Wiener Filter on the PFB

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Abstract

This note we find the optimal Wiener threshold value to pick for the iPFB.

Let w be a row of our Toeplitz matrix, x be the timestream we are trying to retrieve, and q the piece of the IFT'd PFB'd signal, then

$$w * x = q \Rightarrow F\{x\}(\xi) = F\{q\}(\xi)/F\{w\}(\xi)$$

In reality, our signal q is quantized (q is the signal from one channel). It's Fourier transform $Q \equiv F\{q\}$ will be corrupted with Gaussian random noise $n(\lambda) \sim \frac{1}{\sqrt{2\pi\sigma_n}} \exp\{-\lambda^2/\sigma_n^2\}$, therefore the RMSE of $F\{x\}(\xi)$ will be

$$\sqrt{\langle n^2/|W(\xi)|^2\rangle} = \frac{\sigma_n}{|W(\xi)|} \equiv \frac{\sigma_n}{|F\{w\}(\xi)|} \tag{1}$$

Note the dependence on ξ. To mitigate much noise when $W(\xi)$ is large, we apply a Wiener filter to this signal by multiplying $Q(\xi)/W(\xi)$ in fourier space by the function $f(W(\xi))$

$$f_\phi(W(\xi)) = \frac{|W(\xi)|^2 \cdot (1+\phi^2)}{\phi^2 + |W(\xi)|^2} \qquad , \qquad \text{where } \phi \in \mathbb{R}^+ \quad \text{is a threshold}$$

If there where no quantization noise, multiplying by this factor would make our data worse. How much worse? It's useful to know so that we don't select too large a threshold. Assuming no quantization noise, i.e. that this equality holds exactly $F\{x\} = Q/W$, then multiplying by this factor will induce a MSE of

$$\left\langle \left|\frac{Q(\xi)}{W(\xi)}\right|^2 \cdot |f(\xi) - 1|^2 \right\rangle = \sigma_s^2 \langle |f(\xi) - 1|\rangle = \sigma_s^2 \left\langle \left|\phi^2 \frac{1 - |W(\xi)|^2}{\phi^2 + |W(\xi)|^2}\right|^2 \right\rangle = \sigma_s^2 \cdot \phi^4 \cdot \left(\frac{1 - |W(\xi)|^2}{\phi^2 + |W(\xi)|^2}\right)^2$$

Where σ_s is the RMS of our electric field recording. In other words, the RMSE of the noise added to $F\{x\}(\xi)$ by the Wiener filter f_ϕ is

$$\sigma_s \cdot \phi^2 \cdot \left(\frac{1 - |W(\xi)|^2}{\phi^2 + |W(\xi)|^2}\right) \tag{2}$$

Moving on, we examine the effect that the Wiener filter has on the quantization noise. We write

$$\text{MSE quantization} = \left\langle \left| \frac{n}{|W(\xi)|} \frac{|W(\xi)|^2(1+\phi^2)}{\phi^2 + |W(\xi)|^2} \right|^2 \right\rangle = \sigma_n^2 |W(\xi)|^2 \left(\frac{1+\phi^2}{\phi^2 + |W(\xi)|^2}\right)^2$$

Rem: σ_n is the quantization noise and n is a Gaussian random variable. So the RMSE will be

$$\sigma_n |W(\xi)| \left(\frac{1+\phi^2}{\phi^2 + |W(\xi)|^2} \right) \tag{3}$$

This expression is very nice, in particular it has the desirable property that when $W(x) \to 0$ the thing doesn't blow up to infinity as our friend equation (1) does. Combining the two expressions, (2) and (3), we get the expected MSE of our Wiener filtered $F\{x\}(\xi)$ at threshold ϕ

$$(\Delta F\{x\}(\xi))^2 = \sigma_n^2 |W(\xi)|^2 \left(\frac{1+\phi^2}{\phi^2 + |W(\xi)|^2} \right)^2 + \sigma_s^2 \phi^4 \left(\frac{1-|W(\xi)|^2}{\phi^2 + |W(\xi)|^2} \right)^2 \tag{4}$$

Chose the optimal threshold $\phi \in \mathbb{R}^+$ simply amounts to minimizing this expression, which can easily be done with a quick numerical search. To find a good threshold that minimizes everything at once we can search for the variable that minimizes the mean over all ξ and all y, where y is the index of the decoupled problem mentioned above.

$$\int \int dy d\xi (\Delta F\{x^{(y)}\}(\xi))^2$$

It's an integral over the domain of our favourite type of Eigen spectrum plots. Plugging this optimal threshold, we can recover these plots of how much error we can expect to come from each value of ξ (columns).

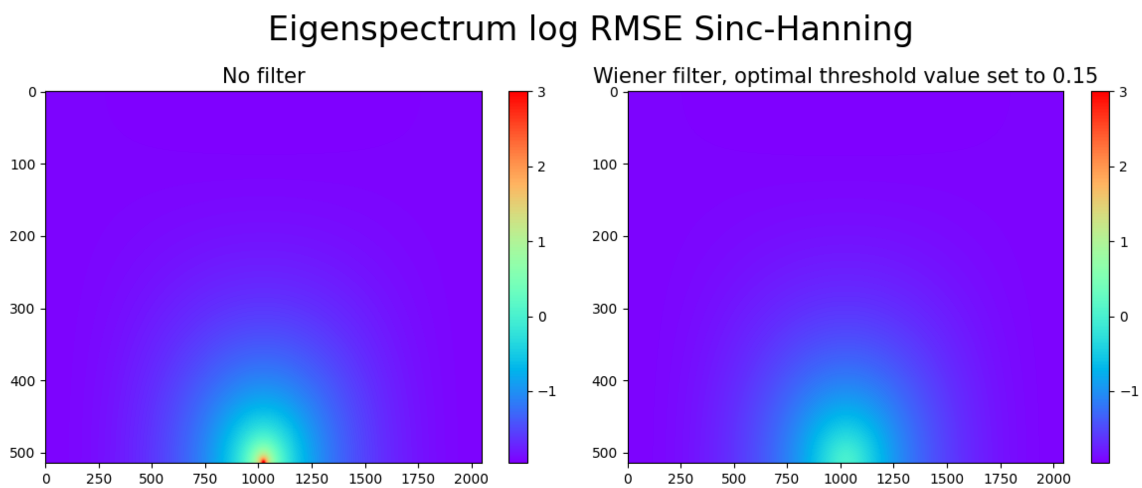

Figure 1: The natural logarithm of the root mean squared error that we would get for each channel and each ϕ from the iPFB.

Finally, we note that the errors will be smeared on each column by the final IFT transform that we need to make to retrieve our original timestream $x(t)$. Each decoupled problem's RMSE will be the average of one of the columns of the above figure.

$$\text{RMSE x(t) for all samples in column y} = \frac{1}{N} \sqrt{\sum_{\xi=1}^{N} (\Delta F\{x^{(y)}\}(\xi))^2}$$