# 4.1 & 4.2 Hmwk check Friday

- 2 vectors are perpindicular if dot product is zero

- orthogonal means perpindicular

## 4.1 Orthogonality of the four subspaces

$N(A)$, $N(A^T)$, $C(A)$, $C(A^T)$

* Row space of A is perpindicular to the null space of A

  every row of A is perpindicular to every solution of $A\vec{x} = \vec{0}$

* Column space of A is perpindicular to the left nullspace of A $N(A^T)$

orthogonal subspaces: $\vec{v}^T\vec{w} = 0$ for all $\vec{v} \in V$ and $\vec{w} \in W$, V & W are subspaces

$\vec{v}^T\vec{w} = 0$ or $\vec{v} \cdot \vec{w} = 0$

Ex) Let $\mathbb{X}$ be the subspace of $\mathbb{R}^3$ spanned by $\vec{e}_1$ and let $\mathbb{Y}$ be the subspace spanned by $\vec{e}_2$. If $\vec{x} \in \mathbb{X}$ and $\vec{y} \in \mathbb{Y}$ these ~~these~~ are of the form

$$\vec{x} = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{y} = \begin{pmatrix} 0 \\ y_2 \\ 0 \end{pmatrix}$$

$$\vec{x}^T \vec{y} = \vec{x} \cdot \vec{y} = \begin{pmatrix} x_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ y_2 \\ 0 \end{pmatrix}$$

$$= 0 \cdot x_1 + 0 \cdot y_2 + 0 \cdot 0 = 0$$

These two subspaces are orthogonal

$$\mathbb{X} \perp \mathbb{Y}$$

$$\mathbb{Y} \neq \mathbb{X}^\perp$$

Note: For example, the floor and wall of a room "look" orthogonal, but, $xy$-plane & $yz$-plane are not orthogonal.

Take: $\vec{x}_1 = (1,1,0)^T$

↑ on xy-plane

$\vec{x}_2 = (0,1,1)^T$

↑ on yz-plane

$$\vec{x}_1^T \vec{x}_2 = (1 \quad 1 \quad 0)\begin{pmatrix}0\\1\\1\end{pmatrix} = 0\cdot1 + 1\cdot1 + 0\cdot1$$

$$= 1$$

<u>Not</u> orthogonal

$\boxed{\text{Ex}}$ Let $A = \begin{bmatrix}1 & 0\\2 & 0\end{bmatrix}$

1 pivot, 1 free variable

pivot $x_1$

Talk about the four subspace of A.

column space: spanned by 1 vector $\begin{bmatrix}1\\2\end{bmatrix}$

$1^{st}$ column of A spans column space

column space of A looks like all vectors of the form $\alpha\begin{bmatrix}1\\2\end{bmatrix}$

$C(A) \perp N(A^T)$

$$A^T = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{Find } N(A^T)$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x_2 \text{ is free}$$

"Special solution" $x_2 = 1$

$$x_1 + 2(1) = 0 \qquad \vec{x}_n = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$x_1 = -2$$

$N(A^T)$ ~~too~~ contains $\beta\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Check: 
$$\begin{array}{ccc} C(A) & \perp & N(A^T) \\ \downarrow & & \downarrow \\ \alpha\begin{bmatrix} 1 \\ 2 \end{bmatrix} & & \beta\begin{bmatrix} -2 \\ 1 \end{bmatrix} \end{array}$$

$$\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = -2 + 2 = 0$$

$$\begin{pmatrix}\alpha & 2\alpha\end{pmatrix}\begin{pmatrix}-2\beta \\ \beta\end{pmatrix} = \alpha(-2\beta) + 2\alpha\beta$$

$$= -2\alpha\beta + 2\alpha\beta = 0$$

Rowspace $(C(A^T))$ is orthogonal to Nullspace of $A$ $(N(A))$

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Find: $C(A^T)$ + $N(A)$

$C(A^T)$ is any vector of the form $\alpha\begin{bmatrix}1 \\ 0\end{bmatrix}$

$N(A) \rightarrow$ $x_2$ is free, $x_2 = 1$

$$\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \vec{x}_n = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$N(A)$ is any vector of the form $\beta\begin{bmatrix}0 \\ 1\end{bmatrix}$

Show... $C(A^T) \perp N(A)$

$$\begin{pmatrix}\alpha & 0\end{pmatrix}\begin{pmatrix}0 \\ \beta\end{pmatrix} = 0 + 0 = 0$$

**Theorem:** If $S$ is a subspace of $\mathbb{R}^n$

then $\dim S + \dim S^\perp = n.$

$\dim S = \#$ of vectors needed to form a basis for $S$

Furthermore, if $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_r\}$ is a basis for $S$ and $\{\vec{x}_{r+1}, \ldots, \vec{x}_n\}$ is a basis for $S^\perp$, then

$\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \ldots, \vec{x}_r, \vec{x}_{r+1}, \ldots, \vec{x}_n\}$ is a basis for $\mathbb{R}^n$.

Ex) $\mathbb{R}^3$

let $S = \{(\alpha, 0, 0)^T \mid \alpha \in \mathbb{R}\}$

basis vector for $S$ is $\begin{pmatrix}1\\0\\0\end{pmatrix}$

$\dim S = 1$, since $\mathbb{R}^3$, $\dim S^\perp = 3-1=2$

basis for $S^\perp$ is $\begin{pmatrix}0\\1\\0\end{pmatrix}$ & $\begin{pmatrix}0\\0\\1\end{pmatrix}$

basis for $\mathbb{R}^3$ is $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$

Ex) Let $S$ be the subspace of $\mathbb{R}^3$ spanned by $\vec{x} = (1, -1, 1)^T$.

Find a basis for $S^\perp$.

$\dim(S) = 1 \implies \dim(S^\perp) = 3 - 1 = 2$

need 2 vectors to span $S^\perp$

$\vec{x} \cdot \vec{y} = 0 \qquad \vec{y} \in S^\perp$

$$\begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \boxed{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}} = 0 \cdot 1 + -1 \cdot 1 + 1 \cdot 1$$

basis $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$$(1, -1, 1) \boxed{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} = 1 \cdot 1 + 0 \cdot -1 + -1 \cdot 1$$

$$= 0$$

Note: $\vec{v} = (-1, 1)^T$ $\qquad$ $\vec{w} = (1, +1)^T$

$$\vec{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} 1 \\ +1 \end{pmatrix}$$

$$\vec{v} \cdot \vec{w} = -1 \cdot 1 + 1 \cdot 1 = 0$$

$$\vec{v}^T \vec{w} = \underbrace{(-1, 1)}_{1 \times 2} \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{2 \times 1} = \underbrace{0}_{1 \times 1}$$

<u>def:</u> The orthogonal complement of a subspace $V$ contains every vector that is perpindicular to $V$ and is called $V^{\perp}$

"$V$ perp"

$\vec{v} \perp \vec{w}$

2 vectors are perpindicular