Elimination Using Matrices

Last time

solve

$$-x + y = 0$$

$$x + y = 2$$

Augmented Matrix

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & 1 & 2 \end{array}\right]$$

$\downarrow R_2 \to R_2 + R_1$

$$\begin{array}{cc} x & y \end{array}$$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 2 & 2 \end{array}\right] \to 2y = 2 \qquad y = 1$$

$-1x + 1 = 0$

$x = 1$

$\uparrow$

$-1x + 1y = 0$

$\uparrow$

$2y = 2$

upper triangular

use back-substitution to solve

soln: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$R_2 \longrightarrow R_2 + R_1$ written in matrix form

Start with identity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Note: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Replace entry $a_{\underline{21}}$ with the multiplier in front of Row 1

changing R_2 by adding R_1

$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ Note: $a_{\text{row, col}}$.

$$E_{21} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

elimination matrix

$$y - x = 0 \qquad -x + y = 0$$
$$y + x = 2 \qquad x + y = 2$$

$$A\vec{x} = \vec{b}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Applying E_{21} to A we will do the elimination of $R_2 \rightarrow R_2 + R_1$

$$A\vec{x} = \vec{b}$$
$$E_{21}A\vec{x} = E_{21}\vec{b}$$

$$\underset{2\times 2}{\overset{E_{21}}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}} \cdot \underset{2\times 2}{\overset{A}{\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}}} = \underset{2\times 2}{\begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}}$$

match for multiplication to work

gives us result

$$E_{21}\vec{b}$$

$$\underset{2\times 2}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}$$

$$E_{21}A\vec{x} = E_{21}\vec{b}$$

$$\begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

<u>Notation:</u> ref $-\ell$ in i,j position (Row i, col j)

E_{ij} subtracts ℓ times Row j from Row i

$$E_{31} \Rightarrow \underset{3\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\ell & 0 & 1 \end{bmatrix}}\underset{3\times 1}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{3\times 1}{\begin{bmatrix} x \\ y \\ -\ell x+z \end{bmatrix}} \quad R_3 \to R_3 - \ell R_1$$

multiply by $\vec{x}$

Purpose of E_{ij} is to produce a zero in the (i,j) position of original matrix

Goal: Find "E"

If we need 3 elimination steps,

$$E = E_{32}E_{21}E_{31}$$

Multiplication of Matrices

$A(BC) = (AB)C$ Associative prop.

$AB \neq BA$ most of time
matrices do not commute

Ex] $E_{21} A \vec{x} = E_{21} \vec{b}$

E_{21} was multiplied on LHS

~~Ex~~ $y = 4x + 5$

$3y = (4x + 5)3$

Ex] Find "E" → Find one matrix that does all elimination in one step.

$2x + y + 3z = 1$

$4x + 3y + 5z = 1$

$6x + 5y + 5z = -3$

write as

$A\vec{x} = \vec{b}$

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 5 \\ 6 & 5 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$$

Need 3 elimination matrices

E_{21} then E_{31} then E_{32}

$$E_{32} E_{31} E_{21} A \vec{x} = E_{32} E_{31} E_{21} \vec{b}$$

eliminate 4 in Row 2 col 1 $\Rightarrow$ want 0 in Row 2 col 1 after we multiply

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ add $-2\,\text{Row}_1$ to Row_2

$$E_{21} A = \underset{3\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 5 \\ 6 & 5 & 5 \end{bmatrix}} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & [illegible] & -1 \\ 6 & 5 & 5 \end{bmatrix}$$

zero out E_{31}

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$ add $-3\,\text{Row}_1$ to Row 3

$$E_{31}(E_{21}A) =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & -1 \\ 6 & 5 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & -1 \\ 0 & 2 & -4 \end{bmatrix}$$

Find E_{32} & then find E