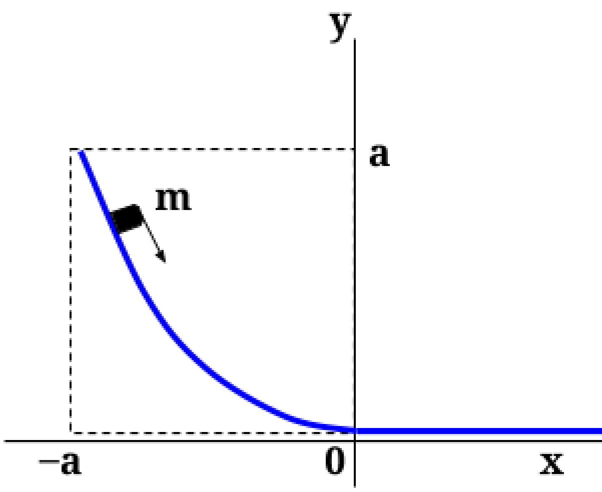
Homework Assignment 9 Name____________________________

due Friday, November 4

Cover sheet : Staple this page in front of your solutions, with answers where indicated.

[41] Problem 4.41 and Problem 4.43

(No answer required here.)



[41x] A mass m slides without friction in Earth's gravity down the track shown in the figure; the equation for the track is $y = x^2/a$ for $x < 0$ and $y = 0$ for $x > 0$. The initial point is $\{x,y\}=\{-a, a\}$ and the initial velocity is 0.

(A) Calculate y' when the height is y, in the form y' = f(y).

(B) Calculate the time when the mass passes the point $\{x,y\}=\{0,0\}$.

Answer: The time in part (B) is

time = 1.874 sqrt(a/g)

---

[42] Problem 5.3.*

Answer: The parameter k is ***k = m g l***

---

[43] Problem 5.5.*

Answer: Express C in terms of B_1 and B_2 ***$C = Sqrt[B_1^2+B_2^2] (B_1/A - i B_2/A) = B_1 - i B_2$***

---

[44] Problem 5.9.*

…Answer: The period is ***1.047 s***

---

[45] Problem 5.12.**

(No answer is required here.)

---

[46] Problem 5.18.*** *Assume $a < l_0$. Show that $\{x,y\}=\{0,0\}$ is an unstable equilibrium, and explain why.*

The coefficient of y^2 is $k(1-l_0/a)$, which is negative if $a < l_0$.
So if the mass moves along the y axis, the potential energy decreases;
i.e., the point {0,0} is an unstable equilibrium.

Homework Assignment #9

[41x] Problem 4.41 and Problem 4.43

$U = k r^n \quad \Rightarrow \quad F_r = -dU/dr = -n k r^{n-1}$

For circular motion, $a_r = -v^2/r$; therefore, $\quad m v^2/r = n k r^{n-1}$.

$T = ½ m v^2 = ½ n k r^n = (n/2) U.$ (*virial theorem*)

Problem 4.43

(a) Given $\mathbf{F}(\mathbf{r}) = f(r)\, \boldsymbol{e}_r = (f(r)/r)\, \mathbf{r} = (f/r)\,(x\, \boldsymbol{e}_x + y\, \boldsymbol{e}_y + z\, \boldsymbol{e}_z)$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \partial_x & \partial_y & \partial_z \\ xf/r & yf/r & zf/r \end{vmatrix} = 0$$

; thus **F** is conservative.

(check this)

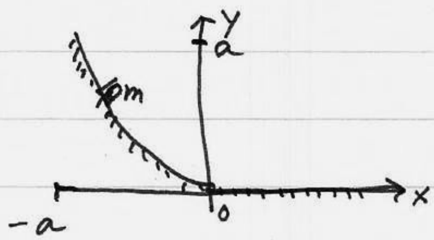
(b) In polar coordinates the curl is

$$= \hat{\mathbf{r}} \frac{1}{r \sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial}{\partial\phi}A_\theta\right] + \hat{\boldsymbol{\theta}}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}A_r - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right] + \hat{\boldsymbol{\phi}}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}A_r\right] \quad \text{[spherical polar]}$$

We have $F_\varphi = F_\theta = 0$; also $\partial F_r / \partial\varphi = \partial F_r / \partial\theta = 0$;

thus $\nabla \times \mathbf{F} = 0$.

Problem [41x]



Track: $y = x^2/a$ for $x < 0$

Initial point: $\{x_0, y_0\} = \{-a, a\}$.

(A) Energy is conserved, so $\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy = mga$

$$x = \sqrt{ay} \quad \text{so} \quad \frac{dx}{dt} = \sqrt{a}\,\tfrac{1}{2}y^{-1/2}\frac{dy}{dt} = \sqrt{\frac{a}{4y}}\,\dot{y}$$

Thus $\frac{1}{2}m\left(1 + \frac{a}{4y}\right)\dot{y}^2 + mgy = mga$

$$\dot{y}^2 = \frac{2g(a-y)}{(1 + a/4y)} = 2ga \cdot \frac{4y(1-y/a)}{4y+a} = 2ga\,\frac{y(1-y/a)}{y+a/4}$$

$$\dot{y} = -\sqrt{2ga}\left(\frac{y(1-y/a)}{y+a/4}\right)^{1/2}$$

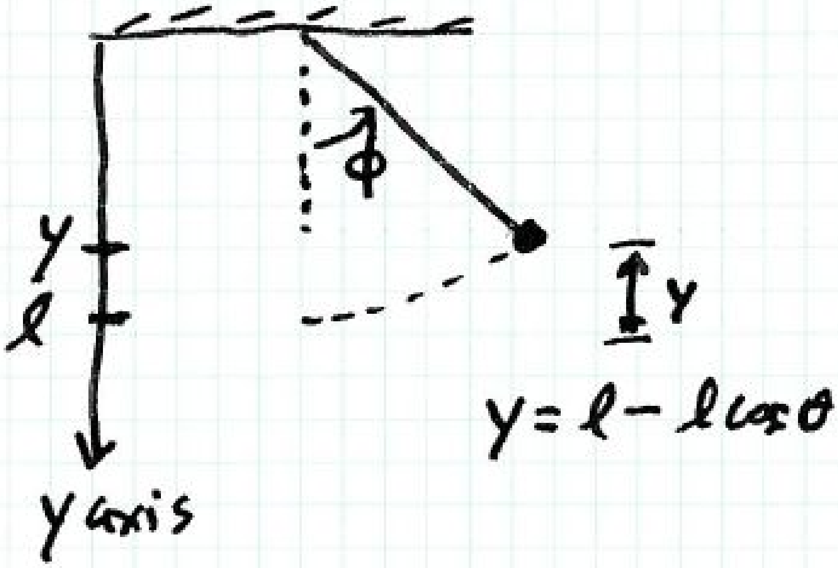
(B) $dt = \frac{dy}{\dot{y}} \Rightarrow \text{time} = \int_a^0 \frac{dy}{-\sqrt{2ga}\sqrt{\frac{y(1-y/a)}{y+a/4}}} = \frac{1}{\sqrt{2ga}}\int_0^a \frac{\sqrt{y+a/4}\,dy}{\sqrt{y(1-y/a)}}$

Let $y = a\xi$; $dy = a\,d\xi$

$$\text{time} = +\underbrace{\int_0^1 \frac{\sqrt{\xi + \frac{1}{4}}\,d\xi}{\sqrt{\xi(1-\xi)}}}_{\text{use Mathematica to do this integral} \;=\; 2.635}\;\frac{a}{\sqrt{2ga}}$$

$$\text{time} = \frac{2.635}{\sqrt{2}}\sqrt{\frac{a}{g}} = 1.874\sqrt{\frac{a}{g}}$$

[42] Problem 5.3 *



$U(\phi) = mgy = mgl(1-\cos\phi)$

For small ϕ, $\cos\phi \approx 1 - \frac{1}{2}\varphi^2$

$U \approx \frac{1}{2} mgl\,\phi^2 = \frac{1}{2} k\,\varphi^2$

where $K = mgl$.

[43] Problem 5.5 *

Given I : $x = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$

$$= (C_1 + C_2)\cos\omega t + i(C_1 - C_2)\sin\omega t$$

$$= B_1 \cos\omega t + B_2 \sin\omega t$$

$$\boxed{\begin{aligned} B_1 &= C_1 + C_2 \\ B_2 &= i(C_1 - C_2) \end{aligned}}$$

Given II : $x = B_1 \cos\omega t + B_2 \sin\omega t$

$$= A\cos\phi \cos\omega t + A\sin\phi \sin\omega t$$

$$= A\cos(\phi - \omega t)$$

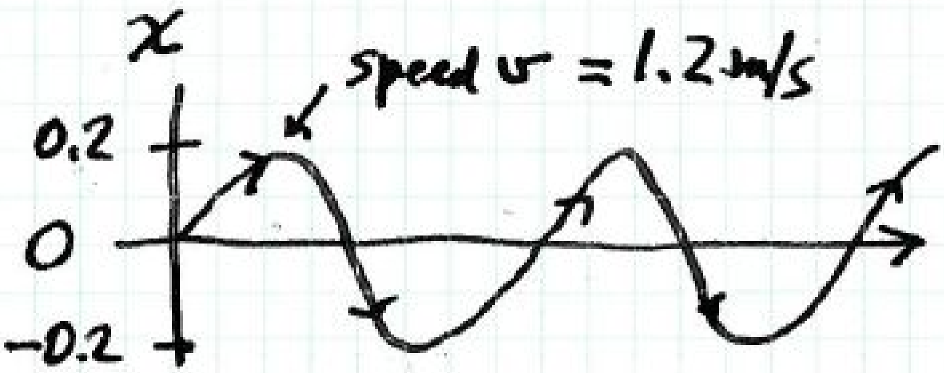
$A\cos\phi = B_1$ & $A\sin\phi = B_2$

Given III : $x = A\cos(\omega t - \phi)$

$$= A\,\mathrm{Re}\, e^{i\omega t} e^{-i\phi} = \mathrm{Re}\, C e^{i\omega t}$$

$C = A e^{-i\phi}$

[44] Problem 5.9 *



Energy is conserved, so

$$E = \frac{1}{2} m v^2 = \frac{1}{2} k A^2$$

The period is

$$\tau = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}}$$

$$\tau = 2\pi \frac{A}{v} = 1.047 \text{ sec.}$$

[45] Problem 5.12 **

Define $\langle f \rangle = \frac{1}{\tau}\int_0^\tau f(t)\,dt$.

Say $x = A\cos\omega t$ and $v = -A\omega\sin\omega t$.

$$\langle T \rangle = \frac{1}{\tau}\int_0^\tau \frac{1}{2}mv^2\,dt$$

$$= \frac{\omega}{2\pi}\frac{m}{2}A^2\omega^2 \underbrace{\int_0^\tau \sin^2\omega t\,dt}_{=\frac{1}{2}\tau\,=\,\pi/\omega}$$

$$\langle T \rangle = \frac{1}{4}mA^2\omega^2 = \frac{1}{4}mA^2\frac{k}{m}$$

$$\langle T \rangle = \frac{1}{4}kA^2 = E/2 \quad \checkmark$$

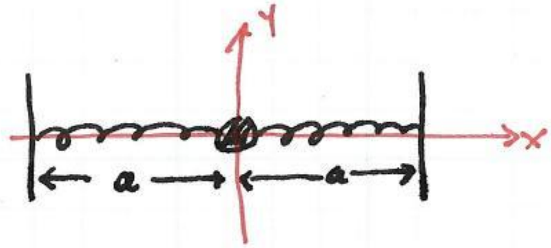
$$\langle U \rangle = \frac{1}{\tau}\int_0^\tau \frac{1}{2}kx^2\,dt \quad \checkmark$$

$$= \frac{\omega}{2\pi}\frac{k}{2}A^2\,\tau/2 = \frac{1}{4}kA^2 = E/2$$

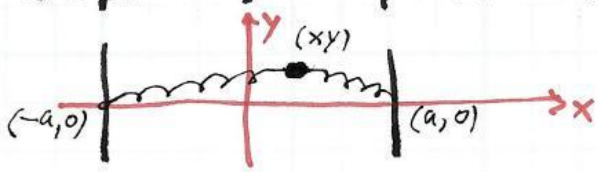
⇧

[46] Problem 5.18 *** 9.7

Equilibrium



Displaced from Equilibrium



$$U = \frac{1}{2}k\Big[\underbrace{\sqrt{(x+a)^2+y^2}}\ - l_0\Big]^2 + \frac{1}{2}k\Big[\sqrt{(x-a)^2+y^2} - l_0\Big]^2$$

$\approx \sqrt{a^2+2ax+x^2+y^2}$ for small x and y

$= a\sqrt{1+\frac{2x}{a}+\frac{x^2+y^2}{a^2}} = a\sqrt{1+\epsilon}$

$\approx a\left(1+\frac{1}{2}\epsilon-\frac{1}{8}\epsilon^2\right)$ by Taylor's theorem

$\approx a\left[1+\frac{x}{a}+\frac{x^2+y^2}{2a^2}-\frac{1}{8}\left(\frac{2x}{a}\right)^2\right]$

$= a + x + y^2/2a$

$$U = \frac{k}{2}\left[a + x + \frac{y^2}{2a} - l_0\right]^2 + \frac{k}{2}\left[a - x + \frac{y^2}{2a} - l_0\right]^2$$

$$\approx \frac{k}{2}\left\{2(a-l_0)^2 + 4(a-l_0)\frac{y^2}{2a} + 2x^2\right\}$$

$$= \text{constant} + \frac{1}{2}(k)x^2 + \frac{1}{2}\left(k[1-l_0/a]\right)y^2$$ ← form of (5.104)

- Now suppose a < l_0 , i.e., the two springs are compressed in the equilibrium configuration.
  Then the coefficient of y^2 = k (1 – l_0/a) *is negative*;
  so the equilibrium is unstable. Why? If the mass moves up or down the y axis, then the springs decompress and the potential energy decreases.