Finm 33000: Homework 3

Problem 1: $B_0 = 1$ $B_T(W_u) = B_T(W_m) = B_T(W_d) = 1.2$

$S_0 = 145$ $S_T(W_u) = 240$, $S_T(W_m) = 120$, $S_T(W_d) = 60$

$C_0 = 10$ $C_T(W_u) = 0$, $C_T(W_m) = 30$; $C_T(W_d) = 30$

a. Find all equivalent martingale measures $[p_u, p_m, p_d]$

Our price system at t=1: $D_1 = \begin{bmatrix} 1.2 & 1.2 & 1.2 \\ 240 & 120 & 60 \\ 0 & 30 & 30 \end{bmatrix}$ Needs to be normalized

and $Z_0 = \begin{bmatrix} 1 \\ 145 \\ 10 \end{bmatrix}$

Such that
$E^{Q}[Z_1] = Z_0$ $Z_1 = \begin{bmatrix} 1 & 1 & 1 \\ 200 & 100 & 50 \\ 0 & 25 & 25 \end{bmatrix} = \begin{bmatrix} 1 \\ 145 \\ 10 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & .4 \\ 0 & 1 & 1.5 & .55 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & .1 \\ 0 & 0 & 1 & .3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & .6 \\ 0 & 1 & 0 & .1 \\ 0 & 0 & 1 & .3 \end{bmatrix}$$

$p_u = .6$; $p_m = .1$; $p_d = .3$

Thus, given the fixed numeraire, the market is free of arbitrage ~~and~~ because a martingale measure exists and the market is complete because the martingale measure Q is unique since one solution only exists.

b.

$X_T(w_u) = 120$; $X_T(w_m) = 60$; $X_T(w_d) = 0$. Can this be done using a portfolio of B, S, and C?

We need to find a portfolio h to solve the system:

$$(D^Z)^T \cdot h = X^Z = \left[\begin{array}{ccc|c} 1 & 200 & 0 & 100 \\ 1 & 100 & 25 & 50 \\ 1 & 50 & 25 & 0 \end{array}\right] \sim \begin{bmatrix} 1 & 200 & 0 & 100 \\ 0 & -100 & 25 & -50 \\ 0 & -150 & 25 & -100 \end{bmatrix}$$

$\frac{1}{4} - \frac{1}{6}$ $\quad -\frac{1}{2} + \frac{2}{3}$

$\frac{6}{24} \; \frac{4}{24}$ $\quad -\frac{3}{6} \; \frac{4}{6}$

$$\sim \begin{bmatrix} 1 & 200 & 0 & 100 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{6} & \frac{2}{3} \end{bmatrix} \sim \begin{bmatrix} 1 & 200 & 0 & 100 \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} \\ 0 & 0 & \frac{1}{12} & \frac{1}{6} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 200 & 0 & 100 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -100 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Thus, the contingent claim X_t can be replicated by $N_0^h = \begin{bmatrix} -100 \\ 1 \\ 2 \end{bmatrix}$

-100 units of the bank, 1 unit of stock, and 2 units of the option.

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c. Find the arbitrage time-0 price of the payoff of X_T using MM and the Value of the replicated portfolio

$$X_0 = q \cdot X_T^Z = [.6 \;\; .1 \;\; .3] \begin{bmatrix} 100 \\ 50 \\ 0 \end{bmatrix} = 60 + 5 = \$65$$

$$N_0^h = h \cdot D_0 = [-100 \;\; 1 \;\; 2] \begin{bmatrix} 1 \\ 145 \\ 10 \end{bmatrix} = -100 + 145 + 20 = \$65.$$

Problem 2 Redo all parts of problem 1 with the change to the definition of C. (3)

$C_0 = 11 \quad C_T(\omega_u) = 0, \; C_T(\omega_m) = 24, \; C_T(\omega_d) = 36$

a) Find the martingale measures $\overset{Q}{E}[Z_1] = Z_0$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 200 & 100 & 50 & 145 \\ 0 & 20 & 30 & 11 \end{array}\right] \sim \left[\begin{array}{cccc} 1 & 0 & \frac{1}{2} & .45 \\ 0 & 1 & \frac{3}{2} & .55 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} p_u &= .45 - .5\,p_d \\ p_m &= .55 - 1.5\,p_d \\ p_d &= p_d \end{aligned} \quad = \quad Q = \begin{bmatrix} .45 \\ .55 \\ 0 \end{bmatrix} + p_d \begin{bmatrix} -.5 \\ -1.5 \\ 1 \end{bmatrix}$$

There are an infinite amount of martingale measures to solve the "market" hence the market is $\boxed{\text{not complete.}}$

b) Try to replicate this payoff of X_T

$$(D^z)^T h = X^z = \left[\begin{array}{ccc|c} 1 & 200 & 0 & 100 \\ 1 & 100 & 20 & 50 \\ 1 & 50 & 30 & 0 \end{array}\right] \sim \begin{bmatrix} 1 & 200 & 0 & 100 \\ 0 & -100 & 200 & -50 \\ 0 & 0 & 0 & -25 \end{bmatrix}$$

$0 \neq -25$ There are no solutions to replicate the payoff of X_T.

This is because the payoff of $X_T \notin$ span of $(D^z)^T$. In other words D^z has a rank of 2, but it needs to have a rank of 3.

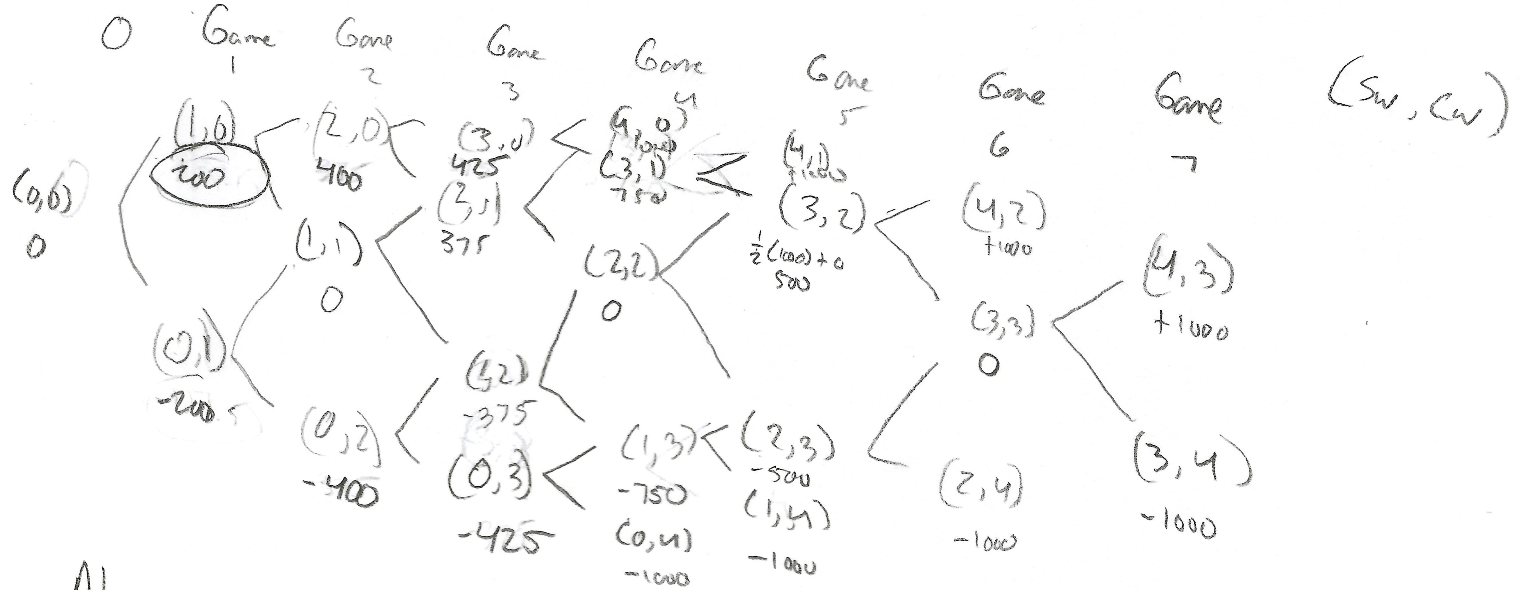
c Since there are an infinite amount of choices for the risk-neutral probabilities there are an infinite time-0 prices on the contingent claim for each martingale measure Q.

White Sox / Cubs

Objective to win \$1000 if sox ~~lose~~ win the series or lose \$1000 if they lose the ~~four~~ seven game series. What bet should you place on game 1?

Use martingale approach: $P[WS = \text{win}] = p$: $p - 1(1-p) = 0$

$p = 1/2$



At game 1 our expected payout if the white sox win ~~is~~ \$200, while if they lose is -\$200. In order to reach the objective of +1000 if they win the series and -1000 if they lose, we need to place a bet equal to \$200 before game 1 starts. Since of the size of the bet is determined after the game concludes we cannot say anything else until the outcome of game 1 is realized.