Cap 6 Campbell, Lo, and Mackinlay

quinta-feira, 29 de outubro de 2020 06:40

- CAPM Beta does not completly explains the cross-section of expected asset returns
- This may suggest that more factors must be added
- Theoretical arguments also suggests that more factors must be added, since the CAPM only apply period by period under strong assumptions
- Two main proposed solutions exists:
  1) Arbitrage pricing theory (APT) of Ross (1976)
  2) Intertemporal Capital asset pricing model (ICAPM) of Merton (1973a)

1. Theoretical Background

- APT - Ross (1976)
- Allows multiple factors
- Does not require the identification of the market portfolio
- Drawback: in its general form the APT provides an approximation for the expected cross-section of returns with an unknown number of unidentified factors $\Rightarrow$ model is not falsefiable and assumptions should be introduced to make the model testable.

- Assumptions:
  a) competitive markets
  b) frictionless
  c) Return generation process is given by:

$$R_i = a_i + b_i^T f + \varepsilon_i \qquad b_{i\,k\times 1}$$
$$E[\varepsilon_i | f] = 0 \qquad f_{k\times 1}$$
$$E[\varepsilon_i^2 | f] = \sigma_i^2 \le \sigma^2 < \infty$$

for N assets stacked we have:

$$R = a + Bf + \varepsilon \qquad R_{N\times 1} \quad B_{N\times k} \quad f_{k\times 1}$$
$$E[\varepsilon | f] = 0 \qquad a_{N\times 1} \quad \varepsilon_{N\times 1}$$
$$E[\varepsilon\varepsilon^T | f] = \Sigma$$

  d) The factors f account for mostly of the common variation in the asset returns $\Rightarrow$ ε vanishes

  (i) This requires that $\varepsilon_i \perp\!\!\!\perp \varepsilon_j$, $i \neq j$

- Given these assumptions, Ross (1976) shows that the absence of arbitrage implies that:

$$\mu \approx \mathbf{1}\lambda_0 + B\lambda_K \qquad (1.7)$$

μ: expected return vector
λ_0: model zero-beta parameter = risk free return if exists
λ_K: vector of factor risk premia

- Note that 1.7 is an approximation $\Rightarrow$ does not produce directly testable restrictions for asset returns
- To obtain these restrictions, we need to add assumptions
- Connor (1984) added assumption:
  a) The market portfolio is well diversified $\rightarrow$ No single asset in the economy accounts for a significant portion of the aggregate wealth
  b) The factors are pervasive / universal $\rightarrow$ permits investors to diversify away idiosyncratic risk without relying in a single factor exposure

- ICAPM of Merton (1973a):
  a) Assumptions on conditional distributions of returns

  $\Rightarrow$ exact multifactor model

- In this chapter we will generally not differentiate the APT from the ICAPM
- We will consider models with exact representation:

$$\mu = \mathbf{1}\lambda_0 + B\lambda_K \qquad (1.8)$$

2. Estimation and testing

- We consider four versions of the exact factor pricing model:
  1) Factors are portfolios of traded assets + risk free asset
  2) Factors are portfolios of traded assets + no risk free asset
  3) Factors are not portfolios of traded assets
  4) Factors are portfolios of traded assets and the factor portfolios span the mean-variance frontier of risky assets

- All four cases use maximum likelihood estimation

- Assumption:
  a) Joint normality for the returns conditional on the factors

- Given this assumption, we can construct a test of any of the four cases using the likelihood ratio

Portfolios of factors with a riskfree Asset

- Factors are traded portfolios
- There exists a risk free asset
- Let Z_t be a $N\times 1$ vector of N assets excess returns
- The k-factor linear model is given by:

$$Z_t = a + B Z_{kt} + \varepsilon_t \quad (2.2), \qquad E[\varepsilon_t] = 0 \qquad E[\varepsilon_t^T \varepsilon_t] = \Sigma$$

$$E[Z_{kt}] = \mu_k \qquad E[(Z_{kt} - \mu_k)^T (Z_{kt} - \mu_k)] = \Omega_k$$

$$\operatorname{Cov}(Z_{kt}, \varepsilon_t^T) = 0_{k\times N}$$

B: $N\times k$ matrix of factor sensitivities
Z_{kt}: $k\times 1$ vector of factor portfolios excess returns
a: $N\times 1$ vector of asset returns intercept
ε_t: $N\times 1$ vector of asset returns error terms

$$\underset{\text{asset returns}}{\begin{bmatrix} Z_{1t} \\ Z_{2t} \\ \vdots \\ Z_{Nt} \end{bmatrix}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} + \begin{bmatrix} \beta_{11} & \cdots & \beta_{1k} \\ \vdots & \ddots & \vdots \\ \beta_{N1} & \cdots & \beta_{Nk} \end{bmatrix} \underset{\text{portfolio returns}}{\begin{bmatrix} Z_t^1 \\ Z_t^2 \\ \vdots \\ Z_t^k \end{bmatrix}} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_N \end{bmatrix}$$

$$\Rightarrow Z_{1t} = a_1 + \beta_{11} Z_t^1 + \beta_{12} Z_t^2 + \dots + \beta_{1k} Z_t^k + \varepsilon_1$$

- Exact factor pricing $\Rightarrow$ $a = 0$
- For the unconstrained model (2.2) the MLE are just the OLS estimators:

$$\hat{a} = \hat{\mu} - \hat{B}\hat{\mu}_k$$

$$\hat{B} = \Bigg[\underbrace{\sum_{t=1}^{T} (Z_t - \hat{\mu})^T (Z_{kt} - \hat{\mu}_k)}_{\text{covariance}}\Bigg] \Bigg[\underbrace{\sum_{t=1}^{T} (Z_{kt} - \hat{\mu}_k)^T (Z_{kt} - \hat{\mu}_k)}_{\text{variance}}\Bigg]^{-1}$$

$$\hat{\Sigma} = \frac{1}{T}\sum_{t=1}^{T} (Z_t - \hat{a} - \hat{B} Z_{kt})(Z_t - \hat{a} - \hat{B} Z_{kt})^T$$

where:

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} Z_t \qquad \hat{\mu}_k = \frac{1}{T}\sum_{t=1}^{T} Z_{kt}$$

$$Z_t = \begin{bmatrix} Z_{1t} \\ Z_{2t} \\ \vdots \\ Z_{Nt} \end{bmatrix} \Rightarrow \begin{matrix} \Sigma \\ \\ \Sigma \end{matrix} \begin{bmatrix} Z_{11} & \cdots & Z_{1T} \\ \vdots & & \vdots \\ Z_{N1} & \cdots & Z_{NT} \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{1}{T}\sum Z_{1t} \\ \vdots \\ \frac{1}{T}\sum Z_{Nt} \end{bmatrix}$$

- For the constrained model (2.2), $a = 0$, we have that:

$$\hat{B}^* = \left[\sum_{t=1}^{T} Z_t Z_{kt}^T\right]\left[\sum_{t=1}^{T} Z_{kt} Z_{kt}^T\right]^{-1}$$

$$\hat{\Sigma}^* = \frac{1}{T}\sum_{t=1}^{T} (Z_t - \hat{B}^* Z_{kt})(Z_t - \hat{B}^* Z_{kt})^T$$

- The Null Hypothesis of $a = 0$ can be tested using the likelihood Ratio test
- Under the null hypothesis the test statistic follows an F-distribution with N degrees of freedom

Portfolios as factors without a riskfree Asset

- In the absence of a riskfree asset, there is a zero-beta model that is a multifactor equivalent to the Black version of the CAPM
- In the multifactor context, the zero-beta portfolio has no sensitivity to any of the factor returns
- Let R_t be an $N\times 1$ vector of real returns for N assets
- For the unconstrained version of the multifactor sharpe model is given by:

$$R_t = a + B R_{kt} + \varepsilon_t \quad (2.14), \qquad E[\varepsilon_t] = 0 \qquad E[\varepsilon_t^T \varepsilon_t] = \Sigma$$

$$E[R_{kt}] = \mu_k \qquad E[(R_{kt} - \mu_k)^T (R_{kt} - \mu_k)] = \Omega_k \qquad \operatorname{Cov}(R_{kt}, \varepsilon_t^T) = 0$$

B: $N\times k$ matrix of factor sensitivities

R_{kt}: $k\times 1$ vector of factor portfolio real returns

- For the unconstrained model (2.14) the MLE is given by:

$$\hat{a} = \hat{\mu} - \hat{B}\hat{\mu}_k$$

$$\hat{B} = \left[\sum_{t=1}^{T} (R_t - \hat{\mu})^T (R_{kt} - \hat{\mu}_k)\right]\left[\sum_{t=1}^{T} (R_{kt} - \hat{\mu}_k)^T (R_{kt} - \hat{\mu}_k)\right]^{-1}$$

$$\hat{\Sigma} = \frac{1}{T}\sum_{t=1}^{T} (R_t - \hat{a} - \hat{B} R_{kt})^T (R_t - \hat{a} - \hat{B} R_{kt})$$

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} R_t \qquad \hat{\mu}_k = \frac{1}{T}\sum_{t=1}^{T} R_{kt}$$

- For the constrained version, real returns enter in excess of the expected zero-beta portfolio return γ_0

$$R_t = \mathbf{1}\gamma_0 + B(R_{kt} - \mathbf{1}\gamma_0) + \varepsilon_t = (\mathbf{1} - B\mathbf{1})\gamma_0 + B R_{kt} + \varepsilon_t$$

which means the MLE is given by:

$$\hat{B}^* = \left[\sum_{t=1}^{T} (R_t - \mathbf{1}\hat{\gamma}_0)(R_{kt} - \mathbf{1}\hat{\gamma}_0)^T\right]\left[\sum_{t=1}^{T} (R_{kt} - \mathbf{1}\hat{\gamma}_0)(R_{kt} - \mathbf{1}\hat{\gamma}_0)\right]^{-1}$$

$$\hat{\Sigma}^* = \frac{1}{T}\sum_{t=1}^{T} \left[R_t - \mathbf{1}\hat{\gamma}_0 - \hat{B}^*(R_{kt} - \mathbf{1}\hat{\gamma}_0)\right]\left[R_t - \mathbf{1}\hat{\gamma}_0 - \hat{B}^*(R_{kt} - \mathbf{1}\hat{\gamma}_0)\right]^T$$

$$\hat{\gamma}_0 = \left[(\mathbf{1} - \hat{B}^*\mathbf{1})^T \hat{\Sigma}^{*-1} (\mathbf{1} - \hat{B}^*\mathbf{1})\right]^{-1}\left[(\mathbf{1} - \hat{B}^*\mathbf{1})^T \hat{\Sigma}^{*-1} (\hat{\mu} - \hat{B}^*\hat{\mu}_k)\right]$$

- The Null Hypothesis of $a = (\mathbf{1} - B\mathbf{1})\gamma_0$ can be tested using the likelihood ratio test

Macroeconomic variables as factors

- Factors need not be traded portfolios of assets, in some cases than can be macroeconomic variables
- For the unconstrained macro version of the multifactor model we have:

$$R_t = a + B f_{kt} + \varepsilon_t \quad (2.28), \qquad E[\varepsilon_t] = 0 \qquad E[\varepsilon_t^T \varepsilon_t] = \Sigma$$

$$E[f_{kt}] = \mu_{fk} \qquad E[(f_{kt} - \mu_{fk})(f_{kt} - \mu_{fk})^T] = \Omega_k \qquad \operatorname{Cov}(f_{kt}, \varepsilon_t^T) = 0$$

(...)

- The constrained model is formulated using (1.8) and (2.28):

$$\left.\begin{aligned} \mu &= \mathbf{1}\lambda_0 + B\lambda_k \\ R_t &= a + B f_{kt} + \varepsilon_t \end{aligned}\right\} \quad \mu = a + B\mu_{fk} \Leftrightarrow a = \mathbf{1}\lambda_0 + B(\lambda_k - \mu_{fk})$$

- Define $\gamma_0 \triangleq \lambda_0$ and $\gamma_1 \triangleq (\lambda_k - \mu_{fk})$, then:

$$R_t = \mathbf{1}\gamma_0 + B\gamma_1 + B f_{kt} + \varepsilon_t \qquad (2.38)$$

6.3 Estimation of risk premia and expected returns

- Recall that the expected return in the multifactor model is given by:

$$\mu = \mathbf{1}\lambda_0 + B\lambda_k$$

- To estimate it, we need to obtain measures of λ_0, λ_k, and estimate B
- B and λ_0 are straightforward to estimate
  - λ_0 $\rightarrow$ riskfree rate returns
  - B $\rightarrow$ MLE for the constrained model

- The appropriate procedure to estimate λ_k varies across four cases of exact factor pricing models

- In the case where factors are the excess returns on traded portfolios, we have that:

$$\hat{\lambda}_k = \hat{\mu}_k = \frac{1}{T}\sum_{t=1}^{T} Z_{kt}$$

$$\operatorname{Var}(\hat{\lambda}_k) = \frac{1}{T}\hat{\Omega}_k$$

- In the same case before but without a riskfree asset, we have:

$$\hat{\lambda}_k = \hat{\mu}_k - \mathbf{1}\hat{\gamma}_0$$

$$\operatorname{Var}(\hat{\lambda}_k) = \frac{1}{T}\hat{\Omega}_k + \operatorname{Var}(\hat{\gamma}_0)\mathbf{1}\mathbf{1}^T$$

- In the case where factors are not traded portfolios, we have:

$$\hat{\lambda}_k = \hat{\mu}_{fk} + \hat{\gamma}_1$$

$$\operatorname{Var}(\hat{\lambda}_k) = \frac{1}{T}\hat{\Omega}_k + \operatorname{Var}(\hat{\gamma}_1)$$

6.4. Selection of factors

- The presented multifactor models assume the the factors are known
- In this section we present methods for specifying the factors:
  1) statistical
  2) Theoretical

Statistical Approaches

- Starting point: linear factor model (APT)

$$R_t = a + B f_t + \varepsilon_t \qquad E[\varepsilon_t] = 0 \qquad E[\varepsilon_t \varepsilon_t^T] = \Sigma$$

1. Factor Analysis

- Two-step procedure:
  1) B and Σ are estimated and then the estimates are used to construct measures of the factors realizations

2. Principal components

Theoretical approaches

6.5. Empirical Results