

LESSON

4.6

STANDARD FORM OF BOOLEAN EXPRESSIONS

INTENDED LEARNING OUTCOME

At the end of the session, the students will be able to:

1. Identify a sum-of-products expression
2. Determine the domain of a Boolean expression
3. Convert any sum-of-products expression to a standard form
4. Evaluate a standard sum-of-products expression in terms of binary values
5. Identify a product-of-sums expression
6. Convert any product-of-sums expression to a standard form
7. Evaluate a standard product-of-sums expression in terms of binary values
8. Convert from one standard form to the other

THE SUM-OF- PRODUCTS (SOP) FORM

- When two or more product terms are summed by Boolean addition, the resulting expression is a sum-of-products (SOP). Some examples are

$$AB + ABC$$

$$ABC + CDE + \overline{B}C\overline{D}$$

$$\overline{A}B + \overline{A}B\overline{C} + AC$$

DOMAIN OF A BOOLEAN EXPRESSION

The **domain of a general Boolean expression** is the set of variables contained in the expression in either complemented or uncomplemented form.

For example,

- *the domain of the expression $AB + ABC$ is the set of variables A, B, C*
- *the domain of the expression $ABC + CDE + BCD$ is the set of variables A, B, C, D, E.*

AND/OR IMPLEMENTATION OF AN SOP EXPRESSION

To implement a Sum-of-Products (SOP) expression, you need to OR the outputs of two or more AND gates. Each product term is generated by an AND operation, and the sum of these product terms is achieved with an OR operation.

For example, Figure 4–22 for the SOP expression $AB + BCD + AC$.

The output X of the OR gate equals the SOP expression.

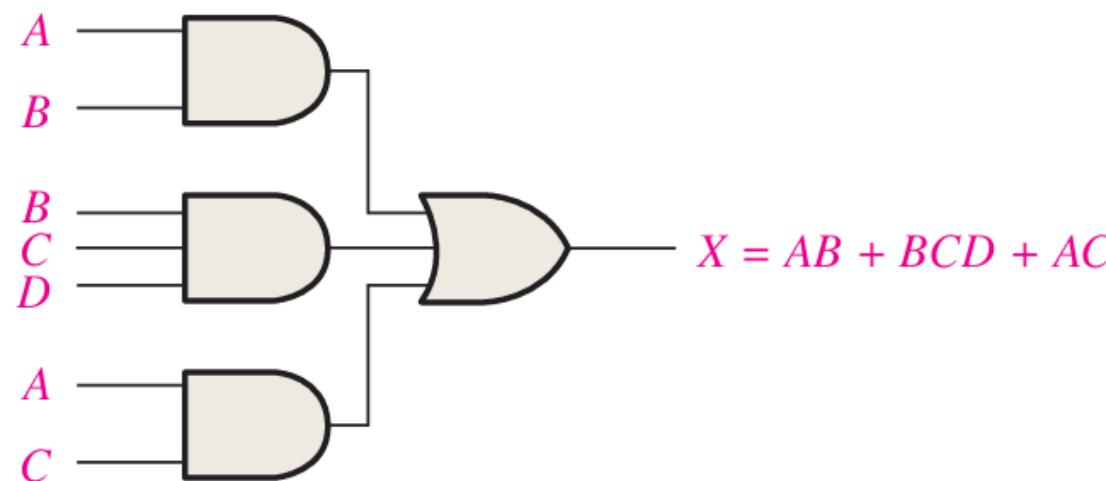


FIGURE 4–22 Implementation of the SOP expression $AB + BCD + AC$.

NAND/NAND IMPLEMENTATION OF AN SOP EXPRESSION

NAND gates can be used to implement SOP expression. By using only NAND gates, an AND/OR function can be accomplished, as illustrated in Figure 4–23. The first level of NAND gates feed into a NAND gate that acts as a negative-OR gate. The NAND and negative-OR inversions cancel and the result is effectively an AND/OR circuit.

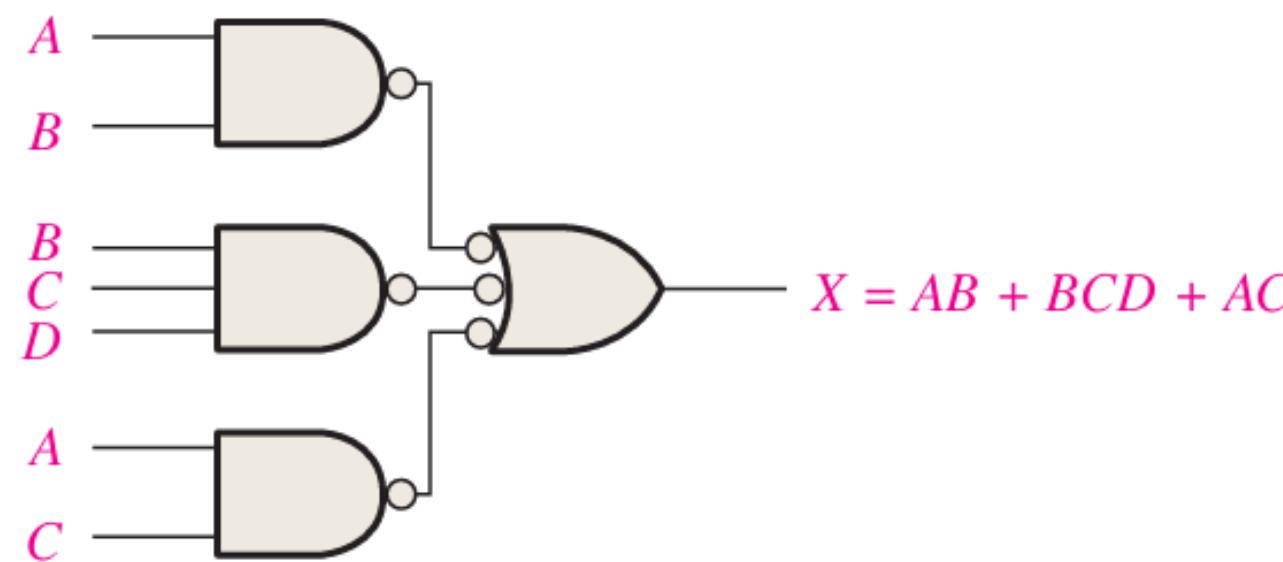


FIGURE 4–23 This NAND/NAND implementation is equivalent to the AND/OR in Figure 4–22.

CONVERSION OF A GENERAL EXPRESSION TO SOP FORM

Any logic expression can be changed into SOP form by applying Boolean algebra techniques.

For example, the expression $A(B + CD)$ can be converted to SOP form by applying the distributive law:

$$A(B + CD) = AB + ACD$$

EXAMPLES WITH SOLUTION:

EXAMPLE 4-14

Convert each of the following Boolean expressions to SOP form:

$$(a) AB + B(CD + EF) \quad (b) (A + B)(B + C + D) \quad (c) \overline{(A + B)} + C$$

Solution

$$(a) AB + B(CD + EF) = AB + BCD + BEF$$

$$(b) (A + B)(B + C + D) = AB + AC + AD + BB + BC + BD$$

$$(c) \overline{(A + B)} + C = \overline{\overline{A} + \overline{B}}\overline{C} = (A + B)\overline{C} = A\overline{C} + B\overline{C}$$

THE STANDARD SOP FORM

- A **standard SOP expression** is one in which all the variables in the domain appear in each product term in the expression. For example, $ABCD + A'BCD + ABC'D + ABCD'$ is a standard SOP expression. Any nonstandard SOP expression (referred to simply as SOP) can be converted to the standard form using Boolean algebra.

THE STANDARD SOP FORM

So far, you have seen SOP expressions in which some of the product terms do not contain all of the variables in the domain of the expression. For example, the expression $\bar{A}\bar{B}\bar{C} + A\bar{B}D + \bar{A}B\bar{C}D$ has a domain made up of the variables A, B, C, and D.

However, notice that the complete set of variables in the domain is not represented in the first two terms of the expression;

- D or \bar{D} is missing from the first term
- C or \bar{C} is missing from the second term.

CONVERTING PRODUCT TERMS TO STANDARD SOP

Each product term in an SOP expression that does not contain all the variables in the domain can be expanded to standard form to include all variables in the domain and their complements. As stated in the following steps, a nonstandard SOP expression is converted into standard form using Boolean algebra rule 6 ($A + \bar{A} = 1$) : A variable added to its complement equals 1

EXAMPLE WITH SOLUTION:

EXAMPLE 4-15

Convert the following Boolean expression into standard SOP form:

$$A\bar{B}C + \bar{A}\bar{B} + A\bar{B}\bar{C}D$$

Solution

The domain of this SOP expression is A, B, C, D . Take one term at a time. The first term, $A\bar{B}C$, is missing variable D or \bar{D} , so multiply the first term by $D + \bar{D}$ as follows:

$$A\bar{B}C = A\bar{B}C(D + \bar{D}) = A\bar{B}CD + A\bar{B}C\bar{D}$$

In this case, two standard product terms are the result.

The second term, $\bar{A}\bar{B}$, is missing variables C or \bar{C} and D or \bar{D} , so first multiply the second term by $C + \bar{C}$ as follows:

$$\bar{A}\bar{B} = \bar{A}\bar{B}(C + \bar{C}) = \bar{A}\bar{B}C + \bar{A}\bar{B}\bar{C}$$

The two resulting terms are missing variable D or \bar{D} , so multiply both terms by $D + \bar{D}$ as follows:

$$\begin{aligned}\bar{A}\bar{B} &= \bar{A}\bar{B}C + \bar{A}\bar{B}\bar{C} = \bar{A}\bar{B}C(D + \bar{D}) + \bar{A}\bar{B}\bar{C}(D + \bar{D}) \\ &= \bar{A}\bar{B}CD + \bar{A}\bar{B}C\bar{D} + \bar{A}\bar{B}\bar{C}D + \bar{A}\bar{B}\bar{C}\bar{D}\end{aligned}$$

In this case, four standard product terms are the result.

The third term, $A\bar{B}\bar{C}D$, is already in standard form. The complete standard SOP form of the original expression is as follows:

$$A\bar{B}C + \bar{A}\bar{B} + A\bar{B}\bar{C}D = A\bar{B}CD + A\bar{B}\bar{C}\bar{D} + \bar{A}\bar{B}CD + \bar{A}\bar{B}\bar{C}\bar{D} + \bar{A}\bar{B}\bar{C}D + \bar{A}\bar{B}\bar{C}\bar{D} + A\bar{B}\bar{C}D$$

Related Problem

Convert the expression $W\bar{X}Y + \bar{X}Y\bar{Z} + W\bar{X}\bar{Y}$ to standard SOP form.

BINARY REPRESENTATION OF A STANDARD PRODUCT TERM

A standard product term is equal to 1 for only one combination of variable values. For example, the product term $A\bar{B}\bar{C}\bar{D}$ is equal to 1 when $A = 1, B = 0, C = 1, D = 0$, as shown below, and is 0 for all other combinations of values for the variables.

$$A\bar{B}\bar{C}\bar{D} = 1 \cdot \bar{0} \cdot 1 \cdot \bar{0} = 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

In this case, the product term has a binary value of 1010 (decimal ten).

Remember, a product term is implemented with an AND gate whose output is 1 only if each of its inputs is 1. Inverters are used to produce the complements of the variables as required.

An SOP expression is equal to 1 only if one or more of the product terms in the expression is equal to 1.

EXAMPLE AND SOLUTION

EXAMPLE 4-16

Determine the binary values for which the following standard SOP expression is equal to 1:

$$ABCD + A\bar{B}\bar{C}D + \bar{A}\bar{B}\bar{C}\bar{D}$$

Solution

The term $ABCD$ is equal to 1 when $A = 1$, $B = 1$, $C = 1$, and $D = 1$.

$$ABCD = 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

Standard Forms of Boolean Expressions

The term $A\bar{B}\bar{C}D$ is equal to 1 when $A = 1$, $B = 0$, $C = 0$, and $D = 1$.

$$A\bar{B}\bar{C}D = 1 \cdot \bar{0} \cdot \bar{0} \cdot 1 = 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

The term $\bar{A}\bar{B}\bar{C}\bar{D}$ is equal to 1 when $A = 0$, $B = 0$, $C = 0$, and $D = 0$.

$$\bar{A}\bar{B}\bar{C}\bar{D} = \bar{0} \cdot \bar{0} \cdot \bar{0} \cdot \bar{0} = 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

The SOP expression equals 1 when any or all of the three product terms is 1.

THE PRODUCT-OF-SUMS (POS) FORM

A sum term was defined in Section 4–1 as a term consisting of the sum (Boolean addition) of literals (variables or their complements). When two or more sum terms are multiplied, the resulting expression is a **product-of-sums (POS)**. Some examples are

$$(\bar{A} + B)(A + \bar{B} + C)$$

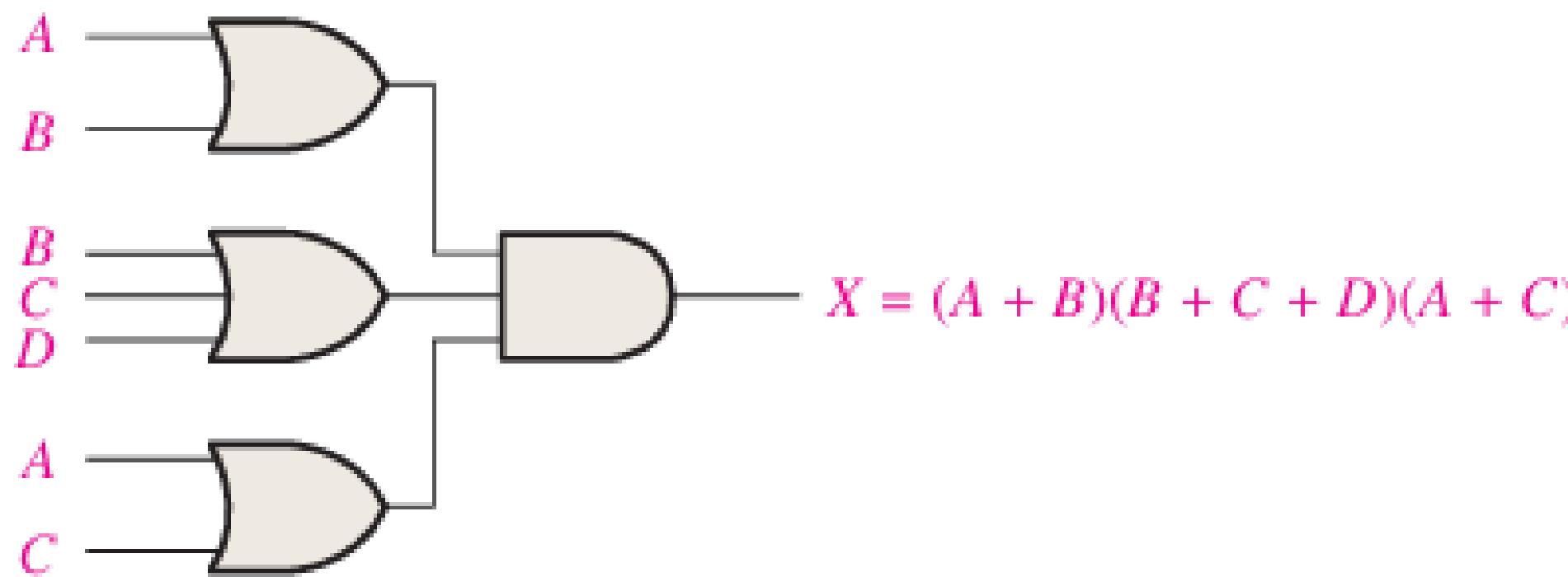
$$(\bar{A} + \bar{B} + \bar{C})(C + \bar{D} + E)(\bar{B} + C + D)$$

$$(A + B)(A + \bar{B} + C)(\bar{A} + C)$$

A POS expression can contain a single-variable term, as in $\bar{A}(A + \bar{B} + C)(\bar{B} + \bar{C} + D)$. In a POS expression, a single overbar cannot extend over more than one variable; however, more than one variable in a term can have an overbar. For example, a POS expression can have the term $\bar{A} + \bar{B} + \bar{C}$ but not $\overline{A + B + C}$.

IMPLEMENTATION OF A POS EXPRESSION

Implementing a POS expression simply requires ANDing the outputs of two or more OR gates. A sum term is produced by an OR operation, and the product of two or more sum terms is produced by an AND operation. Therefore, a POS expression can be implemented by logic in which the outputs of a number (equal to the number of sum terms in the expression) of OR gates connect to the inputs of an AND gate, as Figure 4–24 shows for the expression $(A + B)(B + C + D)(A + C)$. The output X of the AND gate equals the POS expression.



STANDARD POS FORM

- A **standard POS expression** is one in which all the variables in the domain appear in each sum term in the expression. For example,

$$(\bar{A} + \bar{B} + \bar{C} + \bar{D})(A + \bar{B} + C + D)(A + B + \bar{C} + D)$$

is a standard POS expression. Any nonstandard POS expression (referred to simply as POS) can be converted to the standard form using Boolean algebra.

THE STANDARD POS FORM

- POS expressions in which some of the sum terms do not contain all of the variables in the domain of the expression. For example, the expression

$$(A + \bar{B} + C)(A + B + \bar{D})(A + \bar{B} + \bar{C} + D)$$

has a domain made up of the variables A, B, C, and D. Notice that the complete set of variables in the domain is not represented in the first two terms of the expression; that is, D or \bar{D} is missing from the first term and C or \bar{C} is missing from the second term.

CONVERTING A SUM TERM TO STANDARD POS

Each sum term in a POS expression that does not contain all the variables in the domain can be expanded to standard form to include all variables in the domain and their complements. As stated in the following steps, a nonstandard POS expression is converted into standard form using Boolean algebra rule 8 ($A \cdot \bar{A} = 0$) from Table 4–1: A variable multiplied by its complement equals 0.

- Step 1:** Add to each nonstandard product term a term made up of the product of the missing variable and its complement. This results in two sum terms. As you know, you can add 0 to anything without changing its value.
- Step 2:** Apply rule 12 from Table 4–1: $A + BC = (A + B)(A + C)$
- Step 3:** Repeat Step 1 until all resulting sum terms contain all variables in the domain in either complemented or uncomplemented form.

EXAMPLE WITH SOLUTION:

EXAMPLE 4-17

Convert the following Boolean expression into standard POS form:

$$(A + \bar{B} + C)(\bar{B} + C + \bar{D})(A + \bar{B} + \bar{C} + D)$$

Solution

The domain of this POS expression is A, B, C, D . Take one term at a time. The first term, $A + \bar{B} + C$, is missing variable D or \bar{D} , so add $D\bar{D}$ and apply rule 12 as follows:

$$A + \bar{B} + C = A + \bar{B} + C + D\bar{D} = (A + \bar{B} + C + D)(A + \bar{B} + C + \bar{D})$$

The second term, $\bar{B} + C + \bar{D}$, is missing variable A or \bar{A} , so add $A\bar{A}$ and apply rule 12 as follows:

$$\bar{B} + C + \bar{D} = \bar{B} + C + \bar{D} + A\bar{A} = (A + \bar{B} + C + \bar{D})(\bar{A} + \bar{B} + C + \bar{D})$$

The third term, $A + \bar{B} + \bar{C} + D$, is already in standard form. The standard POS form of the original expression is as follows:

$$(A + \bar{B} + C)(\bar{B} + C + \bar{D})(A + \bar{B} + \bar{C} + D) = \\ (A + \bar{B} + C + D)(A + \bar{B} + C + \bar{D})(A + \bar{B} + \bar{C} + D)(\bar{A} + \bar{B} + C + \bar{D})(\bar{A} + \bar{B} + \bar{C} + D)$$

BINARY REPRESENTATION OF A STANDARD SUM TERM

A standard sum term is equal to 0 for only one combination of variable values. For example, the sum term $A + \bar{B} + C + \bar{D}$ is 0 when $A = 0$, $B = 1$, $C = 0$, and $D = 1$, as shown below, and is 1 for all other combinations of values for the variables.

$$A + \bar{B} + C + \bar{D} = 0 + \bar{1} + 0 + \bar{1} = 0 + 0 + 0 + 0 = 0$$

In this case, the sum term has a binary value of 0101 (decimal 5). Remember, a sum term is implemented with an OR gate whose output is 0 only if each of its inputs is 0. Inverters are used to produce the complements of the variables as required.

A POS expression is equal to 0 only if one or more of the sum terms in the expression is equal to 0.

EXAMPLE WITH SOLUTION:

EXAMPLE 4-18

Determine the binary values of the variables for which the following standard POS expression is equal to 0:

$$(A + B + C + D)(A + \bar{B} + \bar{C} + D)(\bar{A} + \bar{B} + \bar{C} + \bar{D})$$

Solution

The term $A + B + C + D$ is equal to 0 when $A = 0, B = 0, C = 0$, and $D = 0$.

$$A + B + C + D = 0 + 0 + 0 + 0 = 0$$

The term $A + \bar{B} + \bar{C} + D$ is equal to 0 when $A = 0, B = 1, C = 1$, and $D = 0$.

$$A + \bar{B} + \bar{C} + D = 0 + \bar{1} + \bar{1} + 0 = 0 + 0 + 0 + 0 = 0$$

The term $\bar{A} + \bar{B} + \bar{C} + \bar{D}$ is equal to 0 when $A = 1, B = 1, C = 1$, and $D = 1$.

$$\bar{A} + \bar{B} + \bar{C} + \bar{D} = \bar{1} + \bar{1} + \bar{1} + \bar{1} = 0 + 0 + 0 + 0 = 0$$

The POS expression equals 0 when any of the three sum terms equals 0.

CONVERTING STANDARD SOP TO STANDARD POS

The binary values of the product terms in a given standard SOP expression are not present in the equivalent standard POS expression. Also, the binary values that are not represented in the SOP expression are present in the equivalent POS expression. Therefore, to convert from standard SOP to standard POS, the following steps are taken:

- Step 1:** Evaluate each product term in the SOP expression. That is, determine the binary numbers that represent the product terms.
- Step 2:** Determine all of the binary numbers not included in the evaluation in Step 1.
- Step 3:** Write the equivalent sum term for each binary number from Step 2 and express in POS form.

Using a similar procedure, you can go from POS to SOP.

EXAMPLE WITH SOLUTION:

EXAMPLE 4-19

Convert the following SOP expression to an equivalent POS expression:

$$\bar{A}\bar{B}\bar{C} + \bar{A}B\bar{C} + \bar{A}BC + A\bar{B}C + ABC$$

Solution

The evaluation is as follows:

$$000 + 010 + 011 + 101 + 111$$

Since there are three variables in the domain of this expression, there are a total of eight (2^3) possible combinations. The SOP expression contains five of these combinations, so the POS must contain the other three which are 001, 100, and 110. Remember, these are the binary values that make the sum term 0. The equivalent POS expression is

$$(A + B + \bar{C})(\bar{A} + B + C)(\bar{A} + \bar{B} + C)$$

LESSON

4.7

BOOLEAN EXPRESSIONS AND TRUTH TABLES

By Maria Shiela Joy Vieja

INTENDED LEARNING OUTCOME

At the end of the session, the students will be able to;

1. Convert a standard SOP expression into truth table format
2. Convert a standard POS expression into truth table format
3. Derive a standard expression from a truth table
4. Properly interpret truth table data

CONVERTING SOP EXPRESSIONS TO TRUTH TABLE FORMAT

A **truth table** is simply a list of the possible combinations of input variable values and the corresponding output values (1 or 0).

- For an expression with a domain of two variables, there are four different combinations of those variables ($2^2 = 4$).
- For an expression with a domain of three variables, there are eight different combinations of those variables ($2^3 = 8$).
- For an expression with a domain of four variables, there are sixteen different combinations of those variables ($2^4 = 16$), and so on.

STEPS IN CONSTRUCTING TRUTH TABLE

- Step 1:** List all possible combinations of binary values of the variables in the expression.
- Step 2:** Convert the SOP expression to standard form if it is not already.
- Step 3:** Place a 1 in the output column (X) for each binary value that makes the standard SOP expression a 1 and place a 0 for all the remaining binary values.

EXAMPLE WITH SOLUTION:

EXAMPLE 4-20

Develop a truth table for the standard SOP expression $\bar{A}\bar{B}C + A\bar{B}\bar{C} + ABC$.

Solution

There are three variables in the domain, so there are eight possible combinations of binary values of the variables as listed in the left three columns of Table 4–6. The binary values that make the product terms in the expressions equal to 1 are

Boolean Expressions and Truth Tables

TABLE 4-6

Inputs			Output	Product Term
A	B	C	X	
0	0	0	0	
0	0	1	1	$\bar{A}\bar{B}C$
0	1	0	0	
0	1	1	0	
1	0	0	1	$A\bar{B}\bar{C}$
1	0	1	0	
1	1	0	0	
1	1	1	1	ABC

$\bar{A}\bar{B}C$: 001; $A\bar{B}\bar{C}$: 100; and ABC : 111. For each of these binary values, place a 1 in the output column as shown in the table. For each of the remaining binary combinations, place a 0 in the output column.

CONVERTING POS EXPRESSIONS TO TRUTH TABLE FORMAT

Recall that a POS expression is equal to 0 only if at least one of the sum terms is equal to 0.

STEPS TO CONSTRUCT A TRUTH TABLE FROM A POS EXPRESSION

- Step 1:** List all the possible combinations of binary values of the variables just as was done for the SOP expression.
- Step 2:** Convert the POS expression to standard form if it is not already.
- Step 3:** Place a 0 in the output column (X) for each binary value that makes the expression a 0 and place a 1 for all the remaining binary values.

EXAMPLE WITH SOLUTION:

EXAMPLE 4-21

Determine the truth table for the following standard POS expression:

$$(A + B + C)(A + \bar{B} + C)(A + \bar{B} + \bar{C})(\bar{A} + B + \bar{C})(\bar{A} + \bar{B} + C)$$

Solution

There are three variables in the domain and the eight possible binary values are listed in the left three columns of Table 4-7. The binary values that make the sum terms in the expression equal to 0 are $A + B + C$: 000; $A + \bar{B} + C$: 010; $A + \bar{B} + \bar{C}$: 011; $\bar{A} + B + \bar{C}$: 101; and $\bar{A} + \bar{B} + C$: 110. For each of these binary values, place a 0 in the output column as shown in the table. For each of the remaining binary combinations, place a 1 in the output column.

TABLE 4-7

Inputs			Output	Sum Term
A	B	C	X	
0	0	0	0	$(A + B + C)$
0	0	1	1	
0	1	0	0	$(A + \bar{B} + C)$
0	1	1	0	$(A + \bar{B} + \bar{C})$
1	0	0	1	
1	0	1	0	$(\bar{A} + B + \bar{C})$
1	1	0	0	$(\bar{A} + \bar{B} + C)$
1	1	1	1	

DETERMINING STANDARD EXPRESSIONS FROM A TRUTH TABLE

To determine the standard SOP expression represented by a truth table, list the binary values of the input variables for which the output is 1. Convert each binary value to the corresponding product term by replacing each 1 with the corresponding variable and each 0 with the corresponding variable complement. For example, the binary value 1010 is converted to a product term as follows:

$$1010 \longrightarrow A\bar{B}C\bar{D}$$

If you substitute, you can see that the product term is 1:

$$A\bar{B}C\bar{D} = 1 \cdot \bar{0} \cdot 1 \cdot \bar{0} = 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

To determine the standard POS expression represented by a truth table, list the binary values for which the output is 0. Convert each binary value to the corresponding sum term by replacing each 1 with the corresponding variable complement and each 0 with the corresponding variable. For example, the binary value 1001 is converted to a sum term as follows:

$$1001 \longrightarrow \bar{A} + B + C + \bar{D}$$

If you substitute, you can see that the sum term is 0:

$$\bar{A} + B + C + \bar{D} = \bar{1} + 0 + 0 + \bar{1} = 0 + 0 + 0 + 0 = 0$$

EXAMPLE WITH SOLUTION:

EXAMPLE 4-22

From the truth table in Table 4–8, determine the standard SOP expression and the equivalent standard POS expression.

TABLE 4-8

Inputs			Output
A	B	C	X
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

EXAMPLE WITH SOLUTION:

Solution

There are four 1s in the output column and the corresponding binary values are 011, 100, 110, and 111. Convert these binary values to product terms as follows:

$$011 \longrightarrow \bar{A}BC$$

$$100 \longrightarrow A\bar{B}\bar{C}$$

$$110 \longrightarrow AB\bar{C}$$

$$111 \longrightarrow ABC$$

The resulting standard SOP expression for the output X is

$$X = \bar{A}BC + A\bar{B}\bar{C} + AB\bar{C} + ABC$$

For the POS expression, the output is 0 for binary values 000, 001, 010, and 101.

Convert these binary values to sum terms as follows:

$$000 \longrightarrow A + B + C$$

$$001 \longrightarrow A + B + \bar{C}$$

$$010 \longrightarrow A + \bar{B} + C$$

$$101 \longrightarrow \bar{A} + B + \bar{C}$$

The resulting standard POS expression for the output X is

$$X = (A + B + C)(A + B + \bar{C})(A + \bar{B} + C)(\bar{A} + B + \bar{C})$$

QUIZ:

I. **Direction:** Answer and comply the following questions in a 1-whole sheet of paper.

1. Identify each of the following expressions as SOP, standard SOP, POS, or standard POS:
 - (a) $AB + \overline{A}BD + \overline{AC}\overline{D}$
 - (b) $(A + \overline{B} + C)(A + B + \overline{C})$
 - (c) $\overline{ABC} + A\overline{B}\overline{C}$
 - (d) $(A + \overline{C})(A + B)$
2. Convert each SOP expression in Question 1 to standard form.
3. Convert each POS expression in Question 1 to standard form.

QUIZ:

II. Direction: Answer and comply the following questions in a 1-whole sheet of paper.

- 1.** If a certain Boolean expression has a domain of five variables, how many binary values will be in its truth table?
- 2.** In a certain truth table, the output is a 1 for the binary value 0110. Convert this binary value to the corresponding product term using variables W , X , Y , and Z .
- 3.** In a certain truth table, the output is a 0 for the binary value 1100. Convert this binary value to the corresponding sum term using variables W , X , Y , and Z .



THANK YOU

