# The Three Body Problem

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## 1 Preamble

The three body problem is my favourite example of chaos theory and the butterfly effect - primarily thanks to Cixin Liu's trilogy by the same name, in which an alien civilisation from the Alpha Centauri binary star system flees for earth out of fear that their home-planet will crash into one of it's suns.

## 2 Problem

This is just a plain old ordinary differential equation with the following setup :)

$$p_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \quad p_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \quad p_3 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} \quad v_1 = \begin{pmatrix} \dot{x_1} \\ \dot{y_1} \\ \dot{z_1} \end{pmatrix} \quad v_2 = \begin{pmatrix} \dot{x_2} \\ \dot{y_2} \\ \dot{z_2} \end{pmatrix} \quad v_3 = \begin{pmatrix} \dot{x_3} \\ \dot{y_3} \\ \dot{z_3} \end{pmatrix} \tag{1}$$

These are the respective positions and velocities of masses $m_1, m_2, m_3$

We're are modeling this with Newton's inverse square law of gravitational attraction. By using classical mechanics our model is entirely deterministic - which is why I love chaos, it shows us that even the simplest deterministic systems are completely unpredictable.

The kinetic energy of a star is (we won't worry about heat or nuclear fusion etc. just 3 bodies in classical mechanics)

$$T = \frac{1}{2}mv^2 \tag{2}$$

The potential energy well of a mass $m$ at position $\vec{p} =< x_1, y_1, z_1 >$ is

$$U(x,y,z) = \frac{Gm}{\sqrt{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2}} = \frac{Gm}{R} \tag{3}$$

So the Lagrangian is (who needs Newton when you have Lagrange)

$$L = T - U = \frac{1}{2}mv^2 + \frac{Gm}{R} \tag{4}$$

And the Euler Lagrange equation

$$\nabla L(x,y,z) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\frac{Gm}{\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}} = \frac{\partial}{\partial t}L_v = \frac{\partial}{\partial t}mv = ma \tag{5}$$

And so the attractive acceleration each star feels from the other two is just $\sum \nabla U$

$$\ddot{p_1} = \dot{v_1} = a_1 = \begin{pmatrix}\ddot{x_1}\\ \ddot{y_1}\\ \ddot{z_1}\end{pmatrix} = G\left[m_2\frac{\vec{p_2}-\vec{p_1}}{||\vec{p_2}-\vec{p_1}||^3} + m_3\frac{\vec{p_3}-\vec{p_1}}{||\vec{p_3}-\vec{p_1}||^3}\right] \tag{6}$$

The norm is the Pythagorean norm (the distance is induced by the norm)

$$||\vec{p_2}-\vec{p_1}|| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} \tag{7}$$

And so our IVP (Initial Value Problem) is $\frac{d}{dt}y = f(y)$, $y(t_0) = y_0$ where $y : \mathbb{R} \to \mathbb{R}^{18}$ and $f : D \times (a,b) \to \mathbb{R}^{18}$. Where $D \subset \mathbb{R}^{18}$ and $(a,b) \subset \mathbb{R}$ are open subsets. Here (a,b) is the time interval and $D$ is the domain of the function that maps $y$ onto $y'$.

$$y \stackrel{def}{=} < x_1, y_1, z_1\dot{x_1}, \dot{y_1}, \dot{z_1}, x_2, y_2, z_2, \dot{x_2}, \dot{y_2}, \dot{z_2}, x_3, y_3, z_3, \dot{x_3}, \dot{y_3}, \dot{z_3} > \tag{8}$$

I will use the following shorthand when referring to $y$ and $f(y)$ so as not to write 18 terms each time:

$$y \stackrel{def}{=} \begin{bmatrix}p_1\\ v_1\\ p_2\\ v_2\\ p_3\\ v_3\end{bmatrix} \qquad f(y) \stackrel{def}{=} \begin{bmatrix}v_1\\ a_1\\ v_2\\ a_2\\ v_3\\ a_3\end{bmatrix} \tag{9}$$

Where $a_1, a_2, a_3$ are the accelerations of the point-like masses $m_1, m_2, m_3$ respectively; their equations are given in equation (6). The point-mass approximation of stars or planets is exact if the objects consist of concentric spherical shells of even mass-density which is quite a good approximation, I will spare you the derivation (it's just an integral).

## 3 Solution

### 3.1 Existence and Uniqueness is not guaranteed!

If we want to solve the IVP we would very much like to know whether a unique solution even exists. The Theorem of Existence and Uniqueness (aka the Picard-Lindelof Theorem) of a solution to and IVP only holds if the function $f : D \to \mathbb{R}^{18}$ is locally Lipschitz Continuous. For $f$ to be Lipschitz continuous on the domain of definition it must satisfy

$$||f(y_1) - f(y_2)|| \leq L||y_1 - y_2|| \quad \text{for some} \quad L < \infty \tag{10}$$

(Note that our ODE is autonomous and so we do not much care about the time term that would otherwise be bothering us - i.e. Newton's laws of gravitational attraction do not change with time.)

However Lipschitz condition breaks (10) breaks if two planets cross paths as $||f(y)|| \to \infty \quad \forall \quad y$ s.t. $p_i - p_j \to 0$ , $i \neq j$

Thus, in order to guarantee existence and uniqueness we must take a closer look at the details of the Picard-Lindelof Theorem, and how we can bend it a little to fit the needs of this specific problem. First - a detour into physics.

### 3.2 Physics

Because this is Newtonian mechanics we can use common sense to help guide us. In the physical reality of the 3 star system we are modeling - before the centre of the stars are able to super-impose their centers of mass on one another, the suns will crash into each other and our our model will no longer be adequate to describe what is going on. To bypass this problem we will restrict our time interval to just before a crash if we find a crash does occur, this allows us to ignore those parts of our domain with unreasonably large values for $L$. Here is the method we will work with to solve the 3 body problem! (and how we will use a computer to *mathematically prove* that our solution is within a small margin of error.)

1. Have a first crack at finding a solution to the IVP via successive approximations (Picard iterations)

2. We inspect the trajectory of each star to make sure they do not come too close to each other, and place an upper bound on the maximal gravitational attraction one star feels at any given time. The upper bound at this stage is not necessarily correct as we have not yet placed an error term on our solution and we are assuming our solution is exact (which it is not)

3. Using this upper bound for the maximal gravitational attraction we can find a preliminary error $b_1$ term for our function. The error term will scale like a large degree polynomial with time (and we will see how this works shortly!). We multiply the error term by some factor $k > 1$ and using this beefed up error term

4. We calculate a new upper-bound for the maximal gravitational attraction.

5. We then verify that this new upper-bound gives us a new error term $b_2$. If $b_2 \leq k \cdot b_1$ all is well. If not we do some more tinkering.

### 3.3 Brief detour into Linear Algebra

This is the subspace where $(x_1 = x_2 \wedge y_1 = y_2 \wedge z_2 = z_3) \vee (x_1 = x_3 \wedge y_1 = y_3 \wedge z_1 = z_3) \vee (x_2 = x_3 \wedge y_2 = y_3 \wedge z_2 = z_3)$. Thus the danger-zone consists of 3, 15 dimensional sub-spaces. $S_1, S_2$ and $S_3$ who's intersection is a 12 dimensional subspace.

*Proof.* $S \subset V$ is a subspace of $V$, a vector space iff

It includes the origin - if all the stars are at $< 0, 0, 0 >$ and have no velocity, our ODE breaks, so our system is broken.

☺

We have a bound system where the total potential gravitational energy (negative, we set $U_{gravity} = 0$ when the stars are infinitely far appart.) is more important than the Kinetic energy of the system. What I mean is : $T + U < 0$. However because we have 3 stars in the system, there may still be enough energy for one star to escape! How is it possible in what appeared to be a bound system? If we give all the kinetic energy to a single star and make the other two relatively slow, we can haveit shoot off at escape velocity :)

We can

### 3.4 Back to Picard-Linderlof

Because this problem does not satisfy the condition that $\exists k < \infty$ s.t. $\sup ||f(y_1) - f(y_2)|| \leq k||y_1 - y_2|| \quad \forall y_1, y_2 \in V$ where V is our 18D vector space

$$\text{Assume} \quad \exists k < \infty \text{ s.t. } \sup ||f(y_1) - f(y_2)|| \leq k||y_1 - y_2|| \quad \forall y_1, y_2 \in D \subset V$$

### 3.5 The Picard Operator - or Convergence onto the solution by successive approximation

#### 3.5.1 The recurrence relation

From $\frac{d}{dt}y = f(y)$ we get the recurrence relation

$$y_{n+1}(t) = \int_{t_0}^{t} f(y_n(s))ds + y_0 \tag{11}$$

Which gives us

$$\ddot{x}_1 = G\left[m_2\frac{x_2 - x_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}^3} + m_3\frac{x_3 - x_1}{\sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2}^3}\right] = f(\dot{x}_1)$$

And

$$f(x_1) = \dot{x}_1$$

## 4 Links and References

- Paper on the 3 body problem - https://arxiv.org/pdf/1508.02312.pdf

Thank you Jean-Phillipe Lessard for giving me the tools necessary to scratch an itch I have wanted to scratch for years!