4.8. Central forces

The most interesting problems in mechanics are about central forces.

Definition of a central force:

(1) The *direction* of the force ***F(r)*** is parallel or antiparallel to ***r*** *;*

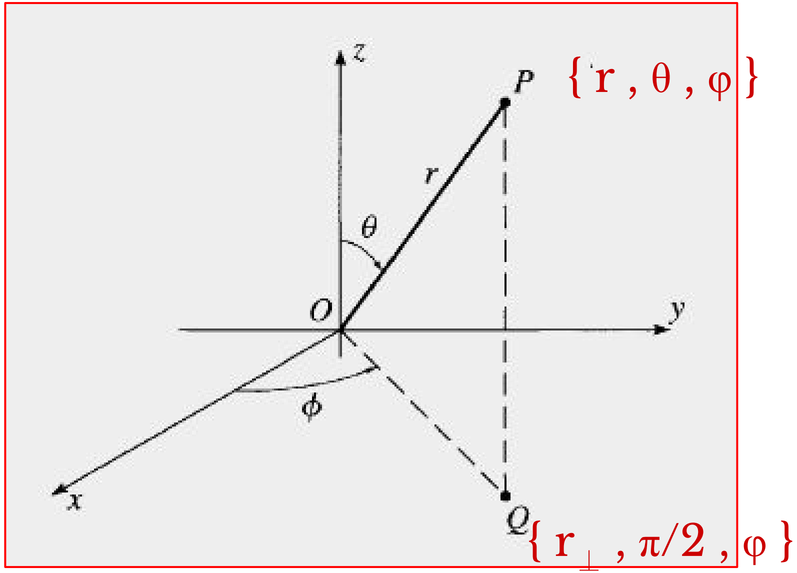
in other words, at any position of the object on which ***F*** is acting, the direction of ***F*** points *toward* or *away from* the origin. ***{attraction to O or repulsion}***

(2) The *magnitude* of the force |***F(r)***| depends only of the distance |***r***| from the origin. ***{spherically symmetric}***

The simplest way to analyze the motion of the object is to use *spherical polar coordinates*.

Spherical Polar Coordinates {r,θ,φ}

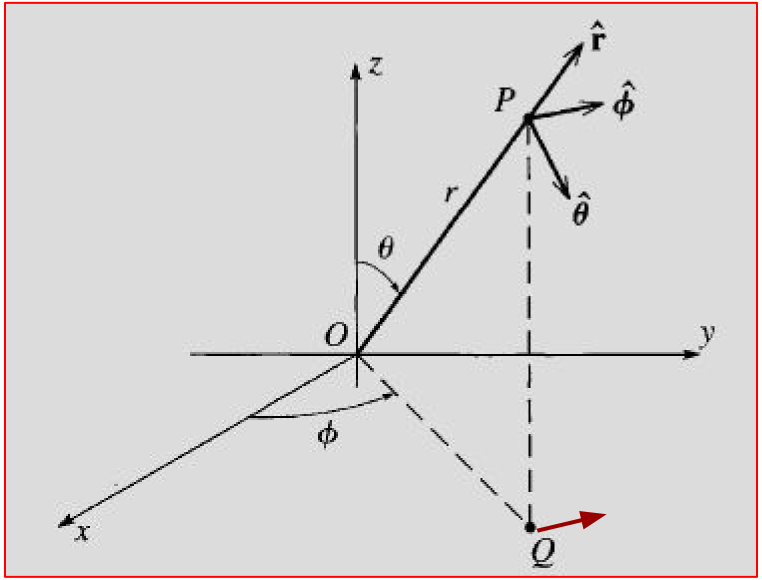
Figure 4.16



<u>Spherical Polar Coordinates</u>

The *direction vectors* for spherical polar coordinates

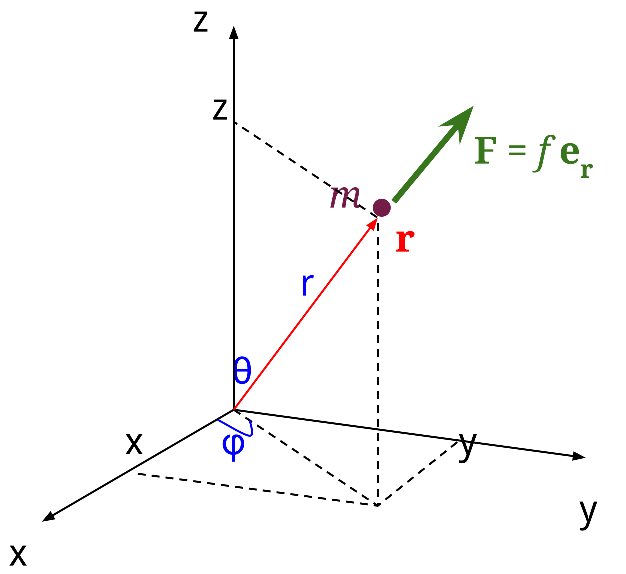
Figure 4.17



A central and spherically symmetric force, using spherical polar coordinates

$$\vec{F}(\vec{r}) = f(r)\,\hat{e}_r$$

$$m\ddot{\vec{r}} = f(r)\,\hat{e}_r \qquad \text{where } \vec{r} = r\,\hat{e}_r$$



"The origin is the center of force."

Conservative central forces

$\boldsymbol{F(r)} = -\nabla U(r)$

Or, handwritten,

$$\vec{F}(\vec{r}) = -\nabla U(r)$$

In words, the potential energy function U (which is a scalar) depends only on the distance from the center of force, *r*.

$U(x,y,z) = U(r)$ where $r = \sqrt{x^2+y^2+z^2}$

U(r) is called a *spherically symmetric potential.*

You should know the gradient operator in spherical polar coordinates,

$$\nabla f = \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

(*study the formulas inside the back cover of the book*)

So, for a spherically symmetric potential, U(r)

$$\nabla U = \hat{r}\frac{dU}{dr}$$

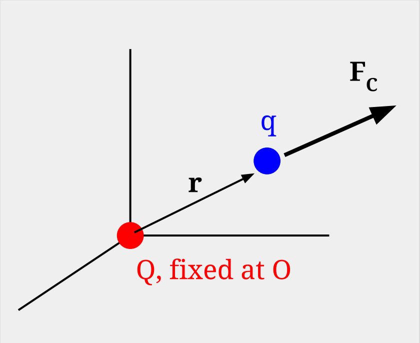
The Coulomb force and the corresponding potential energy

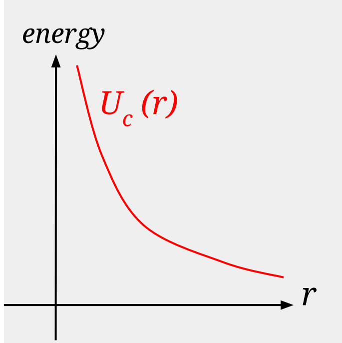
$$\vec{F}_c(\vec{r}) = \frac{kQq}{r^2}\,\hat{e}_r$$

$$k = 8.99 \times 10^9 \ \mathrm{Nm^2/C^2}$$
$$= 8.99 \times 10^9 \ \mathrm{Vm/C}$$

$$\vec{F}_c = -\nabla U_c = -\hat{e}_r \frac{dU_c}{dr}$$

$$\therefore \quad U_c(r) = \frac{kQq}{r}$$





Example. Hydrogen atom.

$$\oplus \xrightarrow{\ r\ } \ominus$$

p, Q   e, q

$$Q = e$$
$$q = -e$$

$$U_c = -8.99\times10^9 \ \frac{\mathrm{Vm}}{\mathrm{C}}\ e\ 1.6\times10^{-19}\,\mathrm{C}\,/\,r$$

$$U_c = \frac{-1.438}{r}\ \mathrm{eV{\cdot}nm}$$

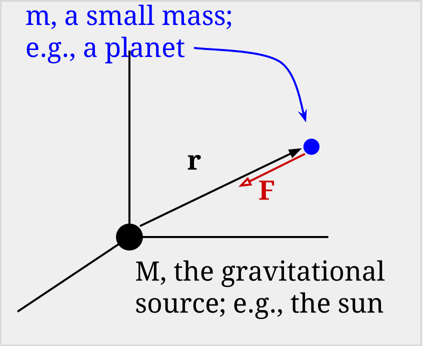
Newton's theory of universal gravitation

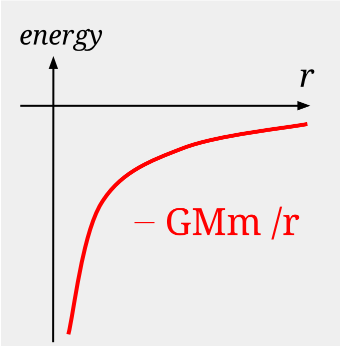
Imagine a large mass M (like the sun; or the Earth) and a smaller mass m (like a planet; or a satellite). The force on mass m is

$$\mathbf{F} = -\frac{GMm}{r^2}\,\mathbf{e}_r$$

More complete …
Both masses are attracted toward the Center of Mass (CoM) point –– Newton's third law.
Both masses revolve around the CoM, and the CoM is fixed.

Chapter 8.





The potential energy is

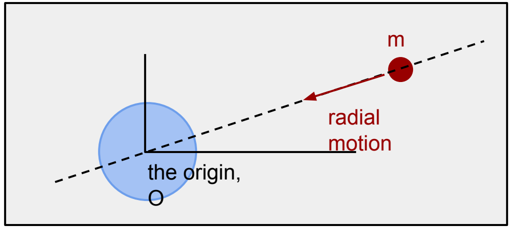
$$U(r) = -\,GMm\,/r$$

where $G = 6.67 \times 10^{-11}\ m^3\ s^{-2}\ kg^{-1}$.

We have a spherically symmetric potential if the bodies are themselves spherically symmetric.

The Earth is oblate, so the potential energy is not truly spherically symmetric.

Radial motion in a spherically symmetric potential



This is an example of "1-dimensional motion" in three dimensions.

If the angular momentum is 0 then the mass m just moves on a radial line through the origin.

Example

An asteroid falls radially toward the Earth, with zero angular momentum relative the Earth.

Take these parameters:

When the asteroid is at distance r_0 from the Earth (r = distance from the center of the Earth) the velocity is 0.
Pick a number : r_0 = earth-moon distance = 384,000 km.

Never mind how this unusual initial condition might be created; just take it as given.

The other parameter:
suppose R_a = 1 km ; then the mass is

$$m = 4/3\ \pi\ R_a^{\ 3}\ .\ (5\times10^3\ \mathrm{kg/m^3}) = 2\times10^{13}\ \mathrm{kg}$$

First question …

Calculate the *kinetic energy* of the asteroid when it hits the Earth.

<u>Solution</u>

We only need an algebraic calculation using conservation of energy …

$$E = \tfrac{1}{2}\, m\, v^2 - GMm/r$$

$$T_{hits} - GMm/R_{\oplus} = -GMm/r_0$$

$$T_{hits} = GMm\,(1/R_{\oplus} - 1/r_0)$$

$GM/R^2 = g$ and $1/r_0 \approx 0$, so

$$T_{hits} = m\,g\,R_{\oplus} = 2E13 \times 9.8 \times 6.4E6 \text{ J}$$

$$T_{hits} = 1.2 \times 10^{21} \text{ J} = 1.2 \times 10^{6} \text{ petajoules}$$

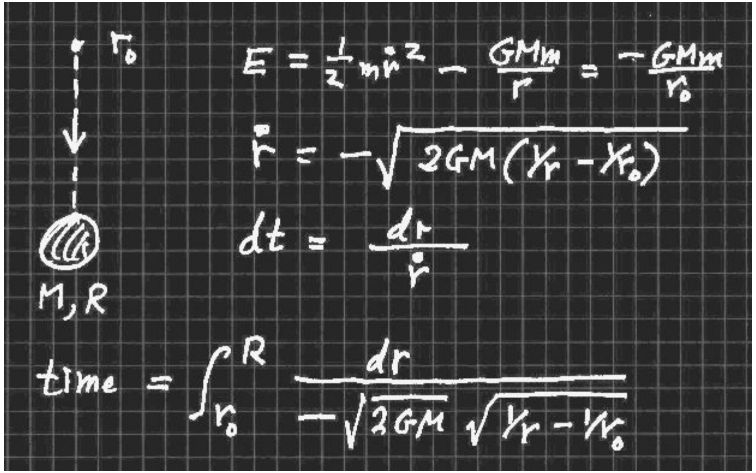
Compare largest H bomb = "Tsar Bomba" = 50 megaton TNT = 210 PJ

Second question ...

Calculate the *time it will take* for the asteroid to fall to the surface of the Earth, from the initial distance r_0 .

<u>Solution</u>

This is a time calculation –– we need to solve a differential equation.



$$\text{time} = \int_{r_0}^{R} \frac{dr}{-\sqrt{2GM}\sqrt{1/r - 1/r_0}}$$

$$= \frac{1}{\sqrt{2GM}} \int_R^{r_0} \sqrt{\frac{r r_0}{r_0 - r}}\, dr$$

$$r = r_0 u$$

$$= \sqrt{\frac{r_0^3}{2GM}} \underbrace{\int_{R/r_0}^{1} \sqrt{\frac{u}{1-u}}\, du}_{\approx \frac{\pi}{2} \text{ because } \frac{R}{r_0} \approx 0}$$

$$\text{time} = \frac{\pi}{2}\sqrt{\frac{r_0^3}{2GM}} = 4.17 \times 10^5 \text{ sec.}$$

$$= 4.83 \text{ days}$$

$$\int \sqrt{\frac{u}{1-u}}\, du = \arcsin(\sqrt{u}) - \sqrt{u(1-u)} + C$$

Compare:
Moon to Earth travel time (from TEI SPS ignition to splashdown):
Apollo 11: 2 days 11 hours 55 minutes
Apollo 12: 3 days 9 minutes
Apollo 14: 2 days 19 hours 26 minutes
Apollo 15: 2 days 23 hours 23 minutes
Apollo 16: 2 days 17 hours 30 minutes
Apollo 17: 2 days 19 hours 49 minutes

Homework Assignment #8
due in class Friday, October 28
[37] Problem 4.26 *
[38] Problem 4.28 ** and Problem 4.29 ** [Computer]
[39] Problem 4.33 ** [Computer]
[40] Problem 4.34 **
[40x] Problem 4.37 *** [Computer]
[40xx] Problem 4.38 *** [Computer]
Use the cover page.
This is a long assignment, so start working on it now.