Homework 10 Solutions

Question 1

K+K Chapter 8, Problem 5. The best case scenario is that the plant operates as a thermodynamic heat engine on the Carnot cycle and has an efficiency given by:

$$\eta_c = \frac{W}{Q_h} = \frac{T_h - T_l}{T_h} \tag{1}$$

Also $Q_l = Q_h T_l / T_h$, which follows from the fact that the process is reversible (hence dS=0). Using this we can write the work in terms of Q_l instead of Q_h as:

$$W = Q_l \frac{T_h - T_l}{T_l} \tag{2}$$

(pay attention to the denominator). We are given that the upper limit to the heat that can be released in one second is $Q_l = 1.5 \times 10^9 J$. Using this we can obtain the maximum amount of work per second

$$W = 1.5 \times 10^9 J \frac{500^o C - 20^o C}{293K} = 2.46 \times 10^9 J \tag{3}$$

or $P = \dot{W} = 2.46 GW$. Note that in the numerator we used temperatures in units of Celsius. This is ok, because the difference of two temperatures is the same in both units. The denominator on the other hand has to be in the absolute temperature scale.

If T_H is raised by $100^o C$, the output will be greater

$$W = 1.5 \times 10^9 J \frac{600^o C - 20^o C}{293K} = 2.97 \times 10^9 J \tag{4}$$

or $P = \dot{W} = 2.97 GW$. This is the best case scenario. In reality inefficiencies will limit the useful work output to smaller values.

Question 2

K+K Chapter 8, Problem 6.

a) The room at temperature T_l gains heat from outdoors at temperature T_h at a rate:

$$\frac{dQ_l}{dt} = A(T_h - T_l) \tag{5}$$

For a refrigerator, the work done to move heat Q_l from the low temperature reservoir to the high temperature reservoir is

$$W = Q_h - Q_l = \frac{T_h - T_l}{T_l} Q_l \tag{6}$$

which is given in Eq(8.12). Hence the rate at which this work is done, namely the power supplied to the cooling unit is

$$P = \frac{dW}{dt} = \frac{T_h - T_l}{T_l}\frac{dQ_l}{dt} \tag{7}$$

With the above expression Eq(5) substituted in, we get:

$$P = \frac{T_h - T_l}{T_l} A(T_h - T_l) = A\frac{(T_h - T_l)^2}{T_l} \tag{8}$$

Solve this for T_l, the temperature of the room:

$$T_l^2 - T_l(2T_h + P/A) + T_h^2 = 0 \tag{9}$$

$$T_l = \frac{(2T_h + P/A) \pm \sqrt{(2T_h + P/A)^2 - 4T_h^2}}{2} \tag{10}$$

We expect $T_l < T_h$, so chooe the "-" sign

$$T_l = T_h + \frac{P}{2A} - \sqrt{\left(T_h + \frac{P}{2A}\right)^2 - T_h^2} \tag{11}$$

b) If $T_h = 37^oC$ and $T_l = 17^oC$ with $P = 2kW$, what is A? From above, we see that

$$A = \frac{T_l P}{(T_h - T_l)^2} = \frac{(290K)(2 \times 10^3 W)}{(20K)^2} = 1.45 \times 10^3 W/K \tag{12}$$

This is the "leak rate" of heat into the room from outdoors.

Question 3

K+K Chapter 8, Problem 7. Consider a 100 W light bulb in the refrigerator and Carnot refrigerator drawing work $W = 100W$. In this case $Q_l = 100W$. Eq (8.12) states $W = \frac{T_h - T_l}{T_l} Q_l$ for a Carnot refrigerator. Hence

$$\frac{T_h - T_l}{T_l} = \frac{W}{Q_l} = \frac{100W}{100W} = 1 \tag{13}$$

Hence the refrigerator can keep up up to $T_l = T_h - T_l$ or $T_l = T_h/2$. If the hot reservoir is at room temperature, $T_h \approx 300K$, then $T_l = 150K$. This is very cold!

The more general case: $Q_l = \alpha W$. Now we have

$$W = \frac{T_h - T_l}{T_l}\alpha W \tag{14}$$

$$\frac{T_l}{\alpha} = T_h - T_l \tag{15}$$

$$T_l = \frac{T_h}{1 + \frac{1}{\alpha}} \tag{16}$$

For $\alpha = 1$, this is half the room temperature (in Kelvins!), as above. For $\alpha = 10$, this ratio is 1.1, which is still pretty good for keeping the milk cold.

Question 4

K+K Chapter 8, Problem 10. Irreversible expansion of a Fermi Gas. Expansion into vacuum means $W = 0$. Similarly no heat is exchanged with the environment. Hence $dU = dQ + dW = 0$, the internal energy of the gas does not change. We know the energy per particle of a degenerate Fermi gas at zero temperature (Eqs. (7.10) and (7.7)):

$$< \varepsilon >= \frac{3}{5}\varepsilon_F = \frac{3}{5}\frac{\hbar^2}{2M}\left(\frac{3\pi^2 N}{V_i}\right)^{2/3} \tag{17}$$

For a dilute ideal gas in 3D we also know the mean energy per particle from equipartition (Eq. (6.32) and page 77):

$$< \varepsilon >= \frac{3}{2}\tau_f \tag{18}$$

Equating these two energies and solving for the final temperature to get

$$\tau_f = \frac{\hbar^2}{5M}\left(\frac{3\pi^2 N}{V_i}\right)^{2/3} \tag{19}$$

We also have to ensure that the final state of the gas is indeed classical. For large enough final volume, the gas will be dilute enough for the classical limit to apply. Our condition for diluteness is $n \ll 2n_Q$ where the factor of 2 comes from the spin 1/2. This condition translates to:

$$\frac{N}{V_f} \ll 2\left(\frac{M\tau_f}{2\pi\hbar^2}\right)^{3/2} \tag{20}$$

Substitution in τ_f from above and solving for the ratio of volumes we get

$$\frac{V_i}{V_f} \ll \frac{6\pi^2}{(10\pi)^{3/1}} = 0.34 \tag{21}$$

Hence if the final volume is only 3 times bigger than the initial volume, the system is nearly in the classical dilute gas limit!

a) Using the mass of the electron in eq (19) we get $7788K$.

b) Using the mass of the nucleon (1 amu, given on the inside back cover of the text) in eq (19) we get $9.21 \times 10^5 K$.