[Section 3.3](#)

The Center of Mass

Read Section 3.3.

■ The center of mass of a system of N particles

The system consists of N particles:

masses = m_α

position vectors = $\boldsymbol{r}_\alpha$

(α = 1 2 3 … N)

The total mass is

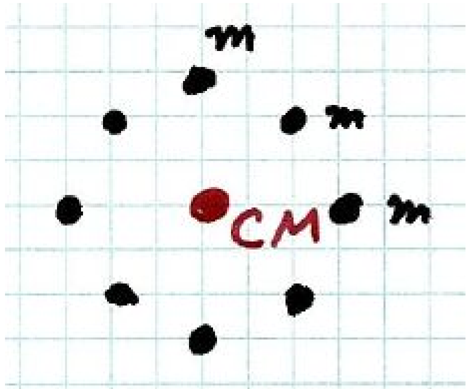
$$M = \sum_{\alpha=1}^{N} m_\alpha \, .$$

Define the center of mass position, ***R***

$$\boldsymbol{R} = \frac{\sum_{\alpha=1}^{N} m_\alpha \boldsymbol{r}_\alpha}{\sum_{\alpha=1}^{N} m_\alpha}$$

$$\boldsymbol{R} = (1/M) \sum_{\alpha=1}^{N} m_\alpha \boldsymbol{r}_\alpha$$

R = the "average position", weighted by the masses



These two theorems shows why $\boldsymbol{R}$ is important:

Theorem 1

$$M\,\mathbf{R}' = \mathbf{P} \qquad \text{(total momentum)}$$

Theorem 2

$$M\,\mathbf{R}'' = \mathbf{F}^{ext}$$

In words, the center of mass position moves in the same way as if all the mass and force was concentrated at the center of mass (assuming $\mathbf{F}^{\mathbf{ext}}$ is independent of the structure)

(prime $'$ or dot $\dot{}$ means d/dt)

Proofs.

Easy …

$$M\dot{\vec{R}} = (m_1+m_2)\frac{d}{dt}\,\frac{m_1\vec{r}_1+m_2\vec{r}_2}{m_1+m_2}$$

$$= m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2$$

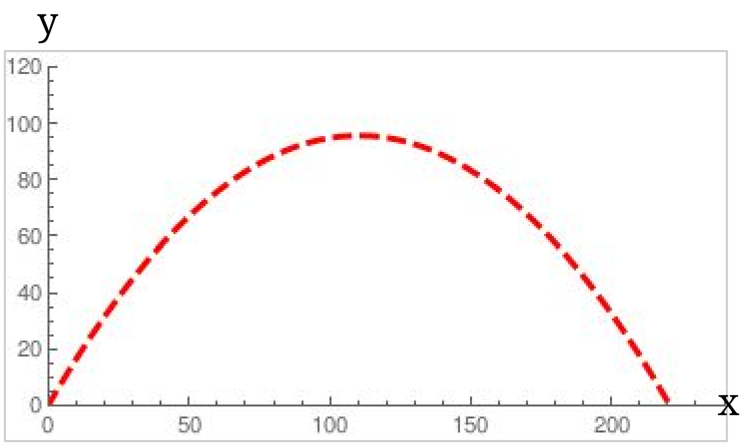
$$= \vec{p}_1 + \vec{p}_2 = \vec{P}$$

$$M\ddot{\vec{R}} = \dot{\vec{P}} = \vec{F}^{ext}$$

Example

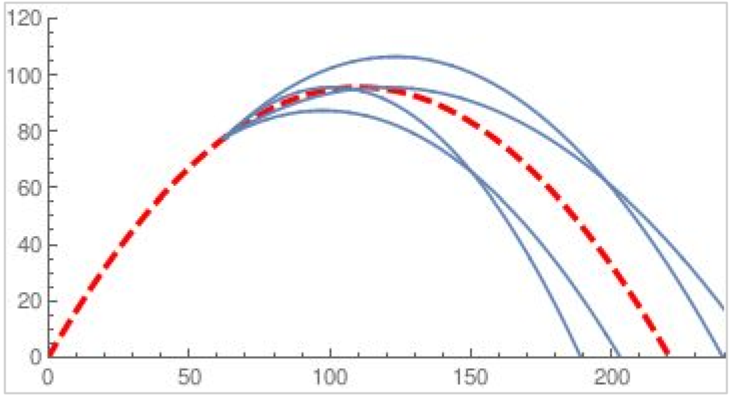
an exploding projectile

Case A. If the projectile does not explode, then the trajectory is a parabola (*ignore air resistance*) .



Case B. If the projectile explodes into fragments, then the center of mass point follows the same parabola as case A.

(F^{ext} = Mg independent of the structure.)



■ The center of mass of two particles

Figure 3.3

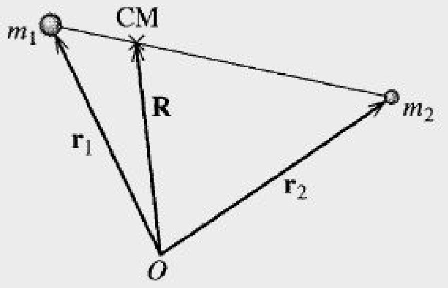
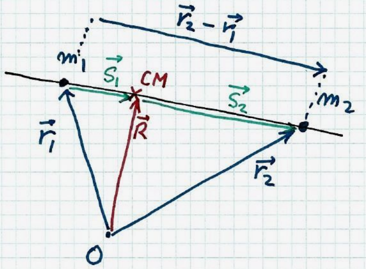


Figure 3.3 The CM of two particles lies at the position $\mathbf{R} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/M$. You can prove that this lies on the line joining m_1 to m_2, as shown, and that the distances of the CM from m_1 and m_2 are in the ratio m_2/m_1.

- ❑ The center of mass of two particles lies on the line joining the two particles.
- ❑ $m_1\ s_1 = m_2\ s_2$

Proof *(Taylor Problem 3.x)*

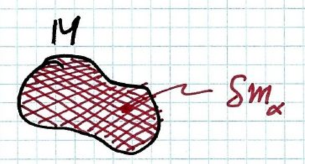


$\vec{r}_1 + \vec{s}_1 = \vec{R}$ so $\vec{s}_1 = \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} - \vec{r}_1 = \dfrac{m_2(\vec{r}_2 - \vec{r}_1)}{m_1+m_2}$

$\vec{R} + \vec{s}_2 = \vec{r}_2$ so $\vec{s}_2 = \vec{r}_2 - \dfrac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} = \dfrac{m_1(\vec{r}_2 - \vec{r}_1)}{m_1+m_2}$

- $\vec{s}_1$ and $\vec{s}_2$ are parallel to $\vec{r}_2 - \vec{r}_1$ so the CM lies on the line from m_1 to m_2
- $\dfrac{|\vec{s}_1|}{|\vec{s}_2|} = \dfrac{m_2}{m_1}$ so $m_1|\vec{s}_1| = m_2|\vec{s}_2|$.

■ **R** for a solid body



➡Imagine the object divided into an infinite number of infinitesimal parts.

➡Recall the definition of an ***integral*** in calculus.

$$\sum_{i=1}^{N} f(x_i)\,\delta x_i \text{ in the limit } N\to\infty$$
$$\longrightarrow \int_a^b f(x)\,dx$$

➡<u>Total mass</u>

$$M = \sum\nolimits^{N}_{\alpha=1} (\delta m_\alpha)$$

Now take the limit $N \to \infty$ and $\delta m \to 0$ to get the continuum,

$$M = \int_B dm \;=\; \int_V \rho(\mathbf{r})\, d^3r$$

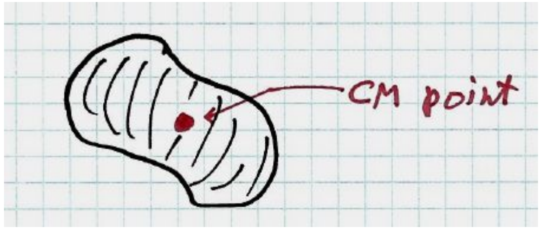
➡<u>Center of Mass position</u>

$$\boldsymbol{R} = (1/M) \sum\nolimits^{N}_{\alpha=1} \boldsymbol{r}_\alpha\, (\delta m_\alpha)$$

Now take the limit $N \to \infty$ and $\delta m \to 0$ to get the continuum,

$$\mathbf{R} = (1/M) \int_B \mathbf{r}\, dm$$
$$= \int_V \mathbf{r}\, \rho(\mathbf{r})\, d^3r \,/\, M\,.$$

I.e., **R** is the mean position weighted by the mass density.



Example 3.2

CoM of a solid cone

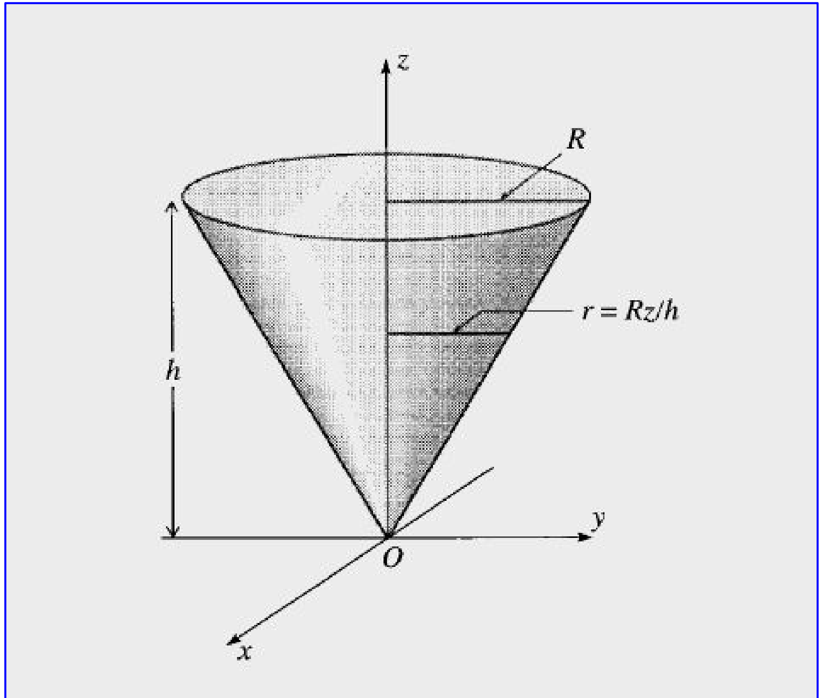
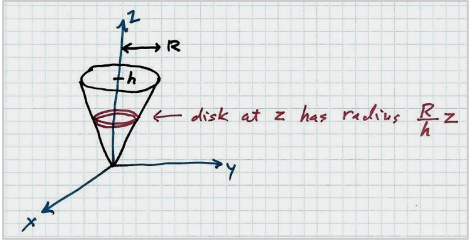


Figure 3.4 A solid cone, centered on the z axis, with vertex at the origin and uniform mass density ϱ. Its height is h and its base has radius R.

Calculate the CoM position.

First, by symmetry the CoM point lies on the z axis; so $R_C = z_c\, \boldsymbol{e_z}$.

Divide the cone into disks and do the integral over z.



$$M = \int \rho \, dV = \rho \int_0^h \pi \left(\frac{Rz}{h}\right)^2 dz$$

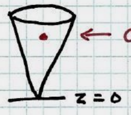
$$= \rho \frac{\pi R^2}{h^2} \frac{h^3}{3} = \frac{\pi}{3} R^2 h \rho$$

(cone = ⅓ cylinder)

$$M z_c = \int z \rho \, dV = \rho \int_0^h \pi \left(\frac{Rz}{h}\right)^2 z \, dz$$

$$= \rho \frac{\pi R^2}{h^2} \frac{h^4}{4} = \frac{\pi}{4} R^2 h^2 \rho$$

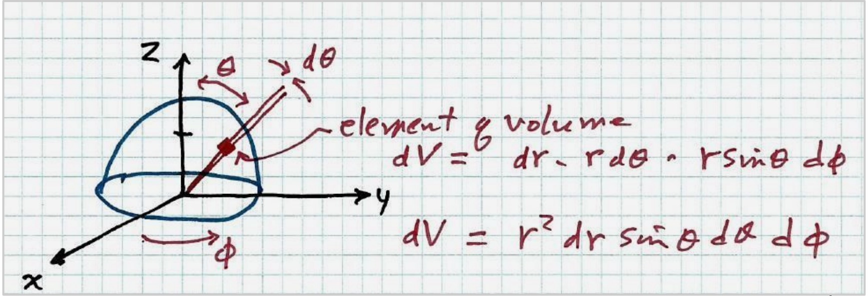
$$z_c = \frac{3}{4} h$$



← CM at $\frac{3}{4}h$

Taylor Problem 3.22

Find the CoM of a hemisphere.



By cylindrical symmetry, the CoM point lies on the z axis. Let z_c = z coordinate of the CoM point.

$$M = \int_B \rho \, dV = \int_0^R \rho \, r^2 dr \int_0^{\pi/2} \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \rho \, \frac{R^3}{3} \cdot 1 \cdot 2\pi = \frac{2\pi R^3}{3} \rho$$

$$M z_c = \int_B z \rho \, dV \quad \text{where} \quad z = r\cos\theta$$

$$= \int_0^R \rho \, r^3 dr \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= \rho \, \frac{R^4}{4} \cdot \frac{1}{2} \cdot 2\pi = \frac{\pi R^4}{4} \rho$$

$$z_c = \frac{3}{8} R$$

Homework Assignment #5
due in class Friday, October 7
[21] Problem 3.4 **
[22] Problem 3.5 **
[23] Problem 3.6 *
[24] Problem 3.10 *
[25] Problem 3.12 **
[26] Problem 3.13 **

Use the cover sheet.

*The first hour exam is in class next Friday (October 7). Do the homework **now** so that you will have some time to study for the exam.*

Study the basic equations and the quiz questions.

www.pa.msu.edu/courses/phy321/