

# Numerically Solving PDE's: Crank-Nicholson Algorithm

June 21, 2019

## 1 PDE

$$\frac{\partial f(x, t)}{\partial t} = a(x, t) \frac{\partial^2 f(x, t)}{\partial x^2} + b(x, t) \frac{\partial f(x, t)}{\partial x} + c(x, t) f(x, t) + d(x, t). \quad (1)$$

## 2 Tridiagonal matrix

### 2.1 equal step

$$x_i = x_{\min} + ih \quad \text{and} \quad t_n = t_{\min} + nk \quad i \in [0, N - 1] \quad (2)$$

$$\begin{aligned} f &= \frac{1}{2} (f_i^n + f_i^{n+1}) \\ \frac{\partial f}{\partial t} &= \frac{1}{k} (f_i^{n+1} - f_i^n) \\ \frac{\partial f}{\partial x} &= \frac{1}{4h} (f_{i+1}^n - f_{i-1}^n + f_{i+1}^{n+1} - f_{i-1}^{n+1}) \\ \frac{\partial^2 f}{\partial x^2} &= \frac{1}{2h^2} (f_{i+1}^n - 2f_i^n + f_{i-1}^n + f_{i+1}^{n+1} - 2f_i^{n+1} + f_{i-1}^{n+1}) \end{aligned} \quad (3)$$

$$\begin{aligned} \frac{1}{k} (f_i^{n+1} - f_i^n) &= \\ &a \left[ \frac{1}{2h^2} (f_{i+1}^n - 2f_i^n + f_{i-1}^n + f_{i+1}^{n+1} - 2f_i^{n+1} + f_{i-1}^{n+1}) \right] \\ &+ b \left[ \frac{1}{4h} (f_{i+1}^n - f_{i-1}^n + f_{i+1}^{n+1} - f_{i-1}^{n+1}) \right] \\ &+ c \left[ \frac{1}{2} (f_i^n + f_i^{n+1}) \right] \\ &+ d \end{aligned} \quad (4)$$

$$\begin{aligned}
A_i f_{i-1}^{n+1} + B_i f_i^{n+1} + C_i f_{i+1}^{n+1} &= D_i, \quad i \in [1, N-2] \\
A_i &= -(2ka - khb) \\
B_i &= 4h^2 + 4ka - 2h^2kc \\
C_i &= -(2ka + khb) \\
D_i &= (2ka - khb)f_{i-1}^n \\
&\quad + (4h^2 - 4ka + 2h^2kc)f_i^n \\
&\quad + (2ka + khb)f_{i+1}^n \\
&\quad + 4h^2kd
\end{aligned} \tag{5}$$

## 2.2 log step

$$\Delta x_{i,j} \equiv x_{i+j} - x_i \tag{6}$$

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{f_i^n - f_{i-1}^n + f_i^{n+1} - f_{i-1}^{n+1}}{2\Delta x_{i-1,1}} \\
\frac{\partial^2 f}{\partial x^2} &= \frac{1}{2} \left( \frac{x_{i+1} + x_i}{2} - \frac{x_i + x_{i-1}}{2} \right)^{-1} \\
&\quad \left( \frac{f_{i+1}^n - f_i^n}{\Delta x_{i,1}} - \frac{f_i^n - f_{i-1}^n}{\Delta x_{i-1,1}} + \frac{f_{i+1}^{n+1} - f_i^{n+1}}{\Delta x_{i,1}} - \frac{f_i^{n+1} - f_{i-1}^{n+1}}{\Delta x_{i-1,1}} \right) \\
&= \frac{f_{i+1}^n + f_{i+1}^{n+1}}{\Delta x_{i,1}\Delta x_{i-1,2}} - \frac{f_i^n + f_i^{n+1}}{\Delta x_{i,1}\Delta x_{i-1,1}} + \frac{f_{i-1}^n + f_{i-1}^{n+1}}{\Delta x_{i-1,1}\Delta x_{i-1,2}}
\end{aligned} \tag{7}$$

$$\begin{aligned}
\frac{1}{k} (f_i^{n+1} - f_i^n) &= \\
&\quad a \left[ \frac{f_{i+1}^n + f_{i+1}^{n+1}}{\Delta x_{i,1}\Delta x_{i-1,2}} - \frac{f_i^n + f_i^{n+1}}{\Delta x_{i,1}\Delta x_{i-1,1}} + \frac{f_{i-1}^n + f_{i-1}^{n+1}}{\Delta x_{i-1,1}\Delta x_{i-1,2}} \right] \\
&\quad + \frac{b}{2} \left[ \frac{f_i^n - f_{i-1}^n + f_i^{n+1} - f_{i-1}^{n+1}}{\Delta x_{i-1,1}} \right] \\
&\quad + \frac{c}{2} [(f_i^n + f_i^{n+1})] \\
&\quad + d
\end{aligned} \tag{8}$$

$$\begin{aligned}
A_i f_{i-1}^{n+1} + B_i f_i^{n+1} + C_i f_{i+1}^{n+1} &= D_i, \quad i \in [1, N-2] \\
A_i &= -\frac{a}{\Delta x_{i-1,1} \Delta x_{i-1,2}} + \frac{b}{2\Delta x_{i-1,1}} \\
B_i &= \frac{a}{\Delta x_{i,1} \Delta x_{i-1,1}} - \frac{b}{2\Delta x_{i-1,1}} + \frac{1}{k} - \frac{c}{2} \\
C_i &= -\frac{a}{\Delta x_{i,1} \Delta x_{i-1,2}} \\
D_i &= \left( \frac{a}{\Delta x_{i-1,1} \Delta x_{i-1,2}} - \frac{b}{2\Delta x_{i-1,1}} \right) f_{i-1}^n \\
&\quad + \left( -\frac{a}{\Delta x_{i,1} \Delta x_{i-1,1}} + \frac{b}{2\Delta x_{i-1,1}} + \frac{c}{2} + \frac{1}{k} \right) f_i^n \\
&\quad + \frac{a}{\Delta x_{i,1} \Delta x_{i-1,2}} f_{i+1}^n \\
&\quad + d
\end{aligned} \tag{9}$$

### 2.3 any step

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{f_{i+1}^n - f_{i-1}^n + f_{i+1}^{n+1} - f_{i-1}^{n+1}}{2\Delta x_{i-1,2}} \\
\frac{\partial^2 f}{\partial x^2} &= \frac{1}{2} \left( \frac{x_{i+1} + x_i}{2} - \frac{x_i + x_{i-1}}{2} \right)^{-1} \\
&\quad \left( \frac{f_{i+1}^n - f_i^n}{\Delta x_{i,1}} - \frac{f_i^n - f_{i-1}^n}{\Delta x_{i-1,1}} + \frac{f_{i+1}^{n+1} - f_i^{n+1}}{\Delta x_{i,1}} - \frac{f_i^{n+1} - f_{i-1}^{n+1}}{\Delta x_{i-1,1}} \right) \\
&= \frac{f_{i+1}^n + f_{i+1}^{n+1}}{\Delta x_{i,1} \Delta x_{i-1,2}} - \frac{f_i^n + f_i^{n+1}}{\Delta x_{i,1} \Delta x_{i-1,1}} + \frac{f_{i-1}^n + f_{i-1}^{n+1}}{\Delta x_{i-1,1} \Delta x_{i-1,2}}
\end{aligned} \tag{10}$$

$$\begin{aligned}
\frac{1}{k} (f_i^{n+1} - f_i^n) &= \\
&\quad a \left[ \frac{f_{i+1}^n + f_{i+1}^{n+1}}{\Delta x_{i,1} \Delta x_{i-1,2}} - \frac{f_i^n + f_i^{n+1}}{\Delta x_{i,1} \Delta x_{i-1,1}} + \frac{f_{i-1}^n + f_{i-1}^{n+1}}{\Delta x_{i-1,1} \Delta x_{i-1,2}} \right] \\
&\quad + b \left[ \frac{f_{i+1}^n - f_{i-1}^n + f_{i+1}^{n+1} - f_{i-1}^{n+1}}{2\Delta x_{i-1,2}} \right] \\
&\quad + c \left[ \frac{1}{2} (f_i^n + f_i^{n+1}) \right] \\
&\quad + d
\end{aligned} \tag{11}$$

$$\begin{aligned}
A_i f_{i-1}^{n+1} + B_i f_i^{n+1} + C_i f_{i+1}^{n+1} &= D_i, \quad i \in [1, N-2] \\
A_i &= -\frac{a}{\Delta x_{i-1,1} \Delta x_{i-1,2}} + \frac{b}{2\Delta x_{i-1,2}} \\
B_i &= \frac{1}{k} + \frac{a}{\Delta x_{i,1} \Delta x_{i-1,1}} - \frac{c}{2} \\
C_i &= -\frac{a}{\Delta x_{i,1} \Delta x_{i-1,2}} - \frac{b}{2\Delta x_{i-1,2}} \\
D_i &= \left( \frac{a}{\Delta x_{i-1,1} \Delta x_{i-1,2}} - \frac{b}{2\Delta x_{i-1,2}} \right) f_{i-1}^n \\
&\quad + \left( \frac{1}{k} - \frac{a}{\Delta x_{i,1} \Delta x_{i-1,1}} + \frac{c}{2} \right) f_i^n \\
&\quad + \left( \frac{a}{\Delta x_{i,1} \Delta x_{i-1,2}} + \frac{b}{2\Delta x_{i-1,2}} \right) f_{i+1}^n \\
&\quad + d
\end{aligned} \tag{12}$$

### 3 Test

$$\begin{aligned}
z &= \ln(x), \quad x = e^z, \quad \frac{dz}{dx} = \frac{1}{x} = \frac{1}{e^z} \\
\frac{\partial}{\partial x} &= \frac{dz}{dx} \frac{\partial}{\partial z} = e^{-z} \frac{\partial}{\partial z} \\
\frac{\partial^2}{\partial x^2} &= e^{-z} \frac{\partial}{\partial z} \left( e^{-z} \frac{\partial}{\partial z} \right) \\
&= e^{-2z} \left( \frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial z} \right)
\end{aligned} \tag{13}$$

#### 3.1 Model 1

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{\partial}{\partial x} \left[ x^2 \frac{\partial f}{\partial x} + af - xf \right] - f + \delta(x - x_0) \Theta(t) \\
&= \frac{\partial^2 f}{\partial z^2} + ae^{-z} \frac{\partial f}{\partial z} - 2f + \delta(e^z - e^{z_0}) \Theta(t)
\end{aligned} \tag{14}$$

#### 3.2 Model 2

$$\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{\partial}{\partial x} \left[ x^2 \frac{\partial f}{\partial x} - xf \right] - \frac{f}{x} + \delta(x - x_0) \Theta(t) \\
&= \frac{\partial^2 f}{\partial z^2} - (1 + e^{-z}) f + \delta(e^z - e^{z_0}) \Theta(t)
\end{aligned} \tag{15}$$

### 3.3 Model 3

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\partial}{\partial x} \left[ x^3 \frac{\partial f}{\partial x} - x^2 f \right] - f + \delta(x - x_0) \Theta(t) \\ &= e^z \frac{\partial^2 f}{\partial z^2} + e^z \frac{\partial f}{\partial z} - (2e^z + 1) f + \delta(e^z - e^{z_0}) \Theta(t)\end{aligned}\tag{16}$$