Physics 410 Midterm Exam 2 Solutions

1. [10] A block of copper in the shape of a perfect cube with edges of length a=1.2 cm is suspended from a thin thread inside a vacuum chamber. Only one face of the cube is illuminated by the radiation from a hot tungsten filament several centimeters away. The power flux from the hot filament is $I=2.5\times10^4$ W/m^2 at the surface of the copper cube. What is the steady-state temperature of the copper cube?

   See solutions to Homework 4, problem 2. The area for P_{in} is a^2, while the area for P_{out} is $6a^2$. So the final answer is $T = (I/6\sigma_B)^{1/4} = 521$ K. Notice that the cube size does not enter into the answer.

2. [12] Consider two boxes filled with electrolytic solutions containing dilute concentrations of K^+ ions at temperature T = 300 K. (There is a fixed background of negative charges in both boxes to keep the systems electrically neutral.) The concentration of K^+ ions in box A is $n_A = 1.5 \times 10^{18}\text{m}^{-3}$, while the concentration of K^+ ions in box B is $n_B = 3.0 \times 10^{18}\text{m}^{-3}$. Now connect the boxes to each other by a narrow tube. You can prevent diffusion of H^+ ions from box B to box A by applying an electrostatic potential difference between the boxes, using a battery. What voltage should you apply to box A relative to box B to prevent diffusion of the H^+ ions? Should that voltage be positive or negative?

   See solutions to Homework 5, problem 5(a).

3. [18] Derive an expression for the chemical potential μ of a 2-dimensional monatomic ideal gas at temperature τ. The derivation has two main parts:

   (a) [12] First derive the partition function Z_1 for a single atom of mass *m* and no spin in a square box of area *A*.

   See solutions to Homework 6, problem 4(a). Several people mixed rectangular coordinates and polar coordinates in their solutions. (If you wrote kdk in the integrand, you were using polar coordinates!) With polar coordinates you can do the exponential integral exactly without looking anything up. With rectangular coordinates you need the Gaussian integral on the 1st page of the formula sheet, with the whole thing squared for the 2D case.

   (b) [6] To get from the single-atom problem to the gas of *N* atoms, there are two ways to proceed.

   For part (b), many of you chose the first method, $Z_N = (Z_1)^N/N!$, even though it is more difficult because you have to find F first, then take the derivative $\mu = \partial F/\partial N$, which requires using the product rule for the NlnN term. The second method is given at the beginning of the solutions to Homework 6, problem 4(a), and is also laid out in Section IX on the last page of the formula sheet, ending with $\mu = \tau \ln(n/n_Q)$.

4. [20] On the homework, you learned that the hemoglobin in your blood binds carbon monoxide (CO) more strongly than it binds Oxygen (O_2). The energies of the hemoglobin with bound CO and O are $\varepsilon_{CO} = -10.6\times10^{-20}$ J and $\varepsilon_{O_2} = -8.4\times10^{-20}$ J, respectively. (The energies are negative relative to free gas molecules.) A misguided teenager starts his car in the garage without opening the garage door. The density of oxygen molecules in the garage is 2.4×10^{24} m^{-3}. Soon the density of CO in the garage rises to 1.2×10^{22} m^{-3}. What fraction of the hemoglobin molecules in his blood becomes poisoned by CO? Take the temperature of the garage as 300 K, and neglect the slight difference between the garage temperature and body temperature. Do this problem in the following steps:

See the solutions to Homework 6, problem 2. I made the exam problem more straightforward by giving you the binding energies rather than having you derive them.

(a) [5] Write down the Grand Partition function $\mathfrak{Z}$ for a single hemoglobin molecule. (A hemoglobin molecule can be empty, or it can hold either a CO molecule or an O_2 molecule, but not both at the same time.)

$$\mathfrak{Z} = 1 + e^{(\mu_{CO}-\varepsilon_{CO})/\tau} + e^{(\mu_{O2}-\varepsilon_{O2})/\tau} = 1 + \lambda_{CO}\, e^{-\varepsilon_{CO}/\tau} + \lambda_{O2}\, e^{-\varepsilon_{O2}/\tau}$$

(b) [5] Write down an algebraic expression for the probability that a hemoglobin molecule holds a CO molecule.

$$\langle N_{CO} \rangle = \frac{e^{(\mu_{CO}-\varepsilon_{CO})/\tau}}{\mathfrak{Z}} = \frac{\lambda_{CO}\, e^{-\varepsilon_{CO}/\tau}}{\mathfrak{Z}}$$

(c) [5] Depending on how you wrote your answer to part (b), calculate either the chemical potentials, μ_{CO} and μ_{O_2}, or the absolute activities, λ_{CO} and λ_{O_2}, of CO and O_2 in the garage. The quantum concentrations are $n_{QCO} = 1.5\times10^{32}\, m^{-3}$ and $n_{QO_2} = 1.8\times10^{32}\, m^{-3}$.

$\lambda_{CO} = n_{CO}/n_{QCO} = 0.80\times10^{-10}$ $\lambda_{O2} = n_{O2}/n_{QO2} = 1.33\times10^{-8}$

(d) [5] Evaluate your expression in part (b) using the numbers given in the problem and your results from part (c).

$\tau = k_B T = 4.14\ 10^{-21}$ J $e^{-\varepsilon_{CO}/\tau} = 1.32\times10^{11}$ $e^{-\varepsilon_{O2}/\tau} = 6.48\times10^{8}$

$\lambda_{CO}\, e^{-\varepsilon_{CO}/\tau} = 10.56$ $\lambda_{O2}\, e^{-\varepsilon_{O2}/\tau} = 8.62$ $\langle N_{CO} \rangle = 10.56/(1+8.62+10.56) = 0.52$.

Hmmm, I guess I should have put a lower concentration of CO in the problem. The important point is that CO is very dangerous even at low concentrations, due to the large binding energy to hemoglobin.