Chapter 3
Momentum and Angular Momentum

Section 3.1
Conservation of Momentum

Read Section 3.1.

First, define momentum for a single particle,

$$\mathbf{p} = m\,\boldsymbol{v} .$$

Now consider a system containing N particles. For each particle there is momentum,

$$\mathbf{p}_\alpha = m_\alpha\,\boldsymbol{v}_\alpha$$

(α = 1 2 3 … N)

The total momentum of the system is **P**

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \ldots + \mathbf{p}_N .$$

$$\mathbf{P} = \sum_{\alpha=1}^{N} \mathbf{p}_\alpha = \sum_{\alpha=1}^{N} m_\alpha \boldsymbol{v}_\alpha$$

Here is the crucial result:

Theorem. $\mathbf{P}' = \mathbf{F}^{ext}$

where $\mathbf{F}^{ext}$ is the sum of all external forces acting on the particles.

(prime ′ or dot ˙ means d/dt)

Proof

$$\vec{P} = \sum_\alpha \vec{p}_\alpha = \sum_\alpha m_\alpha \vec{v}_\alpha$$

$$\frac{d\vec{P}}{dt} = \sum_\alpha m_\alpha \frac{d\vec{v}_\alpha}{dt} = \sum_\alpha \vec{F}_\alpha$$

$$= \sum_\alpha \vec{F}_\alpha^{ext} + \sum_\alpha \sum_{\substack{\beta \\ (\beta \neq \alpha)}} \vec{F}_{\alpha\beta}$$

All the internal forces cancel in pairs by Newton's third law;

$\vec{F}_{12} + \vec{F}_{21} = 0$ or $\vec{F}_{\alpha\beta} + \vec{F}_{\beta\alpha} = 0$

$$\therefore \quad \frac{d\vec{P}}{dt} = \sum_\alpha F_\alpha^{ext} = \vec{F}^{ext}$$

The principle of conservation of momentum

For an *isolated* system of N particles, the total momentum is a constant of the motion.

Proof: Because it is an isolated system, there are no external forces.

Then the theorem states that d**P** /dt = 0.

Hence **P** is constant in time.

Example 3.1

A perfectly inelastic collision

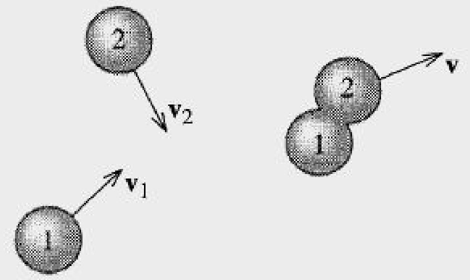


Figure 3.1 A perfectly inelastic collision between two lumps of putty.

The problem is to calculate $\boldsymbol{v}$.

Principle: *The total momentum is conserved.*

Before the collision,

$$\mathbf{P} = m_1 \boldsymbol{v_1} + m_2 \boldsymbol{v_2}$$

After the collision (stuck together)

$$\mathbf{P} = (m_1+m_2)\, \boldsymbol{v}$$

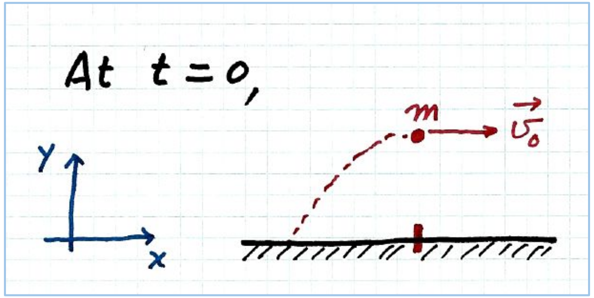
The momentum is conserved (constant) so

$$(m_1+m_2)\, \boldsymbol{v} = m_1 \boldsymbol{v_1} + m_2 \boldsymbol{v_2}$$

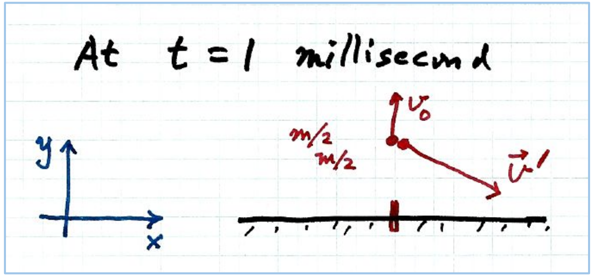
$$\therefore \quad \boldsymbol{v} = (m_1 \boldsymbol{v_1} + m_2 \boldsymbol{v_2}) / (m_1+m_2)$$

Special cases: If $m_1 >> m_2$ then $\boldsymbol{v} \approx \boldsymbol{v_1}$;
If $m_1 << m_2$ then $\boldsymbol{v} \approx \boldsymbol{v_2}$;
If $m_1 = m_2$ then $\boldsymbol{v} \approx \frac{1}{2} (\boldsymbol{v_1} + \boldsymbol{v_2})$.

Another example:
an exploding projectile (Taylor, Problem 3.2)



Now the shell explodes into 2 equal mass fragments, and one fragment goes straight up with speed v_0



▌Calculate the velocity of the second fragment.

By conservation of momentum,

$$m v_0 \hat{e}_x = \frac{m}{2} v_0 \hat{e}_y + \frac{m}{2} \vec{v}'$$

$$\therefore \quad \vec{v}' = 2 v_0 \hat{e}_x - v_0 \hat{e}_y$$

▌How much energy was released in the explosion?

$$T_f - T_i = \frac{1}{2}\frac{m}{2} v_0^2 + \frac{1}{2}\frac{m}{2} v'^2 - \frac{1}{2} m v_0^2$$

$$= \frac{1}{4} m v_0^2 + \frac{1}{4} m \left(4 v_0^2 + v_0^2\right) - \frac{1}{2} m v_0^2$$

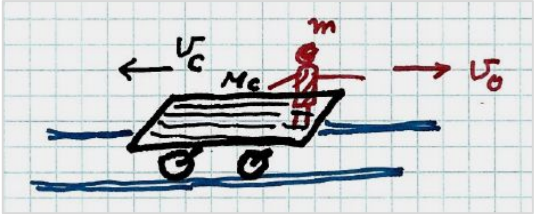
$$= m v_0^2$$

← The explosion released that amount of energy, which was converted into kinetic energy of the fragments.

Another example:

a hobo jumps off a railroad flat car

(Taylor, Problem 3.4)



Assuming the car is initially at rest, calculate the increase of kinetic energy.

Principle: Momentum is conserved.

$$p = 0 = m v_0 + M_c v_c$$

$$v_c = -\frac{m v_0}{M_c}$$

$$\Delta K.E. = \frac{1}{2} m v_0^2 + \frac{1}{2} M_c v_c^2$$

$$= \frac{1}{2} m v_0^2 + \frac{1}{2} M_c \left(-m v_0 / M_c\right)^2$$

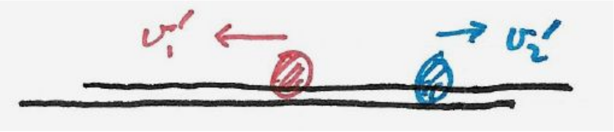
$$= \frac{1}{2} m v_0^2 \left[1 + \frac{m}{M_c}\right] \longleftarrow \begin{cases} \approx \frac{1}{2} m v_0^2 & \text{if } m \ll M_c \\ m v_0^2 & \text{if } m = M_c \end{cases}$$

Comment. In impulsive collisions, momentum is conserved because during the short time of the collision, external forces are negligible.

<u>A head-on elastic collision in 1 dimension with one particle at rest</u>

Before

After



$$m_1 v_1 = m_1 v_1' + m_2 v_2'$$

$$\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$$

ELASTIC

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In[1]:= eqs = {m1 == m1 * a + m2 * b,
          m1 == m1 * a ^ 2 + m2 * b ^ 2}
        Solve[eqs, {a, b}]
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Out[1]= $\{m1 == a\ m1 + b\ m2,\ m1 == a^2\ m1 + b^2\ m2\}$

Out[2]= $\left\{\{a \to 1,\ b \to 0\},\ \left\{a \to \frac{m1 - m2}{m1 + m2},\ b \to \frac{2\ m1}{m1 + m2}\right\}\right\}$

$a = v_1'/v_1$
$b = v_2'/v_1$

$$v_1' = \frac{m_1 - m_2}{m_1 + m_2} v_1 \quad \& \quad v_2' = \frac{2m_1}{m_1 + m_2} v_1$$

Check special cases

- $m_1 = m_2$ — m_1 stops and $v_2' = v_1$
- $m_1 > m_2$ — m_1 continues forward
- $m_1 < m_2$ — m_1 bounces back

Conservation of momentum and Newton's third law

• *Is momentum always conserved?*

((We've already seen that *kinetic energy* is not always conserved.

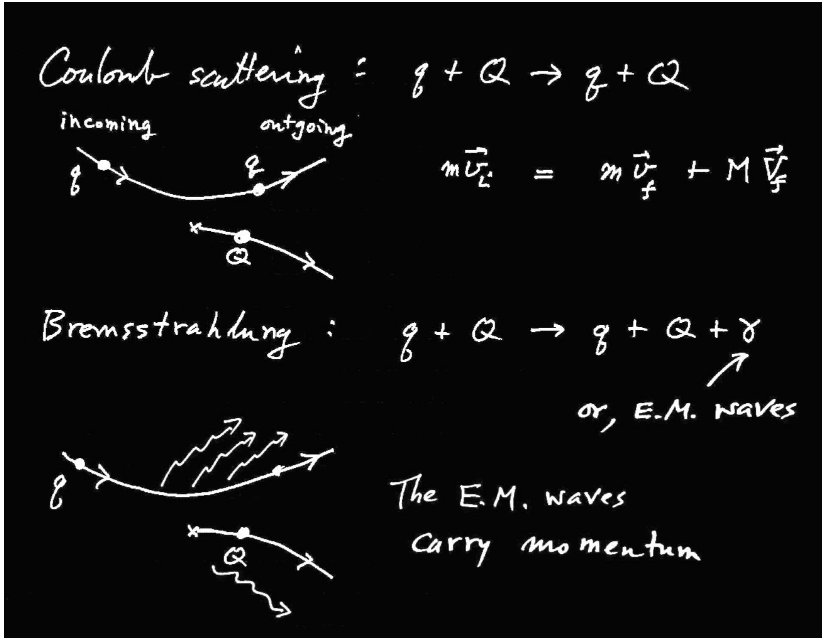
Chapter 4 will introduce potential energy. But *mechanical energy* (T+U) is not always conserved.

THERMODYNAMICS is necessary to understand that *total energy* is always conserved; = the first law of thermodynamics.))

• *Is momentum always conserved?*

Particle momentum is not always conserved because there is field momentum. But *total momentum* is conserved.

• ***Example***



• For PHY 321 !

In *Newtonian mechanics*, particle momentum is always conserved.

Homework Assignment #5

due in class Friday, October 7

[21] Problem 3.4 **

[22] Problem 3.5 **

[23] Problem 3.6 *

[24] Problem 3.10 *

[25] Problem 3.12 **

[26] Problem 3.13 **

Use the cover sheet.