5.3. Continuous Functions on Intervals

Def 5.3.1 A function $f : A \to \mathbb{R}$ is said to be **bounded** on A if $\exists M > 0$ s.t. $|f(x)| \leq M$ for $\forall x \in A$.

- In other words, a function is bounded on a set if its range is bounded
- A function f is **not bounded** on the set A if given any $M > 0$ $\exists x_M \in A$ s.t. $|f(x_M)| > M$

Boundedness Theorem Let $I := [a,b]$ be a closed interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f is bounded on I.

Proof

Suppose f is not bounded on I. Then, for $\forall n \in \mathbb{N}$ $\exists x_n \in I$ s.t. $|f(x_n)| > n$. Since I is bounded, the seq. $X = (x_n)$ is bounded. Therefore, the Bolzano-Weierstrass Theorem implies that there is a subsequence $X' = (x_{n_r})$ of X s.t. $\lim (x_{n_r}) = x$. Since I is closed and $\forall (x_{n_r}) \in I \Rightarrow$ (from Theorem 3.2.6) $x \in I \Rightarrow$ $\Rightarrow$ (from Theorem 4.1.8) $\lim (f(x_{n_r})) = f(x)$ $\Leftrightarrow$ ~~$\lim_{x \to c} f(x) = L$~~ $\Leftrightarrow$ ~~f is continuous at~~ c $\Rightarrow$ (from Theorem 3.2.2) the convergent sequence $(f(x_{n_r}))$ is bounded.

But this is a contradiction since $|f(x_{n_r})| > n_r \geq r$ $\forall r \in \mathbb{N}$. Thus, the supposition must be false

The Maximum-Minimum Theorem

Def 5.3.3 Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. We say that f has an **absolute maximum** on A if $\exists x^* \in A$ s.t.

$$f(x^*) \geq f(x) \quad \forall x \in A$$

We say that f has an **absolute minimum** on A if $\exists x_* \in A$ s.t.

$$f(x_*) \leq f(x) \quad \forall x \in A$$

We say that x^* is an **absolute maximum point** for f on A, and that x_* is an **absolute minimum point** for f on A, if they exist.

Maximum-Minimum Theorem Let $I := [a,b]$ be a closed interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f has an absolute maximum and an absolute minimum

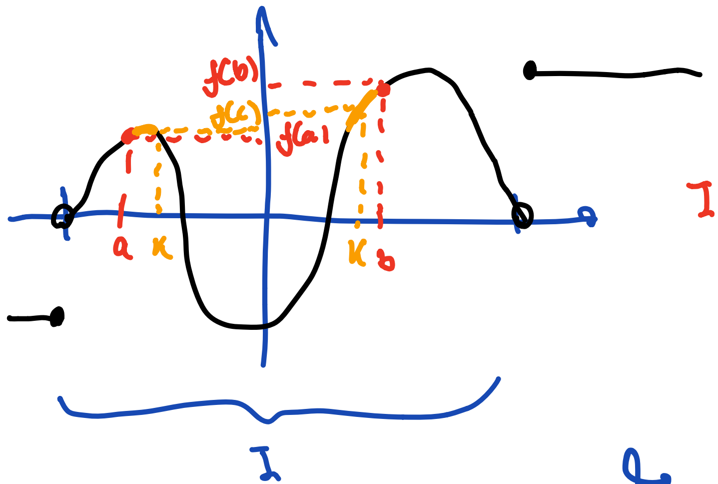
Proof

Location of Roots Theorem Let $I = [a,b]$ and let $f : I \to \mathbb{R}$ be continuous on I. If $f(a) < 0 < f(b)$, or if $f(a) > 0 > f(b)$, then $\exists c \in (a,b)$ s.t. $f(c) = 0$.

Proof

Bolzano's Theorem

Bolzano's Intermediate Value Theorem Let I be an interval and let $f: I \to \mathbb{R}$ be continuous on I. If $a, b \in I$ and if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b) \Rightarrow \exists c \in I$, $a < c < b$, s.t. $f(c) = k$



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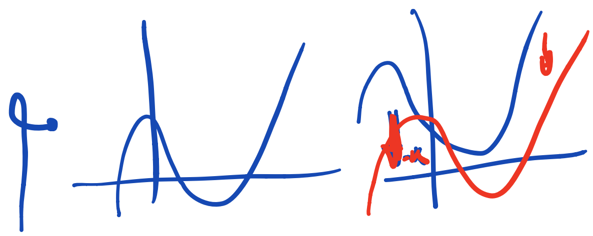
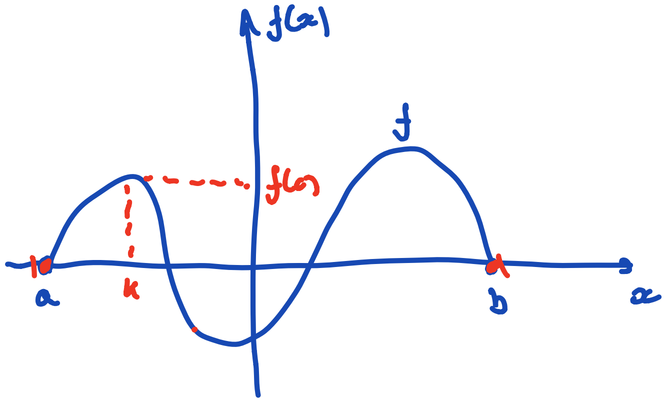


Proof

Suppose $a < b$ and let $g(x) := f(x) - k$; then $g(a) < 0 < g(b)$. By the **location of roots theorem** $\exists c \in]a, b[$ s.t. $0 = g(c) = f(c) - k$. Therefore $f(c) = k$.

If $b < a$, let $h(x) := k - f(x)$ so that $h(b) < 0 < h(a)$. Therefore $\exists c \in]b, a[$ s.t. $0 = h(c) = k - f(c)$, whence $f(c) = k$

Corollary 5.3.8 Let $I = [a, b]$ be a closed, bounded interval and let $f: I \to \mathbb{R}$ be continuous on I. If $k \in \mathbb{R}$ is any number satisfying $\inf f(I) \le k \le \sup f(I) \Rightarrow \exists c \in I$ s.t. $f(c) = k$.



$\mu = \sup S$, $S \subseteq \mathbb{R}$, iff

a) $s \le \mu$ $\forall s \in S$

b) if $v < \mu \Rightarrow \exists s' \in S$ s.t. $v < s'$

$\sup I = a$, $\forall$ $I = [a,b]$?

Proof

It follows from the MAX-MIN Theorem that there are points c_* and $c^* \in I$ s.t.

$$\inf f(I) = f(c_*) \le k \le f(c^*) = \sup f(I)$$

The conclusion follows from the Bolzano's Theorem.

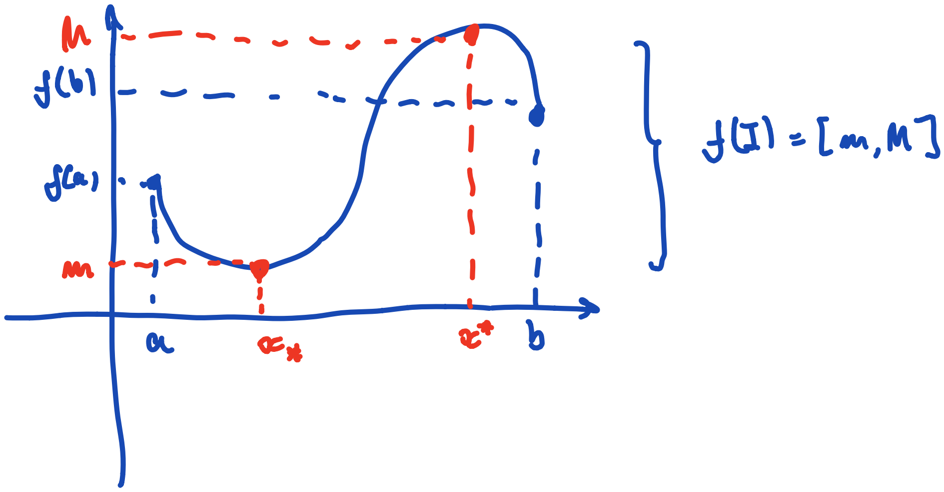
Theorem 5.3.9 Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then the set $f(I) := \{ f(x) : x \in I \}$ is a closed bounded interval.

Proof

$k \in I$ and k $\Rightarrow [m,M] \in I$

If we let $m := \inf f(I)$ and $M := \sup f(I)$, then we know from the MAX-MIN Theorem that $m, M \in f(I)$. Moreover, we have $f(I) \subseteq [m, M]$. If k is any element of $[m, M] \Rightarrow$ $\Rightarrow$ (from Corollary 5.3.8) $\exists c \in I$ s.t. $k = f(c)$. Hence $k \in f(I)$

And we conclude that $[m, M] \subseteq f(I)$. Therefore, $f(I)$ is the interval $[m, M]$



Preservation of Intervals Theorem Let I be an interval and let $f: I \to \mathbb{R}$ be continuous on I. Then the set $f(I)$ is an interval.