Chapter 3 : Vector Spaces & Subspaces

* To understand everything about $A\vec{x}=\vec{b}$ look at vector spaces & spaces

def: The space $\mathbb{R}^n$ consists of all column vectors $\vec{v}$ with n components.

Ex) $\mathbb{R}^2$ ← vector space consists of all column vectors $\vec{v}$ with 2 components

Vector Space Axioms

Let V be a set on which the operations of addition and scalar multiplication are defined.

closure properties

① if $\vec{x} \in V$ and α is a scalar, then $\alpha\vec{x} \in V$

$\in$ belongs

② if $\vec{x}$ and $\vec{y}$ belong to V, then $\vec{x}+\vec{y} \in V$

The set V together with the operations of addition and scalar multiplication is said to form a <u>vector space</u>, if the following axioms are satisfied:

$\forall$ for all

A1. $\vec{x}+\vec{y}=\vec{y}+\vec{x}$ $\quad \forall \underbrace{\vec{x},\vec{y}\in V}_{\text{for all } \vec{x}+\vec{y} \text{ belonging to } V}$

A2. $(\vec{x}+\vec{y})+\vec{z}=\vec{x}+(\vec{y}+\vec{z})$ $\quad \forall \vec{x},\vec{y},\vec{z}\in V$

A3. There exists a zero vector in V
$\exists\, \vec{0}\in V$ such that
$\vec{0}+\vec{x}=\vec{x}$ $\quad \forall \vec{x}\in V$

A4. $\forall \vec{x}\in V$, $\exists\, -\vec{x}\in V \ni \vec{x}+(-\vec{x})=\vec{0}$

Such that

A5. $\alpha(\vec{x}+\vec{y})=\alpha\vec{x}+\alpha\vec{y}$ $\quad \forall \vec{x},\vec{y}\in V$ + any scalar α

A6. $(\alpha+\beta)\vec{x}=\alpha\vec{x}+\beta\vec{x}$ $\quad \forall \vec{x}\in V$ + any scalar α,β

A7. $\alpha(\beta\vec{x})=(\alpha\beta)\vec{x}$ $\quad \forall \vec{x}\in V$, + any scalars α,β

A8. $1\cdot\vec{x}=\vec{x}$ $\quad \forall \vec{x}\in V$

Ex 1 $W = \{(a,1) \mid a \text{ is real}\}$

W is the set of all vectors of the form $(a,1)$ where a is a real #

$(5,1) \in W$ $(3,-1) \notin W$

Is W a vector space?

<u>NO!!</u> $(5,1)$ & $(8,1)$ both are elements of W

$$(5,1) + (8,1) = (13, 2)$$

$(13,2) \notin W$ so W is <u>not</u> a vector space

<u>Three vector spaces other than $\mathbb{R}^n$</u>

M = vector space of all real 2x2 matrices

$$M = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \; a,b,c,d \in \mathbb{R} \right\}$$

F = vector space of all real functions

Z = vector space that consists only of a zero vector

Ex) Let S be the set of all ordered pairs of real numbers. Define scalar multiplication and addition on S by

$$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$$

$$\boxed{(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, 0)}$$

We use $\oplus$ to denote addition in S to not confuse addition with $+$.

Show that S with scalar multiplication and addition operation $\oplus$ is <u>not</u> a vector space.

A3: $\vec{0} + \vec{x} = \vec{x}$

$\vec{x} = (5,5)$

$$(5,5) \oplus (0,0) = (5+0, 0) = (5,0) \neq (5,5)$$

Ex) $S = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$

mult: $\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$

addition: $(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_2, x_2 + y_1)$

Fail: A4: $\vec{x} + (-\vec{x}) = \vec{0}$

$(1,2) \oplus m(-1,-2) = (1-2, 2-1) \neq \vec{0}$

<u>Subspaces</u> Given a vector space V, it is often possible to form another vector space by taking a subset S of V and using the operations of V.

<u>Ex</u> Let $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \ x_2 = 2x_1 \right\}$. S is a subspace of $\mathbb{R}^2$.

If $\begin{bmatrix} c \\ 2c \end{bmatrix}$ is any element of S and α is a scalar, determine if closure properties hold.

① scalar mult. show $\alpha \begin{bmatrix} c \\ 2c \end{bmatrix} \in S$

$$\alpha \begin{bmatrix} c \\ 2c \end{bmatrix} = \begin{bmatrix} \alpha c \\ 2(\alpha c) \end{bmatrix} \in S$$

② Vector addition.

$$\begin{bmatrix} c \\ 2c \end{bmatrix} + \begin{bmatrix} b \\ 2b \end{bmatrix} \in S \ ??$$

$$\underbrace{\begin{bmatrix} c \\ 2c \end{bmatrix} + \begin{bmatrix} b \\ 2b \end{bmatrix}}_{\text{adding 2 general vectors in } S} = \begin{bmatrix} c+b \\ 2c+2b \end{bmatrix} = \underbrace{\begin{bmatrix} c+b \\ 2(c+b) \end{bmatrix}}_{\text{gives a vector in } S}$$

S is a vector space, and a subspace of $\mathbb{R}^2$.

Ex) Let $S = \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} \middle| x \in \mathbb{R} \right\}$

Is a S a subspace of $\mathbb{R}^2$??

$$\alpha \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha \end{pmatrix} \notin S \text{ unless } \alpha = 1$$

Not a subspace

In $\mathbb{R}^2$ $\alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}$

$$2 \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 2 \end{pmatrix} \notin S$$