5.4. Uniform Continuity

- Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$. Def. 5.1.1 states that the following statements are equivalent:

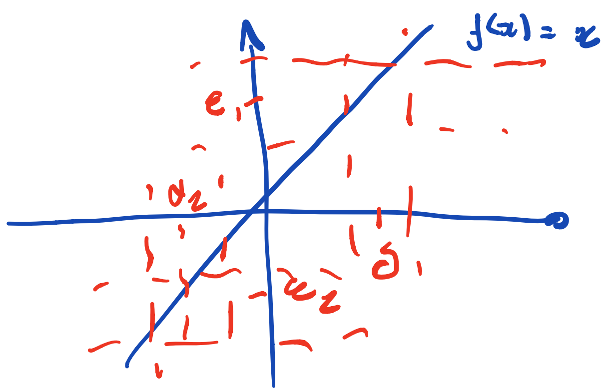
(i) f is continuous on $\forall u \in A$

(ii) Given $\varepsilon > 0$ and $u \in A$, $\exists\, \delta(\varepsilon, u) > 0$ s.t. $\forall x \in A$ and $|x - u| < \delta(\varepsilon, u)$, then $|f(x) - f(u)| < \varepsilon$

- The point the authors wish to emphasize is that δ depends, in general, on both $\varepsilon > 0$ and $u \in A$

↳ f may change rapidly or slowly near certain points

Def 5.4.1 Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$. We say that f is uniformly continuous on A if $\forall \varepsilon > 0\ \exists\, \delta(\varepsilon) > 0$ s.t. if $x, u \in A$ are any numbers satisfying $|x - u| < \delta(\varepsilon)$, then $|f(x) - f(u)| < \varepsilon$



- Note that if f is uniformly continuous $\Rightarrow$ it is continuous at every point of A, but the converse is not true in general $\rightarrow$ $f(x) = 1/x$ on $A = \{x \in \mathbb{R}$ s.t. $x > 0\}$

Nonuniform continuity criteria Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$. Then the following statements are equivalent:

(i) f is not uniformly continuous on A

(ii) $\exists\, \varepsilon_0 > 0$ s.t. $\forall\, \delta > 0$ $\exists\, x_\delta, u_\delta \in A$ s.t. $|x_\delta - u_\delta| < \delta$ and $|f(x_\delta) - f(u_\delta)| \geq \varepsilon_0$

(iii) $\exists\, \varepsilon_0 > 0$ and two sequences (x_n) and (u_n) in A s.t. $\lim(x_n - u_n) = 0$ and $|f(x_n) - f(u_n)| \geq \varepsilon_0$ for $\forall n \in \mathbb{N}$

Uniform continuity Theorem Let I be a closed bounded interval and let $f: I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.

Proof

If f is not uniformly continuous on I, then, by the nonuniform continuity criteria, $\exists\, \varepsilon_0 > 0$ and two sequences (x_n) and (u_n) in I s.t. $|x_n - u_n| < 1/n$ and $|f(x_n) - f(u_n)| \geq \varepsilon_0$ for $\forall n \in \mathbb{N}$. Since I is bounded, the sequence (x_n) is bounded; by the Bolzano-Weierstrass theorem $\exists\, (x_{n_k})$ subsequence of (x_n) that converges to z.

Since I is closed, the limit $z \in I$ by Theorem 3.2.6.
Furthermore, since

Triangular ineq.

$$|u_{n_k} - z| = |u_{n_k} - x_{n_k} + x_{n_k} - z| \le |u_{n_k} - x_{n_k}| + |x_{n_k} - z|$$

then (u_{n_k}) also converges to z.

Now if f is continuous at z, then both the sequences $(f(x_{n_k}))$ and $(f(u_{n_k}))$ must converge to $f(z)$, but this is not possible since $|f(x_n) - f(u_n)| \ge \varepsilon_0 \ \forall n \in \mathbb{N}$.
Thus we conclude that our original hypothesis is false.

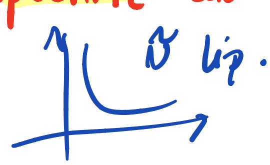
Lipschitz functions

- If a uniformly continuous function is given on a set that is not closed bounded interval, then its difficult in general to establish its uniform continuity
- The following def. establishes a condition for this to happen

$\left|\dfrac{f(x)-f(u)}{x-u}\right| \le K$ coeficiente de Newton

Def 5.4.4 Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. If $\exists K > 0$ s.t. $|f(x) - f(u)| \le K|x-u| \ \forall x, u \in A$, then f is said to be a Lipschitz function (or to satisfy the Lipschitz condition) on A.

Geometrical Interpretation (see...)



Theorem 5.4.5 If $f: A \to \mathbb{R}$ satisfy the Lipschitz condition, then f is uniformly continuous on A.

Proof não precisa ser p/ todo $\varepsilon \geq 0$

If $f: A \to \mathbb{R}$ satisfy the Lipschitz condition, then given $\varepsilon > 0$ we can take $\delta(\varepsilon) := \varepsilon / k$. If $x, u \in A$ satisfy $|x-u| < \delta \Rightarrow |f(x) - f(u)| < k \cdot \frac{\varepsilon}{k} = \varepsilon$.

Thus f is uniformly continuous on A.

Examples

a) $f(x) := x^2$, $f: A \to \mathbb{R}$ s.t. $A = [0, b]$, $b > 0$

$$|f(x) - f(u)| = |x^2 - u^2| = |x+u||x-u|$$

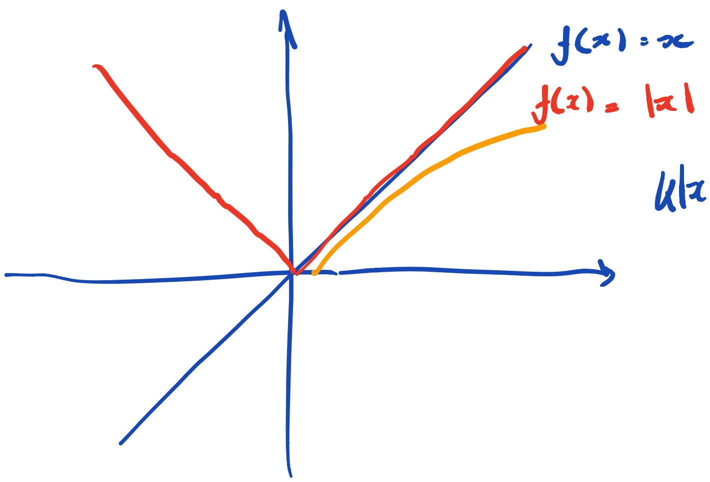
Since $x, u \in [0, b] \Rightarrow |f(x) - f(u)| = |x+u||x-u| \leq 2b|x-u|$

$\Rightarrow f$ satisfy the Lipschitz condition for $k := 2b$. Thus f is uniformly continuous.

b) Not every uniformly continuous function is a Lipschitz function

Let $g(x) := \sqrt{x}$ s.t. $x \in [0, 2]$. Since g is continuous on $I := [0, 2]$, it follows from the Uniformly Continuity Theorem that g is uniformly continuous on I. However, $\nexists\, k > 0$ s.t. $|g(x)| \leq k|x|$ $\forall x \in I$. Thus g is not a Lipschitz function.

// $|g(x)| \leq k|x| \Leftrightarrow |\sqrt{x}| \leq k|x| \Leftrightarrow -k|x| \leq \sqrt{x} \leq k|x|$



$k|x| \geq |x| \geq |\sqrt{x}|$

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THE CONTINUOUS FUNCTION THEOREM

- We have seen that:

continuous functions
(1) on closed bounded intervals $\Rightarrow$ $\Rightarrow$ uniformly continuous U.C.T.
(2) on open bounded intervals may or may not be uniformly continuous

- We will see under what conditions (2) is true

Theorem 5.4.7 If $f: A \to \mathbb{R}$ is uniformly continuous on a subset A of $\mathbb{R}$ and if (x_n) is Cauchy sequence in A, then $(f(x_n))$ is a Cauchy sequence in $\mathbb{R}$

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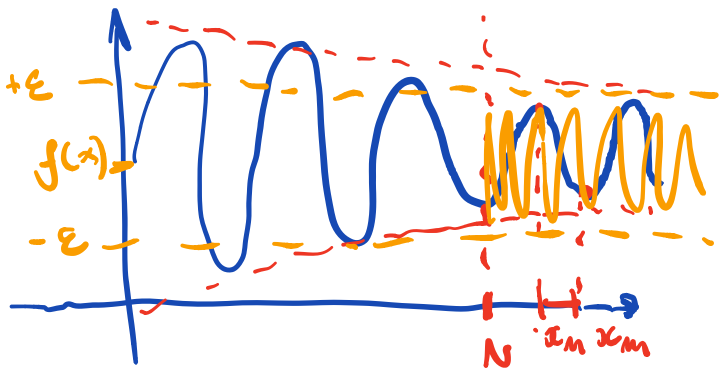
Cauchy sequence A sequence $X = (x_n)$ s.t. $x_n \in \mathbb{R}$ for $n \in \mathbb{N}$ is called a Cauchy sequence if for $\forall \varepsilon > 0$, $\exists N > 0$, $N \in \mathbb{Z}$, s.t. for $\forall m, n > N$ $|x_m - x_n| < \varepsilon$



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Proof

Let (x_n) be a Cauchy sequence in A and let $\varepsilon > 0$ be given. First, choose $\delta > 0$ s.t. if $x, u \in A$ satisfy $|x-u| < \delta \Rightarrow |f(x)-f(u)| < \varepsilon$. Since (x_n) is a Cauchy sequence, $\exists H(\delta)$ s.t. $|x_n - x_m| < \delta$ for $\forall n, m > H(\delta)$. By choice of $\delta \Rightarrow$ for $m, n > H(\delta)$, $|f(x_n) - f(x_m)| < \varepsilon$. Therefore, the sequence $(f(x_n))$ is a Cauchy sequence.

- The preceding result give us an alternative way of seeing that $f(x) := 1/x$, $x \in]0,1[$ is not uniformly continuous by noting that $X = (\frac{1}{n})$ in $]0,1[$ is a Cauchy sequence, but the image sequence $F = (f(x_n)) = (n)$ is not a Cauchy sequence ($\Rightarrow f$ is not uniformly continuous on $]0,1[$)

$f(x) = 1/x$ and $x_n = 1/n \Rightarrow f(x_n) = n$

Continuous Extension Theorem A function f is uniformly continuous on the interval $]a,b[$ $\Leftrightarrow$ it can be defined at the end points a and b s.t. the extended function on $[a,b]$ is continuous.