Flap equation:

$$\int_e^R m(y-e)^2 \ddot{\beta}\, dy + \int_e^R m\Omega^2 y(y-e)\beta\, dy - \int_e^R L(y-e)\, dy = 0 \quad -①$$

$$L = \frac{1}{2}\rho V_{rel}^2 c \left[C_l \cos\phi + C_d \sin\phi\right] \underbrace{\cos\beta}_{1}\, dy \quad -②$$

$$V_{rel} = \Omega y \sec^2\phi \quad -③$$

$$C_l = C_{l_0} + a(\theta - \phi) = C_{l_0} - a(\beta \tan\delta_3 + \phi - \theta_i) \quad -④$$

Substituting,

② , ③ , ④ $\longrightarrow$ ①

$$\int_e^R (y-e)^2 \ddot{\beta}\, dy + \int_e^R \Omega^2 y(y-e)\beta\, dy$$
$$+ \int_e^R \frac{1}{2}\frac{\Omega^2 y^2 \sec^2\phi\, c\, a}{m}(\beta \tan\delta_3 + \phi)\cos\phi\, dy = M_b \quad -⑤$$

Replace,

$y - e = y'$ Integral limits $e \to R$ to $0 \to R'$

$dy = dy'$

$R - e = R'$

⑤ becomes,

$$\int_0^{R'} y'^2 \ddot{\beta}\, dy' + \int_0^{R'} \left[\Omega^2 (y'+e) y' + \frac{1}{2}\rho\Omega^2 c\, a \tan\delta_3\, y'^3\right]\beta\, dy'$$
$$+ \int_0^{R'} \frac{1}{2}\rho\Omega^2 c\, a\, \phi\, y'^3\, dy' = M_b \quad -⑥$$

Integrating (6),

$$\frac{R'^3}{3}\ddot{\beta} + \Omega^2\left(\frac{R'^3}{3} + \frac{eR'^2}{2}\right)\beta + \frac{1}{2}\frac{\rho\,\Omega^2 c a \tan\delta_3\,\beta R'^4}{4m}$$

$$+\frac{1}{2}\frac{\rho\,\Omega\, c a\,\dot{\beta} R'^4}{4m} = \frac{M_b}{m} \quad \text{— (7)}$$

Dividing by $\frac{\Omega^2 R'^3}{3}$

$$\overset{**}{\beta} + \left(1 + \frac{3e}{2R'}\right)\beta + \frac{3\rho c a R' \tan\delta_3}{8m}\beta$$

$$+\frac{3\rho c a R' \overset{*}{\beta}}{8m} = \frac{M_b}{\left(\frac{m\Omega^2 R'^3}{3}\right)} \quad \text{— (8)}$$

Substitute : $\gamma = \frac{3\rho c a R'}{m} = \frac{\rho c a R'^4}{\left(\frac{mR'^3}{3}\right)}$ — (9)

$$\boxed{\overset{**}{\beta} + \frac{\gamma}{8}\overset{*}{\beta} + \left(1 + \frac{3e}{2R'} + \frac{\gamma \tan\delta_3}{8}\right)\beta = \frac{3M_b}{m\Omega^2 R'^3}} \quad \text{— (10)}$$

$$M_b = \int_e^R \left(C_{l_0} y^2 + a(\theta_0 + \theta_{1c}\cos\psi + \theta_{1s}\sin\psi) y^2 + a\left(\frac{V_d - V_i}{\Omega y}\right) y^2 \right) \frac{1}{2} \rho c \Omega^2 (y-e)\, dy \quad -(11)$$

$$M_0 = \frac{1}{2} \rho c \Omega^2 \left(\frac{y^4}{4} - \frac{e y^3}{3}\right)(C_{l_0} + a\theta_0) + \frac{1}{2}\rho c \Omega^2 \left(\frac{y^3}{3} - \frac{e y^2}{2}\right) a \frac{(V_d - V_i)}{\Omega} \Bigg|_e^R \quad -(12)$$

$$M_{1c} = \frac{1}{2} \rho c \Omega^2 \left(\frac{y^4}{4} - \frac{e y^3}{3}\right) a\theta_{1c}\cos\psi \quad -(13)$$

$$M_{1s} = \frac{1}{2} \rho c \Omega^2 \left(\frac{y^4}{4} - \frac{e y^3}{3}\right) a\theta_{1s}\sin\psi \quad -(14)$$

$$\frac{M_0}{I_b \Omega^2} = \frac{1}{2}\frac{\rho c}{I_b} \left[\left(\frac{R^4}{4} - \frac{eR^3}{3} - \frac{e^4}{4} + \frac{e^4}{3}\right)(C_{l_0} + a\theta_0) + \left(\frac{R^3}{3} - \frac{eR^2}{2} - \frac{e^3}{3} + \frac{e^3}{2}\right) a(\lambda_d - \lambda_i) R \right] \quad -(15)$$

$$\frac{M_{1c}}{I_b \Omega^2} = \frac{1}{2}\frac{\rho c}{I_b} \left(\frac{R^4}{4} - \frac{eR^3}{3} - \frac{e^4}{4} + \frac{e^4}{3}\right) a\theta_{1c}\cos\psi \quad -(16)$$

$$\frac{M_{1s}}{I_b \Omega^2} = \underbrace{\qquad\qquad\qquad}_{\text{''}} \theta_{1s}\sin\psi \quad -(17)$$

$$\left.\begin{aligned}
\beta &= \beta_0 + \beta_{1c}\cos\psi + \beta_{1s}\sin\psi \\
\overset{*}{\beta} &= -\beta_{1c}\sin\psi + \beta_{1s}\cos\psi \\
\overset{**}{\beta} &= -\beta_{1c}\cos\psi - \beta_{1s}\sin\psi
\end{aligned}\right\} \quad (18)$$

Substitute (18) $\longrightarrow$ (15), (16) (17)

$$\boxed{\beta_0 = \left[\frac{\gamma}{8}\left(1 - \frac{4e}{3R} + \frac{e^4}{3R^4}\right)\left(\frac{C_{l_0}}{a} + \theta_0\right) + \frac{\gamma}{6}\left(1 - \frac{3e}{2R} + \frac{e^3}{2R^3}\right)(\lambda_d - \lambda_i)\right] \Bigg/ \left(1 + \frac{3e}{2R} + \frac{\gamma}{8}\tan\delta_3\right)} \quad -(19)$$

$$\cancel{\beta_{1c} = \frac{\gamma}{8}\left(1 - \frac{4e}{3R} + \frac{e^4}{3R^4}\right)\theta_{1c}}$$

$$\beta_0 = C + D\theta_0 + E\lambda_d$$

$$\beta_{1c}\left(\frac{3e}{2R} + \frac{\gamma}{8}\tan\delta_3\right) + \beta_{1s}\frac{\gamma}{8} = \frac{\gamma}{8}\left(1 - \frac{4e}{3R} + \frac{e^4}{3R^4}\right)\theta_{1c} \quad -(20)$$

$$\beta_{1s}\left(\frac{3e}{2R} + \frac{\gamma}{8}\tan\delta_3\right) - \beta_{1c}\frac{\gamma}{8} = \frac{\gamma}{8}\left(1 - \frac{4e}{3R} + \frac{e^4}{3R^4}\right)\theta_{1s} \quad (21)$$

$$\cancel{\frac{\beta_{1s}\frac{\gamma}{8}}{\left(\frac{3e}{2R^3}+\frac{\gamma}{8}\tan\delta_3\right)}} = \cancel{\frac{\frac{\gamma}{8}\left(1-\frac{4e}{3R}+\frac{e^4}{3R^4}\right)\theta_{1c}}{}}$$

$$\beta_{1c}\left[\left(\frac{3e}{2R^3}+\frac{\gamma}{8}\tan\delta_3\right)^2+\left(\frac{\gamma}{8}\right)^2\right] = \frac{\gamma}{8}\left(1-\frac{4e}{3R}+\frac{e^4}{3R^4}\right)\left[\left(\frac{3e}{2R^3}+\frac{\gamma}{8}\tan\delta_3\right)\theta_{1c}-\frac{\gamma}{8}\theta_{1s}\right] \quad -(22)$$

$$\boxed{\beta_{1c} = \frac{\frac{\gamma}{8}\left(1-\frac{4e}{3R}+\frac{e^4}{R^4}\right)\left[\left(\frac{3e}{2R^3}+\frac{\gamma}{8}\tan\delta_3\right)\theta_{1c}-\frac{\gamma}{8}\theta_{1s}\right]}{\left(\frac{3e}{2R^3}+\frac{\gamma}{8}\tan\delta_3\right)^2+\left(\frac{\gamma}{8}\right)^2} \quad (23)}$$

$$\boxed{\beta_{1s} = \frac{\frac{\gamma}{8}\left(1-\frac{4e}{3R}+\frac{e^4}{R^4}\right)\left[\left(\frac{3e}{2R^3}+\frac{\gamma}{8}\tan\delta_3\right)\theta_{1s}+\frac{\gamma}{8}\theta_{1c}\right]}{\left(\frac{3e}{2R^3}+\frac{\gamma}{8}\tan\delta_3\right)^2+\left(\frac{\gamma}{8}\right)^2}} \quad (24)$$

$$\beta_{1c} = A\theta_{1c} - B\theta_{1s}$$

$$\beta_{1s} = A\theta_{1s} + B\theta_{1c}$$

Ref. Johnson equations 6.84 and 6.85

$$C_H = \frac{\sigma a}{2}\left[\frac{\theta_0}{3}(-\beta_{1c}) - \frac{\theta_{1c}}{6}\beta_0 + \frac{\theta_{1s}}{4}\lambda + \frac{3}{4}\lambda\beta_{1c} + \frac{\beta_0\beta_{1s}}{6}\right] \quad -(25)$$

$$C_y = -\frac{\sigma a}{2}\left[\frac{\theta_0}{3}\beta_{1s} + \frac{\theta_{1c}\lambda}{4} + \frac{\theta_{1s}\beta_0}{6} - \frac{3\lambda\beta_{1s}}{4} + \frac{\beta_0\beta_{1c}}{6}\right] \quad -(26)$$

$$C_H^* = \frac{\sigma a}{2}\left[\frac{\theta_0}{3}(-A\theta_{1c} + B\theta_{1s}) - \frac{\theta_{1c}}{6}(C + D\theta_0 + E\lambda_d) + \frac{\theta_{1s}}{4}\lambda + \frac{3}{4}\lambda(A\theta_{1c} - B\theta_{1s}) + \frac{(A\theta_{1s} + B\theta_{1c})}{6}(C + D\theta_0 + E\lambda_d)\right] \quad -(27)$$

$$\boxed{C_H = \frac{\sigma a}{2}\left[\theta_0\theta_{1c}\left(\frac{-A}{3} - \frac{D}{6} + \frac{BD}{6}\right) + \theta_0\theta_{1s}\left(\frac{B}{3} + \frac{AD}{6}\right) + \theta_{1c}\left(-\frac{C}{6} - \frac{E\lambda_d}{6} + \frac{3A}{4}\lambda + \frac{BC}{6} + \frac{BE\lambda_d}{6}\right) + \theta_{1s}\left(\frac{\lambda}{4} - \frac{3B}{4}\lambda + \frac{AC}{6} + \frac{AE\lambda_d}{6}\right)\right]} \quad -(28)$$

$$C_y = -\frac{\sigma a}{2}\left[\frac{\theta_0}{3}\left(A\theta_{1s} + B\theta_{1c}\right) + \frac{\theta_{1c}\lambda}{4} + \frac{\theta_{1s}}{6}\left(C + D\theta_0 + E\lambda_d\right) - \frac{3}{4}\lambda\left(A\theta_{1s} + B\theta_{1c}\right) + \frac{\left(C + D\theta_0 + E\lambda_d\right)\left(+A\theta_{1c} - B\theta_{1s}\right)}{6}\right] \quad (29)$$

$$\boxed{C_y = -\frac{\sigma a}{2}\left[\theta_0\theta_{1s}\left(\frac{A}{3} + \frac{D}{6} - \frac{BD}{6}\right) + \theta_0\theta_{1c}\left(\frac{B}{3} + \frac{AD}{6}\right) + \theta_{1s}\left(\frac{C}{6} + \frac{E\lambda_d}{6} - \frac{3A\lambda}{4} - \frac{BC}{6} - \frac{BE\lambda_d}{6}\right) + \theta_{1c}\left(\frac{\lambda}{4} - \frac{3B\lambda}{4} + \frac{AC}{6} + \frac{AE\lambda_d}{6}\right)\right]} \quad (30)$$

$$\boxed{A = \frac{\left(\frac{\gamma}{8}\right)^2\left(1 - \frac{4e}{3R} + \frac{e^4}{R^4}\right)}{\left(\frac{3e}{2R} + \frac{\gamma}{8}\tan\delta_3\right)^2 + \left(\frac{\gamma}{8}\right)^2}} \quad (31A)$$

$$\boxed{B = \frac{\frac{\gamma}{8}\left(1 - \frac{4e}{3R} + \frac{e^4}{R^4}\right)\left(\frac{3e}{2R} + \frac{\gamma}{8}\tan\delta_3\right)}{\left(\frac{3e}{2R} + \frac{\gamma}{8}\tan\delta_3\right)^2 + \left(\frac{\gamma}{8}\right)^2}} \quad (31B)$$

$$\boxed{C = \frac{\frac{\gamma}{8}\left(1 - \frac{4e}{3R} + \frac{e^4}{3R^4}\right)\frac{C_{l_0}}{a} - \frac{\gamma}{6}\left(1 - \frac{3e}{2R} + \frac{e^3}{2R^3}\right)\lambda_i}{1 + \frac{3e}{2R} + \frac{\gamma}{8}\tan\delta_3}} \quad (31C)$$

$$\boxed{D = \frac{\gamma}{8}\left(1 - \frac{4e}{3R} + \frac{e^4}{3R^4}\right) \Big/ \left(1 + \frac{3e}{2R} + \frac{\gamma}{8}\tan\delta_3\right.} \quad (31D)$$

$$\boxed{E = \frac{\gamma}{6}\left(1 - \frac{3e}{2R} + \frac{e^3}{2R^3}\right) \Big/ \left(1 + \frac{3e}{2R} + \frac{\gamma}{8}\tan\delta_3\right.} \quad (31E)$$

$$T = N_b \int_e^R L\,dy = \frac{N_b}{2} \int_e^R \rho\, \Omega^2 y^2 \underbrace{\sec^2\phi}_{1}\, c\, C_l \underbrace{\cos\phi}_{1}\, dy$$

$$= \frac{N_b \rho \Omega^2 c}{2} \int_e^R C_{l_0} - a(\beta \tan\delta_3 + \phi - \theta_i)\, y^2\, dy \quad - (32)$$

$$\boxed{C_T = \frac{\sigma a}{2}\left[\frac{C_{l_0}}{a R^3}\left(\frac{R^3}{3} - \frac{e^3}{3}\right) - \frac{\beta_0 \tan\delta_3}{R^3}\left(\frac{R^3}{3} - \frac{e^3}{3}\right) + \frac{(\lambda_d - \lambda_i)(R^2 - e^2)}{2R^2} + \frac{\theta_d}{R^2}\left(\frac{R^2 - e^2}{2}\right)\right]} \quad - (33)$$

β_0 from (19)

Ref. Leishman pg 206

$$M_H = \frac{N_b}{2\pi}\int_0^{2\pi}\int_0^R y F_z \sin\psi \, d\psi$$

$$= \frac{N_b}{2\pi}\int_0^{2\pi}\int_e^R y \frac{1}{2}\rho V_{rel}^2 c \; C_l \sin\psi \, dy \, d\psi \qquad -(34)$$

$\uparrow$ consider only $\sin\psi$ terms

$$C_{l_{\sin\psi}} = -a\left(\beta_{1s}\sin\psi\tan\delta_3 \underbrace{-\beta_{1c}\sin\psi}_{\text{from } \overset{*}{\beta}} - \theta_{1s}\sin\psi\right) \qquad -(35)$$

$$M_H = \frac{N_b \rho \Omega^2 c a}{4}\left(-\beta_{1s}\tan\delta_3 + \beta_{1c} + \theta_{1s}\right)\left(\frac{R^4 - e^4}{4}\right) \qquad -(36)$$

$$C_{M_H} = \frac{N_b \cancel{\rho} \cancel{\Omega^2} c a \cancel{R^4}}{16 \cancel{\rho} \pi \cancel{R^2} \cancel{\Omega^2} \cancel{R^3}}\Big[\left(-A\theta_{1s} - B\theta_{1c}\right)\tan\delta_3 + \qquad -(37)$$

$$A\theta_{1c} - B\theta_{1s} + \theta_{1s}\Big]\left(1 - \frac{e^4}{R^4}\right)$$

$$\boxed{C_{M_H} = \frac{\sigma a}{16}\left[\theta_{1s}\left(1 - A\tan\delta_3 - B\right) + \theta_{1c}\left(A - B\tan\delta_3\right)\right]\left(1 - \frac{e^4}{R^4}\right)} \qquad (38)$$

$$C_{M_y} = \frac{\sigma a}{16}\left[-\beta_{1c}\tan\delta_3 - \beta_{1s} + \theta_{1c}\right]\left(1 - \frac{e^4}{R^4}\right) \qquad (39)$$

$$\boxed{C_{M_y} = \frac{\sigma a}{16}\left[\theta_{1c}\left(1 - A\tan\delta_3 + B\right) + \theta_{1s}\left(B\tan\delta_3 - A\right)\right]\left(1 - \frac{e^4}{R^4}\right)} \qquad (40)$$