Homework #7 Solutions

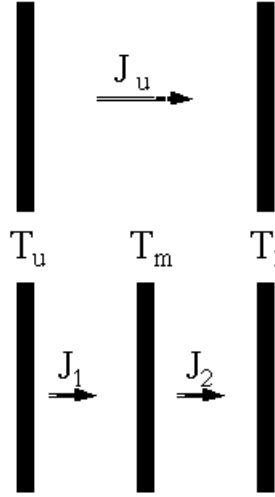**Question 1)** K+K, Chapter 4, Problem 8. See Figure (1) for the setup.

Figure 1: The situation before (top) and after (bottom) a temperature-floating shield is placed between a hot surface (at temperature T_u) and a cold surface (at temperature T_l).

We expect $J_1 = J_2$ in equilibrium, where $J_1 = \sigma_B(T_u^4 - T_m^4)$ and $J_2 = \sigma_B(T_m^4 - T_l^4)$. Solve for T_m:

$$T_m = \left(\frac{T_u^4 + T_l^4}{2}\right)^{1/4} \tag{1}$$

Now the heat current is given by

$$J = \sigma_B(T_u^4 - T_m^4) = \sigma_B\left(\frac{T_u^4 - T_l^4}{2}\right) = \frac{J_u}{2} \tag{2}$$

This is half the original heat current. Note that we would get the same answer if we calculated the flux between the inserted plate and the low temperature plate (up to a possible sign difference). This has to be so in equilibrum for a black body due to energy conservation.

The general case of N floating heat shields reduces the heat current by a factor of (N+1). This is the principle behind super-insulation, in which many layers of Al-coated mylar are wrapped around a cold object to insulate it from radiative heating from nearby warm objects.

Question 2) K+K, Chapter 4, Problem 11 (First part only). Starting with Eq. (41) in K+K we can write the energy of phonons in a solid as

$$U = \frac{9N\tau^4}{(k_B\theta_D)^3} \int_0^{k_B\theta_D/\tau} \mathrm{d}x \frac{x^3}{e^x - 1} \tag{3}$$

We are working in the limit where the upper limit of the integral is small: $k_B\theta_D/\tau \ll 1$. Hence x remains small in the integrand, and we can expand the denominator as $e^x - 1 = 1 + x + \cdots - 1 \approx x$. So,

$$U \approx \frac{9N\tau^4}{(k_B\theta_D)^3} \int_0^{k_B\theta_D/\tau} \mathrm{d}x\, x^2 \tag{4}$$

The integration is now simple and gives $k_B^3\theta_D^3/3\tau^3$. Simplifying the expression we get

$$U = 3N\tau \tag{5}$$

Differentiating this with respect to temperature we get the Dulong-Petit law for the heat capacity of a solid at high temperature ($\theta_D \ll T$):

$$C_V = \left.\frac{\partial U}{\partial T}\right|_U = 3Nk_B \qquad \text{for } \theta_D \ll T \tag{6}$$

Question 3) K+K, Chapter 4, Problem 12. The phonon specific heat at 1 K is

$$C_V^{phonon} = \frac{12\pi^4 Nk_B}{5}\left(\frac{T}{\theta_D}\right)^3 \qquad \text{since } T \ll \theta_D. \tag{7}$$

For photons, we know from Eqs. (25-26) in K+K

$$U/V = 4\sigma_B T^4/c \tag{8}$$

$$C_V^{photon}/V = \frac{1}{V}\frac{\partial U}{\partial T} = 16\sigma_B T^3/c \tag{9}$$

For phonons at T=1K plugging in the numbers

$$C_V^{phonon}/V = 3.23 \times 10^{-5} \frac{J}{K \cdot cm^3} \tag{10}$$

To match this with photons we set Eq. (10) equal to Eq. (9) and solve for T. This gives $T \approx 2.2 \times 10^5$K, showing that phonons carry much more energy per unit volume than photons at the same temperature.

Question 4a) K+K, Chapter 4, Problem 18. In the expansion of the universe the radiation pressure does work and this cools the photon gas. Let $T_d = 3000K$ be the temperature at the moment of de-coupling between photons and matter, and $T_n = 2.73K$ the temperature of the photon gas now (We will

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use these subscripts with other quantities as well to denote their values at the corresponding times). Using the constancy of entropy we can equate the entropy back then and now.

$$\tau_d V_d^{1/3} = \tau_n V_n^{1/3} \tag{11}$$

$$\left(\frac{V_d}{V_n}\right)^{1/3} = \frac{r_d}{r_n} = \frac{T_n}{T_d} = \frac{2.73K}{3000K} = 9.1 \times 10^{-4} \tag{12}$$

If we assume linear increase in radius: $R_n = r_d + vt_d$ and $r_n = vt_u$, where t_d is the time since the decoupling and t_u is the age of the universe.

$$\frac{t_d}{t_u} = \frac{(r_n - r_d)/V}{r_n/V} = 1 - r_d/r_n = 0.9991 \tag{13}$$

This is too recent!

4b) Since there is no heat input (assuming the universe is a closed system) the work done in the expansion is $W = U_i - U_f$. The energy per unit volume is given by Eq. (20) in K+K and can be written as (using the definition of σ_B)

$$U/V = 4\sigma_B T^4/c \tag{14}$$

From this we can calculate the work

$$W = \frac{4\sigma_B}{c}(V_i T_i^4 - V_f T_f^4) \tag{15}$$

The initial and final volumes are related by Eq. (11). Solving for V_f in terms of the other quantities and after some algebra we get the answer:

$$W = \frac{\pi^2 V_i \tau_i^3}{15\hbar^3 c^3}(\tau_i - \tau_f) \tag{16}$$

Question 5) K+K, Chapter 5, Problem 1. We assume the gas has an internal chemical potential $\mu = \mu_{IG}$ with

$$\mu_{IG} = \tau \log\left(\frac{n}{n_Q}\right)$$

In addition there is an external chemical potential μ_{ext}, so

$$\mu_{total} = \mu_{IG} + \mu_{ext}$$

μ_{ext} is due to the rotational energy. This we can calculate from the centrifugal force

$$\vec{F} = \frac{Mv^2}{r}\hat{r}$$

But we know that the velocity at a position r is given by $v(r) = \omega r$. We integrate the force to obtain the potential

$$U(r) = -\int_0^r \vec{F}(r') \cdot d\vec{r'} = -\int_0^r M\omega^2 r' dr' = -\frac{m\omega^2 r^2}{2}$$

Figure 2: Ideal gas of atoms of mass M at temperature τ rotating with angular velocity ω.

Now we can write the total chemical potential

$$\mu_{total}(r) = \tau \log\left(\frac{n(r)}{n_Q}\right) - \frac{M\omega^2 r^2}{2}$$

In thermal and diffusive equilibrium the temperature and the chemical potential should be uniform.

In particular

$$\mu_{total}(r = 0) = \mu_{total}(r) \tag{17}$$

$$\tau \log\left(\frac{n(0)}{n_Q}\right) - 0 = \tau \log\left(\frac{n(r)}{n_Q}\right) - \frac{M\omega^2 r^2}{2} \tag{18}$$

We want to solve this for $n(r)$. Divide through by τ and exponentiate both sides

$$\frac{n(0)}{n_Q} = \frac{n(r)}{n_Q} e^{-M\omega^2 r^2/2\tau}$$

$$\downarrow$$

$$n(r) = n(0) e^{M\omega^2 r^2/2\tau}$$

So the concentration increases exponentially with r^2 as the radius increases.

This calculation was done for one species of mass M. A centrifuge can be used to separate materials of different masses M and $M + \Delta M$, even if ΔM is

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small compared to M (for example ^{235}U and ^{238}U). If the species remain as a gas, the lighter isotope will be found near the axis in an enriched amount.

Question 6) K+K, Chapter 5, Problem 2. From the example on page 125 we know the concentration of molecules of mass M as a function of height h: $n(h) = n(0)e^{-Mgh/\tau}$, assuming that the column is at constant temperature τ. Let g be the gravitational acceleration on the surface of the Earth: r=R. See figure(3).
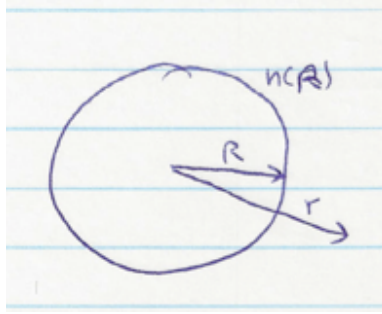

Figure 3: R denotes the radius of the earth. The gravitational acceleration on the surface is denoted by g, the particle density on the surface is $n(R)$.

The gravitational potential of a mass m is given by

$$U(r) = -\frac{mM_E G}{r}$$

where M_E is the mass of the Earth and G is the universal gravitation constant. We want to express this in terms of the known quantity g. For that we consider the gravitational force on mass m on the surface of the Earth:

$$mg = \frac{mM_E G}{R^2}$$
$$M_E G = gR^2$$
$$U(r) = -\frac{mgR^2}{r}$$

Now we can write down the total chemical potential as in question 5.

$$\mu_{total} = \tau \log\left(\frac{n(r)}{n_Q}\right) - \frac{mgR^2}{r}$$

In thermal and diffusive equilibrium the chemical potential is uniform with altitude.

$$\mu_{total}(r) = \mu_{total}(R)$$
$$\tau \log\left(\frac{n(r)}{n_Q}\right) - \frac{mgR^2}{r} = \log\left(\frac{n(R)}{n_Q}\right) - MgR$$

We solve this for $n(r)$ by dividing through by τ and exponentiating, just like we did in question 5. The result is:

$$n(r) = n(R)e^{-\frac{mgR}{\tau}(1-\frac{R}{r})} \tag{19}$$

Now integrate over the entire (spherically symmetric) atmosphere to get the total number of atoms in the atmosphere:

$$N = \int\int\int n(r)dV = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta \int_R^\infty dr r^2 n(r) \tag{20}$$

$$= 4\pi n(R) \int_R^\infty dr r^2 e^{-\frac{mgR}{\tau}(1-\frac{R}{r})} \tag{21}$$

$$= 4\pi n(R)e^{-mgR/r} \int_R^\infty dr r^2 e^{\frac{mgR^2}{\tau r}} \tag{22}$$

This integral does not converge- integrand goes like r^2 for large r. This is taken as evidence that the atmosphere is not in equilibrium with outer space. The atmosphere is leaving! It must be re-supplied by the earth itself.
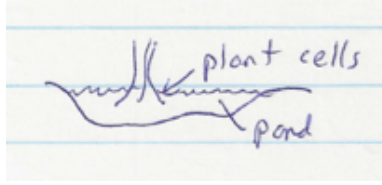

Figure 4: Artist's depiction of plant cells in the pond.

Question 7) K+K, Chapter 5, Problem 4. K^+: ion concentration. The concentration difference of K^+ ions between the sap in a cell and the water outside is given as;

$$\frac{[K^+]_{cell}}{[K^+]_{pond}} = 10^4 = \frac{n_{cell}}{n_{pond}}$$

This difference in concentration must be maintained by a difference in chemical potential between the cell and pond water. The two systems are not in diffusive equilibrium, although they are in thermal equilibrium.

Assume the K^+ ions have an ideal gas chemical potential $\mu = \tau \log(n/n_Q)$. The difference in chemical potentials is given by:

$$\mu_{cell} - \mu_{pond} = \tau \log(n_{cell}/n_Q) - \tau \log(n_{pond}/n_Q) = \tau \log(n_{cell}/n_{pond})$$

At pond temperature of 300K this produces a chemical potential difference of: (note that the logarithms are in basis e: $\log_e(10^4) = 9.21$)

$$\Delta\mu = k_B 300K \log(10^4) = 0.24eV$$

There has to be a 0.24eV potential drop on the cell membrane.