The complete solution

Mon - Today 3.2, $\boxed{3.3}$, 3.4

Wed 3.4 / 3.5 Independence

Hmwk check up to 3.2 (3.3 4th ed)

Fri 3.5 Dimensions of 4 subspace

Mon Finishing

Hmwk Check up to 3.4 (3.5 4th ed)

Wed Review

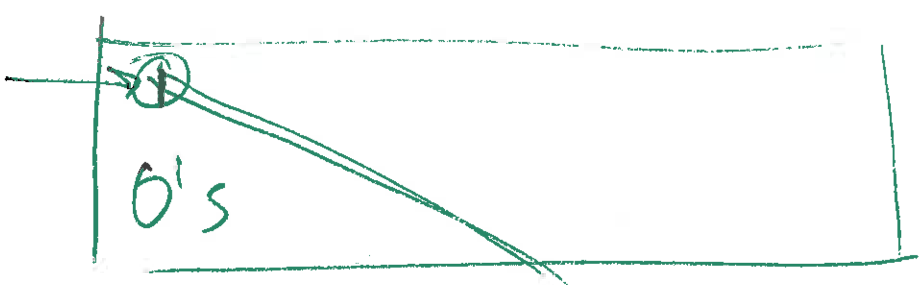
Fri - Exam 2

3.2/3.3

The **Rank** of a matrix A is

the number of pivot variables

(# of lead variables)



If A is $m \times n$ matrix, then

$\text{rank}(A) + N(A) = n$

Rank-Nullity Theorem

Ex) If A is 4×4 matrix and A has 4 pivots.

- A is invertible
- $\text{Rank}(A) = 4$ "full rank"
- $N(A) \Rightarrow N(A)$ only has one ~~solution~~ element: $\vec{0}$ ← the zero vector

3.4 4th Ed. . .

3.3 5th Ed. . . <u>The Complete Solution</u>

Solving $A\vec{x} = \vec{0}$ in the past sections

previously solved $A\vec{x} = \vec{b}$

$\vec{x}_p$ is the solution to $A\vec{x} = \vec{b}$

↑ particular solution

$\vec{x}_n$ is solution to $A\vec{x} = \vec{0}$

↑ null space solution

$$A(\vec{x}_p + \vec{x}_n) = \vec{b}$$

why??

$$A\vec{x}_p + A\vec{x}_n = \vec{b} \quad \text{because } A\vec{x}_n = \vec{0}$$

Ex) Find the complete solution $\vec{x} = \vec{x}_p + \vec{x}_n$ by forward elimination on $[A|\vec{b}]$

↑ A augmented w/ $\vec{b}$.

$$\underset{3\times 4}{\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 4 & 8 \\ 4 & 8 & 6 & 8 \end{bmatrix}} \underset{4\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} = \underset{3\times 1}{\begin{bmatrix} 4 \\ 2 \\ 10 \end{bmatrix}} \leftarrow \vec{b}$$

$$[A|\vec{b}] = \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 4 \\ 2 & 4 & 4 & 8 & 2 \\ 4 & 8 & 6 & 8 & 10 \end{array}\right]$$

← we will find $\vec{x}_p$ when we do row elimin.

$$\begin{array}{c} \\ U= \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 2 & 8 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

x_1 pivot

x_3 pivot

x_2, x_4 free variables

To get R

$\frac{1}{2}R_2 \to R_2$

$R_1 - R_2 \to R_1$

$$\begin{array}{c} \\ R= \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 2 & 0 & -4 & 7 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

- To find particular solution $\vec{x}_p$, set free variables equal to zero

$x_2 = x_4 = 0$

$x_1 + 2x_2 + 0x_3 - 4x_4 = 7$

$x_1 = 7$

$x_3 + 4x_4 = -3$

$x_3 = -3$

$$\vec{x}_p = \begin{bmatrix} 7 \\ 0 \\ -3 \\ 0 \end{bmatrix}$$

- To find the null space solution $\vec{x}_n$

$$[R \mid \vec{0}] \left[\begin{array}{cccc|c} 1 & 2 & 0 & -4 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_2 = 1, x_4 = 0$

$x_1 + 2x_2 - 4x_4 = 0$

$1x_3 + 4x_4 = 0$

$x_1 + 2 = 0 \quad x_1 = -2$

$x_3 = 0 \quad \vec{s}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$x_4 = 1, \quad x_2 = 0$

$x_1 + 2x_2 - 4x_4 = 0$

$x_1 - 4 = 0$

$x_1 = 4$

$x_3 + 4x_4 = 0$

$x_3 = -4$

$\vec{s}_2 = \begin{bmatrix} 4 \\ 0 \\ -4 \\ 1 \end{bmatrix}$

$N(A) = \underbrace{c_1 \vec{s}_1 + c_2 \vec{s}_2}$

all linear combinations of $\vec{s}_1 + \vec{s}_2$

complete solution: $\vec{x}_p + \vec{x}_n$

$$= \begin{bmatrix} 7 \\ 0 \\ -3 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

$\cancel{\text{rank}(A) + \text{nullity}(A)}$

Rank-Nullity Theorem

$$\text{Rank}(A) + \dim(N(A)) = n$$

$A_{m \times n}$ matrix

$\dim(N(A)) = \#$ of free variables

$\text{rank}(A) = \#$ of pivot variables

example from Above: $A_{\underset{m \times n}{3 \times 4}}$ matrix

$\text{rank}(A) = 2 \leftarrow 2$ pivots $\quad x_1$ & x_3

$\dim(N(A)) = 2 \quad$ 2 free variables

How can we tell that

$\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots, \vec{v}_k$ are

linearly independent?

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + \cdots + c_k\vec{v}_k = \vec{0}$$

Only ~~the~~ way to get $\vec{0}$ vector is if $c_1, c_2, \ldots, c_k = 0$