

Section 5.7

Fourier Series

Motivation

Consider the driven damped oscillator, but for an *arbitrary periodic driving force*.

Recall, $D x(t) = f(t)$

where $D x = \ddot{x} + 2\beta \dot{x} + \omega_0^2 x$

and $f = F/m$.

"Periodic" means $f(t + \tau) = f(t)$.

Now, "periodic" does not necessarily mean sinusoidal.

("Sinusoidal" and "harmonic" are the same thing.)

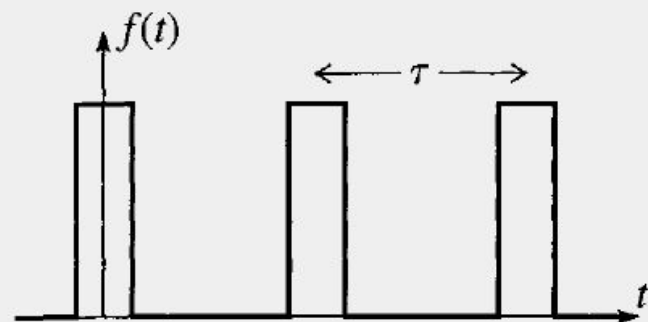
A harmonic function is periodic ;
e.g., $f(t) = f_0 \cos \omega t$ is periodic, with period $\tau = 2\pi / \omega$;

proof :

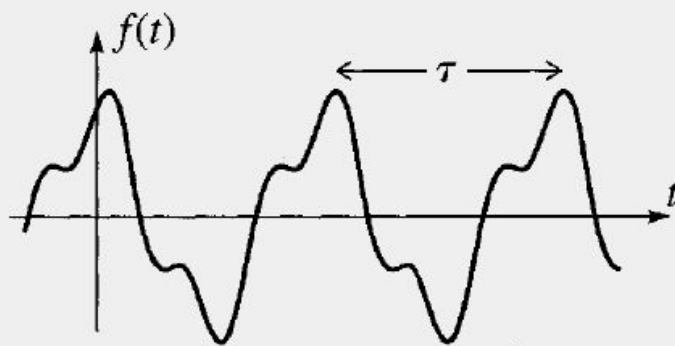
$$\begin{aligned} f(t + \tau) &= f_0 \cos[\omega(t + \tau)] \\ &= f_0 \cos[\omega t + 2\pi] \\ &= f_0 \cos \omega t = f(t). \end{aligned}$$

But a periodic function is not necessarily harmonic.

Figure 5.20 . Two examples of periodic functions



(a)



(b)

Figure 5.20 Two examples of periodic functions with period τ . **(a)** A rectangular pulse, which could represent a hammer hitting a nail with a constant force at intervals of τ , or a digital signal in a telephone line. **(b)** A smooth periodic signal, which could be the pressure variation of a musical instrument.

Motivation

To solve :

$$D^2 x(t) = f(t)$$

where $D^2 x = \ddot{x} + 2\beta \dot{x} + \omega_0^2 x$

and $f(t) = F(t)/m$ is periodic

$$f(t + \tau) = f(t).$$

■ We know the solution if $f(t)$ is harmonic (*from the previous lecture*).

■ Now consider $f = f_1 + f_2$;
then $x = x_1 + x_2$, because the equation is linear.

■ So, if $f(t)$ is a superposition of harmonic functions, then $x(t)$ is the superposition of corresponding solutions, which we already know.

■ Fourier's theorem

If $f(t)$ is a periodic function, then it can be written as a superposition of harmonic functions,

$$f(t) = \sum_{n=0}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

where $\omega = 2\pi/\tau$.

Easy exercise:: Prove that $f(t)$ is periodic.

Figure 5.21

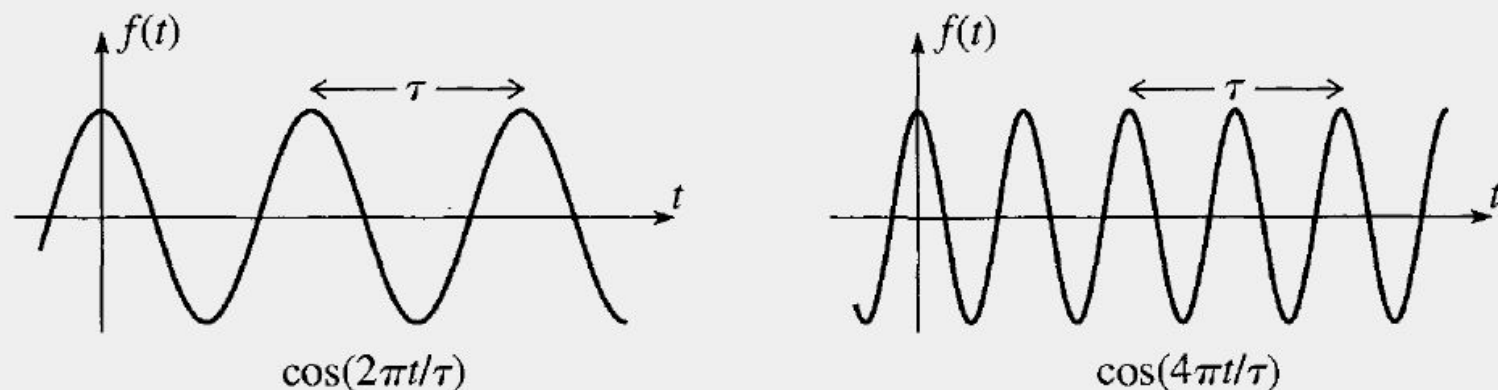


Figure 5.21 Any function of the form $\cos(2n\pi t/\tau)$ (or the corresponding sine) is periodic with period τ if n is an integer. Notice that $\cos(4\pi t/\tau)$ also has the smaller period $\tau/2$, but this doesn't change the fact that it has period τ as well.

■ Fourier's theorem

If $f(t)$ is a periodic function, then it can be written as a superposition of harmonic functions.

$$f(t) = \sum_{n=0}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

where $\omega = 2\pi / \tau$.

Proof: Take a math course.

Now, given $f(t)$ what are the coefficients a_n and b_n ?

Simplifications.

✖ If $f(t)$ is an even function of t , i.e., $f(-t) = f(t)$, then

$$b_1 = 0 = b_2 = b_3 = \dots b_n = \dots = 0.$$

✖ If $f(t)$ is an odd function of t , i.e., $f(-t) = -f(t)$, then

$$a_0 = 0 = a_1 = a_2 = \dots a_n = \dots = 0.$$

✖ Today, assume $f(t)$ is even.

Example 5.4 :

periodic rectangular pulses

Figure 5.22

Important: understand that this $f(t)$ is an even function of t .

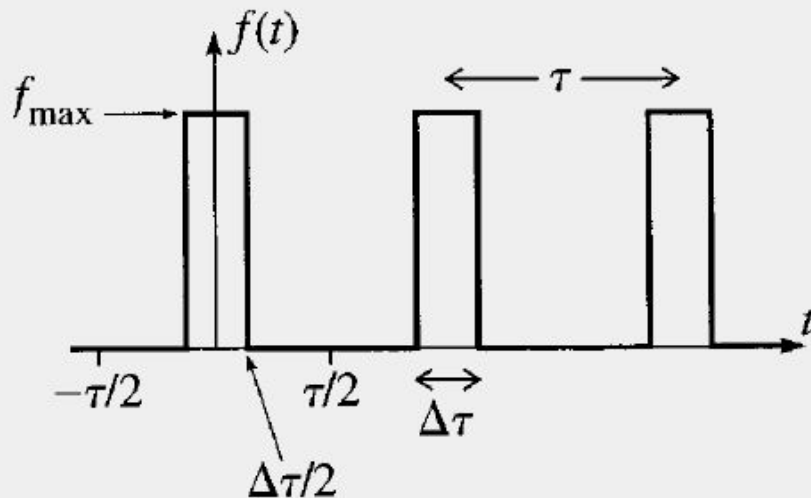


Figure 5.22 A periodic rectangular pulse. The period is τ , the duration of the pulse is $\Delta\tau$, and the pulse height is f_{\max} .

The Fourier coefficients for an even periodic function ...

$$f(t) = \sum_{n=0}^{\infty} a_n \cos(n\omega t) \quad \text{where } \omega = 2\pi/\tau.$$

↑ even function of t ; $f(-t) = f(+t)$.

Now consider

$$\begin{aligned} & \int_{-\tau/2}^{+\tau/2} f(t) \cos(n\omega t) dt \\ &= \sum_{n'=0}^{\infty} a_{n'} \int_{-\tau/2}^{+\tau/2} \cos(n'\omega t) \cos(n\omega t) dt \\ & \quad \rightarrow = \int_{-\tau/2}^{+\tau/2} \left\{ \cos[(n'+n)\omega t] + \cos[(n'-n)\omega t] \right\} \frac{dt}{2} \\ &= \frac{1}{(n'+n)\omega} \sin[(n'+n)\omega t] + \frac{1}{(n'-n)\omega} \sin[(n'-n)\omega t] \bigg|_{-\tau/2}^{+\tau/2} \cdot \frac{1}{2} \\ &= \frac{\sin[(n'+n)\pi]}{(n'+n)\omega} + \frac{\sin[(n'-n)\pi]}{(n'-n)\omega} \end{aligned}$$

$$= 0 + \begin{cases} 0 & \text{if } n' \neq n \\ \int_{-\tau/2}^{+\tau/2} \cos^2(n\omega t) dt & \text{if } n' = n \end{cases}$$

$$\int_{-\tau/2}^{+\tau/2} \cos^2(n\omega t) dt = \tau \quad \text{if } n=0$$

$$\rightarrow = \frac{1}{2}\tau \quad \text{if } n \neq 0$$

$$\text{So, } \int_{-\tau/2}^{+\tau/2} f(t) \cos(n\omega t) dt$$

$$= a_n \left[\tau \delta_{n,0} + \frac{\tau}{2} (1 - \delta_{n,0}) \right]$$

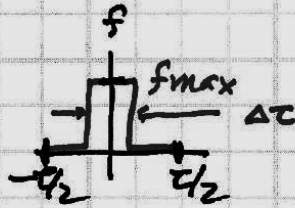
$$= \begin{cases} a_n \tau & \text{if } n=0 \\ a_n \tau/2 & \text{if } n \neq 0 \end{cases}$$

$$a_0 = \frac{1}{\tau} \int_{-\tau/2}^{+\tau/2} f(t) dt$$

$$a_n = \frac{2}{\tau} \int_{-\tau/2}^{+\tau/2} f(t) \cos(n\omega t) dt \quad \text{for } n > 0.$$

For ANY
EVEN, PERIODIC
FUNCTION

The periodic rectangular pulse



$$a_0 = \frac{1}{\tau} \int_{-\Delta\tau/2}^{\Delta\tau/2} f_{\max} dt = \frac{f_{\max} \Delta\tau}{\tau}$$

$$a_n = \frac{2}{\tau} \int_{-\Delta\tau/2}^{\Delta\tau/2} f_{\max} \cos(n\omega t) dt$$

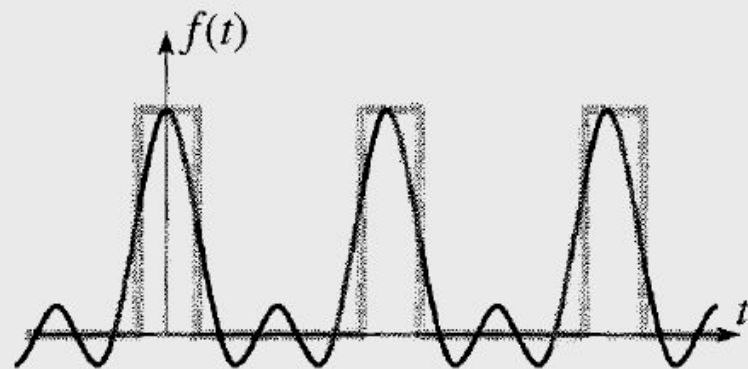
$$= \frac{2f_{\max}}{\tau} \left. \frac{1}{n\omega} \sin(n\omega t) \right|_{-\Delta\tau/2}^{\Delta\tau/2}$$

$$= \frac{4f_{\max}}{n\omega\tau} \sin\left(n\omega \frac{\Delta\tau}{2}\right) = \frac{2f_{\max}}{n\pi} \sin\left(n\pi \frac{\Delta\tau}{\tau}\right)$$

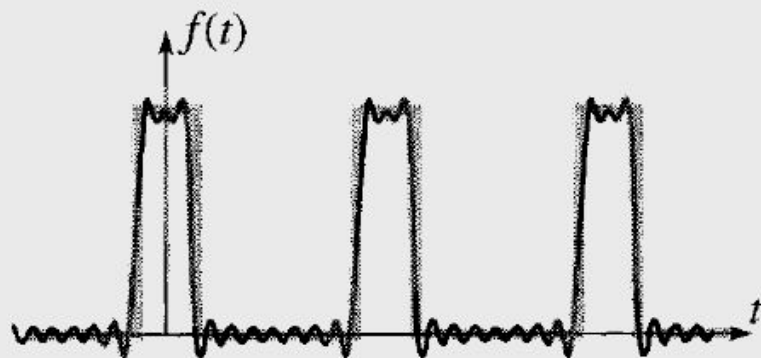
$$f(t) = \sum_{n=0}^{\infty} a_n \cos(n\omega t) \quad \text{where } \omega = 2\pi/\tau.$$

Let's look at that for $\Delta\tau = 0.25\tau$

Figure 5.23 : The Fourier series for the periodic rectangular pulse, truncated to (a) 3 terms , and (b) 11 terms. As $N \rightarrow \infty$, the series approaches f .



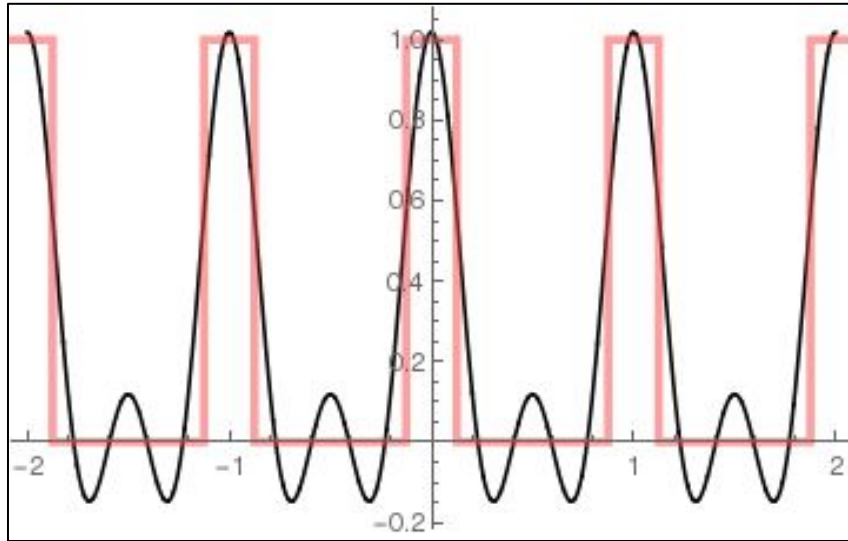
(a) 3 terms



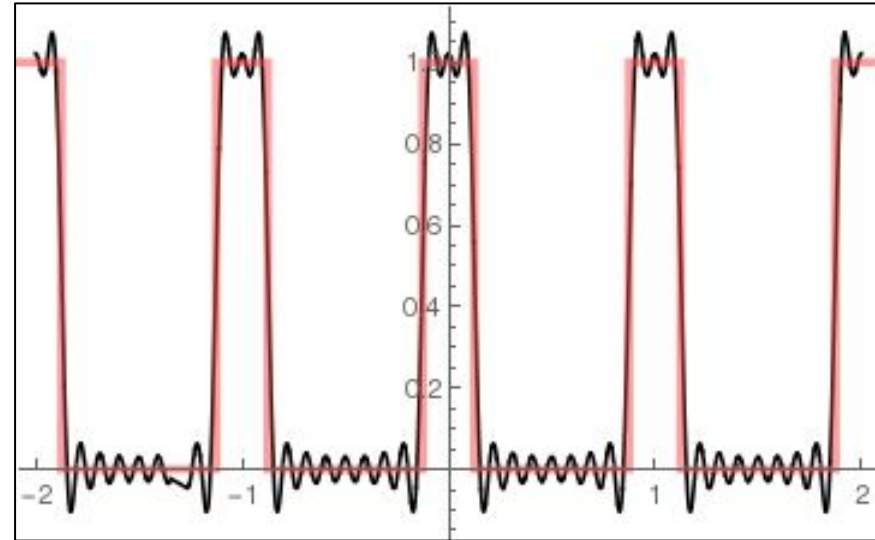
(b) 11 terms

Figure 5.23 (a) The sum of the first three terms of the Fourier series for the rectangular pulse of Figure 5.22. (b) The sum of the first 11 terms.

Fourier series truncated at $n = 2$



truncated at $n = 10$



Comments:

- (1) as $N \rightarrow \infty$ the Fourier series approaches the function $f(t)$;
- (2) at a discontinuity, the *truncated* Fourier series can't reproduce the discontinuity.

Preview of Section 5.8.

Fourier Series Solution for the Driven Oscillator

To solve, $D^2 x = f$.

Let's just obtain the *steady-state solution*; i.e., the particular solution that $x(t)$ approaches as $t \rightarrow \infty$.

We have $f(t) = \sum_n a_n \cos(n\omega t)$
(assuming $f(t)$ is even in t)

By the superposition principle,
$$x(t) = \sum_n a_n A_n \cos [n\omega(t - \delta_n)] .$$

Homework Assignment #10

due in class Friday November 11

[47] Problem 4.53

[48] Problem 5.25 **

[49] Problem 5.30 **

[50] Problem 5.37 **

[50x] Problem 5.44 **

[50xx] Problem 5.52 *** [Computer]

Exam 2 will be Friday November 4.

Make sure you understand:

- (1) Problem [41x] and similar
- (2) Lecture from Friday October 28
- (3) Conservation of Energy

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