PHY 410 Homework 4 Solutions

1. K + K. Ch 4, # 2 Earth - Sun distance $d = 1.5 \cdot 10^{13}$ cm

Energy flux density at Earth from Sun: $I = 0.136 \frac{J}{s\text{-}cm^2}$

a) $P = I \cdot 4\pi d^2 = 0.136 \frac{J}{s\text{-}cm^2} \cdot 4\pi \cdot (1.5 \cdot 10^{13} cm)^2$

$$= \underline{\underline{3.8 \cdot 10^{26} \, J/s}}$$

b) The energy flux density at the surface of the Sun is

$$\frac{P}{4\pi R_\odot^2} = \sigma_B T_\odot^4 \qquad \odot = \text{Sun}$$

$$T_\odot = \left(\frac{P}{4\pi R_\odot^2 \sigma_B}\right)^{1/4} = \left(\frac{3.8 \cdot 10^{26} \, J/s}{4\pi \cdot (7 \cdot 10^{10} cm)^2 \cdot 5.67 \cdot 10^{-12} \frac{J}{s\text{-}cm^2\text{-}K^4}}\right) = \underline{\underline{5700 \text{ K}}}$$

2. K + K. Ch 4, # 5 Surface temperature of the Earth

The Earth receives energy from the Sun over an area of πR_E^2
The Earth emits black-body radiation over its total area $4\pi R_E^2$.

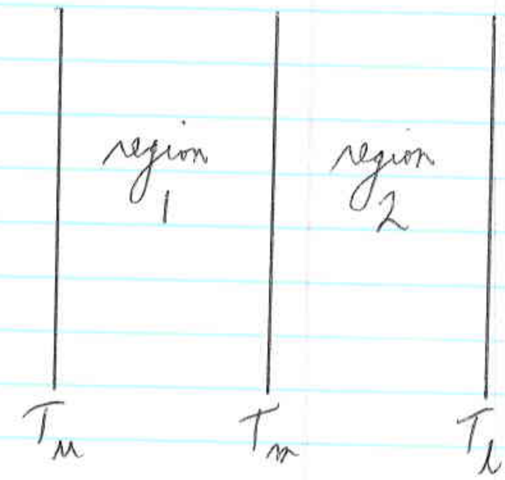
$$P_{in} = P_{out}$$

$$I_{solar} \cdot \pi R_E^2 = \sigma_B T_E^4 \cdot 4\pi R_E^2$$

I will use I_{solar} from the previous problem. But you can derive it from the given numbers.

$$T_E = \left(\frac{I_{solar}}{4\sigma_B}\right)^{1/4} = \left(\frac{0.136 \, J/s\text{-}cm^2}{4 \cdot 5.67 \cdot 10^{-12} \frac{J}{s\text{-}cm^2 \cdot K^4}}\right)^{1/4} = \underline{\underline{280 \text{ K}}}$$

3. K. + K. Ch. 4 # 8



region 1 : $J_u^1 = \sigma_B \left(T_u^4 - T_m^4 \right)$

region 2 : $J_u^2 = \sigma_B \left(T_m^4 - T_l^4 \right)$

Under steady-state conditions,

$$J_u^1 = J_u^2$$

$$T_u^4 - T_m^4 = T_m^4 - T_l^4$$

$$2T_m^4 = T_u^4 + T_l^4$$

$$T_m = \left(\frac{T_u^4 + T_l^4}{2} \right)^{1/4}$$

$$J_u^1 = \sigma_B \left[T_u^4 - \left(\frac{T_u^4 + T_l^4}{2} \right) \right] = \frac{1}{2} \sigma_B \left(T_u^4 - T_l^4 \right)$$

This is half the energy flux we would get if the middle plane were not there.

4. K. + K. Ch. 4, #1 Number of thermal photons in a cavity of volume $V = L^3$.

Method 1, following pages 93 + 94 in K. + K.

$$N = \sum_n \langle s_n \rangle = \underset{\text{polarization}}{2} \cdot \frac{1}{8} \int_0^\infty 4\pi n^2 dn \langle s_n \rangle$$

$$= \pi \int_0^\infty \frac{n^2 dn}{e^{\hbar\omega_n/\tau} - 1} \quad \text{where } \omega_n = \frac{n\pi c}{L}$$

Let $x = \frac{\hbar \pi c}{L\tau} n \qquad dx = \frac{\hbar\pi c}{L\tau} dn$

$$N = \pi \left(\frac{L\tau}{\hbar\pi c}\right)^3 \int_0^\infty \frac{x^2 dx}{e^x - 1}$$

Mathematica says the integral is $2\zeta(3) = 2.404$

$$N = 2.404 \frac{V}{\pi^2} \left(\frac{\tau}{\hbar c}\right)^3$$

Method 2, following my lecture on 2/2/18:

$$N = \sum_n \langle s_n \rangle = 2 \cdot \frac{V}{(2\pi)^3} \cdot \int_0^\infty 4\pi k^2 dk \cdot \frac{1}{e^{\hbar c k/\tau} - 1}$$

let $x = \frac{\hbar c k}{\tau} \qquad dx = \frac{\hbar c}{\tau} dk$

$$N = \frac{V}{\pi^2}\left(\frac{\tau}{\hbar c}\right)^3 \int_0^\infty \frac{x^2 dx}{e^x - 1} = 2.404 \frac{V}{\pi^2}\left(\frac{\tau}{\hbar c}\right)^3$$

5. Maxwell Relation $\quad F = U - \tau\sigma \qquad N$ is fixed, so $dN = 0$

$$dF = dU - d(\tau\sigma) = (\cancel{\tau d\sigma} - p\,dV) - (\cancel{\tau d\sigma} + \sigma\, d\tau)$$

$$dF = -\sigma\, d\tau - p\, dV \qquad (1)$$

We also can write $\quad dF = \left(\frac{\partial F}{\partial \tau}\right)_V d\tau + \left(\frac{\partial F}{\partial V}\right)_\tau dV \qquad (2)$

Compare (1) + (2): $\quad \sigma = -\left(\frac{\partial F}{\partial \tau}\right)_V \qquad p = -\left(\frac{\partial F}{\partial V}\right)_\tau$

Crossed 2^{nd} derivatives: $\quad \frac{\partial^2 F}{\partial V \partial \tau} = \frac{\partial^2 F}{\partial \tau \partial V}$

more explicitly: $\left(\frac{\partial}{\partial V}\left(\frac{\partial F}{\partial \tau}\right)_V\right)_\tau = \left(\frac{\partial}{\partial \tau}\left(\frac{\partial F}{\partial V}\right)_\tau\right)_V$

$$-\left(\frac{\partial \sigma}{\partial V}\right)_\tau = -\left(\frac{\partial p}{\partial \tau}\right)_V$$

$$\underline{\underline{\left(\frac{\partial \sigma}{\partial V}\right)_\tau = \left(\frac{\partial p}{\partial \tau}\right)_V}}$$

6. $$\langle s \rangle = \frac{1}{e^{\hbar\omega/\tau}-1} = \frac{1}{e^{x}-1} \quad \text{where} \quad x \equiv \frac{\hbar\omega}{\tau}$$

a) High temperature expansion : $x \ll 1$

$$e^{x}-1 \simeq x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots = x\left(1+\frac{x}{2}+\frac{x^2}{6}+\ldots\right)$$

$$\langle s \rangle = \frac{1}{x}\,\frac{1}{1+\left(\frac{x}{2}+\frac{x^2}{6}\right)} = \frac{1}{x}\left[1-\left(\frac{x}{2}+\frac{x^2}{6}\right)+\left(\frac{x}{2}+\frac{x^2}{6}\right)^2-\ldots\right]$$

$$= \frac{1}{x}\left[1-\frac{x}{2}-\frac{x^2}{6}+\frac{x^2}{4}+O(x^3)\right]$$

$$= \frac{1}{x}\left[1-\frac{x}{2}+\frac{x^2}{12}+O(x^3)\right]$$

$$\langle s \rangle = \frac{\tau}{\hbar\omega}\left[1-\frac{1}{2}\frac{\hbar\omega}{\tau}+\frac{1}{12}\left(\frac{\hbar\omega}{\tau}\right)^2+O\left(\frac{\hbar\omega}{\tau}\right)^3\right]$$

$$\langle s \rangle_{asymptotic} = \frac{\tau}{\hbar\omega} \qquad a = -\frac{1}{2} \qquad b = \frac{1}{12}$$

b) $$U = \left(\langle s \rangle + \frac{1}{2}\right)\hbar\omega = \langle s \rangle\hbar\omega + \frac{1}{2}\hbar\omega$$

$$= \tau\left[1-\frac{1}{2}\frac{\hbar\omega}{\tau}+\frac{1}{12}\left(\frac{\hbar\omega}{\tau}\right)^2\right] + \frac{1}{2}\hbar\omega$$

$$= \tau - \cancel{\frac{1}{2}\hbar\omega} + \frac{1}{12}\frac{(\hbar\omega)^2}{\tau} + \ldots + \cancel{\frac{1}{2}\hbar\omega}$$

$$U = \tau\left[1+\frac{1}{12}\left(\frac{\hbar\omega}{\tau}\right)^2+\ldots\right]$$