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COLLEGE PHYSICAL SCIENCES

DEPARTMENT OF MATHEMATICS

MTS 101 (MATRICES: LECTURE NOTE)

BY

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COURSE OUTLINE

Matrices ($m, n \leq 3$)

- Matrix Notations
- Algebra of Matrices
- Determinants
- Inverse of Matrix
- Solution of linear system of equations.

MODULE FIVE

Introduction

Matrices play a very crucial role in linear algebra especially in solving systems of equations and other computational problems. This module is basically devoted to the study of matrices with the associated arithmetic operations. We begin with the following definition.

Definition 1.1 A matrix is a rectangular array of numbers. The numbers in the array are called entries or elements.

Example 1 : Some examples of matrices are given below:

$$(i)\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}_{m\times n}.(ii)\begin{pmatrix} 4 & 5 & 6 \\ 2 & 3 & 1 \\ 7 & 8 & 9 \end{pmatrix}_{3\times 3}.(iii)\begin{pmatrix} -5 & 6 \\ \sqrt{2} & \pi \end{pmatrix}_{2\times 2}.(iv)\begin{pmatrix} -1 & 1 \end{pmatrix}_{1\times 2}.(v)\begin{pmatrix} \frac{1}{3} \\ -3 \end{pmatrix}_{2\times 1}.$$

The subscript in each of the above matrices indicates the order of each matrix .

1.1 Matrix Notations

The order of a matrix, or size of a matrix is determined by the numbers of rows (horizontal) and columns (vertical) it contains. If a matrix has m-rows and n-columns, the order is $m \times n$ read as "m by n" matrix. A matrix is usually denoted by a capital letter with boldface font (e.g., **A, B, X**). The elements of the matrix are represented by small letter with double subscripts (e.g., a_{jk}, b_{jk}, x_{jk}). For instance, matrix X, x_{13} is the element in the first row and the third column.

Example 2 : Identify the order of the following matrices:

$$\text{(i)}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix},(ii)\begin{pmatrix} 4 & 5 & 6 \\ 2 & 1 & 3 \end{pmatrix},(iii)\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix},(iv)\begin{pmatrix} 1 & -1 \end{pmatrix},(v)\begin{pmatrix} -5 \end{pmatrix}$$

Solution

(i) 3×3 (ii) 2×3 (iii) 2×2 (iv) 1×2 (v) 1×1

A matrix with only one row is called a row- vector while a matrix with only one column is called column matrix or column-vector.

Example 3: $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \rightarrow$ Column-vector and $H = \begin{pmatrix} h_1 & h_2 & h_3 \end{pmatrix} \rightarrow$ Row- vector

1.1.1 Some Classes (types) of Matrices

A matrix A is called a square matrix if the number of rows and the number of columns are equal.

Example 4 : (i) $\begin{pmatrix} 4 & 5 & 6 \\ 2 & 3 & 1 \\ 7 & 8 & 9 \end{pmatrix}$ (ii) $\begin{pmatrix} -2 & 3 \\ 7 & 5 \end{pmatrix}$ (iii) $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$

1.1.2 Diagonal Matrix : is a matrix in which the non-diagonal elements are all zero.

Example 5 : (i) $\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$ (ii) $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

1.1.3 Upper Triangular Matrix. An upper triangular matrix is a matrix in which all entries below the diagonal are zero.

Example 6 : (i) $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$ (ii) $\begin{pmatrix} 3 & 6 & 8 & 1 \\ 0 & 5 & -1 & -3 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

1.1.4 Lower Triangular Matrix : is a square matrix whose entries above the main diagonal are zero.

Example 7: (i) $\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ (ii) $\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ -3 & -1 & 2 & 0 & 0 \\ 4 & 5 & 8 & 1 & 0 \\ 2 & 6 & 4 & 3 & 7 \end{pmatrix}$

1.1.5 Identity Matrix : is a diagonal matrix in which the diagonal elements are all 1 and zero else where.

Example 8 : (i) $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (ii) $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

1.1.6 Zero or Null Matrix : is matrix in which all it's entries are all zero.

Example 9 : (i) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

1.1.7 Equality of two Matrices : Two matrices are said to be equal if their orders and corresponding entries are equal.

Examples 10 : Consider the following matrices;

$(i)\ A = \begin{pmatrix} 3 & 2 & 1 \\ -2 & -5 & 1 \end{pmatrix}$ $(ii)\ B = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$

If $A = B$, it implies that the order of A and B are equal and their corresponding entries are equal. That is; A is of order 2×3 and B is of order 2×3, $a = 3, b = 2, c = 1, d = -2, e = -5$ and $f = 3$.

1.1.8 Transpose of a Matrix : This simply means to interchange the rows and the columns or vice versa. The transpose of a matrix A is denoted as $A^T orA'$

Example 11 : $(i)\ A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ $Thus,\ A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix}$

More examples are given below;

$(i)\ C = \begin{pmatrix} 3 & 2 & 1 \\ 4 & -1 & 2 \end{pmatrix}$ $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

1.1. 9 Involuntary Matrix

Given a matrix A, if $A^2 = I$, where A and I are of the same order, we say that A is an involuntary matrix.

Example 12. Let

$$A = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A is not an identity matrix.

Now,

$$A^2 = AA = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

i.e., $A^2 = I$. Thus, A is involuntary.

1.1.10 Idempotent Matrix If for any square matrix A, is such that $A^2 = A$, we say that A is an idempotent matrix.

Example 13 Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Solution.

$$A^2 = AA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A$$

Thus, $A^2 = A$

2.1 Algebra of Matrices.

Unlike the set of real numbers, only three algebraic operations are defined in matrix. These are addition, subtraction and multiplication. Under multiplication, we have scalar multiplication. In this case, a matrix is multiplied by a scalar. We also have matrix multiplication where two matrices A and B are multiplied according to some rules to obtain their product AB or BA of both where possible. Division is not defined on matrices. That is, for two matrices A and B $\frac{A}{B}$ *or* $\frac{B}{B}$ is not defined.

2.1.1 Addition and Subtraction of Matrices

When two matrices are of the same order, the corresponding elements can be added to obtain new matrix. The sum of two matrices A and B of the same order is a third matrix C whose elements are formed by adding the corresponding elements of A and B. We say that when two matrices are of the same order of addition, they are conformable.

For instance. Given,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

Then

$$A+B=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{pmatrix}$$

Similarly $A-B$ is a third matrix whose elements are formed by subtracting the corresponding elements of matrix B from that of matrix A. i.e.,

$$A-B=\begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} & a_{13}-b_{13} \\ a_{21}-b_{21} & a_{22}-b_{22} & a_{23}-b_{23} \end{pmatrix}$$

Example 12 : If

$$A=\begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix} \text{ and } B=\begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}$$

Find (i) $A+B$ (ii) $A-B$ (iii) $B-A$

Solution.

(i) $A+B=\begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix}+\begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}=\begin{pmatrix} 3 & 9 \\ 1 & 4 \end{pmatrix}$

(ii) $A-B=\begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix}-\begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}=\begin{pmatrix} 1 & 1 \\ 5 & -2 \end{pmatrix}$

(iii) $B-A=\begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}-\begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix}=\begin{pmatrix} -1 & -1 \\ -5 & 2 \end{pmatrix}$

Note that: $A-B=-(B-A)$

Example 13 : Let

$A = \begin{pmatrix} 7 & 3 \\ -4 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 4 \\ 6 & -5 \end{pmatrix}$ Show that $A+B = B+A$.

Solution.

$$A+B = \begin{pmatrix} 7 & 3 \\ -4 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 4 \\ 6 & -5 \end{pmatrix} = \begin{pmatrix} 8 & 7 \\ 2 & -3 \end{pmatrix}$$

$$B+A = \begin{pmatrix} 1 & 4 \\ 6 & -5 \end{pmatrix} + \begin{pmatrix} 7 & 3 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 7 \\ 2 & -3 \end{pmatrix}$$

Hence, $A+B = B+A$. This property is true of matrix addition in general, and it is called the commutative property of addition of two matrices.

In general, for two matrices A and B of the same order $m \times n$.

(i) $A+B = B+A$ (Commutativity of addition)

(ii) $(A+B)+C = A+(B+C)$ (Associativity of addition)

(iii) $A-B \neq B-A$. Since the operation of subtraction on real numbers is not commutative (i.e., $4-2 \neq 2-4$), then for two matrices $A$and B of the same order $m \times n$, $A-B \neq B-A$.

2.1.2 Matrix Multiplication

Two matrices are conformable for matrix multiplication, if the number of columns of the first matrix is equal to the number of rows of the second matrix. For the product AB, A is the first matrix in the product while B is the second and for the product BA, B is the first matrix in the product while A is the second.

Example 14 : Let

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

Solution.

$$AB = \begin{pmatrix} \lfloor a_{11} & a_{12} \rfloor \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} \rceil & b_{12} \\ b_{21} \rfloor & b_{22} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

where $\begin{pmatrix} c_{11} = a_{11}b_{11} + a_{12}b_{21} & c_{12} = a_{11}b_{12} + a_{12}b_{22} \\ c_{21} = a_{21}b_{11} + a_{22}b_{21} & c_{22} = a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$

Example 15 : Given that

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 1 \\ 5 & -1 \end{pmatrix}, B = \begin{pmatrix} 4 & 3 & -2 \\ 5 & 1 & 4 \end{pmatrix}$$

Find AB. Is BA defined? If yes find BA.

Solution.

$$AB = \begin{pmatrix} 3 & 2 \\ 4 & 1 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} 4 & 3 & -2 \\ 5 & 1 & 4 \end{pmatrix} = \begin{pmatrix} C_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}$$

To obtain AB, multiply 1st row of A with 1st column of matrix B. i.e.,

$$c_{11} = \begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = 3 \times 4 + 2 \times 5 = 12 + 10 = 22$$

$$c_{12} = \begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3 \times 3 + 2 \times 1 = 9 + 2 = 11$$

$$c_{13} = \begin{pmatrix} 3 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ 4 \end{pmatrix} = 3 \times -2 + 2 \times 4 = -6 + 8 = 2$$

$$c_{21} = \begin{pmatrix} 4 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} = 4 \times 4 + 1 \times 5 = 16 + 5 = 21$$

$$c_{22} = \begin{pmatrix} 4 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 4 \times 3 + 1 \times 1 = 12 + 1 = 13$$

$$c_{23} = \begin{pmatrix} 4 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 4 \end{pmatrix} = 4 \times -2 + 1 \times 4 = -8 + 4 = -4$$

Therefore, $AB = \begin{pmatrix} 22 & 11 & 2 \\ 21 & 13 & -4 \end{pmatrix}$

Yes BA is defined, since B is 2×3 matrix and A is 3×2 matrix and the number of columns of B is equal to the number of rows of A, BA is defined with 2×2 matrix.

$$BA = \begin{pmatrix} 4 & 3 & -2 \\ 5 & 1 & 4 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 4 & 1 \\ 5 & -1 \end{pmatrix} = \begin{pmatrix} 14 & 13 \\ 39 & 7 \end{pmatrix}$$

2.1.3 Multiplication of a Matrix by a Scalar

By a scalar we mean any real or complex number. Given a matrix A of any order and a scalar α, the multiplication of A by α is called **Scalar multiplication** and the product is αA. This product αA is called **Scalar product** since αA scales matrix A either up or down depending on the size of the scalar α.

Example 16 : Let

$$A = \begin{pmatrix} 4 & 3 & 1 & 2 \\ -2 & 1 & 0 & -1 \\ 1 & -1 & 2 & 3 \end{pmatrix}$$

(i) For $\alpha = \frac{1}{2}$, find αA

(ii) For $\alpha = -3$, find αA.

Solution.

$$\text{(i) } \alpha A = \alpha \begin{pmatrix} 4 & 3 & 1 & 2 \\ -2 & 1 & 0 & -1 \\ 1 & -1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 4\alpha & 3\alpha & \alpha & 2\alpha \\ -2\alpha & \alpha & 0 & -\alpha \\ \alpha & -\alpha & 2\alpha & 3\alpha \end{pmatrix} = \begin{pmatrix} 2 & \frac{3}{2} & \frac{1}{2} & 1 \\ -1 & \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 1 & \frac{3}{2} \end{pmatrix}$$

(ii) $\alpha A = -3 \begin{pmatrix} 4 & 3 & 1 & 2 \\ -2 & 1 & 0 & -1 \\ 1 & -1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} -12 & -9 & -3 & -6 \\ 6 & -3 & 0 & 3 \\ -3 & 3 & -6 & -9 \end{pmatrix}$

3.1 Determinants of a Matrix

Determinants arose from the solution of simultaneous linear equations. It can only be evaluated for square matrices. For a square matrix A, we denote the determinant by $|A|$ or $detA$.

Example 14 : Find $|A|$ for the 2×2 matrix below:

$$A = \begin{pmatrix} 12 & -4 \\ 3 & 2 \end{pmatrix}$$

So, $|A| = 12 \times 2 - 3 \times -4 = 24 - (-12) = 24 + 12 = 36$

Therefore, $|A| = 36$

Example 15 : Evaluate the determinant of the matrix

$$A = \begin{pmatrix} 3 & 4 & 6 \\ 2 & 1 & -1 \\ -1 & 3 & 5 \end{pmatrix}$$

Solution.

$$|A| = \begin{vmatrix} + & - & + \\ 3 & 4 & 6 \\ 2 & 1 & -1 \\ -1 & 3 & 5 \end{vmatrix} = 3 \begin{vmatrix} 1 & -1 \\ 3 & 5 \end{vmatrix} - 4 \begin{vmatrix} 2 & -1 \\ -1 & 5 \end{vmatrix} + 6 \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix}$$

$= 3(5+3) - 4(10-1) + 6(6+1) = 24 - 36 + 42 = 30$

3.1.1 Singular and non-singular Matrices

If $detA = 0$, we say that the matrix A is a singular matrix, otherwise A is non-singular. The non-singular matrices are useful in application.

3.1.2 The Saurus Rules

The method of Saurus or Saury's rule is a straight forward method for computing determinant of a 3×3 matrix.

For the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

This method of Saurus is as follows:

Step 1: Form a matrix B of order 3×5 by replacing the first two columns

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \lceil a_{11} & a_{12} \rceil \\ a_{21} & a_{22} & a_{23} & \lceil a_{21} & a_{22} \rceil \\ a_{31} & a_{32} & a_{33} & \lceil a_{31} & a_{32} \rceil \end{pmatrix}$$

Step 2: Obtain the product of the right diagonal elements and add them up.

Thus, $S_1 = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$

Step 3: Obtain the product of their left diagonal elements and add them up.

i.e., $S_2 = a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12}$

Step 4: $|A| = S_1 - S_2$.

Example 16 : Consider the matrix

$$A = \begin{pmatrix} 4 & 2 & 3 \\ -1 & 1 & 2 \\ 1 & 0 & 5 \end{pmatrix}$$

Using the Saurus method to compute $|A|$

Solution.

Step 1: Let

$$A = \begin{pmatrix} 4 & 2 & 3 & 4 & 2 \\ -1 & 1 & 2 & -1 & 1 \\ 1 & 0 & 5 & 1 & 0 \end{pmatrix}$$

Step 2: $S_1 = 4 \times 1 \times 5 + 2 \times 2 \times 1 + 3 \times (-1) \times 0 = 24$

Step 3: $S_2 = 1 \times 1 \times 3 + 0 \times 2 \times 4 + 5 \times (-1) \times 2 = 3 + 0 - 10 = -7$

Step 4: $S_1 - S_2 = 24 - (-7) = 31$

3.1.3 Properties of Determinants

(i) The determinant of a matrix and that of its transpose are equal.

Example 17: Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 1 \\ 3 & 2 & -2 \end{pmatrix}, |A| = 14$$

$$A^T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 4 & 1 & -2 \end{pmatrix}, |A^T = 14|$$

(ii) The interchange of two rows and columns changes sign of the determinant without altering its numerical value.

Example 18: Let

$$A = \begin{vmatrix} 4 & -2 \\ 3 & 2 \end{vmatrix}$$

By interchanging rows, we have

$$A = \begin{vmatrix} 3 & 2 \\ 4 & -2 \end{vmatrix}$$

$|A| = 8 + 6 = 14$, $|B| = -6 - 8 = -14, \Rightarrow |A| = -|B|$.

(iii) If every element in a row or in column of a matrix is multiplied by a number k, the determinant is multiplied by k.

(iv) $|kA| = k^n|A|$, where n is the order of A. This is a consequence of (iii).

(v) The determinant of the product of two matrices is the product of their determinant, i.e.

$|AB| = |A||B|$.

The appropriate place signs are given by alternate plus and minus from the top left hand corner which carries a positive (+). The pattern for $n = 3$ and $n = 4$ are given as follows:

$$n=3,\begin{pmatrix}+&-&+\\-&+&-\\+&-&+\end{pmatrix}.n=4,\begin{pmatrix}+&-&+&-\\-&+&-&+\\+&-&+&-\\-&+&-&+\end{pmatrix}$$

3.1.4 Adjoint of a square matrix

The adjoint of a square matrix A denoted by $AdjA$ is the transpose of the matrix formed by substituting each element of the matrix A by its cofactor.

Example 19: Find the adjoint of the matrix

$$A=\begin{pmatrix}3&-2&4\\2&1&5\\3&-1&2\end{pmatrix}$$

Solution.

Step 1. Find the cofactor for each of the element of the matrix.

$$c_{11}=(-1)^{1+1}\begin{vmatrix}1&5\\-1&2\end{vmatrix}=2+5=7$$

$$c_{12}=(-1)^{1+2}\begin{vmatrix}2&5\\3&2\end{vmatrix}=-(4-15)=11$$

$$c_{13}=(-1)^{1+3}\begin{vmatrix}2&1\\3&-1\end{vmatrix}=(-2-3)=-5$$

$$c_{21}=(-1)^{2+1}\begin{vmatrix}-2&4\\-1&2\end{vmatrix}=-(-4+4)=0$$

$$c_{22}=(-1)^{2+2}\begin{vmatrix}3&4\\3&2\end{vmatrix}=6-12=-6$$

$$c_{23}=(-1)^{2+3}\begin{vmatrix}3&-2\\3&-1\end{vmatrix}=-(-3+6)=-3$$

$$c_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 4 \\ 1 & 5 \end{vmatrix} = (-10-4) = -14$$

$$c_{32} = (-1)^{3+2}\begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} = -(15-) = -7$$

$$c_{33} = (-1)^{3+3}\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = 3+4 = 7$$

Step 2: Form the matrix of cofactors C.

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = \begin{pmatrix} 7 & 11 & -5 \\ 0 & -6 & -3 \\ -14 & -7 & 7 \end{pmatrix}$$

Step 3: $AdjA = C^T = \begin{pmatrix} 7 & 0 & -14 \\ 11 & -6 & -7 \\ -5 & -3 & 7 \end{pmatrix}$

4.1 The Inverse of a Matrix

Given that A and B are two non-singular square matrices of the same order, if $AB = BA = I$ where I is identity matrix, then B is called **the inverse of** A and A the inverse of B. The inverse of the matrix A is usually denoted by A^{-1} and

$A^{-1}A = AA^{-1} = I$.

To find the inverse of a square matrix A, we use the following steps,

(i) Evaluate the determinant of A.

If the determinant of A is nonzero, then

(ii) Form a matrix C of the cofactors of the elements of A.

(iii) Write the transpose of C to obtain the adjoint of A.

(iv) Divide each elements of the matrix C^T by $|A|$

(v) The resulting matrix is the inverse, A^{-1} of the original matrix A.

$$A^{-1} = \frac{1}{|A|}AdjA = \frac{1}{|A|}C^T$$

Solution of Linear System of Equation

The linear equation over a real field $\mathbb{R}$ is an expression of the form

$a_1x_1 + a_2x_2 + ... + a_nx_n = b$

Where $a_i, b \in \mathbb{R}$, x_i are the unknowns and a_i are the coefficients.

If the system of equations has a solution, then it is said to be **Consistent** otherwise it is said to be inconsistent. A solution of a system of linear equations where all the x_i are zero is called zero solution or trivial solution. A solution whereby not all x_i's are zero is called non-zero or non-trivial solution. Using the matrix notation, we have,

$AX = B$, where A is the matrix of coefficients of the system of equations, B is the column matrix of the constants and X is the column matrix of the unknown. If A^{-1} exists, then

$$A^{-1}AX = A^{-1}B, \Rightarrow X = A^{-1}B$$

Example 19: Use the matrix method to solve the set of equations

$$\begin{aligned} 3x - 2y + 4z &= 2 \\ 2x + y - 5z &= -1 \\ 3x - y - 2z &= -27 \end{aligned} \tag{1}$$

Solution. We write the equation in matrix form

$AX = B$,

i.e., $\begin{pmatrix} 3 & -2 & 4 \\ 2 & 1 & 5 \\ 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -27 \end{pmatrix}$

where $A = \begin{pmatrix} 3 & -2 & 4 \\ 2 & 1 & 5 \\ 3 & -1 & 2 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 2 \\ -1 \\ -27 \end{pmatrix}$

Next we find the determinant of A.

$|A| = 3\begin{pmatrix} 1 & 5 \\ -1 & 2 \end{pmatrix} - (-2)\begin{pmatrix} 2 & 5 \\ 3 & 2 \end{pmatrix} + 4\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} = 3(2-(-5)) - (-2)(4-15) + 4(-2-3) = 3(7) + 2(-11) + 4(-5) = -23$

Now we find the cofactor of each element in A.

$A_{11} = (-1)^{1+1}\begin{vmatrix} 1 & 5 \\ -1 & 2 \end{vmatrix} = 2+5 = 7$

$A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 5 \\ 3 & 2 \end{vmatrix} = -(4-15) = 11$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = (-2-3) = -5$

$A_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 4 \\ -1 & 2 \end{vmatrix} = -(-4+4) = 0$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 3 & 4 \\ 3 & 2 \end{vmatrix} = 6-12 = -6$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & -2 \\ 3 & -1 \end{vmatrix} = -(-3+6) = -3$

$A_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 4 \\ 1 & 5 \end{vmatrix} = (-10-4) = -14$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} = -(15-) = -7$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = 3+4 = 7$

Form the matrix of cofactors C.

$$C = \begin{pmatrix} 7 & 11 & -5 \\ 0 & -6 & -3 \\ -14 & -7 & 7 \end{pmatrix}$$

$$AdjA = C^T = \begin{pmatrix} 7 & 0 & -14 \\ 11 & -6 & -7 \\ -5 & -3 & 7 \end{pmatrix}$$

$AX = B \Rightarrow X = A^{-1}B$

$$A^{-1} = \frac{1}{-23} \begin{pmatrix} 7 & 0 & -14 \\ 11 & -6 & -7 \\ -5 & -3 & 7 \end{pmatrix}$$

$$X = \frac{1}{-23} \begin{pmatrix} 7 & 0 & -14 \\ 11 & -6 & -7 \\ -5 & -3 & 7 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ -27 \end{pmatrix} = \frac{1}{-23} \begin{pmatrix} 392 \\ 217 \\ -196 \end{pmatrix} = \begin{pmatrix} -\frac{392}{23} \\ -\frac{217}{23} \\ \frac{196}{23} \end{pmatrix}$$

Therefore, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{392}{23} \\ -\frac{217}{23} \\ \frac{196}{23} \end{pmatrix}$.

Hence, $x = -\frac{392}{23}, y = -\frac{217}{23}, z = \frac{196}{23}$