Black-Scholes

- There are 5 parameters in Black-Scholes
  - S_t
  - K
  - σ
  - r
  - τ

A **Call Option** equals:

$$C(K) = e^{-r(T-t)}\mathbb{E}(S_T - K|S_t)^+ = S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \tag{1}$$

A **Put Option** equals:

$$P(K) = e^{-r(T-t)}\mathbb{E}(K - S_T|S_t)^+ = Ke^{-r(T-t)}\Phi(-d_2) - S_t\Phi(-d_1) \tag{2}$$

Where

$$d_1 = (\frac{log(\frac{S_t}{K}) + (\tau)(r + \frac{\sigma^2}{2})}{\sigma\sqrt{\tau}}) \tag{3}$$

$$d_2 = (\frac{log(\frac{S_t}{K}) + (\tau)(r - \frac{\sigma^2}{2})}{\sigma\sqrt{\tau}}) = d_1 - \sigma\sqrt{\tau} \tag{4}$$

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-z^2/2}dz \tag{5}$$

$\Phi(x)$ is the **standard normal cumulative density** function. The probability that $X \leq .3$ is $\mathrm{P}(X \leq .3) = \Phi(.3) = 0.62$

The Greeks

Delta Δ

$\Delta = \frac{\partial V}{\partial S_t}$

- **Delta** measures the change in the option prices per change in the underlying price. It is the slope of the curve option curve with respect to the price.
- The delta of a call option always positive while the delta of a put option is always negative.

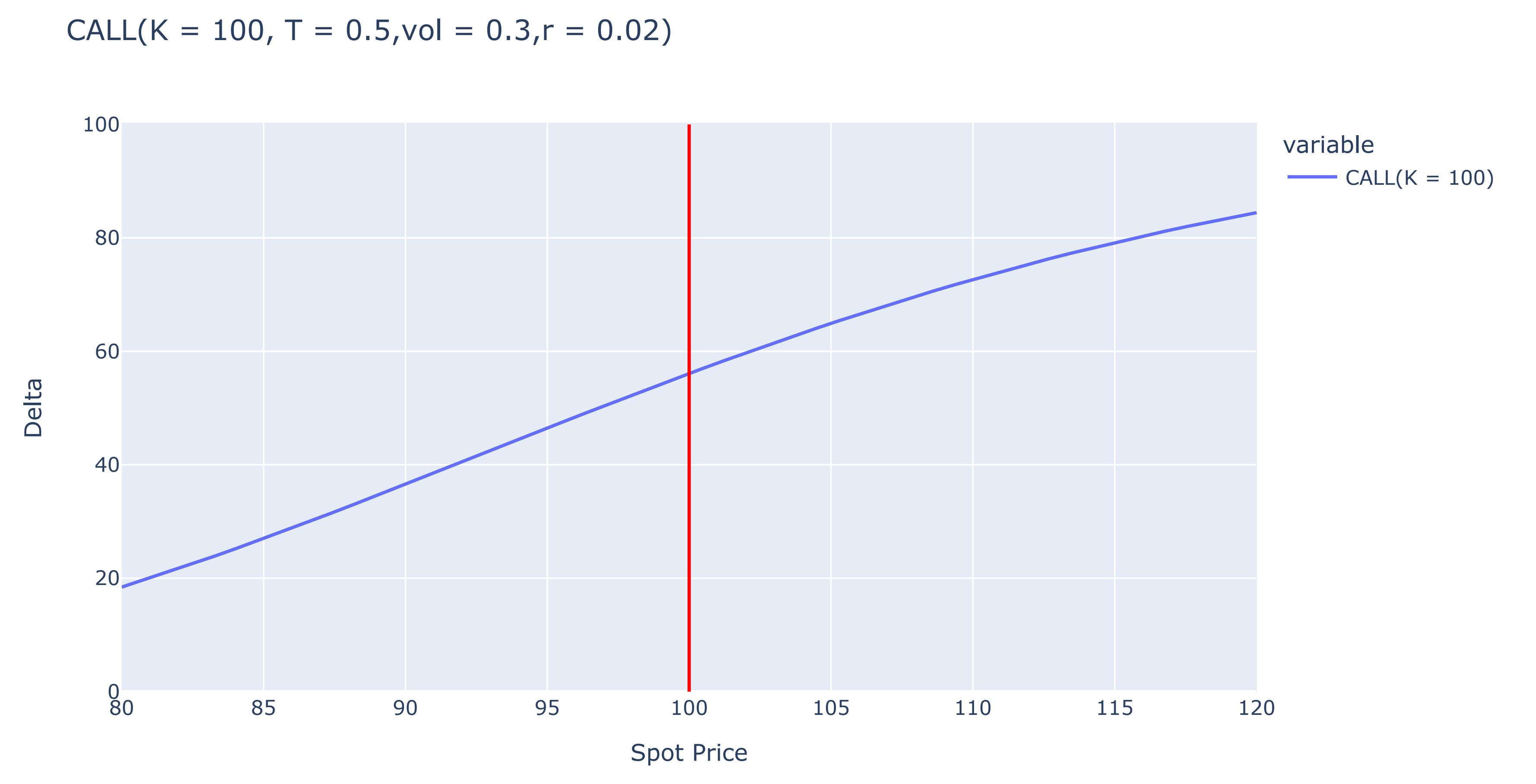
$$\text{Call Option } \frac{\partial C}{\partial S_t} = \Phi(d1) \tag{6}$$

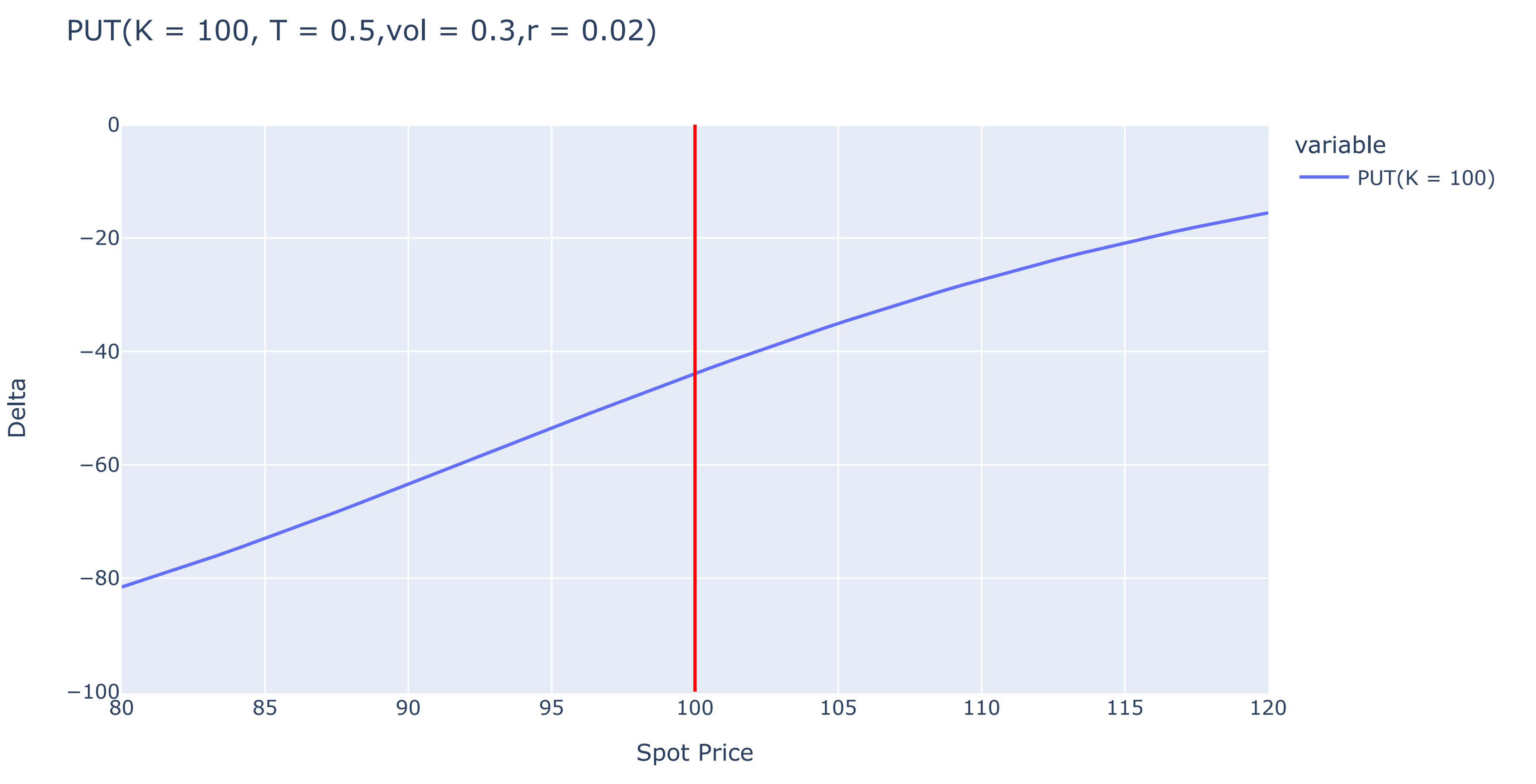
$$\tag{7}$$

$$\text{Put Option } \frac{\partial P}{\partial S_t} = -\Phi(-d1) = \Phi(d1) - 1 \tag{8}$$

If a put contract has a $\Delta = -.5$ this means for every unit increase in the price of the underlying the option price **decreases** by $-.5 \cdot \Delta_s$. If S increases by $\$5$, then the price of the put option theoretically declines by $\$2.5$ or $\$250$ notional.

Below are two plots that show how the delta of a call and put option changes with respect to their spot prices.





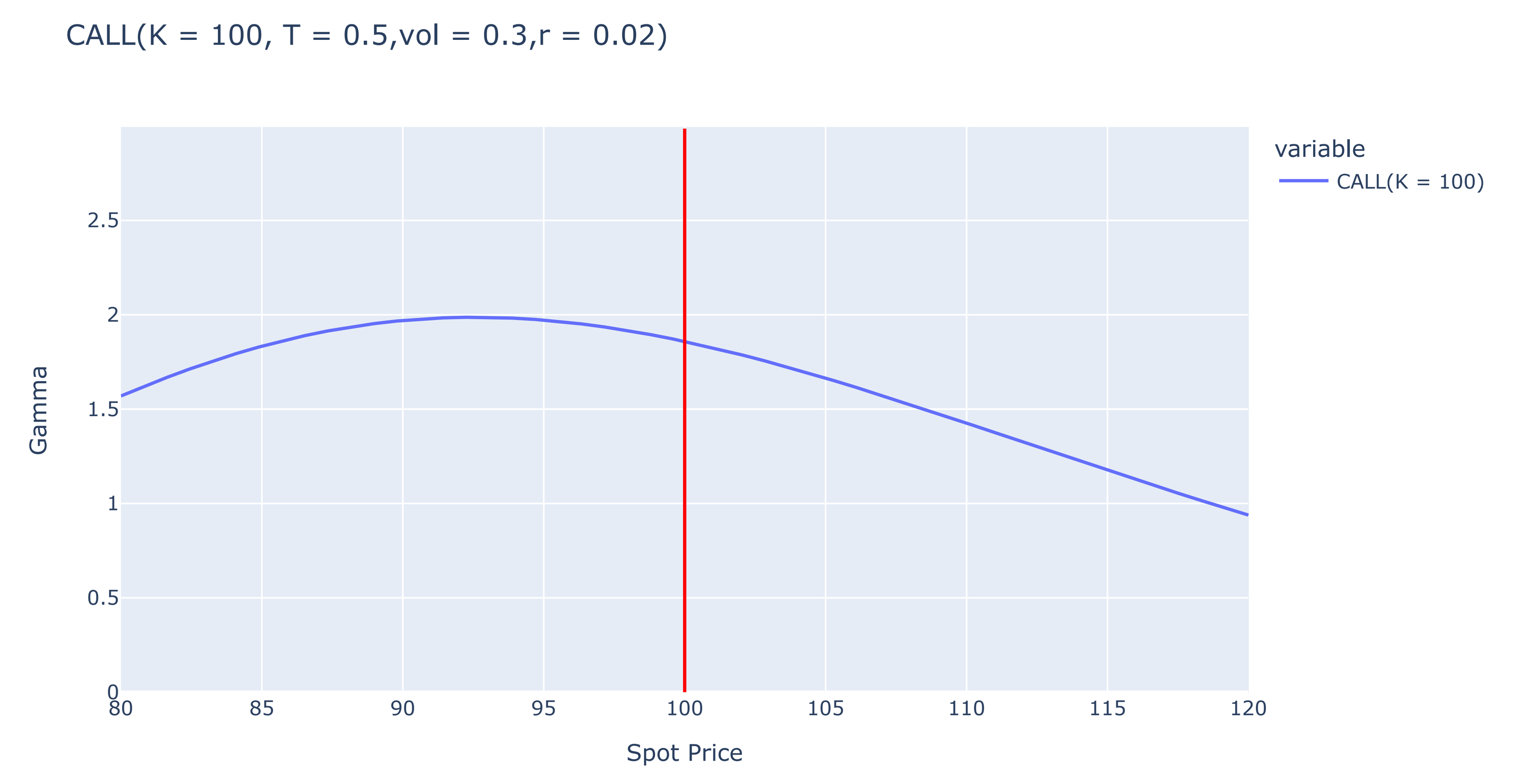
Gamma Γ

- **Gamma** is the second partial derivative of *any derivative contract* with respect to its price. **It essentially measures the change in the option's delta.**
- Delta hedging works well for small price movements and for options with greater time to maturity, however, as the markets become more dislocated and maturity decreases, *gamma can drastically impact profit or loss.*
- The gamma of a long call or put is **always positive**, where as the gamma of a short call or short put is **always negative.**

$$\boxed{\text{Put Gamma} = \text{Call Gamma} = \Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\partial\left(\frac{\partial C}{\partial S}\right)}{\partial S} = \frac{\partial N(d_1)}{\partial S} = \frac{\partial N(d_1)}{\partial d_1}\frac{\partial d_1}{\partial S} = N'(d_1)\frac{\frac{1}{S}}{\sigma\sqrt{t}} = \frac{N'(d_1)}{S\sigma\sqrt{t}}} \tag{9}$$

If a call option has a $\Gamma = .03$ and $\Delta = .55$ and the underlying asset increases by $\$3$, then the theoretical price increase in the call option equals $0.55 \cdot 3 + \frac{0.03 \cdot 3^2}{2} = 1.65 + 0.13 = 1.785$.

- Gamma is always highest near the money and near maturity.



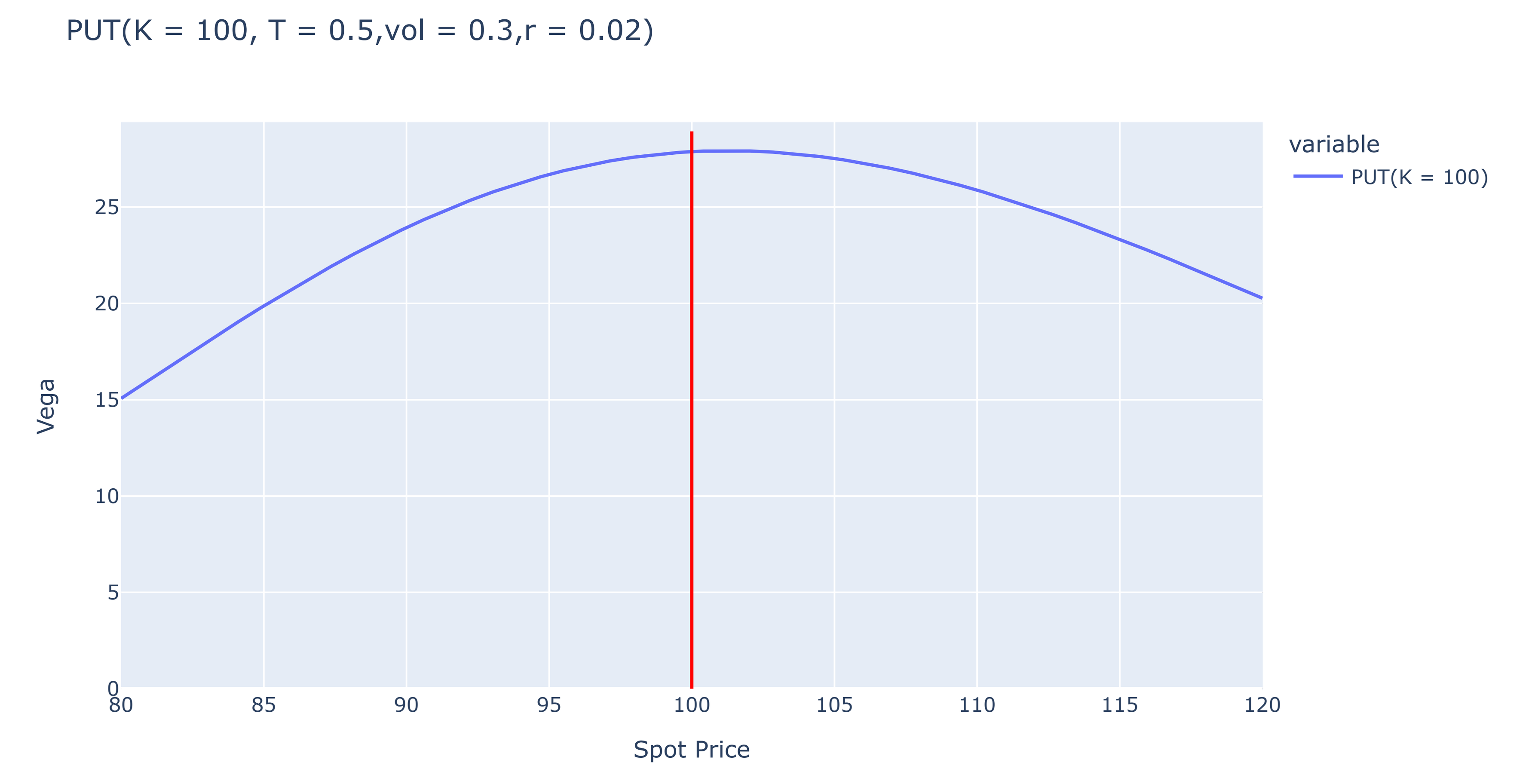
Vega υ

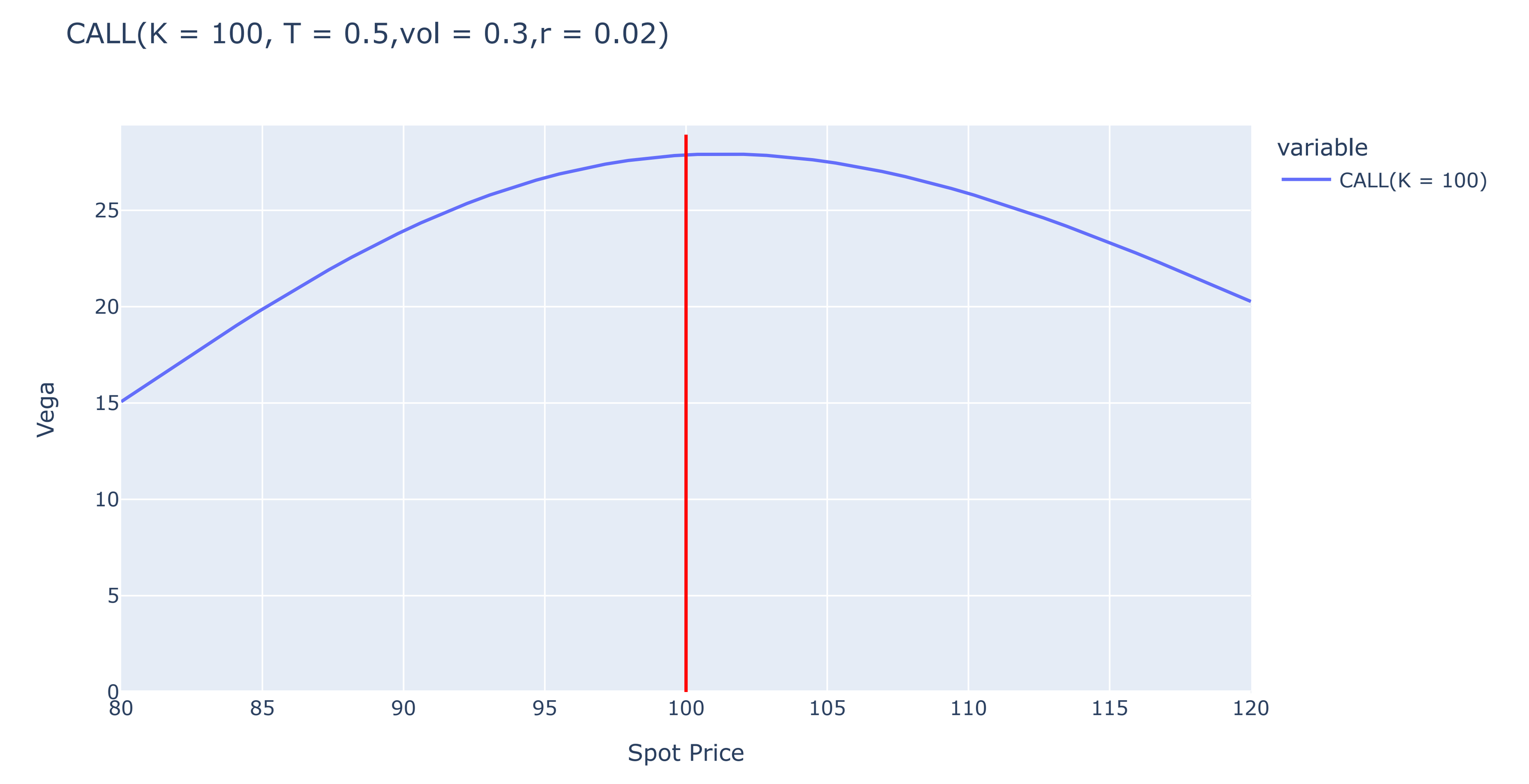
- Measures the change in the option price with respet to a change in the **option's implied probability.**

If $\upsilon = 12.3$, then a 1% increase in the implied volatility increases the option price by $0.01 \cdot 12.3 = .123 \text{ or by } \$12.3 \text{ Notional.}$

$$\boxed{\text{Put Vega} = \text{Call Vega} = \upsilon = \frac{\partial C}{\partial \sigma} = S\sqrt{T}N'(d_1)}$$

The vega for a long call or put option is always positive.





Derivative Strategy

- Let's say we believe implied volatility is trading lower so we buy a **straddle** on the SPY ETF Index. Let's say we want to buy a 0-Delta, $\$1000$ **vega notional** option strategy. This is what a straddle accomplishes because we are hedged against price movements but *long* implied volatility.
- There are two-options being sold right now:
  - Call(S0 = 430,K =426,R = .03,$\sigma = .13$,$\tau = .25$) = $\$15.0159 \cdot 100$.
  - Put(S0 = 430,K =433,R = .03,$\sigma = .13$,$\tau = .25$) = $\$11.028 \cdot 100$

	Price	Δ	Γ	υ
CALL(K = 426)	15.016	0.614729	0.013679	82.2008
PUT(K = 433)	11.0281	-0.483679	0.0142615	85.7008

What combination of the Call and the Put option contracts will delta hedge my position and profit one-thousand dollars when implied volatility increases by one percent? Find the number of puts and calls to buy such that our portfolio will have the these dynamics:

$$\Delta_p = 0$$
$$\upsilon_p = 1000$$

If the implied volatility increase by 1%, then my option portfolio vega increases by $0.01 \cdot 1000 = \$10$ which is $\$1000$ notional value.

This system of equations is equal to the matrix equation:

- We need to multiply Δ by 100 to so it is in the correct units.

$$\begin{bmatrix} \Delta_{\text{CALL}} & \Delta_{\text{PUT}} & 0 \\ \upsilon_{\text{CALL}} & \upsilon_{\text{PUT}} & 1000 \end{bmatrix}$$

$$\begin{bmatrix} 61.47 & -48.36 & 0 \\ 82.2 & 85.7 & 1000 \end{bmatrix}$$

	Price	Delta	Gamma	Vega
CALL(K = 426)	15.015958	61.472942	1.367904	82.200768
PUT(K = 433)	11.028142	-48.367873	1.426148	85.700800

Solution

Solving the problem we get:

	CALL(K = 426)	PUT(K = 433)
Delta	61.4729	-48.3679
Vega	82.2008	85.7008
Position	5.23197	6.65032

We need to buy about 5.23 calls and buy 6.65 puts. Seldomly, you'll ever get a perfect whole number for a solution, so just round to the nearest one!