Chapter 7. Lagrange's Equations

To solve a problem using the Lagrangian method:

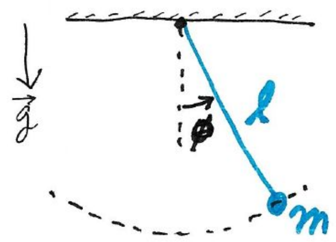
1. Define generalized coordinates.
2. Write T and U in terms of the g.c..
3. $\pounds = T - U$
4. Derive Lagrange's equations.
5. Solve the equations.

Lagrange's equations are

$$\frac{\partial \pounds}{\partial q_i} = \frac{d}{dt} \frac{\partial \pounds}{\partial \dot{q}_i} \qquad \{ i = 1\ 2\ 3\ \ldots \}$$

Section 7.2.
Constrained Systems; an Example

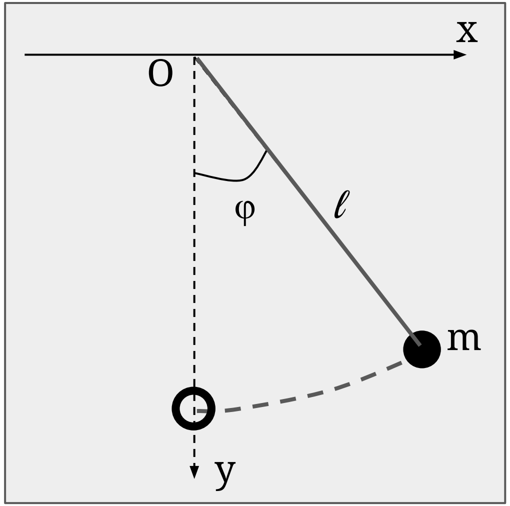
"Constrained motion" means that the particle is not free to move throughout the space; its motion is limited by certain constraints.
For example, consider the pendulum…



The length of the rod (or string) is constant ($= l$) so the mass m can only move on a circle or arc of radius l .

Hamilton's Principle ("least action") still applies and implies Lagrange's equations.

Example: The Plane Pendulum



$x = l \sin \varphi$
$y = l \cos \varphi$

$mgh = mg(l - y)$

The generalized coordinate is φ.

$\mathcal{L} = T - U$

$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m\, l^2 \dot{\varphi}^2$

$U = m g (l - y) = m g\, l (1 - \cos \varphi)$

$\mathcal{L} = \frac{1}{2} m\, l^2 \dot{\varphi}^2 - m g\, l (1 - \cos \varphi)$

Lagrange's equation, in terms of the generalized coordinate, φ …

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right)$$

$$- mg\ell \sin \phi = \frac{d}{dt}\left(m\ell^2 \dot{\phi}\right) = m\ell^2 \ddot{\phi}$$

$$\ddot{\phi} = -\frac{g}{\ell} \sin \phi$$

We are familiar with this equation from earlier calculations.
The solution is an elliptic integral;
Taylor Problem 4.28.

Section 7.3: Constrained Systems in General

To be general, consider a system of N particles:

※ labels $\alpha = \{1\ 2\ 3\ \ldots\ N\}$

※ positions $\mathbf{r}_\alpha = \{\ \mathbf{r}_1\ \ \mathbf{r}_2\ \ \mathbf{r}_3\ \ \ldots\ \ \mathbf{r}_N\ \}$

※ generalized coordinates

$$q_i = \{\ q_1\ \ q_2\ \ q_3\ \ \ldots\ \ q_n\ \} \qquad (i = 1\ \ldots\ n)$$

The number of particle coordinates is 3N (for a three dimensional system). The number of generalized coordinates is smaller, call it n, because of constraints.

※ necessary functional relationships

$\mathbf{r}_\alpha = \mathbf{r}_\alpha\,(\ q_1\ , \ldots , q_n\ ; t\ )$ for $\alpha = \{1\ 2\ 3\ \ldots\ N\}$

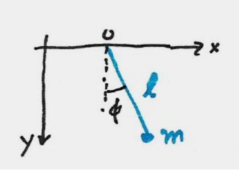
$q_i = q_i\,(\ \mathbf{r}_1\ , \ldots , \mathbf{r}_N\ ; t\ )$ for $i = \{\ 1\ 2\ 3\ \ldots\ n\}$

Note the possible (but not always necessary) time-dependence of the relations.

[[Side comment: Taylor won't use the terms *scleronomous* coordinates and *rheonomous* coordinates; he calls them natural and nonnatural. See footnote 4 on page 249.]]

Taylor gives some examples:

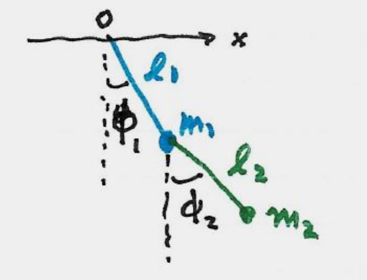
- the plane pendulum



x,y ; N = 2

φ ; n=1

- the double plane pendulum



x_1 , y_1 , x_2 , y_2 ; N = 4

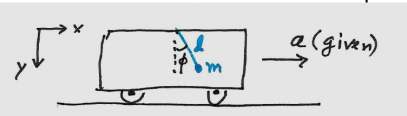
φ_1 , φ_2 ; n=2

- a pendulum in a railroad car with specified acceleration a

x , y ; N =2

φ ; n=1

with time dependent relations



"Degrees of Freedom"

n is the number of degrees of freedom, i.e., the number of coordinates that can vary independently.

3N = the number of mass points.

$n \leq 3N$

For a rigid body, n = 6 while N = infinite.

"Holonomic systems" ≡ n is the number of degrees of freedom *and* n is the number of generalized coordinates.

Nonholonomic systems (Taylor gives a rolling ball as an example) will not be considered in this course.

Section 7.4.
Prove Lagrange's Equations with Constraints

To make it simple, consider a particle that is constrained to move on a surface.

There are two generalized coordinates, q_1 and q_2.

The net constraining for ce is $\mathbf{F}_{cstr}$.
All other forces can be derived from a potential energy function, which may depend on time. So the force on the particle is

$$\mathbf{F}_{total} = \mathbf{F} + \mathbf{F}_{cstr} \quad \text{and} \quad \mathbf{F} = -\nabla U(\mathbf{r},t)$$

Let $\pounds = T - U$.

<u>The action integral</u>

- Let $\mathbf{r}(t)$ = the actual path followed by the particle under the influence of the forces.

- Let $\mathbf{R}(t) = \mathbf{r}(t) + \boldsymbol{\varepsilon}(t)$ where $\boldsymbol{\varepsilon}(t)$ describes a small variation of the path; i.e., infinitesimal; *and* $\mathbf{R}$ obeys the constraints.

- The action integral for $\mathbf{R}(t)$ is

$$S = \int_{t1}^{t2} \pounds\,(\mathbf{R}, \dot{\mathbf{R}}, t)\,dt\ ;$$

and $S_0 = \int_{t1}^{t2} \pounds\,(\mathbf{r}, \dot{\mathbf{r}}, t)\,dt$ = the minimum.

Now $\delta S = S - S_0 = \int_{t1}^{t2} \delta\pounds\ dt$

- $\delta\pounds = \pounds\,(\mathbf{R}, \dot{\mathbf{R}}, t) - \pounds\,(\mathbf{r}, \dot{\mathbf{r}}, t)$

$= \frac{1}{2} m\,[\,(\dot{\mathbf{r}} + \dot{\boldsymbol{\varepsilon}})^2 - \dot{\mathbf{r}}^2\,] - [U(\mathbf{r} + \boldsymbol{\varepsilon}) - U(\mathbf{r})]$

$= m\,\dot{\mathbf{r}} \cdot \dot{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon} \cdot \nabla U$

$= \underbrace{m\,d/dt\,(\dot{\mathbf{r}} \cdot \boldsymbol{\varepsilon})} - m\,\ddot{\mathbf{r}} \cdot \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} \cdot \mathbf{F}$

integrates to 0 because
$\boldsymbol{\varepsilon}(t_1) = 0$ and $\boldsymbol{\varepsilon}(t_2) = 0$.

= drop $- \boldsymbol{\varepsilon} \cdot \mathbf{F}_{\mathbf{cstr}}$

- $$\delta S = -\int_{t1}^{t2} \boldsymbol{\varepsilon} \cdot \mathbf{F}_{\mathbf{cstr}}\,dt$$

- The constraint force is normal to the surface; therefore

$$\boldsymbol{\varepsilon} \cdot \mathbf{F}_{\mathbf{cstr}} = (\mathbf{R} - \mathbf{r}) \cdot \mathbf{F}_{\mathbf{cstr}} = \mathbf{0}.$$

- *The action integral is stationary at the actual path of the particle,* ***r****(t).*

Theorem.

The generalized coordinates obey Lagrange's equations.

Proof.

We just proved that Hamilton's principle ($\delta S = 0$) holds for all variations of the path ***that obey the constraints.***

Any variation of the generalized coordinates, q_1 and q_2 , obeys the constraints.

Write S in terms of the generalized coordinates,

$$S = \int_{t1}^{t2} \pounds (q_1\ q_2\ \dot{q}_1\ \dot{q}_2\ ; t)\ dt$$

Then we have $\delta S = 0$ for any variations of $q_1(t)$ and $q_2(t)$.

By the calculus of variations (Chapter 6) $q_1(t)$ and $q_2(t)$ must obey the Euler-Lagrange equations; i.e.,

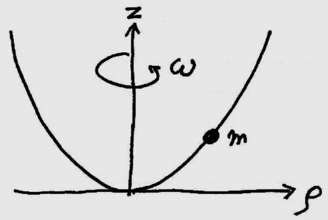
$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right) \quad \text{AND} \quad \frac{\partial \mathcal{L}}{\partial q_2} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2}\right)$$

For a holonomic system,

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) \quad \text{for} \quad i = 1\ 2\ 3 \ldots\ n$$

where $\pounds = T - U$.

Example: Problem 7.41



BEAD ON A FRICTIONLESS WIRE

Cylindrical coordinates of m
(ρ, ϕ, z)

Constraints on m
$z = k\rho^2$
$\phi = \omega t$

■ The Lagrangian $\pounds(\rho, \dot{\rho}; t)$

$$T = \tfrac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \quad \text{where} \quad x = \rho\cos\phi = \rho\cos\omega t, \quad Z = \rho\sin\phi = \rho\sin\omega t$$

$$T = \tfrac{1}{2} m \left[\dot{\rho}^2 + \rho^2\omega^2 + (2k\rho\dot{\rho})^2\right]$$

$$U = mgz = mgk\rho^2$$

$$\mathcal{L} = \tfrac{1}{2} m \left(\dot{\rho}^2 + \rho^2\omega^2 + 4k^2\rho^2\dot{\rho}^2\right) - mgk\rho^2$$

■ The equation of motion

$$\frac{\partial \mathcal{L}}{\partial \rho} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\rho}}\right)$$

$$(1 + 4k^2\rho^2)\ddot{\rho} + 4k^2\rho\dot{\rho}^2 - (\omega^2 - 2gk)\rho = 0$$

■ The equilibrium positions

$\dot{\rho} = 0$ and $\ddot{\rho} = 0$

$\rho = 0$ is an equilibrium point

Also, if $\omega^2 = 2gk$ then <u>any</u> ρ is an equilibrium point.

■ Stability analyses

Stability at $\rho = 0$.
Consider a small variation, $\rho = \epsilon$.
To order ϵ accuracy,

$$\ddot{\epsilon} - (\omega^2 - 2gk)\epsilon = 0.$$

$\rho = 0$ is stable if $\omega^2 - 2gk < 0$
$\rho = 0$ is unstable if $\omega^2 > 2gk$.

The trajectory of a particle moving in a potential obeys Lagrange's equations.
For any set of generalized coordinates,

$$\frac{\partial \pounds}{\partial q_i} = \frac{d}{dt} \frac{\partial \pounds}{\partial \dot{q}_i}$$

n equations;
i = 1 2 3 … n

To solve a problem using the Lagrangian method:

1. Define generalized coordinates.
2. Write T and U in terms of the g.c..
3. $\pounds = T - U$
4. Derive Lagrange's equations.
5. Solve the equations.

Homework Assignment 12
due in class ***Monday November 28***
[61] Problem 7.2 *
[62] Problem 7.3 *
[63] Problem 7.8 **
[64] Problem 7.14 *
[65] Problem 7.21 *
[66] Problem 7.31 **
[67] Problem 7.43 *** [computer]