Ex) Let $A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$

Find an orthonormal basis for the column space of A.

Note: the columns of A are linearly independent

$\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad \vec{a}_2 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ -1 \end{bmatrix}$ $\quad \vec{a}_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}$

Keep $\vec{a}_1$ same direction, but normalize it (make it a unit vector)

$$\|\vec{a}_1\| = \sqrt{\vec{a}_1 \cdot \vec{a}_1} = \sqrt{1^2+1^2+1^2+1^2} = \sqrt{4} = 2$$

$$\vec{q}_1 = \frac{1}{2}\vec{a}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

now we look at $\vec{a}_2 = (-1, 4, 4, -1)^T$

$$\vec{A}_2 = \vec{a}_2 - \underbrace{\text{proj}_{\vec{q}_1} \vec{a}_2}_{\frac{\vec{q}_1^T \vec{a}_2}{\vec{q}_1^T \vec{q}_1} \vec{q}_1}$$

$\vec{q}_1^T \vec{q}_1 = 1$

$$\vec{A}_2 = \vec{a}_2 - \vec{q}_1 \cdot \vec{a}_2 \, \vec{q}_1$$

$$\underbrace{\vec{a}_2 - \langle \vec{q}_1, \vec{a}_2 \rangle \vec{q}_1}_{\text{part of } \vec{a}_2 \text{ that is perp. to } \vec{q}_1}$$

$$\vec{q}_1 \cdot \vec{a}_2 = \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)^T \cdot (-1, 4, 4, -1)^T$$

$$= 3$$

$$\vec{A}_2 = (-1, 4, 4 -1)^T - 3\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$\vec{A}_2 = (-5/2, 5/2, 5/2, -5/2)^T$$

$$\vec{q}_2 = \frac{\vec{A}_2}{\|A_2\|} = \frac{1}{5}(-5/2, 5/2, 5/2, -5/2)^T = (-1/2, \tfrac{1}{2}, \tfrac{1}{2}, -1/2)^T$$

Find $\vec{q}_3$

$$\vec{A}_3 = \vec{a}_3 - \text{proj}_{\vec{q}_1} \vec{a}_3 - \text{proj}_{\vec{q}_2} \vec{a}_3$$

$$= \vec{a}_3 - \underbrace{\vec{a}_3 \cdot q_1 \, q_1}_{\text{part of } \vec{a}_3 \text{ in direction of } \vec{q}_1} - \underbrace{\vec{a}_3 \cdot \vec{q}_2 \, \vec{q}_2}_{\text{part of } \vec{a}_3 \text{ in direction of } \vec{q}_2}$$

$\vec{q}_1 \cdot \vec{a}_3 = 2$ $\qquad$ $\vec{q}_2 \cdot \vec{a}_3 = -2$

$$\vec{A}_3 = (4,-2,2,0)^T - 2\left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right)^T - (-2)\left(-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}\right)^T$$

$$= (2,-2,2,-2)$$

$$\vec{q}_3 = \frac{\vec{A}_3}{\|\vec{A}_3\|} = \frac{1}{4}(2,-2,2,-2)^T$$

$$\vec{q}_3 = \left(\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}\right)^T$$

$$Q = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\begin{array}{cc} & \begin{array}{ccc} & \vec{q}_1\cdot\vec{a}_2 & \vec{q}_1\cdot\vec{q}_3 \end{array} \\ R = & \begin{bmatrix} \underset{11}{2} & \underset{12}{3} & \underset{13}{2} \\ 0 & \underset{22}{5} & \underset{23}{-2} \\ 0 & 0 & \underset{33}{4} \end{bmatrix} \begin{array}{l} \\ \vec{q}_2\cdot\vec{a}_3 \\ \vec{q}_3\cdot\vec{a}_3 \end{array} \end{array}$$

$$A = QR$$

In General

$$\begin{bmatrix} | & | & | \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \vec{q}_1^T\vec{a}_1 & \vec{q}_1^T\vec{a}_2 & \vec{q}_1^T\vec{a}_3 \\ \vec{q}_2^T\vec{a}_1 & \vec{q}_2^T\vec{a}_2 & \vec{q}_2^T\vec{a}_3 \\ \vec{q}_3^T\vec{a}_1 & \vec{q}_3^T\vec{a}_2 & \vec{q}_3^T\vec{a}_3 \end{bmatrix}$$

$$= \begin{bmatrix} | & | & | \\ \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \vec{q}_1^T\vec{a}_1 & \cdot & \cdot \\ & \vec{q}_2\vec{a}_2 & \cdot \\ & & \vec{q}_3\vec{a}_3 \end{bmatrix}$$

Helpful??

$A\hat{x} = \vec{b}$ impossible

$QR\hat{x} = \vec{b}$

Note: $Q^T = Q^{-1}$

$$\underbrace{Q^T Q}_{I} R\hat{x} = Q^T\vec{b}$$

$$R\hat{x} = Q^T\vec{b}$$

upper triangular

use back substitution to solve the system

$$\hat{x} = R^{-1} Q^T \vec{b}$$

Find the least Squares Solution to a problem using $QR = A$

$$\begin{bmatrix} 1 & -2 & -1 \\ 2 & 0 & 1 \\ 2 & -4 & 2 \\ 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$\vec{q}_{t_1} = \left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}, \frac{4}{5}\right)^T$$

$$\vec{q}_{t_2} = \left(-\frac{2}{5}, \frac{1}{5}, -\frac{4}{5}, \frac{2}{5}\right)^T$$

$$\vec{q}_{t_3} = \left(-\frac{4}{5}, \frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)^T$$

$\|\vec{A}\|$

$$QR = \begin{bmatrix} | & | & | \\ \vec{q}_{t_1} & \vec{q}_{t_2} & \vec{q}_{t_3} \\ | & | & | \end{bmatrix} \begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

$\|\vec{A}_2\|$

$\|\vec{A}_3\|$

$\vec{q}_{t_3} \cdot \vec{a}_3$