PHY 410 Homework 5 Solutions

1. Quantum concentration

$$\lambda = \frac{h}{p} \qquad \text{Define } \lambda_{th} \text{ by } \quad \frac{p^2}{2m} = \frac{3}{2}\tau$$

$$\underline{\underline{\lambda_{th} = \frac{h}{\sqrt{3m\tau}}}} \qquad\qquad p = \sqrt{3m\tau}$$

If the average spacing between particles is λ_{th}, then the volume per particle is $\sim \lambda_{th}^3$, so the number density is

$$n = \frac{1}{\lambda_{th}^3} = \frac{(3m\tau)^{3/2}}{h^3} = \left(\frac{3m\tau}{(2\pi\hbar)^2}\right)^{3/2}$$

This is almost the same as $\quad n_Q = \left(\dfrac{m\tau}{2\pi\hbar^2}\right)^{3/2}$

$$\frac{n}{n_Q} = \left(\frac{3/4\pi^2}{1/2\pi}\right)^{3/2} = \left(\frac{3}{2\pi}\right)^{3/2} = \underline{\underline{0.33}}$$

2. K + K. Ch. 3 #11 1D Ideal Gas

$$\psi_n = A \sin\left(n\pi \frac{x}{L}\right) \qquad \varepsilon_n = \frac{\hbar^2}{2m}\left(n\frac{\pi}{L}\right)^2, \quad n = 1, 2, 3 \ldots$$

$$Z_1 = \sum_{n=1}^{\infty} e^{-\varepsilon_n/\tau} = \sum_{n=1}^{\infty} e^{-\hbar^2 n^2 \pi^2 / 2mL^2\tau}$$

$$\approx \int_0^\infty dn\, e^{-\frac{n^2\pi^2\hbar^2}{2mL^2\tau}} = \frac{L\sqrt{2m\tau}}{\pi\hbar} \underbrace{\int_0^\infty dx\, e^{-x^2}}_{\sqrt{\pi}/2} \qquad x = \frac{\pi\hbar}{L\sqrt{2m\tau}}\, n \qquad dx = \frac{\pi\hbar}{L\sqrt{2m\tau}}\, dn$$

$$Z_1 = \frac{L}{\hbar}\sqrt{\frac{m\tau}{2\pi}}$$

N particles: $Z_N = \dfrac{(Z_1)^N}{N!}$ $\qquad \ln Z_N = N \ln Z_1 - \ln N!$

$$\ln Z_N = N \ln \frac{L}{\hbar}\sqrt{\frac{m\tau}{2\pi}} - N\ln N + N = N \ln \frac{L}{N}\sqrt{\frac{m\tau}{2\pi\hbar^2}} + N$$

$$F = -\tau \ln Z = -N\tau\left[\ln \frac{L}{N}\sqrt{\frac{m\tau}{2\pi\hbar^2}} + 1\right]$$

$$\sigma = -\left(\frac{\partial F}{\partial \tau}\right)_{L,N} = N\left[\ln \frac{L}{N}\sqrt{\frac{m\tau}{2\pi\hbar^2}} + 1\right] + N\tau \underbrace{\frac{d}{d\tau}\left(\frac{1}{2}\ln\tau\right)}_{\frac{1}{2\tau}}$$

$$\sigma = N\left[\ln \frac{L}{N}\sqrt{\frac{m\tau}{2\pi\hbar^2}} + \frac{3}{2}\right]$$

$$\underline{\underline{\sigma = N\left[\ln \frac{n_Q}{n} + \frac{3}{2}\right]}} \quad \text{where } n \equiv \frac{N}{L}, \quad n_Q = \sqrt{\frac{m\tau}{2\pi\hbar^2}}$$

2. Alternative method: Use periodic boundary conditions as I did in class:

$$\psi_n = \frac{1}{\sqrt{L}} e^{ik_n x} \qquad k_n = \frac{2\pi}{L} n \quad , n = 0, \pm 1, \pm 2, \dots$$

$$\varepsilon_n = \frac{\hbar^2 k_n^2}{2m}$$

$$Z_1 = \sum_{k_n} e^{-\frac{\hbar^2 k_n^2}{2m\tau}} = \frac{L}{2\pi} \int_{-\infty}^{\infty} dk \, e^{-\frac{\hbar^2 k^2}{2m\tau}} \qquad x = \frac{\hbar k}{\sqrt{2m\tau}}$$

$$dx = \frac{\hbar}{\sqrt{2m\tau}} dk$$

$\frac{L}{2\pi}$ is the "density of modes in k space"

$$Z_1 = \frac{L}{2\pi} \cdot \frac{\sqrt{2m\tau}}{\hbar} \cdot \underbrace{\int_{-\infty}^{\infty} dx \, e^{-x^2}}_{\sqrt{\pi}} = \underline{\frac{L}{\hbar}\sqrt{\frac{m\tau}{2\pi}}}$$

The rest of the problem is on the previous page, starting with

$$Z_N = \frac{(Z_1)^N}{N!}$$

3. K. + K. Ch 5, #4 Ion concentrations in cells

$$n_{in} = 10^4 \, n_{out}$$

$$\mu = \tau \ln n/n_Q$$

$$\mu_{in} - \mu_{out} = \tau\left(\ln \frac{n_{in}}{n_Q} - \ln \frac{n_{out}}{n_Q}\right) = \tau \ln \frac{n_{in}}{n_{out}}$$

$$= k_B T \ln 10^4 = 1.38 \cdot 10^{-23} \frac{J}{K} \cdot 300\,K \cdot 9.2 = 3.8 \cdot 10^{-20}\,J$$

Convert $\Delta\mu$ to a voltage given a charge of $+e$:

$$\Delta V = \frac{\Delta\mu}{e} = \frac{3.8 \cdot 10^{-20}\,J}{1.6 \cdot 10^{-19}\,C} = \underline{\underline{0.24 \text{ Volts}}}$$

4. K. + K. Ch 5, #1 Centrifuge

In the rotating frame, the centrifugal force is $\vec{F} = \frac{mv^2}{r}\hat{r} = m\omega^2 r\,\hat{r}$

The effective potential energy is $U(r) = -\int_0^r \vec{F}\cdot d\vec{r} = -m\omega^2 \int_0^r r\,dr$

$$U(r) = -\tfrac{1}{2} m\omega^2 r^2 \leftarrow \text{This is our } \mu_{ext}(r)$$

$$\mu_{total}(r) = \mu_{int}(r) + \mu_{ext}(r) = \text{constant}$$

$$\tau \ln n(r)/n_Q - \frac{1}{2} m\omega^2 r^2 = \text{constant} = \tau \ln n(0)/n_Q$$

$$\tau\left(\ln \frac{n(r)}{n_Q} - \ln \frac{n(0)}{n_Q}\right) = \frac{1}{2} m\omega^2 r^2$$

$$\ln n(r)/n(0) = \frac{m\omega^2 r^2}{2\tau} \Rightarrow \underline{\underline{n(r) = n(0)\, e^{\frac{m\omega^2 r^2}{2\tau}}}}$$

5. $\mu = \tau \ln \frac{n}{n_Q}$ $\qquad n_A = 1.5 \cdot 10^{18}\ m^{-3} \qquad n_B = 3.0 \cdot 10^{18}\ m^{-3}$

$$\mu_A - \mu_B = \tau\left(\ln \frac{n_A}{n_Q} - \ln \frac{n_B}{n_Q}\right) = \tau \ln \frac{n_A}{n_B} = \tau \ln \frac{1}{2}$$

$$= 1.38 \cdot 10^{-23}\ \frac{J}{K} \cdot 300\ K \cdot \qquad = -2.87 \cdot 10^{-21}\ J$$

a) Box A has a lower chemical potential so we must increase μ_A by adding a positive electrostatic energy

$$\mu_{A\ total} = \mu_{B\ total}$$

$$\mu_{A\ int} + q\,\Delta V = \mu_{B\ int}$$

$$\Delta V = \frac{\mu_{B\ int} - \mu_{A\ int}}{q} = \frac{+2.87 \cdot 10^{-21}\ J}{1.60 \cdot 10^{-19}\ C} = \underline{\underline{0.0179\ V}}$$

positive

b) $\mu_{A\ int} + mgh = \mu_{B\ int}$

$m = 1.67 \cdot 10^{-27}\ kg$ = mass of H^+ ion (proton)

$$h = \frac{\mu_{B\ int} - \mu_{A\ int}}{mg} = \frac{2.87 \cdot 10^{-21}\ J}{1.67 \cdot 10^{-27}\ kg \cdot 9.8\ m/s^2} = \underline{\underline{1.75 \cdot 10^{5}\ m}}$$

that is very high, and totally impractical