Simulating a multistage rocket launch

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Introduction

In this project we will simulate a rocket launch of Saturn V, A three stage rocket used by NASA to successfully land humans on the moon and to launch Skylab[1]. We will investigate if we can launch the rocket into orbit by simulating it's three stage launch using data from Wikipedia. Doing this we will use the Euler and Euler-Cromer methods and compare them to see which one works best. We will also investigate time step sensitivity.

Model, launch stage

For the launch stage we will model in the y-direction with three forces acting on the rocket. Thrust, the force we get from the engine which we will denote T. Gravity from the earth which we denote F_G and the drag from the air which we denote D.

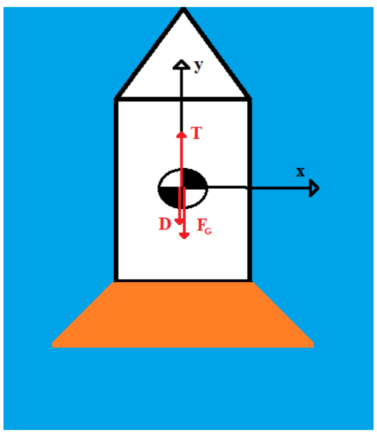

Figure 1: Rocket forces

According to Newtons second law we have that:

$$F = ma \tag{1}$$

Where F is the total force acting on the rocket, m is the rockets mass and a is the acceleration. From Figure 1 we get

$$F = T - F_G - D \tag{2}$$

using equation (1) and (2) we get:

$$a = \frac{1}{m}(T - F_G - D) \tag{3}$$

However it should be mentioned that since the mass of the rocket is not constant, Newtons second law isn't valid since it assumes constant mass. There should be an extra term $-\frac{dm}{dt}v$ on the right hand side of the equation. However this is physically what causes the thrust T so our model works as long as we remember that the mass drops with time. Now we need to decide m, T, D and F_G to calculate the acceleration.

m

The initial mass m_0 for every stage is taken from Wikipedia. The final mass after each stage we get by subtracting the weight of the fuel f used for each stage. Thus by assuming that the mass drops linear with time we get:

$$m(t) = m_0 - \frac{f}{t_{tot}}t \tag{4}$$

T

The thrust T is assumed to be constant during each stage and taken directly from wikipedia.

F_G

The gravity force F_G we get from Newton's law of universal acceleration[2]:

$$F_G = G\frac{mM}{r^2} \tag{5}$$

where $G = 6.67 \cdot 10^{-11}$ Nkg^{-2}m^2 denotes the gravitational constant, m is the mass of the rocket, M is the mass of the earth and r is the distance from the center of the earth. For the launching stage we will only use this in the y-direction. For the orbit phase we are going to use this in both x- and y-direction.

D

To model the drag we use the drag equation[3].

$$D = \frac{1}{2}\rho v^2 C_D A \tag{6}$$

Where ρ is the air density which varies with height above sea level $\rho = \rho(h)$, v is the velocity of the rocket, A is the cross sectional area of the rocket and C_D is the drag coefficient, a good value for the drag coefficient is $C_D = 0.42$[4] is used in this project. The cross-sectional is calculated as $A = \pi \cdot (\text{radius of the rocket})^2$

Next we need to find how the air density changes with height. Using table 1 which is filled with data from Wikipedia[5] and using an exponential curve fit, we get:

$$\rho(h) = 1.228e^{-0.0001168} \tag{7}$$

we see in Figure 2 that this is gives very nice approximation for the air density. Now we only have to take all data for Saturn V from Wikipedia and we can simulate a rocket launch.

h [m]	0	11 000	20 000	32 000	47 000	51 000	70 000
ρ [kg/m^3]	1.2250	0.36391	0.08803	0.01322	0.00143	0.00086	0.000064

Table 1: Air density for different height above ground levels

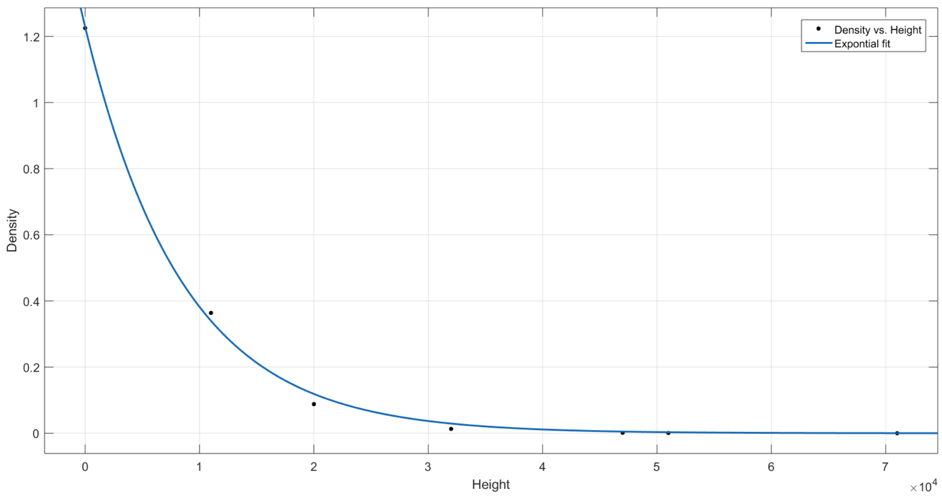

Figure 2: Density vs Height, expontential fit

Data and constants

All data is taken from wikipedia[1]

First stage - S-IC

mass = 2 290 000 kg
fuel = gross mass - empty mass = 2 160 000 kg
Thrust = 35,100 kN
Burn time = 168 s
Radius = 5.5 m

Second stage - S-II

mass = 496 200 kg
fuel = gross mass - empty mass = 456 100 kg
Thrust = 5,141 kN
Burn time = 360 s

Third stage - S-IVB

mass = 123 000 kg
fuel = gross mass - empty mass = 109 500 kg
Thrust = 5,003.1 kN
Burn time = 165 s

Earth data

The earth data is taken from wikipedia[6].
mass of the earth $M = 5.97 \cdot 10^{24}$ kg
mean radius $R = 6371$ km

Numerical methods

Euler:

$$x_{t+1} = x_t + v_t dt \tag{8}$$
$$v_{t+1} = v_t + a_t dt \tag{9}$$

Euler-Cromer:

$$v_{t+1} = v_t + a_t dt \tag{10}$$
$$x_{t+1} = x_t + v_{t+1} dt \tag{11}$$

Result and analysis, launch stage

Using the data above and running simulations for the Euler and Euler-Cromer methods with the timestep $dt = 0.1$ we get:

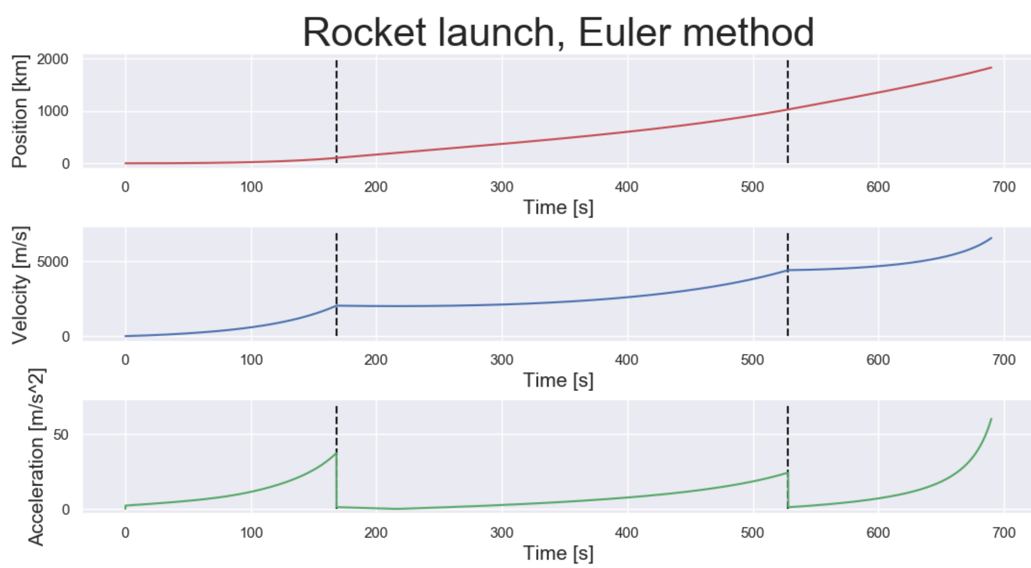

Figure 3: Euler simulation of the rocket launch, the dashed lines separates the stages

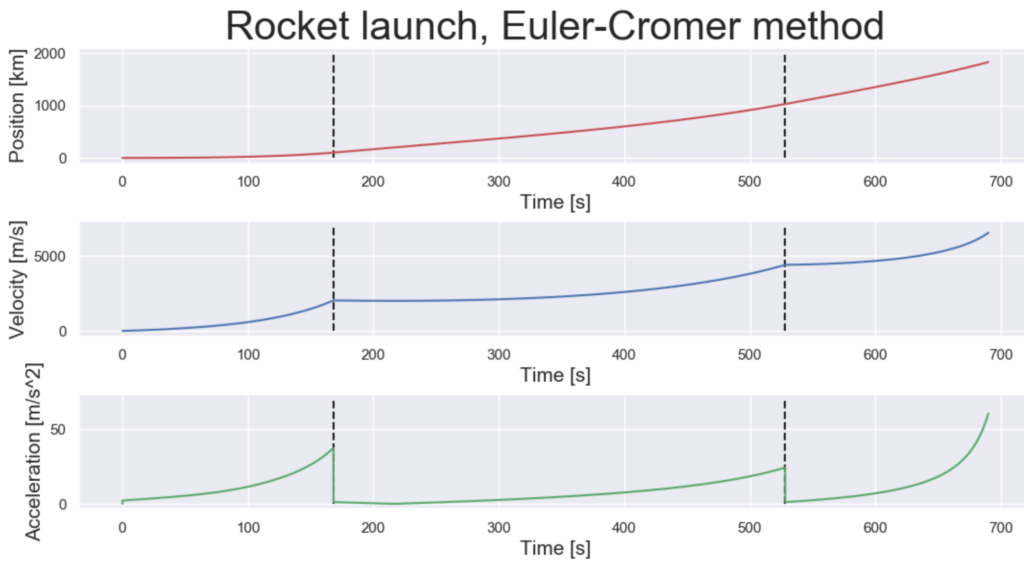

Figure 4: Euler-Cromer simulation of the rocket launch, the dashed lines separates the stages

By studying the graphs we can easy conclude that there is practical no difference at all between these methods for the launch stage of the simulation. This is shown in table 2 which shows the position above sea level, velocity and acceleration for the rocket after all 3 launching stages for both methods. We conclude that both method works fine for this part of the simulation.

	h [km]	v [m/s]	a [m/s^2]
Euler	1828	6556	60.20
Euler-Cromer	1829	6557	60.20

Table 2: Final values of height above ground level, velocity and acceleration for the rocket

Time step study, launch stage

Varying the time step for the Euler-Cromer method from 0.001 to 1 we find that the difference we get in the rockets path for the launch stage is extremely small. This is shown in figure 5 and 6.

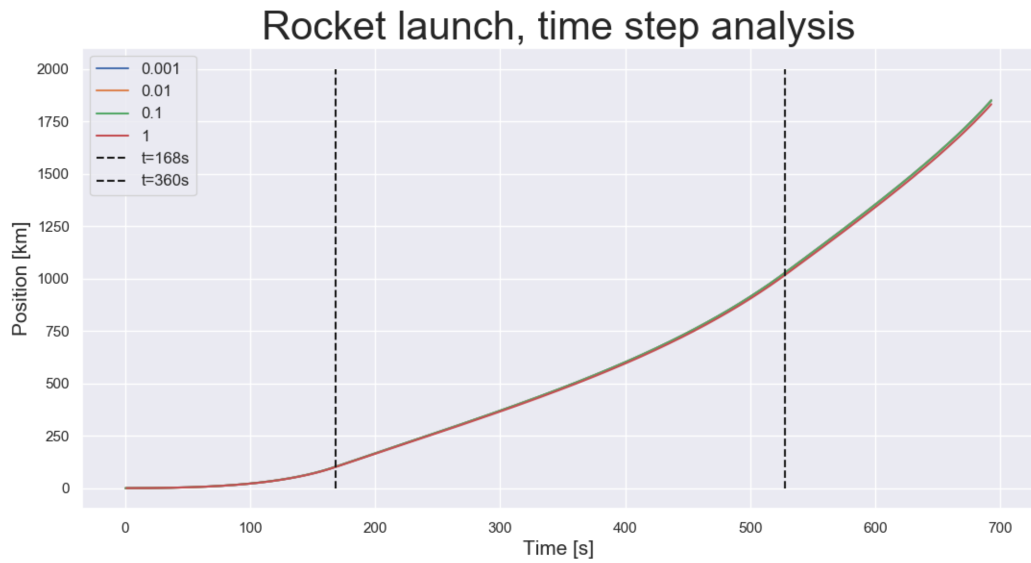

Figure 5: Comparison of different time step for the Euler-Cromer method of the launch

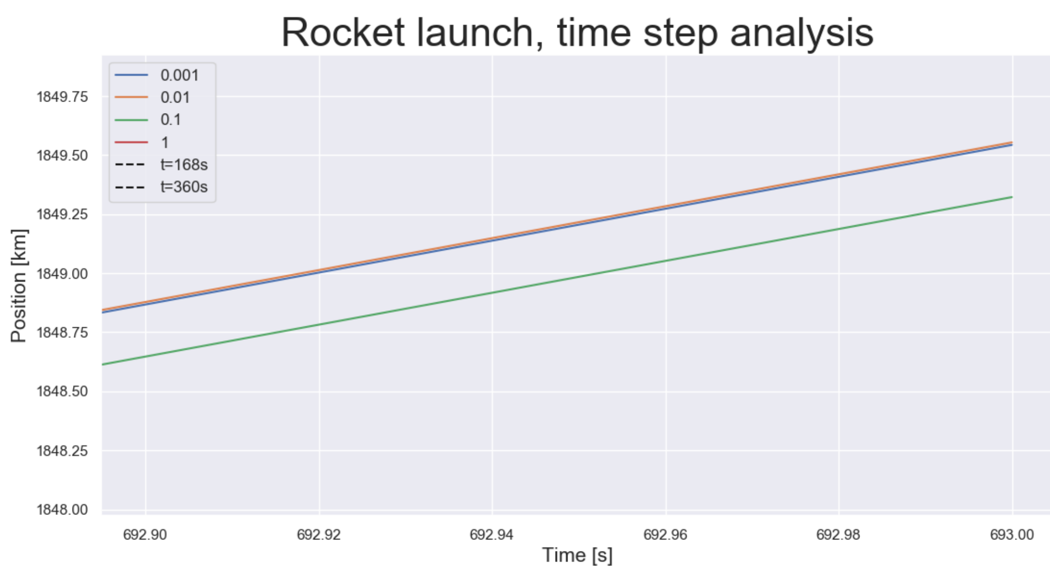

Figure 6: Zoomed plot of time step comparison

From the plots we can conclude that the improved accuracy we get for lowering the time step is very small. Thus to improve the performance we can use $dt = 1$ or at lowest $dt = 0.1$ which barely affects the model.

Model, orbit phase

We will now investigate if we can get the rocket into orbit after the launching phase is complete. After the third stage the rocket is very close to low earth orbit. Which is a distance about 2000 km from the earth where most of the manmade objects in space are[7]. The transverse velocity needed here to get the satellite in orbit can be derived by using the formula for centripetal force[8]. If we assume that the only force acting on the rocket is the gravitational force from the earth we get:

$$F = ma = \frac{mv^2}{R} = G\frac{mM}{R^2} \Rightarrow v = \sqrt{\frac{GM}{R}} \tag{12}$$

Plugging in the the height of the rocket we get:

$$v = \sqrt{\frac{6.67 \cdot 10^{-3} \cdot 5.97 \cdot 10^{24}}{(6371 + 1829) \cdot 10^{-3}}} \approx 6968 \text{ m/s} \tag{13}$$

The rockets velocity along the vertical axis is $v = 6557$ m/s, which is pretty close to the speed of the velocity we need in the transverse direction. A way to try to get the rocket into orbit could be if we somehow would be able to shift the velocity of the rocket by $90°$. In reality this would mean that the rocket needs to brake in vertical direction and accelerate in transverse direction very rapidly. Assuming this is possible to achieve we model that the rocket transmits it's velocity from vertical to horizontal direction by $3°$ every second. Simulating this for 30 seconds the velocity will now almost completely be directed in the transverse direction. Now we turn off all engines and observes what happens when the gravitational force from the earth acts on the rocket.

Result and analysis, orbit phase

Modeling as described in the section above with the time step $dt = 1$ and using Euler and Euler-Cromer we find something interesting shown in the figure 7-10 below. For the Euler method the distance from the rocket to earth increases each turn which can be seen in figure 7 and 8, this shows that the Euler method does not conserve energy, as we learned earlier in the course. The Euler-Cromer method is therefore more suitable to use for simulating the orbit phase since it conserves energy and is therefore able to stay in orbit. Which is shown in figure 9 and 10

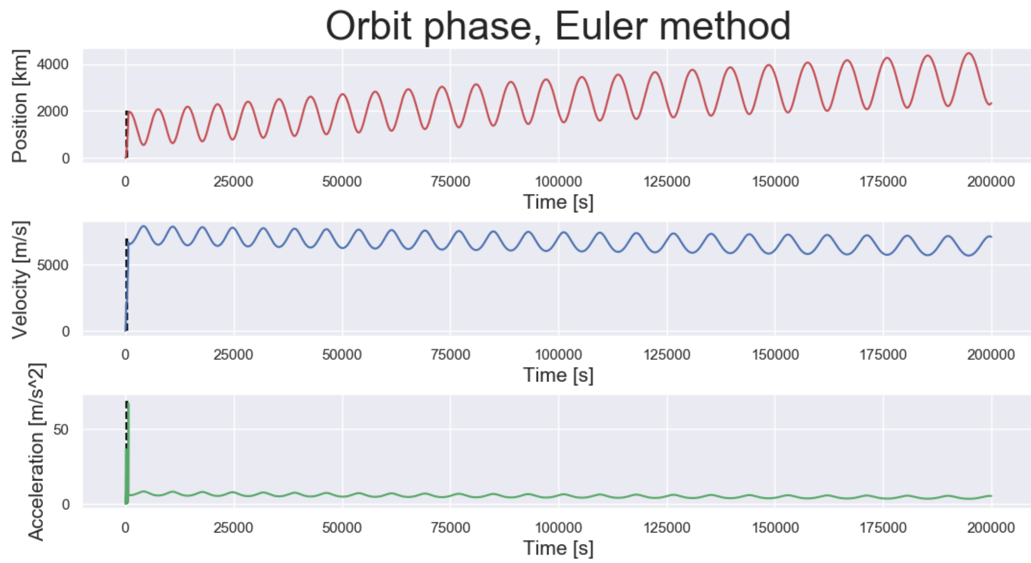

Figure 7: Euler simulation of the orbit phase, shows clearly that Euler does not conserve energy

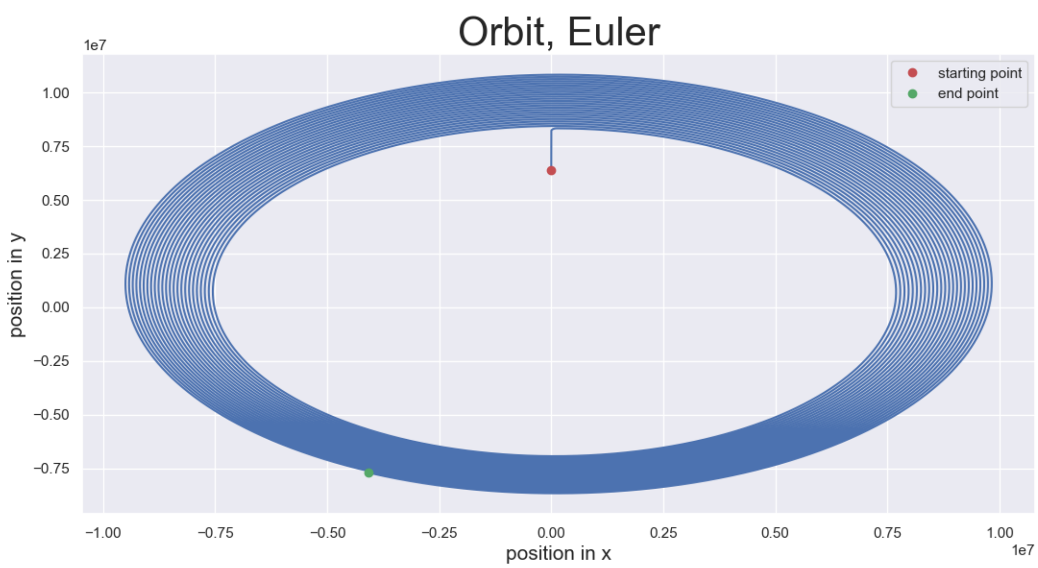

Figure 8: Orbit of the Euler method, we see that the length of the orbit increases by each turn

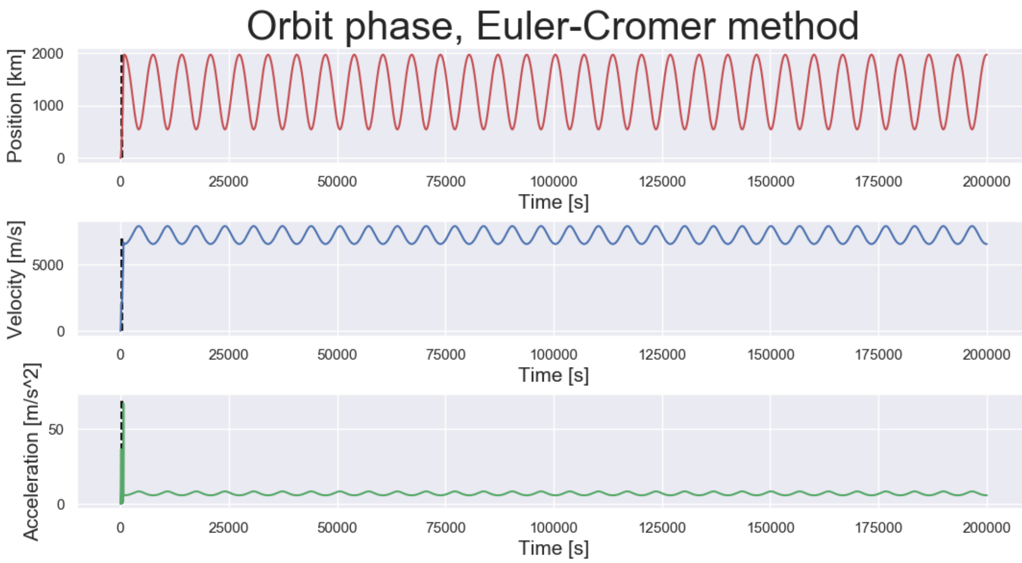

Figure 9: Euler-Cromer simulation of the orbit phase, conserves energy

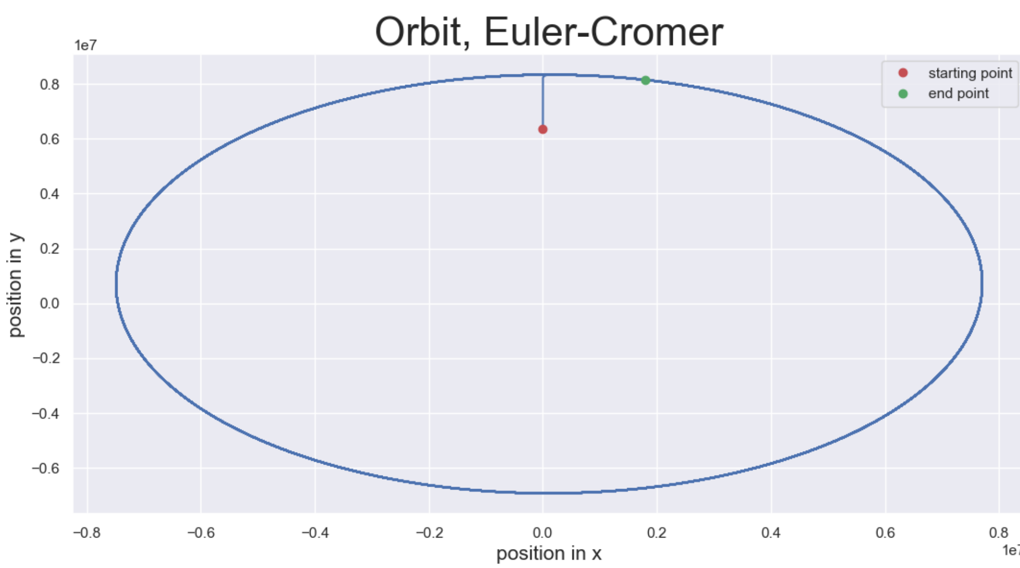

Figure 10: Orbit of the Euler-Cromer method

Time step analysis, orbit phase

Now we vary the time step dt from 0.01 to 1 for Euler-Cromer in the orbit phase. We see that there is now a higher deviation between $dt = 0.1, dt = 1$ than what it was for the launching stage. This is shown in figure 11 and 12 below. We conclude that the orbit phase is more sensitive for the time step and it's better to use $dt = 0.1$ to get a better accuracy. However this will limit how long time we can run the simulation and for a longer simulation $dt = 1$ is a better choice.

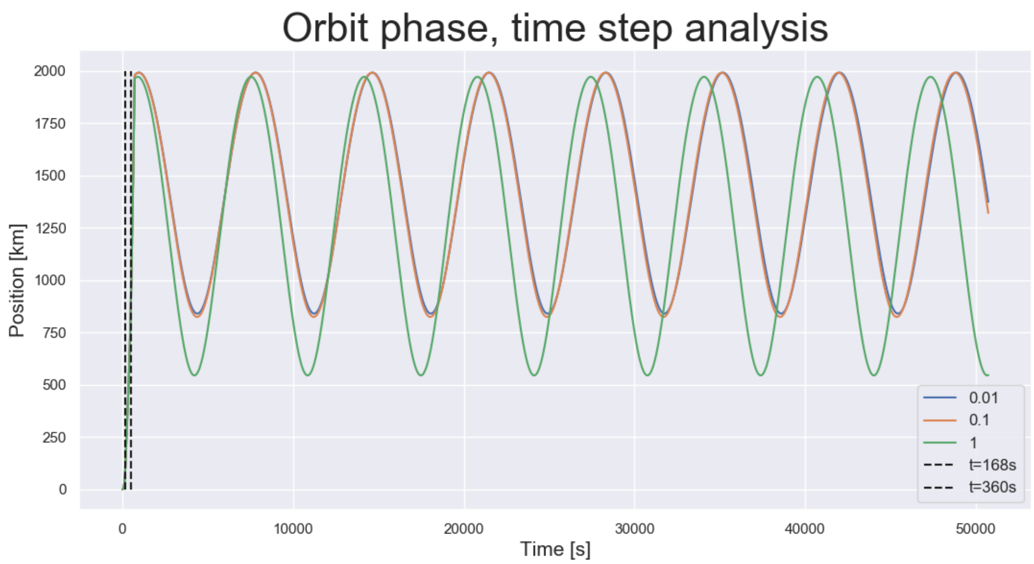

Figure 11: Comparison of different time step for the Euler-Cromer method

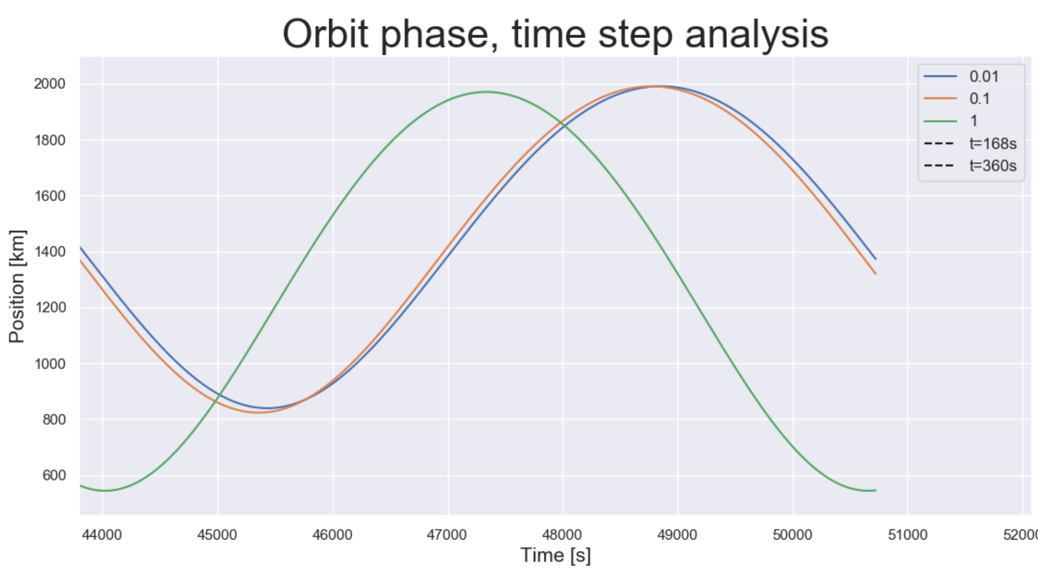

Figure 12: Zoomed plot of time step comparison for the orbit

Conclusion

The launch of the rocket into orbit was successful using data from Wikipedia and modeling the way described in this report. The Euler-Cromer method was clearly the better one for simulating the orbit phase of the simulation. We concluded that $dt = 0.1$ was the optimal time step to use for a simulation to get the best accuracy and $dt = 1$ is better for running a long simulation. It should also be mentioned that the model used to shift the rockets velocity in the orbit phase is very unrealistic. If this project were to be repeated it would be more realistic to give the rocket an acceleration in the transverse direction already in phase 2 and phase 3 of the launch, which would complicate the model a bit but make it more realistic.

References

[1] Wikipedia, *https://en.wikipedia.org/wiki/Saturn_V*

[2] Wikipedia, *https://en.wikipedia.org/wiki/Newton27s_law_of_universal_gravitation*

[3] Wikipedia, *https://en.wikipedia.org/wiki/Drag_(physics)*

[4] Determining the Drag Coefficient Of A Model Rocket Using Accelerometer Based Payloads *https://www.apogeerockets.com/downloads/Cd_determination_using_Coast_Data.pdf?fbclid=IwAR2yTgyb_rVX8mLd29va4kNWWfdHu8Fz4LOcK82439iOXcnrncuZl62Ie-k*

[5] Wikipedia, *https://en.wikipedia.org/wiki/Barometric_formula*

[6] Wikipedia, *https://en.wikipedia.org/wiki/Earth*

[7] Wikipedia, *https://en.wikipedia.org/wiki/Low_Earth_orbit*

[8] Wikipedia, *https://en.wikipedia.org/wiki/Centripetal_force*