Section 4.9.
Energy of interaction for two particles

Read Section 4.9.

Two interacting particles

Particle #1 is located at position $\mathbf{r}_1$;

particle #2 is located at position $\mathbf{r}_2$;

these are vectors from some fixed origin, O.

Also, let $\mathbf{r}$ be the vector from particle #2 to particle #1.

Draw a picture of this system.

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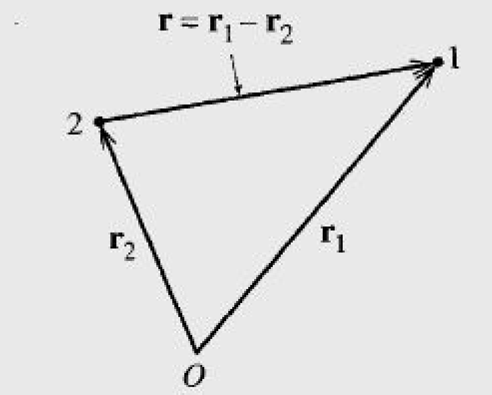


Figure 4.18 The vector $\mathbf{r}$ pointing to particle 1 from particle 2 is just $\mathbf{r} = (\mathbf{r}_1 - \mathbf{r}_2)$.

A comment on *translation in space*

Change the origin, O.

Then $\mathbf{r}_1$ and $\mathbf{r}_2$ change; but $\mathbf{r}$ remains the same.

Translation invariance :: the dynamics only depends on $\mathbf{r}$.

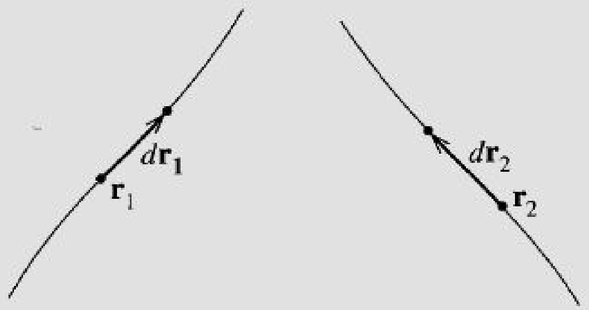


Figure 4.20 Motion of two interacting particles. During a short time interval dt, particle 1 moves from $\mathbf{r}_1$ to $\mathbf{r}_1 + d\mathbf{r}_1$ and particle 2 from $\mathbf{r}_2$ to $\mathbf{r}_2 + d\mathbf{r}_2$.

What is the energy of this translation invariant system of two interacting particles?

The energy of a system of two interacting particles

Assume that the only force on either particle is the force exerted by the other particle.

I.e., the two particles form an "isolated system".

Also, assume the force is conservative.

I.e., there is a potential energy function.

Although there are two forces, $\mathbf{F}_1$ and $\mathbf{F}_2$, there is only one potential energy.

Kinetic energy

$$T_1 = \tfrac{1}{2} m_1 v_1^{\,2} \quad \text{and} \quad T_2 = \tfrac{1}{2} m_2 v_2^{\,2}$$

$$T = T_1 + T_2$$

Potential energy

$$U = U(\mathbf{r}) \text{ where } \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \,.$$

Two forces are implied by *one potential energy* U(**r**);
these are the action-reaction pair required by Newton's third law.

$$\boldsymbol{F_1} = -\nabla_1 U = \text{force on \#1} \quad = \boldsymbol{F_{12}}$$

and

$$\boldsymbol{F_2} = -\nabla_2 U = \text{force on \#2} \quad = \boldsymbol{F_{21}}$$

Newton's third law for a conservative force :

The potential energy function is

$$U = U(\boldsymbol{r}) = U(\boldsymbol{r_1} - \boldsymbol{r_2})$$

$$\boldsymbol{F_1} = -\nabla_1 U = -\nabla_r U$$
because $\boldsymbol{r} = \boldsymbol{r_1} - \boldsymbol{r_2}$; $\quad d\boldsymbol{r} = d\boldsymbol{r_1}$

$$\boldsymbol{F_2} = -\nabla_2 U = +\nabla_r U$$
because $\boldsymbol{r} = \boldsymbol{r_1} - \boldsymbol{r_2}$; $\quad d\boldsymbol{r} = -d\boldsymbol{r_2}$

Thus $\quad \boldsymbol{F_2} = -\boldsymbol{F_1}$.

That's Newton's third law.

Note :: translation invariance implies Newton's third law, which implies that momentum is conserved

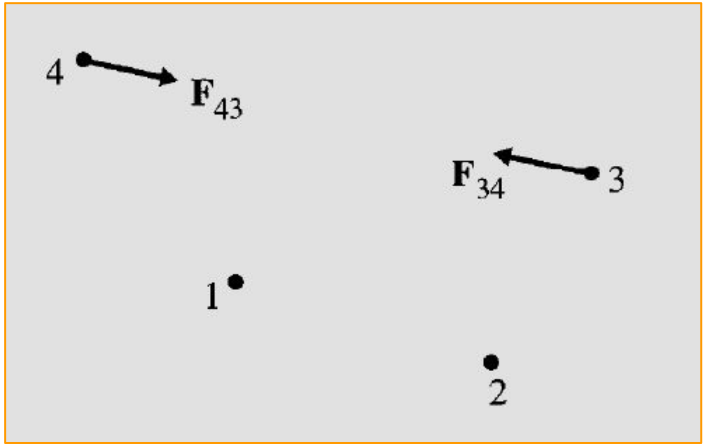
Section 4.10

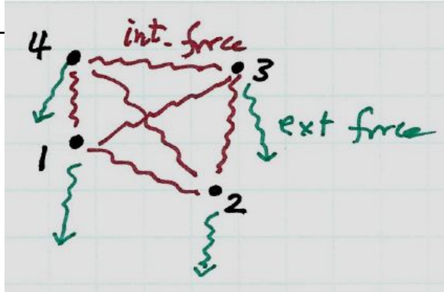
The energy of a multiparticle system

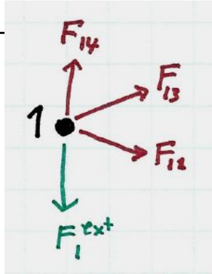
Read Section 4.10.

Four particles

Figure 4.22







The kinetic energies :

$$T_\alpha = \frac{1}{2} m_\alpha v_\alpha^2 \qquad (\alpha = 1\,2\,3\,4)$$

$$T = T_1 + T_2 + T_3 + T_4$$

The internal potential energies :

$$U_{\alpha\beta} = U_{\alpha\beta}(\vec{r}_\alpha - \vec{r}_\beta)$$

$$U^{int} = U_{12} + U_{13} + U_{14} + U_{23} + U_{24} + U_{34}$$

& the external potential energies :

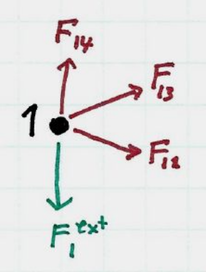
$$U^{ext} = U_1^{ext} + U_2^{ext} + U_3^{ext} + U_4^{ext}$$

N particles – simple generalization

$$T = \sum_{\alpha} \tfrac{1}{2} m_\alpha v_\alpha^2$$

$$U = \sum_{\alpha} \sum_{\substack{\beta \\ (\alpha<\beta)}} U_{\alpha\beta} + \sum_{\alpha} U_\alpha^{ext}$$

The net force on particle #1



$F_1 = F_{12} + F_{13} + F_{14} \ldots + F_{1,ext}$

$= -\nabla_1 (U_{12} + U_{13} + U_{14} \ldots + U_{1,ext})$

The net force on particle α is

$$\vec{F}_\alpha = -\nabla_\alpha U^{int} - \nabla_\alpha U_\alpha^{ext}$$

$$T = \sum_{\alpha=1}^{N} \tfrac{1}{2} m_\alpha v_\alpha^2$$

$$U = \underbrace{\sum_{\alpha} \sum_{\substack{\beta \\ (\alpha<\beta)}}}_{\text{Number of terms } = \frac{1}{2}N(N-1)} U_{\alpha\beta} + \sum_{\alpha} U_\alpha^{ext}$$

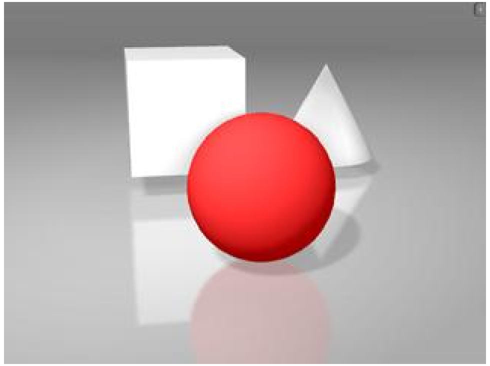
N	# internal
1	0
2	1
3	3
4	6
5	10
etc	

F...

For N particles, most generally …

- there are ½ N (N – 1) potential energies from internal forces;
- there are N potential energies from external forces.

Rigid bodies



A rigid body actually consists of many particles (atoms or molecules).

But it is not reasonable to try to describe the atoms individually.

We should treat the body as a continuum; that should be a good approximation.

DISCRETE AND CONTINUUM

Imagine the body divided into N small parts;

masses $= \delta M_\alpha$ where $\alpha = 1\ 2\ 3 \ldots N$;

$$M = \sum_{\alpha=1}^{N} \delta M_\alpha \, .$$

The continuum limit:

let $N \to \infty$ and $\delta M_\alpha \to 0$, with M fixed;

$$M = \int_B dM = \int_V \rho(\boldsymbol{r})\, d^3r \, .$$

(density)

Rigid bodies

Energies for a rigid body

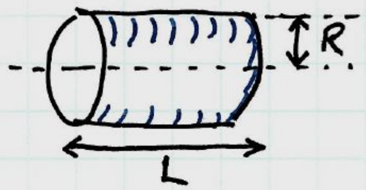
$\mathbf{T}$

$\mathbf{U^{ext}}$

Example

a cylinder rolling down an inclined plane

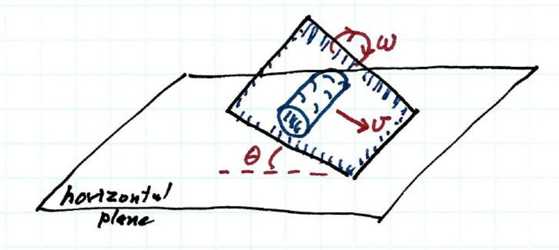
The cylinder: { ρ , R , L }



$$M = \int \rho \, d^3r = \rho \cdot \pi R^2 L$$

Now, the cylinder rolls down an inclined plane {angle θ} ; it *rolls without slipping* so the motion has two components

- translation
- rotation



Calculate the velocity v when the cylinder has descended by height h

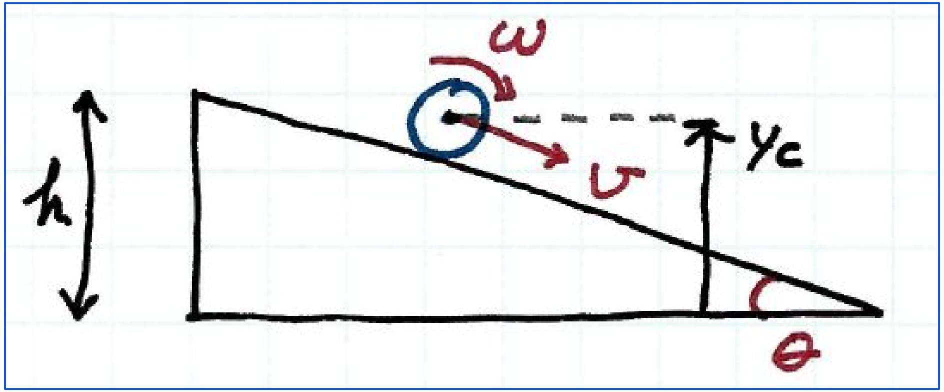
* **Principle:** Energy is conserved.

* The kinetic energy is

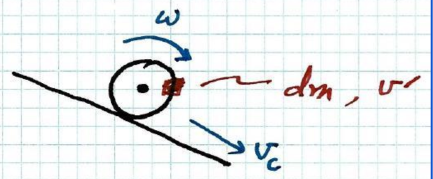
$$T = \frac{1}{2} M v^2 + \frac{1}{2} I \omega^2$$

* The potential energy is

$$U = M g y$$



$$T = \int \frac{1}{2}\, dm\, v'^2$$



It rolls without slipping

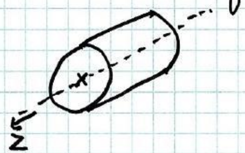
so $\quad ds = R\, d\phi$

$$v_c = R\,\omega$$

$$T = \frac{1}{2} M v^2 + \frac{1}{2} I \omega^2$$

$$= \frac{1}{2}\left(M + I/R^2\right) v^2 = \frac{3}{4} M v^2$$

Moment of inertia of a cylinder



$$I = \int r_\perp^2\, dm = \int r_\perp^2 \rho\, dr_\perp\, r_\perp d\phi\, dz$$

$$r_\perp : 0 \to R \;;\quad \phi : 0 \to 2\pi \;;\quad z : 0 \to \ell$$

$$I = \rho\, \frac{R^4}{4}\, 2\pi \ell$$

density $\rho = \dfrac{M}{\pi R^2 \ell} \quad \Rightarrow \quad I = \frac{1}{2} M R^2$

$$U = \int_{cyl.} dm\, g\, y = M g y_c$$

* Energy is conserved, so

$$T_{bottom} + U_{bottom} = T_{top} + U_{top}$$

$$\frac{3}{4} M v^2 + M g \frac{R}{2} = 0 + M g \left(h + \frac{R}{2} \right)$$

$$v = \sqrt{\frac{4}{3} g h}$$

Result: the velocity when the cylinder has descended by vertical distance h is v = Sqrt[(4/3) gh].

<u>Comment</u>

What would *v* be
if the surface were frictionless?

- Then the cylinder would *slide* down the incline, not rolling.
- ∃ no rotational kinetic energy.
- The final velocity would be

$$v_{sliding} = \sqrt{2 g h}$$

Note that $v_{sliding} > v_{rolling}$.

Test yourself: Explain why $v_{sliding} > v_{rolling}$,
in two different ways –
– in terms of energy and in terms of forces.

Example 4.9 :

a cylinder rolling down an incline

Figure 4.23

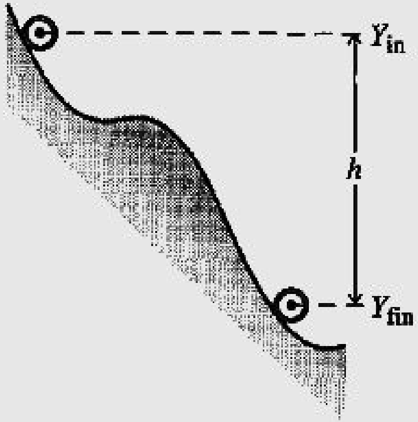


Figure 4.23 A uniform cylinder starts from rest and rolls without slipping down a slope through a total vertical drop $h = Y_{in} - Y_{fin}$ (with the CM coordinate Y measured vertically up).

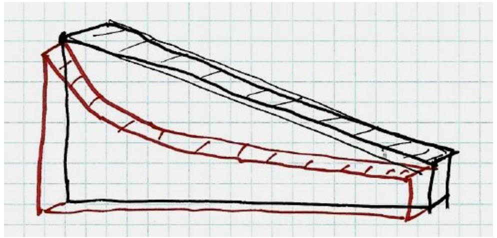
The result is,

v = Sqrt[(4/3) gh] (PAGE 148)

i.e., *just the same as for the inclined plane!*

The velocity as a function of vertical position (y) is the same for any shape of the surface; *that's because energy is conserved*.

However, the velocity as a function of ***time*** is different for different surfaces.



Red track: shorter time although it is a longer distance.

Homework Assignment #8

due in class Friday, October 28

[37] Problem 4.26 *

[38] Problem 4.28 ** and Problem 4.29 ** [Computer]

[39] Problem 4.33 ** [Computer]

[40] Problem 4.34 **

[40x] Problem 4.37 *** [Computer]

[40xx] Problem 4.38 *** [Computer]

Use the cover page.