4.4

1^{st} pages → orthonormal good!

Main reason: makes it easier to find $\vec{p} = A\hat{x}$

orthonormal: if a set of vectors is orthonormal that means the vectors are orthogonal to each other and normal (unit vectors, length is 1)

orthogonal: $v_1, \ldots, v_n$

$v_i \cdot v_j = 0$ if $i \neq j$

normal: $\|v_i\| = 1$ or $v_i \cdot v_i = 1$

If the columns of A are orthonormal, then $A^T A = I$

Book uses Q to represent an orthonormal matrix

* Permutations

Permutation matrix, ~~swit~~ Rows of I are swapped

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \leftarrow \text{orthonormal matrix}$$

Remember: $P^T = P^{-1}$ for permutation matrices

<u>Goal:</u> Given matrix A, normalize and orthogonalize the columns

- this makes least square solution easier to find

$A\vec{x} = \vec{b}$ ← use least squares when no solution ($\vec{b}$ is not in the col. space of A)

$A\hat{x} = \vec{b}$ $\quad$ $\hat{x}$ → projection of $\vec{b}$ onto A, closest thing to $\vec{b}$ in C(A)

$$A^T A \hat{x} = A^T \vec{b}$$

$$(A^T A)^{-1} A^T A \hat{x} = (A^T A)^{-1} A^T \vec{b}$$

$$\hat{x} = (A^T A)^{-1} A^T \vec{b}$$

$$\boxed{\vec{P}} = A\hat{x} = \underbrace{A(A^T A)^{-1} A^T}_{P} \vec{b}$$

projection matrix

If instead we have $Q\hat{x} = \vec{b}$

Q is orthonormal

then $$Q^T Q \vec{x} = Q^T \vec{b}$$
$$\hat{x} = Q^T \vec{b}$$
$$Q\hat{x} = Q Q^T \vec{b}$$

Projection matrix is $Q Q^T = P$

before: Proj. matrix was $\underbrace{A(A^TA)^{-1}A^T}_{P}$

use Gram-Schmidt

- Start with 3 linearly independent vectors $\vec{a}, \vec{b}, \vec{c}$

- We want to construct 3 independent, orthonormal vectors $\vec{A}, \vec{B}, \vec{C}$

Start with $\vec{a} = \vec{A}$ (normalize at end)
First direction is accepted

Then $\vec{B}$ must be orthogonal to $\vec{A}$

$$\vec{B} = \vec{b} - \text{proj}_{\vec{A}} \vec{b}$$

$$= \vec{b} - \underbrace{\frac{\vec{A}^T \vec{b}}{\vec{A}^T \vec{A}}}_{\text{scalar}} \underbrace{\vec{A}}_{\text{vectr}}$$

$= \vec{b} - ($part of $\vec{b}$ that goes in the direction of $\vec{A})$

Now, $\vec{c}$ must be orthogonal to $\vec{A} + \vec{B}$

$$\vec{C} = \vec{c} - \text{proj}_{\vec{A}} \vec{c} - \text{proj}_{\vec{B}} \vec{c}$$

$$\vec{C} = \vec{c} - \frac{\vec{A}^T \vec{c}}{\vec{A}^T \vec{A}} \vec{A} - \frac{\vec{B}^T \vec{c}}{\vec{B}^T \vec{B}} \vec{B}$$

orthonormal set of vectors

$$\frac{\vec{A}}{\|\vec{A}\|}, \quad \frac{\vec{B}}{\|\vec{B}\|}, \quad \frac{\vec{C}}{\|\vec{C}\|} \quad \leftarrow \text{big C}$$