Determinants

* Key: det A = 0, then A is not invertible

determinant of a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

* Note: only square matrices have determinants

det of 3x3 matrix

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$
$$- a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

$$A = \begin{array}{c} \begin{array}{ccccc} & & - & - & - \end{array} \\ \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right] \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{array} \\ \begin{array}{ccccc} & & + & + & + \end{array} \end{array}$$

Ex) Let $A = \begin{bmatrix} 5 & -3 & 2 \\ 1 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix}$

then $\det(A) = ?$

$$\begin{array}{c} \begin{array}{ccc} -0 & +10 & +9 \end{array} \\ \left[\begin{array}{ccc} 5 & -3 & 2 \\ 1 & 0 & 2 \\ 2 & 1 & 3 \end{array}\right] \begin{array}{cc} 5 & -3 \\ 1 & 0 \\ 2 & 1 \end{array} \\ \begin{array}{ccc} 0 & -12 & -2 \end{array} \end{array}$$

$\det A = 0 + 10 + 9 + 0 - 12 + -2 =$

$19 - 12 - 2 = 5$ ✓

Cofactor Expansion

(i,j) - cofactor of A is

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

The Laplace Expansion Theorem

The determinant of an $n \times n$ matrix $A = [a_{ij}]$ where $n \geq 2$ can be computed as

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

$$= \sum_{j=1}^{n} a_{ij} C_{ij}$$

(cofactor expansion along the i^{th} row)

& also

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

$$= \sum_{i=1}^{n} a_{ij} C_{ij}$$

(cofactor expansion along j^{th} column)

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

$A_{ij} \rightarrow$ submatrix of A where we've deleted row i + col. j

Ex) $\begin{bmatrix} 5 & -3 & 2 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix} = A$ $\quad A_{12} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

deleted col 2 + row 1

Using cofactor expansion along row 1

$$\sum_{j=1}^{3} a_{1j} C_{1j} = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

$$= a_{11} (-1)^{1+1} \det A_{11} + a_{12} (-1)^{1+2} \det (A_{12})$$

$$+ a_{13} (-1)^{1+3} \det (A_{13})$$

$$= 5 \begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix} + (-3)(-1) \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + 2 \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = 5$$

can expand on Row 2 or Col 2 (easiest because of the zero)

Ex) Compute the determinant of

$$A = \begin{bmatrix} 2 & -3 & 0 & 1 \\ 5 & 4 & 2 & 0 \\ +1 & -1 & 0 & 3 \\ -2 & 1 & 0 & 0 \end{bmatrix}$$

use 3rd column

$$0 \cdot (-1)^{1+3} \det(A_{13}) + 2(-1)^{2+3} \det(A_{23}) + 0 + 0$$

$$-2 \begin{vmatrix} 2 & -3 & 1 \\ 1 & -1 & 3 \\ -2 & 1 & 0 \end{vmatrix}$$

$$= -2\left(-2(-1)^{3+1} \begin{vmatrix} -3 & 1 \\ -1 & 3 \end{vmatrix} + 1(-1)^{3+2} \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \right)$$

$$= -22$$

Ex)

$$\begin{bmatrix} 2 & 4 \\ 6 & 3 \end{bmatrix} \quad R_2 \to \frac{1}{2}R_1 + \ldots$$

$$R_2 \to \frac{1}{2}R_1 + 3R_2$$

Properties

* the determinant of a triangular matrix is the product of its entries along the main diagonal

ex) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & -5 \end{bmatrix}$

$\det A = 1 \cdot 2 \cdot -5 = -10$

Cramer's Rule

If $A\vec{x}=\vec{b}$ is a system of linear equations in n unknowns such that $\det(A)\neq 0$ then the system has a unique solution

This solution is

$$x_1 = \frac{\det(A_1)}{\det(A)} \quad \cdots\cdots \quad x_n = \frac{\det(A_n)}{\det(A)}$$

Where A_j is the matrix obtained by replacing the entries in the j^{th} ~~column~~ column of A by the entries in the matrix

$$\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Sample Test question: Find x using Cramer's Rule.

$$\begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 30 \\ 8 \end{bmatrix}$$

$$X = \frac{\begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & 2 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix}}$$

use R_1 for cofactor expansion

$$\begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & 2 & 3 \end{vmatrix} = 6\begin{vmatrix} 4 & 6 \\ 2 & 3 \end{vmatrix} - 0\begin{vmatrix} & \\ & \end{vmatrix} + 2\begin{vmatrix} 30 & 4 \\ 8 & 2 \end{vmatrix}$$

Name: ____________________

Please work together to solve the problems. Do not be afraid to ask questions!

Properties

Let A be an $n \times n$ matrix.

(a) If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then det$B$=$k$det$A$.

(b) If B is the matrix that results when two rows or two columns of A are interchanged, then det$B$=-det$A$.

(c) If B is the matrix that results when a multiple of one row of A is added to another row or when a multiple of one column is added to another column, then det$B$=det$A$.

Properties

Let E be an $n \times n$ matrix.

(a) If E results from multiplying a row of I_n by a nonzero number k, then det$E$=k.

(b) If E results from interchanging two rows of I_n, then det$E$=-1.

(c) If E results from adding a multiple of one row of I_n to another, then det$E$=1.

---

1. True or False

   (a) If A is a 3×3 matrix, then det$2A$=2det$A$. FALSE

   (b) A square matrix is invertible if and only if det$A$=0. FALSE

   (c) If a square matrix for the linear system $A\mathbf{x} = \mathbf{b}$ has multiple solutions for $\mathbf{x}$, the det$A$=0. TRUE

   (d) If E is an elementary matrix, then $E\mathbf{x} = \mathbf{0}$ has only the trivial solution. TRUE

2. Verify that det(kA)=k^ndet(A), where A is $n \times n$.

   $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$, $k = 2$

3. Use row reduction to evaluate the determinant of A.

   $A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$

4. Use Cramer's Rule to solve

   $7x - 2y = 3$

   $3x + y = 5$

5. Find the area of a triangle with sides $(3, 2), (1, 4)$ and $(4, 6)$ Draw it.

6. The parallelogram with sides $(2, 1)$ and $(2, 3)$ has the same area as the parallelogram with sides $(2, 2)$ and $(1, 3)$. Find those areas with 2 by 2 determinants and say why they must be equal.

(2)

$A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$ $k=2$ $\det A = -4 - 6 = -10$

$kA = \begin{bmatrix} -2 & 4 \\ 6 & 8 \end{bmatrix}$ $\det kA = -16 - 24 = -40$

yes, $\det 2A = -40$ and $\det A$ is -10, so $-10 \cdot 2^2 = -40$ ✓

(3) $A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$ $\det A = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix}$

$= - \begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$

exchange Rows

$= -3 \begin{vmatrix} 1 & -2 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$

multiply R_1 by $\frac{1}{3}$

$= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$

add $-2R_1$ to R_1, no change in determinant

$$= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$ $-10R_2 + R_3$ no change in determinant

$$= (-3)(-55) \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix}$$ $\frac{1}{55} R_3$, determinant of new matrix is 55 det (old)

$$= (-3)(-55)(1)$$

det of new is product of pivots/diagonal

$$= 165$$

$$\det(A) = 165$$

(4)

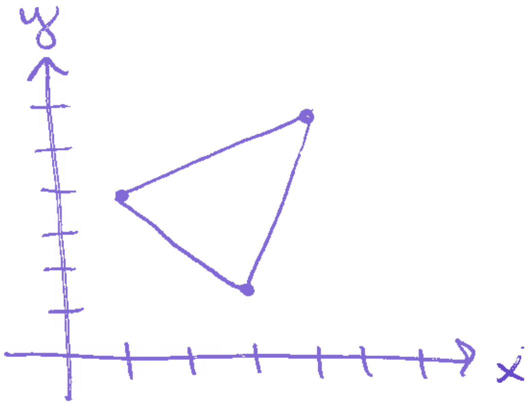
$$7x - 2y = 3$$
$$3x + y = 5$$

$$\begin{bmatrix} 7 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} 3 & -2 \\ 5 & 1 \end{vmatrix}}{\begin{vmatrix} 7 & -2 \\ 3 & 1 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} 7 & 3 \\ 3 & 5 \end{vmatrix}}{\begin{vmatrix} 7 & -2 \\ 3 & 1 \end{vmatrix}} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

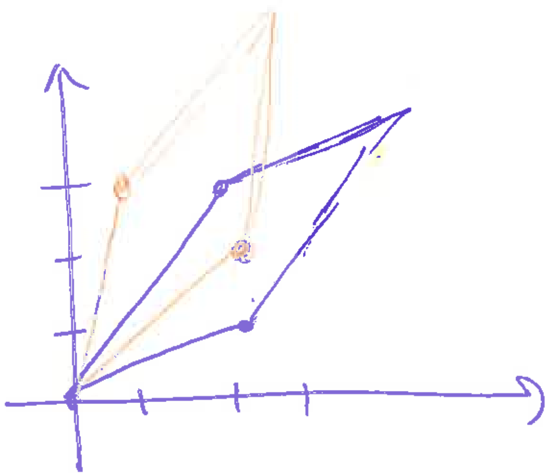
$$x = \frac{3+10}{7+6} = 1 \qquad y = \frac{35-9}{13} = 2$$

5



$$A_{area} = \frac{1}{2}\begin{vmatrix} 1 & 4 & 1 \\ 3 & 2 & 1 \\ 4 & 6 & 1 \end{vmatrix} = 5$$

6



area is

$$\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 6-2 = 4$$

$$\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 6-2 = 4$$