Homework #3 Solutions

Question 1a) Let us start by calculating the entropy using the given multiplicity function

$$\sigma(N,U) = \log[g(N,U)] = \frac{3N}{2}\log(U) + \log(C)$$

The fundamental temperature τ is defined by

$$\frac{1}{\tau} = \left(\frac{\partial \sigma}{\partial U}\right)_N = \frac{3N}{2U} \tag{1}$$

where we have used the fact that the derivative of $\log(U)$ equals 1/U. We can solve this equation for U to get

$$U = \frac{3N\tau}{2} = \frac{3Nk_BT}{2}$$

1b) Take the second derivative of the entropy

$$\left(\frac{\partial^2 \sigma}{\partial U^2}\right)_N = \frac{\partial}{\partial U}\left(\frac{3N}{2U}\right)_N = -\frac{3N}{2U^2} < 0$$

This tells us that the entropy versus energy curve is concave down, suggesting that the temperature of the system τ (inverse slope of σ vs. U) will increase, with increasing U.

Question 2) Let us start by following the hint. The energy is related to the spin excess by $U = -\vec{B}\cdot\vec{M} = -2smB$. Using this relation we can write the entropy as a function of energy as

$$\sigma(U) = \sigma_0 - \frac{U^2}{2m^2B^2N},$$

with $\sigma_0 = \log[g(N,0)]$. Next use the relation between entropy and temperature given by the first equality in equation (**??**) above to get

$$\frac{1}{\tau} = -\frac{U}{m^2B^2N}$$

In order to get the magnetization, recall that U is related to it by $U = -MB$ (considering only the component of magnetic field along magnetization). Putting this into the above formula and solving for M we get

$$M = -\frac{U}{B} = \frac{Nm^2B}{\tau}$$

A quantity of interest is the fractional magnetization which is defined by

$$\frac{M}{Nm} = \frac{mB}{\tau}$$

Finally observe that the magnetic susceptibility obeys the so called Curie-Weiss Law:

$$\chi = \frac{M}{B} \sim \frac{1}{\tau}$$

Question 3) We are going to use the multiplicity function given by eq(1.55) in K+K for $N \gg n$. In this case Stirling's approximation can be used.

$$\sigma(n) = \log[g(N,n)] = \log[(N+n-1)!] - \log(n!) - \log[(N-1)!]$$

Recall Stirling's formula $\log N! \approx N \log N - N$. Also replace $N-1$ by N.

$$\sigma(n) = (N+n)\log(N+n) - (N+n) - n\log n + n - N\log N + N$$

After some more algebraic manipulation one gets

$$\sigma(n) = N\left[\left(1+\frac{n}{N}\right)\log\left(1+\frac{n}{N}\right) - \frac{n}{N}\log\left(\frac{n}{N}\right)\right]$$

Here note that n/N is our small parameter.

3b) The energy is related to n by the simple relation $U = n\hbar\omega$ or $\frac{n}{N} = \frac{U}{\hbar\omega N}$. Using this we can write the entropy as a function of U and N.

$$\sigma(U,N) = N\left[\left(1+\frac{U}{N\hbar\omega}\right)\log\left(1+\frac{U}{N\hbar\omega}\right) - \frac{U}{N\hbar\omega}\log\left(\frac{U}{N\hbar\omega}\right)\right]$$

To get the temperature we again use equation (**??**) above. After some algebra we get

$$\frac{1}{\tau} = \frac{1}{\hbar\omega}\log\left((1+\frac{U}{N\hbar\omega})/(U/N\hbar\omega)\right)$$

Moving the factor of $\hbar\omega$ to the LHS and exponentiating both sides we can get U out of the logarithm and solve for it to get

$$U = \frac{N\hbar\omega}{e^{\hbar\omega/\tau} - 1}$$

This is basically the Planck black body radiation formula for the energy of photons in a box.

Question 4a) The probability of getting each character right is 1/44 assuming the monkeys have no bias, which I am pretty sure is not the case. Since we assume independent probabilities, the probability of getting successive characters right is just the product of individual probabilities.

$$P = \left(\frac{1}{44}\right)^{10^5} = 10^{-164,345}$$

4b) There are 10^{10} monkeys each typing 10^1 characters per second and they have all the time in the universe: 10^{18} seconds. This means there are a total of 10^{29} key strokes made. To a very good approximation we can say there are about 10^{29} sequences of of length 10^5 characters, for which the probability of getting it right was calculated in part a. Hence the probability of having a monkey-Hamlet is

$$P = 10^{29} \times 10^{-164,345} = 10^{-164,316} \approx 0$$

Question 5a) Take $N_1 = N_2 = 10^{22}$, $\delta = 10^{11}$, and $s = 0$, use the Gaussian approximation for the multiplicity,

$$g_1(n_1, \hat{s}_1 + \delta)g_2(N_2, \hat{s}_2 - \delta) = (g_1 g_2)_{max} e^{-\frac{2\delta^2}{N_1} - \frac{2\delta^2}{N_2}} \tag{2}$$

$$\frac{g_1 g_2}{(g_1 g_2)_{max}} = e^{-\frac{2\delta^2}{N_1} - \frac{2\delta^2}{N_2}} = e^{-4} = 0.0183$$

This is much more likely than $\delta = 10^{12}$ for which $P \sim 10^{-174}$.

Question 6) Recall equation (**??**) above. Define $x = \frac{\delta}{N_1} = \frac{\delta}{N_2}$. Then we get

$$g_1(n_1, \hat{s}_1 + \delta)g_2(N_2, \hat{s}_2 - \delta) = (g_1 g_2)_{max} e^{-4N_1 x^2}$$

What is the total number of states counting over all deviations δ ?

$$Total = (g_1 g_2)_{max} \int_{-\infty}^{\infty} e^{-4N_1 x^2} \mathrm{d}x N_1$$

Let $a = 4N_1$ and use standard gaussian integral formula to get

$$Total = (g_1 g_2)_{max} N_1 \sqrt{\frac{\pi}{4N_1}}$$

The number of states from $x_1 = 10^{-10}$ to ∞ is

$$Partial = (g_1 g_2)_{max} \int_{x_1}^{\infty} e^{-4N_1 x^2} \mathrm{d}x N_1$$

Let $y = 2\sqrt{N_1}x$, then $\mathrm{d}y = 2\sqrt{N_1}\mathrm{d}x$. For the limits of integration we have $y_1 = 2\sqrt{N_1}x_1 = 2\delta/\sqrt{N_1}$. But $\delta/N_1 \geq 10^{-10}$ from which it follows that $y_1 \geq 20$. As a result we can write the integral as

$$Partial = (g_1 g_2)_{max} N_1 \frac{1}{2\sqrt{N_1}} \int_{20}^{\infty} e^{-y^2} \mathrm{d}y$$

Now we are in a position to use the results for the complementary error function with x set to 20.

$$Partial = (g_1 g_2)_{max} \frac{\sqrt{N_1}}{2} \frac{e^{-400}}{40}(1 + \varepsilon)$$

The ratio of the number of states with deviation greater than 10^{-10} to the total number of states is

$$\begin{aligned} Ratio &= \frac{(g_1 g_2)_{max} \frac{\sqrt{N_1}}{2} \frac{e^{-400}}{40}(1 + \varepsilon)x^2}{(g_1 g_2)_{max} \frac{\sqrt{\pi N_1}}{2}} \\ &= \frac{2e^{-400}}{40\sqrt{\pi}}(1 + \varepsilon) \\ &\approx 5.4 \times 10^{-176} \end{aligned}$$

not very likely. The message is that the system is very unlikely to show deviations from those states with maximum multiplicity, certainly not more than 1 part in 10^{10}.