The Derivative

Def (Derivative of f) Let $I \subset \mathbb{R}$ be an interval, let $f: I \to \mathbb{R}$, and $c \in I$. We say that a real number L is the derivative of f at c if given any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ s.t. if $x \in I$ satisfies $0 < |x - c| < \delta(\varepsilon)$, then $\left| \frac{f(x) - f(c)}{x - c} - L \right| < \varepsilon$

In this case, we say that **f is differentiable at c**, and we write $f'(c)$ for L (c can be the end point of the interval)

$$g(x) = \frac{f(x) - f(c)}{x - c}, \quad \lim_{x \to c} g(x) = L = f'(c)$$

$$\Leftrightarrow \quad f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

• Note that $f': I' \to \mathbb{R}$ and $f: I \to \mathbb{R}$ is s.t. $I' \subset I$

Theorem 6.1.2 If $f: I \to \mathbb{R}$ has a derivative at $c \in I$, then f is continuous at c.

Differentiability at c is a sufficient condition for continuity at c

continuity at c $\not\Rightarrow$ differentiability at c

$\Leftarrow$

$f(x) = |x|$, $\forall x \in \mathbb{R}$ is continuous at c but not differentiable

Theorem 6.1.3 Let $I \subseteq \mathbb{R}$ be an interval, let $c \in I$, and let $f: I \to \mathbb{R}$ and $g: I \to \mathbb{R}$ be functions that are differentiable at c. Then

a) If $\alpha \in \mathbb{R}$, then the function αf is differentiable at c, and $(\alpha f)'(c) = \alpha f'(c)$

b) The function $f+g$ is differentiable at c, and $(f+g)'(c) = f'(c) + g'(c)$

c) (Product Rule) The function $f \cdot g$ is differentiable at c, and $(f \cdot g)'(c) = f'(c) \cdot g(c) + f(c) \cdot g'(c)$

d) (Quotient Rule) If $g(c) \neq 0$, then the function f/g is differentiable at c, and $(f/g)'(c) = \dfrac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}$.

- Mathematical induction may be used to obtain the following extensions of differentiation rules

Corollary 6.1.4 If $f_1, f_2, \ldots, f_n$ are functions on an interval I to $\mathbb{R}$ that are differentiable at $c \in I$, then:

a) The function $f_1 + f_2 + \ldots + f_n$ is differentiable at c and $(f_1 + f_2 + \ldots + f_n)'(c) = f_1'(c) + f_2'(c) + \ldots + f_n'(c)$

b) The function $f_1 \cdot f_2 \cdot \dots \cdot f_n$ is differentiable at c, and

$$(f_1 \cdot f_2 \cdot \dots \cdot f_n)'(c) = f_1'(c) f_2(c) \dots f_n(c) + f_1(c) f_2'(c) \cdot \dots \cdot f_n(c) + \dots + f_1(c) \cdot f_2(c) \cdot \dots \cdot f_n'(c)$$

If $f_1 = f_2 = \dots = f_n$, then $(f^n)'(c) = n(f(c))^{n-1} f(c)$.

- Note that we usually have the equivalent notation for f':

f' $\quad f'(c)$ $\quad Df$

$(f \cdot g)'$ $\quad (f \cdot g)'(c)$ $\quad D(f \cdot g) = (Df) \cdot g + f(Dg)$

Furthermore, when x is the "independent variable", its common practice in elementary calculus to write:

$$\frac{d(f(x)\,g(x))}{dx} = \left(\frac{df}{dx}(x)\right) g(x) + f(x)\left(\frac{dg}{dx}(x)\right)$$ Leibniz notation

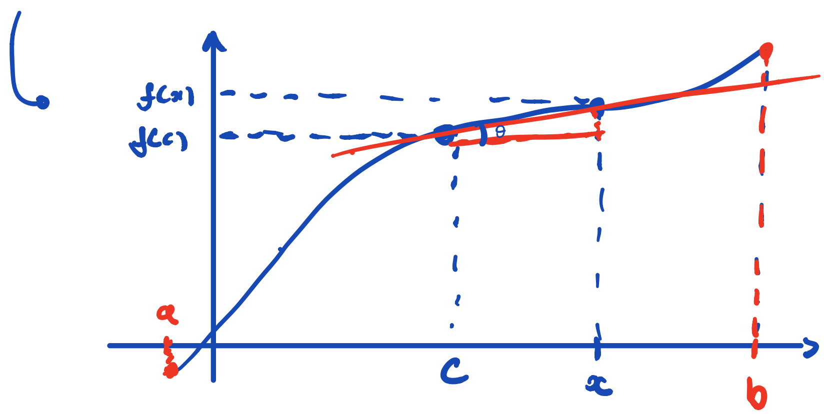
The Chain Rule

- Differentiation of composite functions

Caratheodory's Theorem: Let f be defined on an interval I s.t. $c \in I$. Then f is differentiable at c iff there exists a function φ on I continuous at c and satisfies

$$f(x) - f(c) = \varphi(x)(x - c) \quad \text{for } x \in I$$

In this case, we write $\varphi(c) = f'(c)$.



$$\frac{f(x)-f(c)}{x-c} = \frac{\text{cateto oposto}}{\text{cateto adjacente}} = \varphi(x) = \sin\theta$$

Chain Rule Let I, J be intervals on $\mathbb{R}$, let $g: I \to \mathbb{R}$ and $f: J \to \mathbb{R}$ be functions s.t. $f(J) \subseteq I$, and let $c \in J$. If f is differentiable at c and if g is differentiable at $f(c)$, then the composite function $g \circ f$ is differentiable at c and $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$

Proof

Since $f'(c)$ exists, Carathéodory's Theorem implies that there exists a function φ in J s.t. φ is continuous at c and $f(x) - f(c) = \varphi(x)(x-c)$ for $x \in J$, and where $\varphi(c) = f'(c)$.

Also, since $g'(f(c))$ exists, there exists a function ψ defined on I s.t. ψ is continuous at $d := f(c)$ and $g(y) - g(d) = \psi(y)(y-d)$ for $y \in I$, where

$\varphi(d) = g'(d)$.

Substituting $y = f(x)$ and $d = f(c)$ we get

$$g(f(x)) - g(f(c)) = \varphi(f(x))\,(f(x) - f(c))$$
$$= \left[(\varphi \circ f(x)) \cdot \varphi(x)\right](x - c)$$

for all $x \in J$ s.t. $f(x) \in I$. Since $(\varphi \circ f) \cdot \varphi$ is continuous at c and its value at c is $g'(f(c)) \cdot f'(c)$, Carathéodory's Theorem gives

$$\underbrace{(g \circ f)'(c) = g'(f(c))\, f'(c)}$$

$\square$

$\hookrightarrow (g \circ f)' = g' \circ f \cdot f'$

$$D(g \circ f) = (Dg \circ f) \cdot Df$$

EXAMPLES

1. $f(I) \subseteq \mathbb{R}$
2. g diff. at $f(c)$
3. f diff at c

a) $f : I \to \mathbb{R}$ differentiable on I 1. 3.

$g(y) := y^n$, $y \in \mathbb{R}$, $n \in \mathbb{N}$ $\Rightarrow$ $g'(y) = n y^{n-1}$ 2.

$$(g \circ f)'(x) \overset{\text{CHAIN RULE}}{=} (g' \circ f)(x) \cdot f'(x)$$

$$(g \circ f)(x) = (f(x))^n \Rightarrow (g \circ f)'(x) = n(f(x))^{n-1} f'(x)$$

d) $f: I \to \mathbb{R}$ differentiable on I and $f(x) \neq 0$ and $f'(x) \neq 0$

$h(y) := 1/y$ for $y \neq 0 \Rightarrow h'(y) = -1/y^2$

Therefore, we have that

$$\left(\frac{1}{f}\right)'(x) = (h \circ f)'(x) \overset{\text{CHAIN RULE}}{=} h'(f(x))\, f'(x) = -\frac{f'(x)}{(f(x))^2} \quad x \in I$$

e) $g(x) := |x|$ is differentiable at all $x \neq 0$

$$g'(x) = \operatorname{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

If f is any differentiable function, then the chain rule implies that $g \circ f = |f|$ is also differentiable at all points x where $f(x) \neq 0$, and its derivative is given by:

$$|f|'(x) = \operatorname{sgn}(f(x)) \cdot f'(x) = \begin{cases} f'(x), & f(x) > 0 \\ -f'(x), & f(x) < 0 \end{cases}$$

Inverse functions

- Relate the derivative of a function to the derivative of its inverse function (when this inverse exists)

Theorem 6.1.8 Let I be an interval in $\mathbb{R}$ and let $f: I \to \mathbb{R}$ be strictly monotone and continuous on I. Let $J := f(I)$ and let $g: J \to \mathbb{R}$ be the strictly monotone and continuous function inverse to f. If f is differentiable at $c \in I$ and $f'(c) \neq 0$, then g is differentiable at $d := f(c)$ and

em um ponto

$$g'(d) = 1/f'(c) = 1/f'(g(d))$$

↳ se $f'(c) = 0 \Rightarrow f'(c)\, g'(d) = 1 = 0$, falso

Proof

Given $c \in \mathbb{R}$, we obtain from Carathéodory's Theorem a function φ on I with properties that φ is continuous at c, $f(x) - f(c) = \varphi(x)(x - c)$ for $x \in I$, and $\varphi(c) = f'(c)$. Since $\varphi(c) \neq 0$ by hypothesis, there exists a neighborhood $V :=]c - \delta, c + \delta[$ s.t. $\varphi(x) \neq 0$ for $\forall x \in V \cap I$.

If $U := f(V \cap I)$, then the inverse function g satisfies $f(g(y)) = y$ for all $y \in U$, so that

$$y - d = f(g(y)) - f(c) = \varphi(g(y))(g(y) - g(d))$$

Since $\varphi(g(y)) \neq 0$ for $y \in U$, we have

$$g(y) - g(d) = 1/\varphi(g(y))\,(y - d)$$

Since $1/\varphi \circ g$ is continuous at d, we apply Caratheodory's Theorem to conclude that $g'(d)$ exists and
$g'(d) = 1/\varphi(g(d)) = 1/\varphi(c) = 1/f'(c)$.

Theorem 6.1.9 Let I be an interval and $f : I \to \mathbb{R}$ strictly monotone on I. Let $J := f(I)$ and let $g : J \to \mathbb{R}$ be the function inverse to f. If f is differentiable on I and $f'(x) \neq 0$ for $x \in I$, then g is differentiable on J and

$$g' = \frac{1}{f' \circ g} \quad *$$

em um intervalo

Proof

If f is differentiable on I, then Theorem 6.1.2 implies that f is continuous on I, and by the continuous inverse Theorem, g is continuous on J. From Theorem 6.1.8 we conclude that g is differentiable on J and

$$g' = \frac{1}{f' \circ g}$$

Não precisamos garantir monotonicidade?

Note If f and g are functions defined on Theorem 6.1.9, and if $x \in I$ and $y \in J$ are related by $y = f(x)$ and $x = g(y)$, then * can be written in the form

$$g'(y) = \frac{1}{(f' \circ g)(y)}, \; y \in J \quad \text{or} \quad (g' \circ f)(x) = \frac{1}{f'(x)}, \; x \in I$$