5.6. Monotone and Inverse Function

• Recall that a function $f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}$, can be:

1. increasing: whenever $x_1, x_2 \in A$ and $x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$
2. strictly increasing: whenever $x_1, x_2 \in A$ and $x_1 < x_2 \Rightarrow$
   $\Rightarrow f(x_1) < f(x_2)$.
3. decreasing: whenever $x_1, x_2 \in A$ and $x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$
4. strictly decreasing: whenever $x_1, x_2 \in A$ and $x_1 < x_2 \Rightarrow$
   $\Rightarrow f(x_1) > f(x_2)$
5. monotone $\Leftrightarrow f$ is 1. or 3.
6. strictly monotone $\Leftrightarrow$ 2. or 4.

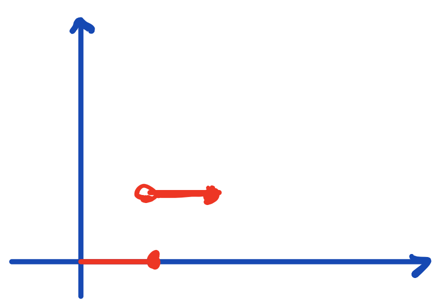
• Furthermore, we note that

1. if $f: A \to \mathbb{R}$ is increasing $\Leftrightarrow -f$ is decreasing in A
2. if $f: A \to \mathbb{R}$ is decreasing $\Leftrightarrow -f$ is increasing in A

• In this section we will be concerned with monotone functions defined on an interval $I \subseteq \mathbb{R}$

• Its important to note that monotone functions are not necessarily continuous

$$f(x) := \begin{cases} 0, & 0 \leq x \leq 1 \\ 1, & 1 < x \leq 2 \end{cases}$$

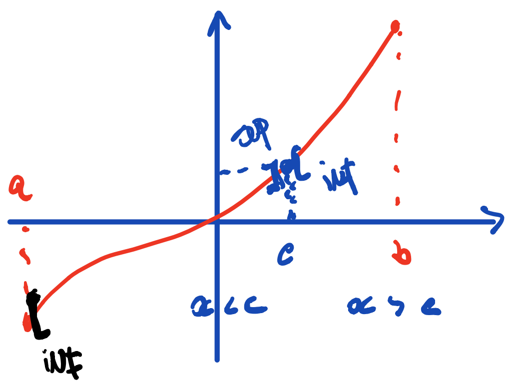


Theorem 5.6.1 Let $I \subseteq \mathbb{R}$ be an interval and let $f: I \to \mathbb{R}$ be increasing on I. Suppose that $c \in I$ is not an endpoint of I. Then

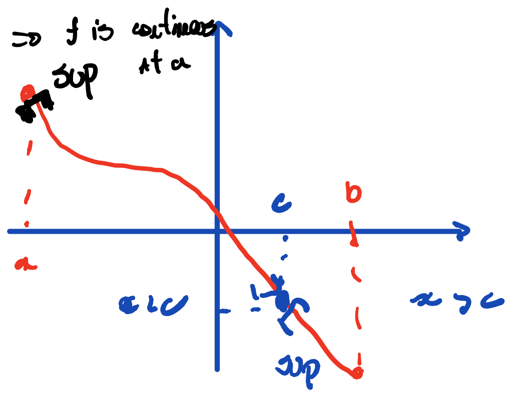
if f is decreasing

(i) $\lim_{x \to c^-} f = \sup\{ f(x) : x \in I, x < c \}$ — inf

(ii) $\lim_{x \to c^+} f = \inf\{ f(x) : x \in I, x > c \}$ — sup



⇒ f continuous at a



Corollary 5.6.2 Let $I \subseteq \mathbb{R}$ be an interval and let $f: I \to \mathbb{R}$ be increasing on I. Suppose that $c \in I$ is not an endpoint of I. Then the following statements are equivalent

(i) f is continuous on c

(ii) $\lim_{x \to c^-} f = \lim_{x \to c^+} f = f(c)$

(iii) $\sup\{ f(x) : x \in I, x < c \} = \inf\{ f(x) : x \in I, x > c \} = f(c)$

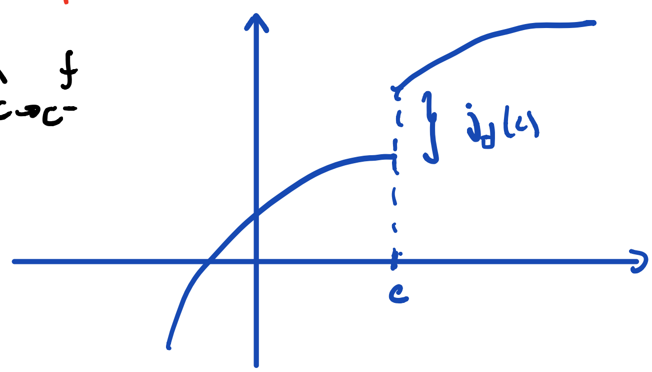
- On the same conditions of the corollary above and if a is a left endpoint of I, then f is continuous at a iff

  f decreasing and a right endpoint

  $f(a) = \inf\{ f(x) : x \in I, a < x \}$ — $\sup\{ f(x) : x \in I, x < a \}$

• If $f : I \to \mathbb{R}$ is increasing on I and if c is not an endpoint of I, we define the jump function at c to be

$$j_f(c) := \lim_{x \to c^+} f - \lim_{x \to c^-} f$$



• It follows from Theorem 5.6.1 that

$$j_f(c) := \inf\{f(x) : x \in I, x > c\} - \sup\{f(x) : x \in I, x < c\}$$

for an increasing function.

• If the right and left endpoint of I, a and b, belongs to I, then we define the jump function at a and b respectively as

jump function at a: $\quad j_f(a) := \lim_{x \to a^+} f - f(a)$

$$= \inf\{f(x) : x \in I, x > c\} - f(a)$$

jump function at b: $\quad j_f(b) := f(b) - \lim_{x \to b^-} f$

$$= f(b) - \sup\{f(x) : x \in I, x < b\}$$

THEOREM 5.6.3 Let $I \subseteq \mathbb{R}$ be an interval and let $f: I \to \mathbb{R}$ be increasing on I. If $c \in I$, then f is continuous at $c \iff \underbrace{j_f(c)}_{} = 0$

↳ Não tem salto no ponto $c \Rightarrow \lim_{x \to c^+} f = \lim_{x \to c^-} f = f(c)$

- We now show that there can be at most a countable set of points at which a monotone function is discontinuous

THEOREM 5.6.4 Let $I \subseteq \mathbb{R}$ be an interval and let $f: I \to \mathbb{R}$ be monotone in I. Then the set of points $D \subseteq I$ at which f is discontinuous is a countable set.

EX 5.2.12: A function $f: \mathbb{R} \to \mathbb{R}$ is said additive if (1) $f(x+y) = f(x) + f(y)$ $\forall x, y \in \mathbb{R}$. Prove that if f is continuous at any $x_0 \Rightarrow$ it is continuous on $\mathbb{R}$.

$$\left.\begin{aligned} & f(x) := cx = f(1)x \\ & f(x+y) = f(x) + f(y) = f(1)x + f(1)y \end{aligned}\right\} \text{monotone}$$

Inverse function

- Recall that a function $f : I \to \mathbb{R}$ has an inverse function iff f is injective (= one-to-one)

  $x, y \in I$ and $x \neq y \Rightarrow f(x) \neq f(y)$

- We note that a strictly monotone function is injective and so has an inverse

Continuous Inverse Theorem Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be strictly monotone and continuous on I. Then the function g inverse to f is strictly monotone and continuous on $J := f(I)$.