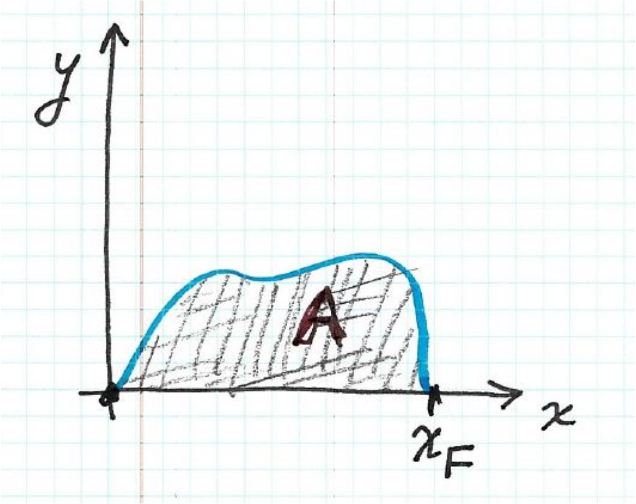
Chapter 6.

Two more examples of the Euler-Lagrange equation



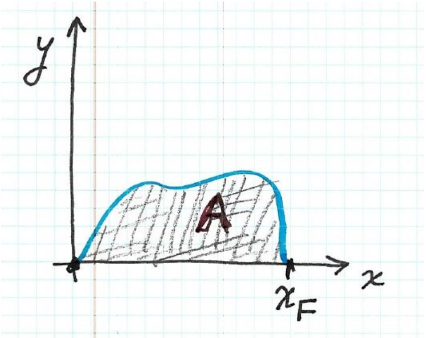
Taylor Problem 6.22 ***

Consider a flexible string with fixed length = ℓ.

One end of the string is pinned at the origin (0,0) in the xy-plane. The other end can be pinned at any point on the x axis. Then the string forms a curve in the xy-plane.

Determine the curve for which the area A is maximum.

(You can probably guess the answer, but can you *prove* it?)



$$A = \int_0^{x_F} y\,dx$$

But this does not have the right form, because the endpoints are not fixed.

Let s = arclength and describe the curve by *y(s)*.

Now the endpoints are fixed , because *y(0) = 0* and *y(l) = 0*.

$$(ds)^2 = (dx)^2 + (dy)^2$$

$$dx = \sqrt{(ds)^2 - (dy)^2} = \sqrt{1 - \left(\frac{dy}{ds}\right)^2}\, ds$$

$$A = \int_0^{l} y(s) \sqrt{1-(y')^2}\, ds \quad \leftarrow y' = \frac{dy}{ds}$$

$$f(y, y', s) = y\sqrt{1-(y')^2}$$

$$\frac{\partial f}{\partial y} = \sqrt{1-(y')^2} \quad \text{and} \quad \frac{\partial f}{\partial y'} = \frac{-yy'}{\sqrt{1-(y')^2}}$$

The Euler Lagrange equation is

$$\sqrt{1-(y')^2} = \frac{d}{ds}\left[\frac{-yy'}{\sqrt{1-(y')^2}}\right]$$

We already know the first integral, because $\frac{\partial f}{\partial s} = 0$.

$$f - y'\frac{\partial f}{\partial y'} = \text{constant} = C$$

$$\frac{d}{ds}\left[f - y'\frac{\partial f}{\partial y'}\right] = \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' - y''\frac{\partial f}{\partial y'} - y'\frac{d}{ds}\left(\frac{\partial f}{\partial y'}\right)$$

$$= y'\left[\frac{\partial f}{\partial y} - \frac{d}{ds}\left(\frac{\partial f}{\partial y'}\right)\right] = 0 \quad \checkmark$$

$$y\sqrt{1-(y')^2} - \frac{-y(y')^2}{\sqrt{1-(y')^2}} = C$$

$$= \frac{y}{\sqrt{1-(y')^2}}\left[1-(y')^2+(y')^2\right] = \frac{y}{\sqrt{1-(y')^2}}$$

To solve : $y = C\sqrt{1-\left(\frac{dy}{ds}\right)^2}$

We can solve it by a change of variables.

Let $y = R\sin\theta$ $\quad$ R = some constant

θ = the new variable

$$\frac{dy}{ds} = R\cos\theta\,\frac{d\theta}{ds}$$

$$R\sin\theta = C\sqrt{1-R^2\cos^2\theta\left(\frac{d\theta}{ds}\right)^2}$$

The solution is $R^2\left(\frac{d\theta}{ds}\right)^2 = 1$ $\quad \longleftarrow \theta = s/R$

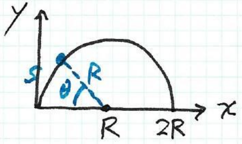
which gives $R\sin\theta = C\sqrt{1-\cos^2\theta}$

$= C\sin\theta$, so $R = C$.

Result $\quad y(s) = R\sin\left(s/R\right)$ and $R = C$.

(So far, R is an unknown constant.)

The result is the equation for a half circle, in terms of arclength s



$$x = R - R\cos\theta$$
$$\left.\begin{array}{l} y = R\sin\theta \\ s(\theta) = R\theta \end{array}\right\} \quad y = R\sin\left(\frac{s}{R}\right) ✔$$

But now, what is R? $\quad \pi R = \ell$.

Taylor Problem 6.23 ***

A plane will fly from O to P; see the figure. Its air speed is v_0 .

Coordinates are x = east; y = north.

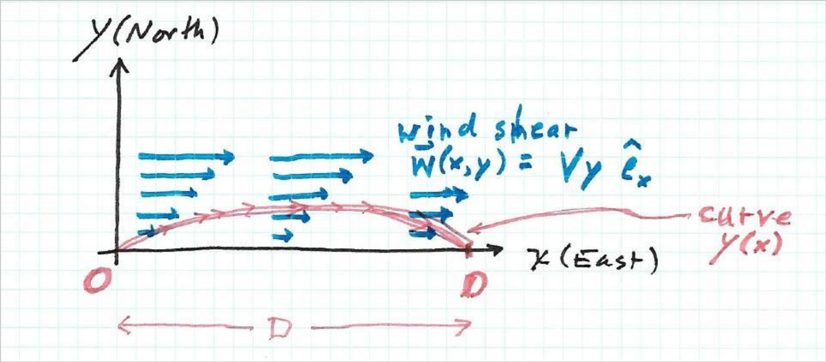
The wind velocity is

$$\boldsymbol{w}(x, y) = V\, y\, \boldsymbol{e}_x \,. \qquad \textit{("wind shear")}$$

The pilot wants to follow the path that minimizes the time to fly from O to P.

The plane will aim slightly north and then slightly south, at angle φ north of east.

Determine the path y(x) that will minimize the flight time.



$$time = \int_\Gamma \frac{ds}{v} \quad \text{where} \quad ds = \sqrt{(dx)^2 + (dy)^2}$$

$$\vec{v} = v_0 \cos\phi\, \hat{e}_x + v_0 \sin\phi\, \hat{e}_y + \vec{w}$$

$$= (v_0 \cos\phi + Vy)\, \hat{e}_x + v_0 \sin\phi\, \hat{e}_y$$

$$v = \sqrt{(v_0 \cos\phi + Vy)^2 + (v_0 \sin\phi)^2}$$

$$= \sqrt{v_0^2 + 2 v_0 V y \cos\phi + V^2 y^2}$$

Approximations: $V \ll \frac{v_0}{D}$; $|y'| \ll 1$; $\phi \ll 1$.

$$v \approx \sqrt{v_0^2 + 2v_0 V y} \approx v_0 + Vy$$

$$ds = \sqrt{1 + (y')^2}\, dx \approx \left(1 + \tfrac{1}{2}(y')^2\right) dx$$

$$\text{time} \approx \int_0^D \frac{1 + (y')^2/2}{v_0 + Vy}\, dx$$

Minimize the time $\Rightarrow$ Euler Lagrange eq.

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \text{ where } f = \frac{1 + (y')^2/2}{v_0 + Vy}$$

Because $\frac{\partial f}{\partial x} = 0$ we know the first integral

$$f - y' \frac{\partial f}{\partial y'} = \text{constant} = C$$

$$\hookrightarrow \frac{1 + (y')^2/2}{v_0 + Vy} - \frac{(y')^2}{v_0 + Vy} = \frac{1 - (y')^2/2}{v_0 + Vy}$$

To solve:

$$C\,(v_0 + Vy) = 1 - \frac{1}{2}\left(\frac{dy}{dx}\right)^2$$

Solution by guesswork

Try $y(x) = \lambda x (D - x)$

Note
$y(0) = 0$
$y(D) = 0$
$y' = \lambda D - 2\lambda x$

so

$$C\left(v_0 + V\lambda x (D - x)\right) = 1 - \tfrac{1}{2}(\lambda D - 2\lambda x)^2$$

$$\left.\begin{aligned} &C v_0 \\ &+ C V \lambda D\, x \\ &- C V \lambda\, x^2 \end{aligned}\right\} = \left\{\begin{aligned} &1 - \tfrac{1}{2}\lambda^2 D^2 \\ &+ 2\lambda^2 D\, x \\ &- 2\lambda^2 x^2 \end{aligned}\right.$$

Solution requires $CV\lambda = 2\lambda^2$ and $C v_0 = 1 - \frac{1}{2}\lambda^2 D^2$ } 2 equations for 2 unknown (C and λ)

Do the rest with Mathematica

Taylor: Assume $D = 2000$ miles

$V_0 = 500$ mi/hr

$V = 0.5$ mph/mi

Calculate y_{max} (how far north does it go?) 366 miles

and $\delta t =$ 4 hours – t (how much time is saved?) 27 minutes

```
(* Parameters *)
Di = 2000 (* miles *);
v0 = 500 (* mi  / hr *);
V = 0.5 (* mph / mi *);
k = V / v0;
Q = Solve[
     2*λ/k == 1-1/2*λ^2*Di^2, λ];
Λ = λ /. Q[[2]];
```

```
(* Trajectory *)
y[x_] := Λ * x * (Di - x)
Plot[y[x], {x, 0, 200},
     PlotRange → {{0, 2000}, {0, 400}},
     AxesLabel → "East [mi]", "North [mi]"]
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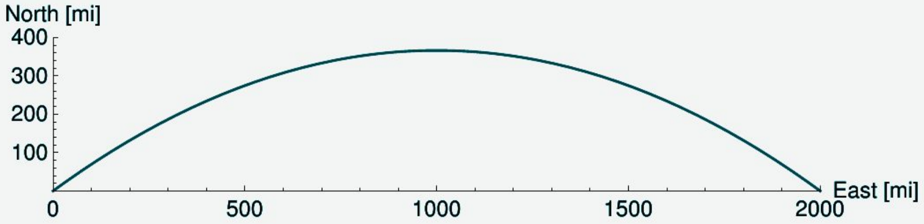
Out[83]=




```
Print ["How far north?  ", y[1000], "  miles"]
time = (1 / v0) * NIntegrate[(1 + 0.5 * y'[x]^2) / (1 + k * y[x]), {x, 0, Di}];
δt = 4 - time;
Print["How much time saved?  ", δt, " hours = ", δt * 60, "  minutes"]
How far north?  366.025  miles
How much time saved?  0.444227 hours = 26.6536  minutes
```

Homework Assignment #11
due in class Friday, November 18
[50] back of the sheet
[51] 6.7
[52] 6.8
[53] 6.10 and 6.20
[54] 6.1 and 6.16
[55] 6.19 *[56] 6.25*
Use the cover sheet.