ for all t and $A(0) = I$, show that the matrix $A'(0)$ has trace zero.

d) Compute: $\dfrac{d^2}{dt^2} \log \det A(t)$.

21 Tridiagonal Matrices

285. Investigate the basic $n \times n$ real tridiagonal matrix:

$$M = \begin{pmatrix} \alpha & \beta & 0 & \dots & 0 & 0 & 0 \\ \beta & \alpha & \beta & \dots & 0 & 0 & 0 \\ 0 & \beta & \alpha & \dots & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & \dots & \alpha & \beta & 0 \\ 0 & 0 & 0 & \dots & \beta & \alpha & \beta \\ 0 & 0 & 0 & \dots & 0 & \beta & \alpha \end{pmatrix} = \alpha I + \beta \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix} = \alpha I + \beta T,$$

where T is defined by the preceding formula. This matrix arises in many applications, such as n coupled harmonic oscillators and solving the Laplace equation numerically. Clearly M and T have the same eigenvectors and their respective eigenvalues are related by $\mu = \alpha + \beta\lambda$. Thus, to understand M it is sufficient to work with the simpler matrix T.

Find the eigenvalues, eigenvectors, and determinant. [I found it simplest to find the eigenvectors first by solving the difference equation $v_{k-1} - \lambda v_k + v_{k+1} = 0$, $k = 1, \dots, n$ with the "boundary conditions" $v_0 = v_{n+1} = 0$. Because this difference equation has constant coefficients, seek a solution having the special form $v_k = r^k$].

Gershgorin's Theorem (Problem 134) gives a quick estimate of the eigenvalues.

286. Investigate the following tridiagonal matrices (eigenvalues, eigenvectors, diagonalize?):

$$A := \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \qquad B := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad C := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

287. The most general real tridiagonal matrix is

$$T := \begin{pmatrix} a_1 & b_1 & 0 & \dots & 0 & 0 & 0 \\ c_1 & a_2 & b_2 & \dots & 0 & 0 & 0 \\ 0 & c_1 & a_3 & \dots & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & \dots & a_{n-2} & b_{n-2} & 0 \\ 0 & 0 & 0 & \dots & c_{n-2} & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \dots & 0 & c_{n-1} & a_n \end{pmatrix}.$$

and let S be an invertible diagonal matrix with elements $s_1, \dots, s_n$ on the diagonal.

a) Compute $S^{-1}TS$, showing that it is also a tridiagonal matrix with $a_1, \dots, a_n$ on the diagonal, b_k replaced by $b_k s_{k+1}/s_k$ and c_k replaced by $c_k s_k/s_{k+1}$.

b) If $b_k c_k > 0$ for all k, show that by an appropriate choice of S the matrix T is similar to a symmetric matrix. Thus its eigenvalues are all real.

22 Block Matrices

The next few problems illustrate the use of block matrices. (See also Problems 124, 182, 185, 243, and 244.)

NOTATION: Let $M = \left(\begin{array}{c|c} \mathrm{A} & \mathrm{B} \\ \hline \mathrm{C} & \mathrm{D} \end{array}\right)$ be an $(n+k) \times (n+k)$ block matrix partitioned into the $n \times n$ matrix A, the $n \times k$ matrix B, the $k \times n$ matrix C and the $k \times k$ matrix D.

Let $N = \left(\begin{array}{c|c} \mathrm{W} & \mathrm{X} \\ \hline \mathrm{Y} & \mathrm{Z} \end{array}\right)$ is another matrix with the same "shape" as M.

288. Show that the naive matrix multiplication

$$MN = \left(\begin{array}{c|c} \mathrm{AW+BY} & \mathrm{AX+BZ} \\ \hline \mathrm{CW+DY} & \mathrm{CX+DZ} \end{array}\right)$$

is correct.

289. [INVERSES]

a) Show that matrices of the above form but with $C = 0$ are a sub-ring.

b) If $C = 0$, show that M in invertible if and only if both A and D are invertible – and find a formula for M^{-1} involving A^{-1}, etc.

c) More generally, if A is invertible, show that M is invertible if and only if the matrix $H := D - CA^{-1}B$ is invertible – in which case

$$M^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BH^{-1}CA^{-1} & -A^{-1}BH^{-1} \\ -H^{-1}CA^{-1} & H^{-1} \end{pmatrix}.$$

d) Similarly, if D is invertible, show that M is invertible if and only if the matrix $K := A - BD^{-1}C$ is invertible – in which case

$$M^{-1} = \begin{pmatrix} K^{-1} & -K^{-1}BD^{-1} \\ -D^{-1}CK^{-1} & D^{-1} + D^{-1}CK^{-1}BD^{-1} \end{pmatrix}.$$

e) For which values of a, b, and c is the following matrix invertible? What is the inverse?

$$S := \begin{pmatrix} a & b & b & \cdots & b & b \\ c & a & 0 & & 0 & 0 \\ c & 0 & a & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ c & 0 & 0 & \cdots & a & 0 \\ c & 0 & 0 & \cdots & 0 & a \end{pmatrix}$$

f) Let the square matrix M have the block form $M := \left(\begin{smallmatrix} A & B \\ C & 0 \end{smallmatrix}\right)$, so $D = 0$. If B and C are square, show that M is invertible if and only if both B and C are invertible, and find an explicit formula for M^{-1}. [ANSWER: $M^{-1} := \begin{pmatrix} 0 & C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{pmatrix}$].

290. [DETERMINANTS]

a) If $B = 0$ *and* $C = 0$, show that $\det M = (\det A)(\det D)$. [SUGGESTION: One approach begins with $M = \left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right)\left(\begin{smallmatrix} I & 0 \\ 0 & X \end{smallmatrix}\right)$ for some appropriate matrix X.]

b) If $B = 0$ *or* $C = 0$, show that $\det M = \det A \det D$. [SUGGESTION: If $C = 0$, compute $\left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right)\left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right)$ for a matrix X chosen cleverly.]

c) If A is invertible, show that $\det M = \det A \ \det(D - CA^{-1}B)$. As a check, if M is 2×2, this reduces to $ad - bc$.

[There is of course a similar formula only assuming D is invertible: $\det M = \det(A - BD^{-1}C) \ \det D$.]

d) Compute the determinant of the matrix S in part e) of the previous problem.

291. Say a $2n \times 2n$ block matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where A, B, C, and D are $n \times n$ matrices that commute with each other. Prove that M is invertible if and only if $AD - BC$ is invertible.

292. Let $M = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$ be a square block matrix, where A is also a square matrix.

a) Find the relation between the non-zero eigenvalues of M and those of A. What about the corresponding eigenvectors?

b) Proof or Counterexample: M is diagonalizable if and only if A is diagonalizable.

293. If a unitary matrix M has the block form $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$, show that $B = 0$ while both A and D must themselves be unitary.

294. Let $L : V \to V$ be a linear map acting on the finite dimensional linear vector space mapping V and say for some subspace $U \in V$ we have $L : U \to U$, so U is an L-invariant subspace. Pick a basis for U and extend it to a basis for V. If in this basis $A : U \to U$ is the square matrix representing the action of L on U, show that in this basis the matrix representing L on V has the block matrix form

$$\begin{pmatrix} A & * \\ 0 & * \end{pmatrix},$$

where 0 is a matrix of zeroes having the same number of columns as the dimension of U and $*$ represent other matrices.

295. Let A, B, and C be $n \times n$ matrices.

a) Find the inverse of the $3n \times 3n$ block matrix

$$\begin{pmatrix} I & A & B \\ 0 & I & C \\ 0 & 0 & I \end{pmatrix}$$

b) Let P, Q, and R be invertible $n \times n$ matrices. Compute

$$\begin{pmatrix} P & A & B \\ 0 & Q & C \\ 0 & 0 & R \end{pmatrix} \begin{pmatrix} P^{-1} & 0 & 0 \\ 0 & Q^{-1} & 0 \\ 0 & 0 & R^{-1} \end{pmatrix}$$

and use the result to find a formula for the inverse of the above matrix on the left.

23 Interpolation

296. a) Find a cubic polynomial $p(x)$ with the properties that $p(0) = 1$, $p(1) = 0$, $p(3) = 2$, and $p(4) = 5$. Is there more than one such polynomial?

b) Given any points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) with the x_i's distinct, show there is a unique cubic polynomial $p(x)$ with the properties that $p(x_i) = y_i$.

297. Let a_0, a_1, $\ldots$, a_n be $n+1$ distinct real numbers and b_0, b_1, $\ldots$, b_n be given real numbers. One seeks an *interpolating polynomial* $p(x) = c_k x^k + \cdots + c_1 x + c_0$ that passes through these points $(a_j,\, b_j)$; thus $p(a_j) = b_j$, $j = 0, \ldots, n$.

a) If $k = n$ show there exists such a polynomial and that it is unique.

b) If p has the special form $p(x) = c_{n+1}x^{n+1} + \cdots + c_1 x$ (so $k = n+1$ and $c_0 = 0$), discuss both the existence and uniqueness of such a polynomial.

c) If p has the special form $p(x) = x^{n+1} + c_n x^n + \cdots + c_1 x + c_0$, discuss both the existence and uniqueness of such a polynomial.

24 Dependence on Parameters

In these problems we explore the dependence of the eigenvalues and eigenvectors of a matrix A on its elements.

298. Show that the eigenvalues of a square matrix depend continuously on the elements of the matrix.

[Suggestion: Because the eigenvalues are the roots of the characteristic polynomial (whose coefficients clearly depend continuously on the elements of the matrix), the problem is show that the – possibly complex – roots of a polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ depend continuously on the coefficients a_j.

It is enough to show that if $p(c) = 0$, then given any $\varepsilon > 0$ there is a $\delta > 0$ so that if $q(z) := z^n + b_{n-1}z^n + \cdots + b_1 z + b_0$ where $|b_j - a_j| < \delta$, then q has a root w that satisfies $|w - c| < \varepsilon$. [Using complex analysis one can prove a stronger assertion if c is a multiple root.]

Without loss of generality, we may assume that $c = 0$ (why?), so $p(0) = 0$, $a_0 = 0$, and $|b_0| < \delta$. If $\gamma_1, \ldots, \gamma_n$ are the roots of q, then $|\gamma_1 \cdots \gamma_n| = |b_0| < \delta$ (why?). Show that by picking δ small enough, then for some k, $|\gamma_k| < \epsilon$.]

299. [Wilkinson[2]] Let $p(x) := (x-1)(x-2)\cdots(x-6) = x^6 - 21x^5 + \cdots$ and let $p(x,t)$ be the perturbed polynomial obtained by replacing $-21x^5$ by $-(21+t)x^5$ (think of t as being small). Let $x(t)$ denote the perturbed value of the root $x = 4$, so $x(0) = 4$.

[2] See the illuminating discussion on pages 943–945 in George E. Forsythe, "Pitfalls in computation, or why a math book isn't enough", *American Mathematical Monthly*, **77**(9), Nov. 1970, pp. 931–956: `https://www.maa.org/sites/default/files/pdf/upload_library/22/Ford/GeorgeForsythe.pdf`

a) If $|t|$ is sufficiently small. Show that $x(t)$ is a smooth function of t.

b) Compute the sensitivity of this root as one changes t, that is, compute $dx(t)/dt)\big|_{t=0}$.

300. Let $A(t)$ be a self-adjoint matrix whose elements depend smoothly on the real parameter t. Say $\lambda(t)$ is an eigenvalue and $v(t)$ a corresponding eigenvector and assume these are differentiable functions of t for t near 0. Show that the derivative, λ' satisfies

$$\lambda' = \frac{\langle v,\, A'v\rangle}{\|v\|^2}.$$

301. a) If $z = c$ is a multiple root of a polynomial $p(z)$, give an easy example where this root is not a differentiable function of the coefficients of p.

b) If $z = c$ is a simple root of p (so $p(c) = 0$ but $p'(c) \neq 0$), show that locally this root depends smoothly on the coefficients. [Suggestion: implicit function theorem.]

c) Let $A(t) = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}$, $-1 \le t \le 1$ and label the eigenvalues so that $\lambda_1(t) \le \lambda_2(t)$. Compute the eigenvalues and corresponding eigenvectors, and graph the eigenvalues. This primitive example illustrates some of the complications if an eigenvalue is not simple.

302. If the elements of a square matrix $A(t)$ depend continuously on the real parameter t, then, as in Problem 298 the eigenvalues also depend contuously on t. Here is an example showing that the eigenvectors need not be continuous, even if the matrix is symmetric and depends smoothly on its elements.

Let $M_+ := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $M_- := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ and let $\varphi(t) := e^{-1/t^2}$ for $t \neq 0$ with $\varphi(0) = 0$. Define

$$A(t) = \begin{cases} \varphi(t)M_- & \text{for } t \le 0 \\ \varphi(t)M_+ & \text{for } t \ge 0 \end{cases}$$

Show that the eigenvectors of $A(t)$, however chosen for $t = 0$, cannot be continuous.

303. Let $A(t)$ be an $n \times n$ matrix whose elements depend smoothly on $t \in \mathbb{R}$ for t near zero. Say λ_0 is a *simple* eigenvalue for $t = 0$ and $\vec{v}_0$ a corresponding eigenvector. Show that for t near 0 the eigenvalue and eigenvector depend smoothly on t in the sense that there is a unique smooth solution, $\vec{v}(t)\ \lambda(t)$, (normalized, say with $\langle \vec{v}_0,\, \vec{v}(t)\rangle = 1$), of $F(\vec{v}, \lambda, t) = 0$ where $F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n+1}$ is defined by

$$F(\vec{v}, \lambda, t) := \begin{pmatrix} f(\vec{v}, \lambda, t) \\ g(\vec{v}, \lambda, t) \end{pmatrix} := \begin{pmatrix} A(t)\vec{v} - \lambda\vec{v} \\ \langle \vec{v}_0,\, \vec{v}\rangle - 1 \end{pmatrix} \tag{5}$$

where $\lambda(0) = \lambda_0$ and $\vec{v}(0) = \vec{v}_0$. Here $\langle \vec{v},\, \vec{w}\rangle = \vec{v}^T\vec{w}$ is the standard inner (dot) product with $\vec{v}^T$ the transpose. This is an outline of a proof[3] using the implicit function theorem.

[3]From `https://www.math.upenn.edu/~kazdan/504/eigenv.pdf`

a) Show that the derivative of F with respect to $\vec{v}$ and λ is

$$F'(\vec{v}, \lambda, t) = \begin{pmatrix} f_{\vec{v}} & f_\lambda \\ g_{\vec{v}} & g_\lambda \end{pmatrix} = \begin{pmatrix} A(t) - \lambda I & -\vec{v} \\ \vec{v}_0^T & 0 \end{pmatrix}$$

Here we used $\langle \vec{v}_0, \vec{v} \rangle = \vec{v}_0^T \vec{v}$ where $\vec{v}_0^T$ is the transpose of $\vec{v}_0$. Thus, at $t = 0$

$$F'(\vec{v}_0, \lambda_0, 0) = \begin{pmatrix} A(0) - \lambda_0 I & -\vec{v}_0 \\ \vec{v}_0^T & 0 \end{pmatrix}.$$

b) Use that λ_0 is a simple eigenvalue of $A(0)$ to show that $\ker(F'(\vec{v}_0, \lambda_0, 0)) = 0$ and hence that $F'(\vec{v}_0, \lambda_0, 0)$ is invertible.

[Suggestion: If if $\vec{w} := \left(\begin{smallmatrix} \vec{z} \\ r \end{smallmatrix}\right) \neq 0$ is in the kernel of $F'(\vec{v}_0, \lambda_0, 0)$, show that $A(0)\vec{z} = \lambda_0 \vec{z} + r\vec{v}_0$ and $\langle \vec{v}_0, \vec{z} \rangle = 0$. Assume $\vec{z} \neq 0$. Write $A(0)$ as a matrix in a basis whose first vector is $\vec{v}_0$ and second is $\vec{z}$ and use that λ_0 was assumed to be a simple eigenvalue of $A(0)$ to contradict $\vec{z} \neq 0$.]

25 Miscellaneous Problems

304. A *tridiagonal matrix* is a square matrix with zeroes everywhere except on the main diagonal and the diagonals just above and below the main diagonal.

Let T be a real anti-symmetric tridiagonal matrix with elements $t_{12} = c_1$, $t_{23} = c_2, \ldots$, $t_{n-1n} = c_{n-1}$. If n is even, compute $\det T$.

305. [Difference Equations] One way to solve a second order linear difference equation of the form $x_{n+2} = ax_n + bx_{n+1}$ where a and b are constants is as follows. Let $u_n := x_n$ and $v_n := x_{n+1}$. Then $u_{n+1} = v_n$ and $v_{n+1} = au_n + bv_n$, that is,

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix},$$

which, in obvious matrix notation, can be written as $U_{n+1} = AU_n$. Consequently, $U_n = A^n U_0$. If one can diagonalize A, the problem is then straightforward. Use this approach to find a formula for the Fibonacci numbers $x_{n+2} = x_n + x_{n+1}$ with initial conditions $x_0 = 0$ and $x_1 = 1$.

Other proofs are in the books: Lax, Peter D., *Linear Algebra and Its Applications*, 2nd. ed., Wiley (2007), pages 131–132, and Evans, L. C. , *Partial Differential Equations*, 2nd. ed., Amer. Math. Soc. (2010).

306. Let P be the vector space of all polynomials with real vcoefficients. For any fixed real number t we may define a linear functional L on P by $L(f) = f(t)$ (L is "evaluation at the point t"). Such functionals are not only linear but have the special property that $L(fg) = L(f)L(g)$. Prove that if L is any linear functional on P such that $L(fg) = L(f)L(g)$ for all polynomials f and g, then either $L = 0$ or there is a c in $\mathbb{R}$ such that $L(f) = f(c)$ for all f.

307. Let $\mathcal{M}$ denote the vector space of real $n \times n$ matrices and let ℓ be a linear functional on $\mathcal{M}$. Write C for the matrix whose ij entry is $(1/\sqrt{2})^{i+j}$. If $\ell(AB) = \ell(BA)$ for all $A, B \in \mathcal{M}$, and $\ell(C) = 1$, compute $\ell(I)$.

308. Let $b \neq 0$. Find the eigenvalues, eigenvectors, and determinant of

$$A := \begin{pmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ b & b & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & a \end{pmatrix}.$$

309. Let $b \neq 0$, $c \neq 0$. Find the eigenvalues, eigenvectors, and determinant of

$$B := \begin{pmatrix} a & b & b & \cdots & b \\ c & a & 0 & \cdots & 0 \\ c & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c & 0 & 0 & \cdots & a \end{pmatrix}.$$

310. Let $S : \mathbb{R}^n \to \mathbb{R}^n$ be the *cyclic permutation map*:

$$S : (x_1, \ldots, x_n) \to (x_n,\, x_1,\, x_2, \ldots, x_{n-1}).$$

a) Show that $S^n = I$. Why does this imply that the eigenvalues of S satisfy $\lambda^n = 1$? Find the eigenvalues of S.

b) Compute the corresponding eigenvectors of S.

c) Show that in the standard basis for R^n

$$S = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 1 & 0 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

311. a) Let $L : V \to V$ be a linear map on the vector space V. If L is nilpotent, so $L^k = 0$ for some integer k, show that the map $M := I - L$ is invertible by finding an explicit formula for $(I - L)^{-1}$.

b) Apply the above result to find a particular solution of $y' - y = 5x^2 - 3$. [Hint: Let V be the space of quadratic polynomials and $L := d/dx$].

c) Similarly, find a particular solution of $y'' + y = 1 - x^2$.

312. [Length of a Day] One year on July 1 in northern Canada I was camping and the sun set around 11 PM. A year later in Hawaii the sun set at around 7 PM. This led to the basic question, given the day of the year and the latitude, when does the sun set? [Since this depends on "time zones", a better question is probably, "how many hours of daylight are there?"

313. [Wine and Water] You are given two containers, the first containing one liter of liquid A and the second one liter of liquid B. You also have a cup which has a capacity of r liters, where $0 < r < 1$. You fill the cup from the first container and transfer the content to the second container, stirring thoroughly afterwords.

Next dip the cup in the second container and transfer k liters of liquid back to the first container. This operation is repeated again and again. Prove that as the number of iterations n of the operation tends to infinity, the concentrations of A and B in both containers tend to equal each other. [Rephrase this in mathematical terms and proceed from there].

Say you now have three containers A, B, and C, each containing one liter of different liquids. You transfer one cup form A to B, stir, then one cup from B to C, stir, then one cup from C to A, stir, etc. What are the long-term concentrations?

314. Snow White distributed 21 liters of milk among the seven dwarfs. The first dwarf then distributed the contents of his pail evenly to the pails of other six dwarfs. Then the second did the same, and so on. After the seventh dwarf distributed the contents of his pail evenly to the other six dwarfs, it was found that each dwarf had exactly as much milk in his pail as at the start.

What was the initial distribution of the milk?
Generalize to N dwarfs.
[From: K. Splinder *Abstract Algebra with Applications*, Vol. 1, page 192, Dekker (1994)]

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`https://www.math.upenn.edu/~kazdan/504/la.pdf`