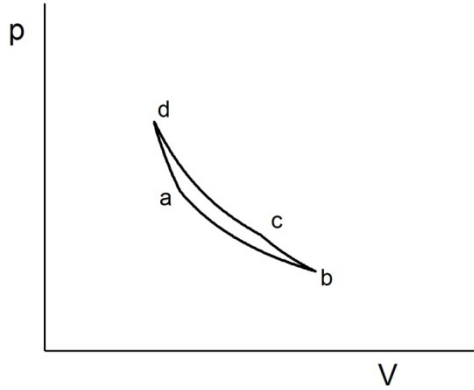# Physics 410 Midterm Exam 3 Solutions

1. [18] Consider a refrigerator consisting of a monatomic ideal gas of N molecules following the Carnot cycle, as shown in the figure. Steps ab and cd are isothermal, while steps bc and da are isentropic. <u>Express all answers in terms of N, V_a, V_b, τ_h, and τ_l.</u>

a) [6] How much work is done by the gas during step ab? (The temperature during this step is τ_l.)

$$W = \int_{V_a}^{V_b} p\, dV = \int_{V_a}^{V_b} \frac{N\tau_l}{V}\, dV = N\tau_l \ln\frac{V_b}{V_a}$$

b) [6] How much heat is absorbed from the cold reservoir during step ab?

Isothermal $\Rightarrow U = $ constant

$$0 = \Delta U = Q - W \Rightarrow Q = W = N\tau_l \ln\frac{V_b}{V_a}$$

Alternative method: $Q = \tau \Delta\sigma = \tau_l \left(N \ln\frac{V_b}{V_a} \right)$, where I used the Sackur-Tedrode equation for σ.

c) [6] Assuming that the refrigerator is reversible, calculate the total work done <u>on</u> the gas during one complete cycle. You may use your knowledge of the Carnot refrigerator efficiency to calculate the total work.

$$\gamma_c = \frac{Q_l}{W} = \frac{\tau_l}{\tau_h - \tau_l}$$ Q_l is in fact the Q we calculated in part (b), so

$$W_{done\,on\,gas} = \frac{Q_l}{\gamma_c} = \left(\frac{\tau_h - \tau_l}{\tau_l}\right) N\tau_l \ln\frac{V_b}{V_a} = N(\tau_h - \tau_l)\ln\frac{V_b}{V_a}.$$

Note that the work done <u>on</u> the gas is positive for a refrigerator.

Alternative method:

$$W_{done\,by\,gas} = W_{ab} + W_{bc} + W_{cd} + W_{da} = W_{ab} + W_{cd} = N\tau_l \ln\frac{V_b}{V_a} + N\tau_h \ln\frac{V_d}{V_c} = N(\tau_l - \tau_h)\ln\frac{V_b}{V_a}$$

Where I used the fact that the work done in the two isentropic steps cancel, and $\frac{V_d}{V_c} = \frac{V_a}{V_b}$. The

problem asks for the work done <u>on</u> the gas, so $W_{done\,on\,gas} = -W_{done\,by\,gas} = N(\tau_h - \tau_l)\ln\frac{V_b}{V_a}$. Keeping

track of all the minus signs using this method is difficult! The first method is much easier.

2. [16] The 2-dimensional electron gas is a topic of active research in condensed matter physics.

a) [10] Derive the following expression for the density of orbitals of a 2-dimensional electron gas in a box of area A:

$$D(\varepsilon) = \frac{Am}{\pi\hbar^2}.$$

See solutions to Homework 8, problem 3.

b) [6] What is the Fermi energy, ε_F, if there are N electrons in the box? Your answer should be expressed in terms of N, A, the electron mass m, and any other fundamental constants you need. Remember that the Fermi energy is a property of the ground state of the system.

If you did part (a) starting from $N(\varepsilon) = 2\left(\dfrac{L}{2\pi}\right)^2 \pi k^2 = \dfrac{A}{2\pi}\dfrac{2m\varepsilon}{\hbar^2}$, then you have your answer just

by setting $N = N(\varepsilon_F) = \dfrac{A}{2\pi}\dfrac{2m\varepsilon_F}{\hbar^2}$, so $\varepsilon_F = \dfrac{\pi\hbar^2 N}{mA}$. Alternatively, you can write:

$$N = \int_0^{\varepsilon_F} D(\varepsilon)d\varepsilon = \frac{Am}{\pi\hbar^2}\varepsilon_F \text{ , so } \varepsilon_F = \frac{\pi\hbar^2 N}{mA}.$$

3. [18] Consider a box of volume V_2 with a partition that separates a third of the box, $V_1 = V_2/3$, from the rest. Initially, V_1 contains N molecules of monatomic ideal gas at temperature τ_1, while the rest of the box is empty. The walls of the box are well insulated.

(a) [3] The partition is suddenly removed. No heat enters or leaves the box during the sudden expansion. What is the final temperature of the gas, τ_2, after the sudden expansion?

A sudden expansion is irreversible, so $Q = 0$ does <u>not</u> imply that $\Delta\sigma = 0$. The gas does no work, so the molecules do not lose any kinetic energy while they fill the box. So $\Delta U = 0$ and $\tau_2 = \tau_1$.

(b) [5] What is the change in entropy of the gas, $\Delta\sigma$, as a result of the sudden expansion?

Method 1: Long ago we showed that the multiplicity increases by a factor 3^N during such an expansion. So $\Delta\sigma = \sigma_2 - \sigma_1 = \ln\dfrac{g_2}{g_1} = \ln 3^N = N\ln 3$.

Method 2: Use the Sackur-Tetrode formula: $\sigma = N\left(\ln\dfrac{n_Q}{n} + \dfrac{5}{2}\right) = N\left(\ln\dfrac{n_Q V}{N} + \dfrac{5}{2}\right)$ to get

$$\sigma_2 - \sigma_1 = N\left(\ln V_2 - \ln V_1\right) = N\ln\dfrac{V_2}{V_1} = N\ln 3$$

(c) [5] How much work would you have to do to compress the gas back to its initial volume? Again, assume that no heat is transferred to the outside world. Express your answer in terms of N and τ_2. (This integral is a bit tricky. To avoid it, do part (d) first, then see if you can figure out an easier way to calculate the work done.)

This is an isentropic compression. You can calculate the work done <u>on</u> the gas the hard way:

$$W_{done\ on\ gas} = -W = -\int_{V_2}^{V_1} pdV = -\int_{V_2}^{V_1}\frac{N\tau_2 V_2^{2/3}}{V^{5/3}}dV = \frac{3}{2}N\tau_2 V_2^{2/3}\left(\frac{1}{V_1^{2/3}} - \frac{1}{V_2^{2/3}}\right) = \frac{3}{2}N\tau_2\left(\frac{V_2^{2/3}}{V_1^{2/3}} - 1\right) = \frac{3}{2}N\tau_2\left(3^{2/3} - 1\right)$$

where I used the isentropic relation $pV^\gamma = p_2 V_2^\gamma = \dfrac{N\tau_2}{V_2}V_2^\gamma = N\tau_2 V_2^{\gamma-1}$ with $\gamma = 5/3$. A much easier

way to do this problem is to do part (d) first to get $\tau_{1'}$, then use the 1st Law of Thermodynamics:

$$\Delta U = Q - W = -W \Rightarrow -W = \Delta U = U_f - U_i = \frac{3}{2}N\tau_{1'} - \frac{3}{2}N\tau_2 = \frac{3}{2}N\tau_2\left(3^{2/3} - 1\right)$$

(d) [5] What is the final temperature τ_1' of the gas after the compression in part (c)? Express your answer in terms of τ_2.

Isentropic compression: $\tau_1 V_1^{2/3} = \tau_2 V_2^{2/3} \Rightarrow \tau_1 = \tau_2\left(\dfrac{V_2}{V_1}\right)^{2/3} = \tau_2 3^{2/3}$

4. [8]
(a) [6] Sketch a graph of the heat capacity per molecule of a diatomic gas (e.g. N_2) versus temperature. Label the vertical axis clearly and show room temperature on the horizontal axis. Explain in a few words the main features of the graph.

See Figure 3.9 in the textbook.

(b) [2] Express the heat capacity per molecule or per mole (please specify which) at low temperature in real units, i.e. in Joules/Kelvin or Joules/mole-Kelvin.

$$C_V = \frac{dU}{dT} = k_B \frac{dU}{d\tau} = \frac{3}{2} k_B = \frac{3}{2} 1.38 \cdot 10^{-23} \frac{J}{K} = 2.07 \cdot 10^{-23} \frac{J}{K}.$$

If you want it per mole, then multiply by N_A, which converts k_B to R:

$$C_V = \frac{3}{2} 8.31 \frac{J}{mole\text{-}K} = 12.5 \frac{J}{mole\text{-}K}.$$