MTS 106 - Calculus for Biological Sciences - 3 Units

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A Virtual Lecture Presented To: Agriculture and Biological Science Students

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Abstract

The objective of this lecture is to teach Functions. Domain and Range of Functions. Graphs of Elementary Functions. One-to-one and Onto Functions. Composition of Functions. Limit and Continuity of Functions. Discontinuity and Removable Discontinuity of Functions. Applications of Functions in Agriculture and Biological Sciences. Several Examples and Practice Problems will be provided.

Introduction, Definitions and Examples

Functions or Mappings are special relations between elements of a set and itself, a set and other one or more sets. Functions may be real-valued or complex-valued depending on the set(s) involved. Functions are very useful in Mathematics and they have wide range of applications in Sciences, Arts and Social Sciences.
Let X and Y be nonempty sets. The rule f which assigns each element x of X to some unique element y of Y is called a mapping or function from X into Y. This rule is usually expressed in the symbol

$$f : X \to Y.$$

X is called the domain of f written $Domf$ and Y is called the co-domain of f written $Codomf$. If the inverse of f exists, it is denoted by f^{-1} and $f^{-1} : Y \to X$.

Functions: Introduction, Definitions and Examples

The unique element y of Y assigned to each element x of X is called the image of x under f. On the other hand, each element x of X is called the pre-image of the unique element y of Y. The set of images of all members of the domain written as Imf is a subset of the co-domain of f.
If $f : X \to Y$ is the mapping from X into Y, then, the unique image $y \in Y$ under f of the element $x \in X$ is given by the equation

$$y = f(x)$$

where $f(x)$ is pronounced as "f of x" that is, f is a function of x. Some well known elementary functions are $e^x, \ln x, \sin x, \cos x, \tan x, \cot x, \sinh x, \cosh x, \tanh x, \coth x, \arcsin x, \arccos x, \arctan x, \cdots$. A polynomial function is of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_0, a_1, a_2, \cdots, a_{n-2}, a_{n-1}, a_n \in \mathbb{R}$ or $\mathbb{C}$.

Definition

Let $f : X \to Y$ be a function. *Domf* is the set of elements of X that are admissible by f that is

$$Domf = \{x \in X : f(x) \text{ is defined}\}$$

Note that $Domf = X$ if all the elements of X are admissible by f.

Definition

Let $f : X \to Y$ be a function. The range of f denoted by *Rangef* is the set of elements of Y that are admissible by f^{-1} that is

$$Range = \{y \in Y : f^{-1}(y) \text{ is defined}\}$$

Note that *Rangef* is a subset of Y and it is possible to have $Rangef = Y$ if all the elements of Y are admissible by f^{-1}.

Functions: Introduction, Definitions and Examples

Definition

Let $f : X \to Y$ be a function. *Imf* is the set of images of elements of X under f that is

$$Imf = \{y \in Y : y = f(x) \ \text{ for some } \ x \in X\}$$

Note that *Imf* is a subset of Y and it is possible to have $Imf = Y$.

Example

Let X and Y be sets given by

$$\begin{aligned} X &= \{x \in \mathbb{Z} : -1 \leq x < 3\}, \\ Y &= \{y \in \mathbb{Z} : -3 \leq y \leq 4\}. \end{aligned}$$

A mapping $f : X \to Y$ is defined by $f(x) = x - 1$. Find:

(i) The domain of f.

(ii) The co-domain of f.

(iii) The image of f.

(iv) Inverse image of f.
(v) The range of f.

Solution:

$$\begin{aligned}(i)\quad Domf &= \{-1,0,1,2\}\\ (ii)\quad Codomf &= \{-3,-2,-1,0,1,2,3,4\}\end{aligned}$$

(iii) Given that $f(x) = x - 1$, then

$$\begin{aligned}f(-1) &= -2, f(0) = -1, f(1) = 0, f(2) = 1\\ \therefore\quad Imf &= \{-2,-1,0,1\}.\end{aligned}$$

(iv) To get the inverse image of f, we need to get $f^{-1}(x)$. Since $y = f(x) = x - 1$, we interchange the role of x and y to have $x = y - 1$ and making y the subject we have $y = x + 1$ which is the $f^{-1}(x)$ that is $f^{-1}(x) = x + 1$. Now,

$$\begin{aligned}f^{-1}(-3) &= -2, f^{-1}(-2) = -1, f^{-1}(-1) = 0, f^{-1}(0) = 1, f^{-1}(1) = 2,\\ &\quad f^{-1}(2) = 3, f^{-1}(3) = 4, f^{-1}(4) = 5\end{aligned}$$

$$\therefore \quad IImf = \{-1, 0, 1, 2\}.$$

(v) From (iv), we deduce that $Rangef = \{-2, -1, 0, 1\}$.

Example

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Find the domain and range of f given that:

(i) $f(x) = 2x - 1$.

(ii) $f(x) = \frac{x-1}{x+1}$.

(iii) $f(x) = \frac{x^2}{x^2+1}$.

(iv) $f(x) = \frac{-x^2}{x^2-1}$.

Solution: (i) $Domf = \mathbb{R}$ since all elements of $\mathbb{R}$ are admissible by f. For the range, let $y = f(x) = 2x - 1$. Interchange the role of x and y to have $x = 2y - 1$ and making y the subject to have $y = f^{-1}(x) = \frac{1}{2}x + \frac{1}{2}$ which can admit all the elements of $\mathbb{R}$. Hence, $Rangef = \mathbb{R}$.

(ii)

Functions: Introduction, Definitions and Examples

$$\begin{aligned} Domf &= \mathbb{R} - \{-1\} \\ f^{-1}(x) &= \frac{x+1}{1-x} \\ \therefore \quad Rangef &= \mathbb{R} - \{1\}. \end{aligned}$$

(iii)

$$\begin{aligned} Domf &= \mathbb{R} \\ f^{-1}(x) &= \pm\sqrt{\frac{x}{1-x}} \\ \therefore \quad Rangef &= \left\{x \in \mathbb{R} : \frac{x}{1-x} \geq 0\right\}. \end{aligned}$$

(iv)

$$Domf = \mathbb{R} - \{-1, 1\}$$

$$
\begin{aligned}
f^{-1}(x) &= \pm\sqrt{\frac{x}{1+x}} \\
\therefore\ Range f &= \left\{x \in \mathbb{R} : \frac{x}{1+x} \geq 0\right\}.
\end{aligned}
$$

Practice Problems 1

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Find the domain and range of f given that:

(i) $f(x) = e^x$.

(ii) $f(x) = \ln x$.

(iii) $f(x) = x^4 - x^3 - x^2 + x - 1$.

(iv) $\frac{1}{x}$.

(v) $\frac{x^3+1}{x^3-1}$.

(vi) $\frac{x^3-1}{x^3+1}$.

2. Find the zeros of all the functions in (1).

Surjective and Injective Functions

Definition

A mapping $f : X \to Y$ is said to be into if every element x of X has its image y for some y in Y. In this type of mapping, two or more elements of the domain may be mapped to a single element of the co-domain. For example, the mapping $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is an into mapping since the elements 2 and -2 of the domain are mapped to a single element of the co-domain which is 4.

Definition

A mapping $f : X \to Y$ is said to be onto (surjective) if every element y of Y has a premage x for some x in X. It should be noted that f will be onto if and only if the range of f is the entire co-domain. For example, the mapping $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{2}(3x - 1)$ is an onto mapping.

Definition

A mapping $f : X \to Y$ is said to be 1-1 (injective) if different elements of X

have distinct images in Y. Alternatively, f will be injective if and only if $f(x) = f(y)$ implies that $x = y$ or $f(x) \neq f(y)$ implies that $x \neq y$ for all $x, y \in X$. For example, the mapping $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 4x + 5$ is a 1-1 mapping.

Definition

A mapping $f : X \rightarrow Y$ is called a bijection or a bijective mapping if it is both 1-1 and onto. For example, the mapping $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x - 1$ is a bijection.

Definition

A mapping $f : X \rightarrow X$ from X onto itself such that $f(x) = x$ for all $x \in X$ is called an identity mapping and it is denoted by I_X. It should be observed that I_X is both surjective and injective that is, I_X is a bijection.

Example

Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}.$$

Show that f is not a bijection.

Solution: The function is not onto because not every element of the co-domain has a preimage in the domain as can be see in

$$Range f = \left\{ x \in \mathbb{R} : \frac{1 + x}{1 - x} \geq 0 \right\}.$$

For 1-1,

$$\begin{aligned} f(a) &= f(b) \\ \Rightarrow \frac{a^2 - 1}{a^2 + 1} &= \frac{b^2 - 1}{b^2 + 1} \end{aligned}$$

$$\begin{aligned}
\Rightarrow a^2 - b^2 &= 0 \\
\Rightarrow a &= \pm b \\
&\therefore f \text{ is not 1-1} \\
&\therefore f \text{ is not a bijection.}
\end{aligned}$$

Practice Problems 2

1. Determine whether or not the functions in **Practice Problems 1** are:

(i) Surjective.

(ii) Injective.

(iii) Bijective.

Composition of Functions

Definition

Let $f : X \to Y$ and $g : Y \to Z$ be two mappings. The mapping $g \circ f : X \to Z$ defined by

$$g \circ f(x) = g(f(x)) \ \ \forall x \in X$$

is called the composition of f followed by g. The mappings f and g are called the factors of the composite $g \circ f$. It should be noted that generally, $g \circ f \neq f \circ g$ except if either f or g is an identity mapping.

Example

Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be functions given by $f(x) = x^2 - 1$, $g(x) = \sin x$ and $h(x) = \ln x$. Find the following:

(i) $f \circ f$.

(ii) $f \circ g$.

(iii) $g \circ f$.

(iv) $h \circ g \circ f$.

Solution: (i)

$$f\circ f(x)=f(f(x))=f(x^2-1)=(x^2-1)^2-1=x^2(x^2-2).$$

(ii)

$$f\circ g(x)=f(g(x))=f(\sin x)=\sin^2 x-1=-\cos^2 x.$$

(iii)

$$g\circ f(x)=g(f(x))=g(x^2-1)=\sin(x^2-1).$$

(iv)

$$h\circ g\circ f(x)=h(g\circ f(x))=h(\sin(x^2-1))=\ln(\sin(x^2-1)).$$

Practice Problems 3

1. Let $f,g,h,k:\mathbb{R}\to\mathbb{R}$ be functions given by $f(x)=2-e^{2x}$, $g(x)=\tan(x-1)$, $h(x)=\frac{1}{2}x^2-\frac{2}{3}x-\frac{2}{5}$ and $k(x)=\sin^{-1}x$. Evaluate:

(i) $f\circ f$.

(ii) $f\circ g$.

(iii) $g\circ h$.

(iv) $h\circ k$.

(v) $f\circ g\circ h\circ k$.

Definition

Let $f(x)$ be a given function. The limit of $f(x)$ as x approaches a point k is the value of $f(x)$ at $x = k$. In symbols we write

$$\lim_{x \to k} f(x) = f(k).$$

Example

Evaluate the following:

(i) $\lim_{x \to -2}[x^2 - x - 1]$.

(ii) $\lim_{x \to 2} \frac{x^2-4}{x-2}$.

(ii) $\lim_{x \to 0} \frac{|x|}{x}$.

Solution: (i)

$$\lim_{x \to -2}[x^2 - x - 1] = 4 + 2 - 1 = 5.$$

(ii) One should be careful in handling this problem. If one rushes to it, one will have

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\frac{4-4}{2-2}=\frac{0}{0}=\infty.$$

This answer is wrong. Simple manipulation will show that another correct answer can be obtained. Now,

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2}[x+2]=2+2=4.$$

In this problem, the point $x=2$ is called a point of removable singularity. If the graph of $y=\frac{x^2-4}{x-2}$ is to be sketched, there will be a jump at the point $x=2$ which is the point of discontinuity.

(iii) This problem involves absolute value of x and must be treated as such. Two cases will be consider.

First Case: If $x>0$, then

$$\lim_{x\to 0}\frac{|\,x\,|}{x}=\lim_{x\to 0}\frac{x}{x}=\lim_{x\to 0}1=1.$$

Second Case: If $x<0$, then

$$\lim_{x\to 0}\frac{|x|}{x}=\lim_{x\to 0}\frac{-x}{x}=\lim_{x\to 0}-1=-1.$$

Conclusion: The limit does not exist because $1\neq -1$. Deeper analysis of this limit will be done at higher levels.

Theorem: Let $\alpha\in\mathbb{R}$ be any real constant. Let $f(x)$ and $g(x)$ be two functions and let

$$\lim_{x\to k}f(x)=M,\lim_{x\to k}f(x)=N,M,N\in\mathbb{R}.$$

Then the following results hold:

$$\begin{aligned}
(1)\ \lim_{x\to k}(f(x)\pm g(x)) &= \lim_{x\to k}f(x)\pm\lim_{x\to k}g(x)=M\pm N.\\
(2)\ \lim_{x\to k}(\alpha f(x)) &= \alpha\lim_{x\to k}f(x)=\alpha M.\\
(3)\ \lim_{x\to k}(f(x)g(x)) &= \left[\lim_{x\to k}f(x)\right]\left[\lim_{x\to k}g(x)\right]=MN.\\
(4)\ \lim_{x\to k}\frac{f(x)}{g(x)} &= \frac{\lim_{x\to k}f(x)}{\lim_{x\to k}g(x)}=\frac{M}{N},\quad N\neq 0.
\end{aligned}$$

Limit and Continuity of Functions

Definition

A function $f(x)$ is said to be continuous at the point $x = k$ if the following conditions hold:

(i) $f(k)$ is defined.

(ii) $\lim_{x \to k} f(x)$ exists.

(iii $f(k) = \lim_{x \to k} f(x)$.

Otherwise, the function $f(x)$ is said to be discontinuous at the point $x = k$.

Example

Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be functions defined by

$$f(x) = 4x - 1, g(x) = \frac{x-1}{2x-1}, h(x) = \ln x.$$

Determine the continuity of:

(i) f at the point $x = 1/2$.

(ii) g at the point $x = 1$.
(iii) h at the point $x = e$.
(iv) $g \circ f$ at the point $x = 3/8$.

Solution: (i)

$$\begin{aligned} f(1/2) &= 2 - 1 = 1, \\ \lim_{x \to 1/2} f(x) &= 2 - 1 = 1, \\ \therefore f(1/2) &= \lim_{x \to 1/2} f(x). \\ &\therefore f \text{ is continuous at the point } x = 1/2. \end{aligned}$$

(ii)

$$\begin{aligned} g(1) &= \frac{1-1}{2-1} = \frac{0}{1} = 0, \\ \lim_{x \to 1} g(x) &= \frac{1-1}{2-1} = \frac{0}{1} = 0 = g(1), \\ &\therefore g \text{ is continuous at the point } x = 1. \end{aligned}$$

(iii)

$$\begin{aligned} h(e) &= \ln e = 1, \\ \lim_{x \to e} h(x) &= \ln(e) = 1 = h(e), \\ &\therefore \quad h \text{ is continuous at the point } x = e. \end{aligned}$$

(iv)

$$g \circ f(x) = \frac{4x - 2}{8x - 3}$$

$$\begin{aligned} g \circ f(3/8) &= \frac{3/2 - 2}{3 - 3} = \text{undefined}, \\ \lim_{x \to 3/8} g \circ f(x) &= \frac{3/2 - 2}{3 - 3} = \infty, \\ &\therefore \quad g \circ f \text{ is not continuous at the point } x = 3/8. \end{aligned}$$

Practice Problems 4

1. Evaluate the following:
   (i) $\lim_{x\to 0} \sin x$.
   (ii) $\lim_{x\to \pi/2} \cos x$.
   (iii) $\lim_{x\to 0} \tan x$.
2. Evaluate the following:
   (i) $\lim_{x\to -10} \frac{x-10}{x^2+100}$.
   (ii) $\lim_{x\to 10} \frac{x+10}{x^2-100}$.
   (iii) $\lim_{x\to -2} \frac{x+2}{|x+2|}$.
3. Determine the continuity of the following functions at the given points.
   (i) $f(x) = \frac{1}{\sqrt{x}}$ at the point $x = 0$.
   (ii) $f(x) = \sqrt{\frac{x-1}{x+1}}$ at the point $x = -1$.
   (iii) $f(x) = x^{\tan x}$ at the point $x = 0$
   (iv) $f(x) = e^{\ln x}$ at the point $x = 1$.

Marine Biology

Marine life is dependent upon the microscopic plant life that exists in the photic zone, a zone that goes to a depth where about 1% of the surface light remains. In some waters with a great deal of sediment, the photic zone may go down only 15 to 20 feet. In some murky habours, the intensity of light d feet below the surface is given approximately by

$$I = I_0 e^{-0.23d}$$

where I_0 is the intensity of light at the surface.

1.
(a) Find d when intensity of light in the water is:
 (i) The same as the intensity of the light at the surface.
 (ii) Half of the intensity of the light at the surface.
 (iii) Two fifth of the the intensity of the light at the surface.
(b) What percentage of the surface light will reach a depth of:
 (i) 10 feet ?
 (ii) 20 feet ?

Real Life Applications of Functions

Solution: (a)(i) When $I = I_0$, we have

$$\begin{aligned} I_0 &= I_0 e^{-0.23d} \\ \Rightarrow \quad e^{-0.23d} &= 1 = e^0 \\ \Rightarrow \quad -0.23d &= 0 \\ \therefore \quad d &= 0. \end{aligned}$$

(ii) When $I = \frac{1}{2} I_0$, we have

$$\begin{aligned} \frac{1}{2} I_0 &= I_0 e^{-0.23d} \\ \Rightarrow \quad e^{-0.23d} &= 1/2 \\ \Rightarrow \quad -0.23d &= \ln 1/2 = -\ln 2 \\ \therefore \quad d &= \frac{\ln 2}{0.23}. \end{aligned}$$

(ii) When $I = \frac{2}{5} I_0$, we have

$$\begin{aligned} \frac{2}{5} I_0 &= I_0 e^{-0.23d} \\ \Rightarrow \quad e^{-0.23d} &= 2/5 \\ \Rightarrow \quad -0.23d &= \ln 2/5 = \ln 2 - \ln 5 \\ \therefore \quad d &= \frac{\ln 5 - \ln 2}{0.23}. \end{aligned}$$

(b)(i) When $d = 10$, we have

$$\begin{aligned} I_{10} &= I_0 e^{-2.3} \\ \therefore \quad \frac{I_{10}}{I} 100\% &= \frac{100}{e^{2.3}} = 10\%. \end{aligned}$$

(ii) When $d = 20$, we have

$$\begin{aligned} I_{20} &= I_0 e^{-4.6} \\ \therefore \quad \frac{I_{20}}{I} 100\% &= \frac{100}{e^{4.6}} = 1\%. \end{aligned}$$