Section 2.4

Quadratic Air Resistance

Read Section 2.4.

Now we'll assume that the air resistance force is proportional to the *square* of the speed; i.e., the linear resistance is negligible compared to the quadratic resistance.

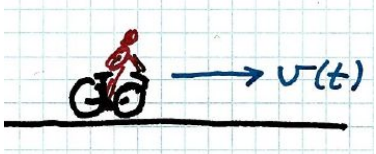
Equations

- $\boldsymbol{f} = -\,c\,v^2\,\boldsymbol{e}_v$

- For a spherical object, with diameter D,

  $c = \gamma\, D^2$, $\gamma = 0.25 \text{ Ns}^2\,\text{m}^{-4}$

Horizontal motion with quadratic air resistance; e.g., *a bicycle coasting on a level road;*



$$m\frac{dv}{dt} = -\,c v^2$$

Separation of variables

$$m\,dv = -\,c v^2\,dt$$

$$\frac{dv}{v^2} = -\frac{c}{m}\,dt$$

Integrations —

$$\int_{v_0}^{v}\frac{dv'}{v'^2} = \frac{-1}{v'}\Big|_{v_0}^{v} = \frac{-1}{v} + \frac{1}{v_0}$$

$$= -\frac{c}{m}\int_0^t dt' = -\frac{c}{m}t$$

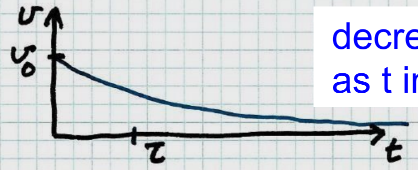
Solve for v —

$$\frac{1}{v} = \frac{1}{v_0} + \frac{ct}{m} \quad \text{or} \quad v = \frac{v_0}{1 + \frac{c v_0 t}{m}}$$

Figure 2.8.

Let $\tau = \dfrac{m}{c v_0}$ "time constant for quadratic resistance"

Then $v(t) = \dfrac{v_0}{1 + t/\tau}$

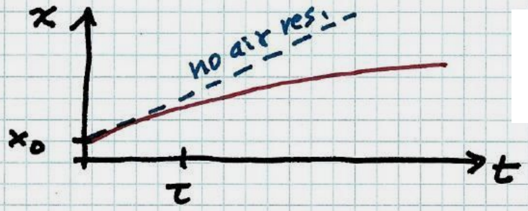


decreasing slowly to 0
as t increases

The position

$$\int_{x_0}^{x} dx' = x - x_0 = \int_0^t v_x(t')\, dt'$$

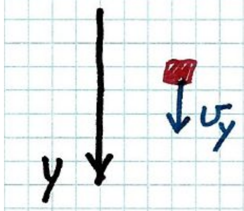
$$= v_0 \tau \ln\left(1 + \frac{t'}{\tau}\right)\Big|_0^t = v_0 \tau \ln\left(1 + \frac{t}{\tau}\right)$$



increasing indefinitely,
but only logarithmically

Vertical motion with quadratic air resistance;

e.g., an object dropped from rest falls straight down;



$$m\frac{dv}{dt} = mg - cv^2$$

It is convenient to let the y axis point downward, so $F_g = + mg$

Before we solve the equation, what is the terminal velocity?

At terminal velocity, F = 0.

Thus $v_{ter} = \sqrt{mg/c}$.

Separation of variables

$$m\,dv = (mg - cv^2)\,dt$$

$$\frac{dv}{(mg - cv^2)} = \frac{dt}{m}$$

$$\int_{v_0}^{v} \frac{dv'}{mg - cv'^2} = \int_0^t \frac{dt'}{m} = \frac{t}{m}$$

$$\longrightarrow = \frac{1}{c}\int_{v_0}^{v} \frac{dv'}{v_{ter}^2 - v'^2} = \frac{1}{c\,v_{ter}} \operatorname{arctanh}\left(\frac{v'}{v_{ter}}\right)\Bigg|_{v_0}^{v}$$

$$= \frac{1}{c\,v_{ter}} \operatorname{arctanh}\left(\frac{v}{v_{ter}}\right) \text{ assuming } v_0 = 0$$

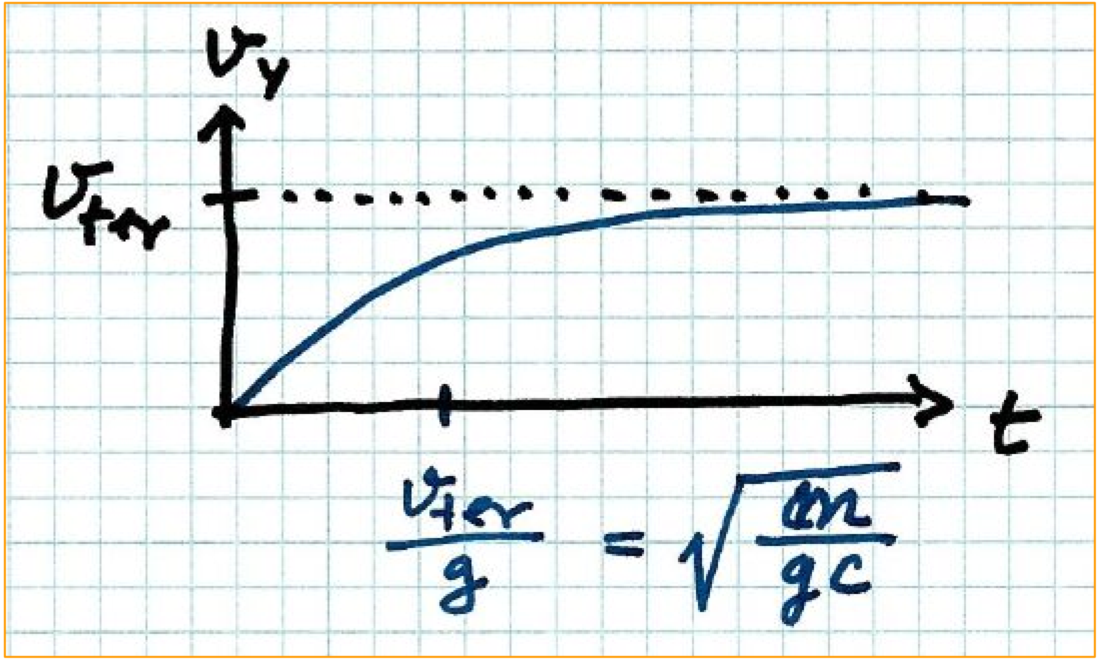
Solve for $v(t)$

$$t = \frac{m}{c\,v_{ter}} \operatorname{arctanh}\left(\frac{v}{v_{ter}}\right) \Rightarrow$$

$$v(t) = v_{ter} \tanh\left(\frac{gt}{v_{ter}}\right)$$

~~Check~~ : $\cancel{gt/v_{ter} = \frac{c\,v_{ter}^2}{m}\,\frac{t}{v_{ter}} = \frac{c\,v_{ter}\,t}{m}}$ ✓

$$v(t) = v_{ter} \tanh\left(\frac{gt}{v_{ter}}\right)$$



Example 2.5

a baseball dropped from a tower

(See Figures 2.9 (a) and (b) .)

Dashed : no air resistance

Black : quadratic air resistance

Mathematica commands; **Note: $c = \gamma D^2$**

```
{mass, diam} = {0.15 , 0.07} (* kg and m *)
{g, gamma} = {9.8,0.25} (* mks units *)
vterm = Sqrt[mass*g/(gamma*diam^2)]
vy[t_] := vterm*Tanh[g*t/vterm]
y[t_] := vterm^2/g*Log[Cosh[g*t/vterm]]
p1 = Plot[{g*t, vy[t]}, {t, 0, 8}]
p2 = Plot[{1/2*g*t^2, y[t]}, {t, 0, 8}]
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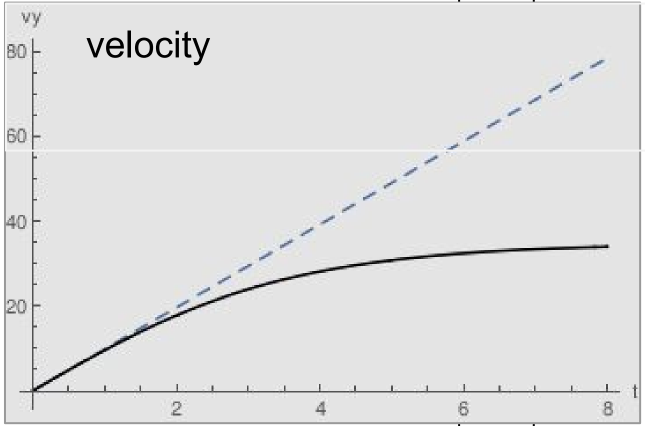
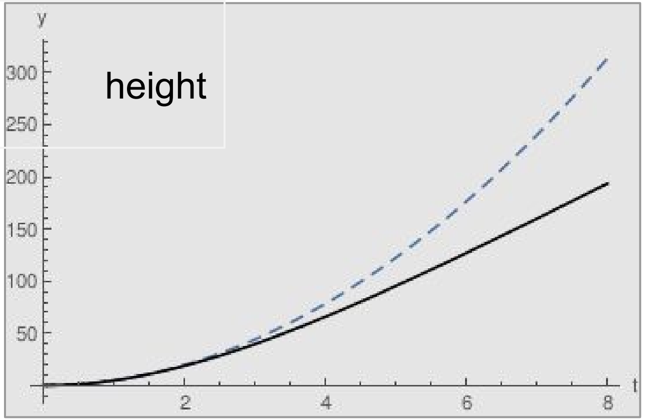




Quadratic drag for a projectile with both x (horizontal) and y (vertical) motion

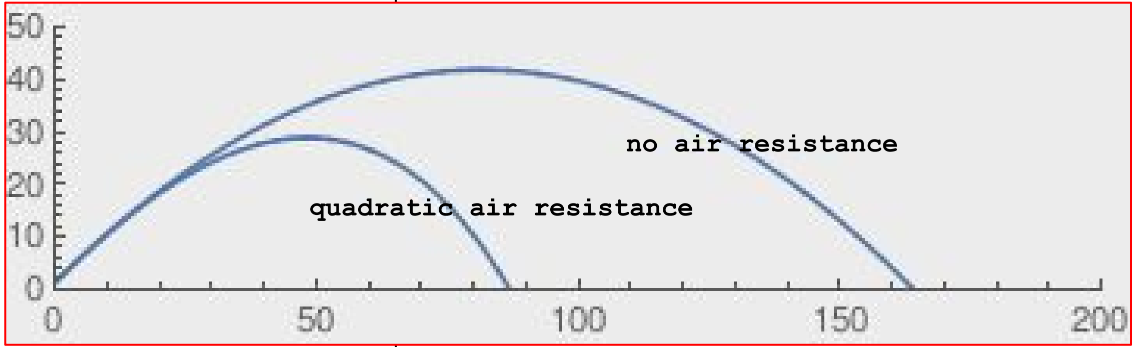
Example. A baseball home run, calculated using Mathematica.

Assume these *initial conditions*:

speed = 40 m/s (~ 90 mi /h)

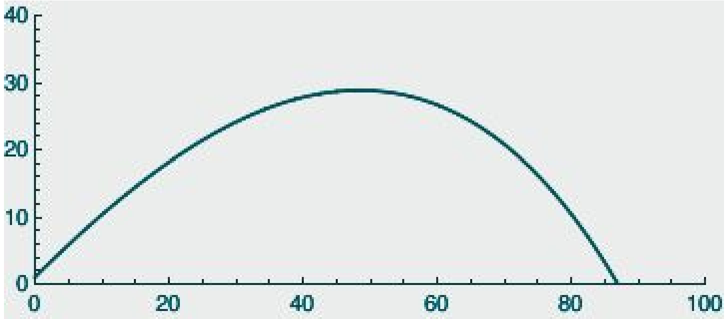
angle = 45 degrees

height = 1 m



Mathematica Program

```
g = 9.8;  vter = 35;     m = 0.15; const = m*g/vter^2;
v0 = 40;  ang=Pi/4;
eqs = {
   m*x''[t] ==    - const*x'[t]*Sqrt[x'[t]^2 + y'[t]^2],
   m*y''[t] ==  -m*g - const*y'[t]*Sqrt[x'[t]^2 + y'[t]^2],
   x[0] == 0, x'[0] == v0*Cos[ang],
   y[0] == 1, y'[0] == v0*Sin[ang]};
Q = NDSolve[eqs, {x[t], y[t]}, {t, 0, 6}];
ParametricPlot[{x[t] /. Q[[1]], y[t] /. Q[[1]]}, {t, 0, 6}]
```



Test yourself:

A bicycle rider coasts down a hill. The angle of the slope is $\theta = 10$ degrees $= 0.174$ radians. (A) Using your knowledge about air resistance, *estimate* the terminal speed of the bicycle, in meters per second. (B) Determine the speed as a function of time, starting from speed v_0 at $t = 0$. Data: Mass = 70 kg ; effective area = 1 m^2 (approximate the shape by a sphere).

Homework Assignment #3

[11] Problem 2.2 *
[12] Problem 2.3 *
[13] Problem 2.10 **
[14] Problem 2.18 *
[15] Problem 2.26 *

Use the cover sheet.