Numerically Solving PDE's: Crank-Nicholson Algorithm

June 20, 2019

1 PDE

$$\frac{\partial f(x,t)}{\partial t} = a(x,t)\frac{\partial^2 f(x,t)}{\partial x^2} + b(x,t)\frac{\partial f(x,t)}{\partial x} + c(x,t)f(x,t) + d(x,t). \tag{1}$$

2 Tridiagonal matrix

2.1 equal step

$$x_i = x_{\min} + ih \quad \text{and} \quad t_n = t_{\min} + nk \quad i \in [0, N-1] \tag{2}$$

$$\begin{aligned}
f &= \frac{1}{2}\left(f_i^n + f_i^{n+1}\right) \\
\frac{\partial f}{\partial t} &= \frac{1}{k}\left(f_i^{n+1} - f_i^n\right) \\
\frac{\partial f}{\partial x} &= \frac{1}{4h}\left(f_{i+1}^n - f_{i-1}^n + f_{i+1}^{n+1} - f_{i-1}^{n+1}\right) \\
\frac{\partial^2 f}{\partial x^2} &= \frac{1}{2h^2}\left(f_{i+1}^n - 2f_i^n + f_{i-1}^n + f_{i+1}^{n+1} - 2f_i^{n+1} + f_{i-1}^{n+1}\right)
\end{aligned} \tag{3}$$

$$\begin{aligned}
\frac{1}{k}\left(f_i^{n+1} - f_i^n\right) = \\
& a\left[\frac{1}{2h^2}\left(f_{i+1}^n - 2f_i^n + f_{i-1}^n + f_{i+1}^{n+1} - 2f_i^{n+1} + f_{i-1}^{n+1}\right)\right] \\
& + b\left[\frac{1}{4h}\left(f_{i+1}^n - f_{i-1}^n + f_{i+1}^{n+1} - f_{i-1}^{n+1}\right)\right] \\
& + c\left[\frac{1}{2}\left(f_i^n + f_i^{n+1}\right)\right] \\
& + d
\end{aligned} \tag{4}$$

$$
\begin{aligned}
&A_i f_{i-1}^{n+1} + B_i f_i^{n+1} + C_i f_{i+1}^{n+1} = D_i, \quad i \in [1, N-2] \\
&A_i = -(2ka - khb) \\
&B_i = 4h^2 + 4ka - 2h^2 kc \\
&C_i = -(2ka + khb) \\
&D_i = (2ka - khb) f_{i-1}^n \\
&\quad + (4h^2 - 4ka + 2h^2 kc) f_i^n \\
&\quad + (2ka + khb) f_{i+1}^n \\
&\quad + 4h^2 kd
\end{aligned}
\tag{5}
$$

2.2 log step

$$
\Delta x_{i,j} \equiv x_{i+j} - x_i \tag{6}
$$

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{f_i^n - f_{i-1}^n}{\Delta x_{i-1,1}} \\
\frac{\partial^2 f}{\partial x^2} &= \frac{f_i^n - f_{i-1}^n}{\left(\Delta x_{i-1,1}\right)^2} - \frac{f_{i-1}^n - f_{i-2}^n}{\Delta x_{i-1,1} \Delta x_{i-2,1}}
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
&\frac{1}{k}\left(f_i^{n+1} - f_i^n\right) = \\
&\qquad \frac{a}{2}\left[\frac{f_i^n - f_{i-1}^n + f_i^{n+1} - f_{i-1}^{n+1}}{\left(\Delta x_{i-1,1}\right)^2} - \frac{f_{i-1}^n - f_{i-2}^n + f_{i-1}^{n+1} - f_{i-2}^{n+1}}{\Delta x_{i-1,1} \Delta x_{i-2,1}}\right] \\
&\qquad + \frac{b}{2}\left[\frac{f_i^n - f_{i-1}^n + f_i^{n+1} - f_{i-1}^{n+1}}{\Delta x_{i-1,1}}\right] \\
&\qquad + \frac{c}{2}\left[\left(f_i^n + f_i^{n+1}\right)\right] \\
&\qquad + d
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
&A_{i-1}f_{i-2}^{n+1}+B_{i-1}f_{i-1}^{n+1}+C_{i-1}f_{i}^{n+1}=D_{i-1}, \quad i\in[2,N-1]\\
&A_{i-1}=-\frac{a}{2\Delta x_{i-1,1}\Delta x_{i-2,1}}\\
&B_{i-1}=\frac{a}{2(\Delta x_{i-1,1})^2}+\frac{a}{2\Delta x_{i-1,1}\Delta x_{i-2,1}}+\frac{b}{2\Delta x_{i-1,1}}\\
&C_{i-1}=\frac{1}{k}-\frac{a}{2(\Delta x_{i-1,1})^2}-\frac{b}{2\Delta x_{i-1,1}}-\frac{c}{2}\\
&D_{i-1}=\frac{a}{2\Delta x_{i-1,1},\Delta x_{i-2,1}}f_{i-2}^{n}\\
&\quad+\left(-\frac{a}{2(\Delta x_{i-1,1})^2}-\frac{a}{2\Delta x_{i-1,1}\Delta x_{i-2,1}}-\frac{b}{2\Delta x_{i-1,1}}\right)f_{i-1}^{n}\\
&\quad+\left(\frac{a}{2(\Delta x_{i-1,1})^2}+\frac{b}{2\Delta x_{i-1,1}}+\frac{c}{2}+\frac{1}{k}\right)f_{i}^{n}\\
&\quad+d
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
&j\equiv i-1\\
&A_{j}f_{j-1}^{n+1}+B_{j}f_{j}^{n+1}+C_{j}f_{i+1}^{n+1}=D_{j}, \quad j\in[1,N-2]\\
&A_{j}=-\frac{a}{2\Delta x_{j,1}\Delta x_{j-1,1}}\\
&B_{j}=\frac{a}{2(\Delta x_{j,1})^2}+\frac{a}{2\Delta x_{j,1}\Delta x_{j-1,1}}+\frac{b}{2\Delta x_{j,1}}\\
&C_{j}=\frac{1}{k}-\frac{a}{2(\Delta x_{j,1})^2}-\frac{b}{2\Delta x_{j,1}}-\frac{c}{2}\\
&D_{j}=\frac{a}{2\Delta x_{j,1},\Delta x_{j-1,1}}f_{j-1}^{n}\\
&\quad+\left(-\frac{a}{2(\Delta x_{j,1})^2}-\frac{a}{2\Delta x_{j,1}\Delta x_{j-1,1}}-\frac{b}{2\Delta x_{j,1}}\right)f_{j}^{n}\\
&\quad+\left(\frac{a}{2(\Delta x_{j,1})^2}+\frac{b}{2\Delta x_{j,1}}+\frac{c}{2}+\frac{1}{k}\right)f_{j+1}^{n}\\
&\quad+d
\end{aligned}
\tag{10}
$$

2.3 any step

$$
\begin{aligned}
\frac{\partial f}{\partial x}&=\frac{f_{i+1}^{n}-f_{i-1}^{n}+f_{i+1}^{n+1}-f_{i-1}^{n+1}}{2\Delta x_{i-1,2}}\\
\frac{\partial^2 f}{\partial x^2}&=\frac{f_{i+1}^{n}+f_{i+1}^{n+1}}{\Delta x_{i,1}\Delta x_{i-1,2}}-\frac{f_{i}^{n}+f_{i}^{n+1}}{\Delta x_{i,1}\Delta x_{i-1,1}}+\frac{f_{i-1}^{n}+f_{i-1}^{n+1}}{\Delta x_{i-1,1}\Delta x_{i-1,2}}
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&\frac{1}{k}\left(f_i^{n+1} - f_i^n\right) = \\
&\qquad a\left[\frac{f_{i+1}^n + f_{i+1}^{n+1}}{\Delta x_{i,1}\Delta x_{i-1,2}} - \frac{f_i^n + f_i^{n+1}}{\Delta x_{i,1}\Delta x_{i-1,1}} + \frac{f_{i-1}^n + f_{i-1}^{n+1}}{\Delta x_{i-1,1}\Delta x_{i-1,2}}\right] \\
&\qquad + b\left[\frac{f_{i+1}^n - f_{i-1}^n + f_{i+1}^{n+1} - f_{i-1}^{n+1}}{2\Delta x_{i-1,2}}\right] \\
&\qquad + c\left[\frac{1}{2}\left(f_i^n + f_i^{n+1}\right)\right] \\
&\qquad + d
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&A_i f_{i-1}^{n+1} + B_i f_i^{n+1} + C_i f_{i+1}^{n+1} = D_i, \quad i \in [1, N-2] \\
&A_i = -\frac{a}{\Delta x_{i-1,1}\Delta x_{i-1,2}} + \frac{b}{2\Delta x_{i-1,2}} \\
&B_i = \frac{1}{k} + \frac{a}{\Delta x_{i,1}\Delta x_{i-1,1}} - \frac{c}{2} \\
&C_i = -\frac{a}{\Delta x_{i,1}\Delta x_{i-1,2}} - \frac{b}{2\Delta x_{i-1,2}} \\
&D_i = \left(\frac{a}{\Delta x_{i-1,1}\Delta x_{i-1,2}} - \frac{b}{2\Delta x_{i-1,2}}\right) f_{i-1}^n \\
&\quad + \left(\frac{1}{k} - \frac{a}{\Delta x_{i,1}\Delta x_{i-1,1}} + \frac{c}{2}\right) f_i^n \\
&\quad + \left(\frac{a}{\Delta x_{i,1}\Delta x_{i-1,2}} + \frac{b}{2\Delta x_{i-1,2}}\right) f_{i+1}^n \\
&\quad + d
\end{aligned}
\tag{13}
$$

3 Test

$$
\begin{aligned}
&z = \ln(x), \quad x = e^z, \quad \frac{dz}{dx} = \frac{1}{x} = \frac{1}{e^z} \\
&\frac{\partial}{\partial x} = \frac{dz}{dx}\frac{\partial}{\partial z} = e^{-z}\frac{\partial}{\partial z} \\
&\frac{\partial^2}{\partial x^2} = e^{-z}\frac{\partial}{\partial z}\left(e^{-z}\frac{\partial}{\partial z}\right) \\
&\ = e^{-2z}\left(\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial z}\right)
\end{aligned}
\tag{14}
$$

3.1 Model 1

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{\partial}{\partial x}\left[x^2\frac{\partial f}{\partial x} + af - xf\right] - f + \delta(x - x_0)\Theta(t) \\
&= \frac{\partial^2 f}{\partial z^2} + ae^{-z}\frac{\partial f}{\partial z} - 2f + \delta(e^z - e^{z_0})\Theta(t)
\end{aligned}
\tag{15}
$$

3.2 Model 2

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{\partial}{\partial x}\left[x^2\frac{\partial f}{\partial x} - xf\right] - \frac{f}{x} + \delta(x - x_0)\Theta(t) \\
&= \frac{\partial^2 f}{\partial z^2} - \left(1 + e^{-z}\right) f + \delta(e^z - e^{z_0})\Theta(t)
\end{aligned}
\tag{16}
$$

3.3 Model 3

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{\partial}{\partial x}\left[x^3\frac{\partial f}{\partial x} - x^2 f\right] - f + \delta(x - x_0)\Theta(t) \\
&= e^z\frac{\partial^2 f}{\partial z^2} + e^z\frac{\partial f}{\partial z} - (2e^z + 1)\, f + \delta(e^z - e^{z_0})\Theta(t)
\end{aligned}
\tag{17}
$$