Homework 4

Problem 1

a) A given contingent claim X is reachable if all claims can be replicated when the market is dynamically complete.

$$\theta_0^{rep} \neq \begin{cases} 0, 0 & \text{if } S_0 \le K \\ 1, -K & \text{if } S_0 > K \end{cases}$$

This argument is flawed because the time-0 price of the contingent claim is not 0 or else arbitrage would exist.

Using the risk neutral probabilities the time-0 price of the call with strike = 105 is given

$C_0 = e^{-rT} \cdot \mathbb{E}^Q[C]$ where Q denotes the martingale measure.

$\boxed{\text{Thus, if } P(\theta_T \geq 0) > 0, \text{ then } \theta_0^{rep.} = e^{-rT} \mathbb{E}^Q[\theta_T^{rep}] > 0.}$

If $S_0 \le K$, and the option is 0 at time-0, you long the option and by defining $\theta_0 = 0$

$P(\theta_T \geq 0) = 1$ then arbitrage exists.

$P(\theta_T > 0) > 0$

b. Find the time-0 value of the call.

$$C_0 = e^{-rT} \cdot \mathbb{E}^Q[C]$$

$$\therefore C_0 = e^{-rT} \cdot \sum_{K=0}^{T} \binom{T}{K} q_u^K q_d^{T-K} \cdot \phi(Su^K d^{T-K})$$

$r = 0$

From the problem, the martingale probabilities are equal to $q_u = q_d = 1/2$.

$u = 1.01$ and $d = .99$. $S_0 = 100$; strike $= \$105$

$$C_0 = \sum_{K=0}^{T} \binom{12}{0} (.5)^0 (.5)^{12} \cdot \underline{0} = 0$$

upmum

$C_0 = 0$ for $K = 0, 1, 2, 3, 4, 5, 6, 7,$ and 8

$$C_0 = \sum_{K=9}^{12} \binom{12}{9} (.5)^9 (.5)^3 \cdot \phi(106.12017)$$

$220(.5)^{12} \cdot 1.12017 = \$0.06016$

$+$

$\binom{12}{10}(.5)^{12} \phi(108.264)$

$66(.5)^{12} \cdot 3.264 = \$0.05259$

$+$

$\binom{12}{11}(.5)^{12} \phi(110.4511) = \$.01597$

$12 \cdot (.5)^{12} \cdot 5.4511$

$+$

$\binom{12}{12}(.5)^{12} \phi(112.682) = \$0.001875$

$(.5)^{12} \cdot 7.682$

$$\boxed{C_0 = \$0.13059}$$

Problem 2

W be a brownian motion and let $Z_t = e^{W_t^2 - 1}$ for $t \geq 0$

Write Z_t in terms of drift and diffusion components.

a. $X_t = W_t$ so $dX_t = 0dt + 1dW_t$

$dZ_t = f'(X_t)\,dX_t + \frac{1}{2} f''(X_t) \cdot d(X_t)^2$

$dZ_t = (2W_t e^{W_t^2 - 1})dW_t + \frac{1}{2} 2(e^{W_t^2-1} + 2W_t e^{W_t^2-1})\,d(W_t)^2$

$= (e^{W_t^2-1} + 2W_t e^{W_t^2-1})dt + 2W_t e^{W_{t-1}^2} dW_t$

$= e^{W_t^2-1}(1 + 2W_t)\,dt + 2W_t e^{W_t^2-1} dW_t$

b) $X_t = W_t^2 - 1$

$dX_t = 0dt + 1d(W_t^2 - 1)$

$= f'(X_t)\,dX_t + \frac{1}{2} f''(X_t)(dX_t)^2$

$= e^{f'(x)} dX_t + \frac{1}{2} e^{f''(X_t)} (dX_t)^2$

$2W_t \cdot e^{W_t^2-1} dW_t + \frac{1}{2}(1 + 2W_t)\,dt$

c) Is Z a martingale?

$$Z = e^{W_t^2 - 1} \qquad \theta = W_t^2 - 1$$

a) Z is F_t measurable

b) $\mathbb{E}[|Z_t|] < \infty$

Need to prove $E[Z_t | F_s] = Z_s$ for $s < t$

$$E[W_t^2 - 1 | F_s] = E[((W_t - W_s) + W_s)^2 | F_s] - t$$

$$= E[(W_t - W_s)^2 | F_s] + 2E[W_s(W_t - W_s) | F_s] + E[W_s^2 | F_s] - t$$

$$= t - s + 0 + W_s^2 - t$$

$$= W_s^2 - s$$

Thus $E[Z_t | F_s] = Z_s$

$\boxed{\text{So } Z \text{ is martingale.}}$