1.1

Linear combination: vector that is the sum of scalar multiples of other vectors is said to be a <u>linear combination</u> of those vectors

$\vec{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad \vec{v} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$2\vec{u} - \vec{v} = \vec{w}$

$\vec{w}$ is a linear combination of $\vec{u}$ & $\vec{v}$

$2\vec{u} + -\vec{v} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \vec{w}$

<u>Def:</u> A vector $\vec{v}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ if there are scalars $c_1, c_2, \ldots, c_k$ such that

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$$

<u>Ex)</u> $\vec{w} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad \vec{u} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ Then is $\vec{p} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ a

linear combination of $\vec{u}$ + $\vec{w}$??

$$c\vec{w} + d\vec{u} = \vec{p}$$

$\vec{p}$ is <u>NOT</u> a linear combination of $\vec{w}$ + $\vec{u}$

Ex) <u>1.1C</u> Find two equations for c + d so that the linear combination $c\vec{v} + d\vec{w} = \vec{b}$. Solve for c + d.

$\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\quad \vec{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$c\begin{bmatrix} 2 \\ -1 \end{bmatrix} + d\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2c \\ -c \end{bmatrix} + \begin{bmatrix} -d \\ 2d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2c - d \\ -c + 2d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$2(2c - d = 1)$$
$$-c + 2d = 0$$

$$4c - \cancel{2d} = 2$$
$$+ \quad -c + \cancel{2}d = 0$$

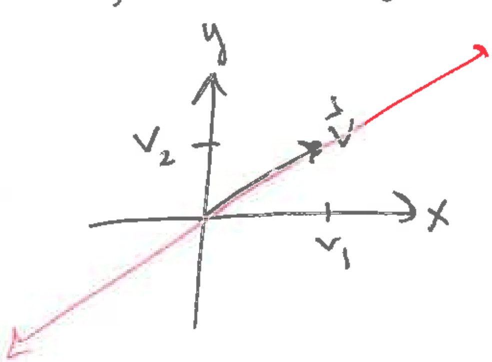
$$3c = 2 \qquad c = 2/3$$

$$-\frac{2}{3} + 2d = 0 \qquad d = 1/3$$

In $\mathbb{R}^2$ (normal x-y plane)

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $\qquad$ $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$c\vec{v}$, think of all possibilities for $c \in \mathbb{R}$



if we think of the picture of all possible $c \cdot \vec{v}$ we get a line

$c\vec{v} + d\vec{u}$, think about all possibilities for any $c, d \in \mathbb{R}$

we get a plane

in $\mathbb{R}^3$ $\quad \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$

the picture of all combinations of

- $c\vec{u}$ is a line through $(0,0,0)$
- $c\vec{u} + d\vec{v}$ fills a plane through $(0,0,0)$
- $c\vec{u} + d\vec{v} + e\vec{w}$ fill the whole space (all of $\mathbb{R}^3$)

1.2 Lengths and dot Products

def: dot product (inner product) of

$\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$

is $\boxed{\vec{v} \cdot \vec{w} = \underbrace{v_1 \cdot w_1 + v_2 w_2}_{\text{Scalar}}}$

Ex) $\vec{v} = (1,1)$ and $\vec{w} = (2,3)$

$$\vec{v} \cdot \vec{w} = 1 \cdot 2 + 1 \cdot 3 = 5$$

$$\boxed{\vec{w} \cdot \vec{v} = w_1 v_1 + w_2 v_2 = v_1 w_1 + v_2 w_2 = \vec{v} \cdot \vec{w}}$$

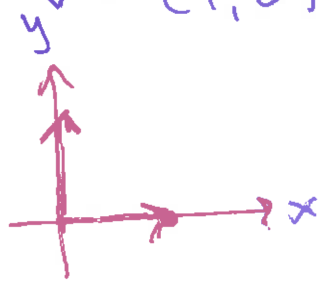
Ex) $\vec{v} = (4,2)$ and $\vec{w} = (-1,2)$

$$\vec{v} \cdot \vec{w} = 4 \cdot -1 + 2 \cdot 2 = -4 + 4 = 0$$

Note: if $\vec{v} \cdot \vec{w}$ is 0, then $\vec{v} \perp \vec{w}$

$\vec{v}$ is perpindicular to $\vec{w}$

Ex) $\vec{v} = (1,0)$ $\quad \vec{w} = (0,1)$



$\vec{v} \cdot \vec{w} = 1 \cdot 0 + 0 \cdot 1 = 0$

For vectors in $\mathbb{R}^n$

$$\vec{v} \cdot \vec{w} = \sum_{i=1}^{n} v_i w_i$$

What is $\vec{v} \cdot \vec{v}$??

$\vec{v} = (1,2,3)$

$\vec{v} \cdot \vec{v} = 1^2 + 2^2 + 3^2 = 14$

$\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$ ← dot product is square of length

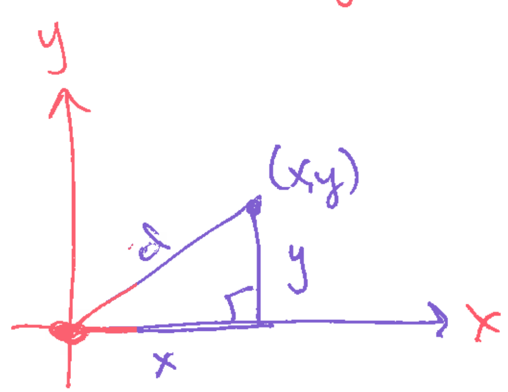
$\|\vec{v}\|$ → notation for the magnitude or length of $\vec{v}$

def: the length $\|\vec{v}\|$ of a vector $\vec{v}$ is the square root of $\vec{v}\cdot\vec{v}$

in $\mathbb{R}^2$ $\vec{v}\cdot\vec{v} = \sqrt{v_1^2 + v_2^2}$

NOTE: distance formula between a point $(0,0)$ & (x,y)

$$d = \sqrt{(x-0)^2 + (y-0)^2}$$



$x^2 + y^2 = d^2$

$d = \sqrt{x^2 + y^2}$

if $\vec{v} \in \mathbb{R}^n$, then $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$

def: a unit vector is a vector whose length $\|\vec{u}\| = 1$

$$\left.\begin{array}{l} \hat{i} = \begin{pmatrix}1\\0\\0\end{pmatrix} \\ \hat{j} = \begin{pmatrix}0\\1\\0\end{pmatrix} \\ \hat{k} = \begin{pmatrix}0\\0\\1\end{pmatrix} \end{array}\right\}$$ some unit vectors

$\vec{u} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ ← unit vector in $\mathbb{R}^4$

How do you check?

show $\|\vec{u}\| = 1$

$\vec{u} \cdot \vec{u} = 1$

Ex) Find a unit vector $\vec{u}$ in the direction of $\vec{v} = (1, 2)$

$$\|\vec{v}\| = \sqrt{1^2 + 2^2} = \sqrt{5} \neq 1$$

$\boxed{\dfrac{\vec{v}}{\|\vec{v}\|}}$ ⇒ should be a unit vector

↑ scalar

$$\vec{u} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$$ ← Should be a unit vector

$$\|\vec{u}\| = \sqrt{\left(\frac{1}{\sqrt{5}}\right)^2 + \left(\frac{2}{\sqrt{5}}\right)^2} = \sqrt{\frac{1}{5} + \frac{4}{5}} = \sqrt{\frac{5}{5}} = 1$$