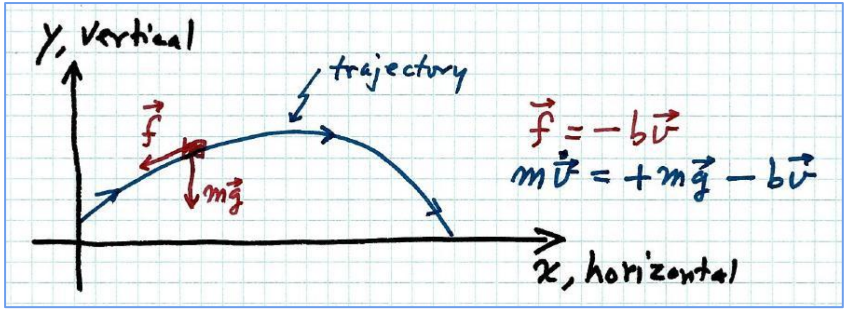
Section 2.3.

Trajectory and range in a linear medium

Read Section 2.3.

Recall the basic equations for projectile motion with linear air resistance.

Let x = the horizontal coordinate;
let y = the vertical coordinate, with the positive direction upward;



Solutions for v_x and v_y as functions of t ...

$$m\dot{v}_x = -b v_x \quad \Rightarrow \quad v_x(t) = v_{0x}\, e^{-bt/m}$$

$$m\dot{v}_y = -mg - b v_y$$

$$\Rightarrow \quad v_y(t) = \left(v_{0y} + \frac{mg}{b}\right) e^{-bt/m} - \frac{mg}{b}$$

Rewrite the solutions ...

$$v_x(t) = v_{0x}\, e^{-t/\tau} \quad \text{where } \underline{\tau = \frac{m}{b}}$$

$$v_y(t) = \left(v_{0y} + v_{ter}\right) e^{-t/\tau} - v_{ter}$$

$$\text{where } \underline{v_{ter} = \frac{mg}{b}} = \tau g$$

"time constant" = τ and

"terminal speed" = v_{ter}

Calculation of the trajectory

The "trajectory" is the curve in space along which the particle moves

▸Horizontal position, x(t)

$$x(t) = \int_0^t v_x(t')\,dt' = v_{0x}(-\tau)\,e^{-t'/\tau}\Big|_0^t = v_{0x}\,\tau\left(1 - e^{-t/\tau}\right)$$

x(t) increases to x_{final} as $\mathbf{t} \to \infty$;

$x_{final} = v_{0x}\,\tau$.

▸Vertical position, y(t); **take y(0) = 0**

$$y(t) = \int_0^t v_y(t')\,dt' = (v_{0y} + v_{ter})(-\tau)\,e^{-t'/\tau}\Big|_0^t - v_{ter}\,t = (v_{0y} + v_{ter})\,\tau\left(1 - e^{-t/\tau}\right) - v_{ter}\,t$$

$v_y(t)$ reaches a maximum downward velocity as $\mathbf{t} \to \infty$;

terminal velocity = $-\,v_{ter}$

▸Trajectory, i.e., the curve in space along which the projectile moves; y as a function of x ;

$$y = (v_{0y} + v_{ter})\frac{x}{v_{0x}} + v_{ter}\,\tau\,\ln\left(1 - \frac{x}{v_{0x}\tau}\right)$$

Figure 2.7

The trajectory of a projectile, assuming linear air resistance

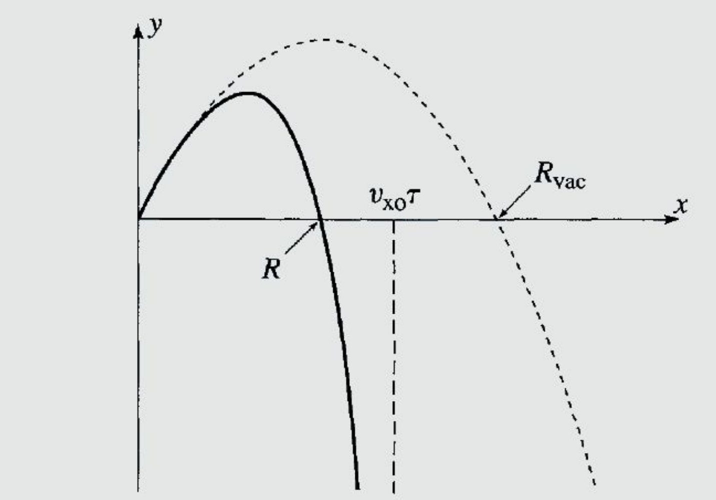


Figure 2.7 The trajectory of a projectile subject to a linear drag force (solid curve) and the corresponding trajectory in a vacuum (dashed curve). At first the two curves are very similar, but as t

Horizontal range ; R and R_{vac}

Recall, without air resistance,

$$R_{vac} = \frac{2 v_{0x} v_{0y}}{g}$$

With linear air resistance,

$$y = 0 \quad \text{where} \quad x = R,$$

$$\therefore \quad (v_{0y} + v_{ter})\frac{R}{v_{0x}} + v_{ter}\tau \ln\left[1 - \frac{R}{v_{0x}\tau}\right] = 0$$

But this is a *transcendental equation* for the range R *(as a function of* v_{0x} *and* v_{0y} *).*

Methods: [i] use a computer, or
[ii] make an approximation.

Example 2.4

RANGE OF SMALL METAL PELLETS

(note: "small" means tiny).

Suppose D = 0.2 mm and { v_{0x} , v_{0y} } = { 1/√2 , 1/√2 } m/s. *Calculate the range.*

To solve:

$$y = 0 \quad \text{where} \quad x = R,$$

$$\therefore \quad (v_{oy} + v_{ter})\frac{R}{v_{ox}} + v_{ter}\,\tau \ln\left[1 - \frac{R}{v_{ox}\tau}\right] = 0$$

Recall, $v_{ter} = mg/b$ and $\tau = m/b = v_{ter}/g$.
We'll calculate m from density ρ,

$$m = (\pi/6)\,\rho\, D^3\,;$$

also, recall $b = (1.6 \times 10^{-4}\ \mathrm{N.s/m^2})\,D$

- If there is no air resistance then the range is

$$R_{vac} = 2\, v_{0x}\, v_{oy}/g = 10.2\ \text{cm}$$

- Assuming linear air resistance, and solving the equation exactly (with *Mathematica*)

material	density (kg m^{-3})	v_{ter} (m/s)	R (cm)
gold	16 ×10^3	20.5	9.74
aluminum	2.7 ×10^3	3.47	7.96

♦ Taylor's approximate method …

$y = 0$ where $x = R$,

$$\therefore \quad (v_{oy} + v_{ter})\frac{R}{v_{ox}} + v_{ter}\tau \ln\left[1 - \frac{R}{v_{ox}\tau}\right] = 0$$

We might anticipate that $R / (v_{ex}\,\tau)$ is small , because these tiny pellets will have a small range.

Let $\varepsilon = R / (v_{ex}\,\tau)$.

Recall the Taylor series for $\ln(1 - \varepsilon)$

$$= -\varepsilon - \varepsilon^2/2 - \varepsilon^3/3 + O(\varepsilon^4)$$

So approximate (anticipating $\varepsilon << 1$)

$$\ln(1-\varepsilon) \approx -\varepsilon - \varepsilon^2/2 - \varepsilon^3/3$$

$$(v_{oy} + v_{ter})\frac{R}{v_{ox}} - v_{ter}\tau\left[\frac{R}{v_{ox}\tau} + \frac{1}{2}\left(\frac{R}{v_{ox}\tau}\right)^2 + \frac{1}{3}\left(\frac{R}{v_{ox}\tau}\right)^3\right] = 0$$

Simplifications : $v_{ter} = g\tau$;

Multiply by v_{ox}^2/R ;

exercise: check this

$$v_{oy}v_{ox} + g\tau v_{ox} - g\tau v_{ox} - \frac{g}{2}\frac{R}{1} - \frac{1}{3}\frac{g}{\tau}\frac{R^2}{v_{ox}} = 0$$

Multiply by $2/g$ $\Longrightarrow$

$$R = \frac{2v_{oy}v_{ox}}{g} - \frac{2}{3}\frac{R^2}{\tau v_{ox}}$$

$$R \approx R_{vac} - \frac{2}{3}\frac{R_{vac}^2}{\tau v_{ox}} \quad \text{where } R_{vac} = \frac{2v_{ox}v_{oy}}{g}$$

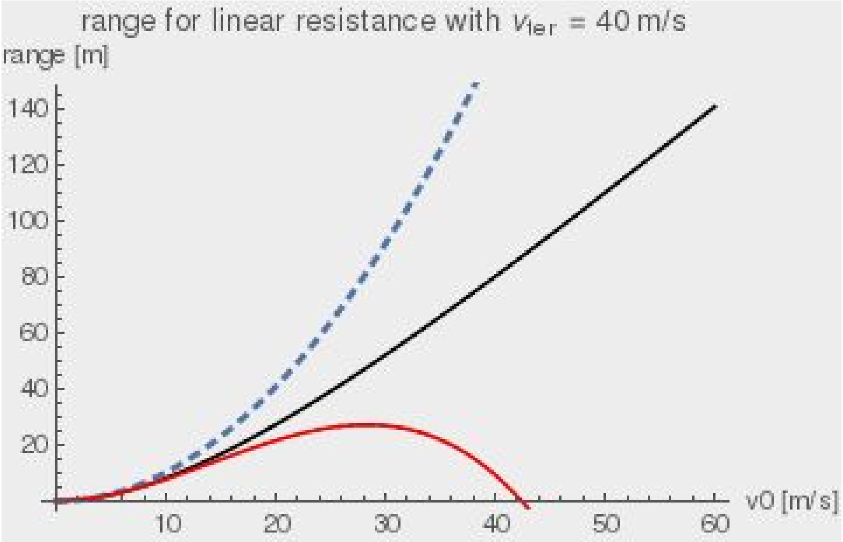
$$R = R_{vac}\left\{1 - \frac{4}{3}\frac{v_{oy}}{v_{ter}}\right\} \quad \text{(eq 2.44)}$$

Results of the approximation

gold	$R \approx 9.73$ cm	[9.74 ⇒ **error =−0.1 %**]
aluminum	$R \approx 7.42$ cm	[7.96 ⇒ **error =−7.3 %**]

Range for linear air resistance

Calculated using Mathematica.
Assume the initial angle of elevation is 45 degrees.
Plot the *range* versus *initial speed*, for v_{ter} = 40 m/s.



Dashed curve = no air resistance

Black curve = linear air resistance exact

Red curve = Taylor's approximation (2.44); breaks down for large v_0

Parametric Plots

Suppose we know *x(t)* and *y(t)* ;

and now we want to make a plot of *y* versus *x* (" *the trajectory* ") .

```
ParametricPlot[ {x[t],y[t]},
 {t,0,5},
 PlotRange->{{0,15},{0,6}}  ]
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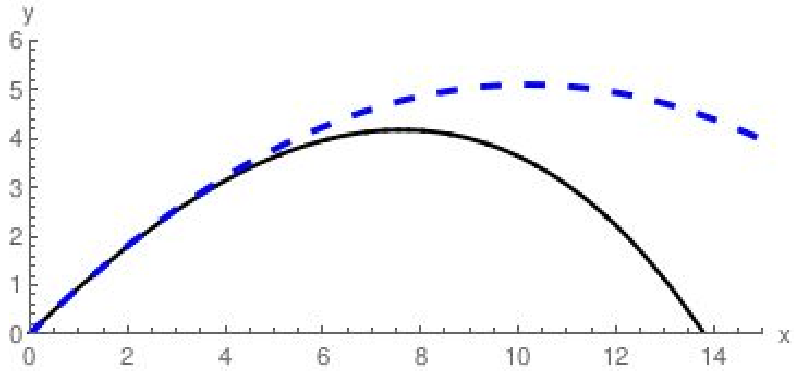
For example, with g = 9.8 m/s^2 ,
v_0 = {10,10} m/s , and τ = 3 s :

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x[t_]:=v0x*τ*(1 - Exp[-t/τ])

y[t_]:=(v0y+vter)*τ*(1 - Exp[-t/τ])
              - vter*t
```

Homework Assignment #3
due in class Friday, September 23
[11] Problem 2.2 *
[12] Problem 2.3 *
[13] Problem 2.10 **
[14] Problem 2.18 *
[15] Problem 2.26 *
[16] Assigned problem
Use the cover sheet.