Homework #5 Solutions

Question 1) K+K Chapter 3, Problem 9. The partition function is a sum of Boltzmann factors over every state of the composite system, without regard to total energy, etc

$$Z = \sum_s e^{-\varepsilon_s/\tau}$$

The combined system has energies $\varepsilon_s = \varepsilon_1 + \varepsilon_2$. These states are the combination of every possible state of system 1 and system 2

$$Z(1+2) \equiv Z = \sum_s e^{-(\varepsilon_1+\varepsilon_2)/\tau} = \sum_{s_1}\sum_{s_2} e^{-(\varepsilon_{s_1}+\varepsilon_{s_2})/\tau} = \sum_{s_1} e^{-\varepsilon_{s_1}/\tau} \sum_{s_2} e^{-\varepsilon_{s_2}/\tau}$$

$$Z(1+2) = Z(1)Z(2)$$

This can be generalized to N independent distinguishable systems that exchange energy at fixed τ

$$Z(1+2+\cdots N) = Z(1)Z(2)\cdots Z(N)$$

This gets more complicated though if we are talking about N indistinguishable particles in the system. In that case we have to worry about not counting states more than once.

Question 2) K+K Chapter 3, Problem 2. With the results of the last problem in mind, start with the partition function of a single spin:

$$Z_1 = e^{mB/\tau} + e^{-mB/\tau} = 2\cosh(mB/\tau)$$

We can get the magnetization by taking the average of the magnetic moment per unit volume

$$\begin{aligned} M &= N\frac{< m >}{V} = \frac{N}{V}\sum_s m(s)P(s) = \frac{N}{VZ}\sum_s m(s)e^{-\varepsilon_s/\tau} \\ &= \frac{N}{VZ}\big(-me^{-mB/\tau} + me^{+mB/\tau}\big) = \frac{Nm}{VZ}2\sinh(mB/\tau) \end{aligned}$$

where N denotes the number of magnetic moments in the system. Recall that $Z = 2\cosh(mB/\tau)$. Let $n = N/V$ be the density of spins. So

$$M = nm\tanh(mB/\tau)$$

The susceptibility is

$$\chi = \frac{\partial M}{\partial B} = \frac{nm^2}{\tau}\big[1 - \tanh^2(mB/\tau)\big]$$

Alternative Solution: The partition function for N distinguishable particles (we can distinguish them by their fixed position):

$$Z_N = Z_1^N = 2^N \cosh^N(mB/\tau)$$

The internal Energy:

$$U = \tau^2 \frac{\partial \ln Z_N}{\partial \tau} = -\overline{M}B$$

Where $\overline{M}$ is the thermal average magnetic moment of the system. The magnetization will be:

$$M = \frac{\overline{M}}{V}$$

Then,

$$\begin{aligned} U &= \tau^2 \frac{\partial \ln Z_N}{\partial \tau} = \tau^2 \frac{\partial}{\partial \tau}[N \ln(2\cosh(mB/\tau))] \\ &= \tau^2 N \frac{\sinh(mB/\tau)}{\cosh(mB/\tau)} \frac{-mB}{\tau} = -NmB\tanh(mB/\tau) \end{aligned}$$

Thus,

$$\overline{M} = Nm\tanh(mB/\tau)$$

$$M = nm\tanh(mB/\tau)$$

Question 3a) K+K Chapter 3, Problem 6. Find the partition function

$$Z = \sum_{j=-1/2}^{\infty} g(j) e^{-\varepsilon(j)/\tau} = \sum_{j=-1/2}^{\infty} (2j+1) e^{-\varepsilon_0 j(j+1)/\tau}$$

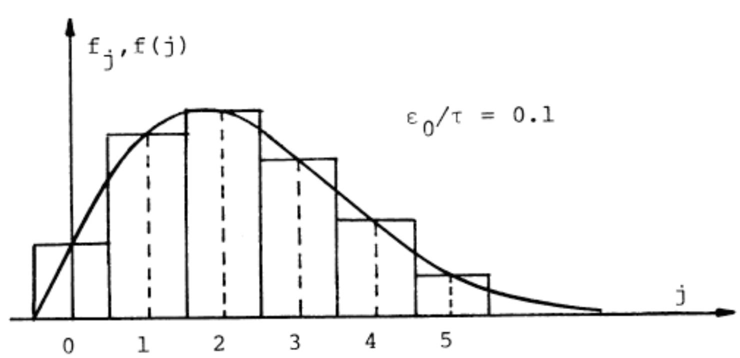

Figure 1: Approximation made in Question 3b)

3b) Evaluate Z in the high temperature limit $\tau \gg \varepsilon_0$. Let $\delta = \varepsilon_0/\tau$ be our small parameter.

$$Z \approx \int_{-1/2}^{\infty} (2j+1) e^{-\delta(j^2+j)} \mathrm{d}j$$

To evaluate the integral let $v = \delta(j^2 + j)$ and note that $\mathrm{d}v = \delta(2j+1)\mathrm{d}j$. This integral is tricky because the sum is dominated by the lowest histogram box.

The lower limit of the integration is now $v_0 = -\varepsilon_0/4\tau$, which we obtained from $j_0 = -1/2$.

$$Z \approx \delta \int_{v_0}^{\infty} e^{-\delta v} \mathrm{d}v = -\delta e^{-v}\big|_{v_0}^{\infty} = \delta e^{-v_0} = \frac{\tau}{\varepsilon_0} e^{-v_0}$$

This we can further approximate by expanding the exponential and keeping to first order in δ.

$$Z \approx \frac{\tau}{\varepsilon_0} + \frac{1}{4}$$

3c) The low temperature limit corresponds to $\eta \equiv \tau/\varepsilon_0 \ll 1$. In this limit only the lowest energy levels will contribute significantly. Hence we can truncate the summation after the second term.

$$Z \approx 1 + 3e^{-2\varepsilon_0/\tau}$$

3d) High temperature results using $Z \approx \frac{\tau}{\varepsilon_0} + \frac{1}{4}$

$$U = \tau^2 \frac{\partial}{\partial \tau} \log\left(\frac{1}{4} + \frac{\tau}{\varepsilon_0}\right) = \frac{\tau^2}{\tau + \varepsilon_0/4}$$

$$c_V \equiv \left.\frac{\partial U}{\partial \tau}\right|_V \approx 1$$

In the last line we didn't include terms second order in δ.

Low temperature results using $Z \approx 1 + 3e^{-2\varepsilon_0/\tau}$

$$U = \tau^2 \frac{\partial}{\partial \tau} \log\left(1 + 3e^{-2\varepsilon_0/\tau}\right) = \frac{6\varepsilon_0}{e^{2\varepsilon_0/\tau} + 3} \approx 6\varepsilon_0 e^{-2\varepsilon_0/\tau}$$

$$c_V = \left.\frac{\partial U}{\partial \tau}\right|_V = 12\frac{\varepsilon_0^2}{\tau^2} e^{-2\varepsilon_0/\tau}$$

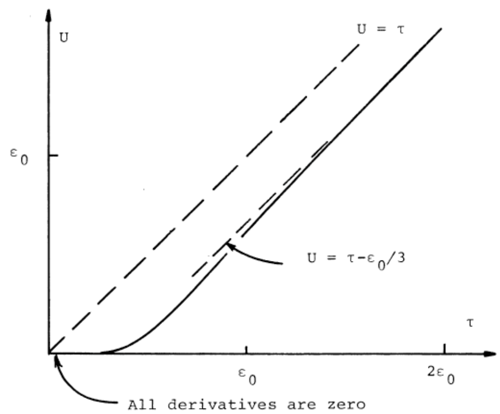

Figure 2: Internal Energy vs Temperature

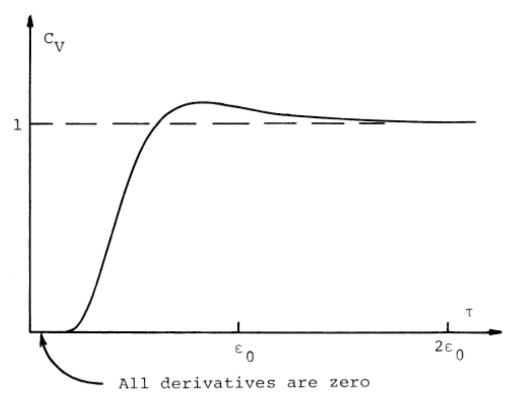

Figure 3: Heat Capacity vs Temperature

Question 4) K+K Chapter 3, Problem 11. As usual start by calculating the partition function for a single particle on the 1D line (the states are those of a particle of mass M in a 1D infinite square well):

$$\begin{aligned} Z_1 &= \sum_{n=1}^{\infty} e^{-n^2\pi^2\hbar^2/(2ML^2)\tau}. \qquad \text{Let } \alpha^2 \equiv \frac{\pi^2\hbar^2}{2ML^2\tau} \\ &\approx \int_0^{\infty} e^{-\alpha^2 n^2} \mathrm{d}n = \frac{\sqrt{\pi}}{2\alpha} = n_{Q1} L \end{aligned}$$

where in the very last step we defined the quantum concentration in 1D $n_{Q_1} = (M\tau/2\pi\hbar^2)^{1/2}$ similar to the one introduced in Eq. (63) of Chapter 3 (note the difference in powers). The N-particle partition function for indistinguishable particles is given by $Z = Z_1^N/N!$, and the free energy can be obtained by $F = -\tau \log Z$. We can use Stirling's approximationfor large N

$$\begin{aligned} F = -\tau\Big(- \log N! + N \log Z_1 \Big) &\approx \tau N \log N - N\tau - N\tau \log(n_{Q1} L) \\ &= \tau N \left[\log\left(\frac{n}{n_{Q1}}\right) - 1 \right] \qquad \text{where } n \equiv N/L \end{aligned}$$

Finally we can compute the entropy. Using $\partial \log n_{Q_1}/\partial\tau = 1/2\tau$ we compute the entropy to be

$$\sigma = -\frac{\partial F}{\partial \tau}\bigg|_L = N\left[\log\left(\frac{n_{Q_1}}{n}\right) + \frac{3}{2} \right]$$

Compare this to the 3D result Eq. (76) on page 77:

$$\sigma_{3D} = N\left[\log\left(\frac{n_{Q_{3D}}}{n}\right) + \frac{5}{2} \right]$$