Chapter 7
Lagrange's Equations

Review

- Generalized coordinates, n

  $q_1 \quad q_2 \quad q_3 \quad \dots \quad q_n$

- Lagrangian $\quad £ = T - U$

- Lagrange's equations

$$\frac{\partial £}{\partial q_i} = \frac{d}{dt} \frac{\partial £}{\partial \dot{q}_i}$$

(n equations; $i = 1\ 2\ 3 \dots n$)

Section 7.5. Examples of Lagrange's Equations

To solve a problem using the Lagrangian method:

1. Define generalized coordinates.
2. Write T and U in terms of the g.c..
3. $£ = T - U$
4. Derive Lagrange's equations.
5. Solve the equations.

Taylor gives 5 examples.

Example 7.3

Atwood's Machine

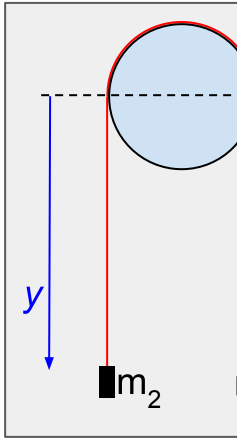


FIGURE 7.6

G.C. : x

Note $\ell = x + y + \pi R$

so $y = \ell - \pi R - x$

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{y}^2$$

$$= \frac{1}{2}(m_1 + m_2)\,\dot{x}^2$$

$$U = -m_1 g x - m_2 g y$$

$$= -(m_1 - m_2) g x + \text{Constant}$$

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2)\,\dot{x}^2 + (m_1 - m_2) g x$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) \Rightarrow \quad (m_1 - m_2) g = \frac{d}{dt}\left\{(m_1 + m_2)\dot{x}\right\}$$

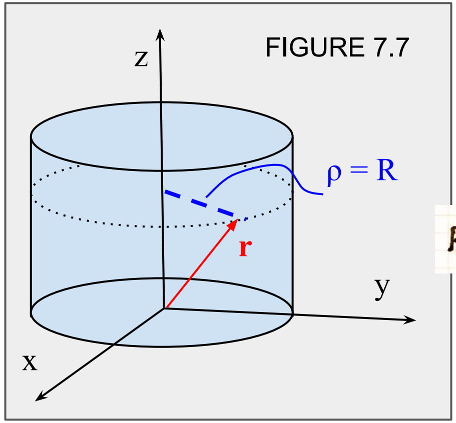
$$\ddot{x} = \frac{m_1 - m_2}{m_1 + m_2}\, g$$

The result is constant acceleration;
m_1 accelerates downward if $m_1 > m_2$; *etc.*

We could derive this from the Newtonian method. We would need to include the string tension in the forces.

Example 7.4

A Particle on a Cylinder †



Cylindrical coordinates

$$\{\rho, \phi, z\}.$$

But $\rho = R$, is constant.

$$x = R\cos\phi$$
$$y = R\sin\phi$$
$$z = z$$

G.C. : ϕ and z

† *Taylor asks, "How can you keep the particle in contact with the surface?" Easy: Lay a hollow cylinder on its side in Earth's gravity.*

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$
$$= \frac{1}{2}m\left(R^2\dot{\phi}^2 + \dot{z}^2\right)$$

$$U = U(\phi, z) \quad \text{(general)}$$

$$\mathcal{L} = \frac{1}{2}m\left(R^2\dot{\phi}^2 + \dot{z}^2\right) - U(\phi, z)$$

$$\frac{\partial\mathcal{L}}{\partial q} = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right)$$

$q = \phi$ $\quad -\frac{\partial U}{\partial\phi} = \frac{d}{dt}\left(mR^2\dot{\phi}\right) = mR^2\ddot{\phi}$

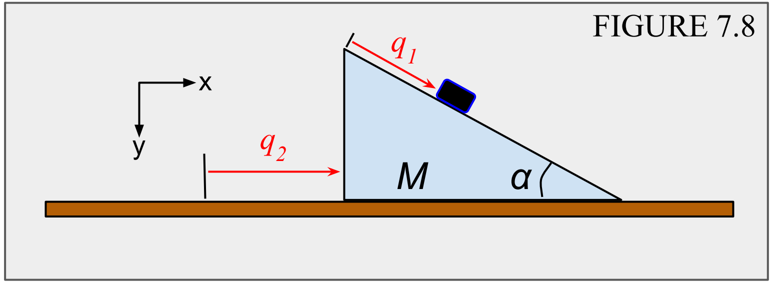
is the same as $\frac{dL_z}{dt} = RF_\phi =$ torque

$q = z$ $\quad -\frac{\partial U}{\partial z} = \frac{d}{dt}\left(m\dot{z}\right) = m\ddot{z}$

is just $m\ddot{z} = F_z$

Example 7.5

A block slides on a sliding wedge



As the block slides down ($\dot{q}_1 > 0$), the wedge slides to the left ($\dot{q}_2 < 0$).
The center of mass does not move horizontally, because there is no horizontal ***external*** force.
Coordinates

$$x_2 = q_2$$

$$x_1 = q_2 + q_1 \cos\alpha \; ; \; y_1 = q_1 \sin\alpha$$

$$T = \tfrac{1}{2}M(\dot{x}_2^2) + \tfrac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2)$$

$$= \tfrac{1}{2}M\dot{q}_2^2 + \tfrac{1}{2}m\left[(\dot{q}_2 + \dot{q}_1\cos\alpha)^2 + (\dot{q}_1\sin\alpha)^2\right]$$

$$= \tfrac{1}{2}(M+m)\dot{q}_2^2 + \tfrac{1}{2}m\dot{q}_1^2 + m\dot{q}_2\dot{q}_1\cos\alpha$$

$$U = -mgy_1 = -mg\,q_1\sin\alpha$$

q_2 equation $\quad \dfrac{\partial \mathcal{L}}{\partial q_2} = 0 = \dfrac{d}{dt}\left[(M+m)\dot{q}_2 + m\dot{q}_1\cos\alpha\right]$

Same as $P_x = (M+m)\dot{q}_2 + m\dot{q}_1\cos\alpha = \text{Constant}$

q_1 equation $\quad \dfrac{\partial \mathcal{L}}{\partial q_1} = mg\sin\alpha = \dfrac{d}{dt}\left[m\dot{q}_1 + m\dot{q}_2\cos\alpha\right]$

$$g\sin\alpha = \ddot{q}_1 + \cos\alpha\left[\frac{-m\ddot{q}_1\cos\alpha}{(M+m)}\right]$$

$$\ddot{q}_1 = \frac{g\sin\alpha}{\left[1 - \frac{m}{M+m}\cos^2\alpha\right]}$$

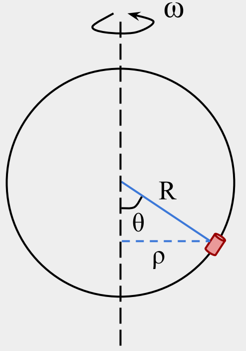
DOES IT MAKE SENSE?

consider m=0 and M=0

Example 7.6

A Bead on a Spinning Wire Hoop

FIGURE 7.9



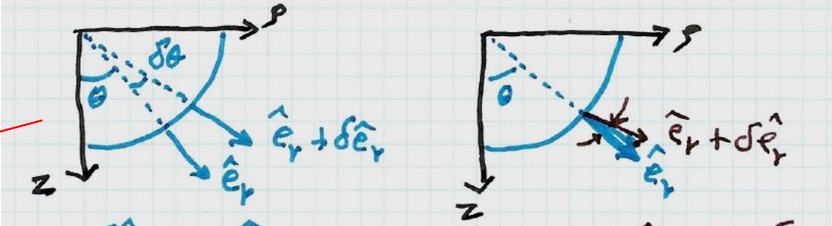
G.C. : θ

Using spherical polar coordinates, $\{r, \theta, \phi\}$

$r = R$ and $\phi = \omega t$

$$T = \frac{1}{2} m \frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt} \quad \text{where } \vec{r} = r\,\hat{e}_r$$

$$\frac{d\vec{r}}{dt} = \dot{r}\,\hat{e}_r + r \frac{d}{dt}\hat{e}_r = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta + r \sin\theta\, \dot{\phi}\,\hat{e}_\phi$$



$$\delta\hat{e}_r = \hat{e}_\theta\, \delta\theta \qquad \delta\hat{e}_r = \hat{e}_\phi \sin\theta\, \delta\phi$$

$$T = \frac{1}{2} m \left\{ \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2 \right\}$$

$$= \frac{1}{2} m \left\{ R^2 \dot{\theta}^2 + R^2 \sin^2\theta\, \omega^2 \right\}$$

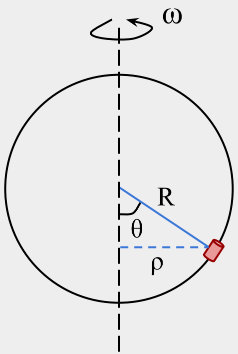
$$U = -mgz = -mgR\cos\theta$$

Lagrange's equation *(gravity & centrifugal force)*

$$\frac{\partial \mathcal{L}}{\partial \theta} = -mgR\sin\theta + mR^2\omega^2 \sin\theta\cos\theta$$

$$= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(mR^2\dot{\theta}\right) = mR^2\ddot{\theta}$$

FIGURE 7.9



Find equilibrium angles of the bead on a spinning wire hoop.

$$\ddot{\theta} = 0 \quad \Rightarrow \quad \sin\theta\left[-\frac{g}{R} + \omega^2\cos\theta\right] = 0$$

Two solutions: $\theta = 0$

$$\cos\theta = \frac{g}{R\omega^2} \quad \text{provided } \omega > \sqrt{\frac{g}{R}}$$

Are they stable or unstable equilibria?

Note $\ddot{\theta} = -\frac{\partial \hat{U}}{\partial\theta}$ where $\hat{U}(\theta) = -\frac{g}{R}\cos\theta + \frac{\omega^2}{4}\cos 2\theta$

Lagrange's equation

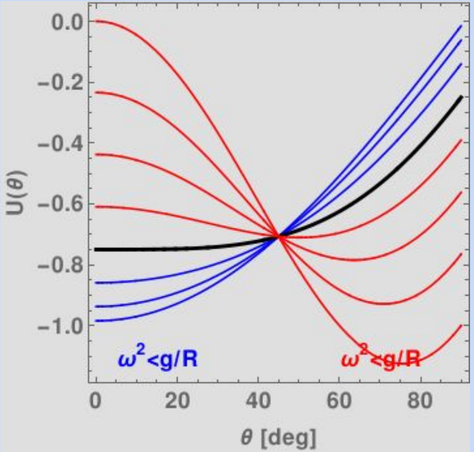
$$\frac{\partial \mathcal{L}}{\partial\theta} = -mgR\sin\theta + mR^2\omega^2\sin\theta\cos\theta$$

$$= \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \frac{d}{dt}\left(mR^2\dot{\theta}\right) = mR^2\ddot{\theta}$$

$$\ddot{\theta} = -\frac{g}{R}\sin\theta + \frac{\omega^2}{2}\sin 2\theta$$

just like a pendulum

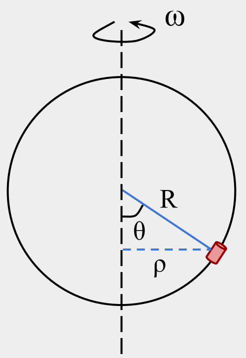
effect of centrifugal force



Example 7.7

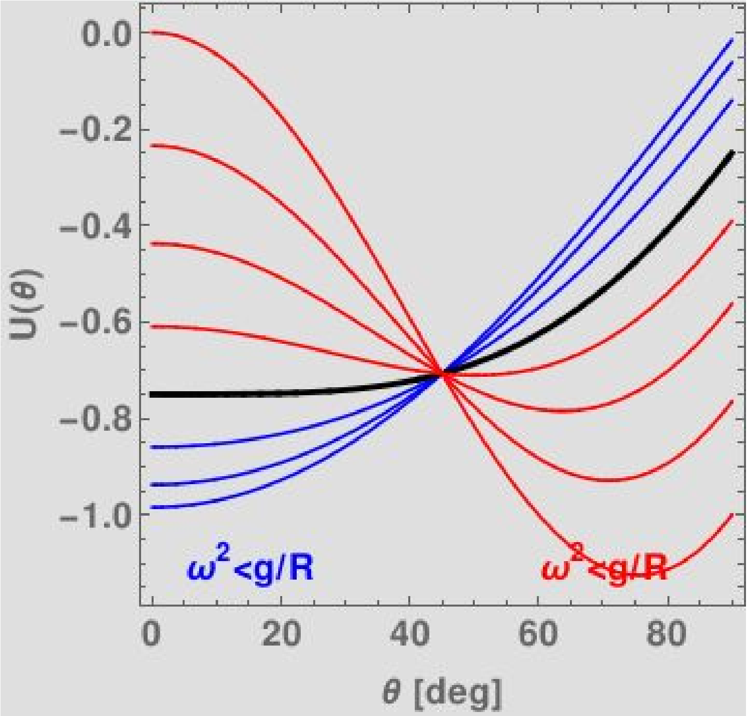
Oscillations of the Bead near Equilibrium

FIGURE 7.9



frequency $\propto$ $d^2U/d\theta^2$

$$U(\theta) = -\,g/R\,\cos(\theta) + \omega^2/4\,\cos(2\theta)$$



Homework

For any set of generalized coordinates, the trajectory obeys Lagrange's equations

$$\frac{\partial \pounds}{\partial q_i} = \frac{d}{dt}\frac{\partial \pounds}{\partial \dot{q}_i}$$

n equations;
i = 1 2 3 ... n

To solve a problem using the Lagrangian method:

1. Define generalized coordinates.
2. Write T and U in terms of the g.c..
3. $\pounds = T - U$
4. Derive Lagrange's equations.
5. Solve the equations.

Homework Assignment 12
due in class Monday November 28

[61] Problem 7.2 *
[62] Problem 7.3 *
[63] Problem 7.8 **
[64] Problem 7.14 *
[65] Problem 7.21 *
[66] Problem 7.31 **
[67] Problem 7.43 *** [computer]

Use the cover sheet.