Name: KEY Section: ____________

Please work together to solve the problems.

1. Reduce these matrices to their ordinary echelon forms $U$:

(a)
$$A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$$

(b)
$$B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}$$

Which are the free variables and which are the pivots?

a) $A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} R_2 \to R_2 - R_1$

$\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} R_3 \to R_3 - R_2$

$U = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Pivots: $x_1, x_3$
(columns 1 + columns 3 have 1st nonzero entries in Rows 1 + 2)

Free variables: $x_2, x_4, x_5$

b) $B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix} R_3 \to R_3 - 2R_2$

$U = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

pivots: $x_1, x_2$

free: $x_3$

2. The matrix below is in reduced row echelon form. The reduced row echelon matrix $R$ has 1's as pivots and has zeros above the pivots as well as below.

$$R = \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

What are the free variables and what are the pivots? Find a special solution for each free variable (set the free variable to 1 and set the other free variables to 0).

Pivots: $x_1, x_2, x_4$

free: $x_3, x_5$

$$[R|0] = \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Special solution #1

$x_3 = 1, x_5 = 0$

$1x_4 + 4x_5 = 0 \Rightarrow x_4 = 0$

$1x_2 + 2x_3 + 3x_5 = 0$

$x_2 + 2 = 0 \Rightarrow x_2 = -2$

$1x_1 + 1x_3 - 1x_5 = 0$

$x_1 + 1 = 0 \Rightarrow x_1 = -1$

$$\begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \vec{s}_1$$

special solution #2

$x_5 = 1, x_3 = 0$

$x_4 + 4x_5 = 0$

$x_4 + 4 = 0 \Rightarrow x_4 = -4$

$1x_2 + 2x_3 + 3x_5 = 0$

$x_2 + 3 = 0 \Rightarrow x_2 = -3$

$1x_1 + x_3 - 1x_5 = 0$

$x_1 - 1 = 0$

$x_1 = 1$

$$\vec{s}_2 = \begin{bmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

so $N(R)$ contains all linear combinations of $\vec{s}_1 + \vec{s}_2$

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3. Put as many 1's as possible in a 4 by 8 *reduced* echelon matrix $R$ so that the free columns are 2, 4, 5, 6.

$$\begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \end{array}$$
$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

free: $x_2, x_4, x_5, x_6$

pivot: $x_1, x_3, x_7, x_8$

4. Construct a matrix $A$ such that its nullspace contains all multiples of $(2, -1, 3, 1)$.

$(2,-1,3,1)$ $\underbrace{x_4 \text{ must be free}}$, $x_1, x_2, x_3$ pivot

"special solution" free variable is 1

$$A = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

5. Suppose an $m$ by $n$ matrix has $r$ pivots. Answer the following questions:

(a) The number of special solutions is $n-r$ ($n-r$ free columns)

(b) The nullspace contains only $\vec{x} = \vec{0}$ when $r =$ $n$ (all rows are pivot rows)