Section 2.6
Complex Exponentials

Read Section 2.6.

Today's subject isn't really classical mechanics.

It's useful mathematics that can be applied in many fields.

So we are going off on a tangent today, to learn about the subject of

Complex Exponentials

The exponential function, *exp*(x)

What is this function?
I.e., how is it defined?

I'm sure that you know

$$exp(x) = e^x$$

where e is a certain number.

But what is e?

e is an irrational number, approximately equal to 2.718.

But what is the *exact* value of e?
And what is so special about 2.718…?

The exponential function, *exp*(x)

What is this function?

The definition of *exp*(x) is that

$$exp'(x) = exp(x) \,,$$

with $exp(0) = 1$.

We'll express *exp*(x) as a power series.

But first we need …

Taylor's theorem

If $f(x)$ is continuous and differentiable then

$$f(x+\delta) = f(x) + f'(x)\,\delta + f''(x)\,\delta^2/2 + f'''(x)\,\delta^3/6 + \ldots + f^{(n)}(x)\,\delta^n/n! + \ldots$$

Proof

Compare LHS and RHS as functions of δ

- set $\delta = 0$: $f(x) = f(x)$ check
- differentiate ***w.r.t. δ*** and set $\delta = 0$:

  $f'(x) = f'(x)$ check
- twice differentiate *"* and set $\delta = 0$:

  $f''(x) = f''(x)$ check
- n times differentiate *"* and set $\delta = 0$:

  $f^{(n)}(x) = f^{(n)}(x)$ check

Q.E.D.

The power series for *exp*(u)

Apply Taylor's theorem,

with $x = 0$ and $\delta = u$.

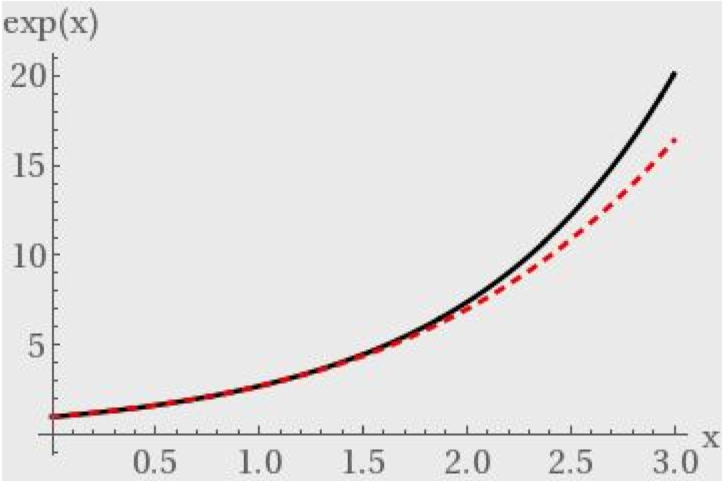
$$\begin{aligned} exp(u) = {} & exp(0) + exp'(0)\ u \\ & + exp''(0)\ u^2/2 \\ & + exp'''(0)\ u^3/6 \\ & + \ldots + exp^{(n)}(0)\ u^n/n! + \ldots \end{aligned}$$

∴ by the definition of *exp*(x)

$$\begin{aligned} exp(u) = {} & 1 + u + u^2/2 + u^3/6 \\ & + \ldots + u^n/n! + \ldots \end{aligned}$$

Result

$$exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$



red dashes :

truncation of the series at n=4

<u>Theorem</u> $\quad exp(x)\, exp(y) = exp(x+y)$

<u>Proof</u>

(This is Taylor's Problem 2.51.)

$$\exp(x+y) = \sum_{p=0}^{\infty} (x+y)^p / p!$$

$$= \sum_{p=0}^{\infty} \underbrace{\sum_{n=0}^{p} x^n y^{p-n} \binom{p}{n}}_{\text{binomial theorem}} \frac{1}{p!}$$

$$= \sum_{p=0}^{\infty} \sum_{n=0}^{p} \sum_{m=0}^{p} \delta_K(m, p-n)\, x^n y^m \frac{p!}{n!\,m!} \frac{1}{p!}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{x^n y^m}{n!\,m!} \qquad \left(\sum_p = 1\right)$$

$$= \exp(x)\, \exp(y)$$

<u>Corollary</u> $\quad exp(x) = e^x$

where e is a certain number.

<u>Proof</u>

$$e^x\, e^y = e^{x+y}$$

e , the base of natural logarithms

e may be expressed in terms of the integers, using e = $exp(1) = e^1 = e$:

By the power series,

e = 1 /0! + 1 /1! +1 /2! + 1 /3! + 1 /4! + …

e ≈ 1 + 1 + 0.5 + 0.1667 + 0.0416 + …

e ≈ 2.718

e = 2.7182818284590450908 …

e is the sum of reciprocal factorials.

Now consider

$$e^{i\theta}$$

where θ is a real number.

It's defined by the power series, so

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

The terms with n odd are imaginary, and the terms with n even are real,

```
i⁰ = 1        n|4   (divisible by 4)
i¹ = i        (n-1)|4
i² = -1       (n-2)|4
i³ = -i       (n-3)|4
i⁴ = 1        n|4
```

So,

$$e^{i\theta} = \sum_{n\ \text{even}} \frac{(-1)^{n/2}}{n!}\theta^n + i \sum_{n\ \text{odd}} \frac{(-1)^{(n-1)/2}}{n!}\theta^n$$

$$\text{Re} = 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 + \dots$$

$$= \cos\theta$$

$$\text{Im} = \theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 + \dots$$

$$= \sin\theta$$

Euler's formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

Magnitude and Phase of a complex number, z

Let z denote a complex number.

We can write $z = x + i\,y$;

$$x = \mathrm{Re}\,z \qquad \text{and} \qquad y = \mathrm{Im}\,z.$$

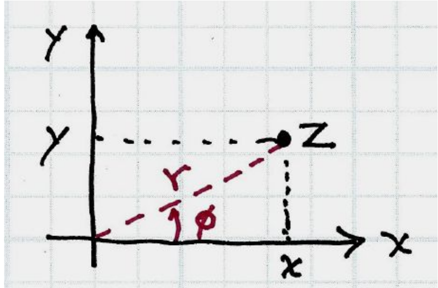
The *magnitude* of z is *r* defined by

$$z^*z = r^2 = (x-iy)\,(x+iy) = x^2 + y^2 \;;$$

the *phase* of z is φ, defined by

$$x = r\cos\varphi\,,$$
$$y = r\sin\varphi\,;$$

$$\text{or,} \quad \tan\varphi = y/x\,.$$



By Euler's formula,

$$z = r\,e^{i\varphi} \quad .$$

This is a crucial trick when we use complex numbers in theoretical physics.

We can write $\boldsymbol{z = x + i\,y}$ ***;***

or we can write $\boldsymbol{z = r\,e^{i\varphi}}$ ***.***

We just use whichever representation is more convenient.

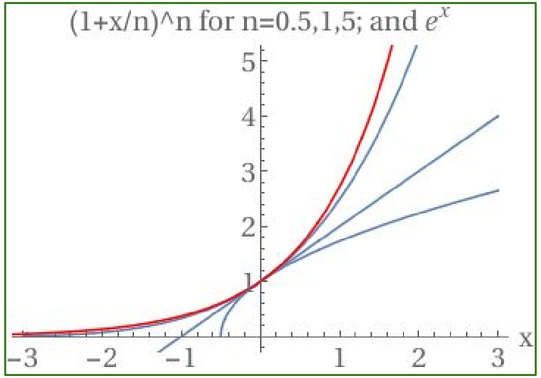
Math. exercise

Define $f_n(x) = (1 + x/n)^n$.

What is the limit as $n \to \infty$?

- Naively, let "$n = \infty$";

  $\Rightarrow (1)^\infty$ = indeterminate.



We have evidence that

$f_n(x) \sim e^x$ as $n \to \infty$.

- Use the binomial expansion for n = an integer;

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + b^n$$

where $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$(1+\epsilon)^n = \sum_{k=0}^{n} \binom{n}{k} \epsilon^k$$

$$\approx 1 + n\epsilon + \frac{n(n-1)}{2} \epsilon^2$$

sort of resembles $\exp(n\epsilon)$

$$\left(1 + \frac{x}{n}\right)^n \approx \exp(x) \ (?)$$

- L'Hôpital's Rule

$$\ln f_n(x) = n \ln(1 + x/n) = \frac{\ln(1 + x/n)}{1/n}$$

$$= \frac{a(n)}{b(n)}.$$

At "$n = \infty$", $\ln f_n = \dfrac{0}{0}$, indeterminate

L'H. Rule: $\lim \ln f_n = \dfrac{da/dn}{db/dn} = \dfrac{\frac{1}{(1+x/n)}\left(\frac{-x}{n^2}\right)}{-1/n^2}$

$$\lim \ln f_n(x) = \lim \frac{x}{1 + x/n} = x$$

$\ln f_\infty(x) = x \Rightarrow \boxed{f_\infty(x) = e^x}$

$\therefore$ Another definition of $e = \lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n =$

n	$(1+\frac{1}{n})^n$
1	2
10	2.594
100	2.705
1000	2.717
10^4	2.718

In the next lecture we'll use the complex exponential function to calculate the motion of a charged particle in a magnetic field.

Homework Assignment #4

due in class Friday

[17] Problem 2.23 *

[18] Problem 2.31 **

[19] Problem 2.41 **

[20] Problem 2.53 *

[20x] Problem 2.43 *** [computer]

Use the cover sheet.