3 Blue 1 Brown challenge

Steve's solution

16^{th} of January, 2019

1 Tracking the Energy

1.1 Variables

- Masses: $\alpha < \beta$, $\alpha = 100^n \cdot \beta$
- Initial velocity of light block : $V_{\alpha 1}$
- Initial velocity of heavy block : $V_{\beta 1}$
- Final velocity of light block : $V_{\alpha 2}$
- Final velocity of heavy block : $V_{\beta 2}$

1.2 Conservation Laws

Energy Let us consider the Kinetic energy of each mass individually and as a whole. As a whole, the energy is conserved after each collision :

$$\alpha V_{\alpha 1}^2 + \beta V_{\beta 1}^2 = \alpha V_{\alpha 2}^2 + \beta V_{\beta 2}^2 = 2E \tag{1}$$

The energy of mass β is $\frac{1}{2}\beta V_{\beta 1}^2 = E_\beta$
The energy of mass α is $\frac{1}{2}\alpha V_{\alpha}^2 = E_\alpha$

Conservation of Momentum

$$\beta V_{\beta 1} - \alpha V_{\alpha 1} = \beta V_{\beta 2} + \alpha V_{\alpha 2} \tag{2}$$

Here we are taking our speed variables to be positive values. Though momentum is conserved upon every $\alpha >< \beta$ collision, momentum is lost (in the first half, gained in the second) by a factor of $2\alpha V_\alpha$ with every collision of α with the wall.

Solving for $V_{\alpha 2}$ we get:

$$V_{\alpha 2} = \frac{\beta(V_{\alpha 1} + 2V_{\beta 1}) - \alpha V_{\alpha 1}}{\alpha + \beta} \tag{3}$$

Similarly for $V_{\beta 2}$ we get:

$$V_{\beta 2} = \frac{\beta V_{\beta 1} - \alpha(2V_{\alpha 1} + V_{\beta 1})}{\alpha + \beta} \tag{4}$$

So

$$V_{\beta 2}^2 = \frac{\beta^2 V_{\beta 1}^2 + \alpha^2(4V_{\alpha 1}^2 + V_{\beta 1}^2 + 4V_{\alpha 1}V_{\beta 1}) - 2\alpha\beta V_{\beta 1}(2V_{\alpha 1} + V_{\beta 1})}{\alpha^2 + \beta^2 + 2\alpha\beta} \tag{5}$$

And since

$$V_{\beta 1}^2 = \frac{\alpha^2 V_{\beta 1}^2 + \beta^2 V_{\beta 1}^2 + 2\alpha\beta V_{\beta 1}^2}{\alpha^2 + \beta^2 + 2\alpha\beta} \tag{6}$$

Conjecture here we will investigate the relation. This is geometrically intuited from the circle graph...

$$\frac{\sqrt{\beta(V_{\beta 2} - V_{\beta 1})^2 + \alpha(V_{\alpha 2} - V_{\alpha 1})^2}}{E = \beta V_{\beta 1}^2 + \alpha V_{\alpha 1}^2} \cong \theta \textit{ is constant} \tag{7}$$

Therefore the expression

$$\beta V_{\beta 2}^2 + \beta V_{\beta 1}^2 - 2\beta V_{\beta 1}V_{\beta 2} + \alpha V_{\alpha 2}^2 + \alpha V_{\alpha 1}^2 - 2\alpha V_{\alpha 1}V_{\alpha 2} = 2E - 2(\beta V_{\beta 1}V_{\beta 2} + \alpha V_{\alpha 2}^2 V_{\alpha 1}) \tag{8}$$

must also be constant where $sin(\theta) \cong \theta$, so in the small angle aproximation, or as $lim\ sin(\theta) \to \theta \to 0 \iff \beta >> \alpha$

Substituting equations (3) and (4) into the above we get

$$\beta V_{\beta 1}V_{\beta 2} + \alpha V_{\alpha 1}V_{\alpha 2} = \beta V_{\beta 1}\frac{\beta V_{\beta 1} - \alpha(2V_{\alpha_1} + V_{\beta 1})}{\alpha + \beta} + \alpha V_{\alpha 1}\frac{\beta(V_{\alpha 1} + 2V_{\beta 1}) - \alpha V_{\alpha 1}}{\alpha + \beta} \tag{9}$$

is a constant

Which gives

$$\beta V_{\beta 1}^2(\beta - \alpha) + \alpha V_{\alpha 1}^2(\beta - \alpha) \cong \beta(\beta V_{\beta 1}^2 + \alpha V_{\alpha 1}^2) = \beta E \tag{10}$$

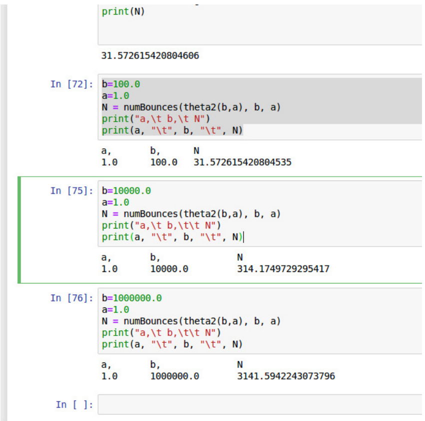
So

$$\theta \cong 2\sqrt{\frac{E - \frac{\beta E}{\alpha + \beta}}{E}} = 2\sqrt{1 - \frac{\beta}{\alpha + \beta}} \tag{11}$$

The number of bounces it does before it can escape is therefore

$$N = \frac{2\pi}{\theta} = \frac{\pi}{\sqrt{1 - \frac{\beta}{\beta + \alpha}}} \tag{12}$$

Plugging in certain values from the 3blue1brwon video to check that I am still sane



```
print(N)

31.572615420804606

In [72]: b=100.0
         a=1.0
         N = numBounces(theta2(b,a), b, a)
         print("a,\t b,\t N")
         print(a, "\t", b, "\t", N)

         a,      b,      N
         1.0     100.0   31.572615420804535

In [75]: b=10000.0
         a=1.0
         N = numBounces(theta2(b,a), b, a)
         print("a,\t b,\t\t N")
         print(a, "\t", b, "\t", N)

         a,      b,              N
         1.0     10000.0         314.1749729295417

In [76]: b=1000000.0
         a=1.0
         N = numBounces(theta2(b,a), b, a)
         print("a,\t b,\t\t N")
         print(a, "\t", b, "\t", N)

         a,      b,              N
         1.0     1000000.0       3141.5942243073796

In [ ]:
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