Section 4.3.

Force as the gradient of potential energy

Section 4.4

The second condition for a force to be conservative

Read Sections 4.3 and 4.4.

The gradient operator (∇, called "del")

$$\nabla = \boldsymbol{e}_x (\partial/\partial x) + \boldsymbol{e}_y (\partial/\partial y) + \boldsymbol{e}_z (\partial/\partial z)$$

The Oxford Dictionary of Physics:

Given a scalar function f and a unit vector ***n***, the scalar product ***n*** . ∇f is the rate of change of f in the direction of ***n***.

4.3. F = – ∇U

A conservative force is equal to *the negative gradient* of the corresponding potential energy function.

Proof

$$\Delta U = -W$$

$$U(\vec{r}) - U(\vec{r}_0) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,'$$

$$\frac{\partial U}{\partial x} = \{ U(\vec{r} + \hat{e}_x \epsilon) - U(\vec{r}) \} / \epsilon$$

$$= \left\{ -\int_{r_0}^{r + \hat{e}_x \epsilon} + \int_{r_0}^{r} \right\} \vec{F} \cdot d\vec{r}\,' / \epsilon$$

$$= -\int_{r}^{r + \hat{e}_x \epsilon} \frac{\vec{F} \cdot d\vec{r}\,'}{\epsilon} = -\vec{F}(r) \cdot \frac{\hat{e}_x \epsilon}{\epsilon}$$

$$\frac{\partial U}{\partial x} = -F_x \quad \text{or} \quad F_x = -\frac{\partial U}{\partial x}$$

Generalize : $\vec{F} = -\nabla U$.

Example 4.4 finding ***F*** from U

- $\boldsymbol{F} = -\nabla U$
- U is a scalar; ***F*** is a vector.
- In *Cartesian coordinates*, the gradient of U is the vector of partial derivatives.

Taylor's example:

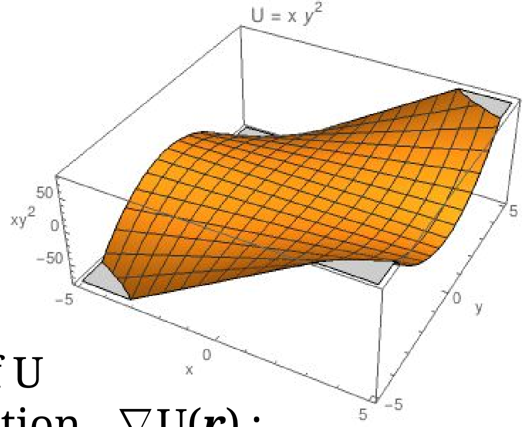
$U(x,y,z) = Axy^2 + B \sin Cz$

$\therefore \quad F_x = -A y^2 \quad ; \quad F_y = -2Ax\,;$

$F_z = -BC \cos Cz$

A 2d example

$U(x,y) = x\,y^2$



The gradient of U is a vector function, $\nabla U(\boldsymbol{r})$;

$dU = \nabla U \cdot d\boldsymbol{r}$;

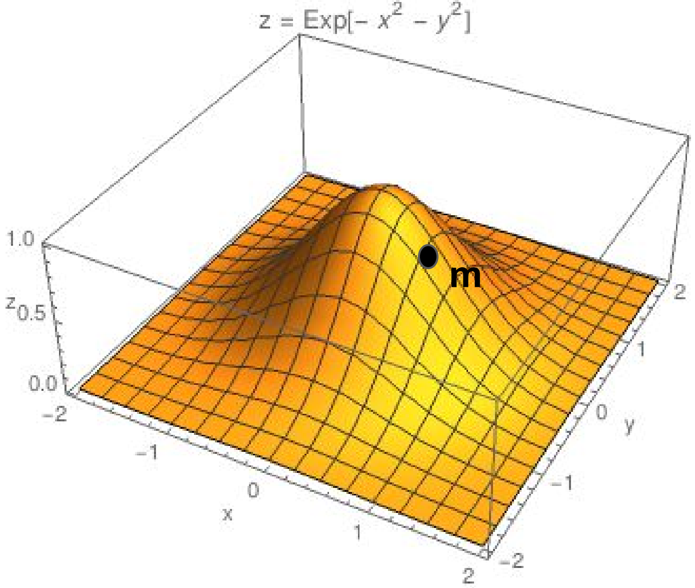
<u>magnitude and direction</u> of ∇U :
The maximum rate of change of U is $|\nabla U|$, and the direction of the max rate of change is parallel to ∇U. *I.e., ∇U points from low U to high U.*

- In *spherical polar coordinates*, see the formulas inside the back cover of the book.

$$= \hat{\mathbf{r}}\frac{\partial f}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

An example that shows the meaning of the gradient.

Consider a mass m on a plane (xy) , repelled from the origin. The picture shows a surface plot of U(x,y) .



(*Not a mass on a hill! That would be 3D.)*

- The height of the potential energy surface is

  $$z = A\, Exp[-x^2 - y^2].$$

- The potential energy of the mass at position {x,y} is

  $$U = A\, z = A\, Exp[-x^2 - y^2]$$

- The gradient of U = U (x,y) is

$$\frac{\partial U}{\partial x} = mg\, e^{-(x^2+y^2)}(-2x)$$

$$\nabla U = mg\, e^{-(x^2+y^2)}\big(\underbrace{-2x\,\hat{e}_x - 2y\,\hat{e}_y}_{-2\vec{r}}\big)$$

$$= -2\, mg\, \vec{r}\, e^{-r^2}$$

- Direction and magnitude

- The force acting on the ball is

  $$\mathbf{F} = -\nabla U = +2\, m\, g\, \mathbf{r}\, e^{-r^2}$$

i.e., ***pointing radially outward in 2D (repelled from 0).***

4.4. The second condition for a force to be conservative.

First, we define another differential operator of vector calculus.

The curl operator , $\nabla \times \boldsymbol{A}$

Oxford Dictionary of Physics: "Curl: The vector product of the gradient operator with a vector function."

$\boldsymbol{A}$ is a vector, and $\nabla \times \boldsymbol{A}$ is also a vector.

$$\vec{\nabla} \times \mathbf{A} = \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right) + \hat{\mathbf{y}}\left(\frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}A_z\right) + \hat{\mathbf{z}}\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right) \quad \text{[Cartesian]}$$

$$= \hat{\mathbf{r}}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial}{\partial\phi}A_\theta\right] + \hat{\boldsymbol{\theta}}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}A_r - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right] + \hat{\boldsymbol{\phi}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}A_r\right] \quad \text{[spherical polar]}$$

Theorem

For a conservative force $\boldsymbol{F}$,

$$\nabla \times \boldsymbol{F} = \boldsymbol{0}\,.$$

Proof:

The curl of a gradient is always 0…

The $\hat{e}_x$ component is $\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \equiv X$

Now suppose $\vec{A} = \nabla\alpha$ where $\alpha = \alpha(x,y,z)$

Then $X = \frac{\partial}{\partial y}\frac{\partial\alpha}{\partial z} - \frac{\partial}{\partial z}\frac{\partial\alpha}{\partial y} = \frac{\partial^2\alpha}{\partial y\,\partial z} - \frac{\partial^2\alpha}{\partial z\,\partial y}$

$= 0.$

Generalize : $\boxed{\nabla\times(\nabla\alpha) = 0.}$

Since $\boldsymbol{F}$ can be written as a gradient, i.e., $\mathbf{F} = -\nabla U$,

$$\nabla \times \boldsymbol{F} = \boldsymbol{0}\,. \qquad \textit{QED}$$

Definition

A force $\vec{F}$ is conservative if

(i) $\vec{F}$ depends only on $\vec{r}$

AND

(ii) $\int_a^b \vec{F} \cdot d\vec{r}$ is independent of the path from a to b

OR

(ii′) $\nabla \times \vec{F} = 0.$

Theorem

If $\vec{F}$ is conservative then we can write $\vec{F} = -\nabla U.$

Example 4.5

Is the Coulomb force conservative?

Figure 4.7

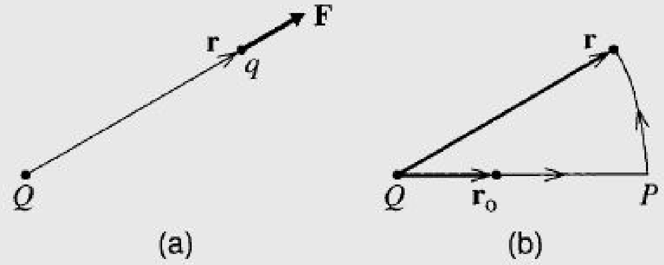


Figure 4.7 **(a)** The Coulomb force $\mathbf{F} = \gamma\hat{\mathbf{r}}/r^2$ of the fixed charge Q on the charge q. **(b)** The work done by $\mathbf{F}$ as q moves from $\mathbf{r}_o$ to $\mathbf{r}$ can be evaluated following a path that goes radially outward to P and then around a circle to $\mathbf{r}$.

$$\boldsymbol{F(r)} = \boldsymbol{e_r}\,\gamma/\mathrm{r}^2 = \mathbf{r}\,\gamma/\mathrm{r}^3$$

What is the potential energy function?

Assuming ***F*** is conservative,

$$W(\vec{r}_0 \to \vec{r}) = \underbrace{\int_{\vec{r}_0}^{P} \frac{\gamma \hat{e}_x}{r'^2}\cdot \hat{e}_x\, dr'}_{\int_{r_0}^{r} \frac{\gamma}{r'^2}\, dr'} + \underbrace{\int_{P}^{\vec{r}} \frac{\gamma \hat{r}}{r^2}\cdot \hat{e}_\phi\, r\, d\phi}_{\hat{r}\cdot\hat{e}_\phi = 0}$$

$$= \gamma\left(1/r_0 - 1/r\right)$$

$$= -\Delta U = -U(r) + U(r_0)$$

$$U(r) = \gamma/r$$

But is the curl equal to 0?

$$\nabla\times\vec{F} = \nabla\times\left(\frac{\gamma}{r^2}\hat{e}_r\right) = \det\begin{bmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\gamma x}{r^3} & \frac{\gamma y}{r^3} & \frac{\gamma z}{r^3} \end{bmatrix}$$

$$= \hat{e}_x \underbrace{\left[\frac{\partial}{\partial y}\left(\frac{\gamma z}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{\gamma y}{r^3}\right)\right]}_{\gamma z\left(\frac{-3}{r^4}\right)\frac{y}{r} - \gamma y\left(\frac{-3}{r^4}\right)\frac{z}{r} = 0} + \hat{e}_y \cdots + \hat{e}_z \cdots$$

$$\nabla\times\vec{F} = 0$$

Easier: use sphericals

The Coulomb force is conservative, and the potential energy is γ/r .

Check: $\boldsymbol{F} = -\nabla U$. ✔

So, we have two criteria …

(ii) The work done by the force
(on the object on which the force acts)
as the object moves from ***a*** to ***b*** is independent of the path from ***a*** to ***b***;

$$W(a \rightarrow b) = \int_{\Gamma} \mathbf{F}.\, d\boldsymbol{r}$$

$$= U(a) - U(b), \quad \text{indep. of path } \Gamma\,;$$

or, (ii') The curl of *F* is 0.

Taylor problem 4.25: "The proof that the condition $\nabla \times \mathbf{F} = 0$ guarantees the path independence of the work $\int_1^2 \mathbf{F}.d\boldsymbol{r}$ done by **F** is unfortunately too lengthy to be included here." *And then Taylor assigns it as a homework problem.*

Stokes's theorem

This is a famous theorem in vector calculus, similar to Gauss's theorem.

Recall Gauss's theorem:

$$\int_V \nabla \cdot \vec{A}\, d^3r = \oint_{S = \text{boundary of } V} \hat{n} \cdot \vec{A}\, dS$$

Stokes's theorem:

$$\int_S (\nabla \times \vec{A}) \cdot \hat{n}\, dS = \oint_{C = \text{boundary of } S} \vec{A} \cdot d\vec{r}$$

Now, suppose $\nabla \times \mathbf{F} = \mathbf{0}$.
Then Stokes's theorem implies $\oint \mathbf{F} \,.\, \mathbf{dr} = 0$, around any closed path.
Therefore $\int_a^b \mathbf{F} \,.\, \mathbf{dr}$ is path independent, because $a \rightarrow b \rightarrow a$ is a closed path;
$(\int_a^b \mathbf{F} \,.\, \mathbf{dr})_1 - (\int_a^b \mathbf{F} \,.\, \mathbf{dr})_2 = 0.$

Check your understanding:

Use both Cartesian coordinates and spherical polar coordinates. Use the formulas in the back cover of the book.

Prove:

$$\nabla r = \mathbf{e}_r$$

$$\nabla \times \mathbf{r} = 0$$

$$\nabla \times (\mathbf{e}_\varphi) = \mathbf{e}_z \,/\, (r \sin\theta)$$

Homework Assignment #7

due in class Friday, October 21

[31] Problem 4.3 **

[32] Problem 4.8 **

[33] Problem 4.9 **

[34] Problem 4.10 *

[35] Problem 4.18 **

[36] Problem 4.23 **

Use the cover page.

This is a pretty long assignment, so allow plenty of time to finish it.