Section 8.6. *Bounded Kepler orbits*
Section 8.7. *Unbounded Kepler Orbits*

Read Sections 8.6 and 8.7.

Review some equations we know ...

$\gamma = Gm_1m_2$

- $u(\phi) = A\cos(\phi - \delta) + \dfrac{\gamma\mu}{\ell^2}$

W.L.O.G. take $\delta = 0$.

$u(\phi) = A\cos\phi + \dfrac{\gamma\mu}{\ell^2}$

$r(\phi) = \dfrac{1}{u} = \dfrac{c}{1+\varepsilon\cos\phi}$

$c = \ell^2 / \gamma\mu$

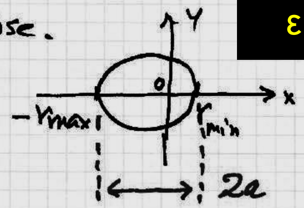
check:

$$\frac{1}{r} = \frac{1}{c} + \frac{\varepsilon\cos\varphi}{c}$$

Exercise: What about $\gamma = 0$?

$\varepsilon < 1$

- $r(\phi) = \dfrac{c}{1+\varepsilon\cos\phi}$ is an ellipse.



$r_{min} = \dfrac{c}{1+\varepsilon}$ and $r_{max} = \dfrac{c}{1-\varepsilon}$

The semimajor axis is

$a = \frac{1}{2}(r_{min} + r_{max}) = \dfrac{c}{1-\varepsilon^2}$

Exercise: Show that $\varepsilon = \dfrac{r_{max} - r_{min}}{r_{max} + r_{min}}$

Use *{a,ε}* to define the ellipse.

Now relate energy (***E***) and angular momentum (***ℓ***) to semimajor axis (***a***) and eccentricity (***ε***).

- Angular momentum $\ell = \mu r^2 \dot{\phi}$

We have $c = \dfrac{\ell^2}{\gamma\mu}$ and $c = a(1-\varepsilon^2)$

So $\ell^2 = \gamma\mu a(1 - \varepsilon^2)$ depends on both a and ε.

- Energy $= \frac{1}{2} m \dot{r}^2 + \frac{\ell^2}{2\mu r^2} - \frac{\gamma}{r}$

Express E in terms of $\{a, \varepsilon\}$.

$$E = \frac{\ell^2}{2\mu r_{min}^2} - \frac{\gamma}{r_{min}} = \frac{\gamma \mu a (1-\varepsilon^2)}{2\mu \left[\frac{c}{1+\varepsilon}\right]^2} - \frac{\gamma}{\left[\frac{c}{1+\varepsilon}\right]}$$

$$= \frac{\gamma c (1+\varepsilon)^2}{2c^2} - \frac{\gamma(1+\varepsilon)}{c} = \frac{\gamma}{2c}\left\{(1+\varepsilon)^2 - 2(1+\varepsilon)\right\}$$

$$= \frac{\gamma}{2} \frac{-1+\varepsilon^2}{a(1-\varepsilon^2)} = \frac{-\gamma}{2a}$$

$$E = \frac{-\gamma}{2a} = \frac{-Gm_1m_2}{2a}$$

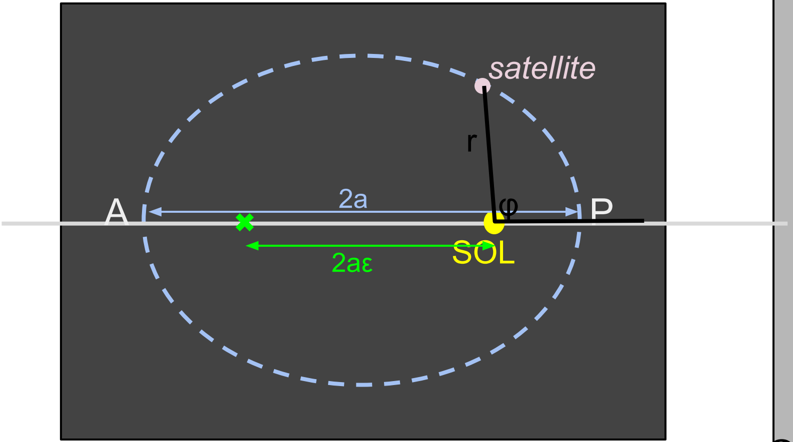
depends on a but not ε

Results

$$\ell^2 = \mu\,\gamma\,a\,(1-\varepsilon^2)$$

$$E = -\gamma/(2a)$$

$$\gamma = G\,m_1\,m_2 = GM\,\mu$$

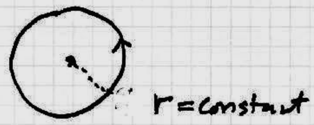


Kepler's third law (1619)

By analyzing Tycho's observations of the planets, Kepler concluded that $\tau^2 \propto a^3$ for all the planets; in other words, $\tau^2 / a^3 =$ **constant.**

It's not precisely true, but it is very close.
(1) Derivation from Newton's theory, for circular orbits.

$$\mu \ddot{\vec{r}} = -\frac{\gamma}{r^2}\hat{r}$$



$$\ddot{\vec{r}} = -\frac{v^2}{r}\hat{r} \quad \text{(circular motion)}$$

$$\frac{\mu v^2}{r} = \frac{\gamma}{r^2} \Rightarrow v^2 = \frac{\gamma}{\mu r}$$

The speed is constant, so the period of revolution is $\tau = \frac{2\pi r}{v} = 2\pi\sqrt{\frac{\mu r}{\gamma}}\, r$

Now, $\gamma = G m_1 m_2 = G M \mu$.

So $\tau^2 = \frac{4\pi^2 r^3}{GM} = \frac{4\pi^2 r^3}{G(M_S + m_{planet})}$

$\tau^2 \approx \frac{4\pi^2 r^3}{G M_S}$ Kepler's third law for a circular orbit

(2) Derivation for elliptical orbits.

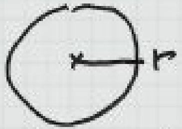
Recall Kepler's 2nd law, which we derived in chapter 3; "equal areas in equal times";

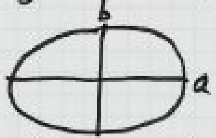
$\frac{dA}{dt} = \frac{\ell}{2\mu}$ for any central force

is constant, so $A = \frac{\ell \tau}{2\mu}$

What is the area of an ellipse?

What is the area of an ellipse?





$A = \pi r^2$

$A = \pi a b$

$$\int dA = \int_{-a}^{a} 2y\, dx \quad \text{and} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$= \int_{-a}^{a} 2b\sqrt{1 - x^2/a^2}\, dx = \pi a b$$

So $\tau = \frac{2\mu}{\ell} \pi a b$ Recall: $a = \frac{c}{1-\varepsilon^2}$

and $b = \frac{c}{\sqrt{1-\varepsilon^2}}$ so $b = a\sqrt{1-\varepsilon^2}$

$$\tau = \frac{2\mu}{\sqrt{\mu \gamma a (1-\varepsilon^2)}} \pi a \cdot a\sqrt{1-\varepsilon^2} = 2\pi a^{3/2} \sqrt{\frac{\mu}{\gamma}}$$

$$\tau = \frac{2\pi a^{3/2}}{\sqrt{GM}} \quad \text{or} \quad \tau^2 = \frac{4\pi^2 a^3}{G(M_s + m_{planet})}$$

same as for a circular orbit

Section 8.6. *Bounded Kepler Orbits*

We have been considering bounded Kepler orbits. These have energy $E < 0$. The orbits are ellipses with eccentricity ε in the range $0 \le \varepsilon < 1$.

(A circular orbit has $\varepsilon = 0$.)

Equations

$$r(\varphi) = \frac{c}{1 + \varepsilon \cos \varphi}$$

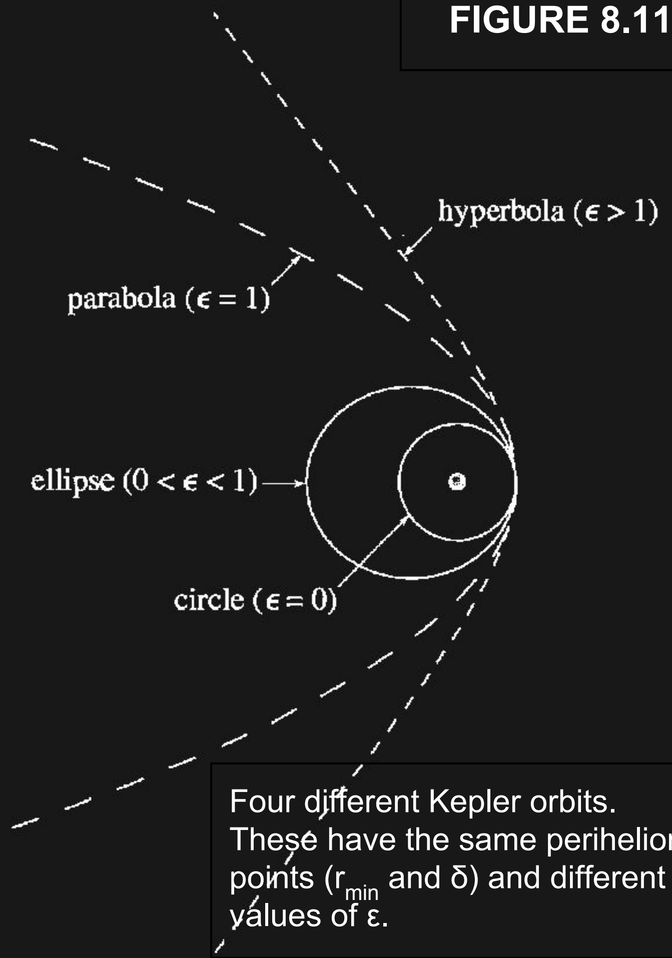
$$r_{min} = c/(1+\varepsilon)$$

$$0 \le \varepsilon < 1$$

$$c = l^2 / (\gamma\mu) \quad \text{and} \quad l = \mu r^2 \dot{\varphi}$$

$$E = \tfrac{1}{2} \mu \dot{r}^2 + \frac{l^2}{2 \mu r^2} - \frac{\gamma}{r}$$

FIGURE 8.11



Four different Kepler orbits. These have the same perihelion points (r_{min} and δ) and different values of ε.

Section 8.7. *Unbounded Kepler Orbits*

Now consider orbits with $E \geq 0$.

We can reuse some of the equations that we had before; they are valid for either $E < 0$ or $E > 0$.

Equations

$$r(\varphi) = \frac{c}{1 + \varepsilon \cos \varphi}$$

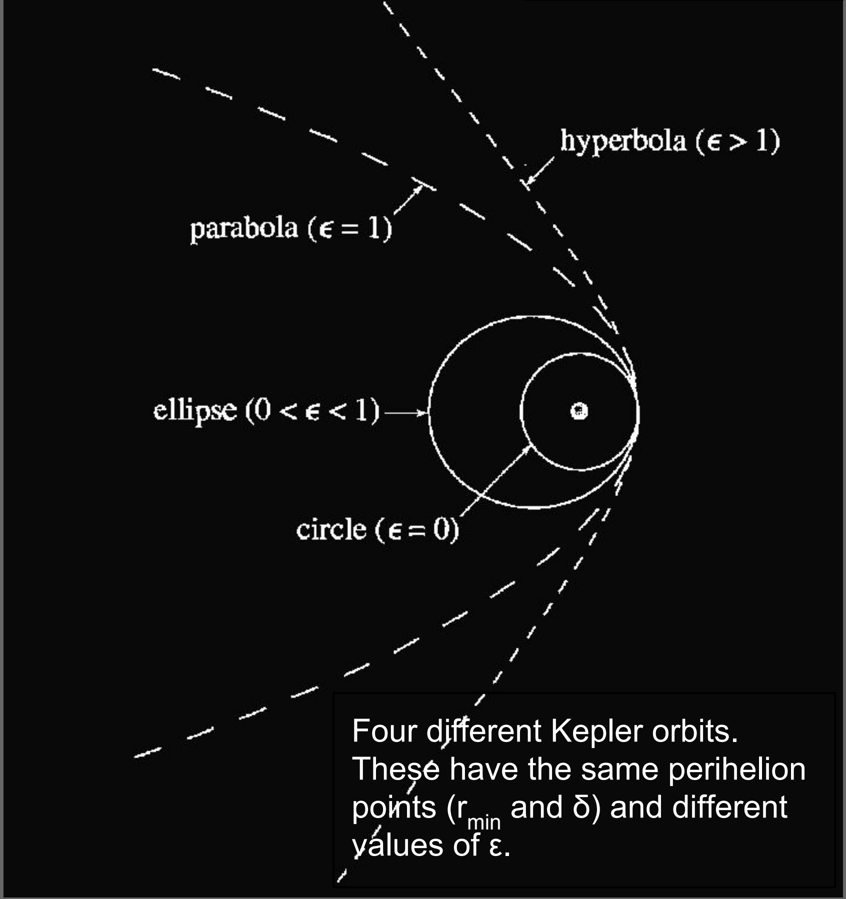
$$r_{min} = c/(1+\varepsilon)$$

$\varepsilon \geq 1$

$$c = l^2/(\gamma\mu) \quad \text{and} \quad l = \mu r^2 \dot{\varphi}$$

$$E = \tfrac{1}{2} \mu \dot{r}^2 + \frac{l^2}{2 \mu r^2} - \frac{\gamma}{r}$$

FIGURE 8.11



Four different Kepler orbits. These have the same perihelion points (r_{min} and δ) and different values of ε.

Parabolic orbits have ε = 1.

$$r(\varphi) = \frac{c}{1 + \cos\varphi}$$

Why is this a parabola?

$x = r \cos \varphi$
$y = r \sin \varphi$

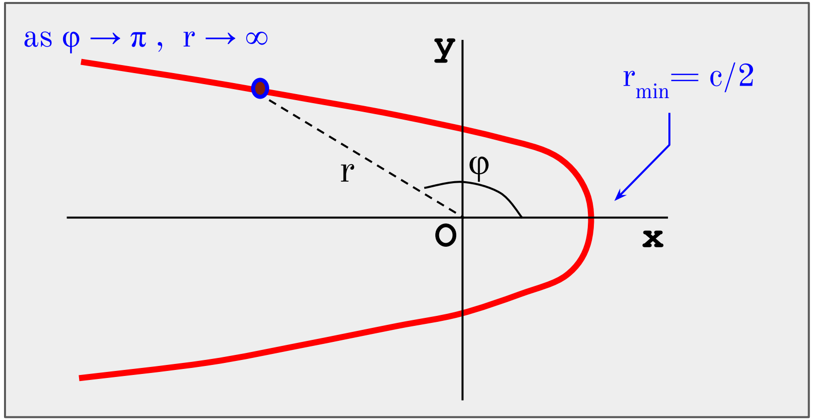
$$(1 + \cos\varphi)\, r = r + x = c$$

$$r^2 = x^2 + y^2 = (c - x)^2 = c^2 - 2cx + x^2$$

$$x = (c^2 - y^2)/(2c)$$

the eq. for a parabola

Or, $y = \pm \text{sqrt}\,(c^2 - 2cx)$



The energy for the parabolic orbit

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} - \frac{\gamma}{r}$$

$$\therefore\ E = \frac{\ell^2}{2\mu r_{min}^2} - \frac{\gamma}{r_{min}} \qquad (\dot{r} = 0 \text{ at peri.})$$

Now recall $c = \frac{\ell^2}{\gamma\mu}$ and $r_{min} = \frac{c}{1+\varepsilon} = \frac{c}{2}$

$$E = \frac{1}{r_{min}}\left(\frac{\ell^2}{2\mu r_{min}} - \gamma\right) = \frac{1}{r_{min}}\left(\frac{\ell^2}{2\mu}\,\frac{2\gamma\mu}{\ell^2} - \gamma\right) = 0$$

$E = 0$

Hyperbolic orbits have $\varepsilon > 1$.

$$r(\varphi) = \frac{c}{1 + \varepsilon \cos\varphi}$$

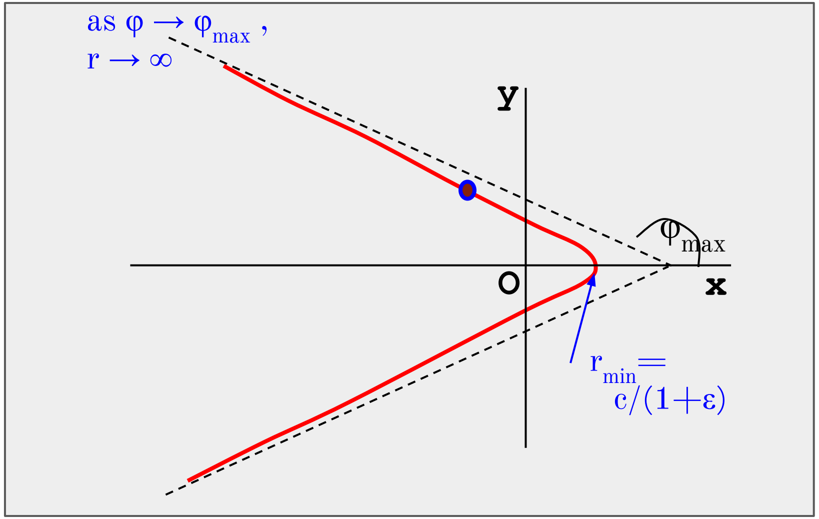
Why is this a hyperbola?

$x = r \cos \varphi$
$y = r \sin \varphi$

Note that $r \to \infty$ as $\varphi \to \varphi_{max}$

where $$\cos \varphi_{max} = -\frac{1}{\varepsilon}$$

which requires $\varepsilon > 1$



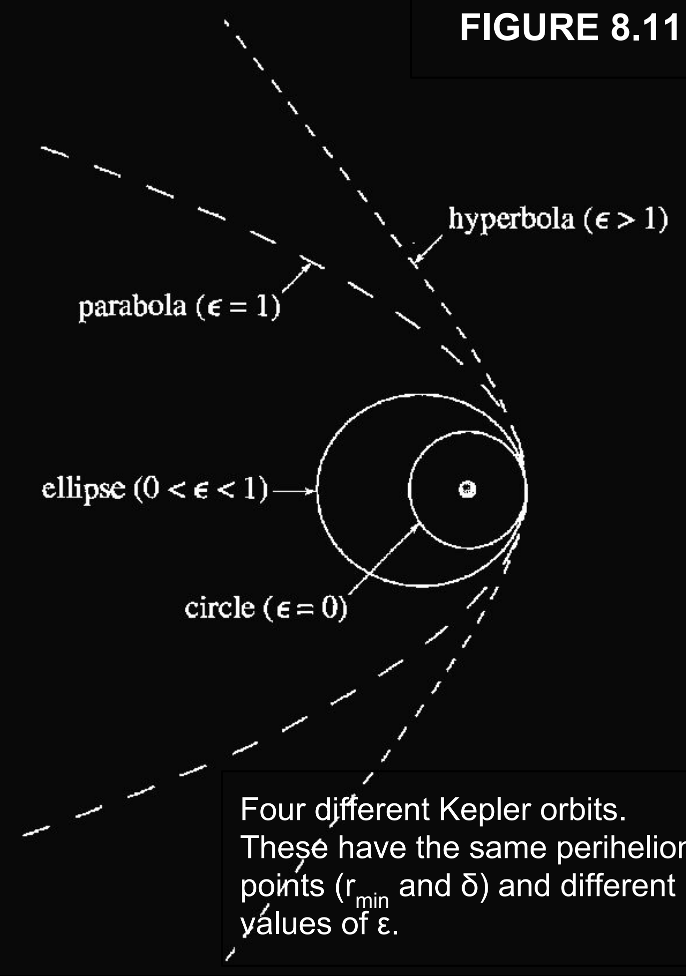
The energy for a hyperbolic orbit.

$$E = \frac{\ell^2}{2\mu r_{min}^2} - \frac{\gamma}{r_{min}} \quad (\dot{r} = 0 \text{ at perih.})$$

Recall $c = \ell^2/\gamma\mu$ and $r_{min} = c/(1+\varepsilon)$

$$E = \frac{\gamma^2 \mu}{2\ell^2}(\varepsilon^2 - 1) \quad \text{so} \quad E > 0.$$

$E > 0$



Four different Kepler orbits. These have the same perihelion points (r_{min} and δ) and different values of ε.

- Given the position and velocity vectors at one point on the orbit, the constants of motion ℓ and E are determined.
- The sign of E determines the curve:

$E < 0$	bounded	elliptical
$E = 0$	unbounded	parabolic
$E > 0$	unbounded	hyperbolic

- Given ℓ and E, the geometric parameters are determined; e.g., $\{r_{min}, \varepsilon\}$.
- These are the Kepler orbits in space; ***but what about the time dependence?***

Exam 3 is Monday.

Homework assignment #14 is due next Friday.