Section 2.5.

Motion of a Charged Particle in a Magnetic Field

The magnetic force on a charged particle is the *Lorentz force*,

$$\mathbf{F} = q\,\boldsymbol{v} \times \mathbf{B}\,. \qquad (1)$$

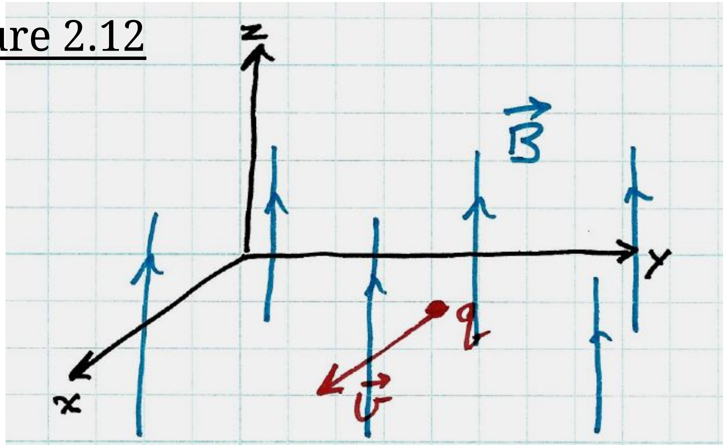
Here **B** is the magnetic field. (PHY 184)

In general, $\mathbf{B} = \mathbf{B}(\mathbf{r},t)$;
in eq. (1) **B** means the field at the position of the charged particle.

We'll keep it simple, and assume that **B** is uniform in space and constant in time.

Read Section 2.5.

Figure 2.12



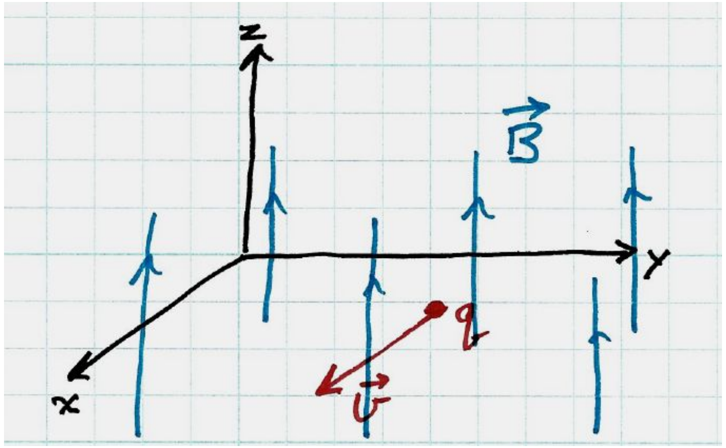
Charge q moves with velocity $\boldsymbol{v}$ in a magnetic field **B.**
Calculate the trajectory.
So, the goal is to solve this equation of motion,

$$m\,\frac{d\mathbf{v}}{dt} = q\,\mathbf{v} \times \mathbf{B}$$

Cartesian coordinates

Assume that **B** is uniform and constant.

Set up a coordinate system such the the z axis is the direction of **B**.



$$\mathbf{B} = B\, \boldsymbol{e}_z$$

The equations of motion

$$m \frac{d\mathbf{v}}{dt} = q\, \mathbf{v} \times \mathbf{B}$$

$$m\, \boldsymbol{v} = m \{ v_x , v_y , v_z \}$$

$$m\, \boldsymbol{v}' = m \{ v_x' , v_y' , v_z' \}$$ *(prime ' means d/dt)*

$$q\, \mathbf{v} \times \mathbf{B} = q \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix} \text{(det)}$$

$$= q \left(\hat{e}_x v_y B - \hat{e}_y v_x B \right)$$

$$q\, \mathbf{v} \times \mathbf{B} = q \{ v_y B , -v_x B , 0 \}$$

Solutions

The z component

$$m \dot{v}_z = 0$$

$$v_z = v_{0z}, \quad \text{constant}$$

$$z(t) = z_0 + v_{0z} t$$

The transverse components

$$m \dot{v}_x = q B v_y$$

$$m \dot{v}_y = -q B v_x$$

The cyclotron frequency

$$\dot{v}_x = +\frac{qB}{m} v_y = +\omega v_y$$

$$\text{where} \quad \omega = qB/m$$

$$\ddot{v}_x = +\omega \dot{v}_y = -\omega^2 v_x$$

$$v_x = c_1 \cos\omega t + c_2 \sin\omega t$$

$$v_y = \frac{1}{\omega} \dot{v}_x = -c_1 \sin\omega t + c_2 \cos\omega t$$

$\vec{v}$ sweeps out a circle of radius $\sqrt{c_1^2 + c_2^2}$.

Results

$$v_z = \text{const.}\ ; \quad z = v_{0z}\, t$$

$$v_x = c_1 \cos\omega t + c_2 \sin\omega t$$

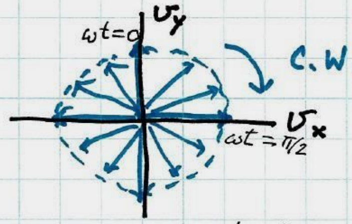
$$x = \frac{c_1}{\omega}\sin\omega t - \frac{c_2}{\omega}\cos\omega t$$

$$v_y = -c_1 \sin\omega t + c_2 \cos\omega t$$

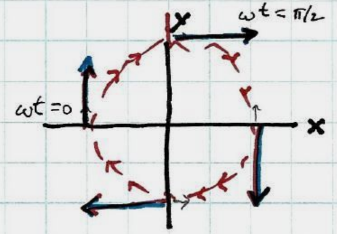
$$y = \frac{c_1}{\omega}\cos\omega t + \frac{c_2}{\omega}\sin\omega t$$

Assume $v_x(0) = 0$; then $c_1 = 0$

$$\vec{v} = c_2\left[\,\hat{e}_x \sin\omega t + \hat{e}_y \cos\omega t\,\right]$$



The speed of q is constant; speed = c_2



The trajectory is a circle;
radius = c_2/ω ;
direction = clockwise

The period is $2\pi/\omega$.
The frequency is $\omega/(2\pi)$.
ω is called the *angular frequency*.

It is interesting to analyze the problem using ***complex numbers***.

Define

$$\eta = v_x + i\, v_y$$

$$i = \sqrt{(-1)}$$

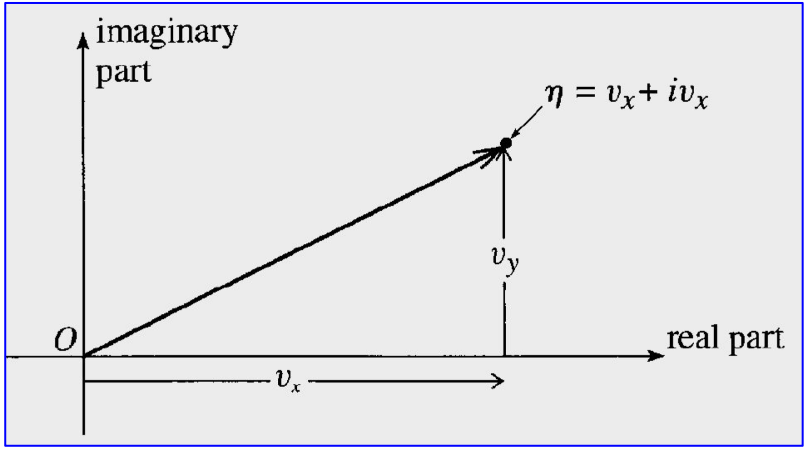
That is,

$$v_x = \mathrm{Re}\ \eta$$

$$v_y = \mathrm{Im}\ \eta$$

Figure 2.13 :

The plane of complex numbers



Now write the equations of motion (transverse components) in terms of η.

Solution using the complex variable

$$\eta = v_x + i\, v_y$$

$$\dot{\eta} = \dot{v}_x + i\, \dot{v}_y$$

$$= \omega v_y + i(-\omega) v_x$$

$$= -i\omega \left(v_x + i v_y \right) = -i\omega\eta$$

$$\dot{\eta} = -i\omega\eta$$

Obviously an <u>exponential</u>.

$$\frac{df}{dx} = \alpha f(x) \Rightarrow f(x) = A\, e^{\alpha x}$$

$$\frac{df}{dx} = A\, e^{\alpha x}\, \alpha = \alpha f.$$

So, $\eta(t)$ is a complex exponential function

$$\eta(t) = A\, e^{-i\omega t}$$

In the next section [Section 2.6] we'll review some important properties of complex numbers and the complex exponential function –– *widely useful in theoretical physics*.

Aside Some related problems:

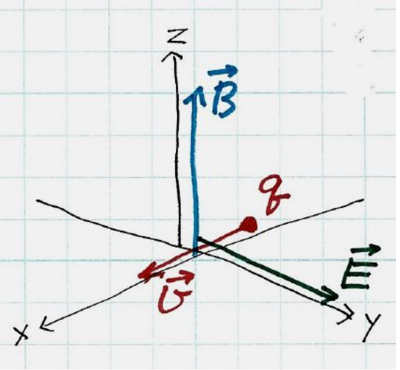
a charge q moving in both magnetic and electric fields

$$\mathbf{F} = q\,(\,\mathbf{E} + \boldsymbol{v} \times \mathbf{B}\,)$$

If **E** and **B** are parallel :

Taylor Problem 2.53 (**easy**)

If **E** and **B** are perpendicular :



If $\boldsymbol{v} = \mathbf{E} \times \mathbf{B}/B^2$ then the charge moves through the fields with constant velocity.

$$\vec{F} = q\vec{E} + q\,\vec{v} \times \vec{B}$$

$$= q\vec{E} + q\,\frac{(\vec{E}\times\vec{B})}{B^2} \times \vec{B}$$

$$(\vec{A}\times\vec{B})\times\vec{C} = \vec{B}\,\vec{A}\cdot\vec{C} - \vec{A}\,\vec{B}\cdot\vec{C}$$

$$= q\vec{E} + \frac{q}{B^2}\Big[\vec{B}\,\underbrace{\vec{E}\cdot\vec{B}}_{\substack{0 \\ (\perp\ \text{fields})}} - \vec{E}\,\underbrace{\vec{B}\cdot\vec{B}}_{B^2}\Big] = 0$$

In the most general case, the charge has a "drift velocity" **E x B** / B^2 ;
PHY 481

Homework Assignment #4
due in class Friday, September 30

[17] Problem 2.23 *

[18] Problem 2.31 **

[19] Problem 2.41 **

[20] Problem 2.53 *

[20x] Problem 2.43 *** [computer]

Use the cover sheet.