1. If $I := [0,4]$, compute the norm of the following partitions

a) $P_1 = (0,1,2,4)$

Note que, para qualquer partição de I temos

$$\|P\| = \max\{x_1 - x_0, x_2 - x_1, \dots, x_{n-1} - x_{n-2}, x_n - x_{n-1}\}$$

Deste modo, se $P_1 = (0,1,2,4)$ então

$$\|P_1\| = \max\{1-0, 2-1, 4-2\} = 2$$

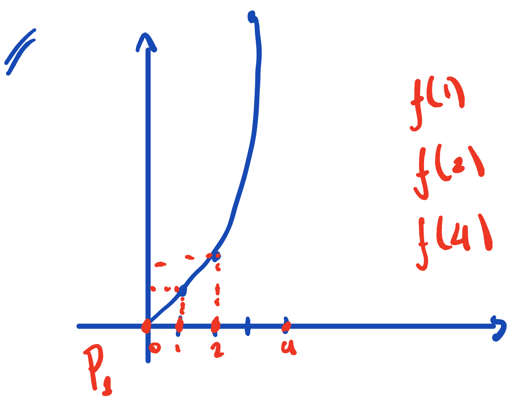
c) $P_3 = (0, 1, 1.5, 2, 3.4, 4)$

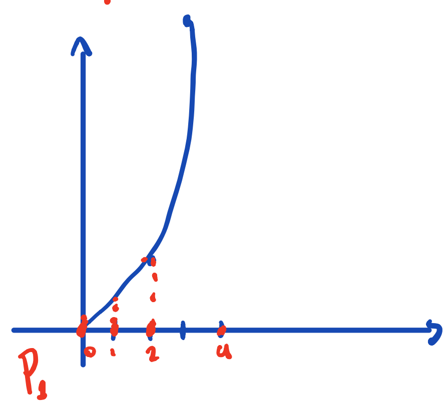
Similarmente, temos

$$\|P_2\| = \max\{1-0, 1.5-1, 2-1.5, 3.4-2, 4-3.4\} = 1.4$$

2. If $f(x) = x^2$, $x \in [0,4]$, calculate the following Riemann sums, where $\dot{P}_i$ is defined as in Ex 1, and the tags are selected as indicated

b) $\dot{P}_1$ with tags at the right endpoints of the subintervals





Seja $\dot{P}_1 = \{(P_1, t_i^R)\}_{i=1}^n$ partição rotulada de $[0,4]$. Então a soma de Riemann respectiva é

$$S(f;\dot{P}_1) = \sum_{i=1}^{n} f(t_i^R)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) =$$

$$= f(1)(1-0) + f(2)(2-1) + f(4)(4-2) = 1 + 4 + 32 = 37$$

c) $\dot{P}_2$ with the tags at the left endpoints

Temos $\dot{P}_2 = \{(P_2, t_i^L)\}_{i=1}^n$ partição rotulada de $[0,4]$. Então

$$S(f;\dot{P}_2) = \sum f(t_i^L)(x_i - x_{i-1}) = \sum f(x_{i-1})(x_i - x_{i-1})$$

$$= 0(1-0) + 1(1.5-1) + 2.25(2-1.5) + 4(3.4-2) + 11.56(4-3.4)$$

6.

a) Let $f(x) = \begin{cases} 2, & 0 \leq x < 1 \\ 1, & 1 \leq x \leq 2 \end{cases}$. Show that $f \in R[0,2]$ and evaluate it's integral.

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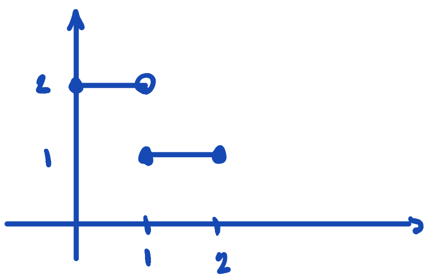

$$L = \int_0^1 f + \int_1^2 f = 2x\Big|_0^1 + x\Big|_1^2 = 2 + 1 = 3$$

$\dot{P}_1 = \{(P_1, t_i)\}_{i=1}^n$ $\quad P_1 = (0, 1, 2) \quad$ t_i ponto mais a direita

$\vdash \exists L \in \mathbb{R}$ t.q. $\forall \varepsilon > 0 \; \exists \delta_\varepsilon > 0$ t.q. se $\|\dot{P}_\varepsilon\| < \delta_\varepsilon \Rightarrow |S(f, \dot{P}_1) - L| < \varepsilon$

$U_1 := \{$ união dos subintervalos ... $\}$

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Tomemos $P_1 = (0, 1)$, $P_2 = (1, 2)$, $\dot{P}_1 = \{(P_1, t_i)\}_{i=1}^n$ e $\dot{P}_2 = \{(P_2, t_i)\}_{i=1}^n$

Isso implica que

$$S(f, \dot{P}) = S(f, \dot{P}_1) + S(f, \dot{P}_2) \quad *$$

Além disso, vamos definir

$$U_i = \{ \text{união dos subintervalos em } \dot{P}_i \}, \quad i = 1, 2$$

Note que

$$[0, 1-\delta] \subset U_1 \subset [0, 1+\delta] \quad e \quad [1+\delta, 2] \subset U_2 \subset [1-\delta, 2]$$

Como para $\forall t_i \in P_1$ temos que $f(t_i) = 2$ e para $\forall t_i \in P_2$ temos $f(t_i) = 1$ então

$$2(1-\delta-0) < S(f;\dot{P}_1) < 2(1+\delta-0)$$

e

$$1(2-1-\delta) < S(f;\dot{P}_2) < 1(2-1+\delta)$$

Usando * temos

$$2(1-\delta) + (1-\delta) < S(f,\dot{P}) < 2(1+\delta) + (1+\delta) \iff$$

$$\iff 2 - 2\delta + 1 - \delta < S(f,\dot{P}) < 2 + 2\delta + 1 + \delta$$

$$\iff 3 - 3\delta < S(f,\dot{P}) < 3 + 3\delta$$

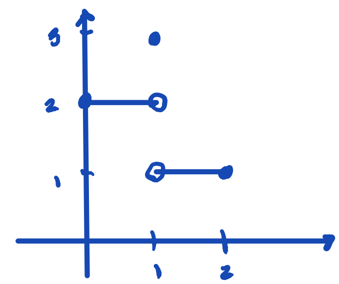
$$\iff |S(f,\dot{P}) - 3| < 3\delta$$

Portanto, tomando $\delta_\varepsilon = \frac{\varepsilon}{3}$ temos

$$|S(f,\dot{P}) - 3| < \varepsilon$$

e provamos que $f \in \mathcal{R}[0,2]$ e tem $\int_0^2 f = 3$.

b) Let $h(x) = \begin{cases} 2, & 0 \leq x < 1 \\ 3, & x = 1 \\ 1, & 1 < x \leq 2 \end{cases}$. Show that $h \in R[0,2]$ and evaluate it's integral.



Similar ao argumento do item a), tomamos

$$P_1 = (0,1) \;;\; P_2 = (3) \;;\; P_3 = (1,2) \quad *$$

e

$U_i = \{$ união dos subintervalos de $P_i \}$, $i = 1,2,3$

Então

$$[0, 1-\delta] \subset U_1 \subset [0, 1+\delta]$$
$$[1-\delta, 2] \subset U_3 \subset [1+\delta, 2] \quad **$$

Além disso, temos que

$$3(1-\delta) < S(h,\dot{P}_1) < 3(1+\delta)$$
$$1(2-1+\delta) < S(h,\dot{P}_3) < 1(2-1-\delta)$$

Portanto, escrevendo a soma de Riemann de h como

$$S(h,\dot{P}) = S(h,\dot{P}_1) + S(h,\dot{P}_2) + S(h,\dot{P}_3)$$

por * e ** temos

$$3-3\delta+1+\delta \leq S(h,\dot{P}) \leq 3+3\delta+1-\delta \iff$$

$$\iff 4-2\delta \leq S(h,\dot{P}) \leq 4+2\delta \iff |S(h,\dot{P})-4| \leq 2\delta$$

Portanto, se tomarmos $\delta_3 = \frac{\varepsilon}{2}$ concluímos que

$$|S(h,\dot{P})-4| \leq \varepsilon$$

o que quer dizer que $h \in R[0,2]$ e $\int h = 4$.

9. If $f \in R[a,b]$ and if $(\dot{P}_n)$ is any sequence of tagged partitions of $[a,b]$ s.t. $\|\dot{P}_n\| \to 0$.
Prove that $\int_a^b f = \lim_n S(f, \dot{P}_n)$.

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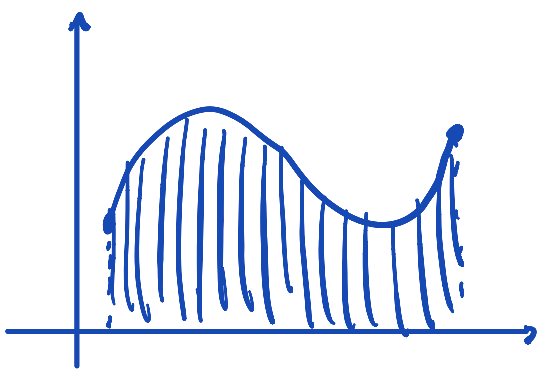
$+\ \int_a^b f = \lim_n S(f, \dot{P}_n) \iff L = \lim_n S(f, \dot{P}_n)$

$$\lim_n S(f, \dot{P}_n) = \lim_n \left[\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \right] =$$



$= f(x_i)(x_i - x_i + \delta)$

$= f(x_i)\,\delta$

$$\lim_n \left[\sum f(t_i)(x_i - x_{i-1}) \right] = \sum f(x_i)(\cancel{x_i} - \cancel{x_i} + \delta) = \delta \sum f(x_i)$$

Teorema da unicidade: Se $g \in R[a,b]$ e se $f(x) = g(x)$ exceto por um número finito de pontos em $[a,b]$ $\Rightarrow f \in R[a,b]$ e $\int_a^b f = \int_a^b g$.

+ Se $g(x) = \lim_n S(f, \dot{P}_n)$, provar que $f(x) = g(x)$ exceto por um número finito de pontos

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① $c \in [a,b]$

② $L = \int_a^b g$

③ ASSUMA QUE $f(x) = g(x)$, $x \neq c$

$\Rightarrow$ $S(f;\dot{P}) = S(g;\dot{P})$, COM EXCEÇÃO DE $x_i = x_{i-1} = c$

$\Rightarrow |S(f;\dot{P}) - S(g;\dot{P})| = \left| \sum [f(x_i) - g(x_i)](x_i - x_{i-1}) \right| \leq$

$\leq 2\left[|g(c)| + |f(c)| \right] \|\dot{P}\|$

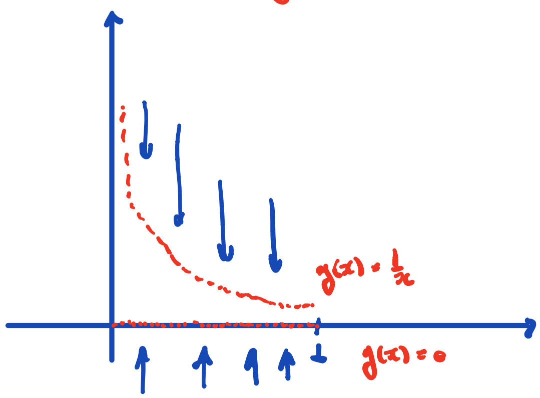
Solução

① $\exists L \in \mathbb{R}$ t.q. $\forall \varepsilon > 0$ $\exists \delta_\varepsilon > 0$ t.q. se $\|\dot{P}_\varepsilon\| < \delta_\varepsilon \Rightarrow |S(f,\dot{P}_1) - L| < \varepsilon$

② $\lim(x_n) = x$ if $\forall \varepsilon > 0$ $\exists K(\varepsilon)$ s.t. $\forall n \geq K(\varepsilon)$ x_n satisfy $|x_n - x| < \varepsilon$ //

10. Let $g(x) = \begin{cases} 0, & x \in [0,1] \cap \mathbb{Q} \\ \frac{1}{x}, & x \in [0,1] \cap \mathbb{R}\setminus\mathbb{Q} \end{cases}$

(i) Show that $g \notin R[0,1]$



$\underline{S}(g, \dot{P}_1) = \sum m_i (x_i - x_{i-1})$

Para $\forall \delta > 0$ e $x \in [x - \delta, x + \delta]$ $\exists m > 0$ e $\exists M \geq 0$ t.q $m = 0$ e $M = \frac{1}{x}$. Desta forma temos que para qualquer $\dot{P}$

$$\underline{S}(g, \dot{P}) = \sum_{i=1}^{M} m_i (x_i - x_{i-1}) = 0$$

$$\overline{S}(g, \dot{P}) = \sum_{i=1}^{M} M_i (x_i - x_{i-1}) = \sum_{i=1}^{M} \frac{1}{x_i} (x_i - x_{i-1})$$

Como $\overline{S}$ só vale se $x_i \neq 0$, então para qualquer $\dot{P}$ $\underline{S} \neq \overline{S}$, o que implica que $g \notin R[0,1]$.

(ii) Show that there exists a sequence $(\dot{P}_n)$ on $[a,b]$ s.t. $\|\dot{P}_n\| \to 0$ and $\lim_n S(g;\dot{P}_n)$ exists

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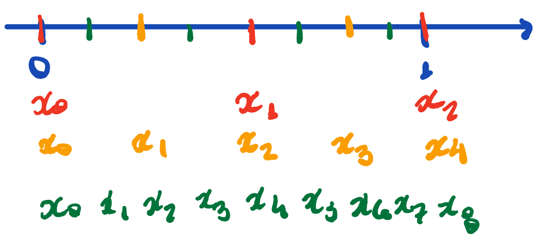
$$t_i = \frac{x_i - x_{i-1}}{2}$$



$$\left.\begin{matrix} \dot{P}_1 \\ \dot{P}_2 \\ \dot{P}_3 \\ \vdots \\ \dot{P}_n \end{matrix}\right\} (\dot{P}_n) = (\dot{P}_1, \dot{P}_2, \dots, \dot{P}_n)$$

Dividindo o intervalo $[a,b]$ em pedaços menores

$$\dot{P}_n = \{([x_{i-1}, x_i], t_n)\}_{i=1}^{n} \Rightarrow \lim(\|\dot{P}_n\|) = 0$$

$$S(g;\dot{P}_n) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

$$\lim_n \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \Rightarrow \| |x_i - x_{i-1}| \| \to 0 \Rightarrow \lim_n S = 0$$

//

Tomemos $\dot{P}_n = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$ uma partição em $[a,b]$. Note que, a medida que $n \to \infty$ a distância entre x_i e x_{i-1} diminui, quer dizer, $\|\dot{P}_n\| \to 0$. Então, como para $i = 1, \dots, n$ $\| |x_i - x_{i-1}| \| \to 0$ temos

$$\lim_{n} \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) = 0$$

Portanto, o limite do enunciado existe.

12. Consider the Dirichlet function

$$f(x) = \begin{cases} 1, & x \in [0,1] \cap \mathbb{Q} \\ 0, & x \in [0,1] \cap \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Show that $f \notin \mathcal{R}[0,1]$.

Similarmente ao exercício 11, note que para $\forall \delta > 0$ $\exists x \in [x-\delta, x+\delta]$ t.q. $m = 0$ e $M = 1$. Isto quer dizer que

$$\underline{S}(f; \dot{P}) = \sum_{i=1}^{n} m (x_i - x_{i-1}) = 0$$

$$\overline{S}(f; \dot{P}) = \sum_{i=1}^{n} M (x_i - x_{i-1}) = \sum_{i=1}^{n} (x_i - x_{i-1})$$

ou seja $\underline{S} \neq \overline{S}$. Portanto $f \notin \mathcal{R}[0,1]$

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