7.6 Generalized Momenta and Ignorable Coordinates

As I have already mentioned, for any system with n generalized coordinates q_i $(i = 1, \cdots, n)$, we refer to the n quantities $\partial\mathcal{L}/\partial q_i = F_i$ as *generalized forces* and $\partial\mathcal{L}/\partial\dot{q}_i = p_i$ as *generalized momenta*. With this terminology, the Lagrange equation,

$$\frac{\partial\mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}, \tag{7.81}$$

can be rewritten as

$$F_i = \frac{d}{dt}p_i. \tag{7.82}$$

portant result (and one that is clear from the Newtonian perspective as well). When the Lagrangian is independent of a coordinate q_i, that coordinate is sometimes said to be **ignorable** or **cyclic**. Obviously it is a good idea, when possible, to choose coordinates so that as many as possible are ignorable and their corresponding momenta are constant. In fact, this is perhaps the main criterion in choosing generalized coor-

invariant, when q_i varies (with all the other q_j held fixed)." Thus we can say that if $\mathcal{L}$ is invariant under variations of a coordinate q_i then the corresponding generalized momentum p_i is conserved. This connection between invariance of $\mathcal{L}$ and certain

7.7 Conclusion

The Lagrangian version of classical mechanics has the two great advantages that, unlike the Newtonian version, it works equally well in all coordinate systems and it can handle constrained systems easily, avoiding any need to discuss the forces of constraint. If the system is constrained, one must choose a suitable set of independent generalized coordinates. Whether or not there are constraints, the next task is to write down the Lagrangian $\mathcal{L}$ in terms of the chosen coordinates. The equations of motion then follow automatically in the standard form

$$\boxed{\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \qquad [i = 1, \cdots, n].}$$

There is, of course, no guarantee that the resulting equations will be easy to solve, and in most real problems they are not, requiring numerical solution or at least some approximations before they can be solved analytically.

Chapter 8.

Two-body Central Force Problems

Section 8.1.

The Problem

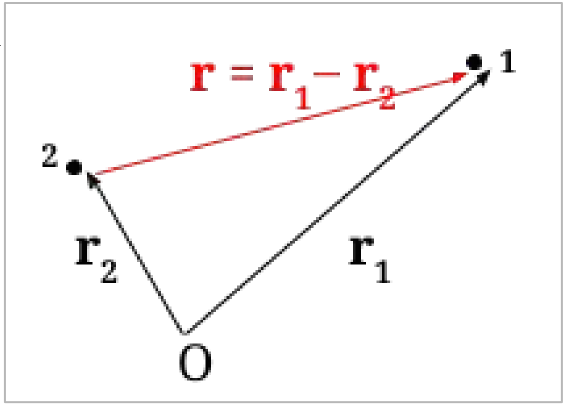
Section 8.2.

CM and Relative Coordinates

Read Sections 8.1 and 8.2.

8.1. The Problem

FIGURE 8.1

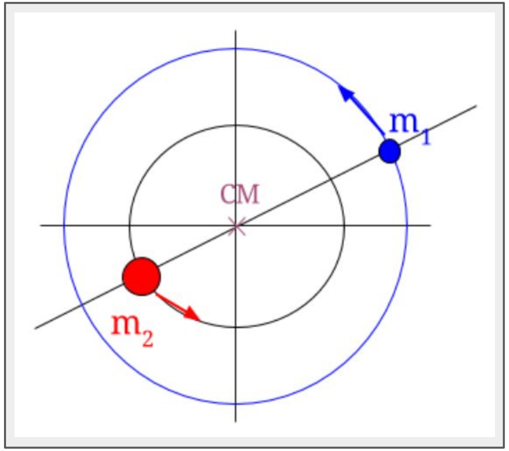


The particles (1 and 2) exert forces on each other, and there are no external forces.

The potential energy is U(|**r**|) where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$;

the forces are central and spherically symmetric ; $r = |\mathbf{r}|$; $U = U(r)$

Astronomical examples



The potential energy is

$$U(r) = -\,G\, m_1\, m_2\ /r\ .$$

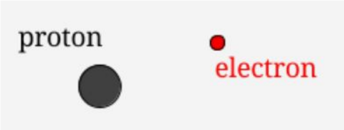
Comments:

- ➡ The orbits are not circular in general.
- ➡ Sun and Earth, or another planet; $m_2 \gg m_1$
- ➡ Earth and Moon, or a satellite; $m_2 \gg m_1$
- ➡ Binary Star; m_2 and m_1 are comparable

Atomic examples

The hydrogen atom

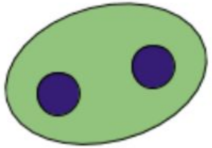
$$U(r) = -\,k\, e^2\ /r$$



Diatomic molecule

O_2

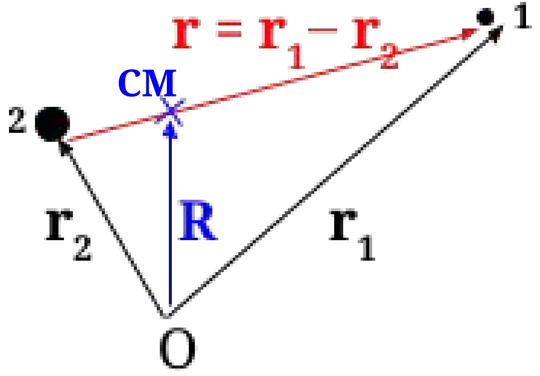
$$U(r) = A_1/r^{12} - A_2/r^6$$



Strictly these would require quantum mechanics, but sometimes semi-classical calculations are interesting.

8.2. CM and Relative Coordinates; Reduced Mass

FIGURE 8.2



CM: $\mathbf{R} = (m_1\, \mathbf{r_1} + m_2\, \mathbf{r_2}) / (m_1 + m_2)$

or, $M\mathbf{R} = m_1\mathbf{r_1} + m_2\mathbf{r_2}$

$(M = m_1 + m_2)$

Relative: $\mathbf{r} = \mathbf{r_1} - \mathbf{r_2}$

Notes:

- ❑ If $m_2 \gg m_1$ then $\mathbf{R} \approx \mathbf{r_2}$.
- ❑ $\mathbf{r_1} = \mathbf{R} + (m_2 /M)\, \mathbf{r}$ and
  $\mathbf{r_2} = \mathbf{R} - (m_1 /M)\, \mathbf{r}$

  Check:

  $m_1 r_1 + m_2 r_2 = (m_1 + m_2)\, R$ ✔

  $r_1 - r_2 = r$ ✔
- ❑ The kinetic energy is

$$T = ½\, m_1\, \dot{\mathbf{r}}_1^{\,2} + ½\, m_2\, \dot{\mathbf{r}}_2^{\,2}$$

$$= ½\, M\, \dot{\mathbf{R}}^2 + ½\, \mu\, \dot{\mathbf{r}}^2$$

where

$$\mu = m_1 (m_2/M)^2 + m_2 (m_1/M)^2 = m_1 m_2/M$$

❑ The reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

or, $$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

<u>Notes</u>

- μ is smaller than either m_1 or m_2
- if $m_1 \ll m_2$ then $\mu \approx m_1$; *sim.* $m_2 \ll m_1$
- if $m_1 = m_2$ then $\mu = m_1/2$.

❑ The Lagrangian

$$\mathcal{L} = T - U$$

$$= \underbrace{\tfrac{1}{2} M \dot{\vec{R}}^2}_{\text{(CM)}} + \underbrace{\tfrac{1}{2} \mu \dot{\vec{r}}^2 - U(r)}_{\leftarrow\text{(relative)}\rightarrow}$$

So, the two-body problem reduces to

1. the center of mass motion (**R**)
   which is trivial; **R**(t) = $\mathbf{V}_{CM}$ t

and

2. the relative motion (**r**)
   which is equivalent to a one-body problem, with reduced mass μ and potential energy U(r).

The Equations of Motion

Generalized coordinates:
R = {X, Y, Z} and **r** = {x,y,z}

The Lagrangian is

$$\mathcal{L} = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$$

Recall Lagrange's equation for a coordinate *q*,

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}}\right) - \frac{\partial\mathcal{L}}{\partial q} = 0$$

- The center of mass coordinates

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{X}}\right) - \frac{\partial\mathcal{L}}{\partial X} = \frac{d}{dt}(M\dot{X}) = 0$$

$$\ddot{X} = 0 \quad ; \quad \ddot{\vec{R}} = 0 \quad ; \quad \vec{R} = \vec{V}_c t$$

- The relative coordinates

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) - \frac{\partial\mathcal{L}}{\partial x} = \frac{d}{dt}(\mu\dot{x}) + \frac{\partial U}{\partial x} = 0$$

$$\mu\ddot{\vec{r}} + \nabla U = 0$$

The two-body problem is reduced to a one-body problem,

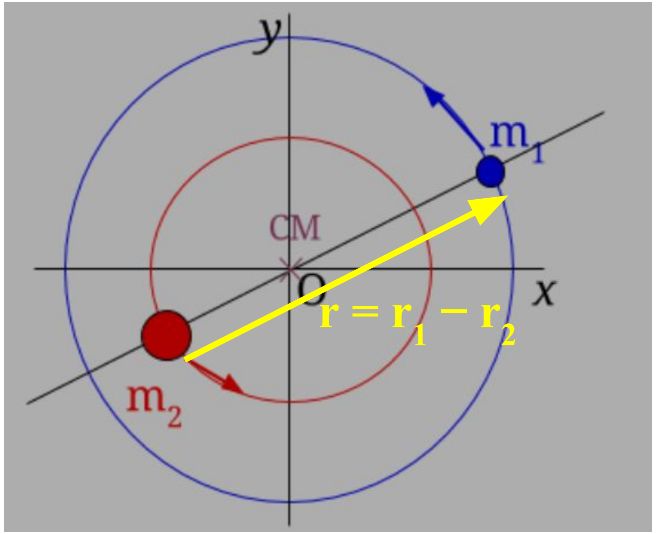
$$\mu\ddot{\vec{r}} = -\nabla U(r) = -\hat{r}\frac{dU}{dr}$$

equivalent to a particle in a central potential.

The center of mass frame of reference

W.L.O.G. take $\mathbf{V}_C = 0$ and $\mathbf{R} = 0$.

I.e., let the origin of the coordinate system be the center of mass.



Note that $\mathbf{r}_1 = (m_2/M)\ \mathbf{r}$

and $\mathbf{r}_2 = -\,(m_1/M)\ \mathbf{r}$

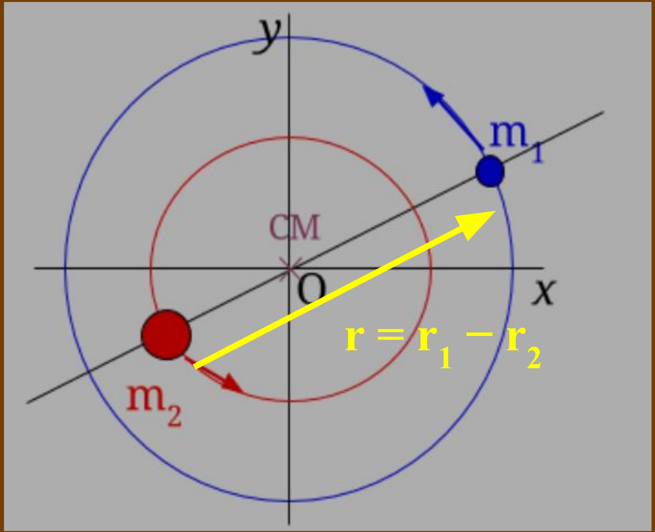
(If $m_2 \gg m_1$ then $\mathbf{r}_1 \approx \mathbf{r}$ and $r_2 \approx \mathbf{0}$.)

For example, consider the solar system …

The center of mass is close to the center of the sun. All the planets revolve around the center of mass. And the whole thing revolves around the center of the Milky Way galaxy. But we can ignore the motion of the center of mass when we calculate the motions of the planets or other satellites.

How to solve the equation of motion,

$$\mu\,\ddot{\mathbf{r}} = -\,G\,m_1 m_2 / r^2\ \ \mathbf{e}_r \qquad ?$$



How to solve the equation of motion,

$$\mu \ddot{\mathbf{r}} = - G\, m_1 m_2 / r^2 \quad \mathbf{e}_r \qquad ?$$

Do not use the second-order differential equation.

Instead, use the equations for conservation of energy and angular momentum.

The orbit is not a circle in general;
it's an ellipse. An ellipse depends on two geometrical parameters, the semi-major axis and the eccentricity. These are determined by two dynamical constants, the energy and the angular momentum.

Homework Assignment #13

due in class Friday December 2

[71] Problem 8.4 ★

[72] Problem 8.6 ★

[73] Problem 8.12 ★★

[74] Problem 8.15 ★

[75] Problem 8.16 ★★

Use the cover sheet.