MTS 101 Lecture

Topic: Sequences and Series
A. P. and G. P., nth term and limits
Binomial Theorem for any index. Binomial series.

1 Sequences and Series

Lemma 1 ***Definition 2*** *A sequence is a set of numbers in some definite order, the succesive terms or numbers of the sequence being formed according to some rule.*

A sequence can be finite or infinite
Example of sequences are: $\{1, 2, 3, 4, 5, ...\}$ t he sequence of positive integers, $\{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, ...\}$ the sequence of positive powers of $-\frac{1}{2}$.
The terms of a sequence are usually listed in braces $\{\}$ and the ellipsis (...) should be read "and so on".

An infinite sequence is a special kind of function, one whose domain is a set of integers extending from some starting integer to infinity. In most cases, the starting integer is usually 1., so the domain is the set of positive integers. The sequence $\{a_1, a_2, a_3, a_4, ...\}$ is the function f that takes the value $f(n) = a_n$ at each positive integer n. A sequence can be specified in three ways:

(i) We can list the first few terms followed by ... if the patterns is obvious.

(ii) We can provide a formula for the general term a_n as a function of n.

(iii) We can provide a formula for calculating the term a_n as a function of earlier terms $a_1, a_2, a_3, a_4, ..., a_{n-1}$ and specify enough of the begining terms so the process of computing higher tems can begin

Example 3 **(i)** $\{n\} = \{1, 2, 3, 4, 5, ...\}$

(ii) $\left\{\left(-\frac{1}{2}\right)^n\right\} = \left\{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, ...\right\}$

(iii) $\left\{\left(1 + \frac{1}{n}\right)^n\right\} = \left\{2, \left(\frac{3}{2}\right)^2, \left(\frac{4}{3}\right)^3, \left(\frac{5}{4}\right)^4, ...\right\}$.

Example 4 *Find the a_n in terms of n of the sequences*

(i) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, ...$

(ii) $-1, 1, -1, 1, -1, 1, ...$

(iii) $1, 4, 3, 16, 5, 36, 7, 64, ...$

(iv) $1, -2, 3, -4, 5, -6, ...$

Solution 5 (*i*) *Observe that the terms can be written as*

$$\frac{1}{1^2,}\frac{1}{2^2},\frac{1}{3^2},\frac{1}{4^2},...$$

$a_n = \frac{1}{n^2}$.

(*ii*) *The odd terms are* -1*, the even terms* $+1$*. Therefore,* $a_{2r+1} = -1$ *and* $a_{2n} = +1$*. This sequence can be described by one formula* $a_n = (-1)^n$*.*

(*ii*) *If* n *is odd,* $a_n = n$*, if* n *is even* $a_n = n^2$*. Since* $2r$ *is always even and* $2r+1$ *is always odd,* $a_{2r+1} = 2r+1, a_{2r} = 4r^2$ *desribe the sequence.*

(*iv*) *By using the results in* (*ii*)*, we have*

$$a_n = (-1)^n(-n) = (-1)^{n+1}n.$$

Example 6 *Find the first four terms of the sequence defined by*

$$a_1 = 1, \quad a_{n+1} = \sqrt{6 + a_n}$$

Solution 7 *In this case*

$a_1 = 1$
$a_2 = \sqrt{6+1} = \sqrt{7}$
$a_3 = \sqrt{6+\sqrt{7}}$
$a_4 = \sqrt{6+\sqrt{6+\sqrt{7}}}$
The first four terms are thus

$$1, \sqrt{7}, \sqrt{6+\sqrt{7}}, \sqrt{6+\sqrt{6+\sqrt{7}}}.$$

(This rule adequately defines a sequence although it would not be easy to find a formula for a_n in terms of n.)

1.1 Series

Definition 8 *A series is the sum of the terms of a sequence.*

Thus a finite series is formed if a finite number of terms of the sequence are summed. The sum of the first n terms of the sequence $a_1, a_2, a_3, ...$ is generally denoted by S_n.

Thus

$$S_n = a_1 + a_2 + a_3 + ... + a_n. \tag{1}$$

Equation (1) above sum can also be denoted by

$$S_n = \sum_{i=1}^{n} a_i \tag{2}$$

Example 9 *The sum of the first n terms of a series is given by the formula*

$$S_n = n^2 + 3n \quad \forall\ n.$$

Find an expression for the r^{th} term of the series.

Solution 10 *If a_n denote the n^{th} term of the series, then*

$$\begin{aligned}
a_n &= S_n - S_{n-1} \\
&= [a_1 + a_2 + a_3 + ... + a_{n-1} + a_n] - [a_1 + a_2 + a_3 + ... + a_{n-1}] \\
&= n^2 + 3n - [(n-1)^2 + 3(n-1)] \\
&= n^2 + 3n - [n^2 + n - 2]^{\cdot}
\end{aligned}$$

Therefore

$$a_n = 2n + 2$$

Example 11 *Evaluate* $\sum_{n=3}^{7} 2^n$

Solution 12

$$\begin{aligned}
\sum_{n=3}^{7} 2^n &= 2^3 + 2^4 + 2^5 + 2^6 + 2^7 \\
&= 8 + 16 + 32 + 64 + 128 \\
&= 248.
\end{aligned}$$

Example 13 *If $u_r = \log_{10} r$, show that*

$$\sum_{r=1}^{10} u_r = \sum_{r=1}^{10} \log_{10} r = \log_{10} 3628800.$$

Solution 14

$$\begin{aligned}
\sum_{r=1}^{10} u_r &= u_1 + u_2 + u_3 + ... + u_{10} \\
&= \log_{10} 1 + \log_{10} 2 + \log_{10} 3 + ... + \log_{10} 10 \\
&= \log_{10} (1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10) \\
&= \log_{10} (3628800) .
\end{aligned}$$

1.2 The Arithmetic sequence and series

Definition 15 *If the consecutive terms of a sequence differ by a constant number, then their terms are said to form an arithmetic sequence or arithmetic progression.*

An arithmetic sequence is completely defined by its first term usually denoted by a and the common difference (the difference between consecutive terms) by d. thus the general arithmetic sequence is given by

$$a, a + d, a + 2d, a + 3d, - - -a + (n - 1)d, ... \tag{3}$$

and the n^{th} term of the sequence by

$$a_n = a + (n - 1)d \tag{4}$$

The sum of terms of an arithmetic sequence form an arithmetic series. For the general sequence (3), the sum of the first n terms is

$$S_n = a + a + d + a + 2d + a + 3d + - - -a + (n - 1)d \tag{5}$$

Thus, (5) can be written in closed formula in terms of n for S_n form as

$$S_n = a + (a + d) + (a + 2d) + (a + 3d) + - - -[a + (n - 2)d + [a + (n - 1)d \tag{6}$$

and

$$S_n = [a + (n - 1)d] + [a + (n - 2)d] + ... + (a + 3d) + (a + 2d) + (a + d) + a \tag{7}$$

Adding (6) and (7), then corresponding pairs add up to $[2a+(n-1)d$, hence

$$\begin{aligned} 2S_n &= [2a + (n - 1)d] + [2a + (n - 1)d] + ... \; n \text{ times} \\ &= n[2a + (n - 1)d]. \end{aligned}$$

Therefore

$$S_n = \frac{n}{2}[2a + (n - 1)d] \tag{8}$$

Thus (8) give the sum of n terms of an arithmetic progression or series. This result can also be written in another form. Since a is the first term, $l = a + (n - 1)d$ is the last term of the arithmetic progression. Then $a + l = 2a + (n - 1)d$ and so (8) can be written as

$$S_n = \frac{n}{2}[a + l \tag{9}$$

Example 16 *Find three numbers in an arithmetic progression whose sum is 3 and whose product is -15.*

Solution 17 *Suppose the numbers are:* $a-d, a, a+d$. *Then* $a-d+a+a+d = 3$. *Therefore* $3a = 3$ *and so* $a = 1$.

But

$$a(a - d)(a + d) = (1 - d)(1 + d) = -15$$

Hence

$$1 - d^2 + 15 = 0$$

That is $d^2 = 16$ *and so* $d = \pm 4$.

Solution 18 *The required numbers are* $-3, 1, 5$.

Example 19 *The first term of an arithmetic progression is* 7, *the last term is* 70 *and the um is* 385. *Find the number of terms in the series and the common difference.*

Solution 20 *Let the number of terms be* n, $a = 7$,

$$l = (a + n - 1)d = 70 \tag{10}$$

and

$$S_n = \frac{n}{2}[a + l] = 383. \tag{11}$$

From (11) we have

$$\frac{n}{2}[7 + 70] = 385$$

That is

$$77n = 770$$

$n = 10$.
Substitute $n = 10$ *into (10) to have*

$$(7 + 9d = 70$$

That is $9d = 63$. *This implies that* $d = 7$.

Example 21 *The* 3^{rd} *of an arithmetic progression is* 18, *the seventh term is* 30. *Find the sum of the first* 33 *terms.*

Solution 22 *Here*

$$a + 2d = 18 \tag{12}$$

and

$$a + 6d = 30 \tag{13}$$

(13)-(12) yields

$$4d = 12 \quad \Longrightarrow d = 3$$

From (12) we have

$$a + 6d = 18 \quad \Longrightarrow a - 12$$

But we are given that $n = 33$, *then using the formula*

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

we obtain the sum of the first 33 *terms to be*

$$\begin{aligned} S_{33} &= \frac{33}{2}[24 + 32 \times 3] \\ &= \frac{33}{2}[24 + 96] \\ &= 33[12 + 48] \\ &= 60 \times 33 \\ &= 1980. \end{aligned}$$

1.3 Geometric Progression

Definition 23 *If the consecutive terms of a sequence are all in the same ratio, then the terms are said to form a geometric sequence or a geometric progression.*

Example 24 *The numbers* $1, 3, 9, 18, ...$ *are in geometric progression, the ratio of any pair of consecutive terms is* 3.

The general geometric sequence is given by

$$a, ar, ar^2, ar^3, ..., ar^{n-1}, ... \tag{14}$$

The n^{th} term of the geometric sequenc is

$$u_n = ar^{n-1} \tag{15}$$

In general the sum of the first n terms of the geometric sequence (\14) is

$$S_n = a + ar + ar^2 + ... + ar^{n-1} \tag{16}$$

As in arithmetic series, we can also obtain a closed formula for S_n in terms of n as follows

$$S_n = a + ar + ar^2 + ... + ar^{n-2} + ar^{n-1} \tag{17}$$

Hence

$$rS_n = ar + ar^2 + ar^3 + ... + ar^{n-1} + ar^n \tag{18}$$

Equation (17)-(18) yields

$$S_n(1 - r) = a - ar^n$$

That is

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1} \tag{19}$$

Example 25 *Find three numbers in a geometric progression whose sum is* 28 *and whose product is* 512.

Solution 26 *Let the numbers be* $\frac{a}{r}, a, ar$, *then*

$$\frac{a}{r}.a.ar = a^3 = 512 \quad \Longrightarrow a = 8$$

Hence

$$\frac{8}{r} + 8 + 8r = 28$$

i.e'

$$8r^2 + 8r + 8 = 28r$$

$$2r^2 - 5r + 2 = 0$$

$$(2r-1)(r-2)=0$$

$$r=\frac{1}{2} \quad or \quad r=2$$

The required numbers are

$$\frac{8}{2}, 8, 8\times 2 \quad i.e. \quad 4, 8, 16.$$

Example 27 *Find the sum of the first* n *terms of the series*

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+...$$

Solution 28 *This is a geometric series with first term* $a=1$ *and common ratio* $r=-\frac{1}{2}$*. Hence by (19) we have*

$$S_n=\frac{1[1-(-\frac{1}{2})^n]}{1-(-\frac{1}{2})}$$

i.e.

$$S_n=\frac{2}{3}\left[1-(-\frac{1}{2})^n\right]=\frac{2}{3}-(-\frac{1}{2})^n.$$

Example 29 *The first and last term of a geometric series are* 2 *and* 2048 *respectively. If the sum of the series is* 2730*. Find the number of terms and the common ratio.*

Solution 30 *Let* n *denote the number of terms and* r *the common ratio. By using the formula for the* n^{th} *term (15), we have*

$$2r^{n-1}=2048$$

i.e.

$$r^{n-1}=1024 \tag{20}$$

But we have by (19) that

$$\frac{2(r^n-1)]}{r-1}=2730$$

i.e.

$$\frac{(r^n-1)]}{r-1}=1365 \tag{21}$$

Substitute (20) into (21) to have

$$\frac{1024r-1}{r-1}=1365$$

1.e.

$$1024r - 1 = 1365r - 1365$$

i.e.

$$1364 = 341r$$

This implies that

$$r = 4$$

Next to find the number of terms n we use (20) to have

$$4^{n-1} = 1024$$

Hence

$$n - 1 = 5$$

This gives

$$n = 6$$

Hence the number of terms is 6 *and the common ratio is* 4.

1.4 Limit of an infinite geometric series

We already know that the general geometric series

$$a + ar + ar^2 + ...$$

has a sum given by

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a}{r - 1} - \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} r^n$$

Now if $-1 < r < 1$, then r^n decreases as n increases and so the limiting value of r^n is zero. Therefore, as n increase, S_n tends to the limiting value denoted by S. We then write

$$\lim_{n \to \infty} S_n = S = \frac{a}{1 - r}. \tag{22}$$

We then say that the series converges to the sum

$$\frac{a}{1 - r}$$

The condition $-1 < r < 1$ is often written in the form $|r| < 1$.

Example 31 *Find the sum of the series*

$$\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625} + ...$$

Solution 32 *This is a geometric series with first term* $\frac{1}{5}$ *and common ratio* $\frac{1}{5}$. *Since the absolute value of the common ratio* $\frac{1}{5}$ *is* $\left|\frac{1}{5}\right| < 1$, *then the sum of the geometric series by (22) is*

$$S = \frac{\frac{1}{5}}{1-\frac{1}{5}} = \frac{\frac{1}{5}}{\frac{4}{5}} = \frac{1}{4}.$$

Example 33 *Express* 0.333 *recurring as fraction and find its sum.*

Solution 34

$$0.333 \textit{ recurring } = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + ...$$

The above form a geometric series with first term $a = \frac{1}{10}$ *and whose common ratio* $r = \frac{1}{10}(< 1)$. *Hence*

$$0.333 \textit{ recurring } = \frac{\frac{3}{10}}{1-\frac{3}{10}} = \frac{\frac{3}{10}}{\frac{7}{10}} = \frac{3}{7}.$$

Example 35 *For what value of* x *does the series*

$$x + \frac{x}{1+x} + \frac{x}{(1+x)^2} + \frac{x}{(1+x)^3} + ...$$

converge? Find the sum it converges to?.

Solution 36 *The series is a geometric series with first term* x *and common ratio* $\frac{1}{1+x}$. *For convergence*

$$\left|\frac{1}{1+x}\right| < 1$$

That is

$$|1+x| > 1$$

Thus

$$x+1 > 1 \quad \textit{i.e.} \quad x > 0$$

or

$$x+1 < -1 \quad \textit{i.e.} \quad x < -2.$$

Hence the series converge if $x > 0$ *or* $x < -2$. *The limit of the sum of the series is*

$$\begin{aligned} \frac{x}{1-\frac{1}{1+x}} &= \frac{x}{\frac{x}{1+x}} \\ &= 1+x \end{aligned}$$

Observe also that the series converge if $x = 0$, *for each of the term is zero. The sum of course also is zero.*

2 Binomial theorem

2.1 The binomial theorem for a positive integral index

In this section we shall be interested in obtaining a formula for the expansion in terms of 1 and x, of $(1+x)^n$, where n is a positive integer.

Consider first the following expansion using ordinary multiplication

$$(1+x) = 1+x$$

$$(1+x)^2 = (1+x)(1+x) = 1+2x+x^2$$

$$(1+x)^3 = (1+2x+x^2)(1+x) = 1+3x+3x^2+x^3$$

$$(1+x)^4 = (1+x)^3(1+x) = 1+4x+6x^2+4x^3+x^4$$

$$(1+x)^5 = (1+x)^4(1+x) = 1+5x+10x^2+10x^3+5x^4+x^5$$

and so on.

Observe from the above that in each case the first and the last coeficients are unity. In addition each of the other coefficients in $(1+x)^{n+1}$ is the sum of the corresponding coeffficient and the preceeding on in the expansion of $(1+x)^n$. These coefficients can be laid out in form of a triangle called the Pascal's triangle as shown below

$$\begin{array}{llllllll}
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 \text{ etc.}
\end{array}$$

In order to obtain the coefficient of x^r in the expansion of $(1+x)^n$, we need a formula which is give as follows:

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{r}x^r + \dots + \binom{n}{n-1}x^{n-1} + x^n \quad (23)$$

The next problem is to find a formular for $\binom{n}{r}$ in terms of n and r. To do this observe that the cofficients in the row corresponding to $(1+x)^n$ follows the pattern

$$1, n, \frac{n(n-1)}{1.2}, \frac{n(n-1)(n-2)}{1.2.3}, \dots$$

For example

$$\binom{6}{3} = \frac{6(6-1)(6-2)-}{1.2.3} = 5.4 = 20$$

$$\binom{5}{2} = \frac{5(5-1)}{1.2} = 5.2 = 10$$

We can also write $\binom{n}{r}$ as

$$\binom{n}{r} = \frac{n(n-1)(n-2)...(n-r+1)}{r(r-1)(r-2)...3.2.1} \tag{24}$$

Note that result (24) can be used to evaluate the coefficients in the above example.

Example 37 *Use (24) to expand* $(1+x)^6$

Solution 38 *Using (23) we have*

$$(1+x)^6 = 1 + \binom{6}{1}x + \binom{6}{2}x^2 + \binom{6}{3}x^3 + \binom{6}{4}x^4 + \binom{6}{5}x^5 + \binom{6}{6}x^6 \tag{25}$$

Here

$$\binom{6}{1} = \frac{6}{1} = 6 \qquad \binom{6}{2} = \frac{6.5}{2.1} = 15 \qquad \binom{6}{3} = \frac{6.5.4}{3.2.1} = 20 \tag{26}$$

$$\binom{6}{4} = \frac{6.5.4.3}{4.3.2.1} = 15 \qquad \binom{6}{5} = \frac{6.5.4.3.2}{5.4.3.2.1} = 6 \qquad \binom{6}{6} = \frac{6.5.4.3.2.1}{6.5.4.3.2.1} = 1 \tag{27}$$

Solution 39 *Substitute (26) and (27) into (25) we obtain*

$$(1+x)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$$

The following result holds

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1} \tag{28}$$

Proof. Consider the rhs of (28) we have

$$\begin{aligned}
\binom{n}{r} + \binom{n}{r-1} &= \frac{n(n-1)(n-2)...(n-r+1)}{r(r-1)(r-2)...3.2.1} + \frac{n(n-1)(n-2)...(n-r+2)}{r(r-1)(r-2)...2.1} \\
&= \frac{n(n-1)(n-2)...(n-r+1) + r(n)(n-1)...(n-r+2)}{r(r-1)...3.2.1} \\
&= \frac{n(n-1)(n-2)...(n-r+2)(n-r+1+r)}{r(r-1)...3.2.1} \\
&= \frac{(n+1)(n+1-1)(n+1-2)...(n+1-r+1)}{r(r-1)...3.2.1} \\
&= \binom{n+1}{r}
\end{aligned}$$

which is required. ■

2.1.1 Expansion of $(a+x)^n$

Suppose we are interested in expanding $(a+x)^n$ in terms of x and a, then we proceed by usig (23) and (24) to have

$$\begin{aligned}(a+x)^n &= \left[a\left(1+\frac{x}{a}\right)\right]^n \\ &= a^n\left(1+\frac{x}{a}\right)^n \\ &= a^n\left[1+\binom{n}{1}\frac{x}{a}+\binom{n}{2}\frac{x^2}{a^2}+...+\binom{n}{r}\frac{x^r}{a^r}+...+\frac{x^n}{a^n}\right]\end{aligned}$$

Hence

$$(a+x)^n = a^n+\binom{n}{1}xa^{n-1}+\binom{n}{2}x^2a^{n-2}+...+\binom{n}{r}x^ra^{n-r}+...+x^n \quad (29)$$

Example 40 *Expand* $(3+x)^4$ *in powers of* x.

Solution 41 *By the above expansion (29) with* $a=3$ *we have*

$$\begin{aligned}(3+x)^4 &= 3^4+\binom{4}{1}x3^{4-1}+\binom{4}{2}x^23^{4-2}+\binom{4}{3}x^33^{4-3}+x^4 \\ &= 3^4+4.3^3.x+6.3^2.x^2+4.3.x^3+x^4 \\ &= 81+108x+54x^2+12x^3+x^4\end{aligned}$$

The following results hold:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \quad (30)$$

In view of (30) we easily obtain that

$$\binom{n}{n-r} = \binom{n}{r} \quad (31)$$

Verify that (30) and (31) hold.

In view of (30) and (31) we easily see that

$$\binom{n}{0} = \binom{n}{n} = 0$$

and so (23) can be rewritten as

$$\begin{aligned}(1+x)^n &= \binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+...+\binom{n}{r}x^r+...+\binom{n}{n}x^n \\ &= \sum_{r=0}^{n}\binom{n}{r}x^r.\end{aligned}$$

Example 42 *Expand* $(1-3x)^4$ *in powers of* x.

Solution 43 *Rewrite* $(1-3x)^4$ *as* $[1+(-3x)]^4$
Therefore

$$\begin{aligned}(1-3x)^4 &= [1+(-3x)]^4\\ &= \binom{4}{0}+\binom{4}{1}(-3x)+\binom{4}{2}(-3x)^2+\binom{4}{3}(-3x)^3+\binom{4}{4}(-3x)^4\\ &= 1-12x+108x^2-108x^3+81x^4\end{aligned}$$

Example 44 *Find the coefficient of* x^8 *in* $\left(x^2+\frac{2y}{x}\right)^{10}$.

Solution 45 *Here the* $(r+1)$ *term in the above expansion is given by*

$$\binom{10}{r}(x^2)^{10-r}\left(\frac{2y}{x}\right)^r = \binom{10}{r}2^r y^r x^{20-2r-r} = \binom{10}{r}2^r y^r x^{20-3r}$$

For the term in x^6, *we have*

$$20-3r=8 \qquad i.e.\ \ r=4$$

Hence the required coefficient is

$$\binom{10}{4}2^4y^4 = \frac{10.9.8.7}{4.3.2.1}16y^4 = 210.16y^4 = 3360y^4$$

Example 46 *Find the values of* a *if the coefficient of* x^2 *in the expansion of* $(1+ax)^4(2-x)^3$ *is* 6.

Solution 47 *Expand each of the terms to have*

$$(1+ax)^4 = 1+4(ax)+6(ax)^2+4(ax)^3+(ax)^4$$

and

$$\begin{aligned}(2-x)^3 &= [2+(-x)]^3 = 2^3+3.2^2(-x)+3.2(-x)^2+(-x)^3\\ &= 8-12x+6x^2-x^3\end{aligned}$$

Therefore, the coefficient of x^2 *in the expansion of* $(1+ax)^4(2-x)^3$ *is the coefficient of* x^2 *in*

$$\left(1+4ax)+6a^2x^2+4a^3x^3+a^4x^4\right)\left(8-12x+6x^2-x^3\right).$$

The required coefficient is $6-48a+48a^2$. *Given that this coefficient equals* 6, *the we have*

$$\begin{aligned}48a^2-48a+6 &= 6\\ 8a^2-8a &= 0\\ 8a(a-1) &= 0\end{aligned}$$

Hence

$$a=0 \quad or \ \ a=1$$

Example 48 *Expand* $(x+5y)^5$. *Hence evaluate* $(1.05)^5$ *correct to three decimal places*

Solution 49 *Observe that*

$$\binom{5}{1} = \binom{5}{4} = 5 \qquad \binom{5}{2} = \binom{5}{3} = 10$$

Therefore

$$\begin{aligned}(x+5y)^5 &= x^5 + 5x^4(5y) + 10x^3(5y)^2 + 10x^2(5y)^3 + 5x(5y)^4 + (5y)^5 \\ &= x^5 + 25x^45y + 250x^3y^2 + 1250x^2y^3 + 3125xy^4 + 3125y^5\end{aligned}$$

Put $x = 1, y = 0.01,$ *we have*

$$\begin{aligned}(1.05)^5 &= 1 + 25(0.01) + 250(0.0001) + 1250(0.000001) \\ &\quad +3125(0.00000001) + 3125(0000000001) \\ &\approx 1 + 0.25 + 0.025 + 0.00125\end{aligned}$$

where the last two terms have been omitted because it does not affect the final result.

Thus

$$(1.05)^5 \approx 1.27625$$

which gives

$$(1.05)^5 \approx 1.276$$

to three decimal places.

2.2 Binomial theorem when n is not a positive integer

If $1 < x < 1$ and n has any value, then we have

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + ... \qquad (32)$$

Expansion (32) is known as the binomial theorem. Now we give some examples.

Example 50 *Obtain the first five terms in the expansion* $(1+x)^{\frac{1}{2}}$. *Hence evaluate* $\sqrt{1.03}$ *to five significant figures.*

Solution 51 *Since* $n = \frac{1}{2}$ *we have by (32) that*

$$\begin{aligned}(1+x)^{\frac{1}{2}} &= 1 + (\frac{1}{2})x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{1.2}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1.2.3}x^3 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{1.2.3.4}x^4 + ... \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + ...\end{aligned}$$

Solution 52 *When $x = 0.03$, which lie between -1 and $+1$,*

$$
\begin{aligned}
(1.03)^{\frac{1}{2}} &= \sqrt{1.03} = 1 + \frac{1}{2}(0.03) - \frac{1}{8}(0.0009) + \frac{1}{16}(0.000027) + ... \\
&\approx 1.0148892 \\
&= 1.0149 \quad \textit{to 5 significant figures.}
\end{aligned}
$$

Example 53 *Expand $\frac{3-x}{(1-2x)(1+x^2)}$ in ascending powers of x as far as the term in x^3.*

Solution 54 *First we resolve $\frac{3-x}{(1-2x)(1+x^2)}$ into partial fractions to obtain*

$$
\begin{aligned}
\frac{3-x}{(1-2x)(1+x^2)} &= \frac{2}{1-2x} + \frac{1+x}{1+x^2} \\
&= 2(1-2x)^{-1} + (1+x)(1+x^2)^{-1} \\
&= 2(1 + 2x + 4x^2 + 8x^3 + ... \\
&\quad +(1+x)(1 - x^2 + x^4 - x^6 + x^8 + ... \\
&= (2 + 4x + 8x^2 + 16x^3 + ...) \\
&\quad +1 + x - x^2 - x^3 + ... \\
&= 3 + 5x + 7x^2 + 15x^3 + ...
\end{aligned}
$$

This expression is valid only if both $-2x$ and x^2 lie between -1 and $+1$. That is

$$
\begin{aligned}
-1 &< -2x < 1 \quad \textit{if} \quad -\frac{1}{2} < x < \frac{1}{2} \\
-1 &< x^2 < 1 \quad \textit{if} \quad -1 < x < 1
\end{aligned}
$$

Hence for the expansion to be valid, x must lie between $-\frac{1}{2}$ and $\frac{1}{2}$ i.e. $|x| < \frac{1}{2}$.