Section 3.5
Angular Momentum for Several Particles

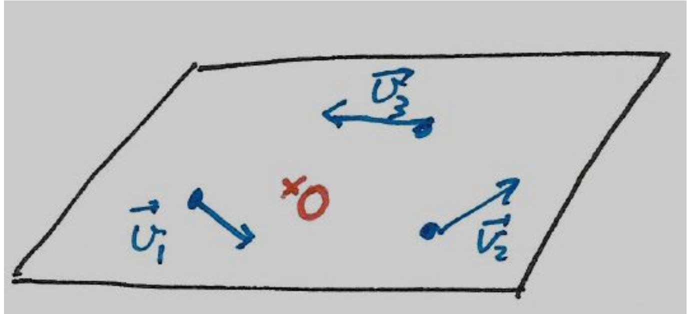
Read Section 3.5.

Consider N particles,

with masses m_α ,

velocities $\boldsymbol{v}_\alpha$ and positions $\boldsymbol{r}_\alpha$.

$\{ \alpha = 1\ 2\ 3 \ldots N \}$



The total angular momentum is

$$\mathbf{L} = \sum_{\alpha=1}^{N} \boldsymbol{\ell}_\alpha$$

$$\mathbf{L} = \sum_{\alpha=1}^{N} m_\alpha \boldsymbol{r}_\alpha \times \boldsymbol{v}_\alpha$$

Remember: $\boldsymbol{\ell}$ and $\boldsymbol{L}$ are defined w.r.t. O.

The importance of **L ...**

[I] If the internal forces are ***central***, then $d\mathbf{L}/dt$ = the external torque.

[I] For an isolated system with central internal forces, $d\mathbf{L}/dt = 0$;
i.e., then **L** is a constant of the motion.

Figure 3.8

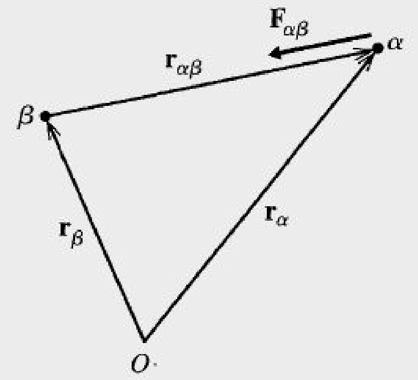


Figure 3.8 The vector $\mathbf{r}_{\alpha\beta} = (\mathbf{r}_\alpha - \mathbf{r}_\beta)$ points to particle α from particle β. If the force $\mathbf{F}_{\alpha\beta}$ is central (points along the line joining α and β), then $\mathbf{r}_{\alpha\beta}$ and $\mathbf{F}_{\alpha\beta}$ are collinear and their cross product is zero.

What is this figure telling us?

For an isolated system with central forces, the total angular momentum is constant.

Consider a system with two isolated particles and a ***central*** force.

$\boldsymbol{F}_{\alpha\beta}$ = the force on α exerted by β

Consider an arbitrary origin :

$\boldsymbol{r}_\alpha$ = position vector of α

$\boldsymbol{r}_\beta$ = position vector of β

The total angular momentum is

$$\mathbf{L} = \boldsymbol{\ell}_\alpha + \boldsymbol{\ell}_\beta$$

$$= m_\alpha\, \boldsymbol{r}_\alpha \times \boldsymbol{v}_\alpha + m_\beta\, \boldsymbol{r}_\beta \times \boldsymbol{v}_\beta \quad ;$$

and so

$$d\mathbf{L}/dt = \boldsymbol{r}_\alpha \times \boldsymbol{F}_{\alpha\beta} + \boldsymbol{r}_\beta \times \boldsymbol{F}_{\beta\alpha}$$

$$= (\boldsymbol{r}_\alpha - \boldsymbol{r}_\beta) \times \boldsymbol{F}_{\alpha\beta} = \boldsymbol{r}_{\alpha\beta} \times \boldsymbol{F}_{\alpha\beta} = \boldsymbol{0}$$

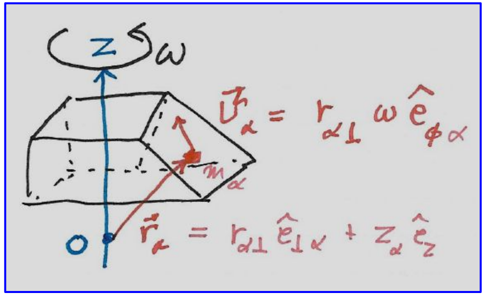
Moment of Inertia I

- You learned this in PHY 183.
- "Moment of inertia" is a property of a solid body.
- I.e., there is a continuum mass density, $\rho(\boldsymbol{r})$.
- Often we'll have $\rho(\boldsymbol{r})$ = *constant*, **called uniform mass density**.
- Moment of inertia is defined with respect to an axis of rotation, which might be a symmetry axis of the body (but not necessarily).

Definition of the moment of inertia

(This is Taylor's Problem 3.30; more complete discussion in Chapter 10.)

Consider rotation about the z axis,



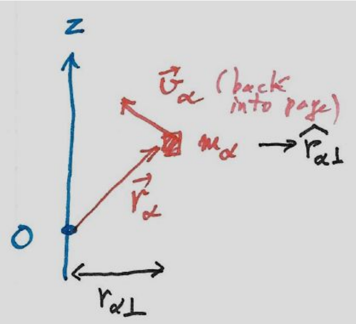
Divide the body into N small parts (treat them as particles) with masses m_α { α = 1 2 3 … N }; then take the continuum limit $N \to \infty$ and $m_\alpha \to 0$ with M constant.

What is the total angular momentum?

Definition of the moment of inertia

Consider rotation about the z axis.

Let m_α be one particle in the system



What is the angular momentum?

$$\vec{l}_\alpha = m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha$$

$$= m_\alpha \left(r_{\alpha\perp} \hat{e}_{\alpha\perp} + z_\alpha \hat{e}_z \right) \times \left(r_{\alpha\perp} \omega \hat{e}_{\phi\alpha} \right)$$

$$= m_\alpha r_{\alpha\perp}^2 \omega \hat{e}_z + m_\alpha z_\alpha r_{\alpha\perp} \omega \left(-\hat{e}_{\alpha\perp} \right)$$

The total angular momentum around the z axis

$$L_z = \sum_\alpha m_\alpha r_{\alpha\perp}^2 \omega$$

$$L_z = I \omega \quad \text{where} \quad I = \sum_\alpha m_\alpha r_{\alpha\perp}^2$$

moment of inertia around the z axis

$$I = \int_{Body} r_\perp^2 \, dm = \int_{Body} r_\perp^2 \rho \, dV$$

For uniform density, $\rho = \dfrac{M}{\int_B dV}$

$$I = M \frac{\int_B r_\perp^2 \, dV}{\int_B dV}$$

[Example 3.3](#)

A lump of putty collides with a turntable

Figure 3.9

h. plane

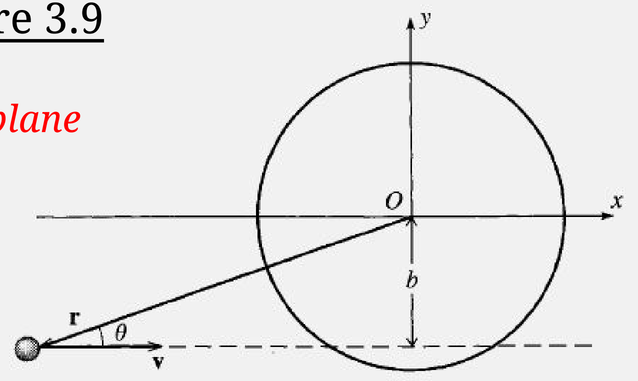


Figure 3.9 A lump of putty of mass m is thrown with velocity $\mathbf{v}$ at a stationary turntable. The putty's line of approach passes within the distance b of the table's center O.

After the collision the lump of putty sticks to the turntable (i.e., it's an *inelastic collision*).

Note the *impact parameter* b defined in the picture.

The problem is to calculate the final angular velocity of the turntable with the lump of putty stuck to it.

§ ***The principle is conservation of angular momentum.***

§ Before the collision ***(anyplace on the dashed line)***

$$\vec{L} = m\,(x\,\hat{e}_x + y\,\hat{e}_y) \times (v_x\,\hat{e}_x)$$

$$= m\,(-b)\,v_0\,(-\hat{e}_z) = m b v_0\,\hat{e}_z$$

§ After the collision:

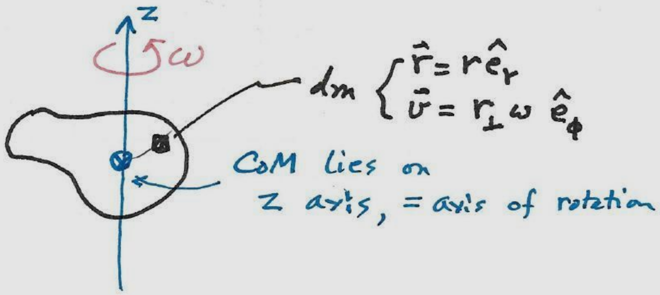
$$L_z = I\omega + mR^2\omega$$

$$= (I + mR^2)\,\omega$$

$$\therefore \quad \omega = \frac{m b v_0}{I + mR^2}$$

For a disk, $I = \frac{1}{2} M R^2 \Rightarrow \omega = \dfrac{b v_0}{R^2\left(\frac{M}{2m} + 1\right)}$

Angular momentum vector of a rigid body that rotates about an axis through the center of mass position



$$\vec L = \int_B \vec r \times \vec v \, dm = \int_B (r_\perp \hat e_\perp + z\hat e_z)\times(r_\perp \omega \hat e_\phi)\, dm$$

$$= \int_B \left(r_\perp^2 \omega \hat e_z + z r_\perp \omega(-\hat e_\perp) \right) dm$$

$= I_{\text{for this axis}}\ \omega\ \mathbf{e}_z$ because we specified that the CoM lies on the axis of rotation; $\oint \mathbf{e}_\perp \, d\varphi = 0$

Example 3.4

A Sliding and Spinning Dumbbell

See Figure 3.10. Kick the sphere on the left—an *impulsive force*—as shown. Calculate the motion.

Figure 3.10

(horizontal plane)

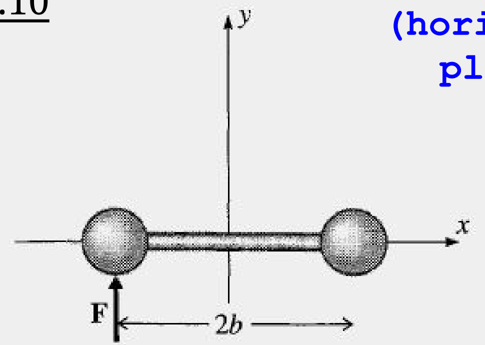


Figure 3.10 The left mass of the dumbbell is given a sharp tap in the y direction.

$M = 2m$ and $I = 2\, m\, b^2$

Principles :

|·| The impulsive force causes the center of mass to accelerate briefly; impulse $F\,\Delta t = \kappa$;

$$d\mathbf{P}/dt = \mathbf{F}^{ext} \quad \Rightarrow \quad \Delta\mathbf{P} = \mathbf{F}\,\Delta t = \kappa\, \boldsymbol{e}_y$$

|·| The impulsive *torque* causes a brief angular acceleration;

$$dL_z/dt = N_z = -bF \quad \Rightarrow \quad \Delta L_z = -bF\,\Delta t = -b\kappa$$

|·| After the impulse, the momentum and angular momentum are constant (∃ no force and ∃ no torque);

$$v_{CM} = P/M = \kappa / (2m)$$

and

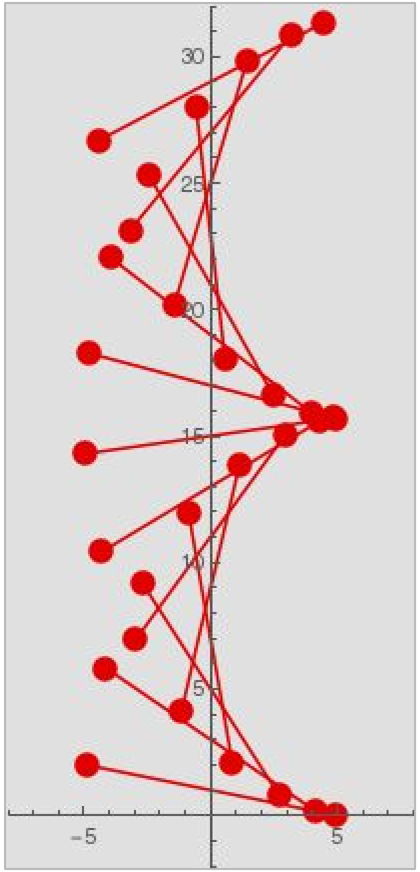
$$\omega = L/I = -b\kappa/I \qquad \text{so} \qquad \omega = -v_{CM}/b$$

Describe the motion after the impulsive force:

- the center of mass moves along the y axis;

- the two spheres revolve round the center of mass;

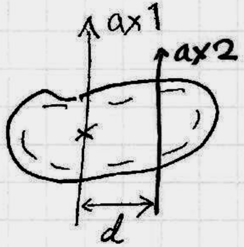
- $v_{cm} = -b\,\omega$



The parallel axis theorem

Let ax1 be an axis that passes through the center of mass point (×); and let

I_{CM} = The moment of inertia around ax1.

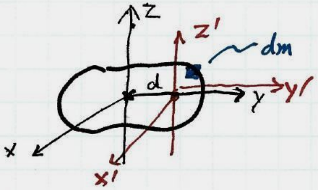


Let ax 2 be a parallel axis at ⊥ distance d from ax 1.

Theorem.

$$I_{ax2} = I_{CM} + Md^2$$

Proof Define some Cartesian axes xyz ; x'y'z'



$$I_{ax2} = \int_{Body} r_\perp'^2 \, dm = \int_B (x_1'^2 + y'^2)\, dm$$

$$= \int_B [\, x^2 + (y-d)^2 \,]\, dm$$

$$= \int_B (x^2+y^2)\, dm - 2d \underbrace{\int_B y\, dm}_{M y_{CM} = 0} + d^2 M$$

$$I_{ax2} = I_{CM} + M d^2$$

Chapter 10 (PHY 422) –
the inertia tensor

Homework Assignment #6
due in class Friday, October 14
[27] Problem 3.16 *
[28] Problem 3.20 **
[29] Problem 3.22 **
[30] Problem 3.27 **
[30x] Problem 3.32 **
[30xx] Problem 3.35 **
Use the cover sheet.