1. Consider a dilute gas of particles in the atmosphere. Near the earth's surface, the force on a particle of mass m may be taken as a constant, $\vec{F} = -mg\hat{j}$, where $\hat{j}$ is a unit vector in the vertical direction.

a.) By enforcing the number constraint on the ideal gas ($\langle N\rangle = \sum_s f(\varepsilon_s, \mu, \tau)$, with $f(\varepsilon_s, \mu, \tau) = \text{Exp}\left(\frac{\mu}{\tau}\right)\text{Exp}\left(-\frac{\varepsilon_s}{\tau}\right)$), <u>derive</u> the expression $\mu_{int} = \tau \log\left(\frac{n}{n_Q}\right)$ for the chemical potential of an ideal gas. The quantum concentration is $n_Q = \left(\frac{m\tau}{2\pi\hbar^2}\right)^{3/2}$. *(5 points)*

b.) Write an expression for the <u>total</u> chemical potential for a gas particle in terms of the internal chemical potential and external chemical potential. *(5 points)*

c.) What conditions must be satisfied for the total chemical potential to be uniform with height? *(5 points)*

d.) The density of particles at height $y = 0$ is given as $n(0)$. Calculate the density $n(y)$ of the dilute gas of atoms of mass m at temperature τ as a function of the height y above the earth's surface. *(10 points)*

a) $N = \sum_s f(\varepsilon_s, \mu, \tau)$ $\quad f(\varepsilon_s, \mu, \tau) = e^{\mu/\tau} e^{-\varepsilon_s/\tau}$

The temperature τ and chemical potential μ are imposed on the system by the reservoir. Hence

$$N = e^{\mu/\tau} \sum_s e^{-\varepsilon_s/\tau}$$

$$= e^{\mu/\tau} Z_1$$ where Z_1 is the partition function for a single particle in a box of volume V and mass m.

We use the shortcut that $Z_1 = n_Q V$, where n_Q is the quantum concentration.

$$N = e^{\mu/\tau} n_Q V$$

Solve for μ: $\mu/\tau = \log(n/n_Q) \Rightarrow \mu = \tau \log(n/n_Q)$ Q.E.D.

b) The <u>total</u> chemical potential is made up of internal (ideal gas) and external (gravitational potential energy) parts: $\mu_{TOTAL} = \mu_{int} + \mu_{ext}$.
The internal part is given in (a). The external part is due to the uniform gravitational field: $U = mgy = \mu_{ext}$

$$\mu_{TOTAL} = \tau \log(n/n_Q) + mgy$$

c) The gas must be in thermal and diffusive equilibrium over its height.

d) μ_{TOTAL} is independent of altitude y. The concentration at the earth's surface is $n(0)$. Comparing $y=0$ with arbitrary y gives:

$$\tau \log(n(0)/n_Q) + 0 = \tau \log(n(y)/n_Q) + mgy$$

Solve for $n(y) = n(0)\, e^{-mgy/\tau}$

2. Consider an ultra-relativistic dilute gas of noninteracting Fermion particles contained in a three-dimensional cube of volume $V = L^3$. The particles have energy $E \gg mc^2$, where m is the rest mass of the particle, so that the energy is given by $E \cong pc$, where p is the momentum and c is the speed of light in vacuum. Note that the momentum $p = \frac{\pi\hbar}{L}\sqrt{n_x^2 + n_y^2 + n_z^2}$, just as for the non-relativistic case.

a.) Calculate the Fermi energy of this gas of N particles. *(15 points)*

b.) Calculate the total energy of the ground state of this gas. *(10 points)*

a) We expect the Fermi energy is given by $E_F = p_F c$, where p_F is the Fermi momentum. The Fermi momentum is the highest occupied momentum state $p_F = \frac{\pi\hbar}{L} n_F$.
We see that each state is labelled by a triplet of positive integers (n_x, n_y, n_z), hence the total number of states up to an including n_F must accomodate all of the particles:

$$\underbrace{N}_{\text{number of particles}} = \underbrace{2}_{\text{spin factor}} \times \frac{1}{8} \times \frac{4\pi}{3} n_F^3 = \frac{\pi}{3} n_F^3$$

Hence $n_F = \left(\frac{3N}{\pi}\right)^{1/3}$, so $p_F = \frac{\pi\hbar}{L}\left(\frac{3N}{\pi}\right)^{1/3}$, and

$$E_F = p_F c = \frac{\pi\hbar c}{L}\left(\frac{3N}{\pi}\right)^{1/3} = \pi\hbar c\left(\frac{3N}{\pi L^3}\right)^{1/3} = \pi\hbar c\left(\frac{3n}{\pi}\right)^{1/3}$$

b) In the ground state $(T=0)$ all states are occupied up to E_F and all higher states are un-occupied. The total energy is

$$U(T=0) = 2\sum_{\substack{n_x, n_y, n_z \\ \leq n_F}} E(n_x, n_y, n_z) = 2 \times \frac{1}{8} 4\pi \int_0^{n_F} dn\, n^2 E(n)$$

Here we used the fact that the number of states ~~with energy~~ between n and $n+dn$ is given by $\frac{1}{8} 4\pi n^2 dn$, the volume of a sphereical shell of thickness dn in n-space. Now use the fact that $E(n) = pc = \frac{\pi\hbar c}{L} n$, to get

$$U(T=0) = \frac{\pi^2\hbar c}{L}\int_0^{n_F} dn\, n^3 = \frac{\pi^2\hbar c}{4L} n_F^4$$

Using n_F from above yields

$$U(T=0) = \frac{3\pi\hbar c}{4L} N\left(\frac{3N}{\pi}\right)^{1/3} = \frac{3N}{4} E_F$$

3. Consider a one-dimensional transmission line of length L on which electromagnetic waves satisfy the one-dimensional wave equation $\frac{\partial^2 E}{\partial t^2} = v^2 \frac{\partial^2 E}{\partial x^2}$, where E is the electric field component and v is the speed of light.

- **a.)** Show that the n[th] mode of the line has energy $\hbar\omega_n = n\pi v/L$. *Hint: The modes correspond to having an integer number of half wavelengths spanning the line. (5 points)*
- **b.)** Find the thermal average occupation number of a mode of energy $\hbar\omega$, assuming that the photons are in thermal equilibrium with a reservoir at temperature τ. *(5 points)*
- **c.)** Find the thermal average energy of photons on the line. Hint: Express the sum over states as an integral on energy, and note that $\int_0^\infty \frac{u\,du}{e^u - 1} = \frac{\pi^2}{6}$. *(10 points)*
- **d.)** Find the heat capacity of photons on the line. *(5 points)*

a) The modes correspond to an integer number of half wavelengths spanning the line: $L = n\frac{\lambda}{2}$. The ~~energy~~ frequency is given by $\omega = vk$

or $\omega_n = \cancel{\;} = \frac{2\pi v}{\lambda} \cancel{\;} = \frac{\pi v}{L} n$

The energy of this mode is $\hbar\omega_n = \frac{\hbar\pi v}{L} n$

b) The photon mode is in thermal equilibrium with a reservoir at temperature τ. The thermal average occupation number is

$$\langle s \rangle = \sum_{s=0}^{\infty} s P(s) = \frac{1}{Z}\sum_{s=0}^{\infty} s\, e^{-s\hbar\omega/\tau} \quad \text{and} \quad Z = \sum_{s=0}^{\infty} e^{-s\hbar\omega/\tau}$$

for a mode with energy $\hbar\omega$. The partition function is

$$Z = \sum_{s=0}^{\infty} z^s = \frac{1}{1-z} = \frac{1}{1-e^{-\hbar\omega/\tau}}.$$

The thermal average $\langle s \rangle = \frac{1}{Z}\sum_{s=0}^{\infty} s e^{-sy}$ $\qquad y \equiv \hbar\omega/\tau$

$$= \frac{-1}{Z}\frac{d}{dy}\sum_{s=0}^{\infty} e^{-sy} = -\frac{d\log Z}{dy}$$

$$\langle s \rangle = \frac{1}{e^{\hbar\omega/\tau} - 1}$$

c) The thermal average energy of photons on the line ~~...~~ is

$$U = \sum_n \langle s(\omega_n)\rangle \hbar\omega_n \qquad \text{where} \quad \langle s(\omega_n)\rangle = \frac{1}{e^{\hbar\omega_n/\tau}-1} \quad \text{was found in (b).}$$

$$U = \sum_{n=1}^{\infty} \frac{\hbar\omega_n}{e^{\hbar\omega_n/\tau}-1}$$

Convert the sum over states to an integral ~~...~~ on n in the limit of a long line:

$$U \approx \int_0^{\infty} \frac{\hbar\omega_n\, dn}{e^{\hbar\omega_n/\tau}-1} \qquad \text{Let} \quad u = \frac{\hbar\omega_n}{\tau} = \frac{\hbar\pi v}{L\tau}\, n$$

↑ no error made by starting at zero

$$du = \frac{\hbar\pi v}{L\tau}\, dn$$

$$U = \frac{L\tau^2}{\hbar\pi v}\int_0^{\infty}\frac{u\, du}{e^u-1} = \frac{L\tau^2}{\hbar\pi v}\,\frac{\pi^2}{6} = \frac{\pi L\tau^2}{6\hbar v}$$

d) The heat capacity is $\quad C_L = \left.\frac{\partial U}{\partial \tau}\right|_L = \frac{\pi L\tau}{3\hbar v}$

4. The distribution function for fermions (+) or bosons (-) is given by

$$f(\varepsilon) = \frac{1}{\exp[(\varepsilon - \mu)/\tau] \pm 1}$$

a) State clearly in words the definition of the distribution function. *(5 points)*
b) Give a physical explanation of why the + sign corresponds to fermions. *(5 points)*
c) Particles confined to a one-dimensional line of length L have orbitals labeled by n with energy

$$\varepsilon_n = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 n^2$$

where the n are positive integers (1,2,3,..). The particles are spin-1/2 fermions, so that each orbital can have at most two particles in it. If N particles are placed into this system, what is the Fermi energy ε_f? *(5 points)*
d) Derive the density of states $D(\varepsilon)$ for this one-dimensional Fermi gas. *(5 points)*
e) Write general finite-temperature expressions for <N>, U, and C_V involving $D(\varepsilon)$; do not evaluate them. *(5 points)*

a) The distribution $f(\varepsilon)$ is the thermal average occupation of a state of energy ε in thermal and diffusive equilibrium with a reservoir at temperature τ and chemical potential μ.

b) At $\tau = 0$ the distribution function has filled states ($f=1$) for all energies below the chemical potential, and empty states ($f=0$) for all states with energy greater than the chemical potential μ. This is the zero-temperature Fermi sea, and the energy of the highest occupied state is the Fermi energy.

c) The n^{th} orbital accomodates a total of $2n$ particles. For the system to hold N particles, the orbitals are filled up to n_F, determined by $N = 2n_F$. The Fermi energy is given by $\varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 n_F^2 = \frac{\hbar^2}{8m}\left(\frac{\pi N}{L}\right)^2$

d) The density of states in 1D can be found from the total number of states up to energy ε. Solving the equation for n in terms of ε from part (b), we have $n = \left(\frac{2m\varepsilon}{\hbar^2}\right)^{1/2}\frac{L}{\pi}$. The total number of states up to energy ε is then $N = 2n = \frac{2L}{\pi}\sqrt{\frac{2m\varepsilon}{\hbar^2}}$. The density of states is the rate at which states are added with increasing energy $D(\varepsilon) = dN(\varepsilon)/d\varepsilon$

$$D(\varepsilon) = \frac{L}{\pi}\sqrt{\frac{2m}{\hbar^2}}\frac{1}{\sqrt{\varepsilon}}$$

e) The general finite temperature expressions are

$$\langle N\rangle = \int_0^\infty d\varepsilon\, D(\varepsilon)\, f(\varepsilon,\tau,\mu) = \frac{L}{\pi}\sqrt{\frac{2m}{\hbar^2}}\int_0^\infty d\varepsilon\, \frac{\varepsilon^{-1/2}}{e^{(\varepsilon-\mu)/\tau}\pm 1}$$

$$U = \int_0^\infty d\varepsilon\, D(\varepsilon)\,\varepsilon\, f(\varepsilon,\tau,\mu) = \frac{L}{\pi}\sqrt{\frac{2m}{\hbar^2}}\int_0^\infty d\varepsilon\, \frac{\varepsilon^{1/2}}{e^{(\varepsilon-\mu)/\tau}\pm 1}$$

$$C_V = \left.\frac{\partial U}{\partial \tau}\right|_V = \frac{L}{\pi}\sqrt{\frac{2m}{\hbar^2}}\int_0^\infty d\varepsilon\, \varepsilon^{1/2}\frac{\partial}{\partial\tau}\frac{1}{e^{(\varepsilon-\mu)/\tau}\pm 1}$$

Both $\mu(\tau)$ and τ are involved in the derivative. Let's ignore the temperature dependence of $\mu(\tau)$ for now.

$$\frac{\partial}{\partial\tau}\frac{1}{e^{(\varepsilon-\mu)/\tau}\pm 1} = \frac{\varepsilon-\mu}{\tau^2}\frac{e^{(\varepsilon-\mu)/\tau}}{\left(e^{(\varepsilon-\mu)/\tau}\pm 1\right)^2}$$

so

$$C_V \simeq \frac{L}{\pi}\sqrt{\frac{2m}{\hbar^2}}\frac{1}{\tau^2}\int_0^\infty d\varepsilon\, \sqrt{\varepsilon}(\varepsilon-\mu)\frac{e^{(\varepsilon-\mu)/\tau}}{\left(e^{(\varepsilon-\mu)/\tau}\pm 1\right)^2}$$