PHY 410 - Homework 2 Solutions

1. K. + K. Ch. 2, # 1 $\quad g(U) = C\,U^{3N/2}$

a) $\sigma = \ln g = \ln C + \frac{3N}{2}\ln U$

$$\frac{1}{\tau} = \left(\frac{\partial\sigma}{\partial U}\right)_N = \frac{3N}{2}\frac{1}{U} \Rightarrow \underline{\underline{U = \frac{3}{2}N\tau}}$$

b) $\left(\frac{\partial^2\sigma}{\partial U^2}\right)_N = \frac{3N}{2}\left(\frac{-1}{U^2}\right) < 0$

2. K. + K. Ch. 2, # 2 $\quad$ N spins of moment m. $\quad 2s \equiv N_\uparrow - N_\downarrow$

In class we showed that

$$\sigma(s) = \ln g(N, s=0) - \frac{2s^2}{N} \quad \text{for } s \ll N$$

$$U(s) = -2smB, \quad \text{so} \quad s = \frac{-U}{2mB}$$

$$\sigma(U) = \ln g(N,0) - \frac{U^2}{2m^2B^2N} = \sigma_0 - \frac{U^2}{2m^2B^2N}$$

$$\frac{1}{\tau} = \left(\frac{\partial\sigma}{\partial U}\right)_N = \frac{-U}{m^2B^2N} = \frac{2s}{mBN}$$

$$\frac{M}{Nm} = \frac{2\langle s\rangle m}{Nm} = \underline{\underline{\frac{mB}{\tau}}}$$

The $\frac{1}{\tau}$ dependence is called the "Curie Law."

3. K. & K. Ch. 2 # 3 Quantum harmonic oscillator

N oscillators, n quanta: $g(N,n) = \dfrac{(N+n-1)!}{n!\,(N-1)!}$

a) $$\sigma = \ln g = \ln (N+n-1)! - \ln n! - \ln (N-1)!$$

$$\approx (N+n-1)\ln(N+n-1) - (N+n-1)$$

$$- (n \ln n - n) - \left[(N-1)\ln(N-1) - (N-1)\right]$$

$$= (N+n-1)\ln(N+n-1) - n\ln n - (N-1)\ln(N-1)$$

For large N we can replace $N-1$ by N:

$$\left.\begin{aligned} \sigma &\approx (N+n)\ln(N+n) - n\ln n - N\ln N \\ &= N\ln\left(\frac{N+n}{N}\right) + n\ln\left(\frac{N+n}{n}\right) \end{aligned}\right\}$$ either form is O.K.

b) $U = \underline{n\hbar\omega} \rightarrow n = \dfrac{U}{\hbar\omega}$

$$\sigma(U) = N\ln\left(1+\frac{U}{N\hbar\omega}\right) + \frac{U}{\hbar\omega}\ln\left(1+\frac{N\hbar\omega}{U}\right)$$

$$\frac{1}{\tau} = \left(\frac{\partial\sigma}{\partial U}\right)_N = \frac{N}{1+\frac{U}{N\hbar\omega}}\cdot\frac{1}{N\hbar\omega} + \frac{U/\hbar\omega}{1+\frac{N\hbar\omega}{U}}\cdot\left(-\frac{N\hbar\omega}{U^2}\right)$$

$$+ \frac{1}{\hbar\omega}\ln\left(1+\frac{N\hbar\omega}{U}\right)$$

product rule

$$\frac{1}{\tau} = \cancel{\frac{N}{N\hbar\omega + U}} - \cancel{\frac{N}{U + N\hbar\omega}} + \frac{1}{\hbar\omega}\ln\left(1 + \frac{N\hbar\omega}{U}\right)$$

$$\frac{\hbar\omega}{\tau} = \ln\left(1 + \frac{N\hbar\omega}{U}\right) \Rightarrow e^{\frac{\hbar\omega}{\tau}} = 1 + \frac{N\hbar\omega}{U}$$

$$\Rightarrow e^{\hbar\omega/\tau} - 1 = \frac{N\hbar\omega}{U} \Rightarrow \boxed{U = \frac{N\hbar\omega}{e^{\hbar\omega/\tau} - 1}}$$

This is the famous Planck distribution

4. 2 spin systems, $N_1 = N_2 = 10$

Initially, S_1 has all 10 spins up, and S_2 has 2 up and 8 down. Here is a table of all possible configurations and their multiplicities:

$N_{\uparrow 1}$	s_1	g_1	$N_{\uparrow 2}$	s_2	g_2	$g_1 g_2$
10	5	1	2	-3	45	45
9	4	10	3	-2	120	1200
8	3	45	4	-1	210	9450
7	2	120	5	0	252	30240
6	1	210	6	1	210	44100
5	0	252	7	2	120	30240
4	-1	210	8	3	45	9450
3	-2	120	9	4	10	1200
2	-3	45	10	5	1	45

$$\sum g_1 g_2 = 125{,}970$$

a) Initial multiplicity is $\underline{\underline{45}}$

b) see the table on the previous page.

$$\sum g_1 g_2 = \underline{\underline{125,970}}$$

$$\sigma_i = \ln 45 = 3.81 \qquad \sigma_f = \ln 125970 = 11.74$$

$$\sigma_f - \sigma_i = \underline{\underline{7.93}}$$

c) $P(\hat{s}_1) + P(\hat{s}_1 + 1) + P(\hat{s}_1 - 1)$

$$= \frac{44,100 + 30,240 + 30,240}{125,970} = \underline{\underline{0.83}}$$

$$P_{initial} = \frac{45}{125,970} = \underline{\underline{3.6 \cdot 10^{-4}}}$$

5. 2 large spin systems: $N_1 = 10^4$, $N_2 = 2 \cdot 10^4$

a) Initially $s_1 = 0$, $s_2 = 1500$ $\qquad U = -25\,mB$

$$\underline{\underline{U_1^0 = 0}} \quad , \quad \underline{\underline{U_2^0 = -3000\,mB}}$$

$$g(N,s) \approx 2^N e^{-2s^2/N} \qquad \text{ignore } \left(\frac{2}{\pi N}\right)^{1/2} \text{ term}$$

$$\sigma(N,s) \approx N \ln 2 - \frac{2s^2}{N}$$

$$\sigma_1^0 = N_1 \ln 2 = \underline{\underline{0.69 \cdot 10^4}} \qquad \sigma_2^0 = 1.39 \cdot 10^4 - 225 = \underline{\underline{1.36 \cdot 10^4}}$$

b) $\sigma(N, U) = N \ln 2 - \frac{2}{N}\left(\frac{U}{2mB}\right)^2$ from $U = -2mBs$

$$\frac{1}{\tau} = \left(\frac{\partial \sigma}{\partial U}\right)_N = \frac{-U}{Nm^2B^2} = \frac{2mBs}{Nm^2B^2} = \frac{2s}{NmB}$$

$$\tau = \frac{NmB}{2s} \Rightarrow \underline{\underline{\tau_1^0 = \infty}} \qquad \underline{\underline{\tau_2^0 = 6.67\, mB}}$$

c) $\tau \propto \frac{N}{s}$ means that the ratio $\frac{N}{s}$ should be the same in the two systems when they are in thermal equilibrium, which we showed in class.

$$\frac{\hat{s}_1}{N_1} = \frac{\hat{s}_2}{N_2} = \frac{s}{N} = \frac{1500}{3 \cdot 10^4} = 0.05$$

$$\Rightarrow \underline{\underline{\hat{s}_1 = 500}}, \quad \underline{\underline{\hat{s}_2 = 1000}}, \quad \tau = \frac{NmB}{2s} = \underline{\underline{10\, mB}}$$

d) Ignore the $N \ln 2$ terms in $\sigma(N, s)$; look at $-\frac{2s^2}{N}$:

$$\sigma_{1i} = 0, \quad \sigma_{2i} = -225 \Longrightarrow \sigma_i = -225$$

$$\sigma_{1f} = -50, \quad \sigma_{2f} = -100 \Longrightarrow \underline{\sigma_f = -150}$$

$$\Delta\sigma = \sigma_f - \sigma_i = \underline{\underline{75}}$$

$$\frac{g_f}{g_i} = e^{(\sigma_f - \sigma_i)} = e^{75} = \underline{\underline{3.7 \cdot 10^{32}}}$$

The system will never return to the initial state.