

Homework 9 Solutions

Question 1

K+K Chapter 6, Problem 9

a) We can calculate the chemical potential using eq(6.48):

$$\mu = \tau (\log(n/n_Q) - \log Z_{int}) \quad (1)$$

$$Z_{int} = \sum_{\text{int. states}} e^{-\epsilon_{int}/\tau} = 1 + e^{-\Delta/\tau} \quad (2)$$

$$\mu = \tau \log \left(\frac{n/n_Q}{1 + e^{-\Delta/\tau}} \right) \quad (3)$$

b) The partition function in the presence of internal degrees of freedom is $Z = Z_{IG} \times Z_{int}$. As a result the free energy, which is proportional to the logarithm of Z is additive: $F = F_{IG} + F_{int}$. F_{IG} is the known ideal gas result. Let us calculate the contribution from the internal degrees of freedom.

$$F_{int} = -N\tau \log Z_{int} \quad (4)$$

$$F_{int} = -N\tau \log (1 + e^{-\Delta/\tau}) \quad (5)$$

$$F = F_{IG} + F_{int} = N\tau \log \left[\frac{n/n_Q}{1 + e^{-\Delta/\tau}} \right] - N\tau \quad (6)$$

c) The entropy can be easily calculated using the free energy obtained in part b using the formula:

$$\sigma_{int} = \left. \frac{\partial F_{int}}{\partial \tau} \right|_V \quad (7)$$

$$= -\frac{\partial}{\partial \tau} \left(-N\tau \log (1 + e^{-\Delta/\tau}) \right) \quad (8)$$

$$= N \left\{ \log (1 + e^{-\Delta/\tau}) + \frac{\Delta/\tau}{e^{\Delta/\tau} + 1} \right\} \quad (9)$$

Just like the free energy, entropy is additive as well:

$$\sigma = \sigma_{IG} + \sigma_{int} \quad (10)$$

$$= N \left\{ \log (n_Q/n) + 5/2 + \log (1 + e^{-\Delta/\tau}) + \frac{\Delta/\tau}{e^{\Delta/\tau} + 1} \right\} \quad (11)$$

$$= N \left\{ \log \left(\frac{n_Q(1 + e^{-\Delta/\tau})}{n} \right) + \frac{5}{2} + \frac{\Delta/\tau}{e^{\Delta/\tau} + 1} \right\} \quad (12)$$

d) Pressure is not effected by the existence of the internal degrees of freedom:

$$p = -\left.\frac{\partial F}{\partial V}\right|_{\tau, N} \quad (13)$$

$$= -\frac{\partial}{\partial V} \left\{ N \left\{ \log \left(\frac{n_Q(1 + e^{-\Delta/\tau})}{n} \right) + \frac{5}{2} + \frac{\Delta/\tau}{e^{\Delta/\tau} + 1} \right\} \right\} \quad (14)$$

$$= \frac{N\tau}{V} \quad (15)$$

e) Heat capacity at constant pressure

$$C_p = \tau \left.\frac{\partial \sigma}{\partial \tau}\right|_p = \left.\frac{\partial U}{\partial \tau}\right|_p + p \left.\frac{\partial V}{\partial \tau}\right|_p \quad (16)$$

$$U = F + \tau \sigma = \dots = \frac{3}{2}N\tau + \frac{N\Delta}{e^{\Delta/\tau} + 1} \quad (17)$$

$$\left.\frac{\partial U}{\partial \tau}\right|_p = \frac{3}{2}N + N \frac{(\Delta^2/\tau^2)e^{\Delta/\tau}}{(e^{\Delta/\tau} + 1)^2} \quad (18)$$

$$C_p = \frac{3}{2}N + N \frac{(\Delta^2/\tau^2)e^{\Delta/\tau}}{(e^{\Delta/\tau} + 1)^2} + p \left(\frac{N}{p} \right) \quad (19)$$

where in the last term of the last equation we used the ideal gas law to obtain $\frac{\partial V}{\partial \tau}$. Hence the final answer reads

$$C_p = \frac{5}{2}N + N \frac{(\Delta^2/\tau^2)e^{\Delta/\tau}}{(e^{\Delta/\tau} + 1)^2} \quad (20)$$

Question 2

K+K Chapter 6, Problem 12

a) In K+K Chapter 7, Problem 1, part b we calculate the density of states for an electron in 2 dimensions. We can borrow that result here but we need to divide it by 2 because electrons have spin 1/2 and 2 degenerate states associated with every energy. For the ideal gas we will assume zero spin hence no degeneracy factor. Hence from Eq.(39) below:

$$D(\varepsilon) = \frac{Am}{2\pi\hbar^2} \quad (21)$$

where A is the area of the container. Using this we can calculate the one particle partition function

$$Z_1 = \int d\varepsilon D(\varepsilon) e^{-\varepsilon/\tau} = \frac{mA\tau}{2\pi\hbar^2} \equiv n_Q^{2D} A \quad (22)$$

where the last equality defines the two dimensional quantum density. Chemical potential is found by enforcing the number constraint:

$$N = \lambda Z_1 = \lambda n_Q^{2D} A \quad (23)$$

$$\lambda = \frac{N/A}{n_Q^{2D}} = \frac{n^{2D}}{n_Q^{2D}} \quad (24)$$

$$\mu = \tau \log \left(\frac{n^{2D}}{n_Q^{2D}} \right) \quad (25)$$

b) What is the energy of the gas? Find the free energy first. This can be done in two ways. One can integrate $\mu = \frac{\partial F}{\partial N}|_{\tau, V}$. I will calculate F directly from the partition function for N particles. Since we are considering classical indistinguishable particles $Z = Z_1^N / N!$ using Stirlings formula for large N we get

$$F = -\tau \log Z = -\tau \log \left(\frac{Z_1^N}{N!} \right) = \tau N \left[\log \left(\frac{n^{2D}}{n_Q^{2D}} \right) - 1 \right] \quad (26)$$

once we have the free energy the internal energy is obtained by partial differentiation wrt temperature:

$$U = -\tau^2 \left(\frac{\partial F}{\partial \tau} \right)_{V, N} \quad (27)$$

$$= -\tau^2 \frac{\partial}{\partial \tau} \{ N \log (n^{2D} / n_Q^{2D}) - N \} \quad (28)$$

$$= N\tau \quad (29)$$

c) Entropy is also simply obtained from the free energy

$$\sigma = - \frac{\partial F}{\partial \tau} \Big|_{N, V} \quad (30)$$

$$\sigma = N \left[\log \left(\frac{n^{2D}}{n_Q^{2D}} \right) + 2 \right] \quad (31)$$

Question 3

K+K Chapter 7, Problem 1

a) In 1D the energy levels are $E_n = \frac{n_x^2 \pi^2 \hbar^2}{2mL^2}$, $n_x = 1, 2, 3, \dots$. The Fermi energy is defined as

$$\varepsilon_F = \frac{n_F^2 \pi^2 \hbar^2}{2mL^2} \quad (32)$$

The energy levels are equally spaced in n_x space. The total number of orbitals filled up to ε_F are given by

$$n_F = 2 \times \sqrt{\frac{2mL^2\varepsilon_F}{\pi^2\hbar^2}} \quad (33)$$

where the factor of 2 in front of the square root comes from the spin degeneracy. Electrons, having spin 1/2, can have two spin states having the same energy. So

$$N(\varepsilon) = 2\sqrt{\frac{2mL^2}{\pi^2\hbar^2}}\varepsilon^{1/2} \quad (34)$$

and the density of states is

$$D(\varepsilon) = \frac{dN}{d\varepsilon} = \frac{L}{\pi} \sqrt{\frac{2m}{\hbar^2}} \varepsilon^{-1/2} \quad (35)$$

b) Free electrons in 2D. The energy levels are given by $E_n = \frac{(n_x^2 + n_y^2)\pi^2\hbar^2}{2mL^2}$, $n_x, n_y = 1, 2, \dots$. The energy levels are equally spaced in the (n_x, n_y) space. The total number of states up to energy ε_F by definition add up to the number of particles (at zero temperature):

$$N = 2 \times \frac{1}{4} \times \pi n_F^2 = \pi n_F^2/2 \quad (36)$$

so $n_F = \left(\frac{2N}{\pi}\right)^{1/2}$. Above the factor 2 again originates from spin degeneracy. The factor of 1/4 is necessary because we want to add states with positive (n_x, n_y) pairs. The negative values do not represent different states. Another way of saying this is that we only consider the first quadrant in the (n_x, n_y) plane. The Fermi energy can be calculated using n_F :

$$\varepsilon_F = \frac{\hbar^2}{2m} \frac{\pi^2 n_F^2}{L^2} \quad (37)$$

$$= \frac{N\pi\hbar^2}{mL^2} \quad (38)$$

Again thinking of N as the number of states having energies up to ε_F we can solve it for N as a function of ε_F . This gives $N(\varepsilon) = \frac{mL^2\varepsilon}{\pi\hbar^2}$. The density of states is obtained by taking derivative wrt energy

$$D(\varepsilon) = \frac{dN}{d\varepsilon} = \frac{mL^2}{\pi\hbar^2} = \frac{mA}{\pi\hbar^2} \quad (39)$$

which is a constant. In the last line $A = L^2$ is the area of the box.

Question 4

K+K Chapter 7, Problem 5

a) ${}^3\text{He}$ has spin $1/2$ and is a Fermion. $\rho = 0.081\text{gm/cm}^3$. Mass per atom is $3\text{amu} \approx 4.98 \times 10^{-24}\text{gm}$. Using these we can calculate the concentration of atoms:

$$n = \frac{N}{V} = 0.081 \frac{\text{gm}}{\text{cm}^3} \times \frac{1}{4.98 \times 10^{-24}\text{gm/atoms}} = 1.63 \times 10^{22} \frac{\text{atoms}}{\text{cm}^3} \quad (40)$$

So the Fermi energy is

$$\varepsilon_F = \frac{\hbar^2}{2m} \left(3\pi^2 \frac{N}{V} \right)^{2/3} = \dots = 6.87 \times 10^{-23}\text{J} \quad (41)$$

$$= 429\mu\text{eV} \quad (42)$$

$$T_F = \frac{\varepsilon_F}{k_B} = 4.98\text{K} \quad (43)$$

$$v_F = \sqrt{\frac{2\varepsilon_F}{m}} = 166.1 \text{ m/s} \quad (44)$$

b) Use the expression for degenerate free electrons eq(7.38)

$$C_V = \frac{N\pi^2 k_B T}{2T_F} = \frac{\pi^2}{24.98\text{K}} N k_B T \quad (45)$$

$$= 0.99 N k_B T \quad (46)$$

The observed heat capacity is $C_V = 2.89 N k_B T$.

The enhancement comes from the fact that the liquid ${}^3\text{He}$ is not an ideal gas. When one ${}^3\text{He}$ atom moves, others have to get out of its way making the effective mass of the ${}^3\text{He}$ atom larger. It appears to be about 3 times larger than the bare mass of the ${}^3\text{He}$ atom.

Question 5

a) The heat capacity

$$C_v = \left. \frac{\partial u}{\partial \tau} \right|_V = \frac{\pi^2}{3} D(\epsilon_F) \tau \quad (47)$$

$$= \frac{\pi^2}{3} \frac{3N}{2\epsilon_F} \tau \quad (48)$$

$$= \frac{\pi^2}{2} N \frac{\tau}{\tau_F} \quad (49)$$

where $\epsilon_F \equiv \tau_F$, from Eq(36) in Chapter 7.

b) The heat capacity can be written as

$$C_V = \tau \left. \frac{\partial \delta}{\partial \tau} \right|_V$$

Hence,

$$\delta = \int_0^\tau \frac{C_V}{\tau'} d\tau' \quad (50)$$

$$= \int_0^\tau \frac{\pi^2}{2} N \frac{\tau'}{\tau_F} \frac{1}{\tau'} d\tau' \quad (51)$$

$$= \frac{\pi^2}{2} \frac{N}{\varepsilon_F} \int_0^\tau d\tau' \quad (52)$$

$$= \frac{\pi^2}{2} \frac{N}{\varepsilon_F} \tau \quad (53)$$

c) The Holmholtz free energy:

$$F = u - \tau \delta \quad (54)$$

$$= u_0 + \frac{\pi^2}{6} D(\varepsilon_F) \tau^2 - \frac{\pi^2}{2} \frac{N}{\varepsilon_F} \tau^2 \quad (55)$$

$$= u_0 - A \tau^2 \quad (56)$$

where

$$A = D(\varepsilon_F) \left[\frac{\pi^2}{3} - \frac{\pi^2}{6} \right] \quad (57)$$

$$= \frac{\pi^2}{6} D(\varepsilon_F) \quad (58)$$

$$= \frac{N \pi^2}{4 \varepsilon_F} \quad (59)$$

Question 6

a) The chemical potential can be found from the Helmholtz free energy as

$$\mu = \left. \frac{\partial F}{\partial N} \right|_{\tau, V}$$

Solve for the free energy:

$$F = \int_0^N \mu dN \quad (60)$$

$$= \int_0^N \left(\varepsilon_F - \frac{1}{3} \left(\frac{\pi}{2} \right)^2 \frac{\tau^2}{\varepsilon_F} \right) dN' \quad (61)$$

Note that $\varepsilon_F = \frac{\hbar^2}{2m} \left(\frac{3\pi^2 N}{V} \right)^{2/3}$, hence

$$F = \int_0^N \left[\frac{\hbar^2}{2m} \left(\frac{3\pi^2}{V} \right)^{2/3} N'^{2/3} - \frac{1}{3} \left(\frac{\pi}{2} \right)^2 \frac{1}{\frac{\hbar^2}{2m} \left(\frac{3\pi^2}{V} \right)^{2/3}} N'^{-2/3} \right] dN' \quad (62)$$

$$= \frac{3}{5} \frac{\hbar^2}{2m} \left(\frac{3\pi^2}{V} \right)^{2/3} N^{2/3} N - \frac{1}{3} \left(\frac{\pi}{2} \right)^2 \frac{1}{\frac{\hbar^2}{2m} \left(\frac{3\pi^2 N}{V} \right)^{2/3}} 3N \quad (63)$$

$$= \frac{3}{5} N \varepsilon_F - \left(\frac{\pi}{2} \right)^2 \frac{N}{\varepsilon_F} \tau^2 = u_0 - A \tau^2 \quad (64)$$

where $u_0 = \frac{3}{5} N \varepsilon_F$ and $A = \left(\frac{\pi}{2} \right)^2 \frac{N}{\varepsilon_F}$.

b) The pressure can be obtained from Helmholtz free energy as

$$P = - \left. \frac{\partial F}{\partial V} \right|_{\tau, V} \quad (65)$$

$$= - \frac{3}{5} N \left. \frac{\partial \varepsilon_F}{\partial V} \right|_{\tau, V} - \left(\frac{\pi}{2} \right)^2 \frac{N}{\varepsilon_F^2} \tau^2 \left. \frac{\partial \varepsilon_F}{\partial V} \right|_{\tau, V} \quad (66)$$

And

$$\left. \frac{\partial \varepsilon_F}{\partial V} \right|_{\tau, V} = \frac{\partial}{\partial V} \frac{\hbar^2}{2m} \left(\frac{3\pi^2 N}{V} \right)^{2/3} \quad (67)$$

$$= - \frac{2}{3} \frac{\varepsilon_F}{V} \quad (68)$$

Substitute it into the above equation:

$$P = \frac{2}{5} \frac{N \varepsilon_F}{V} + \frac{2}{3} \left(\frac{\pi}{2} \right)^2 \frac{N}{\varepsilon_F^2 V} \tau^2 \quad (69)$$

When $\tau \rightarrow 0$, the pressure:

$$P_0 = \frac{2}{5} \frac{N \varepsilon_F}{V} \quad (70)$$

c) From the above equation, the pressure will reach a finite value when temperature approaches 0. This is because the Pauli Principle force the fermions into higher energy states as they are added into the "box", creating an effective pressure even at $\tau = 0$. This degeneracy pressure is what prevents white dwarfs and neutron stars from collapsing under gravitational attraction.