PHY410 Homework 11 solutions

1. K. + K. Ch 14, #1 Mean speeds in Maxwellian distrib.

Speed distribution: $P(v) = 4\pi \left(\frac{m}{2\pi\tau}\right)^{3/2} v^2 e^{-mv^2/2\tau}$

a) $\langle v^2 \rangle = \int_0^\infty v^2 P(v)\, dv = 4\pi \left(\frac{m}{2\pi\tau}\right)^{3/2} \int_0^\infty v^4 e^{-mv^2/2\tau}\, dv$

put exponent in dimensionless form: $u = \sqrt{\frac{m}{2\tau}}\, v \quad du = \sqrt{\frac{m}{2\tau}}\, dv$

$$\langle v^2 \rangle = 4\pi \left(\frac{m}{2\pi\tau}\right)^{3/2} \left(\frac{2\tau}{m}\right)^{5/2} \int_0^\infty u^4 e^{-u^2}\, du$$

From Appendix A, equations (5), (6), (9):

$$\int_0^\infty u^4 e^{-u^2}\, du = \frac{1}{2}\Gamma\left(\frac{5}{2}\right) = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} = \frac{3\sqrt{\pi}}{8}$$

$$\langle v^2 \rangle = 4\pi \cdot \frac{2\tau}{m} \cdot \frac{1}{\pi^{3/2}} \cdot \frac{3\sqrt{\pi}}{8} = \underline{\underline{\frac{3\tau}{m}}}$$

$$v_{rms} = \sqrt{\langle v^2 \rangle} = \underline{\underline{\sqrt{\frac{3\tau}{m}}}}$$

b) The most probable speed is where $P(v)$ has its maximum

$$0 = \frac{d}{dv} P(v) = 4\pi \left(\frac{m}{2\pi\tau}\right)^{3/2} \left[2v e^{-\frac{mv^2}{2\tau}} + v^2 \left(-\frac{mv}{\tau}\right) e^{-\frac{mv^2}{2\tau}}\right]$$

$$0 = 2v - \frac{mv^3}{\tau}$$

$$2 = \frac{mv^2}{\tau} \Rightarrow \underline{\underline{v_{mp} = \sqrt{\frac{2\tau}{m}}}}$$

c) Mean speed $\bar{c} = \int_0^\infty v P(v)\, dv$

$$\bar{c} = 4\pi \left(\frac{m}{2\pi\tau}\right)^{3/2} \cdot \int_0^\infty v^3 e^{-\frac{mv^2}{2\tau}}\, dv$$

$u = \sqrt{\frac{m}{2\tau}}\, v$ as before

$$= 4\pi \left(\frac{m}{2\pi\tau}\right)^{3/2} \cdot \left(\frac{2\tau}{m}\right)^2 \cdot \int_0^\infty u^3 e^{-u^2}\, du$$

Appendix A again: $\int_0^\infty u^3 e^{-u^2}\, du = \frac{1}{2}\Gamma(2) = \frac{1}{2}$

$$\bar{c} = 4\pi \left(\frac{2\tau}{m}\right)^{1/2} \frac{1}{\pi^{3/2}} \cdot \frac{1}{2} = 2\left(\frac{2\tau}{\pi m}\right)^{1/2} = \underline{\underline{\left(\frac{8\tau}{\pi m}\right)^{1/2}}}$$

d) From class, $P(v_z) = \frac{1}{n} a(v_z) = \left(\frac{m}{2\pi\tau}\right)^{1/2} e^{-\frac{mv_z^2}{2\tau}}$

$$\langle |v_z| \rangle = \int_{-\infty}^\infty |v_z| P(v_z)\, dv_z = 2\int_0^\infty v_z P(v_z)\, dv_z$$

$$= 2\left(\frac{m}{2\pi\tau}\right)^{1/2} \int_0^\infty v_z e^{-\frac{mv_z^2}{2\tau}}\, dv_z$$

again $u = \sqrt{\frac{m}{2\tau}}\, v_z$

$$= 2\left(\frac{m}{2\pi\tau}\right)^{1/2} \cdot \left(\frac{2\tau}{m}\right) \int_0^\infty u e^{-u^2}\, du$$

$\int_0^\infty u e^{-u^2}\, du = \frac{1}{2}$

Appendix A again

$$= 2 \cdot \left(\frac{2\tau}{m}\right)^{1/2} \frac{1}{\pi^{1/2}} \cdot \frac{1}{2} = \underline{\underline{\left(\frac{2\tau}{\pi m}\right)^{1/2}}}$$

2. If a molecule travels a distance L in a gas, the probability not to scatter is

$$P(L) = e^{-L/\ell}$$, where ℓ = mean free path

For this problem, $L = 20\text{ cm} = 0.20\text{ m}$, $P(L) = 0.9$

$$\ln P(L) = -\frac{L}{\lambda}$$

$$\ell = \frac{-L}{\ln P(L)} = \frac{-0.20\text{ m}}{\ln 0.9} = \frac{-0.20\text{ m}}{-0.105} = 1.9\text{ m}$$

In class we showed that $\ell = \frac{1}{\sqrt{2}\, n \sigma_c}$

$$n = \frac{1}{\sqrt{2}\,\ell\sigma_c} = \frac{1}{\sqrt{2}\cdot 1.9\text{ m}\cdot 10^{-19}\text{ m}^2} = 3.7\cdot 10^{18}\text{ m}^{-3}$$

Ideal gas: $pV = N\tau \quad \rightarrow \quad p = \frac{N}{V}\tau = n k_B T$

$$p = 3.7\cdot 10^{18}\text{ m}^{-3}\cdot 1.38\cdot 10^{-23}\text{ J/K}\cdot 300\text{ K} = \underline{\underline{1.5\cdot 10^{-2}\text{ Pa}}}$$

Many experimentalists still measure pressure in Torr

$10^5\text{ Pa} = 760\text{ Torr}$

$$p = 1.5\cdot 10^{-2}\text{ Pa}\left(\frac{760\text{ Torr}}{10^5\text{ Pa}}\right) = 1.1\cdot 10^{-4}\text{ Torr}$$

$$p_1 = N\frac{\tau_1}{V_1}$$

	τ_2	p_2	ΔU	$\Delta\sigma$	W	Q
Isothermal	τ_1	$\frac{1}{3}p_1$	0	$N\ln 3$	$N\tau_1 \ln 3$	$N\tau_1\ln 3$
Isentropic	$\tau_1/3^{2/3}$	$p_1/3^{5/3}$	$\frac{3}{2}N\tau_1\left(\frac{1}{3^{2/3}}-1\right)$	0	$\frac{3}{2}N\tau\left(1-\frac{1}{3^{2/3}}\right)$	0
Sudden	τ_1	$\frac{1}{3}p_1$	0	$N\ln 3$	0	0

a) Isothermal: $\tau_2 = \tau_1$ $\quad U = U(\tau) \Rightarrow \Delta U = 0$

Ideal gas $pV = N\tau$, $V_2 = 3V_1 \Rightarrow p_2 = \frac{1}{3}p_1$

$$W = \int_{V_1}^{V_2} p\,dV = \int_{V_1}^{V_2} \frac{N\tau_1}{V}\,dV = N\tau_1 \ln\frac{V_2}{V_1} = N\tau_1 \ln 3$$

1st Law: $\Delta U = Q - W$, $\Delta U = 0 \Rightarrow Q = W = N\tau_1 \ln 3$

$$\sigma = N\left[\ln\frac{n_Q}{n} + \frac{5}{2}\right] = N\left[\ln\left(\frac{V n_Q}{N}\right) + \frac{5}{2}\right]$$

$\sigma_2 - \sigma_1 = N\ln\frac{V_2}{V_1} = N\ln 3$ since τ is constant

Alternative method: $d\sigma = \frac{dQ}{\tau} \Rightarrow \Delta\sigma = \frac{Q}{\tau} = N\ln 3$

no integral necessary since τ is constant

b) Isentropic: $\sigma_2 = \sigma_1 \Rightarrow \Delta\sigma = 0$

$dQ = \tau\,d\sigma = 0 \Rightarrow Q = 0$

continued

Monatomic gas has $\gamma = \frac{5}{3}$, $\tau V^{\gamma-1} = \tau V^{2/3} = \text{constant}$

$$\tau_1 V_1^{2/3} = \tau_2 V_2^{2/3} \rightarrow \tau_2 = \tau_1 \left(\frac{V_1}{V_2}\right)^{2/3} = \frac{\tau_1}{3^{2/3}}$$

$$pV^{\gamma} = \text{constant} \rightarrow p_2 = p_1 \left(\frac{V_1}{V_2}\right)^{5/3} = \frac{p_1}{3^{5/3}}$$

$$\Delta U = U_f - U_i = \frac{3}{2}N\tau_2 - \frac{3}{2}N\tau_1 = \frac{3}{2}N\tau_1\left(\frac{1}{3^{2/3}} - 1\right)$$

$$\Delta U = Q - W, \quad Q = 0 \Rightarrow W = -\Delta U$$

c) Sudden: The gas does no work and absorbs no heat: $W = 0$, $Q = 0$

$$\Delta U = Q - W = 0 \Rightarrow \tau \text{ is constant}$$

Ideal gas law $p = \frac{N\tau}{V} \Rightarrow p_2 = \frac{1}{3} p_1$

$$\sigma = N\left[\ln\left(\frac{V n_Q}{N}\right) + \frac{5}{2}\right] \qquad \tau \text{ doesn't change}$$

$$\Delta\sigma = N \ln \frac{V_2}{V_1} = N \ln 3$$

Alternative method: Multiplicity increases by 3^N

$$\Delta\sigma = \ln \frac{g_2}{g_1} = \ln 3^N = N \ln 3$$

4. Initial state: 10 g water at $T_{1i} = 10^\circ C$

30 g water at $T_{2i} = 90^\circ C$

a) The two systems exchange heat, ending at a common temperature T_f.

$$Q_1 = -Q_2$$

$$m_1 c_p (T_f - T_{1i}) = -m_2 c_p (T_f - T_{2i})$$

$$T_f (m_1 + m_2) = m_2 T_{2i} + m_1 T_{1i}$$

$$T_f = \frac{m_1 T_{1i} + m_2 T_{2i}}{m_1 + m_2} = \frac{10g \cdot 10^\circ C + 30g \cdot 90^\circ C}{40 g} = \underline{\underline{70^\circ C}}$$

b) Let the systems exchange heat slowly, so we can use $dS = \frac{dQ}{dT}$ with $dQ = m c_p \, dT$

$$\Delta S = \Delta S_1 + \Delta S_2 = \int_{T_{1i}}^{T_f} \frac{dQ_1}{T} + \int_{T_{2i}}^{T_f} \frac{dQ_2}{T}$$

$$= c_p \left[m_1 \int_{10^\circ C}^{70^\circ C} \frac{dT}{T} + m_2 \int_{90^\circ C}^{70^\circ C} \frac{dT}{T} \right]$$

Must convert temperatures to <u>Kelvin</u>

$$= 4.19 \frac{J}{gK} \left[10g \cdot \ln \frac{343}{283} + 30g \ln \frac{343}{363} \right]$$

$$= 4.19 \frac{J}{gK} \left[1.923g \quad 1.700g \right] = \underline{0.93 \ J/K}$$

5. Spins in a $\vec{B}$ field; 2 states for each spin

$$\vec{m}_{\downarrow} = -m_0 \hat{e}_z \qquad m_{\uparrow} = m_0 \hat{e}_z$$

$$\mathcal{E}_{\downarrow} = +m_0 B \qquad \mathcal{E}_{\uparrow} = -m_0 B$$

a) Probabilities: $P_{\uparrow} = \dfrac{e^{+m_0 B/\tau}}{e^{m_0 B/\tau} + e^{-m_0 B/\tau}}$ $\quad P_{\downarrow} = \dfrac{e^{-m_0 B/\tau}}{e^{m_0 B/\tau} + e^{-m_0 B/\tau}}$

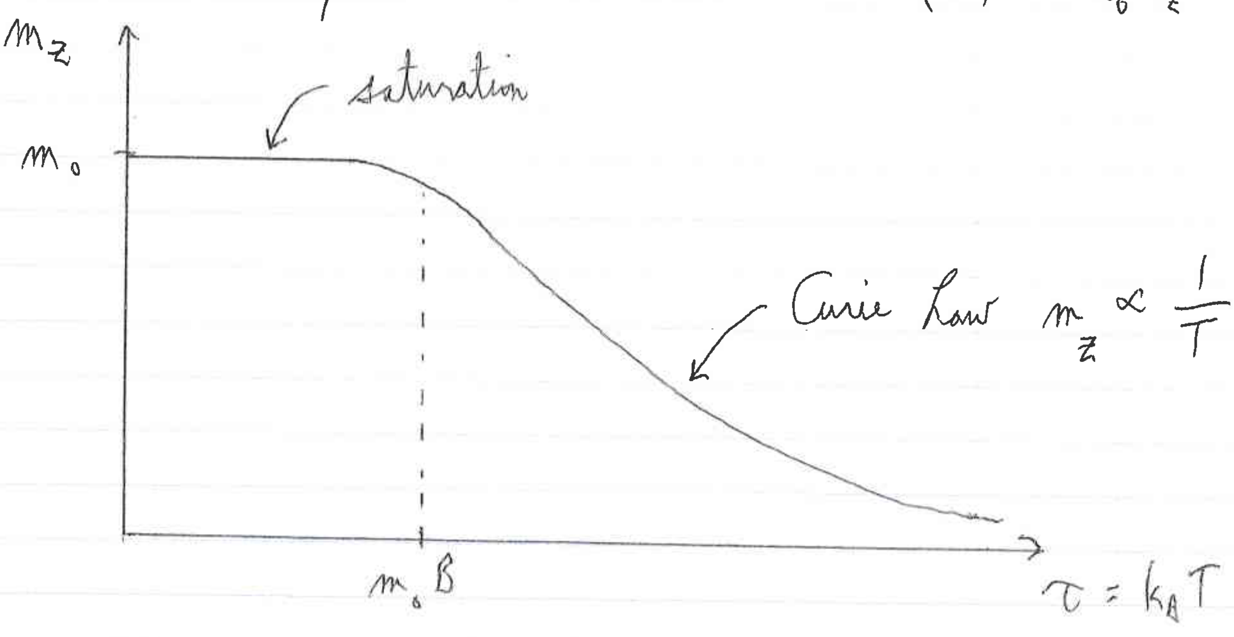
$$\langle \vec{m} \rangle = \sum_{j=1}^{2} \vec{m}_j P_j = \frac{m_0 \hat{e}_z e^{m_0 B/\tau} - m_0 \hat{e}_z e^{-m_0 B/\tau}}{e^{m_0 B/\tau} + e^{-m_0 B/\tau}}$$

$$\underline{\underline{\langle \vec{m} \rangle = m_0 \hat{e}_z \tanh\left(\frac{m_0 B}{\tau}\right)}}$$

b) This part is tricky:

high temperature: $\tanh x \sim x$ $\quad \langle \vec{m} \rangle \approx m_0 \hat{e}_z \cdot \dfrac{m_0 B}{k_B T}$

low temperature: $\tanh x \sim 1$ $\quad \langle \vec{m} \rangle = m_0 \hat{e}_z$



6. $S = A V^{1/3} U^{1/2}$ I am using U instead of E for "internal energy".

With N fixed, the Thermodynamic Identity is

$$dU = T\,dS - p\,dV$$

or $$dS = \frac{1}{T}\,dU + \frac{p}{T}\,dV$$

The latter form is for $S = S(U, V)$, which is the situation for a Microcanonical Ensemble. You can see the derivatives directly:

$$(1): \frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_V \qquad (2): \frac{p}{T} = \left(\frac{\partial S}{\partial V}\right)_U$$

$$(1): \frac{1}{T} = \frac{1}{2} A V^{1/3} U^{-1/2} \qquad (2): \frac{p}{T} = \frac{1}{3} A V^{-2/3} U^{1/2}$$

Solve (1) for $U^{1/2}$: $U^{1/2} = \frac{1}{2} A V^{1/3} T$

Plug into (2): $$\frac{p}{T} = \frac{1}{3} A V^{-2/3} \cdot \frac{1}{2} A V^{1/3} T$$

$$p = \frac{1}{6} A^2 V^{-1/3} T^2 \quad \text{or} \quad \underline{\underline{p V^{1/3} = \frac{A^2}{6} T^2}}$$

7. Ideal Gas with $S = Nk_B \ln\left(VT^{5/2}\right) + \text{constant}$

Gas expands adiabatically from T_1, V_1 to T_2, V_2.

a) Adiabatic = isentropic $\Rightarrow \Delta S = 0$

b) $S = \text{constant}$

$$\Rightarrow \quad V_1 T_1^{5/2} = V_2 T_2^{5/2}$$

$$\left(\frac{T_2}{T_1}\right)^{5/2} = \frac{V_1}{V_2}$$

$$\frac{T_2}{T_1} = \left(\frac{V_1}{V_2}\right)^{2/5} \Rightarrow T_2 = T_1\left(\frac{V_1}{V_2}\right)^{2/5}$$

Optional: $\Delta T = T_2 - T_1 = T_1\left[\left(\frac{V_1}{V_2}\right)^{2/5} - 1\right]$

c) Ideal gas $pV = Nk_BT$ $\qquad p = \frac{Nk_BT}{V} \propto \frac{T}{V}$

$$\frac{p_2}{p_1} = \frac{\left(T_2/V_2\right)}{\left(T_1/V_1\right)} = \frac{T_2}{T_1}\frac{V_1}{V_2} = \left(\frac{V_1}{V_2}\right)^{2/5}\cdot\left(\frac{V_1}{V_2}\right) = \left(\frac{V_1}{V_2}\right)^{7/5}$$

d) S is extensive: If $N \to 2N$, $V \to 2V$, $T \to T$, we must have $S \to 2S$! We can satisfy that by writing

$$\underline{\underline{S = Nk_B \ln\left(\frac{V}{N}T^{5/2}\right) + \text{constant}}}$$

The missing term was $-Nk_B \ln N$