1. Let $f(x) = \cos ax$ for $x \in \mathbb{R}$ where $a \neq 0$. Find $f^{(n)}(x)$ for $n \in \mathbb{N}$, $x \in \mathbb{R}$

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$$\left.\begin{aligned} f^{(0)}(x) &= \cos ax \\ f^{(1)}(x) &= -a \sin ax \\ f^{(2)}(x) &= -a^2 \cos ax \\ f^{(3)}(x) &= a^3 \sin ax \\ f^{(4)}(x) &= a^4 \cos ax \\ &\vdots \end{aligned}\right\} \qquad \begin{aligned} f^{(2n)}(x) &= a^{2n} \cos(ax) \\ f^{(2n+1)} &= -a^{2n+1} \sin(ax) \end{aligned}$$

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Usaremos o princípio da indução para mostrar que para todo $k \in \mathbb{N}$

$$f^{(2k)}(x) = a^{2k} \cos(ax) \quad \text{e} \quad f^{(2k+1)}(x) = -a^{2k+1} \sin(ax)$$

Note que, para $k = 1$ temos

$$f^{(2)}(x) = -a^2 \cos(ax) = -a^{2(1)} \cos(ax)$$

Vamos assumir que $f^{(2k)} = a^{2k} \cos(ax)$ é verdadeira, ou seja

$$f^{(2k)}(x) = a^{2k} \cos(ax)$$

Para $k+1$ temos

$$f^{(2k+k+1)}(x) = a^{(2k+k+1)}\cos(ax) = a^{2k+1}\, a^{k}\cos(ax) =$$

$$= a^{2k+2}\, a^{k-1}\cos(ax) = a^{k-1}\, a^{2(k+1)}\cos(ax)$$

$$f^{(2k+(k+1))}(x) = a^{k+1} f^{(2k)}(x)$$

2. Let $g(x) = |x^3|$ for $x \in \mathbb{R}$. Find $g^{(1)}(x)$ and $g^{(2)}(x)$ for $x \in \mathbb{R}$, and $g^{(3)}(x)$ for $x \neq 0$. Show that $g^{(3)}(0)$ does not exists

Tomemos

$$g^{(0)}(x) = x^2|x| = \begin{cases} x^3, & x > 0 \\ -x^3, & x < 0 \\ 0, & x = 0 \end{cases}$$

Então

$$g^{(1)}(x) = \begin{cases} 3x^2, & x > 0 \\ -3x^2, & x < 0 \\ 0, & x = 0 \end{cases}$$

$$g^{(2)}(x) = \begin{cases} 6x, & x > 0 \\ -6x, & x < 0 \\ 0, & x = 0 \end{cases}$$

$$g^{(3)}(x) = \begin{cases} 6, & x > 0 \\ -6, & x < 0 \end{cases}$$

Note que $g^{(3)}(0)$ existe se $g^{(2)}(x)$ for diferenciável em $x = 0$. Mas

$$g^{(2)}(x) = 6|x|$$

E, ALÉM DISSO, SABEMOS QUE $g(x) = |x|$ É NÃO DIFERENCIÁVEL EM $x=0$. PORTANTO $g^{(3)}(0)$ NÃO EXISTE.

4. Show that if $x > 0$, então $1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x} \leq 1 + \frac{1}{2}x$

Notemos que se

$$f^{(0)}(x) = \sqrt{1+x}$$

então

$$f^{(1)}(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}} \qquad f^{(2)}(x) = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$$

$$f^{(3)}(x) = +\frac{3}{8}(1+x)^{-\frac{5}{2}}$$

Se $f: I \to \mathbb{R}$ é uma função t.q. $f^{(1)}, f^{(2)}, \dots, f^{(n)}$ são contínuas em $I$, $f^{(n+1)}$ existe em $]a,b[$, e $x_0 \in I$ então, pelo Teorema de Taylor, para $\forall x \in I$ $\exists c \in [x_0, x]$ t.q.

$$f(x) = f(x_0) + \frac{f^{(1)}(x_0)(x-x_0)^1}{1!} + \dots + \frac{f^{(n)}(x_0)(x-x_0)^n}{n!} + R_n(x)$$

onde $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$.

Seja $x_0 = 0$ e $f(x) = \sqrt{1+x}$, então

$$f(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + R_2(x)$$

onde

$$R_2(x) = \frac{f^{(3)}(c)}{3!}x^3 = +\frac{3}{48}(1+c)^{-\frac{5}{2}}x^3, \quad c \in [0, x]$$

Por Hipótese, se $x > 0$ então $R_2(x) > 0$ o que implica

$$\sqrt{1+x} \geq 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

e além disso, como $-\frac{1}{8}x^2 < 0$ então

$$1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x} \leq 1 + \frac{1}{2}x \quad , \quad x > 0$$

7. If $x > 0$ show that $\left|(1+x)^{1/3} - \left(1 + \frac{1}{3}x - \frac{1}{9}x^2\right)\right| \leq (5/81)\, x^3$

$$f(x) = \frac{5}{81}x^3 \qquad f^{(1)}(x) = \frac{15}{81}x^2$$

$$f^{(2)}(x) = \frac{30}{81}x \qquad f^{(3)}(x) = \frac{30}{81}$$

$$f(x) = \frac{5}{81}x_0^3 + \frac{5}{81}x_0^2\,(x - x_0) + \frac{\phantom{0}}{81}\left(\frac{1}{2}\right)x_0\,(x-x_0)^2 + \frac{30}{81}\left(\frac{1}{6}\right)(x-x_0)^3 + R_3$$

$x_0 = 1$

$$= \frac{15}{81} + \frac{15}{81}(x-1) + \frac{15}{81}\left(\frac{1}{2}\right)(x-1)^2 + \frac{15}{81}\left(\frac{1}{6}\right)(x-1)^3 + R_3(x)$$

$$= \frac{15}{81}\left[1 + (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3\right] + R_3(x)$$

$$= 1 + x - 1 + \frac{1}{2}x^2 - 1 + \frac{1}{2} + \frac{1}{6}(x^2 - 2 + 1)(x - 1)$$

$$= x + \frac{1}{2}x^2 - 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{6}(x^3 - x^2 - 2x + 2 + x - 1)$$

$$= x + \frac{1}{2}x^2 \cancel{-1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{6}x^3 - \frac{1}{6}x^2 - \frac{1}{\cancel{3}}x + \frac{1}{3} + \frac{1}{6}x - \frac{1}{6}$$

$$= -(1 + \frac{1}{3}x$$

$$x^2\left(\frac{1}{2} - \frac{1}{6}\right) = x^2\left(\frac{6-2}{12}\right) = x^2\left(\frac{4}{12}\right) = x^2\,\frac{1}{3}$$

8. If $f(x) = e^x$ show that the remainder term of Taylor's Theorem converges to zero as $n \to \infty$, for each fixed $x_0$ and $x$

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$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

$$f^{(n+1)}(c) = e^c \Rightarrow R_n(x) = \frac{e^c}{(n+1)!}(x - x_0)^{n+1}$$

$$\lim_{n \to \infty} \frac{e^c}{(n+1)!} = e^c \lim_{n \to \infty} \frac{1}{(n+1)!} = 0$$

$$\lim_{n \to \infty} (x - x_0)^{n+1} = \infty$$

Apply L'Hopital Rule to get

$$\lim_{n \to \infty} \frac{e^c}{(n+1)!}(x - x_0)^{n+1} = \lim_{n \to \infty} \frac{d}{dn}\left[\frac{e^c}{(n+1)!}\right] \frac{d}{dn}\left[(x - x_0)^{n+1}\right]$$

$$\frac{d}{dn}\left[(x - x_0)^{n+1}\right] = (x - x_0)^{n+1} \ln(x - x_0)$$