2. Find the points of relative extrema and the intervals on which the following functions are increasing / decreasing

a) $f(x) = \frac{x+1}{x}$, $x \neq 0$

//

$$f'(x) = \frac{1(x) - (x+1)1}{x^2} = \frac{x-x-1}{x^2} = -\frac{1}{x^2} < 0, \quad x \in \mathbb{R}$$

(ii)

$$f'(x) = 0 \iff -\frac{1}{x^2} = 0 \iff x^2 = -1 \iff x = \sqrt{-1}$$

$$\lim_{x \to \infty} \frac{x+1}{x} = \lim_{x \to \infty} 1 + \frac{1}{x} = 1$$

(i)

$$\lim_{x \to -\infty} \frac{x+1}{x} = \lim_{x \to -\infty} 1 + \frac{1}{x} = 1$$

//

b) $g(x) = \dfrac{x}{(x^2+1)}$, $x \in \mathbb{R}$

//

$$g'(x) = \frac{1(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

$$g'(x) = 0 \Leftrightarrow \frac{1-x^2}{(x^2+1)^2} = 0 \Leftrightarrow x = 1$$

(ii)

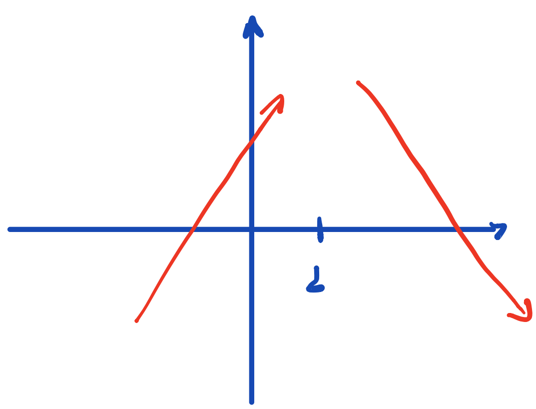
$(x^2+1)^2 > 0$, $x \in \mathbb{R}$

$$\begin{cases} (1-x^2) \leq 0 , & x \geq 1 \\ (1-x^2) > 0 , & x < 1 \end{cases} \Rightarrow g(x) \begin{cases} \geq 0 , & x \leq 1 \\ < 0 , & x > 1 \end{cases}$$

$\Downarrow$

$x = 1$ É MÁXIMO LOCAL

(i)



$$\lim_{x \to \infty} \frac{x}{x^2+1} = \lim_{x \to \infty} \frac{1}{2x} = 0$$

$$\lim_{x \to -\infty} \frac{x}{x^2+1} = 0$$

6. Use the Mean Value Theorem to prove that

$$|\sin x - \sin y| \leq |x-y| \quad , \quad x,y \in \mathbb{R}$$

//

T.V.I $\begin{cases} \text{Hipóteses: (i) } f:[a,b] \to \mathbb{R} \text{ (ii) contínua em } [a,b] \\ \text{(iii) derivável em } (a,b) \\ \text{Tese: } \exists c \in (a,b) \text{ t.q. } f'(c) = \dfrac{f(b)-f(a)}{b-a} \end{cases}$

$$\left|\frac{\sin x - \sin y}{x-y}\right| \leq 1 \qquad f(x) = \sin x : \mathbb{R} \to \mathbb{R}$$

$\vdash \exists c \in \mathbb{R}$ t.q. $f'(c) = 1$

Tome $c=0$, então $f'(c) = 1$. Portanto pelo T.V.I temos

$$f'(c) = 1 = \frac{\sin x - \sin y}{x-y} \Leftrightarrow |1| = 1 \geq \left|\frac{\sin x - \sin y}{x-y}\right|$$

$$\Leftrightarrow |\sin x - \sin y| \leq |x-y|$$

//

7. Use the mean value Theorem to prove that

$$\frac{(x-1)}{x} < \ln x < x-1 \quad , \quad x > 1$$

// $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x} \quad , \quad > 0$

$$\frac{1}{x} < \frac{\ln x - \ln 1}{x-1} = \frac{\ln x}{x-1} < 1$$

9. Let $f:\mathbb{R}\to\mathbb{R}$ s.t. $f(x) = 2x^4 + x^4\sin\left(\frac{1}{x}\right)$, $x \neq 0$ and $f(0)=0$

$$f(x) = \begin{cases} 2x^4 + x^4\sin\left(\frac{1}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

Show that f has an absolute minimum at $x=0$, but that its derivative has both positive and negative values in every neighbourhood of 0.

$$f'(x) = 8x^3 + 4x^3\sin\left(\frac{1}{x}\right) - \frac{x^4}{x^2}\cos\left(\frac{1}{x}\right) = 8x^3 + 4x^3\sin\left(\frac{1}{x}\right) - x^2\cos\left(\frac{1}{x}\right)$$

$f'(0) = 0 + 0 - 0 = 0 \Rightarrow f$ têm máximo ou mínimo local em $x=0$

Para qualquer $\delta > 0$ e $x \in \mathbb{R}$ temos

$$f(x+\delta) = 8(x+\delta)^3 + 4(x+\delta)^3\sin\left(\frac{1}{x+\delta}\right) - (x+\delta)^2\cos\left(\frac{1}{x+\delta}\right)$$

$$= (x+\delta)^2\left[8(x+\delta) + 4(x+\delta)\sin\left(\frac{1}{x+\delta}\right) - \cos\left(\frac{1}{x+\delta}\right)\right]$$

onde

$(x+\delta)^2 > 0$

Note que se $x > 0$ então

$(x+\delta) > 0$

10.