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Name__________grading key_____________________

Homework Assignment #11 due in class Friday November 18

Staple this cover sheet in front of your solutions.

Write the requested answers on this sheet, and do the detailed calculations on your own paper.

[50] The problem on the back of the page.

Answer: The time can be written in the form x.xxx sqrt(a/g). What is x.xxx?

1.863 3 Points

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[51] Problem 6.7 *

Answer: Explain why the geodesic has the form it does.

Unwrap the cylinder and lay it out flat;

then the geodesic is a straight line on the rectangle. 1 point

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[52] Problem 6.8 *

Answer: Explain why.

$v = \sqrt{(2gy)}$ because energy is conserved. 1 point

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[53] Problems 6.10* and 6.20**

No answer is required here. 2 points

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[54] Problems 6.1* and 6.16**

Answer: Describe the corresponding geodesics if point 1 is on the z axis.

If point 1 is on the z axis, then the geodesic is a ***portion of a line of longitude***

(portion of a great circle with constant longitude) 2 points

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[55] Problem 6.19 **

Answer: Sketch a picture of the surface.

Check the drawing of the surface. 2 points

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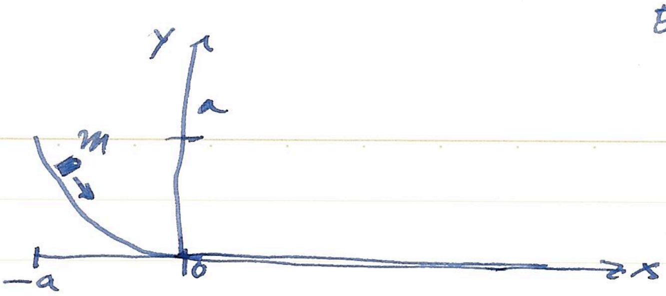
[56] Problem 6.25 ***

Answer: Explain qualitatively how this surprising result can possibly be true.

If θ_0 is made smaller then the speed will be smaller, and the distance will be smaller; but the time will remain the same. 3 points

[50]



$$y = \frac{x^2}{a}$$

$\{x(0), y(0)\} = \{-a, a\}$ and $\{\dot{x}(0), \dot{y}(0)\} = \{0, 0\}$.

$E = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + mgy$ is constant; $E = mga$

$$\dot{y} = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{2x}{a}\dot{x}$$

$$\dot{x} = \frac{a}{2x}\dot{y} = \frac{a}{2\sqrt{ay}}\dot{y} = \sqrt{\frac{a}{4y}}\,\dot{y}$$

$$E = \frac{1}{2} m \left[\frac{a}{4y} + 1\right] \dot{y}^2 + mgy = mga$$

$$\dot{y}^2 = \frac{2g(a-y)}{\left[\frac{a}{4y} + 1\right]} = \frac{2gay\,(1 - y/a)}{\left[\frac{a}{4} + y\right]}$$

(A) $$\dot{y} = -\sqrt{2ga}\,\sqrt{y}\,\sqrt{1 - y/a} \Big/ \sqrt{y + a/4}$$

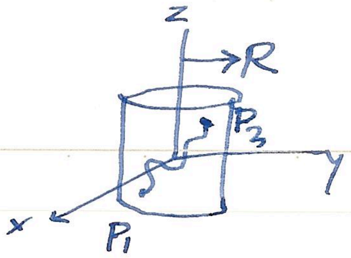
(B) Time integral: $dt = \frac{dy}{\dot{y}} \Rightarrow t = \int_a^0 \frac{dy}{\dot{y}(y)}$

$$t = \int_0^a \frac{dy\,\sqrt{y + a/4}}{\sqrt{2ga}\,\sqrt{y}\,\sqrt{1 - y/a}} = \sqrt{\frac{a}{2g}} \int_0^a \frac{d(y/a)\,\sqrt{y/a + 1/4}}{\sqrt{y/a}\,\sqrt{1 - y/a}}$$

$$= \sqrt{\frac{a}{g}}\,\frac{1}{\sqrt{2}}\underbrace{\int_0^1 \frac{du\,\sqrt{u + 1/4}}{\sqrt{u}\,\sqrt{1-u}}}_{2.635}$$ (change variable of integration from y to u where $y = au$)

$$= \sqrt{\frac{a}{g}}\; 1.863$$

(51) Problem 6.7



Cylindrical coordinates (ρ, ϕ, z)

$$x = \rho\cos\phi = R\cos\phi$$
$$y = \rho\sin\phi = R\sin\phi$$
$$z = z$$

$\rho = R$; determine $\phi(z)$

Length of the curve $= \int \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

$$= \int \sqrt{R^2(d\phi)^2 + (dz)^2}$$

$$= \int_{z_1}^{z_2} \sqrt{R^2\left(\frac{d\phi}{dz}\right)^2 + 1}\; dz = \int_{z_1}^{z_2} f(\phi, \phi')\, dz$$

The Euler Lagrange equation

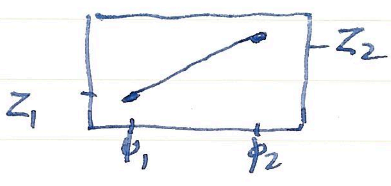
$$\frac{\partial f}{\partial \phi} = \frac{d}{dz}\left(\frac{\partial f}{\partial z'}\right) \Rightarrow 0 = \frac{d}{dz}\left[\frac{1}{2}\frac{\cancel{2}R^2\phi'}{\sqrt{R^2\phi'^2+1}}\right]$$

Thus $\dfrac{R^2\phi'}{\sqrt{R^2\phi'^2+1}}$ is constant.

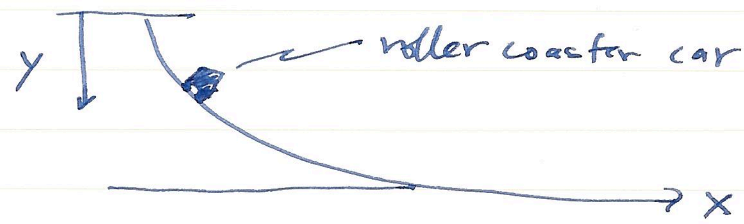
That is, ϕ' is constant.

$$\phi(z) = az + b \text{ where } \begin{cases} \phi_1 = a z_1 + b \\ \phi_2 = a z_2 + b \end{cases}$$

Unwrap the cylinder and lay it out flat;
the geodesic is a straight line on the rectangle



[52] Problem 6.8



Energy is conserved so $\frac{1}{2}mv^2 - mgy = \text{constant} = 0$

$$v = \sqrt{2gy}$$

[53] Problem 6.10 Suppose $f = f(y', x)$.

Then $\frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$ implies $0 = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$

So $\frac{\partial f}{\partial y'}$ is a constant. $\frac{\partial f}{\partial y'} = \text{const.}$

Problem 6.20 Suppose $f = f(y, y')$.

Then $\frac{df}{dx} = \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx}$

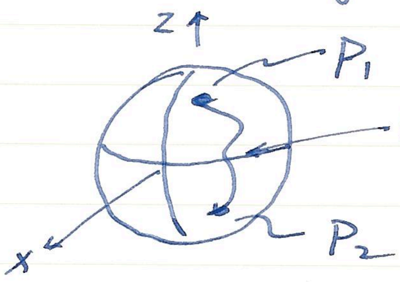
By the Euler Lagrange equation, $\frac{\partial f}{\partial y} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$; therefore

$$\frac{df}{dx} = \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) y' + \frac{\partial f}{\partial y'} y'' = \frac{d}{dx}\left\{\frac{\partial f}{\partial y'} y'\right\}$$

$\Rightarrow$ the first integral

$$\frac{d}{dx}\left[f - y'\frac{\partial f}{\partial y'}\right] = 0 \text{ so } f - y'\frac{\partial f}{\partial y'} = \text{constant.}$$

[54] <u>Problem 6.1</u> Paths that join two points on the surface of a sphere.



(r, θ, ϕ) where $r = R$

$x = R\sin\theta\cos\phi$
$y = R\sin\theta\sin\phi$
$z = R\cos\theta$

The length of the path $= L = \int \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

$$= \int \sqrt{R^2 (d\theta)^2 + R^2 \sin^2\theta (d\phi)^2}$$

$$= R\int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2\theta \left(\frac{d\phi}{d\theta}\right)^2}\; d\theta$$

<u>Problem 6.16</u> Minimize $L \Rightarrow$ the Euler Lagrange equation

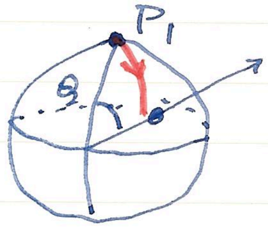
$$\frac{\partial f}{\partial \phi} = \frac{d}{d\theta}\left(\frac{\partial f}{\partial \phi'}\right) \Longrightarrow 0 = \frac{d}{d\theta}\left\{\frac{1}{2}\left(1 + \sin^2\theta\, \phi'^2\right)^{-1/2} 2\sin^2\theta\, \phi'\right\}$$

$$0 = \frac{d}{d\theta}\left\{ \underbrace{\frac{\sin^2\theta}{\sqrt{1 + \sin^2\theta \left(\frac{d\phi}{d\theta}\right)^2}} \frac{d\phi}{d\theta}}_{} \right\}$$

This must be a constant so $\frac{d}{d\theta}\{\} = 0$.

$$\sin^2\theta \frac{d\phi}{d\theta} = \text{const.} \sqrt{1 + \sin^2\theta \left(\frac{d\phi}{d\theta}\right)^2}$$

W.L.O.G. Let P_1 be the north pole; $\theta_1 = 0$ and $P_2 = (\theta_2, \phi_2)$.

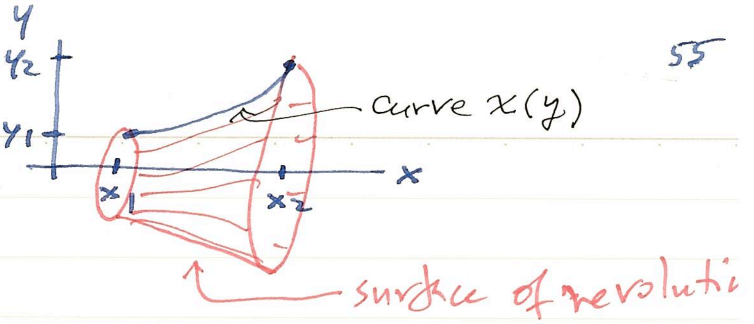


So $\sin^2\theta_1 \frac{d\phi}{d\theta_1} = \text{const.} \sqrt{1 + \sin^2\theta_1 \left(\frac{d\phi}{d\theta_1}\right)^2}$ implies $0 = \text{const.}$.

Then $\sin^2\theta \frac{d\phi}{d\theta} = 0$ along the geodesic;

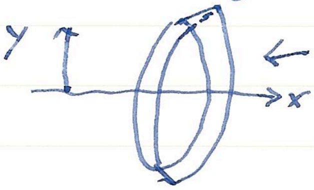
i.e. $\frac{d\phi}{d\theta} = 0$; the solution is $\phi = \text{const.} = \phi_2$

<u>The geodesic is a part of a great circle; part of a line of longitude.</u>



[55] Problem 6.19

Minimize the surface area,



← this ring has surface area

$$= 2\pi y \sqrt{(dx)^2 + (dy)^2}$$

$$= 2\pi y \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\; dy$$

$$A = \int_{y_1}^{y_2} 2\pi y \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\; dy$$

$\delta A = 0$ implies the Euler Lagrange equation

$$\frac{\partial f}{\partial x} = \frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right) \implies 0 = \frac{d}{dy}\Big[\underbrace{y\, \tfrac{1}{2}\left((x')^2+1\right)^{-1/2} 2x'}_{}\Big]$$

This must be a constant because $\frac{d}{dy}[\ldots] = 0$.

$$\frac{y\, dx/dy}{\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}} = C$$

$$y^2 \left(\frac{dx}{dy}\right)^2 = C^2\left[\left(\frac{dx}{dy}\right)^2 + 1\right]$$

$$(y^2 - C^2)\left(\frac{dx}{dy}\right)^2 = C^2 \;; \text{ thus } \frac{dx}{dy} = \frac{C}{\sqrt{y^2 - C^2}}$$

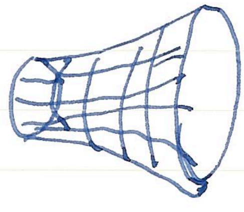
$$\cancel{\text{Integrate} \Rightarrow x - x_1 = \int_{y_1}^{y} \frac{C\, dy}{\sqrt{y^2 - C^2}}}$$

Using indefinite integrals, $x = \displaystyle\int \frac{C\, dy}{\sqrt{y^2 - C^2}} = C \operatorname{arccosh}\frac{y}{C} + C_1$

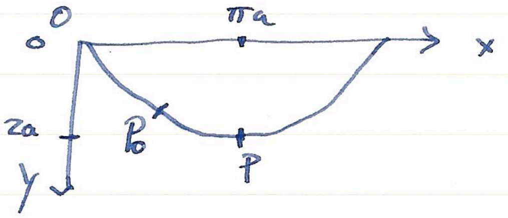
constant of integration

Thus $\dfrac{x - x_0}{y_0} = \operatorname{arccosh}\dfrac{y}{y_0}$ $\quad \left(\begin{array}{l} x_0 = C_1 \\ y_0 = C \end{array}\right)$

$$\boxed{y = y_0 \cosh\left(\frac{x - x_0}{y_0}\right)}$$



[56] Problem 6.25 — The TAUTOCHRONE



Calculate the time to slide from rest down the curve from P_0 to P.
The parametric functions for the curve are

$$x(\theta) = a(\theta - \sin\theta)$$
$$y(\theta) = a(1 - \cos\theta).$$
$$\left.\right\} \quad dx = a(1-\cos\theta)\,d\theta = y\,d\theta$$

P_0 is at θ_0; and P is at $\theta = \pi$.

$$\text{time} = \int \frac{dx}{dx/dt} = \int_{\theta_0}^{\pi} \frac{d\theta}{d\theta/dt}$$

Energy is conserved so $\frac{1}{2}m(\dot{x}^2+\dot{y}^2) - mgy = -mg\,y_0$

$$\frac{1}{2}\left[a^2(1-\cos\theta)^2\dot{\theta}^2 + a^2\sin^2\theta\,\dot{\theta}^2\right] = g(y-y_0)$$

$$\frac{1}{2}a^2\left[\dot{\theta}^2 + \dot{\theta}^2 - 2\cos\theta\,\dot{\theta}^2\right] = ga(\cos\theta_0 - \cos\theta)$$

$$a^2\dot{\theta}^2\left[1-\cos\theta\right] = ga(\cos\theta_0 - \cos\theta)$$

$$\dot{\theta} = \sqrt{\frac{g}{a}}\left(\frac{\cos\theta_0 - \cos\theta}{1-\cos\theta}\right)^{1/2}$$

$$\text{time} = \sqrt{\frac{a}{g}}\,\underbrace{\int_{\theta_0}^{\pi} d\theta\left(\frac{1-\cos\theta}{\cos\theta_0 - \cos\theta}\right)^{1/2}}$$

$\boxed{\text{time} = \pi\sqrt{\frac{a}{g}} \text{ independent of } \theta_0}$

Use Mathematica to calculate this integral; or, use the change of variables $\theta = \pi - 2\alpha$ and $u = \sin\alpha$.

$$\text{integral} = \int_{\frac{\pi}{2}-\frac{\theta_0}{2}}^{0} (-2)\,d\alpha\left(\frac{1+\cos 2\alpha}{\cos\theta_0 \cancel{-1} + \cos 2\alpha}\right)^{1/2} = \int_0^{\pi/2 - \frac{\theta_0}{2}} 2\,d\alpha\left[\frac{1+1-2\sin^2\alpha}{\cos\theta_0 \cancel{-1} + 1 - 2\sin^2\alpha}\right]^{1/2}$$

$$= \int_0^{\pi/2-\theta_0/2} 2\,d\alpha\left[\frac{2-2\sin^2\alpha}{1+\cos\theta_0 - 2\sin^2\alpha}\right]^{1/2} = \int_0^{\sin\left(\frac{\pi}{2}-\frac{\theta_0}{2}\right)} \frac{2\,du}{\sqrt{1-u^2}}\left[\frac{2(1-u^2)}{1+\cos\theta_0 - 2u^2}\right]^{1/2}$$

$$= 2\sqrt{2}\int_0^{\cos(\theta_0/2)} \frac{du}{\sqrt{1+\cos\theta_0 - 2u^2}} = 2\sqrt{2}\int_0^{\cos(\theta_0/2)} \frac{du}{\sqrt{1+\cos\theta_0 - 2u^2}}$$

$$= 2\sqrt{2}\int_0^{\cos(\theta_0/2)} \frac{du}{\sqrt{2\cos^2(\theta_0/2) - 2u^2}} = 2\int_0^{A} \frac{du}{\sqrt{A^2-u^2}} = 2\int_0^1 \frac{du'}{\sqrt{1-u'^2}} = \pi$$