

Project Euler Problem 153 - Gaussian Integers

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1 Problem

As we all know the equation $x^2 = -1$ has no solutions for real x .

If we however introduce the imaginary number i this equation has two solutions: $x = i$ and $x = -i$.

If we go a step further the equation $(x - 3)^2 = -4$ has two complex solutions: $x = 3 + 2i$ and $x = 3 - 2i$.

$x = 3 + 2i$ and $x = 3 - 2i$ are called each others' complex conjugate.

Numbers of the form $a+bi$ are called complex numbers.

In general $a + bi$ and $a - bi$ are each other's complex conjugate.

A Gaussian Integer is a complex number $a + bi$ such that both a and b are integers.

The regular integers are also Gaussian integers (with $b = 0$).

To distinguish them from Gaussian integers with $b \neq 0$ we call such integers "rational integers."

A Gaussian integer is called a divisor of a rational integer k if the result is also a Gaussian integer.

Let $s(k)$ denote the sum of all the positive real parts of the Gaussian divisors of k

Find

$$\gamma = \sum_{k=1}^{10^8} s(k) \tag{1}$$

Helper:

$$\sum_{k=1}^{10^5} s(k) = 17924657155 \tag{2}$$

2 Some Math behind the Algorithm: Explanation and Proofs

Note: I mainly use the symbol \mathbb{N} instead of \mathbb{Z} because this problem doesn't concern it's self with negative parts.

2.1 \mathbb{N}

The first thing to do is to solve the problem considering only the integer factors. We want to find an efficient way of calculating the following sum where $n = 10^8$

$$\sum_{k=1}^n S_{\mathbb{Z}}(k) \quad \text{where} \quad S_{\mathbb{Z}}(k) = S(k) \quad \text{restricted to the integers}$$

Rather than factoring each $k \in [1, n]$ and summing the factors we instead consider, for each number $q < n$, what is q a factor of? Which is simple to answer, we just do the q times table; we then add the number of positive numbers in the q times table that are smaller or equal to n ; we do this for every $q < n$ to find κ

$$\kappa := \sum_{k=1}^n S_{\mathbb{Z}}(k) = \sum_{q=1}^n q \cdot \lfloor n/q \rfloor \quad (3)$$

This reformulation of the sum lends it's self to much speedier computation, in python it can be written in a single line of code:

```
sum([q*(n//q) for q in range(1,n+1)])
```

2.2 \mathbb{C}

All we have to do is now is to find the sum of the positive real parts of the non-integer complex factors of all the numbers smaller than n . We will use the same trick as with the integer factors: instead of trying to factor each natural number smaller than $n = 10^8$ we will look at each Gaussian integer smaller than n and determine how many natural numbers (also smaller than n) it is a factor of. All it takes to reduce this problem to something my computer can tackle in under 2 minuets is to a) draw some straight lines and b) exploit the symmetries of the complex plane. We will start with b), the symmetries:

Observation 1. $(a + ib)|k$ with $a, b, k \in \mathbb{N} \Rightarrow (a \pm ib)|k$ and $(b \pm ia)|k$

Proof.

$$\begin{aligned} (a + ib)|k &\Rightarrow \frac{k}{a + ib} = c + id \quad \text{for some } c, d \in \mathbb{Z} \\ \Rightarrow \frac{k}{a + ib} &= \frac{k(a - ib)}{a^2 + b^2} \Rightarrow \frac{a}{a^2 + b^2} \in \mathbb{Z} \text{ and } \frac{b}{a^2 + b^2} \in \mathbb{Z} \end{aligned}$$

Hence

$$\frac{k}{a - ib} = \frac{k \cdot a}{a^2 + b^2} + i \frac{k \cdot b}{a^2 + b^2} \in \mathbb{Z}[i] \quad \text{and} \quad \frac{k}{b \pm ia} = \frac{k \cdot b}{a^2 + b^2} \mp \frac{k \cdot a}{a^2 + b^2} \in \mathbb{Z}[i]$$

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This can be made intuitive if you consider that multiplication in the complex plane as multiplying absolute values and summing angles; if there is one Gaussian integer that lies on the circle centered at the origin of radius $r = \sqrt{a^2 + b^2}$, then there are at the very least 3 others. And if $a \neq b$ then there are at least 7 others, 4 of these 8 divisors have positive real components and are therefore candidates for consideration. This has the effect of cutting in 4 the number of Gaussian integers we need to inspect

Further, we notice that each Gaussian factor (and it's conjugate pair) $(a \pm ib)|k$ with $a, b, k \in \mathbb{N}$ and $k \leq n$ is part of a pair since

$$\frac{k}{a \pm ib} = \frac{k(a \mp ib)}{a^2 + b^2} \in \mathbb{Z}[i] \Rightarrow \frac{ka}{a^2 + b^2} \in \mathbb{Z} \text{ and } \frac{kb}{a^2 + b^2} \in \mathbb{Z}$$

Is also a factor of k

Intuition: considering multiplication between complex numbers to be a binary operation whereby you multiply the lengths and sum the arguments of pointy arrows in the complex plane and consider $u = (a + ib) = ze^{i\theta}$ such that $\gcd(a, b) = 1$. Now draw a straight line that passes through 0 and u . If you follow the line from the origin outwards, the first Gaussian integer you encounter will be u , and all the other Gaussian integers you encounter will be integer multiples of u . Because of the way our complex binary operator behaves, dividing any real number by u will give you a complex number that lies on the line that is the reflection of this line on the real line, in other words the argument is negated: $\frac{r}{ze^{i\theta}} = \frac{r}{z}e^{-i\theta}$. So any integer k s.t. $u|k$ is the product of u and v where $\arg(v) = -\theta$, so $v = \alpha(a - ib)$

Corollary 2. If $\gcd(a, b) = 1$ and $a^2 + b^2 > n$ then $a + ib$ is not a Gaussian factor of *any* natural number smaller than n .

We are now ready to design an efficient algorithm.

3 The Algorithm

3.1 The core calculation loop

Consider $u = ze^{i\theta} = a + ib \in \mathbb{Z}[i]$ with $a, b > 0$ and $\gcd(a, b) = 1$. Let λ be the number of natural numbers $k < n$ are there such that $u|k$? Applying observation 2, $k \in \{m \cdot z^2 : m \in \mathbb{N}\}$. So $\lambda = \lfloor n/z^2 \rfloor$. That's it! So the net contribution of u to the sum is $\lambda \cdot a$. But we can take this further : employing the symmetry of observation 1, we know that for λ is the same lambda for $u^* = a - ib$ and also $b \pm ia$. So with only a, b we can add to our final sum

$$\gamma += 2(a + b) \cdot \left\lfloor \frac{n}{a^2 + b^2} \right\rfloor$$

But we can take this further still! From $u = a + ib$ alone we can find quickly all the contributions given by multiples of u and it's reflected counter-parts – What λ is associated with $q \cdot u$? $\lambda = \lfloor n/(q \cdot z^2) \rfloor$. Put concisely: the total of the contributions of all the Gaussian integers whereby the ratio of the real and complex components is either a/b or b/a with $\gcd(a, b) = 1$ to $\gamma = \sum_{k=1}^n s(k)$ is

$$\sum_{q=1}^{\lfloor n/z^2 \rfloor} 2 \cdot (a + b) \cdot \left\lfloor \frac{n}{q \cdot z^2} \right\rfloor = 2(a + b) \cdot \sum_{q=1}^{\lfloor n/z^2 \rfloor} \left\lfloor \frac{n}{q \cdot z^2} \right\rfloor$$

If $a \neq b$. If $a = b$ it is:

$$2a \cdot \sum_{q=1}^{\lfloor n/z^2 \rfloor} \left\lfloor \frac{n}{q \cdot z^2} \right\rfloor$$

In python this translates to

```
zsq = a**2+b**2
if a==b: the_sum += 2 * a * sum([zsq * (n//(q * zsq)) for q in range(1,n//zsq +1)])
else: the_sum += 2 * (a + b) * sum([zsq * (n//(q * zsq)) for q in range(1,n//zsq +1)])
```

3.2 Main loop

All we need now is a couple of for loops to find us every relatively prime a, b such that $a^2 + b^2 < n$ and apply the above formula to them. The symmetries reduces our domain to an eighth of a circle of the complex plane, for the sake of the illustration we will pick the 'bottom' half of the 1st quadrant.

```
for b in range(1, int(0.5*(np.sqrt(2*n-1)-1)+1)):
    for a in relatively_primes(b, n): eta += factors_sum_line(a, b, n)
```

`relatively_primes(b, n)` returns all the natural numbers $a < n$ s.t. $\gcd(a, b) = 1$
`factors_sum_line(a, b, n)` returns what is added to `the_sum` in the code box above this one.

4 Links

[Project Euler Problem 153](#)

[Code](#)

[My post about this problem](#)

Check out the relative run-times of each ‘row’ of complex numbers:

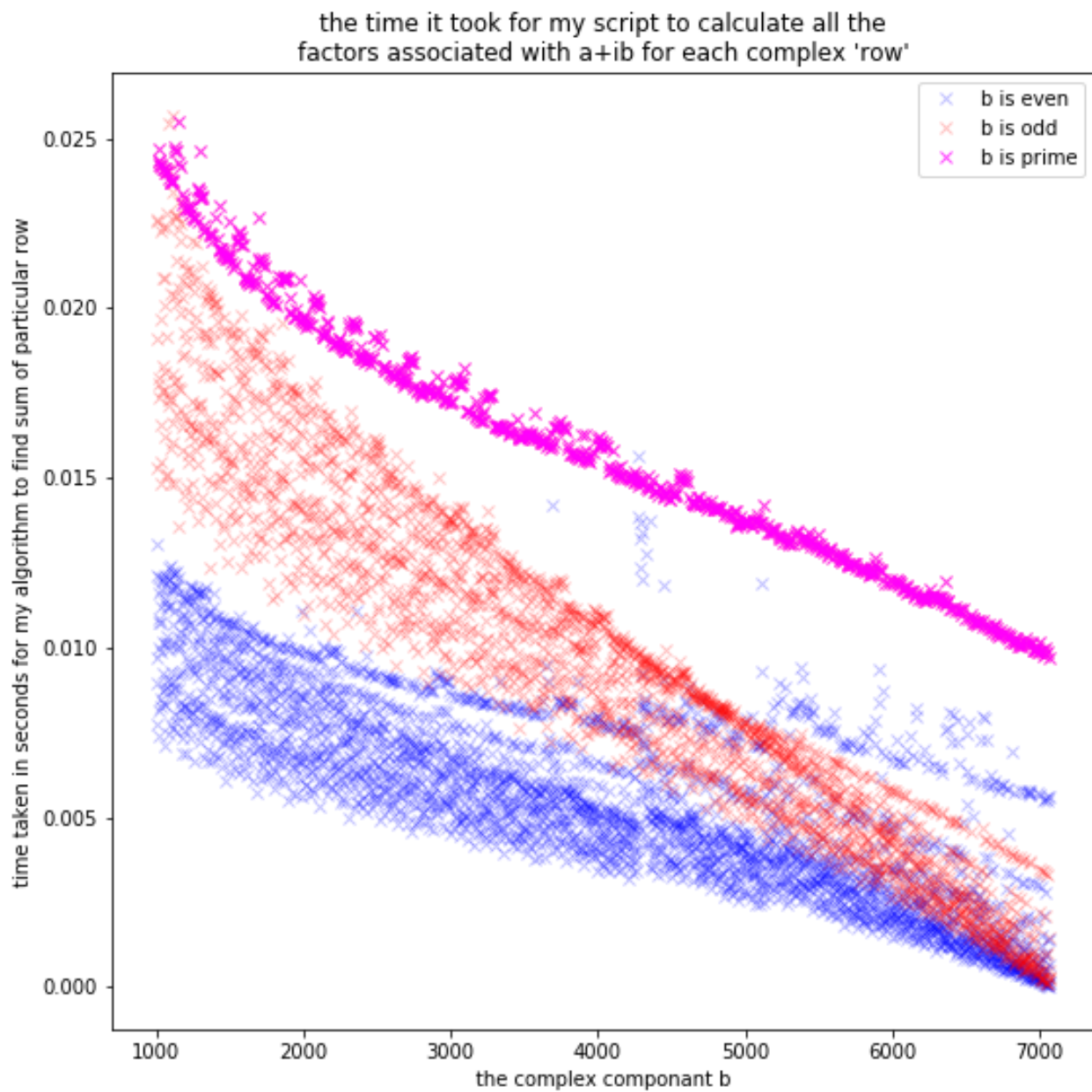


Figure 1: run time