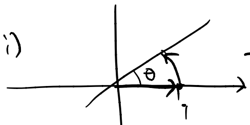
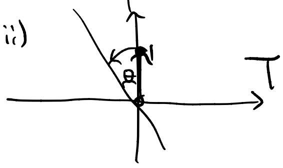
Example) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the map which rotates a vector counter-clockwise by θ. Find A that represents T with the standard basis.

$\to$ $\mathbb{R}^2$ has $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

i)  $T: \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

ii)  $T: \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

$\therefore A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Example) Let $T: \mathbb{P}_3 \to \mathbb{P}_2$ be the map given by

$$T(p(t)) = \frac{d}{dt} p(t)$$

What is the matrix A that represents T w.r.t. the standard basis.

$\to$ $T(\alpha p_1(t) + \beta p_2(t)) = \alpha T(p_1(t)) + \beta T(p_2(t))$

$\to$ The bases for $\mathbb{P}_3 = \{1, t, t^2, t^3\}$ and $\mathbb{P}_2 = \{1, t, t^2\}$

$A \in \mathbb{R}^{3\times4}$

i) $T(1) = 0 \longrightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to A = \begin{bmatrix} 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix}$

ii) $T(t) = \frac{d}{dt} t = 1 \to \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to A = \begin{bmatrix} 0 & 1 & \sim \\ 0 & 0 & \\ 0 & 0 & \end{bmatrix}$

iii) $T(t^2) = \frac{d}{dt} t^2 = 2t \to \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \to A = \begin{bmatrix} 0 & 1 & 0 & \vdots \\ 0 & 0 & 2 & \vdots \\ 0 & 0 & 0 & \vdots \end{bmatrix}$

iv) $T(t^3) = \frac{d}{dt} t^3 = 3t^2 \to \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \to A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

* We can do differentiation (of polynomials) with matrix multiplications

* Differentiate $7t^3 - t + 3$ using A.

$\to \begin{bmatrix} 3 \\ -1 \\ 0 \\ 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 21 \end{bmatrix} \longrightarrow -1 + 21t^2$.

$\therefore \frac{d}{dt}(7t^3 - t + 3) = 21t^2 - 1$

* What is the nullspace of A?

$\mathrm{Nul}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \in \mathbb{P}_3 \to$ it corresponds to $p(t) = C$

$\to \frac{d}{dt}(p(t)) = 0$

- Orthogonality.

The inner product and distances.

Definition) The innerproduct of $\vec{v}, \vec{w} \in \mathbb{R}^n$

$$\vec{v} \cdot \vec{w} \triangleq v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

Definition) The norm (or length) of a vector $\vec{v} \in \mathbb{R}$ is

$$\|\vec{v}\| \triangleq \sqrt{\vec{v} \cdot \vec{v}} \quad \left(= \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}\right)$$

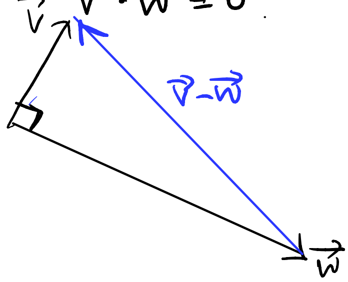
$* \ \|\vec{v}\|_p = \sqrt[p]{|v_1|^p + |v_2|^p + \cdots}$

distance between $\vec{v}$ and $\vec{w}$

$$\mathrm{dist}(\vec{v}, \vec{w}) = \|\vec{v} - \vec{w}\|$$

Definition) $\vec{v}, \vec{w} \in \mathbb{R}^n$ are orthogonal if

$$\vec{v} \cdot \vec{w} = 0.$$



Pythagoras: $\vec{v}$ and $\vec{w}$ are orthogonal

$\overset{\text{iff}}{\Longleftrightarrow} \ \|\vec{v}\|^2 + \|\vec{w}\|^2 = \|\vec{v} - \vec{w}\|^2$

$\Longleftrightarrow \ \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} - 2\vec{v} \cdot \vec{w}$

$\Longleftrightarrow \ \vec{v} \cdot \vec{w} = 0$

Theorem) Suppose that $\vec{v}_1, \cdots, \vec{v}_n$ are non zero and pairwise orthogonal.
Then, they are independent.

$\Rightarrow$ Suppose that

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_n \vec{v}_n = \vec{0} \qquad (*)$$

i) Take the dot product of $\vec{v}_1$ on $(*)$ with both sides.

$c_1 \vec{v}_1 \cdot \vec{v}_1 + 0 + 0 \cdots = 0$

$c_1 \|\vec{v}_1\|_2^2 = 0 \quad \therefore c_1 = 0$

ii) It remains the same for $\vec{v}_2, \cdots \vec{v}_n$

$\therefore c_1 = c_2 = \cdots = c_n = 0$

$\therefore \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are linearly independent.