Homework #6 Solutions

Question 1) Kittel+Kroemer, Chapter 4, Problem 1. The total number of thermal photons is the sum of photons in each mode:

$$N = \sum_n < s_n >= \sum_n \frac{1}{e^{\hbar\omega_n/\tau} - 1} \tag{1}$$

Replace the sum over n with an integral, similar to the calcualtion in the text on pages 93 and 94.

$$\sum_n \to 2 \times \frac{1}{8} \int_0^\infty \mathrm{d}n 4\pi n^2 \tag{2}$$

The factor of 2 takes care of the two polarizations of light. The factor of 1/8 comes from integrating only over one quadrant (in 3 dimensional n-space). In the integrand $\omega_n = n\pi c/L$. To bring the integral to a dimensionless form we make the change of variables $x = \frac{\pi\hbar c}{L\tau}n$. Thus the integral becomes:

$$N = \pi \left(\frac{L\tau}{\hbar\pi c}\right)^3 \int_0^\infty \mathrm{d}x \frac{x^2}{e^x - 1} \tag{3}$$

We note that the integral is a known one and equals $2\zeta(3) = 2.404$. So the final answer reads

$$N = 2.404 \frac{V}{\pi^2} \left(\frac{\tau}{\hbar c}\right)^3 \tag{4}$$

Question 2a) Kittel+Kroemer, Chapter 4, Problem 2. The solar constant gives the power per unit area, which is placed a distance $R_{e-s} = 1.5 \times 10^{23}$cm away from the sun (the distance from the sun to earth). To get the total power output of the sun we have to integrate over the area over which this radiation is spread: $4\pi R_{e-s}^2$.

$$P_{sun} = 4\pi R_{e-s}^2 \times 0.136 \frac{J}{\mathrm{s} \cdot \mathrm{cm}^2} = 3.85 \times 10^{26} J/s \tag{5}$$

b) To use the Stefan-Boltzmann Law we need the power radiated per area. This can be easily obtained using the radius of the sun $r_s = 7 \times 10^{10} cm$.

$$J_{sun} = \frac{P_{sun}}{4\pi r_s^2} = 6.2 \times 10^3 \frac{J}{\mathrm{s} \cdot \mathrm{cm}^2} \tag{6}$$

The Stefan-Boltzmann Law reads $J_{sun} = \sigma_B T_{sun}^4$. Solving for the temperature we get

$$T_{sun} = \left(\frac{J_{sun}}{\sigma_B}\right)^{1/4} = 5750K \tag{7}$$

Question 3) Kittel+Kroemer, Chapter 4, Problem 5. For this question we could readily use some results from the previous problem. But we prefer to do it from scratch in order to get familiarity with the methods. The Earth is first assumed to be a black body absorber in equilibrium with the radiation from the sun. This means that the earth will radiate the same amount of energy it receives from the sun. We will start by calculating the power of the radiation from the sun using Stefan-Boltzmann law

$$P_\odot = J_\odot \times 4\pi R_\odot^2 = (\sigma_B T_\odot^4) \times 4\pi R_\odot^2 \tag{8}$$

Note that the circle with a dot in it ($\odot$) is a symbol for the sun. The flux density at earth's orbit is

$$J_E = \frac{P_\odot}{4\pi R_{e-s}^2} = J_\odot \left(\frac{R_\odot}{R_{e-s}} \right)^2 \tag{9}$$

The flux absorbed by earth is this value multiplied by the cross section of the Earth (which is just πR_E^2): $\Phi_{in} = J_E \times \pi R_E^2$. In equilibrium this has to be equal to the energy radiated by earth, which again is given by the Stefan-Boltzmann law: $\Phi_{out} = (\sigma_B T_E^4) \times (4\pi R_E^2)$. Equating the two: $\Phi_{in} = \Phi_{out}$ gives,

$$\pi \sigma_B T_\odot^4 \left(\frac{R_\odot R_E}{R_{e-s}} \right)^2 = \sigma_B T_E^4 4\pi R_E^2 \tag{10}$$

Simplifying and solving for T_E we get

$$T_E = \frac{T_\odot}{4^{1/4}} \sqrt{\frac{R_\odot}{R_{e-s}}} \approx 280K \tag{11}$$

Question 4a) Kittel+Kroemer, Chapter 4, Problem 7. The partition function of a single photon mode is

$$Z_1 = \frac{1}{1 - e^{-\hbar\omega_n/\tau}} \qquad \text{(3) on page 90 of K+K.}$$

The partition function for all the modes in the box is the product of the partition functions for each mode:

$$Z_{Total} = Z_{\omega_1} Z_{\omega_2} \cdots Z_{\omega_n} \cdots \tag{12}$$

Note that each mode is distinguishable by its mode frequency ω_n, is independent and non-interacting, hence there is no concern about indistinguishability, as there was for the ideal gas of N particles. The total partition function can be written compactly as,

$$Z_{Total} = \prod_n \frac{1}{1 - e^{-\hbar\omega_n/\tau}} \tag{13}$$

where the product is over every mode n of the system (there are an infinite of modes for the photons in a box).

4b) The free energy is given by $F = -\tau \log Z_{Total}$:

$$F = -\tau \log \prod_n \frac{1}{1 - e^{-\hbar\omega_n/\tau}} = +\tau \sum_n \log(1 - e^{-\hbar\omega_n/\tau}) \tag{14}$$

, where we used the fact that the log of a product (the big Π is a product symbol) is the sum of the logs. Next we transform the sum into an integral as we did in question 1. If you compare equations (1) and (3) with equation (14) you should be able to see that the sum in equation (14) transforms into the following integral

$$F = \tau\pi \left(\frac{L\tau}{\pi\hbar c}\right)^3 \int_0^\infty \mathrm{d}y y^2 \log(1 - e^{-y}) \tag{15}$$

We will integrate this by parts to bring it into a more manageble form. For this we let $u = \log(1 - e^{-y})$ and $\mathrm{d}v = y^2 \mathrm{d}y$. This gives for the integral:

$$\frac{1}{3} y^3 \log(1 - e^{-y})\Big|_0^\infty - \frac{1}{3} \int_0^\infty \mathrm{d}y \frac{y^3 e^{-y}}{1 - e^{-y}} \tag{16}$$

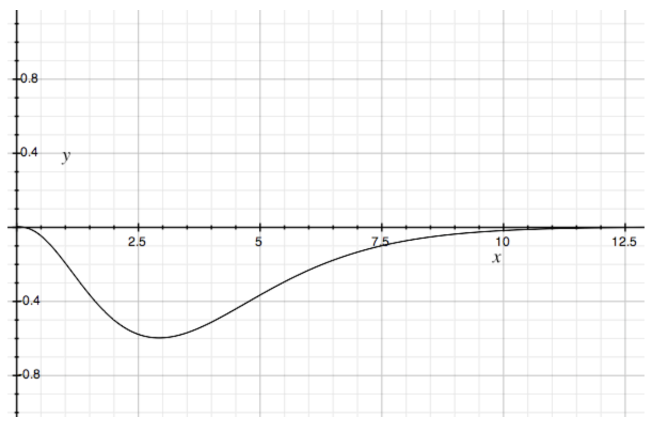

Figure 1: A plot of the function $f(y) = y^3 \log(1 - e^{-y})$

The crucial point is that both limits in the first term give zero. To see that you have to take the $y \to 0$ and $y \to \infty$ limits carefully. The sketch of the function in the first term is given in Figure (1).

So we end up with the second term only

$$F = -\frac{\pi V \tau^4}{(\pi\hbar c)^3} \frac{1}{3} \int_0^\infty \mathrm{d}y \frac{y^3}{e^y - 1} \tag{17}$$

The integral is known to have the numerical value $\pi^4/15$. Hence the final answer is obtain after simplifications to be:

$$F = -\frac{\pi^2 V \tau^4}{45\hbar^3 c^3} \tag{18}$$

Question 5) This is effectively one dimensional motion. We can describe the system completely by the angle θ and its rate of change $\dot{\theta} = \omega$. In statistical physics Hamiltonian is the object that is most useful. It appears in the exponential in the Boltzmann factor. The Hamiltonian is a function of generalized position and momenta. In this problem θ is the generalized position and $\dot{\theta}$ is the generalized velocity. To switch from generalized velocity to momentum we need to introduce the Lagrangian (This subject is usually covered in classical mechanics class phys410).

$$\mathcal{L}(\theta, \omega) = T - V \tag{19}$$

Here $\mathcal{L}$ denotes the Lagrangian, T is the kinetic energy and V is the potential energy. Note that $\mathcal{L}$ is a function of position and velocity. The momentum conjugate to θ is given by

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \omega} \tag{20}$$

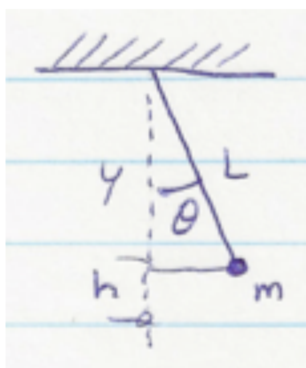

Figure 2: Pendulum Problem

Now we have all what it takes to build the Hamiltonian. Let us start by writing the kinetic energy in terms of position and velocity

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(L\omega)^2 \tag{21}$$

where I used $v = L\omega$. Next write down the potential energy. I will set the lowest point of the pendulum as the zero of my energy scale. Then using simple geometry we determine the height of the mass from this point to be $h = L(1 - \cos\theta)$. The potential energy is:

$$U = mgL(1 - \cos\theta) \tag{22}$$

To switch to the Hamiltonian we need the momentum. Using eq(20) with eq(21) we obtain

$$p_\theta = mL^2\omega \tag{23}$$

In physical systems where energy is conserved the Hamiltonian of the system can be obtained by writing the total energy in terms of the generalized position and momentum. This is a very important point. The Hamiltonian is a function of position and momenta. Replacing $\omega = p_\theta/(mL^2)$ in eq(21) we have

$$H(\theta, p_\theta) = \frac{p_\theta^2}{2mL^2} + mgL(1 - \cos\theta) \tag{24}$$

In passing we also note that p_θ is nothing but the angular momentum of the pendulum and mL^2 is the moment of inertia.

Now we have the Hamiltonian but it has the awkward cosine term. For small angles we can approximate this term using $\cos\theta = 1 - \theta^2/2 + \cdots$. Then eq(24) becomes

$$H(\theta, p_\theta) = \frac{p_\theta^2}{2mL^2} + \frac{mgL\theta^2}{2} \tag{25}$$

Now we can calculate averages using the standard Boltzmann factors.

$$< \theta >= \frac{\frac{1}{h}\int_\infty^\infty dp_\theta \int_\infty^\infty d\theta e^{-\frac{1}{\tau}\left(\frac{p_\theta^2}{2mL^2} + \frac{mgL\theta^2}{2}\right)}\theta}{\frac{1}{h}\int_\infty^\infty dp_\theta \int_\infty^\infty d\theta e^{-\frac{1}{\tau}\left(\frac{p_\theta^2}{2mL^2} + \frac{mgL\theta^2}{2}\right)}}$$

Note that we let the integrals over θ go from minus infinity to plus infinity. In reality the angles vary from $-\pi$ to π, however the error introduced by extending the integral is small because of the exponential dependence on θ^2. The integrand gets very small and doesn't contribute significantly to the integral over this region. This way we obtain gaussian integrals that you are by now familiar with. First observe that the momentum integrations are the same in the numerator and denominator. So they cancel out. The θ integral in the numerator gives zero, because the integrand is an odd function of θ integrated from minus to plus infinity.

For the average $< \theta^2 >$ we write (after canceling the momentum integrals):

$$< \theta^2 >= \frac{\int_\infty^\infty d\theta e^{-\frac{1}{\tau}\left(\frac{mgL\theta^2}{2}\right)}\theta^2}{\int_\infty^\infty d\theta e^{-\frac{1}{\tau}\left(\frac{mgL\theta^2}{2}\right)}}$$

The gaussian integrals can be converted to dimensionless quantities by the change of variables: $u = \sqrt{\frac{mgL}{2\tau}}\theta$, which leads to

$$< \theta^2 >= \frac{2\tau}{mgL} \times \frac{\int_{\infty}^{\infty} due^{-u^2}u^2}{\int_{\infty}^{\infty} due^{-u^2}} = \frac{\tau}{mgL}$$

In the last step we used the well known values of the gaussian integrals. Note that the variance is proportional to the temperature. The pendulum will have thermally induced oscillations.

For the $< v >$ and $< v^2 >$, the methods are similar with the above. First, equation (23) tells us

$$v = Lw = P_\theta / mL$$

Thus,

$$< v >=< P_\theta > /mL$$

$$< v^2 >=< P_\theta^2 > /m^2L^2$$

The same process as $\langle\theta\rangle$ and $\langle\theta^2\rangle$:

$$< P_\theta >= \frac{\int_{-\infty}^{\infty} dP_\theta e^{-\frac{1}{\tau}\left(\frac{P_\theta^2}{2mgL}\right)} P_\theta}{\int_{-\infty}^{\infty} dP_\theta e^{-\frac{1}{\tau}\left(\frac{P_\theta^2}{2mgL}\right)}} = 0$$

$$< v >=< P_\theta > /mL = 0$$

And for $\langle v^2\rangle$:

$$< P_\theta^2 >= \frac{\int_{-\infty}^{\infty} dP_\theta e^{-\frac{1}{\tau}\left(\frac{P_\theta^2}{2mgL}\right)} P_\theta^2}{\int_{-\infty}^{\infty} dP_\theta e^{-\frac{1}{\tau}\left(\frac{P_\theta^2}{2mgL}\right)}} = mL^2\tau$$

$$< v^2 >=< P_\theta^2 > /m^2L^2 = \frac{\tau}{m}$$

Question 6) Suppose we want $\sqrt{< \theta^2 >} = 0.001$ radians at room temperature (300K). What value of mL is required?

From problem 1 we have

$$\sqrt{< \theta^2 >} = \sqrt{\frac{\tau}{mgL}}$$

Solving for mL

$$\begin{aligned} mL &= \frac{\tau}{g < \theta^2 >} \\ &= \frac{k_B 300K}{9.8m/s^2(10^{-3}Rad)^2} \\ &= 4.2 \times 10^{-16} kg \cdot m \end{aligned}$$

This is a very small number.
Suppose the pendulum is $10\mu m$ long. Then its mass would have to be $42ng$, which is very small too. The lesson to learn is that observable fluctuations occur only in very small systems. For everyday life objects we can generally neglect thermal fluctuations.

Question 7) Consider the N-particle system with the Hamiltonian given by

$$\mathcal{H}(\vec{p},\vec{q}) = \sum_{i=1}^{N} \frac{p_{x_i}^2 + p_{y_i}^2 + p_{z_i}^2}{2m_i} + V(\vec{r}_1, \vec{r}_2 \cdots, \vec{r}_N)$$

This assumes that the potential of interaction V between the particles is a function of their coordinates only.
The partition function becomes:

$$Z = \frac{1}{h^{3N}} \int \cdots \int (d\vec{p})^N \int \cdots \int (d\vec{r})^N e^{-\sum_{i=1}^{N} \frac{\vec{p}_i \cdot \vec{p}_i}{2m_i \tau}} \times e^{-V(\vec{r}_1, \vec{r}_2 \cdots, \vec{r}_N)/\tau}$$

Note that the two parts of the integral depend only on the momenta and coordinates separately, so they can be separated into a product as:

$$Z = \left(\frac{V^N}{h^{3N}} \int \cdots \int (d\vec{p})^N e^{-\sum_{i=1}^{N} \frac{\vec{p}_i \cdot \vec{p}_i}{2m_i \tau}} \right) \times \left(\frac{1}{V^N} \int \cdots \int (d\vec{r})^N e^{-V(\vec{r}_1, \vec{r}_2 \cdots, \vec{r}_N)/\tau} \right)$$

Note that we multiplied and divided by V^N. The first part is the partition function for an ideal gas of N particles (without the N! - we assume each particle is distinguishable by its mass m_i).

The second part depends only on the coordinates of the particle, and is due to the potential. Hence $Z = Z_{IG} Z_{pot}$. This enables us to treat the complications due to the potential separately.