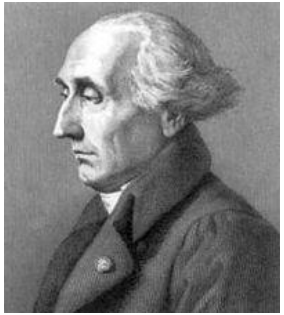
Chapter 7. Lagrange's Equations

1 Historical Introduction



Joseph-Louis Lagrange

(1736-1813)

Berlin; Paris;

Mécanique analytique

William Rowan Hamilton

(1805-1865)

Dublin;

"On a General Method in Dynamics"

Section 7.1. Lagrange's Equations for Unconstrained Motion

What do we mean by "unconstrained" motion?

The particle moves in 3 dimensions with a conservative net force **F(r).**

The potential energy is U(**r**).

$$T = \tfrac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right)$$

$$U = U(\mathbf{r})$$

The *Lagrangian* is $\mathcal{L} = T - U.$

(Notation: Script L)

(An example of "constrained motion" would be something like curvilinear motion of a bead on a wire, or planar motion.)

2 Lagrange's equations

We define $\pounds = T - U$.

Think of this as a function of {x,y,z} and $\{\dot{x}, \dot{y}, \dot{z}\}$; i.e.,

$$\pounds = \pounds(\mathbf{r}, \dot{\mathbf{r}}) \qquad \mathbf{r} = \{x,y,z\}$$

Now consider the *partial derivative*

$\partial / \partial x$ meaning

vary x but keep the all other 5 variables $\{ y, z, \dot{x}, \dot{y}, \dot{z} \}$ fixed;

$$\partial \pounds / \partial x = - \partial U / \partial x = F_x(x)$$

Now consider the partial derivative

$\partial / \partial \dot{x}$ meaning

vary $\dot{x}$ but keep the all other 5 variables $\{ x, y, z, \dot{y}, \dot{z} \}$ fixed;

$$\partial \pounds / \partial \dot{x} = \partial T / \partial \dot{x} = m \dot{x}$$

Newton's second law: $F_x(x) = m\ddot{x} \Rightarrow$

$$\boxed{\frac{\partial \pounds}{\partial x} = \frac{d}{dt} \frac{\partial \pounds}{\partial \dot{x}}}$$

Similarly for *y* and *z*.

$$\frac{\partial \pounds}{\partial y} = \frac{d}{dt} \frac{\partial \pounds}{\partial \dot{y}} \qquad \frac{\partial \pounds}{\partial z} = \frac{d}{dt} \frac{\partial \pounds}{\partial \dot{z}}$$

These are Lagrange's equations.

Lagrange's equations

For unconstrained motion,

$$\frac{\partial \pounds}{\partial \mathbf{r}} = \frac{d}{dt}\frac{\partial \pounds}{\partial \dot{\mathbf{r}}} \qquad \text{(3 eqs.)}$$

for $\mathbf{r} = \{x, y, z\}$,

where $\pounds = T - U$.

Remember the meanings of the partial derivatives!

$\partial/\partial x$ means vary x but keep the other 5 variables fixed;

$\partial/\partial \dot{x}$ means vary $\dot{x}$ but keep the other 5 variables fixed.

Do you see that the equation looks like the Euler -Lagrange equation. Then what is the variational problem?

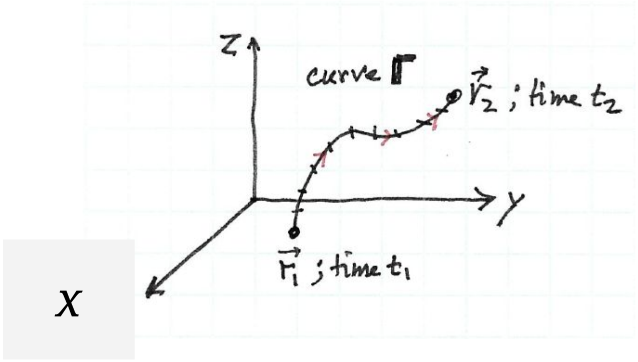
Hamilton's action integral

Define the action integral S by

$$S(\Gamma) = \int_{t_1}^{t_2} \pounds(\mathbf{r}, \dot{\mathbf{r}})\, dt$$

where:

- Γ is a path is space from $\mathbf{r}_1$ to $\mathbf{r}_2$;
- $\mathbf{r}(t)$ is a function of time that traverses the path as $t : t_1 \rightarrow t_2$;
- Important: $\mathbf{r}(t_1) = \mathbf{r}_1$ and $\mathbf{r}(t_2) = \mathbf{r}_2$.

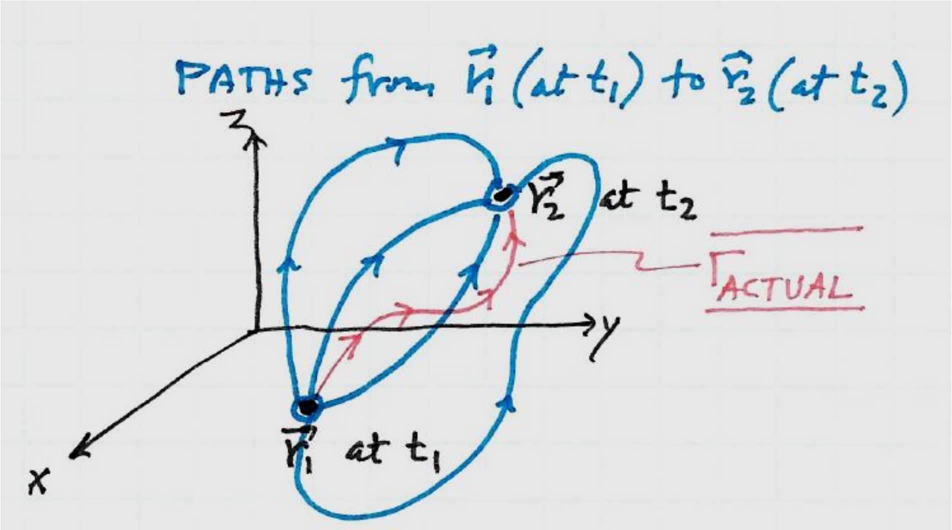


3

Hamilton's Principle

The actual path taken by m under the influence of the force $-\nabla U$, from (t_1, $\mathbf{r}_1$) to (t_2, $\mathbf{r}_2$), will be the path Γ_{actual} for which S is minimum.

"least action"



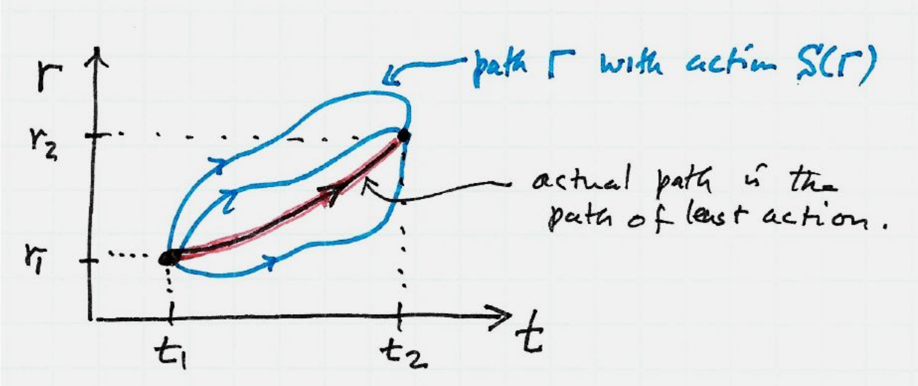
Hamilton's Principle

Suppose the particle moves from (t_1 , $\mathbf{r_1}$) to (t_2 , $\mathbf{r_2}$), under the influence of the force $\mathbf{F} = -\nabla U$. The trajectory of the particle is $\mathbf{r}(t)$, which defines a path Γ_{actual} .

Hamilton's Principle states

$$\min_{\{\Gamma\}} \quad S(\Gamma) \quad = \quad S(\Gamma_{actual})$$

Of all the paths from (t_1 , $\mathbf{r_1}$) to (t_2 , $\mathbf{r_2}$), the particle follows the *path of least action.*



Note:
The endpoints are fixed in both space and time.

Proof of Hamilton's Principle

$$S(\Gamma) = \int_{t_1}^{t_2} \pounds\,(\,\mathbf{r}, \dot{\mathbf{r}}\,)\,dt$$

What do I need to prove?

min S(Γ) occurs when **r**(t) obeys Lagrange's equations __

$$\frac{\partial \pounds}{\partial \mathbf{r}} = \frac{d}{dt}\,\frac{\partial \pounds}{\partial \dot{\mathbf{r}}}$$

- The minimum over all paths Γ [from $(t_1 , \mathbf{r}_1)$ to $(t_2 , \mathbf{r}_2)$] has δS = 0.
- The calculus of variations; consider δ**r**

$$\delta S = \int_{t1}^{t2} \{\; (\partial \pounds/\partial \mathbf{r}) \cdot \delta\mathbf{r} + (\partial \pounds/\partial \dot{\mathbf{r}}) \cdot \delta\dot{\mathbf{r}} \;\}\; dt$$

$$\text{2nd term} = \; d/dt\,[\,(\partial \pounds/\partial \dot{\mathbf{r}}) \cdot \delta\mathbf{r}\,] - d/dt\,[\,(\partial \pounds/\partial \dot{\mathbf{r}})\,] \cdot \delta\mathbf{r}$$

We require δ**r** = 0 at the endpoints, so the integral of d/dt [...] is zero.

$$\therefore\; \delta S = \int_{t1}^{t2} \{\; (\partial \pounds/\partial \mathbf{r}) - d/dt\,(\partial \pounds/\partial \dot{\mathbf{r}})\;\} \cdot \delta\mathbf{r}\;\; dt$$

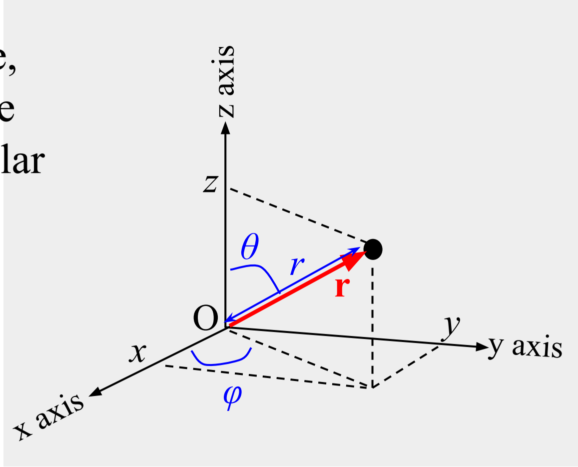
- δS must be = 0 for any variation of the path, i.e., for any function δ**r**(t). The only way that can be true is if the function in {} brackets is 0.
- For the least action, **r**(t) obeys Lagrange's equation.

4 Generalized coordinates

We can always use Cartesian coordinates {x, y, z} to specify the trajectory of the particle.

But now suppose some other coordinates could be used, say, $\{ q_1 , q_2 , q_3 \}$.

For example, we could use spherical polar coordinates $\{ r, \theta, \varphi \}$.



We would have a 1-to-1 correspondence between $\{ q_1 , q_2 , q_3 \}$ and $\{ x , y , z \}$.
That is, $\exists$ functions

$$q_i = q_i (\mathbf{r}) \qquad \text{for} \quad i = 1, 2, 3$$

or

$$\mathbf{r} = \mathbf{r} (q_1 , q_2 , q_3) .$$

Then we could write

$$\pounds = \pounds (q_1,q_2,q_3 , \dot{q}_1,\dot{q}_2,\dot{q}_3)$$

and

$$S = \int_{t_1}^{t_2} \pounds (q_1\ q_2\ q_3\ \dot{q}_1\ \dot{q}_2\ \dot{q}_3)\, dt$$

The actual path of the particle has least action, $\delta S = 0$; that's Hamilton's principle.

The equation $\delta S = 0$ gives us Lagrange's equations, but now in terms of $\{ q_1 , q_2 , q_3 \}$.

The actual path obeys these equations, in terms of any set of generalized coordinates,

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

3 equations; i = 1 2 3

To solve a problem using the Lagrangian method:

1. Define generalized coordinates.
2. Write T and U in terms of the g.c..
3. $\mathcal{L} = T - U$
4. Derive Lagrange's equations.
5. Solve the equations.

Example 7.2 from Taylor

Plane Polar Coordinates

FIGURE 7.1

$x = r\cos\phi$

$y = r\sin\phi$

$\dot{x} = \dot{r}\cos\phi - r\dot{\phi}\sin\phi$

$\dot{y} = \dot{r}\sin\phi + r\dot{\phi}\cos\phi$

z = 0

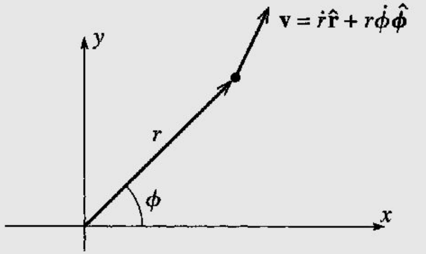


Figure 7.1 The velocity of a particle expressed in two-dimensional polar coordinates.

$$\mathcal{L} = T - U = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U_c(x,y)$$
$$= \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r,\phi)$$

The r equation $\quad \frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right)$

$$m r\dot{\phi}^2 - \frac{\partial U}{\partial r} = m\ddot{r} \quad = F_r + m r\dot{\phi}^2$$

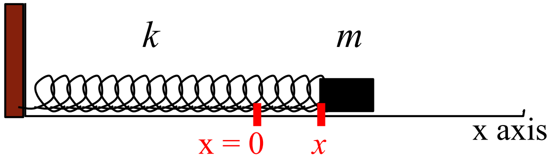
$$a_r = \ddot{r} - r\dot{\phi}^2$$

The ϕ equation $\quad \frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right)$

$$-\frac{\partial U}{\partial \phi} = \underbrace{\frac{d}{dt}\left(m r^2\dot{\phi}\right)}_{\text{ang. momentum } L_z} = \underbrace{r F_\phi}_{\text{torque } N_z}$$

Problem 7.2 from Taylor

"Write down the Lagrangian for a one-dimensional particle moving along the x axis and subject to a force $F = -k\,x$ (with k positive). Find the Lagrange equation of motion and solve it."



$$\pounds = \tfrac{1}{2}\, m\, \dot{x}^2 - \tfrac{1}{2}\, k\, x^2$$

$$\partial \pounds / \partial x = (\, d/dt\,)\ \partial \pounds / \partial \dot{x}$$

$$-\,k\,x = (\, d/dt\,)\, m\,\dot{x} = m\,\ddot{x}$$

$$\ddot{x} = -\,\omega^2\, x \quad \Rightarrow \quad x(t) = A\cos(\omega t - \delta)$$

Homework Assignment 12
due in class Friday December 2

[61] Problem 7.2 *
[62] Problem 7.3 *
[63] Problem 7.8 **
[64] Problem 7.14 *
[65] Problem 7.21 *
[66] Problem 7.31 **
[67] Problem 7.43 *** [computer]

Use the cover sheet.