Homework #2 Solutions

Question 1) Adiabatic Process for an ideal gas. We are given:

$$\Delta U = Q + U$$

$$U = \frac{f}{2} N k_B T$$

Adiabatic Process means: $Q = 0$. Hence, from the lst Law,

$$\Delta U = W$$

And compressive work can be written as $W = -P\Delta V$. With the above information:

$$\frac{f}{2} N k_B \Delta T = -P\Delta V$$

Now use the ideal gas Law $PV = Nk_BT$ to eliminate Pressure P,

$$\frac{f}{2} N k_B \Delta T = -\frac{Nk_BT}{V}\Delta V$$

Divide both sides by T,

$$\frac{f}{2}\frac{\Delta T}{T} = -\frac{\Delta V}{V}$$

Integrate both sides from the initial state (T_i, V_i) to the final state (T_f, V_f):

$$\frac{f}{2}\int_{T_i}^{T_f}\frac{dT}{T} = -\int_{V_i}^{V_f}\frac{dV}{V}$$

$$\frac{f}{2}\ln\frac{T_f}{T_i} = -\ln\frac{V_f}{V_i}$$

After exponentiating both sides of the equation,

$$(\frac{T_f}{T_i})^{\frac{f}{2}} = \frac{V_i}{V_f}$$

Multiply both sides by $V_f T_i^{\frac{f}{2}}$, then

$$V_f T_f^{\frac{f}{2}} = V_i T_i^{\frac{f}{2}}$$

So, in the other words,

$$VT^{\frac{f}{2}} = C$$

where C is a constant. Now, we can use the ideal gas law to eliminate T in favor of P:

$$V(\frac{PV}{Nk_B})^{\frac{f}{2}} = C$$

Both N and k_B are constant, so we can write

$$P^{\frac{f}{2}}V^{\frac{f}{2}+1} = C'$$

Where C' is another constant. Now raise both sides to the power $2/f$ to get

$$PV^{\frac{2}{f}(1+\frac{f}{2})} = C''$$

and C'' is also a constant.

Define $\gamma \equiv \frac{f+2}{f}$, then we have

$$PV^{\gamma} = constant$$

Discussion: Alternative Method:

$$dW = -PdV$$

$$dU = d(\frac{f}{2}NK_BT) = d(\frac{f}{2}PV) = \frac{f}{2}(VdP + PdV)$$

For an adiabatic process, $Q = 0$, then

$$dU = dW \Rightarrow -PdV = \frac{f}{2}(VdP + PdV)$$

This can be written as

$$\gamma\frac{dV}{V} = -\frac{dP}{P}$$

Integrate both sides from the initial state (V_i, P_i) to the final state (V_f, P_f) to get:

$$PV^{\gamma} = constant$$

Question 2a) We know that (from the above question), $P_iV_i{}^{\gamma} = P_fV_f{}^{\gamma}$. Hence

$$V_f = (\frac{P_i}{P_f})^{\frac{1}{\gamma}}V_i$$

where $\gamma \equiv \frac{f+2}{f} = \frac{7}{5}$. Hence,

$$V_f = (1Liter)(\frac{1atm}{7atm})^{\frac{5}{7}} = 0.25L$$

b) The word done in compression is

$$W = -\int_{V_i}^{V_f} PdV$$

From the equation

$$PV^{\gamma} = constant$$

we can get

$$P = \frac{C}{V^{\gamma}}$$

Hence

$$W = -C\int_{V_i}^{V_f} \frac{dV}{V^\gamma} = \frac{C}{\gamma-1}[V_f^{1-\gamma} - V_i^{1-\gamma}]$$

Where $C = P_f V_f^\gamma = P_i V_i^\gamma$, so that

$$W = \frac{1}{\gamma-1}[P_f V_f^\gamma V_f^{1-\gamma} - P_i V_i^\gamma V_i^{1-\gamma}] = \frac{1}{\gamma-1}[P_f V_f - P_i V_i]$$

Before the calculation, we should change the units to SI,

$$1atm = 1.013\times 10^5 N/m^2$$

$$1Liter = 10^{-3}m^3$$

Then we can get the result

$$W = 189.9N\cdot m = 189.9J$$

Discussion *Alternative method:*

$$W = \frac{f}{2}NK_B(T_f - T_i) = \frac{f}{2}(P_f V_f - P_i V_i)$$

c) We know from the last problem that $VT^{\frac{f}{2}} = constant$. Hence,

$$V_f T_f^{f/2} = V_i T_i^{f/2}$$

So

$$T_f = (\frac{V_i}{V_f})^{\frac{2}{f}} T_i = 300K(\frac{1L}{0.25L})^{2/5} = 522K = 249^oC$$

which is pretty hot!

Discussion *Alternative methods:* ***Method 1)*** *From the ideal gas law,*

$$\frac{T_f}{T_i} = \frac{P_f V_f}{P_i V_i} \approx \frac{7atm\times 0.25L}{1atm\times 1L} = \frac{7}{4}$$

$$T_f \approx \frac{7}{4}T_i = 525K$$

Method 2) *From the result of question 2b,(adiabatic process $Q = 0$)*

$$dU = W$$

you get the change of internal energy, then

$$dU = \frac{f}{2}Nk_B(T_f - T_i)$$

where $Nk_B = \frac{P_i V_i}{T_i}$.

Question 3) Note that this probability is nothing but $g(N_h, N_t)/2^N$ where $g(N_h, N_t) = \frac{N!}{N_h! N_t!}$ is the multiplicity function for spins that you calculated in class. I use N_h to denote the number of heads and N_t for the number of tails.

3a) $P(5,5) = \frac{10!}{5!5!}\frac{1}{2^{10}} = 0.246$ (i.e. 24.6%)

3b) $P(4,6) = \frac{10!}{6!4!}\frac{1}{2^{10}} = 0.205$ (i.e. 20.5%)

3c) The probabilities and the corresponding average is given in Table 1. Taking the square root of the

Table 1:

N_h	Probability	$(\frac{N}{2} - N_h)^2$
0	0.001	25
1	0.010	16
2	0.044	9
3	0.117	4
4	0.205	1
5	0.246	0
6	0.205	1
7	0.117	4
8	0.044	9
9	0.010	16
10	0.001	25
	sum=1	average=2.508

average of $(\frac{N}{2} - N_h)^2$ we get $\delta N_{rms} = 1.58$ and $\delta N_{rms}/N = 0.158$.

Discussion *How to get the root-mean squared deviation? The formula should take into consideration the probability of each deviation from the mean. And "$\langle\rangle$" represents the average.*

$$\delta N_{rms} = \langle(\frac{1}{2}N - N_h)^2\rangle^{1/2} = [\sum_{N_h=0}^{N_h=10}(\frac{1}{2}N - N_h)^2 P(\frac{1}{2}N - N_h)]^{1/2}$$

Where $P(\frac{1}{2}N - N_h)$ *is the probability of each* N_h *as shown in the table above.*

Let us compare this to the gaussian approximation. Remember that the spin excess $2s$ is equal to $N_h - N_t$ and the probability in terms of s is given by $P(s) = \sqrt{\frac{2}{\pi N}} e^{-\frac{2s^2}{N}}$ and $< (\frac{N}{2} - N_h)^2 >=< s^2 >$, valid for large N and $|s| << N$.

$$< (\frac{N}{2} - N_h)^2 >=< s^2 >= \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi N}} e^{-\frac{2s^2}{N}} s^2 ds = \frac{N}{4}$$

Hence $\delta N_{rms} = \sqrt{N}/2$ and $\delta N_{rms}/N = 1/(2\sqrt{N})$. Putting N=10 we get 0.158, which is very close to the exact solution. Remember for this problem N is pretty small so $1/\sqrt{N}$ is not very small. In the next problem the approximation will work much better.

Question 4) Again we will use the probability distribution function $g(N_h, N_t)/2^N$ as in question 3. However now the numbers are so big that we have to use the gaussian approximation (Try typing 5000! in your calculator!)

4a) $P(5000, 5000) = P(s = 0) = \sqrt{\frac{2}{\pi N}} e^0 = \sqrt{\frac{2}{\pi 10000}} = 8 \times 10^{-3}$

4b) $P(6000, 4000) = P(s = 1000) = \sqrt{\frac{2}{\pi 10000}} e^{-2\frac{1000^2}{10000}} = P(s = 0) \times e^{-200}$

4c) We did the hard work for this problem in question 3c. Set N=10000 there to get $\delta N_{rms} = 50$ and $\delta N_{rms}/N = 5 \times 10^{-3}$.