PHY 410 - Homework 8 solutions

1. K. + K. Ch. 6 #1

$$f_{FD}(\varepsilon) = \frac{1}{e^{(\varepsilon-\mu)/\tau}+1} \qquad -\frac{\partial f}{\partial \varepsilon} = \frac{\frac{1}{\tau}\, e^{(\varepsilon-\mu)/\tau}}{\left(e^{(\varepsilon-\mu)/\tau}+1\right)^2}$$

$$-\frac{\partial f}{\partial \varepsilon}(\varepsilon=\mu) = \frac{1/\tau}{(1+1)^2} = \underline{\underline{\frac{1}{4\tau}}}$$

2. K. + K. Ch. 6 #2

$$f_{FD}(\varepsilon=\mu+\delta) = \frac{1}{e^{(\mu+\delta-\mu)/\tau}+1} = \frac{1}{e^{\delta/\tau}+1}$$

$$f_{FD}(\varepsilon=\mu-\delta) = \frac{1}{e^{-\delta/\tau}+1} = \frac{e^{\delta/\tau}}{1+e^{\delta/\tau}}$$

$$1-f(\mu-\delta) = 1-\frac{e^{\delta/\tau}}{e^{\delta/\tau}+1} = \frac{\left(e^{\delta/\tau}+1\right)-e^{\delta/\tau}}{e^{\delta/\tau}+1} = \frac{1}{e^{\delta/\tau}+1}$$

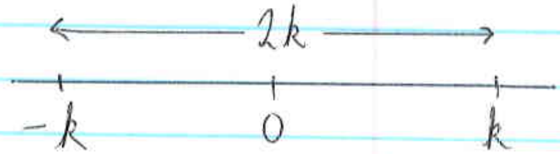
$$1-f(\mu-\delta) = f(\mu+\delta) \quad \checkmark$$

The probability of an orbital at energy $\varepsilon=\mu+\delta$ to be filled is the same as the probability for an orbital at $\varepsilon=\mu-\delta$ to be empty.

3. K. + K. Ch. 7 #1 1D, 2D density of orbitals

I'll use Method 2 from lecture. $\varepsilon = \frac{\hbar^2 k^2}{2m} \rightarrow |k| = \frac{\sqrt{2m\varepsilon}}{\hbar}$

a) 1D: The number of orbitals inside a "sphere" of radius $k = \frac{\sqrt{2m\varepsilon}}{\hbar}$ is:



$$N(\varepsilon) = \underset{\text{spin}}{2} \cdot \underset{\substack{\text{density of orbitals} \\ \text{in } k\text{-space}}}{\frac{L}{2\pi}} \cdot 2k = \frac{2L}{\pi} k = \frac{2L}{\pi} \cdot \frac{\sqrt{2m\varepsilon}}{\hbar}$$

$$D(\varepsilon) = \frac{dN(\varepsilon)}{d\varepsilon} = \frac{2L}{\pi\hbar} \cdot \sqrt{2m} \cdot \frac{1}{2}\varepsilon^{-1/2} = \underline{\underline{\frac{L}{\pi\hbar}\sqrt{\frac{2m}{\varepsilon}}}}$$

b) 2D:

$$N(\varepsilon) = 2 \cdot \left(\frac{L}{2\pi}\right)^2 \cdot \pi k^2 = \frac{A}{2\pi} k^2 = \frac{A}{2\pi} \cdot \frac{2m\varepsilon}{\hbar^2}$$

$$D(\varepsilon) = \frac{dN(\varepsilon)}{d\varepsilon} = \underline{\underline{\frac{Am}{\pi\hbar^2}}}$$

$D(\varepsilon)$ is independent of ε in 2D.

4. K. + K. Ch. 7, #11 Fluctuations in a Fermi gas.

Start with the Grand Canonical probability distribution:

$$P(N,\varepsilon) = \frac{e^{(N\mu-\varepsilon)/\tau}}{\mathcal{Z}} \quad ; \quad \mathcal{Z} = \sum_N \sum_{\varepsilon(N)} e^{(N\mu-\varepsilon)/\tau}$$

We showed in class that

$$\langle N \rangle = \sum_N \sum_{\varepsilon(N)} N\, P(N,\varepsilon) = \frac{\sum_N \sum_\varepsilon N\, e^{(N\mu-\varepsilon)/\tau}}{\mathcal{Z}}$$

$$= \frac{e^{(\mu-\varepsilon)/\tau}}{1 + e^{(\mu-\varepsilon)/\tau}} = \frac{1}{e^{(\varepsilon-\mu)/\tau} + 1} = f_{FD}(\varepsilon)$$

Similarly:

$$\langle N^2 \rangle = \frac{\sum_N \sum_\varepsilon N^2 e^{(N\mu-\varepsilon)/\tau}}{\mathcal{Z}} = \frac{e^{(\mu-\varepsilon)/\tau}}{1+e^{(\mu-\varepsilon)/\tau}} = \langle N \rangle$$

We have shown that

$$\langle (\Delta N)^2 \rangle = \langle N^2 \rangle - \langle N \rangle^2$$

$$= \langle N \rangle - \langle N \rangle^2 \quad \text{from above}$$

$$= \langle N \rangle \left(1 - \langle N \rangle\right)$$

It makes sense that fluctuations in the occupation number of an orbital go to zero when the orbital is always empty ($\langle N \rangle \to 0$) or always full ($\langle N \rangle \to 1$).

5. K. & K. Ch 7, # 12 Fluctuations in a Bose gas

Performing the sum as we did in the previous problem is difficult, so instead we'll use the derivative trick derived in Eqn. 5.83:

$$\langle (\Delta N)^2 \rangle = \tau \frac{\partial \langle N \rangle}{\partial \mu}$$

where $\langle N \rangle = f_{BE}(\varepsilon) = \dfrac{1}{e^{(\varepsilon-\mu)/\tau} - 1}$

$$\frac{\partial \langle N \rangle}{\partial \mu} = \frac{-e^{(\varepsilon-\mu)/\tau} \cdot \left(-\frac{1}{\tau}\right)}{\left(e^{(\varepsilon-\mu)/\tau} - 1\right)^2}$$

$$\langle (\Delta N)^2 \rangle = \frac{e^{(\varepsilon-\mu)/\tau}}{\left(e^{(\varepsilon-\mu)/\tau} - 1\right)^2} = \frac{1}{\left(e^{(\varepsilon-\mu)/\tau} - 1\right)} \frac{e^{(\varepsilon-\mu)/\tau}}{\left(e^{(\varepsilon-\mu)/\tau} - 1\right)}$$

$$= \langle N \rangle \left(\langle N \rangle + 1 \right)$$

So, the fluctuations in the number of bosons in an orbital get larger as the average number of bosons in that orbital increases.

For large $\langle N \rangle$, we get $\delta N \equiv \sqrt{\langle (\Delta N)^2 \rangle} \approx \langle N \rangle$

as compared to the Classical gas result: $\delta N \approx \sqrt{N}$.

6. K. + K. Ch. 7, # 3, part (a)

Method 1: $p = -\left(\frac{\partial U}{\partial V}\right)_{\sigma, N} = \frac{-\partial U_0}{\partial V}$

In the ground state the entropy is constant, so this is O.K.

In class we derived:

$$U_0 = \frac{3}{5} N \varepsilon_F = \frac{3}{5} N \cdot \frac{\hbar^2 k_F^2}{2m} = \frac{3}{5} N \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$$

write this as $U_0 = a V^{-2/3}$

$$p = \frac{-\partial U_0}{\partial V} = \frac{-\partial}{\partial V}\left(a V^{-2/3}\right) = \frac{2}{3} a V^{-5/3} = \frac{2}{3}\frac{U_0}{V}$$

$$p = \frac{2}{3}\frac{1}{V}\cdot\frac{3}{5} N \frac{\hbar^2}{2m}\left(\frac{3\pi N}{V}\right)^{2/3} = \underline{\underline{\frac{\hbar^2}{5m}\left(3\pi^2\right)^{2/3}\left(\frac{N}{V}\right)^{5/3}}}$$

Method 2: $p = \int_0^\infty d\varepsilon\, D(\varepsilon) f(\varepsilon)\, p(\varepsilon)$

Since the energy of an orbital $\varepsilon \propto V^{-2/3}$, the pressure due to an orbital is given by

$$p(\varepsilon) = -\frac{\partial}{\partial V}\left(b V^{-2/3}\right) = \frac{2}{3} b V^{-5/3} = \frac{2}{3}\frac{\varepsilon}{V}$$

So $p = \int_0^\infty d\varepsilon\, D(\varepsilon) f(\varepsilon) \cdot \frac{2}{3}\frac{\varepsilon}{V} = \frac{2}{3}\frac{1}{V}\cdot\int_0^\infty d\varepsilon\, D(\varepsilon) f(\varepsilon)\, \varepsilon$

$p = \frac{2}{3}\frac{U_0}{V}$ which agrees with Method 1.