1. Let $I := [a,b]$ and $f : I \to \mathbb{R}$ be a continuous function s.t. $f(x) > 0$ for $\forall x \in I$.

Prove that $\exists\, \alpha > 0$ s.t. $f(x) \geq \alpha$ $\forall x \in I$

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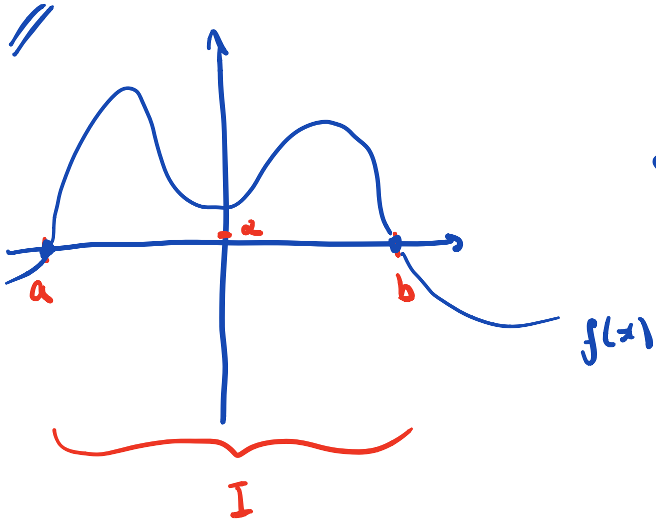

α must be equal to the absolute minimum of the function

↳ Hypothesis: I bounded; f continuous on I

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Let $I := [a,b]$, $a, b \in \mathbb{R}$, then I is a bounded subset of $\mathbb{R}$. Furthermore, we know that $f : I \to \mathbb{R}$ is continuous on I s.t. $f(x) > 0$. From the MAX-MIN THEOREM we know that f has an absolute minimum, that is, $\exists\, x_* \in I$ s.t. $f(x_*) \leq f(x)$, $\forall x \in I$. Take $\alpha := f(x_*) \Rightarrow f(x) \geq \alpha$. Furthermore, since $f(x) > 0$ $\forall x \in I$ and $f(x) \geq \alpha \Rightarrow f(x) \geq \alpha > 0$

$\square$

3. Let $I := [a,b]$ and let $f : I \to \mathbb{R}$ continuous on I s.t for $\forall x \in I$ $\exists\, y \in I$ s.t. $|f(y)| \le \frac{1}{2}|f(x)|$.

Prove that $\exists c \in I$ s.t. $f(c) = 0$.

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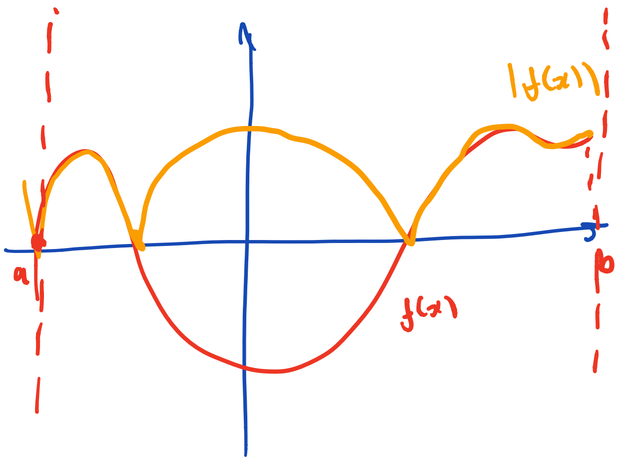
$|f(y)| \le \frac{1}{2}|f(x)| \iff -\frac{1}{2}|f(x)| \le f(y) \le \frac{1}{2}|f(x)|$



$0 \le |f(y)| \le \frac{1}{2}|f(x)|$

Hypothesis $\begin{cases} \forall I, \ I := [a,b] \checkmark \\ f \text{ continuous on } I \checkmark \end{cases}$

5. Show that the polynomial $p(x) := x^4 + 7x^3 - 9$ has at least two real roots

//

$p(x) : \mathbb{R} \to \mathbb{R}$

+ find at least two intervals $]a,b[\in \mathbb{R}$, $]c,d[\in \mathbb{R}$ s.t.
for $I = \mathbb{R}$ and f is continuous, if $f(a) < 0 < f(b)$ and
$f(c) < 0 < f(d) \Rightarrow \exists\, c_1 \in]a,b[$ and $c_2 \in]c,d[$ s.t.
s.t. $f(c_1) = 0$ and $f(c_2) = 0$

$p(x) : x^4 + 7x^3 - 9$

$$\frac{dp}{dx} = 4x^3 + 21x^2 = x^2(4x + 21) = 0 \iff x = 0 \text{ or } x = -\frac{21}{4}$$

$$\frac{dp}{dx} > 0 \iff 4x + 21 > 0 \iff x > -\frac{21}{4}$$

//

6. Let $f : I \to \mathbb{R}$, $I := [0,1]$, be continuous on I s.t. $f(0) = f(1)$.

Prove that $\exists c \in [0, \frac{1}{2}]$ s.t. $f(c) = f(c + \frac{1}{2})$

Hint: $g(x) = f(x) - f(x + \frac{1}{2})$ $\longrightarrow$ $g(c) = 0$

//

$I = [a,b]$, $a = 0$, $b = 1$ $\Rightarrow$ $a < b$

$g(x) = f(x) - f(x + \frac{1}{2})$ $\Rightarrow$ $g(a) < 0 < g(b)$

$\Rightarrow$ (Location of Roots) if g is continuous on I, then $\exists c \in I$ s.t. $g(c) = 0$ $\Leftrightarrow$ $f(c) = f(c + \frac{1}{2})$ //

Let $I := [a,b]$, $a = 0$, $b = 1$, and let $g(x) = f(x) - f(x + \frac{1}{2})$. Since f is continuous on I, by the Theorem 5.2.2 then g is continuous on I. Since $a < b$, then $g(a) < 0 < g(b)$ which implies, by the Location of Roots Theorem that $\exists c \in I$ s.t. $g(c) = 0 \Leftrightarrow f(c) = f(c + \frac{1}{2})$

$\square$

g crescente?

7. Show that $x = \cos x$ has a solution on $I := [0, \pi/2]$

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$\cos 0 = 1$, $\cos \frac{\pi}{2} = -\frac{1}{2}$

$a = 0$, $b = \frac{\pi}{2}$, and $f(x) = \cos x \Rightarrow f(b) < 0 < f(a)$

$\Rightarrow$ (Location of Roots) if f is continuous on I, then $\exists\, c \in I$ s.t. $f(c) = 0$

13. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and that $\lim_{x \to -\infty} f = 0$ and $\lim_{x \to \infty} f = 0$.

alcança

Prove that f is bounded on $\mathbb{R}$ and attains either a maximum or minimum on $\mathbb{R}$.

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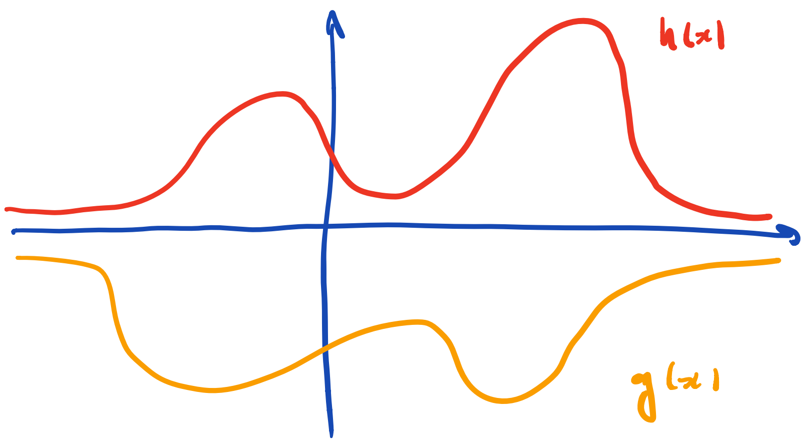
i) f is bounded on $\mathbb{R}$ $\Leftrightarrow$ $\exists M > 0$ s.t. $|f(x)| \leq M \quad \forall x \in \mathbb{R}$



Não importa o quão maluca ela seja, ela sempre vai para zero nas caudas!!

$h(x) = f(x), \quad \forall x \in \mathbb{R}^+ \Rightarrow |h(x)| \leq \max h(x)$

$g(x) = f(x), \quad \forall x \in \mathbb{R}^- \Rightarrow |g(x)| \leq \min g(x)$

negative?