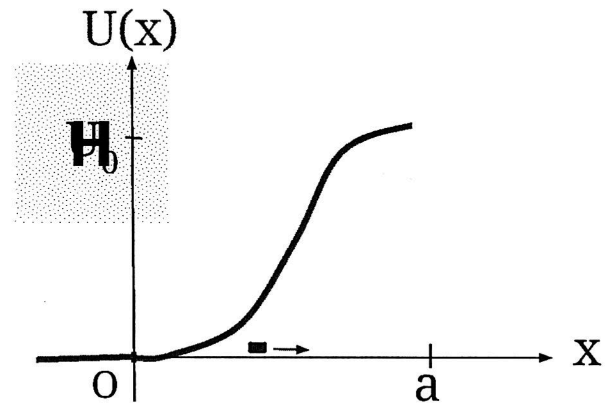
Name *Grading Key*

Extra Credit for Exam 2 : due in class Friday November 11

Write your solutions on this paper, and staple the computer plot for 1(c).

(1) A particle (mass = m) moves in one dimension (x) under the influence of a conservative force. The potential energy is U(x) = H $\sin^2(\pi x/(2a))$ for $x \geq 0$. The initial position and velocity are x(0) = 0 and v(0) = v_0 . { x is horizontal, so there is no gravitational force. }



(a) Suppose the particle energy E is equal to H. Prove that v_0 = sqrt(2H / m).

(b) For E = H , calculate the time it will take to move from 0 to x .

(c) Now assume m = 1 kg, a = 1 meter, and H = 5 joules. *Use a computer* to plot t(x) (= the time to move from 0 to x) versus x. Label the axes clearly in seconds and meters.

$$\boxed{\int \frac{d\theta}{\cos(\theta)} = \ln\left[\frac{1+\sin(\theta)}{\cos(\theta)}\right] + C}$$

Energy $E = \frac{1}{2} m \dot{x}^2 + H \sin^2\left(\frac{\pi x}{2a}\right)$ is constant.

Initial conditions $\Rightarrow E = \frac{1}{2} m v_0^2$.

(a) Now let E = H. Then $v_0 = \sqrt{2H/m}$. 1pt

(b) Time calculation: $dt = \frac{dx}{\dot{x}}$, or $t = \int_0^x \frac{dx'}{v(x')}$

We have $\frac{1}{2} m v^2 = H\left[1 - \sin^2 \frac{\pi x}{2a}\right] = H \cos^2\left(\frac{\pi x}{2a}\right)$

So $t = \int_0^x \frac{dx'}{\sqrt{2H/m}\, \cos\left(\frac{\pi x'}{2a}\right)} = \sqrt{\frac{m}{2H}} \int_0^{\pi x/2a} \frac{\frac{2a}{\pi} d\theta}{\cos\theta}$

$\theta = \frac{\pi x'}{2a}$
$d\theta = \frac{\pi}{2a} dx'$

$= \sqrt{\frac{m}{2H}}\, \frac{2a}{\pi} \ln\left[\frac{1+\sin(\pi x/2a)}{\cos(\pi x/2a)}\right]$. 1 pt

(c) Assume m = 1 kg, a = 1 m and H = 5 J. Then in MKS units,

$t = \frac{1}{\sqrt{10}} \frac{2}{\pi} \ln\left[\frac{1+\sin(\pi x/2)}{\cos(\pi x/2)}\right]$ ← Plot t versus x.

1pt
(See the plot)

(2) A damped oscillator is driven by a harmonic driving force. $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$

The parameters are $\omega_0 = 2\pi\ \text{sec}^{-1}$, $\beta = 0.2\ \pi\ \text{sec}^{-1}$, $f_0 = 10\ \text{m/s}^2$; the angular frequency of the driving force is ω.

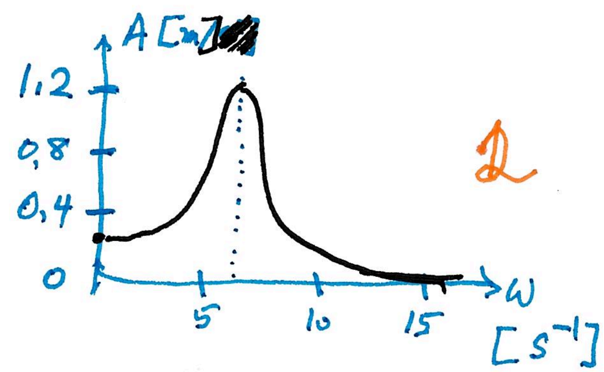
(A) A = amplitude of the steady-state oscillations.
Sketch a reasonably accurate hand-drawn graph of A versus ω ;
label the axes carefully—A in m/s^2 and ω in sec^{-1} .

(B) δ = phase shift of the steady-state oscillations.
Sketch a reasonably accurate hand-drawn graph of δ versus ω ;
label the axes carefully—δ in radians and ω in sec^{-1} .

The steady state solution in complex form is $z = \dfrac{f_0\, e^{i\omega t}}{\omega_0^2 - \omega^2 + 2\beta i\omega}$

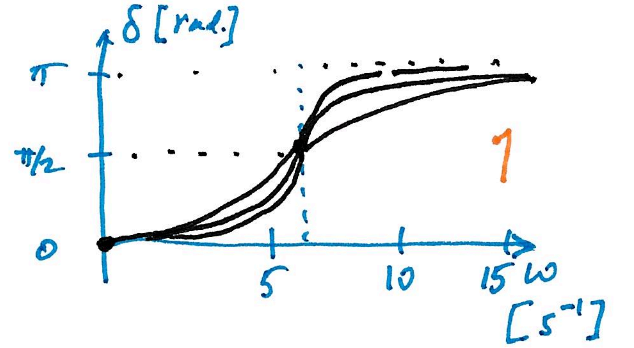
$A = \dfrac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}}$ and $\tan\delta = \dfrac{2\beta\omega}{\omega_0^2 - \omega^2}$

(A)



$A(\omega_0) = \dfrac{f_0}{2\beta\omega_0} = \dfrac{10\text{ m}}{0.8\pi^2}$
≈ 1.2 m
$A(0) = \dfrac{f_0}{\omega_0^2} = \dfrac{10\text{ m}}{4\pi^2}$
≈ 0.25 m

(B)



$\delta(0) = 0^+$
$\delta(\omega_0) = \pi/2 = 1.57$
$\delta(\omega \gg \omega_0) = 0^-$ or π
(?) (?)

(?) Which branch of arctan is appropriate?
Suppose $\omega \gg \omega_0$ and β is small. Then $\delta \approx \pi$:
$\ddot{x} = f_0 \cos(\omega t) \Rightarrow x = -\dfrac{f_0}{\omega^2}\cos(\omega t) = \dfrac{f_0}{\omega^2}\cos(\omega t - \pi)$.

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In[13]:= bs = {FontFamily → "Helvetica", FontSize → 14, FontWeight → "Bold"}

Out[13]= {FontFamily → Helvetica, FontSize → 14, FontWeight → Bold}
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In[26]:= {m, a, H} = {1, 1, 5};
         t[x_] := Sqrt[m / 2 / H] * (2 * a / Pi) *
           Log[(1 + Sin[Pi * x / 2 / a]) / Cos[Pi * x / 2 / a]]
         Plot[t[x], {x, 0, 1},
          PlotRange → {{0, 1.1}, {0, 3}},
          Frame → True, FrameLabel → {"x [m]", "t [sec]"},
          PlotStyle → {Black, Thickness[0.0075]},
          BaseStyle → bs]
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Out[28]=