Numerically Solving PDE's: Crank-Nicholson Algorithm

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1 PDE

$$\frac{\partial f(x,t)}{\partial t} = a(x,t)\frac{\partial^2 f(x,t)}{\partial x^2} + b(x,t)\frac{\partial f(x,t)}{\partial x} + c(x,t)f(x,t) + d(x,t). \tag{1}$$

2 Tridiagonal matrix

2.1 equal step

$$x_i = x_{\min} + ih \quad \text{and} \quad t_n = t_{\min} + nk \quad i \in [0, N-1] \tag{2}$$

$$\begin{aligned}
f &= \frac{1}{2}\left(f_i^n + f_i^{n+1}\right) \\
\frac{\partial f}{\partial t} &= \frac{1}{k}\left(f_i^{n+1} - f_i^n\right) \\
\frac{\partial f}{\partial x} &= \frac{1}{4h}\left(f_{i+1}^n - f_{i-1}^n + f_{i+1}^{n+1} - f_{i-1}^{n+1}\right) \\
\frac{\partial^2 f}{\partial x^2} &= \frac{1}{2h^2}\left(f_{i+1}^n - 2f_i^n + f_{i-1}^n + f_{i+1}^{n+1} - 2f_i^{n+1} + f_{i-1}^{n+1}\right)
\end{aligned} \tag{3}$$

$$\begin{aligned}
&\frac{1}{k}\left(f_i^{n+1} - f_i^n\right) = \\
&\qquad a\left[\frac{1}{2h^2}\left(f_{i+1}^n - 2f_i^n + f_{i-1}^n + f_{i+1}^{n+1} - 2f_i^{n+1} + f_{i-1}^{n+1}\right)\right] \\
&\qquad + b\left[\frac{1}{4h}\left(f_{i+1}^n - f_{i-1}^n + f_{i+1}^{n+1} - f_{i-1}^{n+1}\right)\right] \\
&\qquad + c\left[\frac{1}{2}\left(f_i^n + f_i^{n+1}\right)\right] \\
&\qquad + d
\end{aligned} \tag{4}$$

$$
\begin{aligned}
&A_i f_{i-1}^{n+1} + B_i f_i^{n+1} + C_i f_{i+1}^{n+1} = D_i, \quad i \in [1, N-2] \\
&A_i = -(2ka - khb) \\
&B_i = 4h^2 + 4ka - 2h^2 kc \\
&C_i = -(2ka + khb) \\
&D_i = (2ka - khb) f_{i-1}^n \\
&\quad + (4h^2 - 4ka + 2h^2 kc) f_i^n \\
&\quad + (2ka + khb) f_{i+1}^n \\
&\quad + 4h^2 kd
\end{aligned} \tag{5}
$$

2.2 log step

$$
\Delta x_{i,j} \equiv x_{i+j} - x_i \tag{6}
$$

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{f_i^n - f_{i-1}^n + f_i^{n+1} - f_{i-1}^{n+1}}{2\Delta x_{i-1,1}} \\
\frac{\partial^2 f}{\partial x^2} &= \frac{1}{2}\left(\frac{x_{i+1} + x_i}{2} - \frac{x_i + x_{i-1}}{2}\right)^{-1} \\
&\quad \left(\frac{f_{i+1}^n - f_i^n}{\Delta x_{i,1}} - \frac{f_i^n - f_{i-1}^n}{\Delta x_{i-1,1}} + \frac{f_{i+1}^{n+1} - f_i^{n+1}}{\Delta x_{i,1}} - \frac{f_i^{n+1} - f_{i-1}^{n+1}}{\Delta x_{i-1,1}}\right) \\
&= \frac{f_{i+1}^n + f_{i+1}^{n+1}}{\Delta x_{i,1}\Delta x_{i-1,2}} - \frac{f_i^n + f_i^{n+1}}{\Delta x_{i,1}\Delta x_{i-1,1}} + \frac{f_{i-1}^n + f_{i-1}^{n+1}}{\Delta x_{i-1,1}\Delta x_{i-1,2}}
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
&\frac{1}{k}\left(f_i^{n+1} - f_i^n\right) = \\
&\qquad a\left[\frac{f_{i+1}^n + f_{i+1}^{n+1}}{\Delta x_{i,1}\Delta x_{i-1,2}} - \frac{f_i^n + f_i^{n+1}}{\Delta x_{i,1}\Delta x_{i-1,1}} + \frac{f_{i-1}^n + f_{i-1}^{n+1}}{\Delta x_{i-1,1}\Delta x_{i-1,2}}\right] \\
&\qquad + \frac{b}{2}\left[\frac{f_i^n - f_{i-1}^n + f_i^{n+1} - f_{i-1}^{n+1}}{\Delta x_{i-1,1}}\right] \\
&\qquad + \frac{c}{2}\left[\left(f_i^n + f_i^{n+1}\right)\right] \\
&\qquad + d
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
&A_i f_{i-1}^{n+1} + B_i f_i^{n+1} + C_i f_{i+1}^{n+1} = D_i, \quad i \in [1, N-2] \\
&A_i = -\frac{a}{\Delta x_{i-1,1} \Delta x_{i-1,2}} + \frac{b}{2\Delta x_{i-1,1}} \\
&B_i = \frac{a}{\Delta x_{i,1} \Delta x_{i-1,1}} - \frac{b}{2\Delta x_{i-1,1}} + \frac{1}{k} - \frac{c}{2} \\
&C_i = -\frac{a}{\Delta x_{i,1} \Delta x_{i-1,2}} \\
&D_i = \left(\frac{a}{\Delta x_{i-1,1} x_{i-1,2}} - \frac{b}{2\Delta x_{i-1,1}} \right) f_{i-1}^n \\
&\quad + \left(-\frac{a}{\Delta x_{i,1} \Delta x_{i-1,1}} + \frac{b}{2\Delta x_{i-1,1}} + \frac{c}{2} + \frac{1}{k} \right) f_i^n \\
&\quad + \frac{a}{\Delta x_{i,1} \Delta x_{i-1,2}} f_{i+1}^n \\
&\quad + d
\end{aligned} \tag{9}
$$

2.3 any step

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{f_{i+1}^n - f_{i-1}^n + f_{i+1}^{n+1} - f_{i-1}^{n+1}}{2\Delta x_{i-1,2}} \\
\frac{\partial^2 f}{\partial x^2} &= \frac{1}{2} \left(\frac{x_{i+1} + x_i}{2} - \frac{x_i + x_{i-1}}{2} \right)^{-1} \\
&\quad \left(\frac{f_{i+1}^n - f_i^n}{\Delta x_{i,1}} - \frac{f_i^n - f_{i-1}^n}{\Delta x_{i-1,1}} + \frac{f_{i+1}^{n+1} - f_i^{n+1}}{\Delta x_{i,1}} - \frac{f_i^{n+1} - f_{i-1}^{n+1}}{\Delta x_{i-1,1}} \right) \\
&= \frac{f_{i+1}^n + f_{i+1}^{n+1}}{\Delta x_{i,1} \Delta x_{i-1,2}} - \frac{f_i^n + f_i^{n+1}}{\Delta x_{i,1} \Delta x_{i-1,1}} + \frac{f_{i-1}^n + f_{i-1}^{n+1}}{\Delta x_{i-1,1} \Delta x_{i-1,2}}
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
&\frac{1}{k} \left(f_i^{n+1} - f_i^n \right) = \\
&\qquad a \left[\frac{f_{i+1}^n + f_{i+1}^{n+1}}{\Delta x_{i,1} \Delta x_{i-1,2}} - \frac{f_i^n + f_i^{n+1}}{\Delta x_{i,1} \Delta x_{i-1,1}} + \frac{f_{i-1}^n + f_{i-1}^{n+1}}{\Delta x_{i-1,1} \Delta x_{i-1,2}} \right] \\
&\qquad + b \left[\frac{f_{i+1}^n - f_{i-1}^n + f_{i+1}^{n+1} - f_{i-1}^{n+1}}{2\Delta x_{i-1,2}} \right] \\
&\qquad + c \left[\frac{1}{2} \left(f_i^n + f_i^{n+1} \right) \right] \\
&\qquad + d
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
&A_i f_{i-1}^{n+1} + B_i f_i^{n+1} + C_i f_{i+1}^{n+1} = D_i, \quad i \in [1, N-2] \\
&A_i = -\frac{a}{\Delta x_{i-1,1}\Delta x_{i-1,2}} + \frac{b}{2\Delta x_{i-1,2}} \\
&B_i = \frac{a}{\Delta x_{i,1}\Delta x_{i-1,1}} - \frac{c}{2} + \frac{1}{k} \\
&C_i = -\frac{a}{\Delta x_{i,1}\Delta x_{i-1,2}} - \frac{b}{2\Delta x_{i-1,2}} \\
&D_i = \left(\frac{a}{\Delta x_{i-1,1}\Delta x_{i-1,2}} - \frac{b}{2\Delta x_{i-1,2}}\right) f_{i-1}^n \\
&\quad + \left(-\frac{a}{\Delta x_{i,1}\Delta x_{i-1,1}} + \frac{c}{2} + \frac{1}{k}\right) f_i^n \\
&\quad + \left(\frac{a}{\Delta x_{i,1}\Delta x_{i-1,2}} + \frac{b}{2\Delta x_{i-1,2}}\right) f_{i+1}^n \\
&\quad + d
\end{aligned}
\tag{12}
$$

3 Test

$$
\begin{aligned}
&z = \ln(x), \quad x = e^z, \quad \frac{dz}{dx} = \frac{1}{x} = \frac{1}{e^z} \\
&\frac{\partial}{\partial x} = \frac{dz}{dx}\frac{\partial}{\partial z} = e^{-z}\frac{\partial}{\partial z} \\
&\frac{\partial^2}{\partial x^2} = e^{-z}\frac{\partial}{\partial z}\left(e^{-z}\frac{\partial}{\partial z}\right) \\
&\quad = e^{-2z}\left(\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial z}\right)
\end{aligned}
\tag{13}
$$

3.1 Model 1

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{\partial}{\partial x}\left[x^2\frac{\partial f}{\partial x} + af - xf\right] - f + \delta(x - x_0)\Theta(t) \\
&= \frac{\partial^2 f}{\partial z^2} + ae^{-z}\frac{\partial f}{\partial z} - 2f + \delta(e^z - e^{z_0})\Theta(t)
\end{aligned}
\tag{14}
$$

boundary condition:

$$
F(x) = x^2\frac{\partial f}{\partial x} + af - xf = 0 \tag{15}
$$

3.2 Model 2

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= \frac{\partial}{\partial x}\left[x^2\frac{\partial f}{\partial x} - xf\right] - \frac{f}{x} + \delta(x - x_0)\Theta(t) \\
&= \frac{\partial^2 f}{\partial z^2} - \left(1 + e^{-z}\right) f + \delta(e^z - e^{z_0})\Theta(t)
\end{aligned}
\tag{16}
$$

boundary condition:

$$F(x) = x^2 \frac{\partial f}{\partial x} - xf = 0 \tag{17}$$

3.3 Model 3

$$\begin{aligned} \frac{\partial f}{\partial t} &= \frac{\partial}{\partial x}\left[x^3 \frac{\partial f}{\partial x} - x^2 f\right] - f + \delta(x - x_0)\Theta(t) \\ &= e^z \frac{\partial^2 f}{\partial z^2} + e^z \frac{\partial f}{\partial z} - (2e^z + 1) f + \delta(e^z - e^{z_0})\Theta(t) \end{aligned} \tag{18}$$

boundary condition:

$$F(x) = x^3 \frac{\partial f}{\partial x} - x^2 f = 0 \tag{19}$$