Chap. 2 : The 2-body central force

Section 8.3. *The Equations of Motion*
Section 8.4. *The equivalent one-body problem*

Read Sections 8.3 and 8.4.

Review:
the two-body problem reduces to

(1) center of mass motion; $£_{CM} = ½ M \dot{\mathbf{R}}^2$;
$\Rightarrow$ M d**R**/dt = constant ; $\mathbf{R} = \mathbf{V}_C\, t$.

and
(2) relative motion; $£_{rel} = ½ \mu \dot{\mathbf{r}}^2 - U(r)$;
$\Rightarrow$ conservation laws .

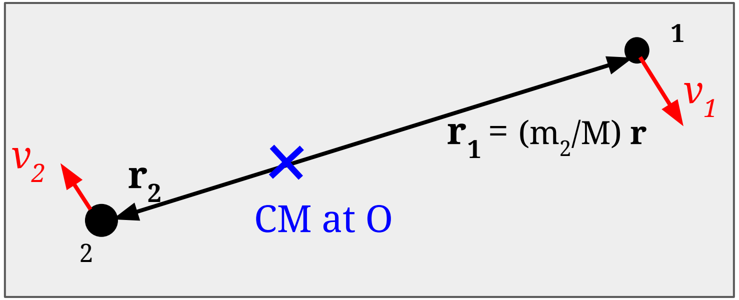
For astronomical examples,

$$U(r) = -G\, m_1\, m_2 /r$$

8.3. *The Equations of Motion*

The *center of mass frame of reference* is illustrated in FIG. 8.3; ***R* = *0* *is fixed.***

FIGURE 8.3



The Lagrangian is

$$£ = ½ \mu \dot{\mathbf{r}}^2 - U(r) .$$

Lagrange's equation is

$$\mu \ddot{\mathbf{r}} + \nabla U = 0.$$

Section 8.3. *The Equations of Motion*

The Lagrangian is

$$\pounds = \tfrac{1}{2}\,\mu\,\dot{\mathbf{r}}^2 - U(r)\,.$$

Lagrange's equation is

$$\mu\,\ddot{\mathbf{r}} + \nabla U = 0.$$

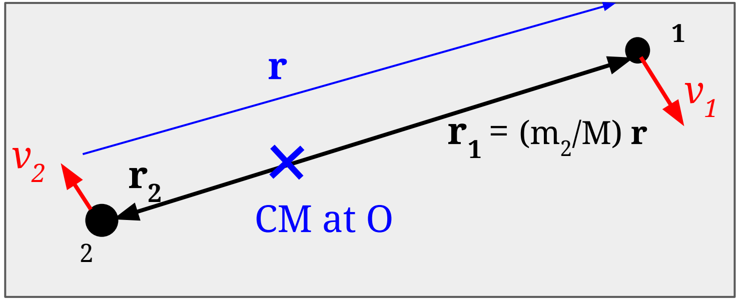
CONSERVATION OF ANGULAR MOMENTUM

Recall: *the total angular momentum is conserved,* because there are no external forces and the internal force is central.

$$\vec{L} = \vec{r}_1 \times m_1\vec{v}_1 + \vec{r}_2 \times m_2\vec{v}_2$$

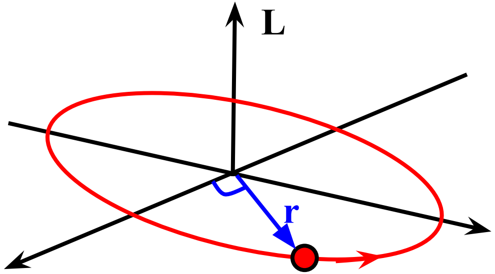
$$= m_1\left(\frac{m_2}{M}\right)^2 \vec{r}\times\dot{\vec{r}} + m_2\left(\frac{m_1}{M}\right)^2 \vec{r}\times\dot{\vec{r}}$$

$$= \mu\,\vec{r}\times\dot{\vec{r}} \qquad \text{where } \mu = \frac{m_1 m_2}{M}$$



Theorem. The orbit lies in a plane.

Proof. Because the vector **L** is perpendicular to the orbit plane, and **L** is constant.



Exercise: Prove that d**L** /dt = 0.

SPHERICAL POLAR COORDINATES

➔ Set up a coordinate system.

➔ Define the xy-plane to be the orbit plane.

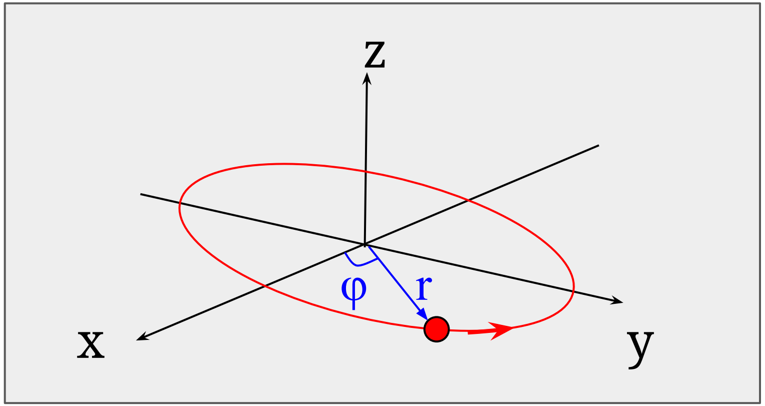
➔ Use spherical polar coordinates $\{r, \theta, \varphi\}$.

➔ The xy-plane is $\theta = \pi/2$.

➔ The Lagrangian for two coordinates, r and φ , is

$$\boxed{\pounds = \tfrac{1}{2}\,\mu\,(\dot{r}^2 + r^2\,\dot{\varphi}^2) - U(r)}$$

$$d/dt\,(\partial \pounds / \partial \dot{q}\,) - \partial \pounds / \partial q = 0$$



- The angular coordinate ($q = \varphi$)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}}\right) - \frac{\partial \mathcal{L}}{\partial \varphi} = \frac{d}{dt}\left(\mu r^2 \dot{\phi}\right) = 0$$

$$\mu r^2 \dot{\phi} = \text{a constant} = \ell$$

φ is ignorable; the constant *(generalized momentum)* is ℓ .
Exercise: Show that $\ell = |\mathbf{L}|$.

$$\mu r^2 \dot{\varphi} = \ell$$

- The radial coordinate, r

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \frac{d}{dt}(\mu \dot{r}) - \mu r \dot{\phi}^2 + \frac{dU}{dr}$$

$$= \mu \ddot{r} - \mu r \left(\frac{\ell}{\mu r^2}\right)^2 + \frac{dU}{dr}$$

$$= \mu \ddot{r} - \frac{\ell^2}{\mu r^3} + \frac{dU}{dr}$$

$$= \mu \ddot{r} + \frac{d}{dr}\left[U_{cf}(r) + U(r)\right] = 0$$

$$\text{where } U_{cf}(r) = \frac{\ell^2}{2\mu r^2}$$

We define $U_{CF}(r) = \ell^2 / (2\mu r^2)$.

This is called the ***CentriFugal potential energy***. It is not really a potential energy; it's really part of the kinetic energy. But it combines with U(r), so ...

- The energy

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \dot{\phi}^2 + U(r)$$

$$= \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\mu r^2 \left(\frac{\ell}{\mu r^2}\right)^2 + U(r)$$

$$= \frac{1}{2}\mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r)$$

$$= \frac{1}{2}\mu \dot{r}^2 + U_{cf}(r) + U(r)$$

- The energy is a constant of the motion; prove it ...

$$\frac{dE}{dt} = \frac{1}{2}\mu\, 2\dot{r}\ddot{r} + \frac{d}{dr}\left[U_{CF} + U\right]\dot{r}$$

$$= \dot{r}\left\{\mu \ddot{r} + \frac{d}{dr}\left[U_{CF} + U\right]\right\}$$

$$= 0 \quad \text{equivalent to the radial equation}$$

So these are the equations of motion …

(1) $\ell = \mu\, r^2\, \dot{\varphi}$

(2) $E = \frac{1}{2}\, \mu\, \dot{r}^2 + U_{eff}(r)$

where $U_{eff}(r) = U_{CF}(r) + U(r)$ "EFFECTIVE POTENTIAL ENERGY"

and $U_{CF}(r) = \ell^2 / (2\, \mu\, r^2)$ "CENTRIFUGAL POTENTIAL ENERGY"

ℓ and E are constants, which would be determined from the initial conditions or other information.

One Strategy: First solve (2) [*which only depends on r(t)*] ;
then integrate (1) to get φ(t).

Better strategy: Combine (1) and (2) to eliminate t, and get r(φ);
then integrate (1) to get the relation between φ and t .

Section 8.4.

The equivalent one-dimensional problem

THE RADIAL EQUATION

$$E = \tfrac{1}{2}\,\mu\,\dot{r}^{\,2} + U_{CF}(r) + U(r)$$

It's a one dimensional problem;
to find r(t) .

Recall the graphical analysis of potential energy. Kinetic energy is positive, so E must be greater than $U_{eff}(r)$; or, rather, ***r is limited to have $U_{eff}(r) < E$.***

Also, where $U_{eff}(r) = E$ is a turning point.

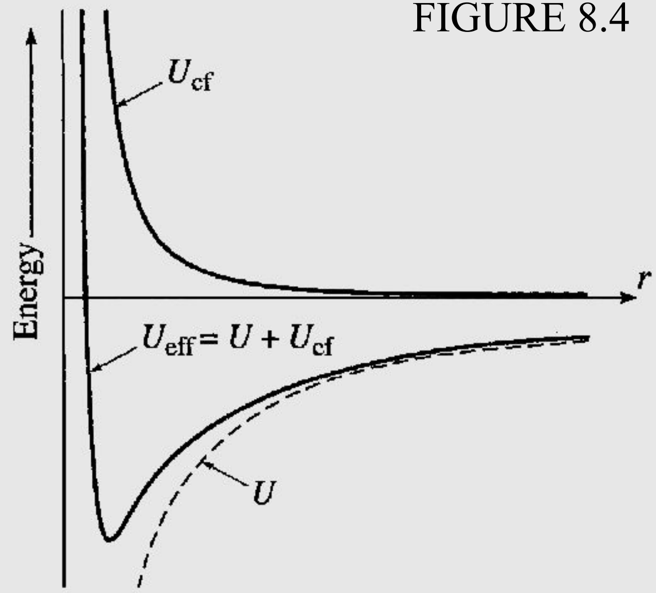
The effective potential energy

$$U_{eff}(r) = U(r) + \ell^2 / (2\mu r^2)$$

$U(r) = -\,G\, m_1\, m_2 /r = -\,GM\mu /r$ for satellites

$U_{CF}(r) = \ell^2 / (2\mu r^2)$ "centrifugal potential"

FIGURE 8.4



Example 8.2.
Energy considerations for a comet or planet

Look at FIGURE 8.5.

☒ If $E < 0$ then there are two turning points, at $r = r_{min}$ and $r = r_{max}$. This is a bounded orbit.

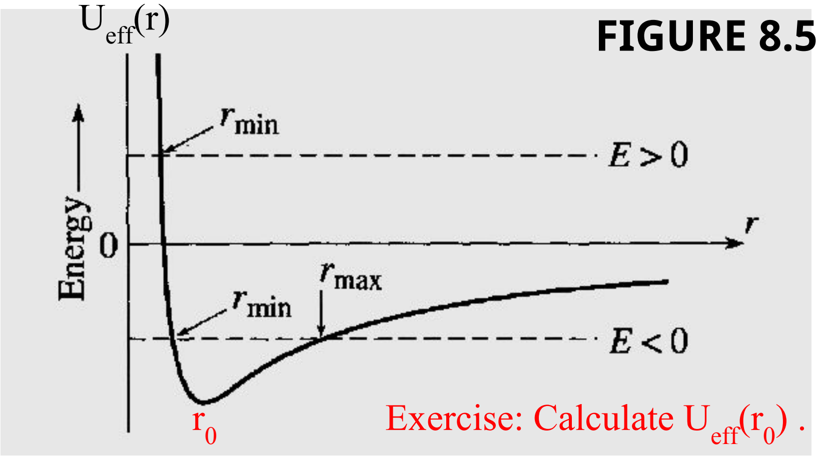
As the satellite revolves around the sun, it never gets closer than r_{min} and it never gets farther away than r_{max}. At some time, $r = r_{min}$; then r increases until $r = r_{max}$; then r decreases back to r_{min}; *etc.*

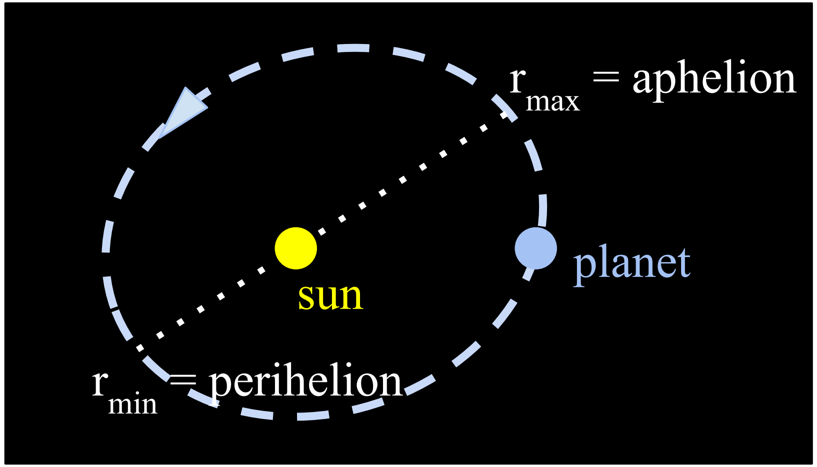
☒ $r_{min} = r_{max} = r_0$ is a circular orbit. ***Calculate E.***

☒ If $E > 0$ then there is only one turning point, at $r = r_{min}$. This is an unbounded orbit.

The satellite will escape from the sun ($r \to \infty$) .

FIGURE 8.5





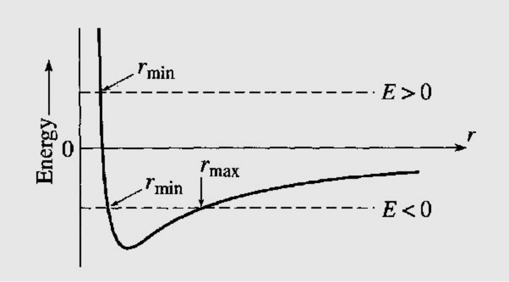
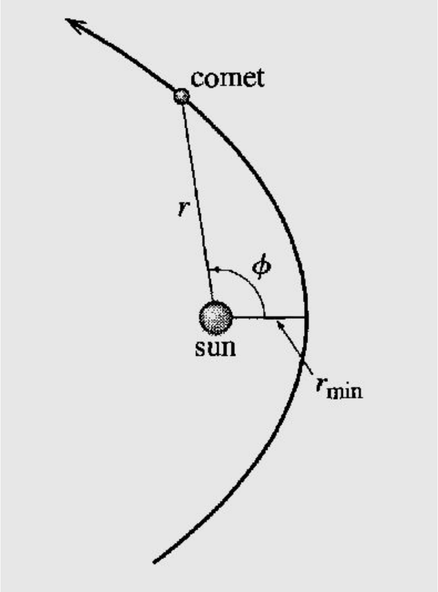


FIGURE 8.6 : A typical unbounded orbit



Calculate r(t) using a computer

$$\dot{r}^2 = \frac{2}{\mu}\left[E - \frac{\ell^2}{2\mu r^2} + \frac{GM\mu}{r}\right]$$

and $r(0) = r_p$; then $E = \frac{\ell^2}{2\mu r_p^2} - \frac{GM\mu}{r_p}$

$$\text{time} = \int_{r_p}^{r} \frac{dr'}{\sqrt{\frac{2}{\mu}\left[E - \frac{\ell^2}{2\mu r'^2} - \frac{GM\mu}{r'}\right]}}$$

Calculate the integral numerically; then plot r versus t.

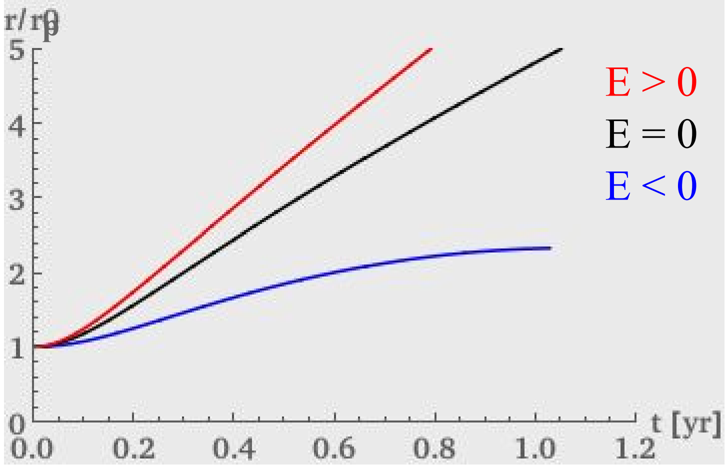
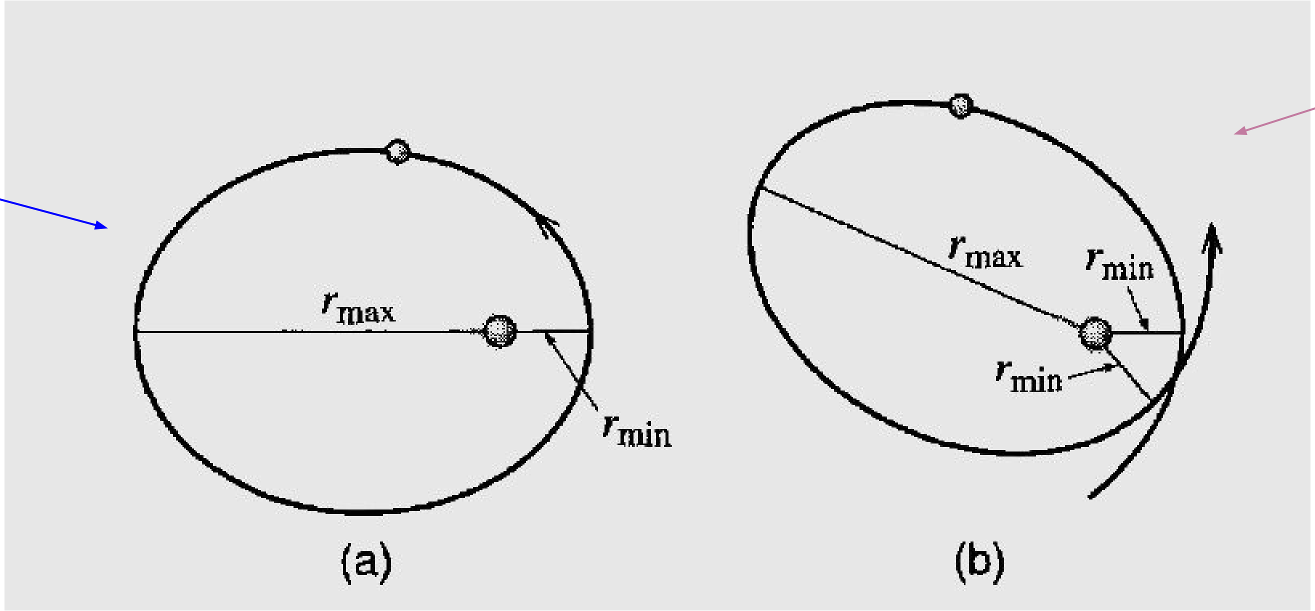


FIGURE 8.7. Typical bounded orbits

(a) A closed orbit: the orbit is a closed curve because when r varies from r_{min} to r_{max} to r_{min} , φ varies from 0 to 2π; i.e., the *radial period* is equal to the *angular period;* for example, an ellipse.

(b) An unclosed orbit: the orbit is bounded but not closed; in this figure the radial period is less than the angular period; for example, a precessing ellipse.



Homework Assignment 13

due in class Friday December 2

[71] Problem 8.4 ★

[72] Problem 8.6 ★

[73] Problem 8.12 ★★

[74] Problem 8.15 ★

[75] Problem 8.16 ★★

Use the cover sheet.