# Section 3.2
## ***Rockets***

Read Section 3.2.

The rocket contains a fuel that burns rapidly. As the exhaust gas is expelled from the combustion chamber, there is a reaction force on the rocket; ***"thrust".***

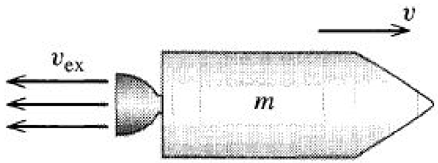



Figure 3.2 A rocket of mass $m$ travels to the right with speed $v$ and ejects spent fuel with exhaust speed $v_{ex}$ relative to the rocket.

Now analyze the motion, using the principle of momentum, i.e.,

$$\mathbf{P}' = \mathbf{F}^{\text{ext}} \quad .$$

But be careful to identify **P** correctly.

( prime ′ or dot ˙ means d/dt )

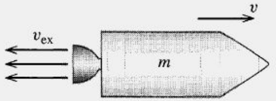



Figure 3.2 A rocket of mass $m$ travels to the right with speed $v$ and ejects spent fuel with exhaust speed $v_{ex}$ relative to the rocket.

<u>Derive the equation of motion for the rocket.</u>

What I mean by the "rocket" is the metal cylinder plus the fuel *<u>inside</u>*.

The mass of the rocket decreases as fuel is expelled out the back.

Let m(t) = the mass of the rocket at time t;

$$m(t) = M_{cyl} + M_{\text{fuel inside}}(t) \qquad ;$$

The equation of motion will depend on two parameters of the rocket engine.

■ Let $v_{ex}$ = the *relative speed* of the exhaust gas. Understand this:

**The velocity of the exhaust gas at time t is $v(t) - v_{ex}$ , where $v(t)$ is the velocity of the rocket.**

■ Let K = the *mass rate* of the exhaust gas. *K is positive.* Understand this:

$$K = - dM_F / dt = - dm/dt$$

**I.e., K δt = the mass of fuel expelled by the engine during the time δt**

**= the *decrease* of the rocket mass during δt. (units of K: kg/s)**

Consider the change of momentum of the system = *rocket and fuel*, from time $t$ to time $t + \delta t$.

The total momentum at time $t$ is

$$P(t) = m(t)\, v(t) + P_{\text{fuel already expelled before t}}$$

(I'm only considering one-dimensional motion of the rocket.)

The total momentum at time $t + \delta t$ is

$$P(t+\delta t) = m(t+\delta t)\, v(t+\delta t) \qquad \leftarrow\text{rocket at t+dt}$$

$$+ (K\,\delta t)\,[\, v(t) - v_{ex}\,] \qquad \leftarrow\text{fuel expelled during dt}$$

$$+ P_{\text{fuel already expelled before t}}$$

$$= (m - \delta m)(v + \delta v) + (K\,\delta t)\,(v - v_{ex}) + P_{\text{already}}$$

*check it yourself*

(Because I will take δt to be infinitesimal, it does not matter if I take the exhaust speed during δt to be $v(t)-v_{ex}$ or $v(t+\delta t)-v_{ex}$ or something in between.)

The *change* of momentum = $\delta P$

$$= P(t+\delta t) - P(t)\,.$$

Now, the equation of motion follows from the theorem, $dP/dt = F^{ext}$.

$$\therefore\ F^{ext} = \lim\,[P(t+\delta t) - P(t)]/\delta t \qquad (\delta t \to 0)$$

$$= [\,(m - K\,\delta t)\,(v+\delta v) + K\,\delta t\,(v - v_{ex}) - mv\,]/\delta t$$

$$= [\, m\,\delta v - K\,\delta t\, v + K\,\delta t\,(v - v_{ex}) + O(\delta^2)]/\delta t$$

$$= m\, dv/dt - K\, v_{ex} \qquad \text{in the limit } \delta t \to 0.$$

***Note the cancellations!***

## The rocket equations

For a rocket moving in one dimension, the velocity v(t) obeys

$$m \frac{dv}{dt} = K v_{ex} + F^{ext} \qquad (1)$$

where $v_{ex}$ = the relative speed of the exhaust and K = the mass rate of the exhaust. Here m(t) is the mass of the rocket including enclosed fuel, so

$$\frac{dm}{dt} = -K \qquad (2)$$

Thrust force $= K v_{ex}$ .

## Example. A rocket in deep space with constant values of $v_{ex}$ and K ...

Let the initial velocity be $v_0$ and the initial mass be $m_0$.

Calculate the velocity at time t.

By eq.(2), $m(t) = m_0 - K t$

Then eq.(1) is

$$(m_0 - K t)(dv/dt) = K v_{ex}$$

Using separation of variables,

$$dv = K v_{ex} / (m_0 - K t) \ dt ;$$

integrate both sides of the equation,

$$v - v_0 = - v_{ex} [ \, ln (m_0 - K t) - ln(m_0) \, ]$$

$$v = v_0 + v_{ex} \, ln [ \, m_0 / (m_0 - K t)]$$

Example. A rocket in deep space with constant $v_{ex}$ and any time dependence, K(t)

Let the initial velocity be $v_0$ and the initial mass be $m_0$.

Calculate the velocity at time t.

Result: $v = v_0 + v_{ex} \, ln \, [m_0 \, / m(t) \,]$

Proof:

$$v = v_0 + v_{ex} \, ln \, [m_0] - v_{ex} \, ln \, [\, m(t) \,]$$

Calculate $dv/dt$

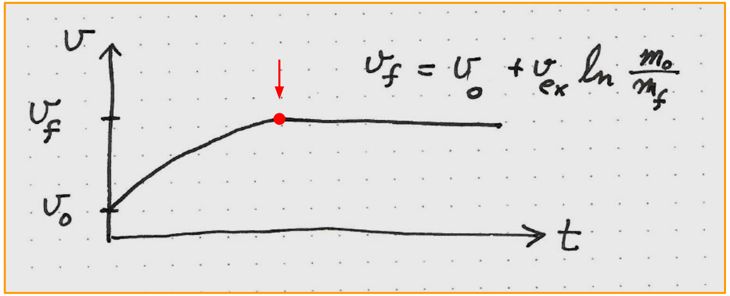
$$= -v_{ex} \frac{1}{m} \frac{dm}{dt}$$

Thus

$$m \frac{dv}{dt} = K \cancel{v}_{ex}$$

which is correct because $F^{ext} = 0$ in deep space.

Graph of velocity versus time for constant K .



# TAKE OFF FROM EARTH'S SURFACE

We have

$$m\,(dv/dt) = v_{ex}\,K - mg$$

Approximate

$g$ = constant and $v_{ex}$ = constant.

How to integrate the diff. eq.?

$$m\,dv = v_{ex} K\,dt - mg\,dt$$

but m = m(t) !

$$= -\,v_{ex}\,dm - m\,g\,dt$$

$$dv = -\,v_{ex}\,(dm/m) - g\,dt$$

Integrate

$$\int_0^v dv' = -\,v_{ex}\int_{m0}^{m} dm'/m' - g\int_0^t dt'$$

$$v(t) = v_{ex}\,\ln\,[m_0/m(t)] - gt$$

Also, $m(t) = m_0 - Kt$ *(assuming K is constant)*

Graph of v as a function of t

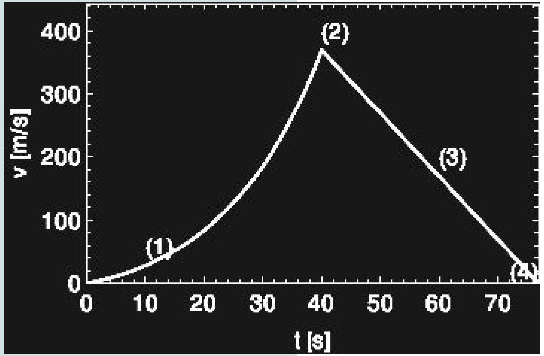
(1) Slope ~ $(v_{ex}K - m_0 g)/m_0$

(2) rocket runs out of fuel

(3) slope = $-\,g$

(4) rocket starts to fall downward

Area under the curve = height



A more difficult example:

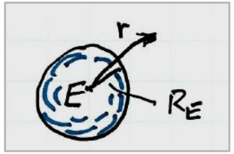
***Fire a rocket to the stars***

Now g is not constant. As the distance from the earth increases, g decreases;

$$g(r) = g_s R_E^2 / r^2 \qquad \text{for} \quad r > R_E \,.$$

The equation of motion is

$$m \frac{dv}{dt} = K v_{ex} - m g_s \frac{R_E^2}{r^2}$$




***Example: Saturn V rocket***

1st stage: burn time = 165 sec;
fuel = 2,003,000 *kg* ; thrust = 35,000 *kN*

2nd stage: burn time = 360 sec;
fuel = 456,000 *kg* ; thrust = 5,000 *kN*

3rd stage: burn time = 500 sec;
fuel = 110,000 *kg* ; thrust = 1,000 *kN*

Test yourself :
Given $m_{\text{rocket body}}$ = 100 kg,
$m_{\text{fuel}}$ = 1000 kg, and $v_{\text{exhaust}}$ = 2000 m/s.
*Neglecting gravity*, calculate the final velocity of the rocket. Would it escape from Earth's gravity?

Homework Assignment #5
due in class Friday, October 7
[21] Problem 3.4 **
[22] Problem 3.5 **
[23] Problem 3.6 *
[24] Problem 3.10 *
[25] Problem 3.12 **
[26] Problem 3.13 **

***Use the cover sheet.***

# Take off from Earth's surface

Approximat

Separatio

v

d

Integrate,

$\int_{v_0=0}^{v} d$

$v(t) =$

Also, $m(t) = m_0 - Kt$ if $K$ is constant.

A successful take off requires

$v_{ex} K > m_0 g$ i.e. thrust > weight

t