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HW 5

1) Let W be Brownian Motion. Define X by $\boxed{dX_t = k(\theta - X_t)dt + \sigma dW_t}$
where k, θ, σ, and X_0 are constants.

a) Write $e^{kT}X_t$ as the sum of a constant, a time integral, and an Itô integral w.r.t to Brownian motion, such that both integrands may depend on t but not on X.

$$d(e^{kt}X_t) = ke^{kt}X_t\,dt + e^{kt}(dX_t) + \frac{1}{2}k^2e^{kt}(dt)^2 + O$$

$$d(e^{kt}X_t) = ke^{kt}X_t\,dt + e^{kt}X_t(k(\theta - X_t) + \sigma dW_t)$$

$$d(e^{kt}X_t) = k\theta e^{kt}\,dt + \sigma e^{kt}\,dW_t$$

$$\int_t^T e^{kt}\,dX_s = \int_t^T k\theta e^{ks}\,ds + \int_t^T \sigma e^{ks}\,dW_s$$

$$\boxed{X_T = X_t e^{-k(T-t)} + \theta\left[1 - e^{-k(T-t)}\right] + \int_t^T \sigma e^{-k(T-s)}\,dW_s}$$

b) $\mathbb{E}[X_T \mid X_t = x] = x e^{-\kappa(T-t)} + \theta[1 - e^{-\kappa(T-t)}]$

$$\mathrm{Var}(X_T \mid X_t = x) = \mathbb{E}\left[\left(\int_t^T \sigma e^{-\kappa(T-s)} dW_s\right)^2\right] =$$

$$= \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(T-t)}\right)$$

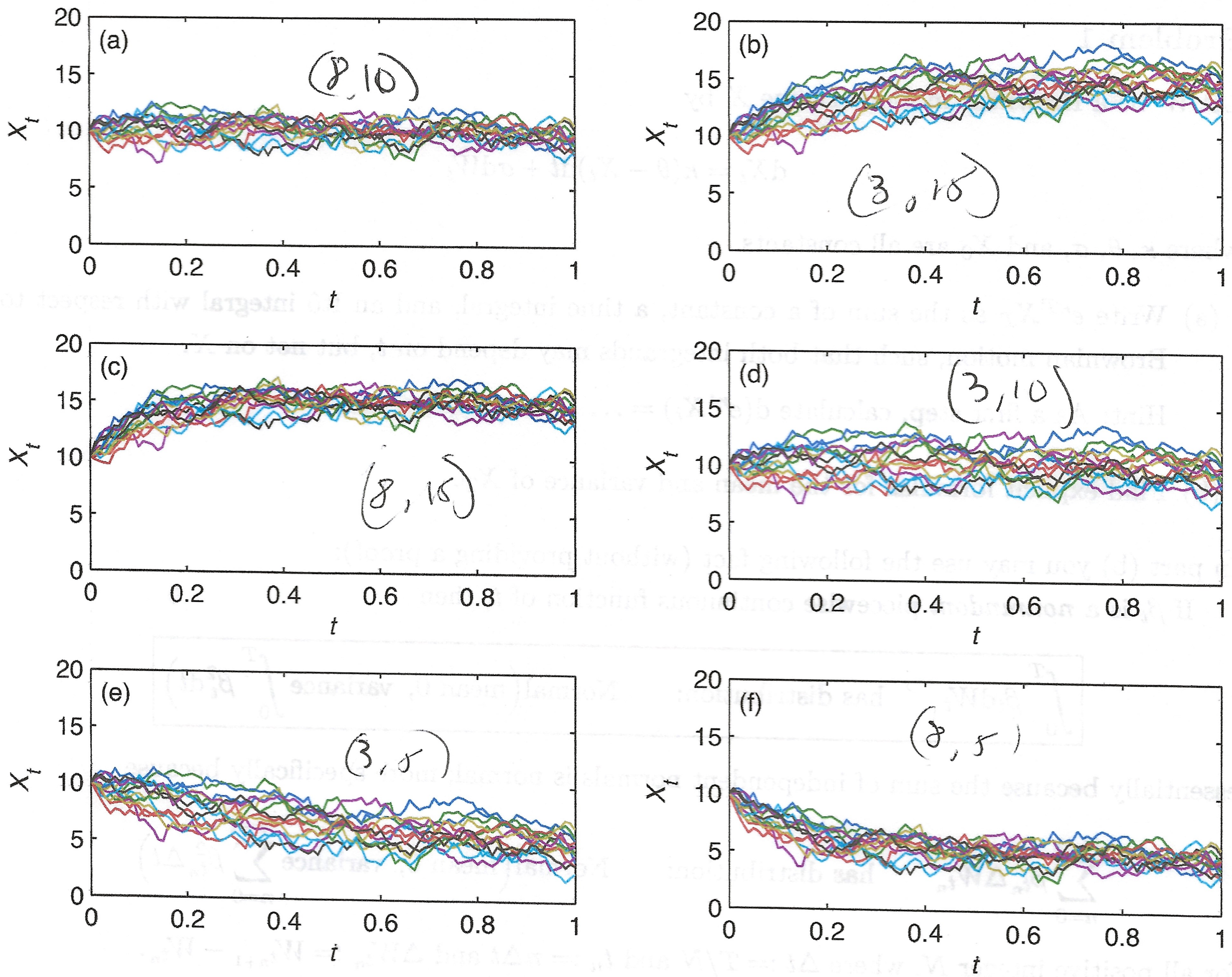
Problem 2

Here are 6 examples of the dynamics in Problem 1. All have the form

$$dX_t = \kappa(\theta - X_t)dt + \sigma dW_t$$

where $\sigma = 4$ and $X_0 = 10$ in all cases. For each of the $2 \times 3 = 6$ combinations of choices $\kappa \in \{3, 8\}$ and $\theta \in \{5, 10, 15\}$, I plotted some sample paths.

Where (κ, θ)



For each process (a,b,c,d,e,f), state the κ and the θ that I used to generate that process. No justification necessary.

Hint: In each case the process is *mean-reverting*. Intuitively, θ is the mean reversion "level" or the "long-term mean"; and κ is the mean reversion "rate" or "speed"