1. Prove the sequential criterion for continuity

// **Seq. criterion for continuity** A function $f: A \to \mathbb{R}$ is continuous at $c \in \mathbb{R}$ iff for $\forall (x_n) \in A$ s.t. $\lim(x_n) = c$, $\lim(f(x_n)) = f(c)$ //

(i) if $f: A \to \mathbb{R}$ is continuous at $c \in \mathbb{R} \Rightarrow \forall (x_n) \in A$ s.t. $\lim(x_n) = c$, $\lim(f(x_n)) = f(c)$.

If $f: A \to \mathbb{R}$ is continuous at $c \in A \Rightarrow f(c) = \lim_{x \to c} f(x)$. Furthermore, from the sequential criterion we have $\lim_{x \to c} f(x) = f(c) \Leftrightarrow$ $\forall (x_n) \in A$ s.t. $\lim(x_n) = c$, $x_n \neq c$, for $\forall n \in \mathbb{N}$, $\lim(f(x_n)) = f(c)$
Thus, we conclude that (i) is true.

(ii) If $\forall (x_n) \in A$ s.t. $\lim(x_n) = c$, $\lim(f(x_n)) = f(c) \Rightarrow f: A \to \mathbb{R}$ is continuous at $c \in A$.

If for $\forall (x_n) \in A$ s.t. $\lim(x_n) = c$, $\lim(f(x_n)) = f(c)$ $\Rightarrow$ (from the seq. criterion) $\lim_{x \to c} f(x) = f(c)$. Furthermore, $\lim_{x \to c} f(x) = f(c)$ $\Rightarrow f$ is continuous at c. Thus, we conclude that (ii) is true.

3. Let $a < b < c$, and suppose that f and g are continuous on $[a,b]$ and $[b,c]$ respectively, and $f(b) = g(b)$. Define

$$h(x) := \begin{cases} f(x), & x \in [a,b] \\ g(x), & x \in [b,c] \end{cases}$$

Prove that h is continuous on $[a,c]$.

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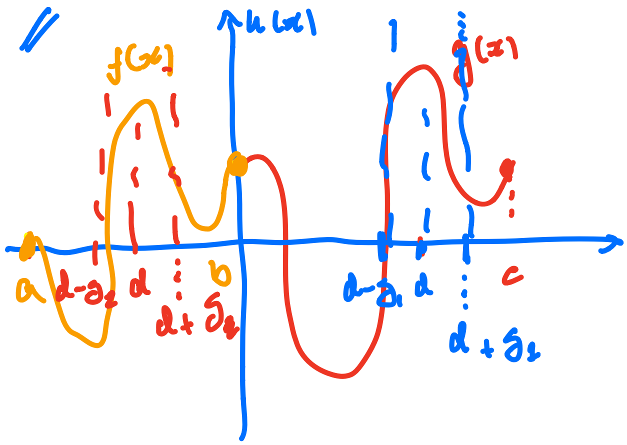

f is continuous on $[a,b]$ $\Leftrightarrow$

$\Leftrightarrow$ for any $\varepsilon > 0$ $\exists \delta_1(\varepsilon) > 0$ s.t. $|x-d| < \delta_1(\varepsilon) \Rightarrow |f(x) - f(d)| < \varepsilon$ $\forall d \in [a,b]$

Take $\delta = \min\{\delta_1, \delta_2\} \Rightarrow \delta \le \delta_1$ and $\delta \le \delta_2$

$\Rightarrow |x - c| < \delta$ $\quad x \in [a,c]$ //

Since f is continuous on $A = [a,b] \Rightarrow$ for $\forall \varepsilon > 0$, $\exists \delta_1(\varepsilon) > 0$ s.t. $|x - d| < \delta_1(\varepsilon) \Rightarrow |f(x) - f(d)| < \varepsilon$, $\forall d \in A$. Furthermore, since g is continuous on $B = [b,c] \Rightarrow$ for $\forall \varepsilon > 0$, $\exists \delta_2(\varepsilon) > 0$ s.t. $|x - d| < \delta_2(\varepsilon) \Rightarrow |g(x) - g(d)| < \varepsilon$, $\forall d \in B$.

Let $\delta = \min\{\delta_1(\varepsilon), \delta_2(\varepsilon)\}$ and $C = A \cup B$, then for $\forall \varepsilon > 0$ and $d \in C$, $|x - d| < \delta \Rightarrow |h(x) - h(d)| < \varepsilon$.
Thus, we conclude that h is continuous on C.

4. Define $f(x) := [[x]]$, $x \in \mathbb{R}$ to be the greatest integer $n \in \mathbb{Z}$ s.t. $n \le x$.

Determine the continuity of the following functions

a) $f(x) := [[x]]$, $x \in \mathbb{R}$

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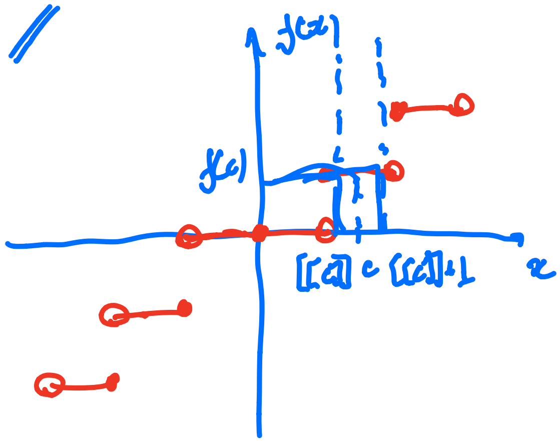

Claim: i) f is continuous on $\forall c \in \mathbb{R} \setminus \mathbb{Z}$
but ii) discontinuous on $\forall c \in \mathbb{Z}$

i) ⊢ f is continuous on $\mathbb{R}\setminus\mathbb{Z} \iff \forall c \in \mathbb{R}\setminus\mathbb{Z}$
$\lim_{x \to c} f(x) = f(c)$

ii) ⊢ f is discontinuous on $\mathbb{Z} \iff$
$\forall c \in \mathbb{Z}$ $\lim_{x \to c^+} f \ne \lim_{x \to c^-} f$

~~$\delta \ge [[c]]$ and $\delta < [[c]]+1 \iff [[c]] \le \delta < [[c]]+1$~~ ?

~~Take $\delta = \min(|c - [[c]]|, |c - ([[c]]+1)$~~ $|)$

Lấy như thế
phạm ngoài?

//

⊢ f is continuous on $\mathbb{R}\setminus\mathbb{Z} \iff \lim_{x\to c} f(x) = f(c)$ for $\forall c \in \mathbb{R}\setminus\mathbb{Z}$

Let $\delta = \min(|c - [[c]]|, |c - [[c]]+1|)$, then

$$|x - c| < \delta \iff -\delta < x - c < \delta \iff c - \delta < x < \delta + c$$

$$\iff [[c]] < x < [[c]]+1 .$$

Therefore, $f(x) = [[c]]$ $\forall x \in \mathbb{R}\setminus\mathbb{Z}$. Thus, for $\forall \varepsilon > 0$ if
$|x - c| < \delta \Rightarrow |f(x) - f(c)| = 0 < \varepsilon$.

$\therefore \lim_{x \to c} f(x) = [[c]]$, $\forall x \in \mathbb{R}\setminus\mathbb{Z} \Rightarrow f$ is continuous on $\mathbb{R}\setminus\mathbb{Z}$

$\vdash$ f is discontinuous on $\mathbb{Z}$ $\iff$ $\forall c \in \mathbb{Z}$, $\lim\limits_{x \to c^+} f \neq \lim\limits_{x \to c^-} f$

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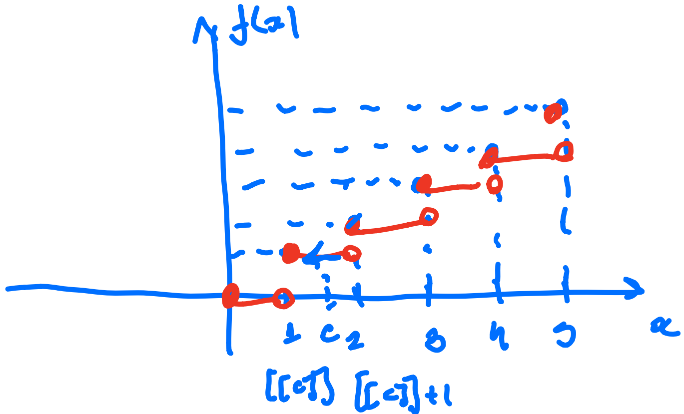


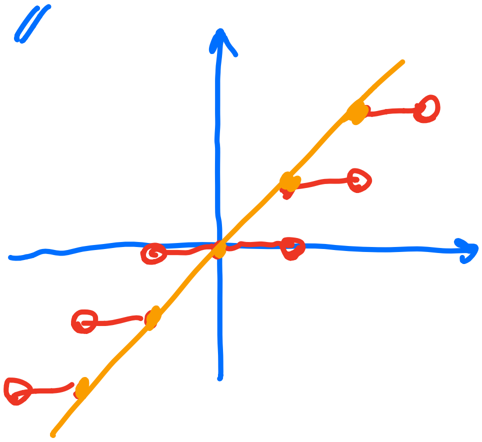
$c \in \mathbb{Z}$, $\lim\limits_{x \to c^+} f(x) = L = [[c]] = c$

$\vdash$ Given $\forall \varepsilon > 0$ s.t. $\delta(\varepsilon) > 0$ $\quad 0 < |x - c| < \delta(\varepsilon) \Rightarrow |f(x) - L| < \varepsilon$

$f(x) = [[x]]$, $\forall x \in \mathbb{R}$ $\Rightarrow$ $[[x]] - 1 \leq c \leq [[x]]$ //

Let $c \in \mathbb{Z} \Rightarrow \cancel{\forall} x \in \mathbb{R}$, $c - 1 < \cancel{[[x]]} < c$. Therefore, $\lim\limits_{x \to c^+} f = c$ and $\lim\limits_{x \to c^-} f = c - 1$, and we conclude that f is discontinuous on $\mathbb{Z}$

b) $g(x) := x[[x]]$



$f(x) := [[x]]$ $\quad h(x) := x$, $x \in \mathbb{R}$

(i) $c \in \mathbb{R} \setminus \mathbb{Z}$

$$\lim_{x \to c} f(x) = [[c]] \qquad \lim_{x \to c} h(x) = c$$

$$\Rightarrow \lim f \cdot g = c[[c]]$$

(ii) $c \in \mathbb{Z}$

$$\lim_{x \to c^+} h(x) = \lim_{x \to c^-} h(x) = c \qquad \lim_{x \to c^+} g(x) = c \qquad \lim_{x \to c^-} g = c-1$$

Let $f(x) := [[x]]$ and $h(x) := x$. Suppose $c \in \mathbb{R} \setminus \mathbb{Z}$, then

$$\lim_{x \to c} f(x) = [[c]] \quad \text{and} \quad \lim_{x \to c} h(x) = c$$

On the other hand, suppose $c \in \mathbb{Z}$, then

$$\lim_{x \to c^+} f = c \ , \quad \lim_{x \to c^-} f = c - 1 \ , \quad \lim_{x \to c^+} h = \lim_{x \to c^-} h = c$$

Then by Theorem 4.2.4 we have that

$$\lim_{x \to c} g(x) = c[[c]] \ , \quad c \in \mathbb{R} \setminus \mathbb{Z}$$

$$\lim_{x \to c^-} g(x) = c(c-1) \ , \quad c \in \mathbb{R} \setminus \mathbb{Z}$$

$$\lim_{x \to c^+} = c^2 \ , \quad c \in$$

5. Let $f(x) := (x^2+x-6)/(x-2)$, $x \in \mathbb{R}$, $x \neq 2$,

Define f at $x=2$ s.t. it becomes continues at 2

Let $f: A \to \mathbb{R}$ and define $\bar{f}: A \cup \{2\} \to \mathbb{R}$ s.t.

$$\bar{f}(x) := \begin{cases} (x^2+x-6)/(x-2) & , x \neq 2 \\ \lim_{x \to 2} f(x) & , x = 2 \end{cases}$$

Since $\lim_{x \to 2} f(x) = 0$, then $\bar{f}(x)$ exists. Furthermore, $\lim_{x \to 2} \bar{f}(x) = 0$, then $\bar{f}(x)$ is continues at 2.

como isso é verdade $\Rightarrow$ posso pegar $\varepsilon > \ldots$

7. $f: \mathbb{R} \to \mathbb{R}$ continues at c and $f(c) > 0$,

Show that $\exists V_\delta(c)$ s.t. if $x \in V_\delta(c) \Rightarrow f(x) > 0$

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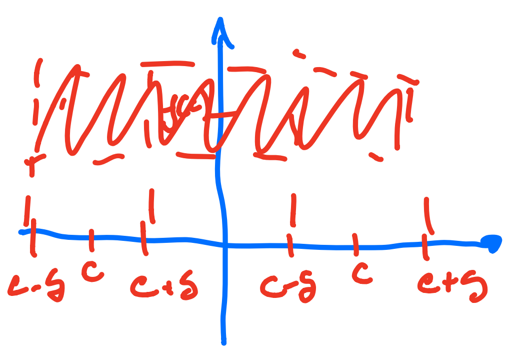
f is continues at $c \iff$ for $\forall \varepsilon > 0$ $\exists \delta(\varepsilon) > 0$ s.t. $|x-c| < \delta$ $\Rightarrow |f(x) - f(c)| < \varepsilon \iff f(c) - \varepsilon < f(x) < f(c) + \varepsilon$

Choose $\varepsilon > 0$ s.t. $f(c) - \varepsilon > 0$ and $f(c) > \varepsilon$



$f(c) > 0$

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Let $0 < \varepsilon < f(c)$. Note that $f: \mathbb{R} \to \mathbb{R}$ is continuous at c and $f(c) > 0$. If $x \in V_\delta(c) \Rightarrow |f(x) - f(c)| < \varepsilon \Leftrightarrow$

$f(c) - \varepsilon < f(x) < f(c) + \varepsilon \Rightarrow f(x) > 0$.

11. Let $k > 0$ and $f: \mathbb{R} \to \mathbb{R}$ satisfy $|f(x) - f(y)| \leq k|x-y|$ $\forall x, y \in \mathbb{R}$.

Show that f is continuous at $c \in \mathbb{R}$.

//
$|f(x) - f(y)| \leq k|x-y| < \varepsilon \Leftrightarrow |f(x) - f(y)| < \varepsilon/k$

Take $\delta(\varepsilon/k) > 0$ //

For any $\varepsilon > 0$, take $\delta(\varepsilon/k) > 0$ s.t. $|x-y| < \delta(\varepsilon/k)$. Since

$|f(x) - f(y)| \leq k|x-y| < \varepsilon \Leftrightarrow |f(x) - f(y)| < \varepsilon/k$

Thus, we conclude that for $c = y$ f is continuous at c.

12. Suppose $f: \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and that $f(r) = 0$, $\forall r \in \mathbb{Q}$.

Prove that $f(x) = 0$ $\forall x \in \mathbb{R}$

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D.T. If $\forall (x_m) \in \mathbb{Q} \Rightarrow \lim(x_m) = x$, $x \in \mathbb{R}$

$\lim(f(x_m)) = f(x) = 0$, $x \in \mathbb{R}$ //

Since $f:\mathbb{R}\to\mathbb{R}$ is continuous on $\mathbb{R}$ $\Rightarrow \exists f(c)\in\mathbb{R}$ s.t. $\lim\limits_{x\to c} f(x) = f(c)$. From the density theorem we have that if $\cancel{\forall}\ \exists\ (x_n)\in\mathbb{Q}$ $\cancel{\Rightarrow}$ s.t. $\lim(x_n)=x$, $x\in\mathbb{R}$. But $f(x_n)=0$, $\forall x_n\in\mathbb{Q}$ $\Rightarrow$ $\lim(x_n)=0$ $\Leftrightarrow$ $\lim(f(x_n)) =$
$= f(0) = 0$

$0 = \lim f(x_n) = f(\lim x_n) = f(x)$

13. $g(x) := \begin{cases} 2x, & x\in\mathbb{Q} \\ x+3, & x\in\mathbb{R}\setminus\mathbb{Q} \end{cases}$

Show where $g(x)$ is continuous

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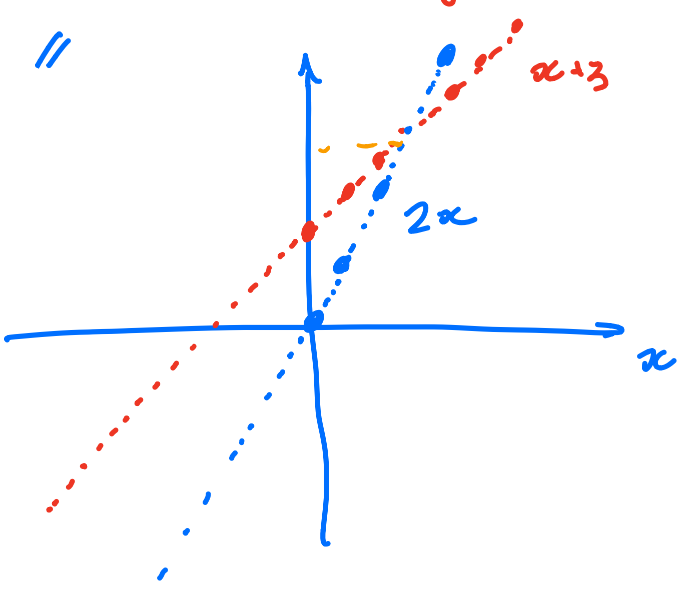

Claim: g is not continuous anywhere

Let $c\in\mathbb{Q}$, $\cancel{\forall}\ \exists\ (x_n)\in\mathbb{R}\setminus\mathbb{Q}$
$\lim(x_n) = c$. Since
$f(x_n) = x+3$, $\lim(f(x_n))$
$= x+3$ but $f(c) = 2c$

Let $g(x) := \begin{cases} 2x, & x \in \mathbb{Q} \\ x+3, & x \in \mathbb{Q}\setminus\mathbb{R} \end{cases}$

We claim that g is not continuous anywhere.

Density theorem

Indeed, let $c \in \mathbb{Q}$. For $\forall (x_n) \in \mathbb{R}\setminus\mathbb{Q}$, $\lim(x_n) = c$.
Since $g(x_n) = x_n + 3$, $\forall n \in \mathbb{N} \Rightarrow \lim(g(x_n)) = c+3$.
But $g(c) = 2c$, $\forall c \in \mathbb{Q}$. Thus, we conclude that g is not continuous on $\mathbb{Q}$.

The same reasoning applies to $c \in \mathbb{Q}\setminus\mathbb{R}$.