5.1 Capital Asset Pricing Model

- One of the most important theorems in modern financial economics is the **two-fund** theorem and its expected return
- CAPM says the best fund is combo this tangentialy of risk free the return for account
- The main implication of the CAPM is that the expected return on item is not based on its own variance relative to the economy of the between risk free return or risk premium

5.1 Review of the CAPM

- Markowitz (1959) lays the groundwork for the CAPM
- Begins an investors portfolio selection problem in terms of expected return and variance of return
- An investor that optimizes an investor chooses this the mean-variance efficient portfolio, which is the portfolio that has the lowest expected return for a certain level of variance
- Sharpe (1964) and Lintner (1965) derive an analytic form to create standard asset pricing
- Starts by making some assumptions:
  - a homogeneous expectations
  - a risk-free asset investing
  - All asset bids the mean-variance portfolio
  - portfolio of all risky assets (market portfolio) will study be a mean-variance efficient portfolio
  - so CAPM
- Strong and linear implications of the CAPM theorem:
  - the existence of linear the relationship of a particular risk or between
  - the relationship holds in non-stochastic
- In this case, the expected return on asset i is given by:

$$E[R_i] = R_f + \beta_{im}(E[R_m] - R_f) \quad (1a)$$

$$= R_f + \frac{\mathrm{Cov}(R_i, R_m)}{\mathrm{Var}(R_m)}(E[R_m] - R_f) \quad (1b)$$

  - R_m = return on the market portfolio
  - R_f = return on the riskfree asset
- We can also express equation (1a) in terms of return in excess, that is, $E_i = R_i - R_f$:

$$E[E_i] = \beta_{im} E[E_m] \Rightarrow \frac{\mathrm{Cov}(E_i, E_m)}{\mathrm{Var}(E_m)} \quad (2)$$

  - E_m = excess return on the market portfolio or excess.
- Note that we are only talk to risk equation (2) because no risk-free asset is non-stochastic
- Empirical tests of Sharpe-Lintner CAPM focus on testing these implications of equation (2)
  - The intercept is zero
  - has complete relation the relationship be the cross-section of returns
  - Market risk premium $E[E_m]$ is positive
- In the economy as a whole the market stock (1973) derives a more general version of the CAPM
- In this version, no risk-free asset exists and a zero-beta return to a linear function of the form:

$$E[R_i] = E[R_{zm}] + \beta_{im}(E[R_m] - E[R_{zm}]) \quad (3)$$

  - $\beta_{im} = \mathrm{Cov}(R_i, R_m) / \mathrm{Var}(R_m)$
  - where z denotes the asset m is random the this return $(r_{1m}, r_{2m}, \ldots, r_{Nm})$ are the residuals excess returns on time period t the asset i and the market portfolio, respectively.
- This, and CAPM typically refers to a proxy for the market portfolio and the its stock bid pursue the inequality return

5.2 Results from empirical tests of Sharpe-Lintner

- Intuition
  - $\tilde{R}_i$ = random variable
  - $\tilde{R}_m$ = random variable of the of market
  - σ_{im} = covariance between the the of market
  - σ_m^2 = variance of the market
  - $\beta_{im} = \sigma_{im}^2 / \sigma_m^2$ = measure of portfolio i relative
  - $\sigma_m = \sigma_m^2 / \sigma_m$ = measure of portfolio m relative
  - $\sigma_{im} = \sigma_{im}^2 / \sigma_m$ = covariance between one item portfolio m and
- Assumptions:
  - bid stock investment, instead
  - the following of bid two assets exposure
- **(1)** (mean-variance portfolio) Portfolio p is the mean-variance portfolio of all portfolios, while mean return μ_p is the portfolio variance equal to the combination of the portfolio two between return return portfolio

$$\min_{\tilde{R}_p} \sigma_p^2 = \sigma_p^2 \quad (4a)$$

$$\text{s.t.} \quad E[\tilde{R}_p] = \mu_p \quad (4b)$$

$$E[\tilde{R}_i] = \mu_i \quad (4c)$$

- The above formulation and the above means the Lagrangian:

$$L = \sigma_p^2 + \lambda_0 (\mu_p - E[\tilde{R}_p]) - \lambda_i (\mu_i - E[\tilde{R}_i])$$

$$\frac{\partial L}{\partial \tilde{R}_i} = 2\sum_{j=1}^{N} \lambda_i \tilde{R}_j - \lambda_0 \bar{R}_i = 0$$

$$\Rightarrow \tilde{R}_i = \frac{1}{2}(\lambda_0 \bar{R}_i + \lambda_i) \sum^{-1}$$

$$\Rightarrow \tilde{R}_p = \frac{1}{2}\lambda_0 \bar{R} + \frac{1}{2}\lambda_i \bar{R}$$

$$\tilde{\gamma}_0 = \frac{1}{2}\lambda_0 \left[b\,(\sigma^2 I) - a\,(\sigma^2 \bar{R}) \right]$$

$$\tilde{\gamma}_1 = \frac{1}{2}\lambda_1 \left[c\,(\sigma^2 \bar{R}) - a\,(\sigma^2 I) \right]$$

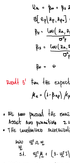
$$a = \bar{R}\,\sigma^2 \bar{R} \quad b = \bar{R}^T \sigma^2 \bar{R} \quad c = I^T \sigma^2 I \quad D = bc - a^2$$

- Result 1: The mean-variance frontier can be constructed from any two distinct mean-variance portfolios.
- Result 2: Any portfolio of mean-variance portfolios is also a mean-variance portfolio
- Result 3: Define portfolio g as the global mean-variance portfolio. Let portfolio g and have

$$\mu_g = \frac{b}{c} \quad \sigma_g^2 = \frac{1}{c}$$

- Result 4: Let each mean-variance portfolio p, except for the global mean-variance portfolio g, is a unique mean-variance portfolio that has zero covariance with g. This portfolio is called the zero-beta portfolio with respect to g.
- Result 5: The covariance of the return on this global mean-variance portfolio g with any asset or portfolio q satisfies a

$$\mathrm{Cov}(\tilde{r}_q, \tilde{r}_g) = \frac{1}{c}$$



- Result 6: Consider a portfolio benchmark $\tilde{r}_b$ the return on any asset or portfolio $\tilde{r}_q$ (except for the global mean-variance) has the balance of the relationship from their portfolio $\tilde{r}_b$:

$$\tilde{r}_q = \tilde{r}_b + \beta_{qb}\,\tilde{r}_b + \tilde{\varepsilon}_q \quad \text{s.t.}$$

$$E[\tilde{\varepsilon}_q] = E[\tilde{r}_b \tilde{\varepsilon}_q] = 0$$

$$\beta_{qb} = \frac{\mathrm{Cov}(\tilde{r}_q, \tilde{r}_b)}{\sigma_b^2} = \frac{\beta_{qg}}{\sigma_b^2}$$

$$\beta_{qg} = \frac{\mathrm{Cov}(\tilde{r}_q, \tilde{r}_g)}{\sigma_g^2} = \frac{1}{c}\,\beta_{qg}$$

$$\beta_{qb} = 0$$

- Result 7: For the expected return of a risk asset:

$$\mu_q = (1-\beta_{qb})\,\mu_b + \beta_{qg}\,\mu_g$$

- We now proceed the cross-sections but assume a non-zero risk-free portfolio $\tilde{r}_f$ or μ_f
- The constraints maximization problem becomes:

$$\min_{\tilde{R}_p} \sigma_p^2 = \sigma_p^2$$

$$\text{s.t.} \quad E[\tilde{R}_p] = (1-E[\tilde{R}_f])\,\mu_f = \mu_p$$

- Result 1a: Given a benchmark portfolio:

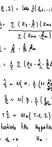
$$L = \sigma_p^2 + \lambda_0 (\mu_p - E[\tilde{R}_p]) - \lambda_i (1-E[\tilde{R}_f])$$

$$\frac{\partial L}{\partial \tilde{R}_i} = 2\sum_{j=1}^{N} \lambda_i \tilde{R}_j - \lambda_0 \bar{R}_i = 0$$

$$\Rightarrow \tilde{R}_i = \frac{(\mu_p - \lambda_0)}{(\bar{R}-\lambda_0 I)} \frac{\sigma^2 (\bar{R} - \lambda_0 I)}{\sigma^2 (\bar{R} - \lambda_0 I)}$$

- Then with a benchmark model of mean-variance portfolio are a combination of a risk-free asset portfolio and the tangent portfolio

$$\tilde{R}_p = \frac{\mu_p - \mu_f}{\sigma^2 (\bar{R} - \lambda_0 I)} \sigma^2 (\bar{R} - \lambda_0 I)$$



- The tangency portfolio is called the tangency portfolio since the maximum slope point of all portfolios of risky stocks μ_p / σ_p is maximized

$$\mu_p / \sigma_p = (\mu_p - \mu_f) / \sigma_p = \mu_p / \sigma_p$$

- Result 1b: Given a benchmark portfolio $\tilde{r}_b$ the return on any asset or portfolio $\tilde{r}_q$ (except for the global mean-variance) has the balance of the relationship from their portfolio $\tilde{r}_b$:

$$\tilde{r}_q = \tilde{r}_b + \beta_{qb}\,\tilde{r}_b + \tilde{\varepsilon}_q \quad \text{s.t.}$$

$$E[\tilde{\varepsilon}_q] = E[\tilde{r}_b \tilde{\varepsilon}_q] = 0$$

$$\beta_{qb} = \frac{\mathrm{Cov}(\tilde{r}_q, \tilde{r}_b)}{\sigma_b^2} = \frac{\beta_{qg}}{\sigma_b^2}$$

$$\beta_{qg} = \frac{\mathrm{Cov}(\tilde{r}_q, \tilde{r}_g)}{\sigma_g^2} = \frac{1}{c}\,\beta_{qg}$$

$$\beta_{qb} = 0$$

- Result 7: For the expected return of a risk asset:

$$\mu_q = (1-\beta_{qb})\,\mu_b + \beta_{qg}\,\mu_g$$

- We now proceed the cross-sections but assume a non-zero risk-free portfolio $\tilde{r}_f$ or μ_f
- The constraints maximization problem becomes:

$$\min_{\tilde{R}_p} \sigma_p^2 = \sigma_p^2$$

$$\text{s.t.} \quad E[\tilde{R}_p] = (1-E[\tilde{R}_f])\,\mu_f = \mu_p$$

- Result 1a: Given a benchmark portfolio:

$$L = \sigma_p^2 + \lambda_0 (\mu_p - E[\tilde{R}_p]) - \lambda_i (1-E[\tilde{R}_f])$$

$$\frac{\partial L}{\partial \tilde{R}_i} = 2\sum_{j=1}^{N} \lambda_i \tilde{R}_j - \lambda_0 \bar{R}_i = 0$$

$$\Rightarrow \tilde{R}_i = \frac{(\mu_p - \lambda_0)}{(\bar{R}-\lambda_0 I)} \frac{\sigma^2 (\bar{R} - \lambda_0 I)}{\sigma^2 (\bar{R} - \lambda_0 I)}$$

- Then with a benchmark model of mean-variance portfolio are a combination of a risk-free asset portfolio and the tangent portfolio

$$\tilde{R}_p = \frac{\mu_p - \mu_f}{\sigma^2 (\bar{R} - \lambda_0 I)} \sigma^2 (\bar{R} - \lambda_0 I)$$

- The tangency portfolio is called the tangency portfolio since the maximum slope point of all portfolios of risky stocks μ_p / σ_p is maximized

$$\mu_p / \sigma_p = (\mu_p - \mu_f) / \sigma_p = \mu_p / \sigma_p$$

5.3 Statistical inference for intercept on beta

- Sometimes one has no choice of the tangency beta

Sharpe-Lintner version $\eta = \mathrm{const}$

- Let $\tilde{\varepsilon}_i$ be the error relative of time t. The errors below market model is given by

$$\tilde{\varepsilon}_i = \tilde{\varepsilon}_i + \beta_i \tilde{\varepsilon}_m + \tilde{\varepsilon}_i \quad \text{s.t.} \quad \eta(\tilde{\varepsilon}_i) = 0 \quad \Rightarrow \quad \mathrm{var}(\tilde{\varepsilon}_i) = \sigma_i^2$$

$$E[\tilde{\varepsilon}_i \tilde{\varepsilon}_m] = \beta_i \qquad E[\tilde{\varepsilon}_i \tilde{\varepsilon}_m \tilde{\varepsilon}_m] = \sigma_m^2 \qquad \mathrm{cov}(\tilde{\varepsilon}_m, \tilde{\varepsilon}_i) = 0$$

- Then **(1)** and **(2)** or $\alpha_i = 0$
- **(3)** $\eta = \alpha$ or $\alpha = \eta$ the best portfolio
- We can alternate with alternative to estimate $\eta = \alpha$ β
- Assuming that $\tilde{\varepsilon}_m, \tilde{\varepsilon}_m, \ldots, \tilde{\varepsilon}_m$ are jointly normal, we have that

$$\tilde{\beta}_i \left(\tilde{\varepsilon}_m \tilde{\varepsilon}_m \right) \sim \left(\sigma_i^2 \right) \frac{\sigma_i^2}{\sigma_m^2} \sim \chi^2_{(1)} \left(\frac{(\eta - \alpha)\,\mathrm{var}(\tilde{\varepsilon}_i)}{\sigma_i^2} \right)$$

- $\tilde{\varepsilon}_i = \left[\tilde{\varepsilon}_i \tilde{\varepsilon}_m \ldots \tilde{\varepsilon}_m \right]$

$$\tilde{\beta}_i \left(\tilde{\varepsilon}_m \ldots \tilde{\varepsilon}_m \right) \sim \left(\frac{\sigma_i^2}{\sigma_m^2} \right) \frac{(\sigma_i^2)}{\sigma_i^2} \sim \chi^2_{(1)} \left(\frac{(\eta - \alpha)\,\mathrm{var}(\tilde{\varepsilon}_i)}{\sigma_i^2} \right) \sim \chi^2_{(1)} \left(\frac{(\eta - \alpha)\,\mathrm{var}(\tilde{\varepsilon}_i)}{\sigma_i^2} \right)$$

$$\tilde{\beta}_i \left(\tilde{\varepsilon}_m \tilde{\varepsilon}_m \right) \sim \mathrm{var}\left(\tilde{\varepsilon}_m \tilde{\varepsilon}_m \right) \sim \left(\frac{\sigma_i^2}{\sigma_m^2} \right) \frac{(\sigma_i^2)}{\sigma_i^2} \sim \chi^2_{(1)} \left(\frac{(\eta - \alpha)\,\mathrm{var}(\tilde{\varepsilon}_i)}{\sigma_i^2} \right)$$

$$\Rightarrow \tilde{\beta}_i \sim \frac{\sum (\tilde{\varepsilon}_i \tilde{\varepsilon}_m)}{\sum (\tilde{\varepsilon}_m \tilde{\varepsilon}_m)^2} \left(\frac{\sigma_i^2}{\sigma_m^2} \right)$$

$$\tilde{\beta}_i = \frac{\sigma_i^2}{\sigma_m^2} \tilde{\beta}_i$$

$$\tilde{\beta}_i = \frac{1}{T} \sum_{t=1}^{T} \left(\tilde{\varepsilon}_i - \frac{\sigma_i^2}{\sigma_m^2} \tilde{\varepsilon}_m \right)^2$$

$$\tilde{\eta} = \alpha + \left(\eta - \frac{\sigma_i^2}{\sigma_m^2} \right) \tilde{\varepsilon}_m$$

- $\tilde{\eta} = \alpha + \left(\eta - \frac{\sigma_i^2}{\sigma_m^2} \right) \tilde{\varepsilon}_m$