Homework Assignment #8 Name_________grading___________

due in class Friday, October 28

Cover sheet : Staple this page in front of your solutions.

Write the *answers* (without calculations) on this page;
write the detailed *solutions* on your own paper.

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[37] Problem 4.26.* *Answer: What is dE/dt?*

$dE/dt = m\, y\, dg/dt$ ← 1 point

$\boxed{\text{1 point}}$

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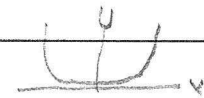
[38] Problem 4.28 and 4.29.***[computer]



For #4.29, hand in the computer program and any plots. Check plot. ← 1 pt.

Answer: What is the period for #4.29 part (d)?

3.708 time units ← 2 pt

$\boxed{\text{3 points}}$

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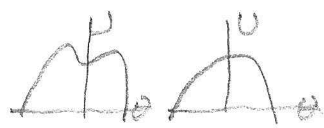
[39] Problem 4.33.**[computer]



Hand in the computer program and any plots.

Answer: Did you hand in the computer results?

Check the plots of U(θ) for b = 0.9 r and b = 1.1 r. ← 2 points

$\boxed{\text{2 points}}$

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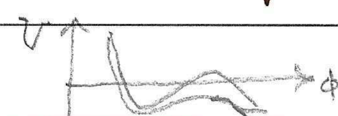
[40] Problem 4.34.**

Answer: What is the period if the length is 1 m?

2.007 seconds ← 2 points

$\boxed{\text{1 points}}$

[40x] Problem 4.37.***[computer]



Hand in the computer program and any plots.

Answer: Did you hand in the computer results? *Check the plots of U vs* . 2 pt.

Answer: What is the critical ratio m/M ? ~~0.7426~~ ← 1 pt

0.7246

$\boxed{\text{3 points}}$

[40xx] Problem 4.38.***[computer]

Hand in the computer program and any plots.

Answer: Did you hand in the computer results? *Check the plot of* τ *vs* Φ. 2 pt

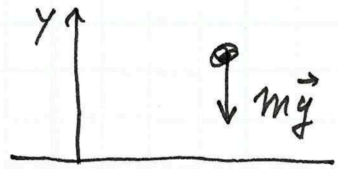
Answer: ***Explain*** *what becomes of* τ *as the amplitude* Φ *of oscillation approaches* π.

The period approaches infinity because $\Phi = \pi$ *is a point of (unstable) equilibrium.* ↑ 1 pt

$\boxed{\text{3 points}}$

14 total

[37] Problem 4.26



and $g = g(t)$

(A) Let $U = mgy$. Then $-\nabla U = -mg\,\hat{e}_y = \vec{F}$

(B) Let $E = \frac{1}{2}mv^2 + mgy$.

Then $$\frac{dE}{dt} = \frac{1}{2}m\,2v\,\frac{dv}{dt} + m\dot{g}\,y + mg\frac{dy}{dt}$$

$$= mv_y\,\frac{dv_y}{dt} + mg(+v_y) + m\dot{g}\,y$$

$$= v(\underbrace{ma_y + mg}_{=0 \text{ by Newton's second law}}) + m\dot{g}y$$

$ma_y = F_y = -mg$

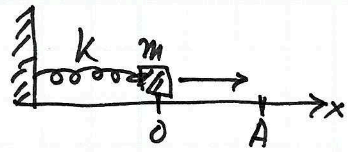
$$\frac{dE}{dt} = m\,\dot{g}\,y$$

So E is not constant if $\dot{g} \neq 0$.

[38] 1

[38] Problem 4.28

Mass on a spring. The mass is given a kick at $t=0$ and then moves out to maximum displacement $x_{max} = A$.



$$\boxed{\omega^2 = \frac{k}{m}}$$

(A) Conservation of energy

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 = \frac{1}{2} k A^2$$

Thus $\dot{x} = \pm \sqrt{\omega^2 (A^2 - x^2)}$

(B) As m moves from 0 to A, the time integral is

$$t = \int_0^x \frac{dx'}{\dot{x}'} = \int_0^x \frac{dx'}{\sqrt{\omega^2 (A^2 - x'^2)}}$$

$$= \frac{1}{\omega} \arcsin\left(\frac{x}{A}\right)$$

(C) Thus $x(t) = A \sin \omega t$ ← simple harmonic motion

with period $T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}}$

Problem 4.29 Mass in the potential $U(x) = Kx^4$.

(A) Plot U versus x. ← Mathematica output

(B) The mass is given a kick, and then oscillates between $x_{max} = +A$ and $x_{min} = -A$; by conservation of energy,

$$\frac{1}{2} m \dot{x}^2 + K x^4 = K A^4 \Rightarrow \dot{x} = \pm \sqrt{\frac{2K}{m} (A^4 - x^4)}$$

Time $t = \int_0^x \frac{dx'}{\dot{x}'} = \sqrt{\frac{m}{2K}} \int_0^x \frac{dx'}{\sqrt{A^4 - x'^4}}$

The period of oscillation is ~~τ~~ τ where $\tau/4$ = the time to go from 0 to A.

$$\tau = 4\int_0^A \sqrt{\frac{m}{2k}}\,\frac{dx'}{\sqrt{A^4-x'^4}} = \sqrt{\frac{8m}{k}}\int_0^A \frac{dx}{\sqrt{A^4-x^4}}$$

(C) Change the variable of integration to

$u = x/A \Rightarrow$

$$\tau = \sqrt{\frac{8m}{k}}\,\frac{1}{A}\int_0^1 \frac{du}{\sqrt{1-u^4}} = \frac{C}{A}$$ is proportional to $1/A$

(D) Calculate C using Mathematica. ← Mathematica output

$\int_0^1 \frac{du}{\sqrt{1-u^4}} = 1.311$; for $m=1, k=1, A=1$, $C = 3.708$.

[39] and [40x]

[39] Problem 4.33

(A) Equation (4.59) was derived in class.

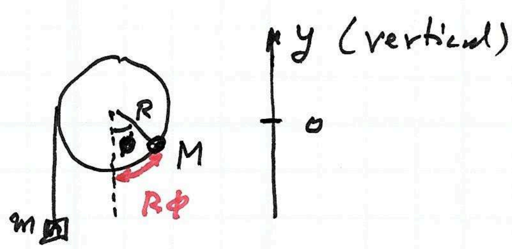
$$U(\theta) = mg\left\{(r+b)\cos(\theta) + r\theta\sin(\theta)\right\}$$

(B) Plot U for $b=0.9r$ and $1.1r$. ← Mathematica output

(C) Interpretation:

- For $b=0.9r$ the configuration $\theta=0$ is a stable equilibrium; also there are two unstable equilibrium points at $\theta=\pm 0.54$.
- For $b=1.1r$, the only equilibrium point is $\theta=0$, and it is unstable.

[40x] Problem 4.37



(A) The potential energy

$$U = Mg\,y_M + mg\,y_m \quad \text{where} \quad \begin{cases} y_M = -R\cos\phi \\ y_m = -R - R\phi \end{cases}$$

$$U(\phi) = -MgR\cos\phi - mgR - mgR\phi$$

The answer on page 756 has a different constant.

(B) Equilibrium positions occur where $\frac{dU}{d\phi} = 0$.

$$\frac{dU}{d\phi} = MgR\sin\phi - mgR = 0$$

$$\therefore \sin\phi = m/M$$

$\sin\phi = m/M$

— If $m > M$ then there is no equilibrium position.

— If $m = M$ ~~than~~ there is equilibrium at $\phi = \pi/2$

— If $m < M$ then there are two equilibrium points, at $\phi = \pi/2 \pm \alpha$ where $\cos\alpha = m/M$;

$\phi = \frac{\pi}{2} + \alpha$ is unstable and $\phi = \frac{\pi}{2} - \alpha$ is unstable.

(C) Plot $U(\phi)$ for $m = 0.7M$ and $m = 0.8M$.

Mathematica output

If the masses are released from rest at $\phi = 0$: then for $m = 0.7M$ they undergo oscillations; and for $m = 0.8M$ they will not oscillate, and m will fall until it hits the floor.

(D) The critical value of m/M is the value such that the equilibrium point at $\phi = \frac{\pi}{2} + \alpha$ occurs with $U(\phi) = -1$.

I.e. solve

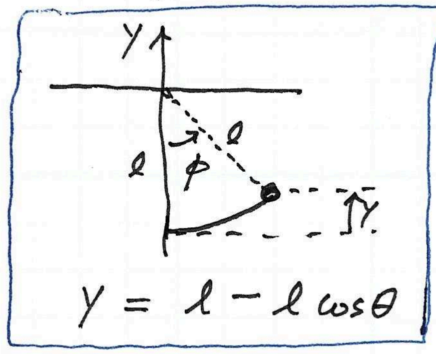
$$\underset{(\frac{\pi}{2}+\alpha)}{U(\phi)} = MgR\sin\alpha - mgR - mgR\left(\tfrac{\pi}{2}+\alpha\right)$$

$$= MgR\sin\alpha - mgR\left(1 + \tfrac{\pi}{2} + \alpha\right)$$

$$= -1 \quad \text{and } \cos\alpha = m/M.$$

Use Mathematica $\Rightarrow$ $\frac{m}{M} = 0.7246$ Mathematica output

[40] Problem 4.34 (The simple pendulum)



(A) The potential energy $U(\phi)$ is

$$U(\phi) = mgy = mgl(1-\cos\phi)$$

The energy is

$$E = \frac{1}{2}mv^2 + U = \frac{1}{2}ml^2(\dot\phi^2) + mgl(1-\cos\phi)$$

(B) The energy is constant, so $dE/dt = 0$;

$$ml^2\dot\phi\ddot\phi + mgl\sin\phi\,\dot\phi = 0$$

Thus $\ddot\phi = -\frac{g}{l}\sin\phi$

This is the same as $I\alpha = \Gamma$ (torque about the suspension pt.)
because $I = ml^2$, $\alpha = \ddot\phi$, and $\Gamma = -lmg\sin\phi$

(C) For small ϕ, approximate $\sin\phi \approx \phi$

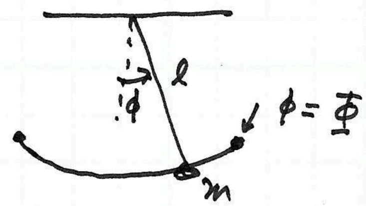
Then $\ddot\phi = -\frac{g}{l}\phi = -\omega^2\phi$ where $\omega = \sqrt{\frac{g}{l}}$

Small oscillations are harmonic, with
period $= \tau = \frac{2\pi}{\omega} = 2\pi\sqrt{l/g}$.

[40xx] **Problem 4.38** (Simple Pendulum)
↳ not limited to small angles

(A) Let Φ = amplitude of oscillation.

Calculate τ = period of oscillation.



By conservation of energy,

$$\frac{1}{2} m l^2 \dot{\phi}^2 + mgl(1-\cos\phi) = mgl(1-\cos\Phi)$$

$$\dot{\phi}^2 = \frac{2g}{l}(\cos\phi - \cos\Phi)$$

The time calculation

$$dt = \frac{d\phi}{\dot{\phi}} \Rightarrow t = \int_0^{\phi} \frac{d\phi'}{\pm\sqrt{\frac{2g}{l}(\cos\phi' - \cos\Phi)}}$$

Consider one quarter cycle, as ϕ goes from 0 to Φ.

$$\frac{\tau}{4} = \sqrt{\frac{l}{2g}} \int_0^{\Phi} \frac{d\phi'}{\sqrt{\cos\phi' - \cos\Phi}} = \sqrt{\frac{l}{4g}} \int_0^{\Phi} \frac{d\phi'}{\sqrt{\sin^2\Phi/2 - \sin^2\phi'/2}}$$

(note: $\cos\phi = \cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2} = 1 - 2\sin^2\phi/2$)

Also, $\sqrt{\frac{l}{4g}} = \frac{1}{2}\frac{\tau_0}{2\pi} = \frac{\tau_0}{4\pi}$ where τ_0 = the period of small oscillations $= 2\pi\sqrt{\frac{l}{g}}$

$$\tau = \frac{\tau_0}{\pi} \int_0^{\Phi} \frac{d\phi'}{\sqrt{\sin^2\Phi/2 - \sin^2\phi'/2}}$$

Change the variable of integration. Let $u = \frac{\sin\phi/2}{\sin\Phi/2}$

$$du = \frac{1}{2}\frac{\cos\phi/2}{Q} d\phi = \frac{1}{2Q}\sqrt{1-\sin^2\phi/2}\, d\phi = \frac{1}{2Q}\sqrt{1-Q^2u^2}\, d\phi \qquad \boxed{Q = \sin\Phi/2}$$

denominator $= \sqrt{Q^2 - Q^2u^2} = Q\sqrt{1-u^2}$

$$\therefore \tau = \frac{\tau_0}{\pi}\int_0^1 \frac{2Q\, du}{\sqrt{1-Q^2u^2}} \frac{1}{Q\sqrt{1-u^2}} = \frac{2\tau_0}{\pi} \underbrace{\int_0^1 \frac{du}{\sqrt{1-u^2}\sqrt{1-Q^2u^2}}}_{= K(Q^2)} \qquad (4.103)$$

[40xx]

(c) Use *Mathematica* to get $K(Q^2)$

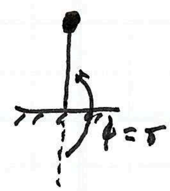
where $Q = \sin \Phi/2$.

Plot $\tau/\tau_0 = \frac{2}{\pi} K\left(\sin^2 \frac{\Phi}{2}\right)$ versus Φ. Mathematica output

The limit of τ/τ_0 as Φ approaches π is ∞,

because $\phi = \pi$ is an equilibrium point,

(The inverted pendulum) although unstable.

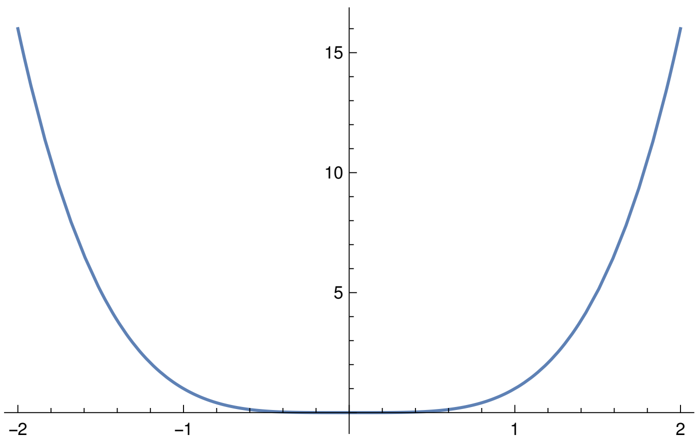


Problem 4.29

Mass in the potential U(x) = k x^4 .

(a)

```
Plot[x^4, {x, -2, 2}]
```



If the mass starts at x = 0 and is given a kick toward positive x, it will oscillate between x_{max} = +A and x_{min} = -A.

(d) Use *Mathematica* to calculate the integral

```
theintegral = NIntegrate[1 / Sqrt[1 - u^4], {u, 0, 1}]
SetPrecision[theintegral, 10]
theperiod = Sqrt[8 * m / k] * theintegral / A
theperiod /. {m → 1, k → 1, A → 1}
```

1.31103

1.311028777

$$\frac{3.70815 \sqrt{\frac{m}{k}}}{A}$$

3.70815

Problem 4.33

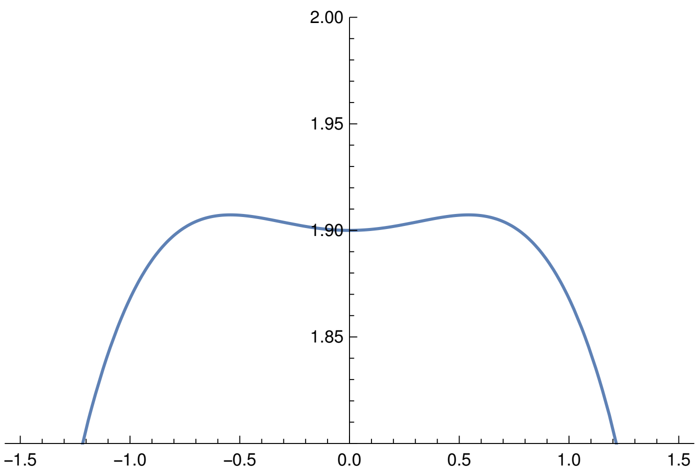
- (a) Eq 4.59 was derived in class

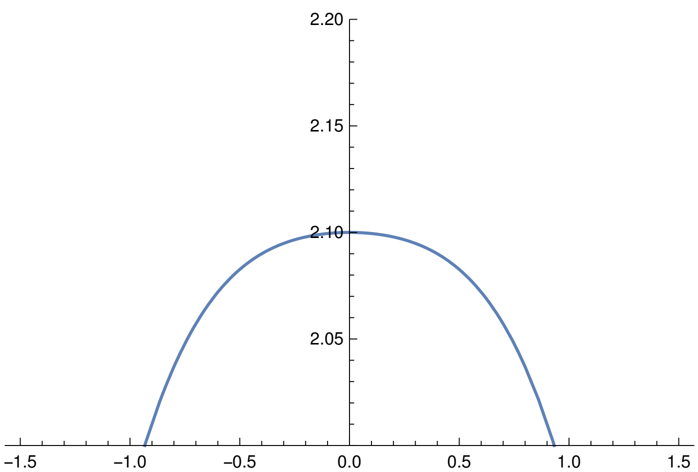
$$U(\theta) = mg\{(r+b)\,\mathrm{Cos}[\theta] + r\,\theta\,\mathrm{Sin}[\theta]\}$$

(b) Plot U for b = 0.9 r and 1.1 r

```
{r, m, g} = {1, 1, 1}
U[θ_] := m * g * ((r + b) * Cos[θ] + r * θ * Sin[θ])
p1 = Plot[
  U[θ] /. b → 0.9 r,
  {θ, -Pi / 2, Pi / 2}, PlotRange → {{-Pi / 2, Pi / 2}, {1.8, 2.0}}]
p2 = Plot[
  {U[θ] /. b → 1.1 r},
  {θ, -Pi / 2, Pi / 2}, PlotRange → {{-Pi / 2, Pi / 2}, {2.0, 2.2}}]
```

```
{1, 1, 1}
```





(c) Interpretations

For b = 0.9 r :

the configuration θ = 0 is a stable equilibrium;

also there are two unstable equilibrium points
at $\theta = +0.54$ and -0.54

For $b = 1.1\,r$:

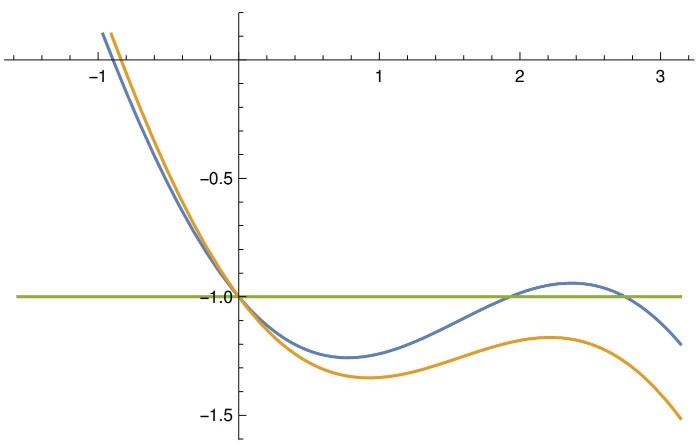
the only equlibrium point is $\theta = 0$, and it is unstable.

Problem 4.37

- (c) Plot $U(\phi)$ for m = 0.7 M and m = 0.8 M.
Assume the masses are released from rest at $\phi = 0$;
i.e., the energy is U(0).

```
Remove["Global`*"]
{M, g, R} = {1, 1, 1}
U[ϕ_] := -M*g*R*Cos[ϕ] - m*g*R*ϕ
Plot[
 {U[ϕ] /. m → 0.7 M, U[ϕ] /. m → 0.8 M, U[0]},
 {ϕ, -Pi/2, Pi}]
```

```
{1, 1, 1}
```



Interpretations:

The case m = 0.7 M (blue curve) will undergo oscillations
between $\phi = 0$ and $\phi = 1.8$.

The case m = 0.8 M (orange curve) will run away as m falls down ultil it hits the ground.

- (d) The critical value of m/M is the value such that the equilibrium point at $\phi = \pi/2+\alpha$
occurs with U[ϕ] = -1.0.

```
ϕeq = Pi/2 + ArcCos[m/M]
val[ratio_] := U[ϕeq] /. m/M → ratio
FindRoot[val[r] == -1.0, {r, 0.7}]
```

$\frac{\pi}{2} + \text{ArcCos}[m]$

{r → 0.724611}

Critical ratio is m/M = 0.7246 .

Problem 4.38 : the simple pendulum

The period is given by Equation 4.103:

$$\tau = \tau_0 \frac{2}{\pi} \int_0^1 \frac{du}{\sqrt{1-u^2}\sqrt{1-A^2 u^2}}$$

where $\tau_0 = 2\pi \sqrt{l/g}$ and A = Sin[Φ/2]

The integral is the complete elliptic integral of the first kind, $K(A^2)$.

(b) Plot τ / τ_0 for amplitudes $0 \leq \Phi \leq 3$.

```
τratio[Φ_] := 2 / Pi * EllipticK[(Sin[Φ / 2]) ^ 2]
Plot[τratio[Φ], {Φ, 0, 3},
 PlotRange → {{0, Pi}, {0.5, 3}}]
```