Section 5.5

Driven damped oscillations

Section 5.6

Resonance

Read Sections 5.5 and 5.6.

Topics:

•particular and homogeneous solutions;

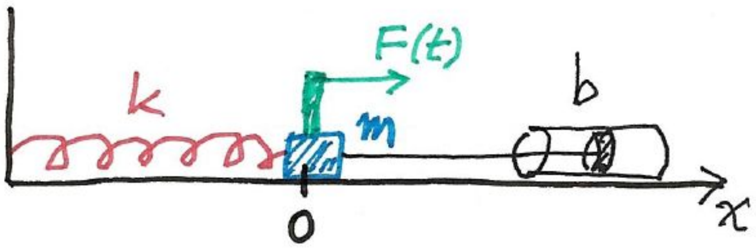
•complex solutions for a sinusoidal driving force;

•resonance.

can be included on the Exam November 4.

5.5. Driven damped oscillations

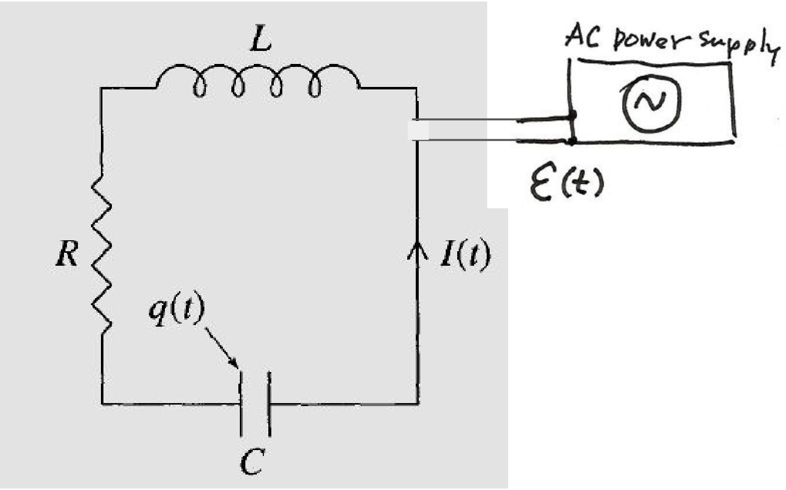
Generic picture



Equation

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

The equivalent LRC circuit



$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = \mathcal{E}(t)$$

the math is the same for the mechanical system and the electric circuit.

$$m\,x'' + b\,x' + k\,x = F(t) \qquad (*)$$

This is called an *inhomogeneous* linear differential equation; **the inhomogeneous term is the F(t).**

There is a general method for solving this kind of equation (**MTH 235**).

The general solution of (*) is

$$x(t) = x_P(t) + x_H(t)$$

where $x_P(t)$ is any *particular* solution, and $x_H(t)$ is a general solution of the *homogeneous* equation.

We already know the homogeneous equation, so the problem is just $x_P(t)$.

A linear differential operator

▌ Taylor introduces some mathematical formalism. Define this differential operator,

$$D = d^2/dt^2 + 2\beta\, d/dt + \omega_0^{\,2}\ .$$

▌ "Particular solution and solution of the homogeneous equation;"

The equation is $\quad D\,x = F/m = f$

The particular solution is *any solution*,

$$D\,x_{\mathbf{P}} = f.$$

The homogeneous equation is $D\,x_{\mathbf{H}} = 0$, and its general solution is

$$x_{\mathbf{H}}(t) = c_1 \exp(p_1\, t) + c_2 \exp(p_2\, t),$$

or

$$x_{\mathbf{H}}(t) = \exp(-\beta t)\,[\,A \cos \omega_1 t + B \sin \omega_1 t\,].$$

▌ The most interesting case is a *harmonic driving force; $f(t) = f_0 \cos \omega t$.*

Use complex numbers; write

$$x(t) = \mathrm{Re}\, z(t) \quad ;$$

$$f(t) = \mathrm{Re}\ f_0\, e^{i\omega t} \quad ;$$

$$D\, z = f_0\, e^{i\omega t} \quad .$$

Now, we need a ***particular*** solution of $\quad D\,z(t) = f_0\, e^{i\omega t}$.

The ***steady-state*** solution is

$$z(t) = C\, e^{i\omega t}$$

where

$$(\,-\omega^2 + 2\beta\, i\omega + \omega_0^{\,2}\,)C = f_0$$

$$C = \frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega}$$

$$C = \frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega}$$

<u>Amplitude</u> and <u>Phase Angle</u>

Write $C = A e^{-i\delta}$.

$$A^2 = CC^* = \frac{f_0}{\omega_0^2-\omega^2+2i\beta\omega} \; \frac{f_0}{\omega_0^2-\omega^2-2i\beta\omega}$$

$$A^2 = \frac{f_0^2}{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}$$

$$e^{i\delta} = \frac{A}{C} = \frac{\omega_0^2-\omega^2+2i\beta\omega}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}$$

$$= \cos\delta + i\sin\delta$$

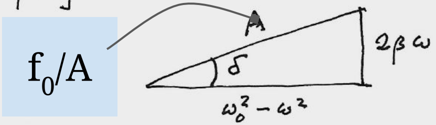
$$\therefore i\tan\delta = \frac{2i\beta\omega}{\omega_0^2-\omega^2} \qquad \left(\frac{Im}{Re}\right)$$

The real part is the PARTICULAR solution

$$x_p(t) = \mathrm{Re}\, z = \mathrm{Re}\, Ce^{i\omega t} = \mathrm{Re}\, Ae^{i(\omega t-\delta)}$$

$$x_p(t) = A\cos(\omega t-\delta)$$

where $A = \dfrac{f_0}{\left[(\omega_0^2-\omega^2)^2+4\beta^2\omega^2\right]^{1/2}}$ and $\tan\delta = \dfrac{2\beta\omega}{\omega_0^2-\omega^2}$



Summary

$$m x'' + b x' + kx = F(t) \qquad (*)$$

$$x'' + 2\beta x' + \omega_0^2 x = F(t)/m = f(t) = f_0 \cos \omega t$$

The GENERAL solution for Eq. (*) with $F(t) = m f_0 \cos \omega t$

$$x(t) = A \cos(\omega t - \delta) + C_1 e^{p_1 t} + C_2 e^{p_2 t}$$

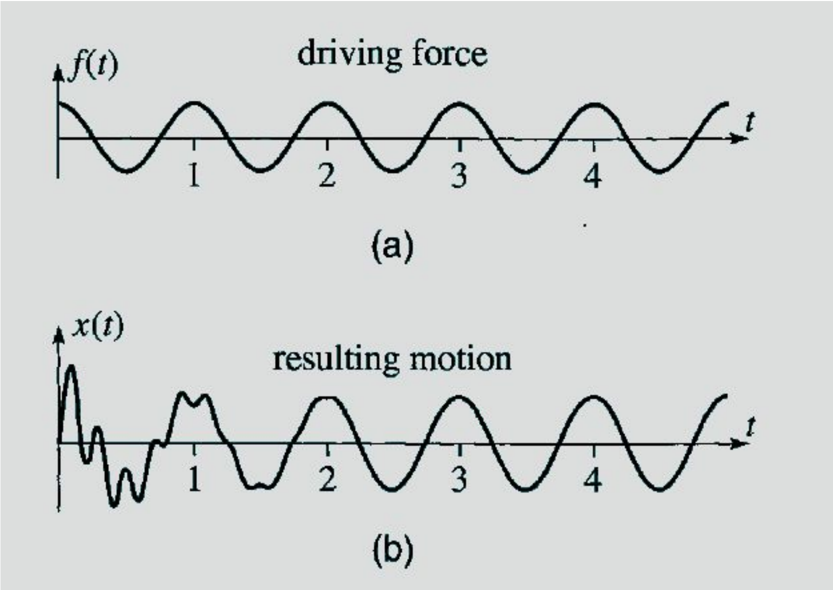
particular sol.
or
steady-state sol.

general sol. of the homogeneous eq.
or
"transients" (they $\longrightarrow$ 0 as $t \longrightarrow \infty$)

Example 5.3

graphing a driven damped oscillator

FIGURE 5.15



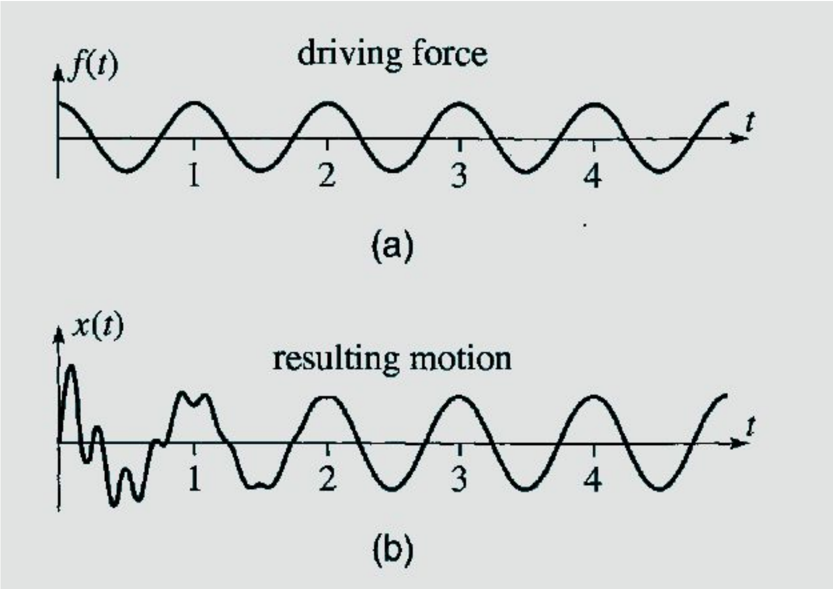
transients depend on
the initial conditions

Let's reproduce that figure, using Mathematica.

parameters and equations

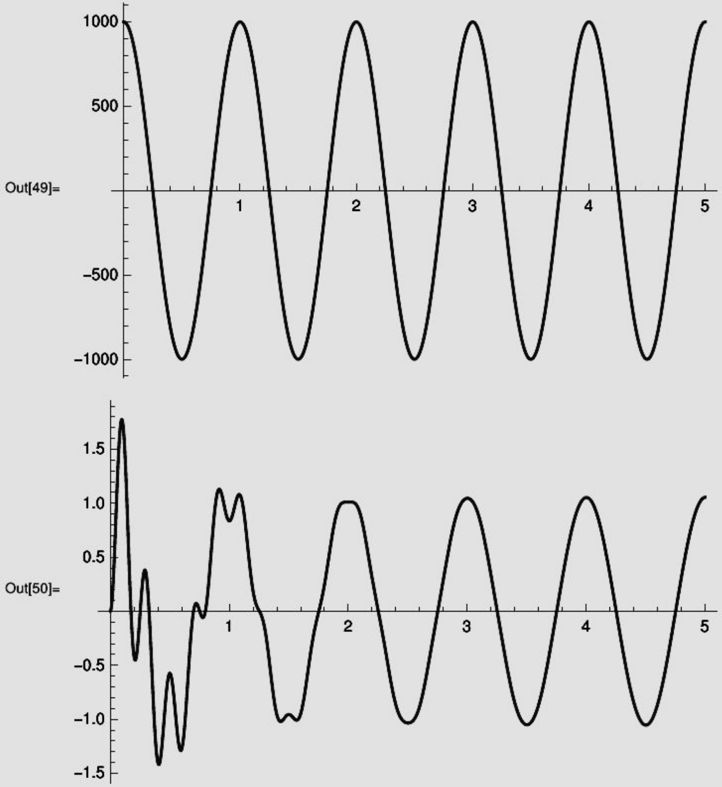
```
In[37]:= {ω, ω0, β, f0} = {2. π, ω0 = 10. π, β = π / 2., f0 = 1000.}
         A = f0 / Sqrt[(ω0^2 - ω^2)^2 + (2 * β * ω)^2]
         δ = ArcTan[(2 * β * ω) / (ω0^2 - ω^2)]
         ω1 = Sqrt[ω0^2 - β^2]
         {x0, v0} = {0, 0}
         {B1, B2} = {x0 - A * Cos[δ], (v0 - ω * A * Sin[δ] + β * B1) / ω1}
```

```
In[45]:= f[t_] := f0 * Cos[ω * t]
         x[t_] := A * Cos[ω * t - δ] +
           Exp[-β * t] * (B1 * Cos[ω1 * t] + B2 * Sin[ω1 * t])
```



Plots

```
In[49]:= Plot[f[t], {t, 0, 5}]
         Plot[x[t], {t, 0, 5}]
```



5.6. Resonance

This is the solution for the driven damped oscillator, with a harmonic driving force :

Amplitude A

$$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}$$

Phase Angle δ

$$\tan\delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

FIGURE 5.16

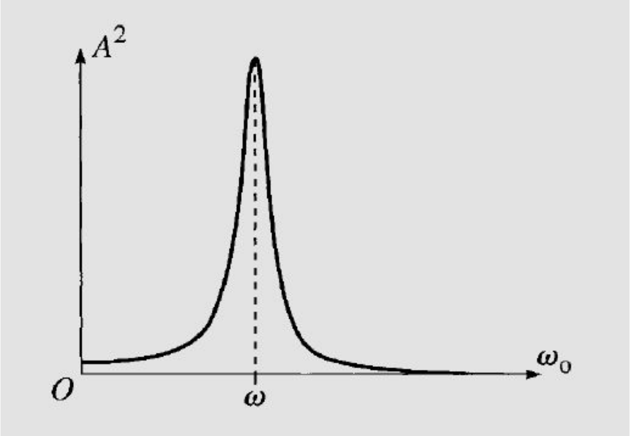
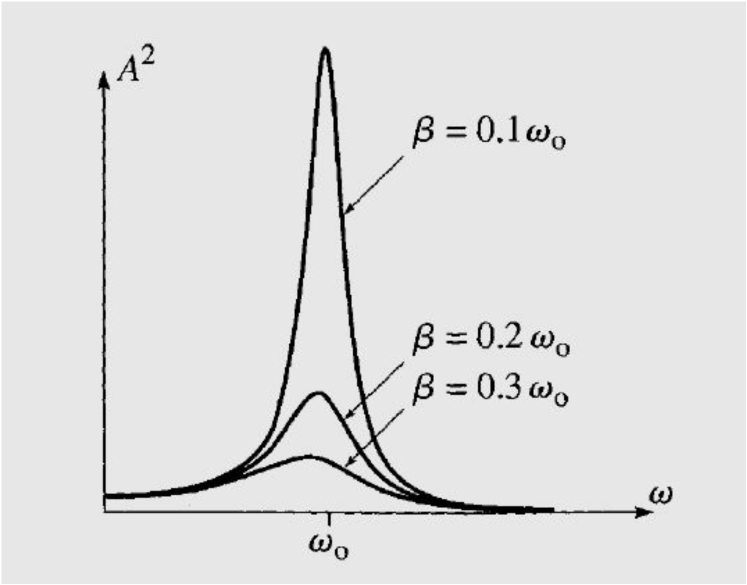


Figure 5.16 The amplitude squared, A^2, of a driven oscillator, shown as a function of the natural frequency ω_0, with the driving frequency ω fixed. The response is dramatically largest when ω_0 and ω are close.

This is for some small value of β
How does the resonance depend on β ?

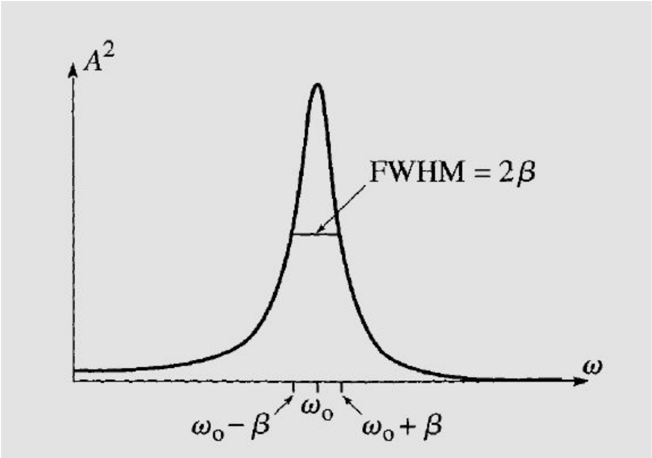
FIGURE 5.17 : *cases with weak damping*



As β decreases, the resonant peak becomes sharper.

<u>Width and Q factor</u>

FIGURE 5.18 :



FWHM = **F**ull **W**idth at **H**alf **M**aximum

Quality factor

$$Q = \frac{\omega_0}{2\beta} = \frac{\text{decay time}}{\text{period}} \times \pi$$

The Phase at Resonance

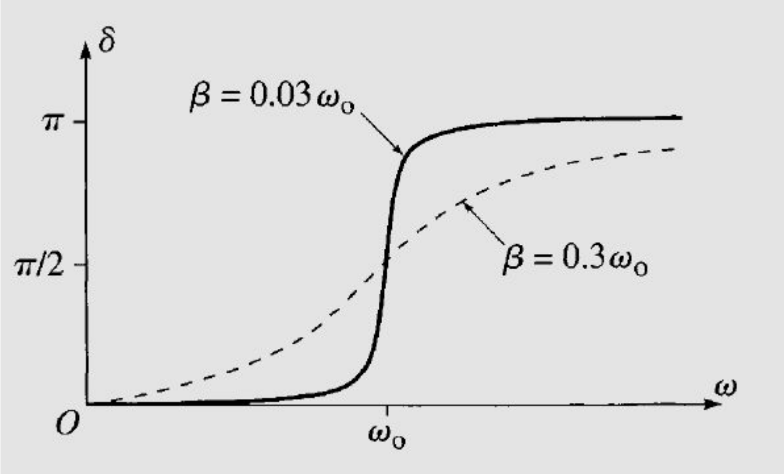


Figure 5.19 The phase shift δ increases from 0 through $\pi/2$ to π as the driving frequency ω passes through resonance. The narrower the resonance, the more suddenly this increase occurs. The solid curve is for a relatively narrow resonance ($\beta = 0.03\omega_o$ or $Q = 16.7$), and the dashed curve is for a wider resonance ($\beta = 0.3\omega_o$ or $Q = 1.67$).

Taylor comment …

In the resonances of classical mechanics, the behavior of the phase (as in Figure 5.19) is usually less important than that of the amplitude (as in Figure 5.18).[14] In atomic and nuclear collisions, the phase shift is often the quantity of primary interest. Such collisions are governed by quantum mechanics, but there is a corresponding phenomenon of resonance. A beam of neutrons, for example, can "drive" a target nucleus. When the energy of the beam equals a resonant energy of the system (in quantum mechanics energy plays the role of frequency) a resonance occurs and the phase shift increases rapidly from 0 to π.

Homework Assignment #9
due in class Friday month date

[41] Problem 4.41 and Problem 4.43
[41X] ASSIGNED PROBLEM
[42] Problem 5.3 *
[43] Problem 5.5 *
[44] Problem 5.9 *
[45] Problem 5.12 **
[46] Problem 5.18 ***

Use the cover sheet.