The Euler-Lagrange Equation

Suppose

$$S = \int_{x1}^{x2} f(\, y(x), y'(x), x \,) \; dx$$

where $y(x_1) = y_1$ and $y(x_2) = y_2$.

To minimize S *(with the endpoints fixed)*
y(x) obeys

$$\frac{\partial f}{\partial y(x)} = \frac{d}{dx} \; \frac{\partial f}{\partial y'(x)}$$

Example 6.2.

The brachistochrone

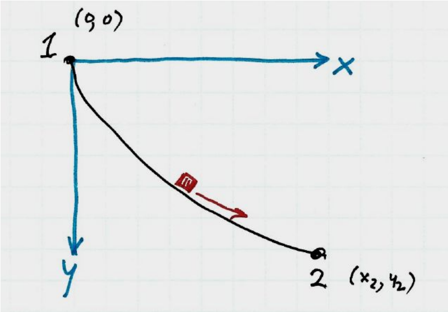
Read Chapter 6.

We'll spend only one week on Chapter 6.

The *brachistochrone problem* was posed by Johann Bernoulli in 1696. He sent a copy of the problem to Isaac Newton as a challenge; he thought maybe Newton wouldn't be able to solve it. Newton solved the problem overnight and sent the solution back to Bernoulli anonymously, as a kind of insult, to say "this is easy".

THE BRACHISTOCHRONE

A small mass (ice cube, say) slides without friction down a curve.



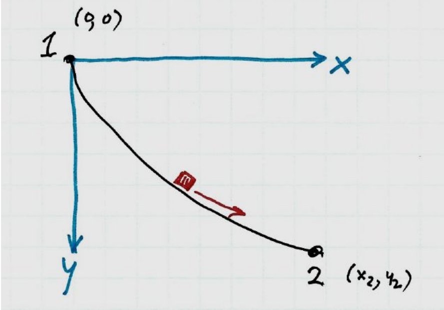
What is the shape of the curve such that the ice cube slides to the bottom in the shortest time?

"brachisto – chrone"

translates from Greek

as "shortest – time"

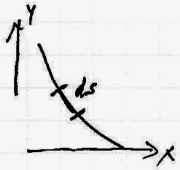
– the curve of fastest descent –



First,
we need a formula for the time of descent.

Using Taylor's notations

$$\text{time}(1 \to 2) = \int_1^2 \frac{ds}{v}$$



$$ds = \sqrt{(dx)^2 + (dy)^2}$$
$$= \sqrt{(dx/dy)^2 + 1}\; dy$$

$$v = \sqrt{2gy} \quad \text{by conservation of energy}$$
$$-mgy + \tfrac{1}{2}mv^2 = \text{constant}$$
$$0 = -mgy + \tfrac{1}{2}mv^2$$
$$v = \sqrt{2gy}$$

$$t_{12} = \frac{1}{\sqrt{2g}} \int_0^{y_2} \frac{\sqrt{(dx/dy)^2 + 1}}{\sqrt{y}}\, dy$$

The function we need to determine is x(y).

That requires we make some changes in the equations from last time:

- ❑ the independent variable
  last time = "x" → today = y ;
- ❑ the dependent variable
  last time = "y" → today = x ;

$$t_{12} = \frac{1}{\sqrt{2g}} \int_0^{y_2} f(x, x', y)\, dy \quad \text{where } x' = \frac{dx}{dy}$$

$$f(x, x', y) = \frac{\sqrt{x'^2 + 1}}{\sqrt{y}}$$

An important point is that the endpoints (x_1,y_1) and (x_2,y_2) are fixed.

$$t_{12} = \int_{y_1}^{y_2} \underbrace{\frac{\sqrt{1+x'^2}}{\sqrt{y}}}_{f(x,x',y)} dy \quad /\sqrt{2g}$$

Find the minimum of t_{12} .

Second,

apply the Euler-Lagrange equation.

$$\frac{\partial f}{\partial x} = \frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right)$$

$$0 = \frac{d}{dy}\left[\frac{1}{\sqrt{y}}\,\frac{1}{2}(1+x'^2)^{-\frac{1}{2}}\,2x'\right]$$

$$= \frac{d}{dy}\left[\frac{x'}{\sqrt{y}\,\sqrt{(1+x'^2)}}\right]$$

Third,

solve the differential equation.

We already have a first integral, because f(x,x',y) does not depend on x!

$$\frac{x'}{\sqrt{y}\,\sqrt{1+x'^2}} = \text{Constant} = \frac{1}{\sqrt{2a}}$$

call the constant 1 /√(2a); interpret " a " later

$$\frac{(x')^2}{y} = (1+x'^2)\frac{1}{2a} \Rightarrow (x')^2\left(\frac{1}{y}-\frac{1}{2a}\right) = \frac{1}{2a}$$

$$x' = \frac{dx}{dy} = \sqrt{\frac{y}{2a-y}}$$

This we can solve by direct integration.

Do the indefinite integral; put in the end points later.

Integration

$$x = \int \sqrt{\frac{y}{2a-y}}\, dy$$

Change the variable of integration from y to θ, related by $y = a(1-\cos\theta)$

$$x = \int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\; a\sin\theta\, d\theta$$

$$= \int (1-\cos\theta)\, a\, d\theta$$

$$= a(\theta - \sin\theta)$$

$\Rightarrow$ Parametric Equations for the Brachistochrone Curve

Figure 6.5

$$x = a(\theta - \sin\theta)$$

$$y = a(1-\cos\theta)$$

when θ goes from 0 $((x_1, y_1) = (0,0))$ to θ_2 where $\begin{cases} x_2 = a(\theta_2 - \sin\theta_2) \\ y_2 = a(1-\cos\theta_2) \end{cases}$

Note that the boundary (x_2, y_2) determines the constants (a, θ_2).

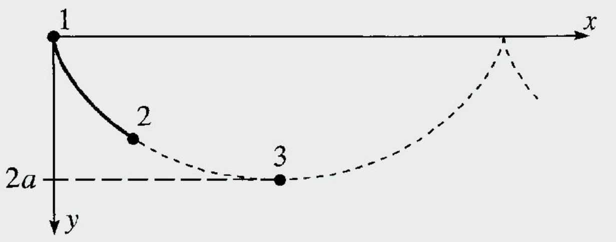


Figure 6.5 The path for a roller coaster that gives the shortest time between the given points 1 and 2 is part of the cycloid with a

Final result,

The brachistochrone is a segment of a cycloid curve.

__Parametric equations__

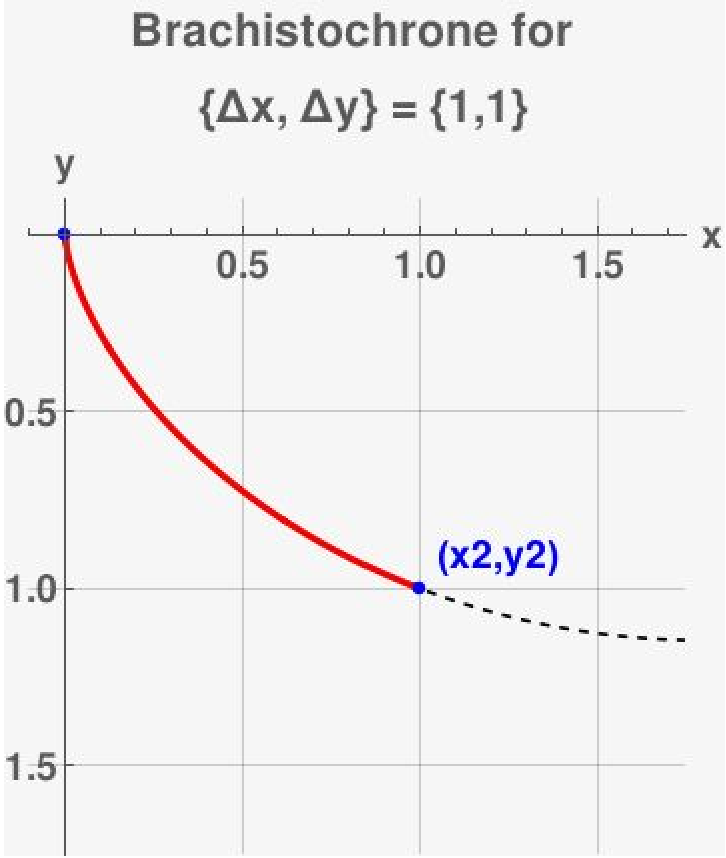
$$x(\theta) = a\,(\,\theta - \sin\theta\,)$$

$$y(\theta) = a\,(\,1 - \cos\theta\,)$$

There are two unknown constants (a and θ_2) which are determined from the coordinates of the final point (x_2 and y_2) :

$$x_2 = a\,(\theta_2 - \sin\theta_2)$$

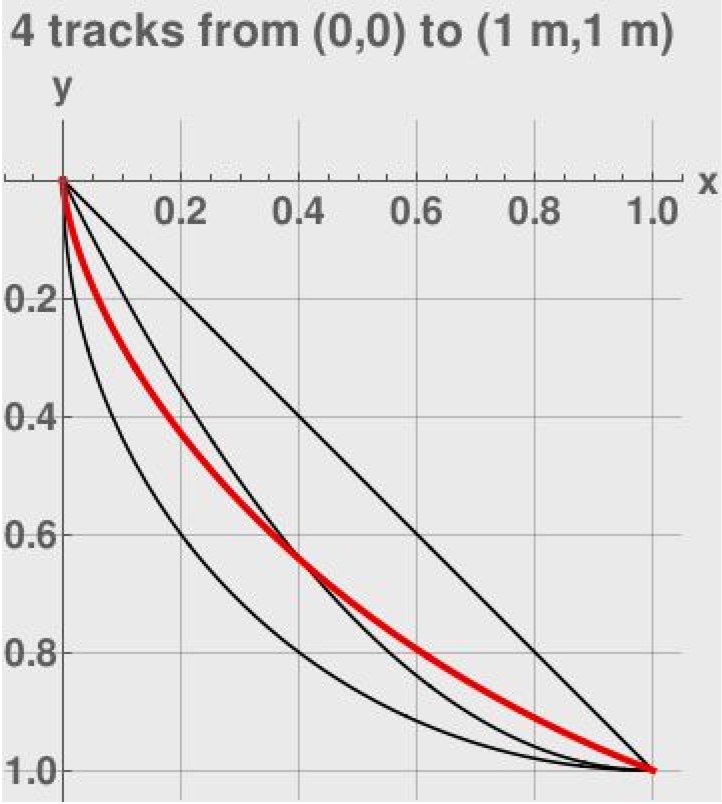
$$y_2 = a\,(1 - \cos\theta_2)$$



Comparisons

Consider high point = $(x_1 , y_1) = (0 , 0)$ and low point = $(x_2 , y_2) = \{ 1\ m, 1\ m \}$. ($g = 9.8\ m/s^2$)

shape of the track	time {0,0} −> {1 m, 1 m}
straight line	0.6389 s
parabola	0.5952 s
circular arc	0.5923 s
brachistochrone	0.5832 s

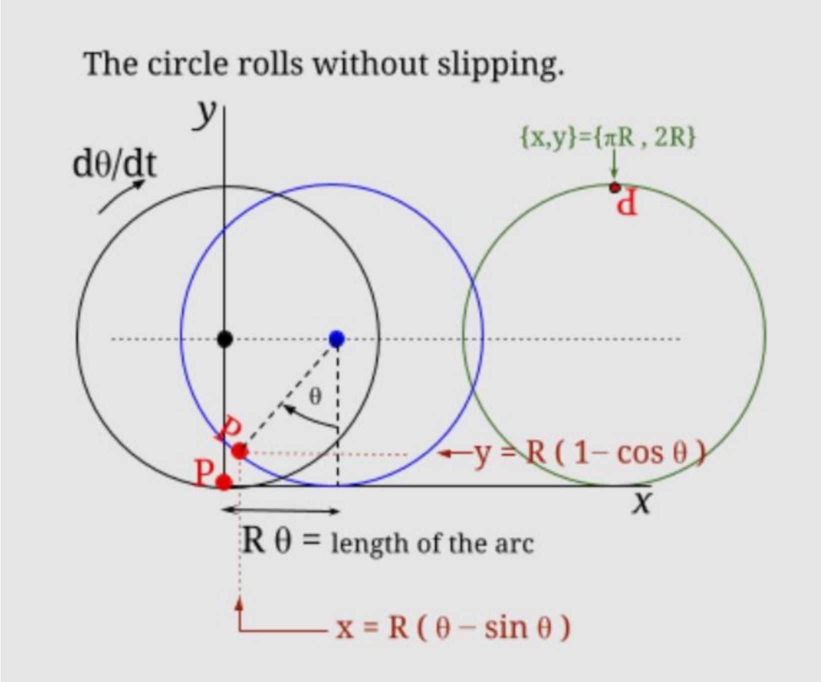


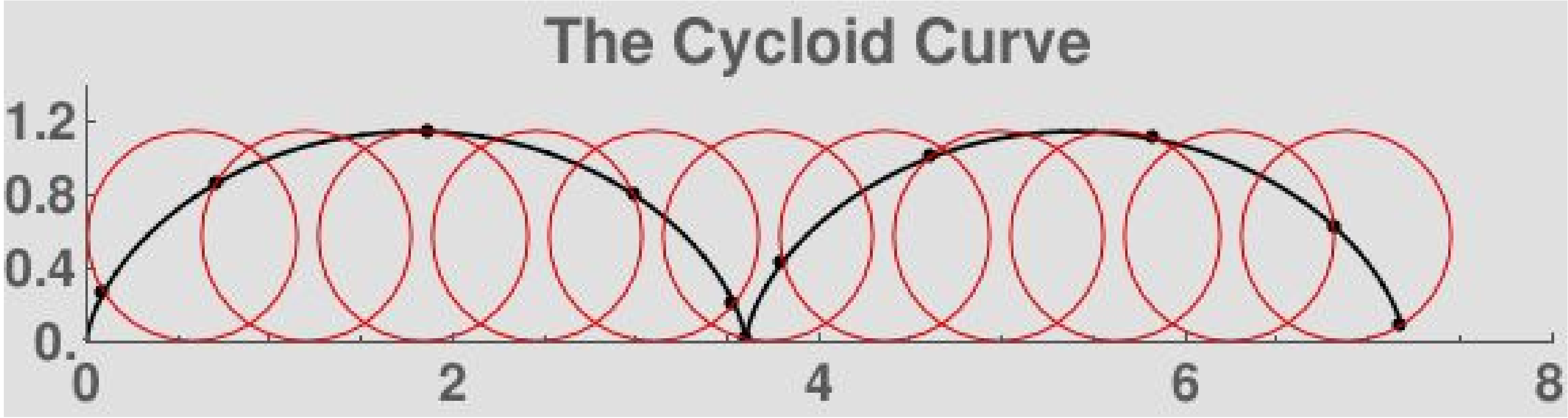
Mathematics of the Cycloid Curve

A circle (radius = R) rolls without slipping on the x axis.

What is the curve traced out by P, a point on the circle?

$$x(\theta) = R\,(\,\theta - \sin\theta\,)$$
$$y(\theta) = R\,(\,1 - \cos\theta\,)$$




The Cycloid Curve
1.2
0.8
0.4
0.
0
2
4
6
8

A related problem

The tautochrone problem –

– identify the curve such that the time of descent is the same for any initial point;

– solved by Christiaan Huygens. He proved, in his book *Horologium Oscillatorium*, published in 1673, that the curve is a cycloid.

= Taylor Problem 6.25.

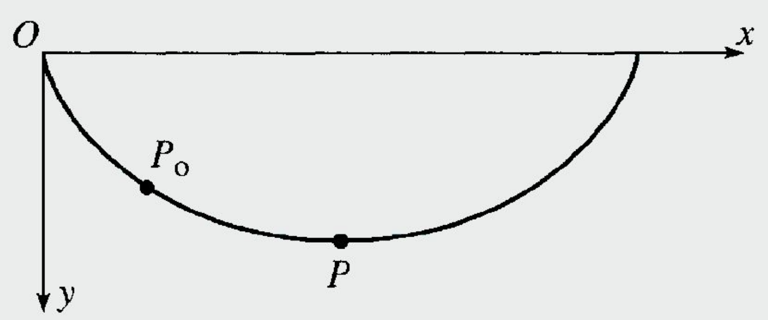


Figure 6.11 Problem 6.25

Homework Assignment #11
due in class Friday November 18
[50] back of the page
[51] Prob. 6.7*
[52] Prob. 6.8*
[53] Probs. 6.10* and 6.20**
[54] Probs. 6.1* and 6.16**
[55] Prob. 6.19**
[56] Prob. 6.25***
Use the cover sheet.