Section 5.3

Two dimensional oscillators

Section 5.4

Damped oscillations

Read Sections 5.3 and 5.4.

Figure 5.7 **(a)** A restoring force that is proportional to $\mathbf{r}$ defines the isotropic harmonic oscillator. **(b)** The mass at the center of this arrangement of springs would experience a net force of the form $\mathbf{F} = -k\mathbf{r}$ as it moves in the plane of the four springs.

for small oscillations

5.3. Two dimensional oscillators

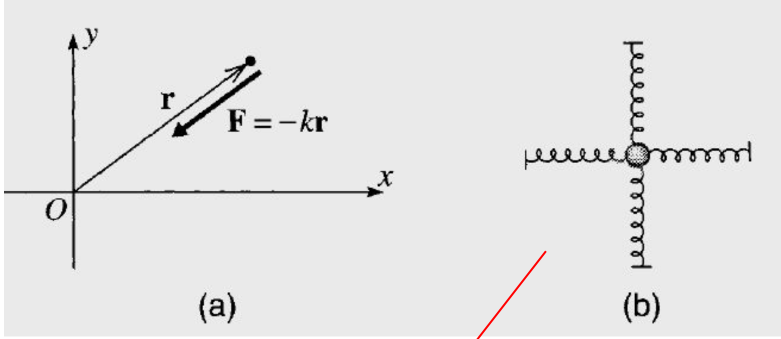
The definition of an "isotropic" oscillator is 2 or 3 dimensions is

$$\boldsymbol{F} = -k\,\boldsymbol{r}$$

$$U = \tfrac{1}{2} k r^2 = \tfrac{1}{2} k (x^2 + y^2 + z^2)$$

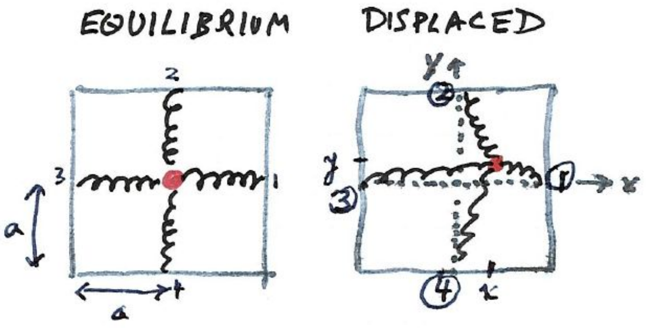
in 3 dimensions

Figure 5.7 shows a 2d example; the particle (mass m) attached to the springs moves in the xy plane.



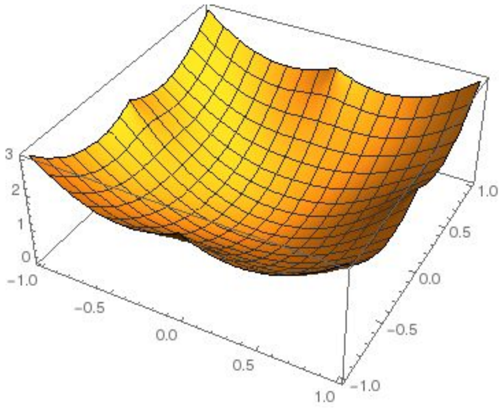
Comments about Figure 5.7.

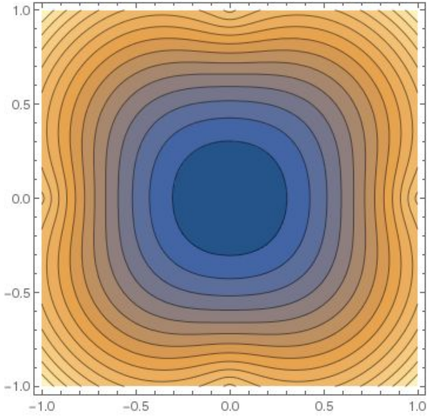
The particle (mass = m) attached to the springs moves in the xy plane.



What is the potential energy when the particle is displaced to { x , y }?

Assume that the equilibrium length of each spring is a, and the spring constant is k/2. Also, the size of the square is $2a \times 2a$.





$$U = \frac{k}{4}(r_1-a)^2 + \frac{k}{4}(r_2-a)^2 + \frac{k}{4}(r_3-a)^2 + \frac{k}{4}(r_4-a)^2$$

$$= \frac{k}{4}\left\{ r_1^2 + r_2^2 + r_3^2 + r_4^2 + 4a^2 - 2a(r_1 + r_2 + r_3 + r_4) \right\}$$

$$\left.\begin{matrix} r_1 \\ r_3 \end{matrix}\right\} = \sqrt{(a \mp x)^2 + y^2} \approx a \mp x + \frac{y^2}{2a}$$

etc.

Result: For $x^2 + y^2 << a^2$ we have
$U \approx$ **½** $k\,(x^2 + y^2) =$ **½** $k\,r^2$ and $\boldsymbol{F} \approx -\,k\,\boldsymbol{r}$.

[5] It is perhaps worth pointing out that one does *not* get a force of the form (5.17) by simply attaching a mass to a spring whose other end is anchored to the origin.

Figure 5.8. Three examples of isotropic oscillations in 2d:

$U = \frac{1}{2} k x^2 + \frac{1}{2} k y^2$

$x(t) = A \cos(\omega t)$

$y(t) = B \cos(\omega t - \delta)$

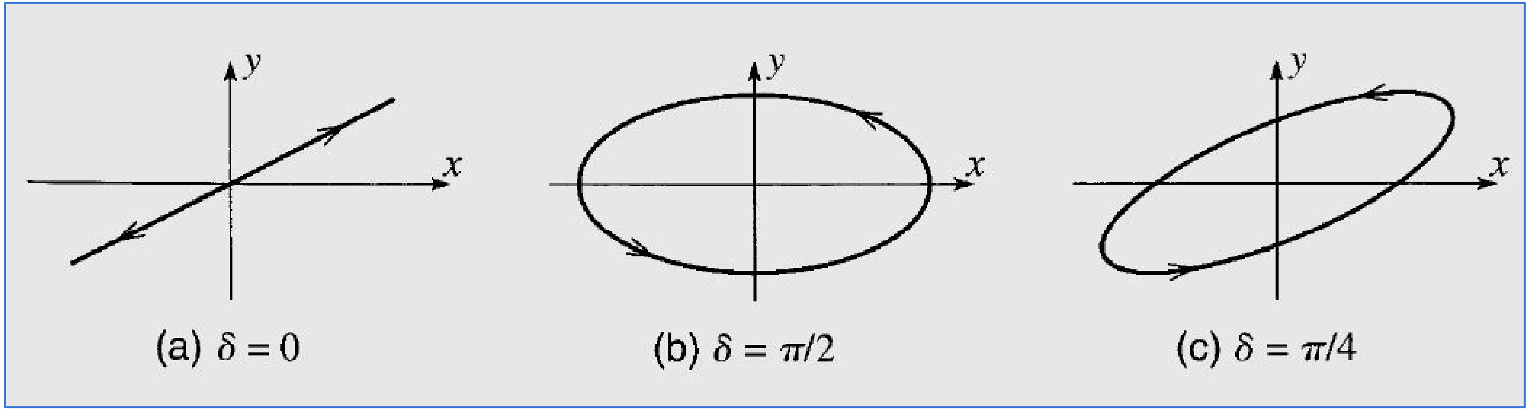


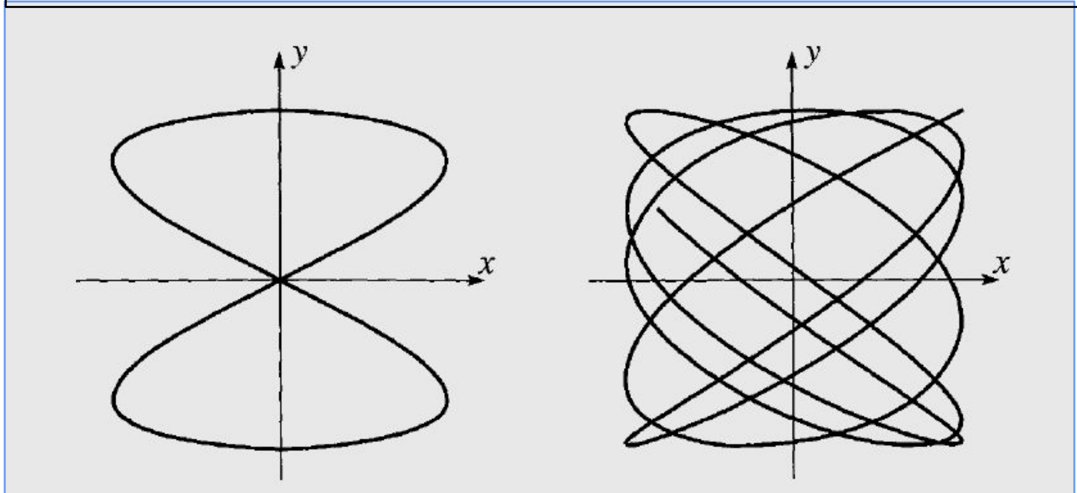
Figure 5.9. Two examples of anisotropic oscillations

$U = \frac{1}{2} k_x x^2 + \frac{1}{2} k_y y^2$

$x(t) = A \cos(\omega_x t)$

$y(t) = B \cos(\omega_y t - \delta)$

(a) 2x1 Lissajous fig. (b) quasi periodic



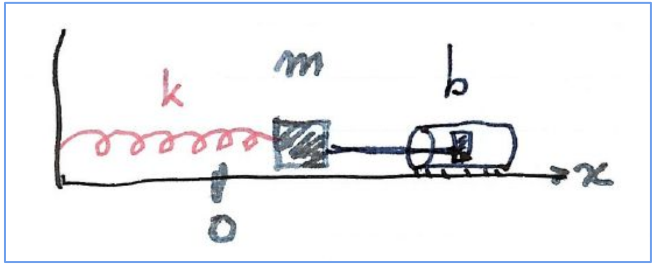
(a) $\omega_x = 2\,\omega_y$ (b) $\omega_x = \sqrt{2}\,\omega_y$

5.4. Damped oscillations

Sometimes in everyday life, oscillations may create problems.
For example, that's why a car has shock absorbers –– to damp out the oscillations when the wheels hit a bump in the road or a pothole.

Go back to 1-dimensional oscillations, but now add damping.

Generic picture



The equation of motion is

$$m\,a = -b\,v - k\,x$$

Note the assumption of "linear damping"; i.e., $F_{damping} \propto -v$;

Or we can write it this way,

$$m\,x'' + b\,x' + k\,x = 0\,.$$

It is useful to rescale the parameters to write the equation in a standard form ;

$$x'' + 2\beta\,x' + \omega_0^2\,x = 0$$

where

$$2\beta = b/m \quad \text{and} \quad \omega_0^2 = k/m\,.$$

Figure 5.10

THE EQUIVALENT LRC CIRCUIT

Recall from circuit theory

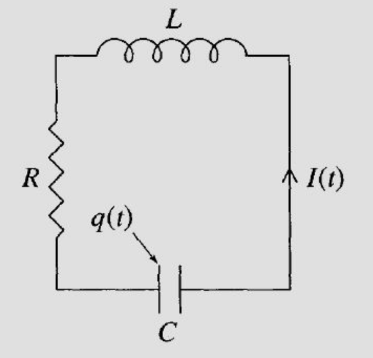


Figure 5.10 An LRC circuit.

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = 0.$$

so the math is the same as for the mechanical system.

$$\boxed{x'' + 2\,\beta\, x' + \omega_0^{\,2}\, x = 0}$$

This is an example of a *"homogeneous linear differential equation with constant coefficients"*.

There is a standard method to solve this kind of diff.eq. **(MTH 234)**

First, try $x(t) = e^{pt}$.

$\dot{x} = p\,e^{pt}$ and $\ddot{x} = p^2 e^{pt}$, so

$$p^2 + 2\beta p + \omega_0^2 = 0$$

$$p_\pm = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0}$$

■ We have two solutions, $\exp(p_+t)$ and $\exp(p_-t)$ where

$$p_\pm = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

■ The equation is second order, so the *general solution* depends on two constants. The equation is linear so we can write the general solution as

$$x(t) = c_+ e^{p_+ t} + c_- e^{p_- t}$$
$$= e^{-\beta t} \left\{ c_+ e^{\sqrt{\beta^2-\omega_0^2}\,t} + c_- e^{-\sqrt{\beta^2-\omega_0^2}\,t} \right\}$$

■ The 2 constants, c_+ and c_- , must be determined from the initial conditions or some other information.

■ <u>Overdamped oscillator ;</u> $\beta > \omega_0$

This is the case of *strong* damping.

In this case p_+ and p_- are *real*.

$$x(0) = c_+ + c_- \quad \text{and} \quad v(0) = p_+ c_+ + p_- c_-$$

$$c_\pm = \left[p_\mp x(0) - v(0) \right] / \left(p_\mp - p_\pm \right)$$

■ <u>Underdamped oscillator ;</u> $\beta < \omega_0$

This is the case of *weak* damping.

In this case p_1 and p_2 are complex numbers.

Recall $e^{\pm i\theta} = \cos\theta \pm i \sin\theta$ **(Euler)**

$$x(t) = e^{-\beta t}\left[A\cos\omega_1 t + B \sin\omega_1 t \right]$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$.

$$x(0) = A \quad \text{and} \quad \dot{x}(0) = -\beta A + \omega_1 B$$

■ <u>The critically damped oscillator</u>

$\beta = \omega_0$

In this case p_+ and p_- are *equal*,
$p_+ = p_- = \omega_0$; so exp(pt) is only one solution. To get the general solution we need another solution.

<u>Exercise</u>: Show that ***x(t)* = *t exp(pt)*** is also a solution for the critically damped oscillator ($\beta = \omega_0$).

$$x(t) = e^{-\beta t}\left[A + B t \right]$$

$$x(0) = A \quad \text{and} \quad \dot{x}(0) = -\beta A + B$$

<u>Example</u>. Consider these initial conditions: *x(0)* = *1* and *v(0)* = *0*.

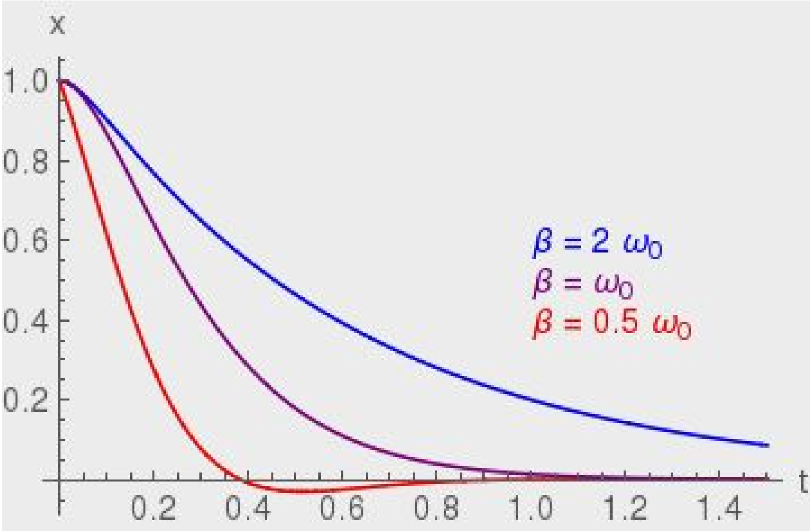


Figure 5.11
Underdamped oscillator

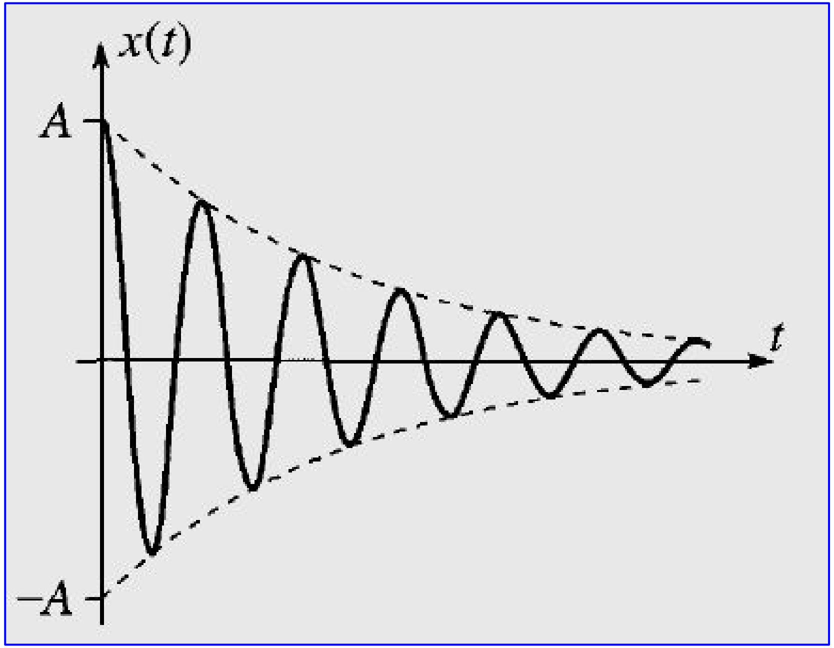


Figure 5.12
Overdamped oscillator

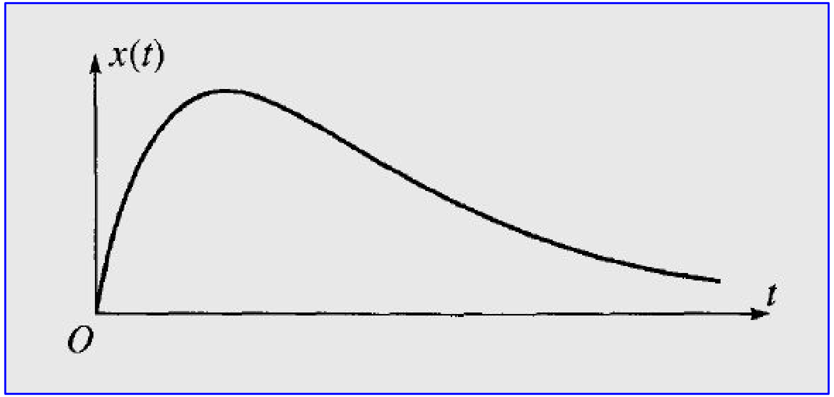


Fig. 5.12 corresponds to these initial conditions: $x(0) = 0$ and $v(0) > 0$; i.e., the mass is kicked in the +x direction , reaches a maximum displacement , and returns to equilibrium monotonically.

Critical damping ($\beta = \omega_0$)

This special case has the most rapid return to equilibrium …

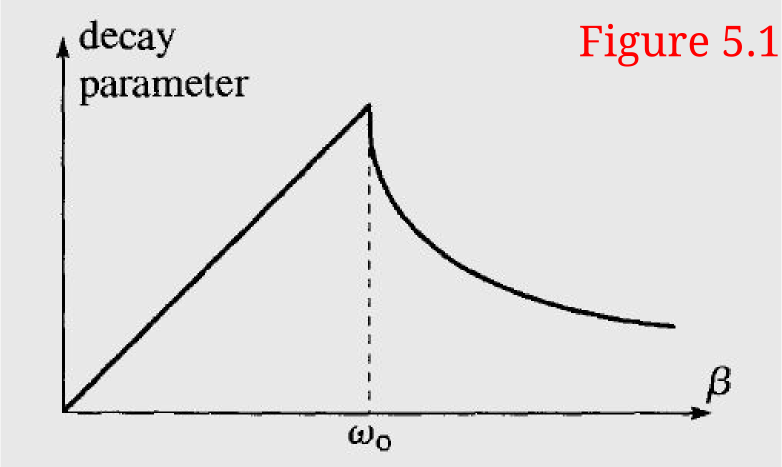
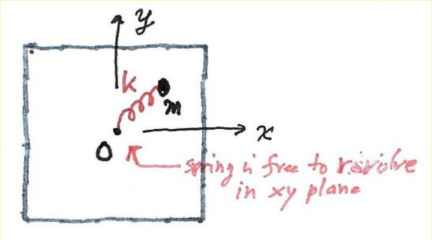


Figure 5.13

The "decay parameter" p versus β. The decay parameter is largest, so the motion dies out most quickly, for critical damping $\beta = \omega_0$.

damping	β	decay parameter
none	$\beta = 0$	0
under	$\beta < \omega_0$	β
critical	$\beta = \omega_0$	β
over	$\beta > \omega_0$	$\beta - \sqrt{\beta^2 - \omega_0^2}$

Test yourself:



Mass m moves in the xy-plane, attached to a spring as shown. According to a footnote in Taylor, the force on m is not $-k\boldsymbol{r}$.

What ***is*** the force?

Homework Assignment #9
due in class Friday November 4
[41] Problem 4.41 and Problem 4.43
[42] Problem 5.3 *
[43] Problem 5.5 *
[44] Problem 5.9 *
[45] Problem 5.12 **
[46] Problem 5.18 ***

Use the cover sheet.