Name: Key **Section:**

Please work together to solve the problems.

1. Find the LU-decomposition of the coefficient matrix, and then use it to solve the system.

(a)

$$\begin{bmatrix} -5 & -10 \\ 6 & 5 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 19 \end{bmatrix}$$

$$E_{11} = \begin{bmatrix} -1/5 & 0 \\ 0 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} -5 & 0 \\ 6 & 1 \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 2 \\ 0 & -7 \end{bmatrix}$$

$E_{21}E_{11}A = U$

and $E_{11}^{-1}E_{21}^{-1} = L$

$L\vec{y} = \vec{b}$

$$\begin{bmatrix} -5 & 0 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 19 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

$U\vec{x} = \vec{y}$

$$\begin{bmatrix} 1 & 2 \\ 0 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 0 & -1 \end{bmatrix} \times \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} -3 & 12 & -6 \\ 1 & -2 & 2 \\ 0 & 0 & -1 \end{bmatrix} \quad -\tfrac{1}{3}R_1 \to R_1 \quad (\ell_{11} = -3)$$

$$\begin{bmatrix} 1 & -4 & 2 \\ 1 & -2 & 2 \\ 0 & 0 & -1 \end{bmatrix} \quad R_2 \to R_2 - R_1 \quad (\ell_{21} = 1)$$

$$U = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad L = \begin{bmatrix} -3 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$A\vec{x} = \vec{b}$

$LU\vec{x} = \vec{b}$

$U\vec{x} = \vec{y}$

$L\vec{y} = \vec{b}$

$U\vec{x} = \vec{y}$

$$\begin{bmatrix} 1 & -4 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \\ -1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$L\vec{y} = \vec{b}$

$$\begin{bmatrix} -3 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 7 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \\ -1 \end{bmatrix}$$

2. **True or False** Determine whether the statement is true or false, and justify your answer.

(a) It is impossible for a system of linear equations to have exactly two solutions.

TRUE: possibilities are 0, 1, or infinitely many

(b) If the linear system $A\vec{x} = \vec{b}$ has a unique solution, then the linear system $A\vec{x} = \vec{c}$ also must have a unique solution.

False: $A\vec{x} = \vec{c}$ could have 0, 1, or infinitely many

(c) If A and B are $n \times n$ matrices such that $AB = I_n$, then $BA = I_n$.

TRUE

(d) If A and B are row equivalent matrices, then the linear system $A\vec{x} = \vec{0}$ and $B\vec{x} = \vec{0}$ have the same solution.

TRUE

(e) Let A be an $n \times n$ matrix. The linear system $A\vec{x} = 4\vec{x}$ has a unique solution if and only if $A - 4I$ is an invertible matrix.

TRUE $A\vec{x} = 4\vec{x}$

$A\vec{x} - 4\vec{x} = \vec{0}$ → $(A - 4I)\vec{x} = \vec{0}$

so $A - 4I$ must be invertible

(f) Let A and B be $n \times n$ matrices. If A or B (or both) are not invertible, then neither is AB.

TRUE

(g) A diagonal matrix is invertible if and only if all of its diagonal entries are positive.

FALSE → $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ is invertible

(h) If A and B are $n \times n$ matrices such that $A + B$ is upper triangular, then A and B are upper triangular.

FALSE $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

(i) The sum of an upper triangular matrix and a lower triangular matrix is a diagonal matrix.

FALSE $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

(j) If each component of a vectore in R^3 is doubled, the norm of that vector is doubled.

TRUE

(k) If $||\vec{u}|| = 2$, $||\vec{v}|| = 1$, and $\vec{u} \cdot \vec{v} = 1$, then the angle between $\vec{u}$ and $\vec{v}$ is $\pi/3$ radians.

TRUE $\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| \, ||\vec{v}||} = \frac{1}{2 \cdot 1} = \frac{1}{2}$ $\cos\theta = 1/2$ $\theta = \pi/3$

(l) In R^2, if $\vec{u}$ lies in the first quadrant and $\vec{v}$ lies in the third quadrant, then $\vec{u} \cdot \vec{v}$ cannot be positive.

TRUE

(m) Every square matrix has a LU-decomposition.

False

3. Solve the matrix equation for X.

$$X \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 1 & 5 \end{bmatrix}$$

3x3 2x3

X must be a 2x3 matrix

can set up 2 systems of equations or use guess & check

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 1 & 5 \end{bmatrix}$$

$-1x_1 + x_2 + 3x_3 = 1$

$x_2 + x_3 = 2$

$x_1 + -x_3 = 0$

$-y_1 + y_2 + 3y_3 = -3$

$y_2 + y_3 = 1$

$y_1 - y_3 = 5$

& solve the system