PSET 1 PHYS 512

Problem 1 If we eval at four pts $x \pm \delta$, $x \pm 2\delta$

a)

$$f(x\pm\delta) = f(x) \pm \delta f'(x) + \frac{\delta^2}{2} f''(x) \pm \frac{\delta^3}{3!} f'''(x) + \frac{\delta^4}{4!} f^{iv}(x) \pm \frac{\delta^5}{5!} f^{v}(x) + O(\delta^6)$$

$$f(x\pm 2\delta) = f(x) \pm 2\delta f'(x) + 4\frac{\delta^2}{2} f''(x) \pm 8\frac{\delta^3}{3!} f'''(x) + 16\frac{\delta^4}{4!} f^{iv}(x) \pm 32\frac{\delta^5}{5!} f^{v}(x) + O(\delta^6)$$

$$\Rightarrow b_1 f(x+2\delta) + a_1 f(x+\delta) + a_2 f(x-\delta) + b_2 f(x-2\delta)$$

$$= a_1 f(x) + a_1\delta f'(x) + a_1\frac{\delta^2}{2} f''(x) + a_1\frac{\delta^3}{3!} f'''(x) + a_1\frac{\delta^4}{4!} f^{iv}(x) + a_1\frac{\delta^5}{5!} f^{v}(x) + O(\delta^6)$$
$$+ a_2 f(x) - a_2\delta f'(x) + a_2\frac{\delta^2}{2} f''(x) - a_2\frac{\delta^3}{3!} f'''(x) + a_2\frac{\delta^4}{4!} f^{iv}(x) - a_2\frac{\delta^5}{5!} f^{v}(x) + O(\delta^6)$$
$$+ b_1 f(x) + 2b_1\delta f'(x) + 4b_1\frac{\delta^2}{2} f''(x) + 8b_1\frac{\delta^3}{3!} f'''(x) + 16b_1\frac{\delta^4}{4!} f^{iv}(x) + 32b_1\frac{\delta^5}{5!} f^{v}(x) + O(\delta^6)$$
$$+ b_2 f(x) - 2b_2\delta f'(x) + 4b_2\frac{\delta^2}{2} f''(x) - 8b_2\frac{\delta^3}{3!} f'''(x) + 16b_2\frac{\delta^4}{4!} f^{iv}(x) - 32b_2\frac{\delta^5}{5!} f^{v}(x) + O(\delta^6)$$

We want as many low order terms to be zero as possible.

$a_1 + a_2 + b_1 + b_2 = 0$ (δx^0) ; $a_1 - a_2 + 2b_1 - 2b_2 = 1$ (δx)

$a_1 + a_2 + 4b_1 + 4b_2 = 0$ (δx^2) $\Rightarrow \boxed{b_1 = -b_2}$ & $\boxed{a_1 = -a_2}$

(δx^3): $a_1 - a_2 + 8b_1 - 8b_2 = 0 \Rightarrow -6b_1 + 6b_2 = 1 \Rightarrow 12 b_2 = 1 \Rightarrow$ $\boxed{\begin{matrix} b_1 = -1/12 \\ b_2 = 1/12 \end{matrix}}$

(δ^4): $a_1 + a_2 + 16b_1 + 16b_2 = 0$ We get this for free!

~~(δ^5): $a_1 - a_2 + 32b_1 - 32b_2 = 0 \Rightarrow 2a_1 + 64b_1 = 0 \Rightarrow a_1 = 32/12$~~ ×

$\rightarrow \boxed{a_1 = 4/6 \quad a_2 = -4/6}$

$$\Rightarrow f'(x) = -\frac{1}{12} f(x+2\delta) + \underbrace{\frac{4}{6}}_{=\frac{2}{3}} f(x+\delta) - \underbrace{\frac{4}{6}}_{=\frac{2}{3}} f(x-\delta) + \frac{1}{12} f(x-2\delta)$$

b) Let g_i be a random variable of $O(1)$. Then

$$\delta f'(x) \simeq -\frac{1}{12} f(x+2\delta) + \frac{4}{6} f(x+\delta) - \frac{4}{6} f(x-\delta) + \frac{1}{12} f(x-2\delta) + O\left(\frac{\delta^5}{5!}\right)$$

$$\approx \left(f(x)(g\varepsilon) + \delta f'(x) + \delta g_2 \varepsilon f'(x) + \frac{\delta^2}{2} g_3 f''(x) + \frac{\delta^3}{3!} g_4 f'''(x) + \frac{\delta^4}{4!} g_5 f^{iv}(x) + O\left(\frac{\delta^5}{5!} f^{v}(x)\right) \right) / \delta$$

Bit error

taylor error

For an optimal δ, the largest error comes from both of these:

Minimize $\frac{\varepsilon}{\delta} = \frac{\delta^4}{5!}$

$\Rightarrow \boxed{\delta = \sqrt[5]{5!\,\varepsilon}}$

If $\varepsilon = 10^{-7} \Rightarrow \delta \approx 0.10$

If $\varepsilon = 10^{-16} \Rightarrow \delta \approx 0.0016$

ε is the bit precision