Lista 0

sábado, 16 de janeiro de 2021 09:10

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4. Prove the second De Morgan law

Theorem: If A, B, C are sets, then (b) $A\setminus(B\cap C) = (A\setminus B)\cup(A\setminus C)$

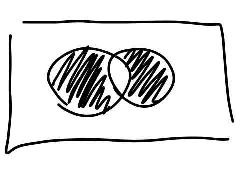
(i) $A\setminus(B\cap C) \subset (A\setminus B)\cup(A\setminus C)$

For $\forall x$, if $x \in A\setminus(B\cap C) = (B^c\cup C^c) \Rightarrow x \notin B$ or $x \notin C$ or $x \notin B$ e $x \notin C \Rightarrow$
$\Rightarrow x \in B^c = A\setminus B$ or $x \in C^c = A\setminus C$ or $x \in B^c$ and $x \in C^c$. Therefore, we have that, for $\forall x$, $x \in A\setminus(B\cap C) \Rightarrow x \in (A\setminus B)\cup(A\setminus C)$

(ii) $(A\setminus B)\cup(A\setminus C) \subset A\setminus(B\cap C)$

For $\forall x$, if $x \in (A\setminus B)\cup(A\setminus C) \Rightarrow x \in A\setminus B = B^c$ or $x \in A\setminus C = C^c$ or $x \in B^c$ and $x \in C^c \Rightarrow x \in B^c\cup C^c$. Therefore, for $\forall x$, $x \in (A\setminus B)\cup(A\setminus C) \Rightarrow x \in A\setminus(B\cap C)$.

6. Symmetric difference: $A \Delta B = (A-B)\cup(B-A) = D$



(a) Show that $D = (A\setminus B)\cup(B\setminus A)$

We know that D is the set of all elements that belong to either A or B but not both.

(i) $D \subset (A\setminus B)\cup(B\setminus A)$ $A\setminus B = B^c$ $B\setminus A = A^c \Rightarrow (A\setminus B)\cup(B\setminus A) = B^c \cup A^c$

If, for $\forall x$, $x \in D \Rightarrow x \in A$ or $x \in B$ or $x \notin A\cap B \Rightarrow x \in A-(A\cap B)$ or $x \in B-(A\cap B)$ or $x \notin A\cap B$. Therefore, for $\forall x$, if $x \in D \Rightarrow x \in (A\setminus B)\cup(B\setminus A)$

(ii) $(A\setminus B)\cup(B\setminus A) \subset D$

If, for $\forall x$, $x \in (A\setminus B)\cup(B\setminus A) \Rightarrow x \in (A\setminus B)$ or $x \in (B\setminus A)$ or $x \in (A\setminus B)$ and $x \in (B\setminus A)$ $\Rightarrow x \in A-(A\cap B)$ or $x \in B-(A\cap B)$ or $x \notin A\cap B$. Therefore, for $\forall x$, if $x \in (A\setminus B)\cup(B\setminus A) \Rightarrow x \in D$

(b) Show that D is also given by $D = (A\cup B)\setminus(A\cap B)$

$(A\cup B)\setminus(A\cap B) = A\cup B - [(A\cup B)\cap(A\cap B)] = (A\cup B)-(A\cap B) = (A\setminus B)\cup(B\setminus A)$

7. For each $m \in \mathbb{N}$, let $A_m = \{(m+1)k : k \in \mathbb{N}\}$

(a) What is $A_1 \cap A_2$

$A_1 = \{(1+1)k : k \in \mathbb{N}\} = \{2k : k \in \mathbb{N}\}$: set of the even numbers

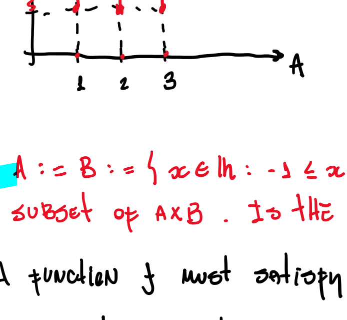
$A_2 = \{(2+1)k : k \in \mathbb{N}\} = \{3k : k \in \mathbb{N}\}$: set of numbers divisible by 3

$A_1 \cap A_2 = \{6k : k \in \mathbb{N}\}$

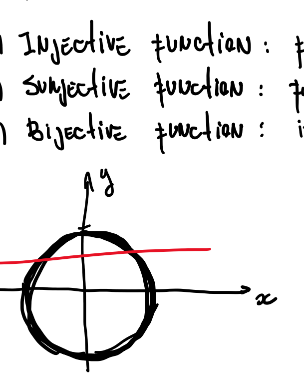
(b) Determine the sets $\cup\{A_m : m \in \mathbb{N}\}$ and $\cap\{A_m : m \in \mathbb{N}\}$

8. Draw diagrams in the plane of the cartesian products $A \times B$ for the given sets A and B

(a) $A = \{x \in \mathbb{R} : 1 \le x \le 2$ or $3 \le x \le 4\}$ $B = \{x \in \mathbb{R} : x = 1$ or $x = 2\}$



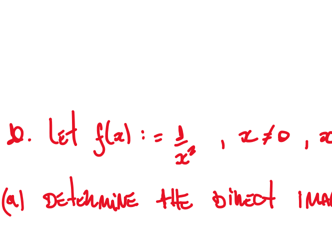
(b) $A = \{1, 2, 3\}$ $B = \{x \in \mathbb{R} : 1 \le x \le 3\}$



9. $A := B := \{x \in \mathbb{R} : -1 \le x \le 1\}$, $C := \{(x,y) : x^2+y^2 = 1\}$, which is a subset of $A \times B$. Is the set C a function?

A function f must satisfy one of the definitions:

a) Injective function: for $\forall x_1, x_2$, if $x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$
b) Surjective function: for $\forall A, B$, $f(A) = B$, that is, $R(f) = B$
c) Bijective function: its a function that is both injective and surjective



$y = \frac{1}{2} \Rightarrow x = \pm\sqrt{\frac{1}{2}}$

$\nvdash$ surjective

$\forall b \in B, \exists x \in A$ s.t. $f(x) = b$

$f(x,y) = x^2+y^2 = 1$

$D(f) = \{(x,y) : x, y \in \mathbb{R}\}$

$R(f) = \{1\}$

$\forall b \in R(f), \exists x, y \in D(f)$ s.t. $f(x,y) = b$

$x = 1$ and $y = 0 \Rightarrow f(x,y) = 1$ $\therefore f$ is surjective

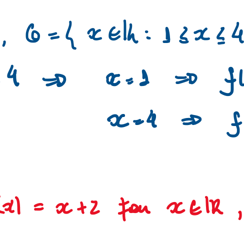
10. Let $f(x) := \frac{1}{x^2}$, $x \neq 0$, $x \in \mathbb{R}$

(a) Determine the direct image $f(E)$ where $E := \{x \in \mathbb{R} : 1 \le x \le 2\}$

If $E \subseteq A$, then the direct image of E under f is the subset $f(E) = \{f(x) : x \in E\}$.

We have that $E \subseteq \mathbb{R}$, then, for $E := \{x \in \mathbb{R}, x \neq 0 : 1 \le x \le 2\}$, the direct image of E under f is the subset $f(E) = \{f(x) = \frac{1}{x^2} : 1 \le x \le 2, x \in \mathbb{R}\}$

$x = 1 \Rightarrow f(x) = 1$
$x = 2 \Rightarrow f(x) = \frac{1}{2}$



(b) Determine the inverse image $f^{-1}(G)$ where $G := \{x \in \mathbb{R} : 1 \le x \le 4\}$

If $H \subseteq B$, then the inverse image of A under f is the subset $f^{-1}(H) = \{x \in A : f(x) \in H\}$.

We have that $G \subseteq R(f)$, then the inverse image of G under f is the subset $f^{-1}(G) = \{x \in \mathbb{R}, x \neq 0 : f(x) \in G\}$

$x \in \mathbb{R}, x \neq 0 : f(x) \in G$, $G = \{x \in \mathbb{R} : 1 \le x \le 4\}$
$f(x) \in G \Leftrightarrow 1 \le x \le 4 \Rightarrow$ $x = 1 \Rightarrow f(x) = 1$
$x = 4 \Rightarrow f(x) = \frac{1}{4}$

11. Let $g(x) = x^2$ and $f(x) = x+2$ for $x \in \mathbb{R}$, and let $h = g \circ f$

$D(g) = D(f) = \mathbb{R}$

$\left.\begin{array}{l} R(g) = \{y \in \mathbb{R} : 0 \le y < \infty\} \\ R(f) = \{y \in \mathbb{R} : -\infty < y < \infty\} \end{array}\right\} \Rightarrow R(g) \subseteq R(f)$

$\Rightarrow R(f) \subseteq D(g)$

Let $A = D(f) = \mathbb{R}$, $B = R(f) = \mathbb{R} \subseteq D(g) = \mathbb{R}$, and let $C = R(g) = \mathbb{R}^+$, then $h = g \circ f : \mathbb{R} \to \mathbb{R}^+$

(a) Find the direct image $h(E)$ of $E = \{x \in \mathbb{R} : 0 \le x \le 1\}$

$E \subseteq D(f) = \mathbb{R} \Rightarrow h(E) \subseteq R(f)$, $E = \{x \in \mathbb{R} : 0 \le x \le 1\}$

$h(E) : E \to \mathbb{R}^+$

$h(x) = g(f(x)) = (x+2)^2 \Rightarrow$ $x = 0 \Rightarrow h(x) = 4$
$x = 1 \Rightarrow h(x) = 9$

$\therefore h(E) = \{y \in \mathbb{R}^+ : 4 \le y \le 9\}$

(b) Find the inverse image $h^{-1}(G)$ of $G = \{x \in \mathbb{R} : 0 \le x \le 4\}$

$h^{-1}(G) \subseteq D(h) = D(f) = \mathbb{R} \Rightarrow h^{-1}(G) : G \to \mathbb{R}$

$h(x) = (x+2)^2 \Leftrightarrow \sqrt{y} = x+2 \Leftrightarrow x = \sqrt{y}-2 \Rightarrow h^{-1}(y) = \sqrt{y}-2$

Let $G = \{y \in \mathbb{R} : 0 \le y \le 4\} \Rightarrow$ $y = 0 \Rightarrow h^{-1}(y) = -2$
$y = 4 \Rightarrow h^{-1}(y) = 0$

$\therefore h^{-1}(G) = \{x \in G : -2 \le x \le 0\}$

12. Let $f(x) = x^2$, $x \in \mathbb{R}$, and let $E = \{x \in \mathbb{R} : -1 \le x \le 0\}$ and $F = \{x \in \mathbb{R} : 0 \le x \le 1\}$. Show that:

(a) $E \cap F = \{0\}$

$\forall x \in \mathbb{R}$, if $x \in E \Rightarrow -1 \le x \le 0$. Furthermore, if $x \in F \Rightarrow 0 \le x \le 1$. Therefore, if $x \in E \cap F \Rightarrow x = 0$.

(b) $f(E \cap F) = \{0\}$

Let $f(E \cap F)$ be a function s.t. $f : E \cap F \to \mathbb{R}$. Since $E \cap F = \{0\}$, then $f(0) = 0$. Therefore, $f(E \cap F) = \{0\}$.

(c) $f(E) = f(F) = \{y \in \mathbb{R} : 0 \le y \le 1\}$

We know that $E = \{x \in \mathbb{R} : -1 \le x \le 0\}$. If, $x = -1 \Rightarrow f(x) = 1$ and $x = 0 \Rightarrow f(x) = 0$. Therefore, $f(E) = \{y \in \mathbb{R} : 0 \le y \le 1\}$

We also know that $F = \{x \in \mathbb{R} : 0 \le x \le 1\}$. If $x = 0 \Rightarrow f(x) = 0$ and $x = 1 \Rightarrow f(x) = 1$. Therefore, $f(F) = \{y \in \mathbb{R} : 0 \le y \le 1\}$

$\therefore f(E) = f(F)$

(d) $f(E \cap F)$ is a proper subset of $f(E) \cap f(F)$

We know that, for any two sets A and B, we say that A is a proper subset of B if $A \subset B$ but $\exists b \in B$ s.t. $b \notin A$.

Since $f(E) \cap f(F) = \{y \in \mathbb{R} : 0 \le y \le 1\}$ and $f(E \cap F) = \{0\}$. Choose $y = 1$. We know that $y \in f(E) \cap f(F)$ but $y \in f(E \cap F)$. Therefore, $f(E \cap F) \subset f(E) \cap f(F)$.

16. Show that the function $f(x) = x/\sqrt{x^2+1}$, $x \in \mathbb{R}$, is a bijection of $\mathbb{R}$ onto $\{y : -1 < y < 1\}$

We know that a function f is said to be a bijection if its injective and surjective. Let $B = \{y : -1 < y < 1\}$.

(i) injective

For $\forall x_1, x_2 \in \mathbb{R}$, take $x_1 = x_2$ and assume $f(x_1) = f(x_2)$. Then:

$\frac{x_1}{\sqrt{x_1^2+1}} = \frac{x_2}{\sqrt{x_2^2+1}} \Leftrightarrow x_1\sqrt{x_2^2+1} = x_2\sqrt{x_1^2+1} \Leftrightarrow x_1^2(x_2^2+1) = x_2^2(x_1^2+1)$

$\Leftrightarrow \cancel{x_1^2x_2^2} + x_1^2 = \cancel{x_2^2x_1^2} + x_2^2 \Leftrightarrow x_1 = x_2$

$\therefore f$ is injective

(ii) Bijection

We know that a function has an inverse $\Leftrightarrow$ its bijective.

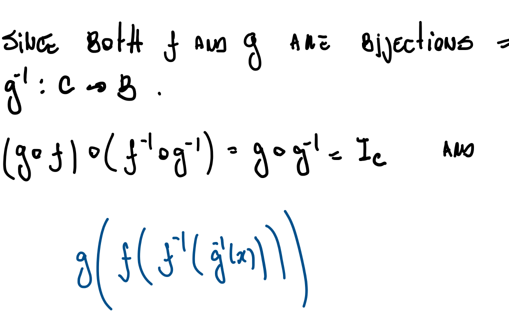
$y = \frac{x}{\sqrt{x^2+1}} \Leftrightarrow y\sqrt{x^2+1} = x \Leftrightarrow y^2(x^2+1) = x^2 \Leftrightarrow \frac{x^2+1}{x^2} = \frac{1}{y^2}$

$\Leftrightarrow 1 + \frac{1}{x^2} = \frac{1}{y^2} \Leftrightarrow \frac{1}{x^2} = \frac{1-y^2}{y^2} \Leftrightarrow x^2 = \frac{y^2}{1-y^2} \Leftrightarrow x = \sqrt{\frac{y^2}{1-y^2}}$

$y \neq 1 \Rightarrow R(f) = \{y \in \mathbb{R} : y \neq 1\}$

17. For $a, b \in \mathbb{R}$ s.t. $a < b$, find a bijection of $A = \{x : a < x < b\}$ onto $B = \{y : 0 < y < 1\}$

An invertible function $f : A \to B$



18.

21. Prove that if $f : A \to B$ is bijective and $g : B \to C$ is bijective $\Rightarrow$ $g \circ f : A \to C$ is bijective

Since both f and g are bijections $\Rightarrow$ there exists $f^{-1} : B \to A$ and $g^{-1} : C \to B$.

$(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ g^{-1} = I_C$ and $(f^{-1} \circ g^{-1}) \circ (g \circ f) = I_A$

$g(f(f^{-1}(g^{-1}(x))))$

$\therefore (g \circ f)$ has an inverse and thus is bijective.

$(g \circ f)(x)$

22. Let $f : A \to B$ and $g : B \to C$

(a) Show that if $g \circ f$ is injective $\Rightarrow f$ is injective

If $g \circ f$ is injective $\Rightarrow \forall x_1, x_2$, if $x_1 = x_2 \Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2)$
$\Leftrightarrow g(f(x_1)) = g(f(x_2))$. Suppose that f is surjective $\Rightarrow f(A) = B \Rightarrow$
$\Rightarrow g(A) = B$, which is false, because, by hypothesis, $g(x)$ is injective

$\therefore$ if $g \circ f$ is injective $\Rightarrow f$ is injective

(b) Show that if $g \circ f$ is surjective $\Rightarrow g$ is surjective

If $g \circ f : A \to C$ is surjective $\Rightarrow \exists (g \circ f)^{-1}$, $(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A)) \Rightarrow \exists g^{-1}$.

$\therefore g$ is surjective