Exam 2 Friday November 4

Name ________________________________

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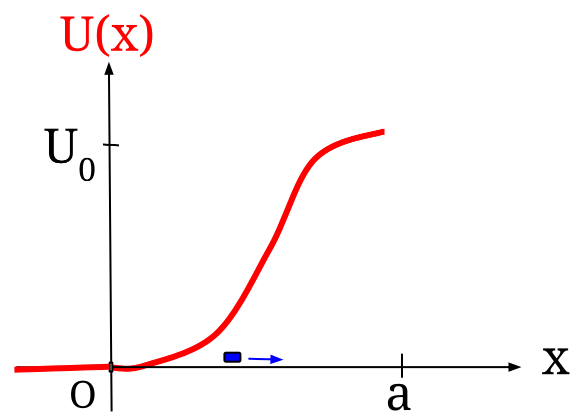
(1.) A particle (mass = m) moves in one dimension (x) under the influence of a conservative force. The potential energy is $U(x) = U_0 \sin^2(\pi x/(2a))$ for $x \geq 0$. The initial position and velocity are $x(0) = 0$ and $v(0) = v_0$ .




(a) Determine $v_0$ such that the particle energy is $U_0$.
(b) For $E = U_0$ , calculate the time it will take to move from 0 to x .

$$\int \frac{d\theta}{\cos(\theta)} = \ln\left[\frac{1+\sin(\theta)}{\cos(\theta)}\right] + C$$

(a) $E = \frac{1}{2} m \dot{x}^2 + U(x) = \text{constant}.$

From initial conditions, $E = \frac{1}{2} m v_0^2 + U(x_0) = \frac{1}{2} m v_0^2.$

For $E = U_0$ we require $U_0 = \frac{1}{2} m v_0^2$

$\therefore \quad v_0 = \sqrt{2U_0/m}$ ← 1 point

(b) $dt = \frac{dx}{\dot{x}}$ , or $t = \int_0^x \frac{dx'}{v(x')}$

$\frac{1}{2} m \dot{x}^2 + U(x) = E = U_0$

$\dot{x}^2 = \frac{2}{m}\left(U_0 - U_0 \sin^2\frac{\pi x}{2a}\right) = \frac{2U_0}{m}\left[1 - \sin^2\frac{\pi x}{2a}\right]$

$\dot{x} = \sqrt{\frac{2U_0}{m}} \sqrt{1-\sin^2} = \sqrt{\frac{2U_0}{m}} \cos\left(\frac{\pi x}{2a}\right)$

So

$t = \int_0^x \frac{dx'}{\sqrt{\frac{2U_0}{m}} \cos\left(\frac{\pi x'}{2a}\right)} = \sqrt{\frac{m}{2U_0}} \int_0^x \frac{dx'}{\cos(\theta)}$

$\boxed{\begin{aligned} &\text{Let } \theta = \frac{\pi x'}{2a} \\ &dx' = \frac{2a}{\pi} d\theta \end{aligned}}$

$= \sqrt{\frac{m}{2U_0}} \frac{2a}{\pi} \int_0^{\pi x/2a} \frac{d\theta}{\cos\theta}$

$= \sqrt{\frac{m}{2U_0}} \frac{2a}{\pi} \ln\left[\frac{1+\sin\frac{\pi x}{2a}}{\cos\frac{\pi x}{2a}}\right]$ ← 4 points

(2.) The equation of motion for a driven oscillator is

$\ddot{x} + 2\beta \dot{x} + \omega_0^2 x = f_0 \cos(\omega t)$ . Assume $\beta = 0.1\,\omega_0$ .

The steady-state oscillations have $x(t) = A \cos(\omega t - \delta)$.

(A) Sketch a graph of $A$ versus $\omega$ ; label the axes. Calculate $A$ for $\omega = \omega_0$ .

(B) Sketch a graph of $\delta$ versus $\omega$ ; label the axes. Calculate $\delta$ for $\omega = \omega_0$ .

(A) The easiest way to determine $A$ and $\delta$ is to use the complex exponential function.

$x = \text{Re}\, z$ where $\ddot{z} + 2\beta \dot{z} + \omega_0^2 z = f_0 e^{i\omega t}$

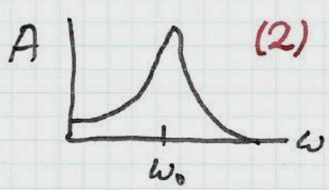
The steady state solution has

$z = C e^{i\omega t}$ , with $-\omega^2 C + 2i\beta\omega C + \omega_0^2 C = f_0$

$$C = \frac{f_0}{\omega_0^2 - \omega^2 + 2i\beta\omega} = A e^{-i\delta}$$

$\uparrow$ so $x = \text{Re}\, A e^{i(\omega t - \delta)}$
$= A \cos(\omega t - \delta)$

(A) $$A^2 = \frac{f_0^2}{(\omega_0^2 - \omega^2)^2 + (2\beta\omega)^2}$$



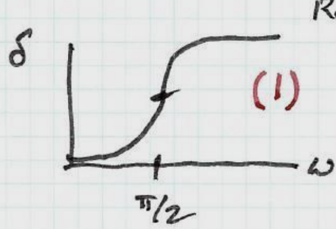

(2)

(1)

and $A(\omega_0) = \dfrac{f_0}{2\beta\omega_0} = 5\,\dfrac{f_0}{\omega_0^2}$

} 5 points

(B) $$\tan\delta = -\frac{\text{Im}\, C}{\text{Re}\, C} = \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$



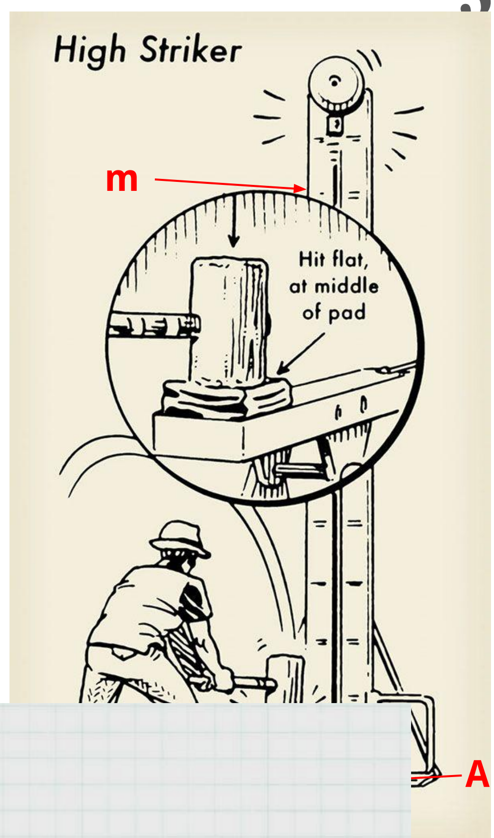

(1)

(1)

$\delta(\omega_0) = \arctan(\infty) = \pi/2$

(3.) The mass **m** ( = 1 kg) is initially at y = 0. An impulsive force occurs when the hammer hits the lever at point **A**. (The lever has no mechanical advantage.)
(A) Show how to calculate the maximum height that the mass will reach *using conservation of energy*.
(B) Make a reasonable numerical estimate for the impulse, and calculate the maximum height. Is your answer reasonable?

1 kg weight = 2.2 pounds




(A) The impulse:

$$I = \int F\,dt = \int m a\,dt = m(\Delta v) = m v_0$$

The mass goes up and energy is conserved

$$E = \frac{1}{2} m v^2 + m g y$$

$$E_0 = \frac{1}{2} m v_0^2 = E_f = m g y_{max}$$

$$y_{max} = \frac{v_0^2}{2g} \quad \text{and} \quad v_0 = \frac{F\,\delta t}{m}$$

$(m = 1\ \text{kg})$

2 points

(B) Numerical estimates:

$F \approx 20$ pounds $= 89$ N and $\delta t \approx 0.1$ s

$I = 8.9$ kg m/s and $v_0 = 8.9$ m/s

$$y_{max} = \frac{v_0^2}{2g} = 4.04\ \text{m} = 13.3\ \text{feet}$$

3 points

which seems reasonable
(a little more than 2 × height of a strongman.)