1. Show that the function $f(x) := 1/x$ is uniformly continuous on the set $A := [a, \infty)$, where $a > 0$

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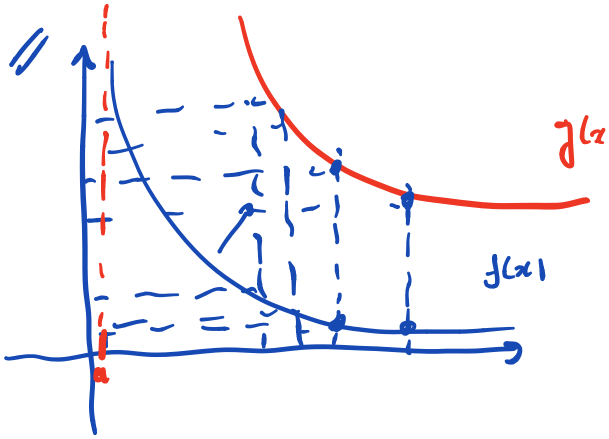

Corollary: If $f : [a, \infty[\to \mathbb{R}$ is continuous and $\lim_{x \to \infty} f(x) = 0$, then f is uniformly continuous on $[a, \infty)$

Def
- Lipschitz function *
- Cauchy seq.

P: $f(x) := \frac{1}{x}$, $x \in [a, \infty)$, $a > 0$

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y-x}{xy}\right| = \frac{1}{xy}|y-x| = \frac{1}{xy}|x-y|$$

$z := xy$ s.t. $z\,[a^2, \infty[$, let $K := \frac{1}{a^2} > 0$ //

Let $f(x) := \frac{1}{x}$ s.t. $x \in [a, \infty[$, $a > 0$. Note that

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y-x}{xy}\right|$$

Define $z := xy$ so that $z \in [a^2, \infty[$, then

$$\left|\frac{y-x}{xy}\right| = \frac{1}{xy}|y-x| = \frac{1}{xy}|x-y|$$

Furthermore, let $k := 1/a^2$, then we have that

$$\left| \frac{1}{x} - \frac{1}{y} \right| \le k\,|x-y| \Rightarrow f \text{ is lipschitz on } [a,\infty[$$

From the Theorem of the lipschitz function we conclude that f is uniformly continuous on $[a,\infty[$, $a>0$.

2. Show that $f(x) := 1/x^2$ is uniformly continuous on $A := [1,\infty[$ but not on $B :=]0,\infty[$

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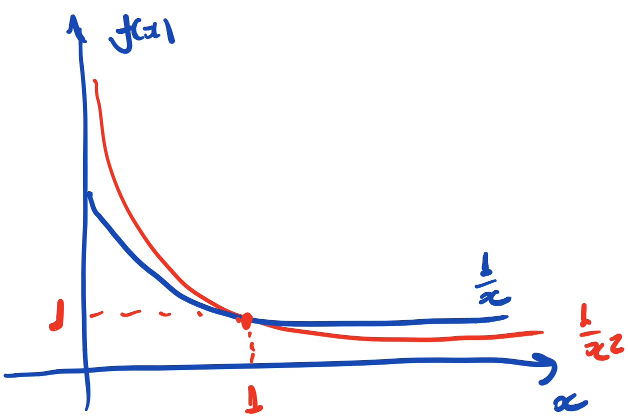

Def
- lipschitz
- Cauchy

$$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{y^2-x^2}{x^2y^2}\right| = \frac{1}{x^2y^2}\,|y^2-x^2| = \frac{1}{x^2y^2}\,|x^2-y^2|$$

$$|x^2-y^2| \ge |x-y| \Leftrightarrow k\,|x^2-y^2| \ge k\,|x-y|, \quad k>0$$

$$x \ge 1 \Rightarrow \left|\frac{1}{x^2} - \frac{1}{y^2}\right| \le \left|\frac{1}{x} - \frac{1}{y}\right| \le k\,|x-y| \quad //$$

From exercise 1, we know that $f(x) = 1/x$, $x \in [a,\infty[$ is a lipschitz function, that is,

$$\left|\frac{1}{x}-\frac{1}{y}\right| \leq k|x-y|$$

which implies from the Theorem of the Lipschitz function that f is uniform continuous on $[a,\infty[$, $a>0$.
Note that, for $x \geq 1$

$$\left|\frac{1}{x^2}-\frac{1}{y^2}\right| \leq \left|\frac{1}{x}-\frac{1}{y}\right| \leq k|x-y|$$

Thus, the same reasoning applies to $g(x) := 1/x^2$, $x \in [1,\infty[$ and g is uniform continuous on its domain.

On the other hand, let $h(x) := 1/x^2$, $x \in]0,\infty[$.
Note that

$$\left|\frac{1}{x^2}-\frac{1}{y^2}\right| \geq \left|\frac{1}{x}-\frac{1}{y}\right| \quad , \text{ for } x \in]0,1[$$

Then $\exists \delta(\epsilon) > 0$ s.t. for $\forall \epsilon > 0$ if x satisfy

$|x-y| < \delta(\epsilon)$, then $|f(x)-f(y)| < \epsilon$.

8. Prove that if f and g are each uniformly continuous on $\mathbb{R}$, then $f \circ g$ is uniformly continuous on $\mathbb{R}$

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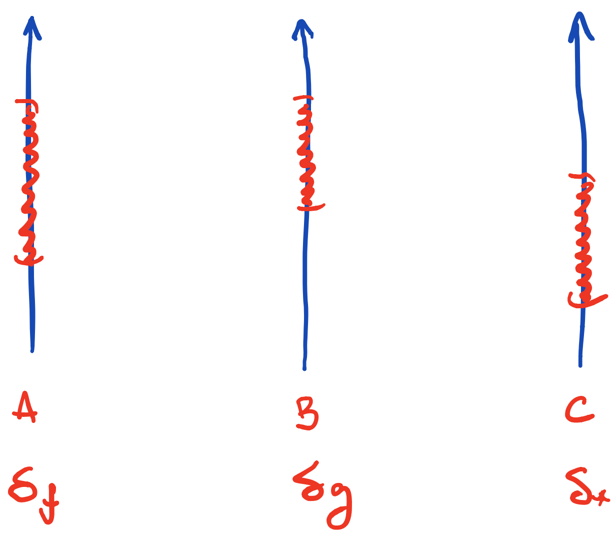
$f : A \to \mathbb{R}$ $\qquad$ $g : B \to \mathbb{R}$

$f \circ g : C \to \mathbb{R}$



$|f(g(x)) - f(g(y))| < \varepsilon$

$\Leftrightarrow \quad B \subset A$

$\forall \varepsilon > 0 \ \exists \delta_f(\varepsilon) > 0$ s.t. y satisfy $|y - v| < \delta_f \Rightarrow |f(y) - f(v)| < \varepsilon$

$\exists \delta_g(\varepsilon) > 0$ s.t. x satisfy $|x - u| < \delta_g \Rightarrow |g(x) - g(u)| < \varepsilon$

?

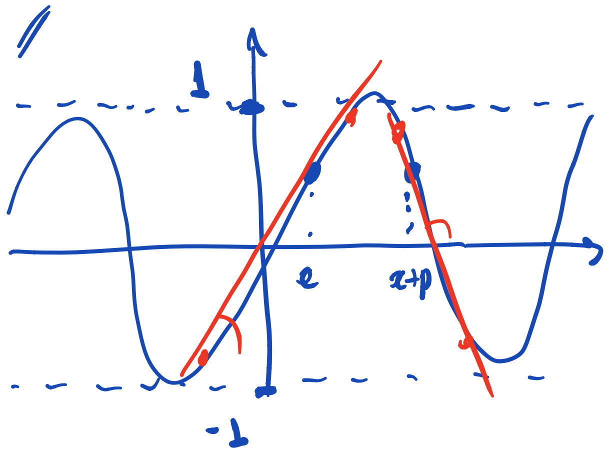
Let $y = g(x)$ and $v = g(u) \Rightarrow \forall \varepsilon > 0 \ |g(x) - g(u)| < \delta_f$

$\Rightarrow |f(g(x)) - f(g(u))| < \varepsilon$

//

14. $f:\mathbb{R}\to\mathbb{R}$ is said to be periodic on $\mathbb{R}$ if $\exists p>0$ s.t. $f(x+p)=f(x)\ \forall x\in\mathbb{R}$.

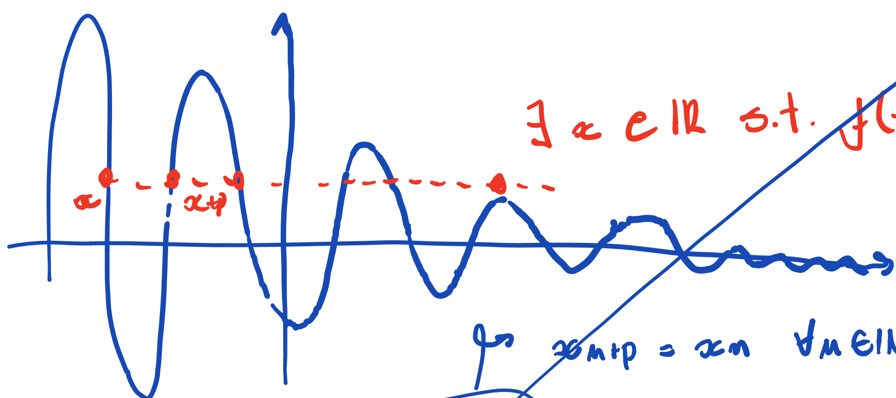
Prove that a continuous periodic function on $\mathbb{R}$ is bounded and uniformly continuous on $\mathbb{R}$.



DEF

~~Cauchy~~ *

Lipschitz *



$\exists x\in\mathbb{R}$ s.t. $f(x+p)\neq f(x)$

$x_{m+p}=x_m\ \forall m\in\mathbb{N}$

Bounded

Divergent

Claim: Every periodic seq. is Cauchy seq.

$\forall \varepsilon>0,\ \exists N>0,\ N\in\mathbb{Z},\ \forall m,n>N,\ |x_m-x_n|<\varepsilon$

Claim: Every periodic function satisfies the Lipischitz condition

$f: A \to \mathbb{R}$, $\exists k > 0$ s.t. $|f(x) - f(y)| \leq k|x - y| \Rightarrow f$ Lipischitz

$f(x) = f(x+p) \quad \forall x \in \mathbb{R} \quad \Rightarrow |f(x) - f(x+p)| = 0$

$|x - (x+p)| = |-p| = p$, $p > 0$. Let $y := x+p$, then

$0 < |x - y| \leq p|x-y| \Rightarrow |f(x) - f(x+p)| < p|x+y|$