Lista 1

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1 A 11, 13 A 20 SEÇÃO 1.2 BARTLE (2011)

1. Prove that $\frac{1}{1.2}+\frac{1}{2.3}+\dots+\frac{1}{n(n+1)}=\frac{n}{(n+1)}$, for $\forall n\in\mathbb{N}$

Let $P(n)$ be the statement $\frac{1}{1.2}+\frac{1}{2.3}+\dots+\frac{1}{n(n+1)}=\frac{n}{(n+1)}$, $\forall n\in\mathbb{N}$.

If $n=1 \Rightarrow \frac{1}{1(1+1)}=\frac{1}{2} \Rightarrow P(1)$ is true.

Suppose $P(k)$ is true. For $P(k+1)$, we have:

$\frac{k+1}{k+2}$

$$P(k+1)=\frac{1}{1.2}+\frac{1}{2.3}+\dots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{(k+1)^{\cancel{2}}}{\cancel{(k+1)}(k+2)}=\frac{(k+1)}{[(k+1)+1]}$$

PMI

$\Rightarrow P(n)$ is true for $\forall n\in\mathbb{N}$.

2. Prove that $1^3+2^3+\dots+n^3=\left[\frac{1}{2}n(n+1)\right]^2$, $\forall n\in\mathbb{N}$

Let $P(n)=1^3+2^3+\dots+n^3=\left[\frac{1}{2}n(n+1)\right]^2$, $\forall n\in\mathbb{N}$.

For $n=1 \Rightarrow P(1)=1^3=\left[\frac{1}{2}.1.(1+1)\right]^2=1 \Rightarrow P(1)$ is true.

Suppose $P(k)$ is true for $\forall k\in\mathbb{N}$. For $P(k+1)$ we have:

$$P(k+1)=1^3+2^3+\dots+k^3+(k+1)^3=\left[\frac{1}{2}k(k+1)\right]^2+(k+1)^3$$

$$=\left[\frac{1}{2}(k^2+k)\right]^2+(k+1)^3=$$

$$\left[\frac{1}{2}(k+1)(k+2)\right]^2$$
$$=\left[\frac{1}{2}(k^2+2k+k+2)\right]^2$$
$$=\left[\frac{1}{2}(k^2+3k+2)\right]^2$$
$$=\left(\frac{1}{2}k^2+\frac{3}{2}k+1\right)^2$$

$$P(k+1)=1^3+2^3+\dots+k^3+(k+1)^3=\left[\frac{1}{2}k(k+1)\right]^2+(k+1)^3$$
$$=\left(\frac{1}{2}\right)^2k^2(k+1)^2+(k+1)^3=(k+1)^2\left[\left(\frac{1}{2}\right)^2k^2+k+1\right]=$$

3. Prove that $3+11+\dots+(8n-5)=4n^2-n$, $\forall n\in\mathbb{N}$ $\left[4(k+1)^2-(k+1)\right]$

$$P(k+1)=3+11+\dots+(8k-5)+[8(k+1)-5]=4k^2-k+[8(k+1)-5]$$
$$=4k^2-k+8k+8-5=4k^2-k+8k+3=\left[4(k+1)^2-(k+1)\right]$$

PMI

$\Rightarrow P(n)$ true for $\forall n\in\mathbb{N}$

4. Prove that $1^2+3^2+\dots+(2n-1)^2=\frac{(4n^3-n)}{3}$, $\forall n\in\mathbb{N}$ $\frac{4(k+1)^3-(k+1)}{3}$

$$P(k+1)=1^2+3^2+\dots+(2k-1)^2+[2(k+1)-1]^2=\frac{4k^3-k}{3}+[2(k+1)-1]^2$$
$$=\frac{4k^3-k+3[2k+2-1]^2}{3}=\frac{4k^3-k+3(2k+1)^2}{3}$$
$$=\frac{4k^3-k+3(4k^2+4k+1)}{3}=\frac{\cancel{4k^3}-\cancel{k}+\cancel{12k^2}+\cancel{12k}+\cancel{3}}{3}$$

$$4(k^2+2k+1)(k+1)=4(k^3+k^2+2k^2+2k+k+1)$$
$$=4(k^3+3k^2+3k+1)=4k^3+12k^2+12k+4$$

$$=\frac{4k^3+12k^2+12k+4-k-1}{3}=\frac{4(k+1)^3-(k+1)}{3}$$

PMI

$\Rightarrow P(n)$ is true, $\forall n\in\mathbb{N}$

5. Prove that $1^2-2^2+3^2+\dots+(-1)^{n+1}n^2=\frac{(-1)^{n+1}n(n+1)}{2}$, for $\forall n\in\mathbb{N}$

$$P(k+1)=1^2-2^2+3^2+\dots+(-1)^{k+1}k^2+(-1)^{(k+1)+1}(k+1)^2=\frac{(-1)^{k+1}k(k+1)}{2}+(-1)^{(k+1)+1}(k+1)^2$$

$$\frac{(-1)^{(k+1)+1}(k+1)[(k+1)+1]}{2}$$

$$=\frac{(-1)^{k+1}k(k+1)+2(-1)^{(k+1)+1}(k+1)^2}{2}=\frac{(-1)^{(k+1)+1}(k+1)[(-1)^{-1}k+2(k+1)]}{2}$$

$$=\frac{(-1)^{(k+1)+1}(k+1)[-k+2k+2]}{2}=\frac{(-1)^{(k+1)+1}(k+1)[(k+1)+1]}{2}$$

PMI

$\Rightarrow$ $P(n)$ is true for $\forall n\in\mathbb{N}$

6. Prove that n^3+5n is divisible by 6 for all $n\in\mathbb{N}$

Let $P(n)=n^3+5n=3j$, $\forall n,j\in\mathbb{N}$

Given $n=1 \Rightarrow [1+5(1)]\frac{1}{3}=2$, which is divisible by 6 $\Rightarrow P(1)$ is true for $n=1$.

Suppose $P(k)$ is true, for $\forall k\in\mathbb{N}$. Given $P(k+1)$, we have:

$$P(k+1)=(k+1)^3+5(k+1)=(k^2+2k+1)(k+1)+5k+5=\cancel{k^3}+\cancel{k^2}+\cancel{2k^2}+\cancel{2k}+\cancel{k}+\cancel{1}+\cancel{5k}+\cancel{5}$$
$$=k^3+3k^2+8k+6=k^3+5k+3k^2+3k+6=6n+3k^2+3k+6$$

$P(k)=k^3+5k=3n$, $\forall n\in\mathbb{N}$

$$=3(2n+k^2+k+2)=3J, \forall J\in\mathbb{N}$$

PMI

$\Rightarrow$ $P(n)$ is true, for $\forall n\in\mathbb{N}$

7. Prove that $5^{2n}-1$ is divisible by 8 for $\forall n\in\mathbb{N}$

$P(k)=5^{2k}-1=8n$, for $\forall k,n\in\mathbb{N}$.

$$P(k+1)=5^{2(k+1)}-1=5^{2k+2}-1=5^{2k}.5^2-1=25(5^{2k})-1$$
$$=5^{2k}(24+1)-1=24(5^{2k})+5^{2k}-1=24(5^{2k})+8n=8[3(5^{2k})+n]$$
$$=8J, \text{ for } \forall J\in\mathbb{N}$$

PMI

$\Rightarrow$ $P(k)$ is true for $\forall n\in\mathbb{N}$

8. Prove that 5^n-4n-1 is divisible by 16 for $n\in\mathbb{N}$

$P(k)=5^k-4n-1=16n$, $\forall k,n\in\mathbb{N}$.

$$P(k+1)=5^{k+1}-4(k+1)-1=5^k5^1-4k-4-1=5^k(4+1)-4k-4-1$$
$$=4(5^k)+5^k-4k-4-1=16n+4(5^k)-4=4[5^k+4n-1]$$

$=$

9. Prove that $n^3+(n+1)^3+(n+2)^3$ is divisible by 9 for $\forall n\in\mathbb{N}$

$P(k)=k^3+(k+1)^3+(k+2)^3=9n$, $\forall k,n\in\mathbb{N}$.

$$P(k+1)=(k+1)^3+[(k+1)+1]^3+[(k+1)+2]^3=$$

13. Prove that $n<2^n$ for $\forall n\in\mathbb{N}$

Let $P(n)$ be the statement $n<2^n$. For $n=1 \Rightarrow 1<2 \Rightarrow n<2^n$ is true. Suppose $k<2^k$, $\forall k\in\mathbb{N}$. For $P(k+1)$ we have that $k+1<2^{k+1}$. We know that:

$k<2^k \Leftrightarrow 2k<2^{k+1}$ and for $\forall k\in\mathbb{N}$, $2k>k+1$

$\Rightarrow k+1<2k<2^{k+1} \Rightarrow k+1<2^{k+1}$

PMI

$\Rightarrow$ $P(n)$ is true for $\forall n\in\mathbb{N}$.

14. Prove that $2^n<n!$ for all $n\geq 4$, $n\in\mathbb{N}$

Let $P(n)$ be the statement $2^n<n!$ for $\forall n\geq 4$, $n\in\mathbb{N}$. Given $n=4$ $\Rightarrow 2^4<4! \Leftrightarrow 16<24 \Rightarrow P(1)$ is true. Suppose $P(k)$ is true for $\forall k\geq 4$, $k\in\mathbb{N}$. Given $P(k+1)$, we have that $2^{k+1}<(k+1)!$. We know that:

$2^k<k! \Leftrightarrow 2^{k+1}<2(k!)$ and, for $\forall n\geq 4$, $n\in\mathbb{N} \Rightarrow (k+1)!>2k!$

$\Rightarrow (k+1)!>2k!>2^{k+1} \Rightarrow (k+1)!>2^{k+1}$

PMI

$\Rightarrow$ $P(n)$ is true for $\forall n\geq 4$, $n\in\mathbb{N}$.