6▪0 Complex Numbers

The roots of the quadratic equation in the form $ax^2 + bx + c = 0$ is given by the formulae,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

⇒ the roots of the equation, $5x^2 - 6x + 5 = 0$ can be obtained using the formula above to obtain:

$$x = \frac{-6 \pm \sqrt{36 - 100}}{10} = \frac{-6 \pm \sqrt{-64}}{10}.$$

The problem arising here is that $\sqrt{-64}$ cannot be obtained directly because $\sqrt{-64} \notin \mathbb{R}$, so we cannot conclude that the solution to $\sqrt{-64}$ is either 8 or -8. We can express -64 as -1 x 64 such that $\sqrt{-64} = \sqrt{-1 \text{ x } 64} = 8 \text{ x } \sqrt{-1}$.

We can represent $\sqrt{-1}$ by a letter say, i such that $\sqrt{-64} = 8i$. The expression 8i is called a complex number due to the presence of the letter i.

A complex number, $\mathbb{Z}$ has the form $\mathbb{Z} = a + bi$ such that $a, b \in \mathbb{R}$, a is called the real part and b is known as the imaginary part of the complex number, i is called the imaginary unit with the following properties:

(i) $i = \sqrt{-1} \Rightarrow i^2 = -1$

(ii) $i^3 = i^2 i = -1 \cdot i = -i$

(iii) $i^4 = i^2 \cdot i^2 = -1 \times -1 = 1$

(iv) $i^7 = (i^2)^3 i = (-1)^3 \cdot i = (-1) \times i = -i$

Note: The complex number $a + bi$ is said to be pure imaginary if $a = o$ and $b \neq 0$ e.g. 10i also $a + bi$ is said to be pure real if $b = 0$.

6▪1 Algebra of Complex Numbers

(a) Addition and Subtraction : If a, b, c and d are real numbers, then

(i) $(a + bi) + (c + di) = a + c + bi + di = (a + c) + (b + d)i$

(ii) $(a + bi) – (c + di) = a + bi – c – di = a - c + bi – di = (a – c) + (b – d)i$

Example 1.

Write each of the following in the form $a + bi$ and simplify

(1.) $(8 + i) + (2 + 3i)$ (2) $(7 + 8i) - (- 4 - 3i)$

<u>Solution.</u>

(1) $(8 + i) + (2 + 3i) = (8 + 2) + i + 3i = (8 + 2) + (1 + 3)i = 10 + 4i$

(2) $(7 + 8i) - (- 4 - 3i) = 7 - (- 4) + 8i - (- 3i) = 7 + 4 + 8i + 3i = 11 + (8 + 3)i = 11 + 11i.$

##

(b) Multiplication: If a, b, c and d are real numbers, then $(a + bi) (c + di) = ac + adi + cbi + bdi^2$

But $i^2 = -1$, $(a + bi) (c + di) = ac + (ad + cb)i + bdi^2$

$= (ac - bd) + (ad + cb)i.$

Example 2.

Obtain the product of the following and simplify:

(a.) $(2 - 7i) (2 + 5i)$ (b) $-5i (4 + 6i)$

<u>Solution</u>

(a) $(2 - 7i) (2 + 5i) = 4 + 10i - 14i - 35i^2$

$= 4 + (10 - 14)i - 35 (-1)$

$= 4 + (-4)i + 35$

$= 4 + 35 - 4i = 39-4i.$

(b) $-5i (4 + 6i) = -20i - 30i^2$

$= -20i - 30(-1)$

$= -20i + 30 = 30 - 20i.$

6▪2 Conjugate of a Complex Number

If $\mathbb{Z}$ = a + bi is a complex number, a, b ∈ $\mathbb{R}$ then the complex conjugate of $\mathbb{Z}$ denoted as $\overline{\mathbb{Z}} = a - bi$

Note: $\mathbb{Z} = \overline{\mathbb{Z}}$ if b = 0.

(c) Division: To divide a complex number by another complex number, we multiply by the conjugate of the denominator.

Consider the complex numbers $\mathbb{Z}_1$ and $\mathbb{Z}_2$ of the form a + bi and c + di respectively, then

$$\frac{\mathbb{Z}_1}{\mathbb{Z}_2} = \frac{a+bi}{c+di} = \frac{a+bi}{c+di} \times \frac{c-di}{c-di}$$

$$= \frac{ac-a\,di+bci-b\,di^2}{c^2-c\,di+c\,di-d^2i^2}$$

$$= \frac{ac+bci-a\,di-b\,d(-1)}{c^2-d^2(-1)}$$

$$= \frac{ac+bci-a\,di+b\,d}{c^2+d^2}$$

$$= \frac{(ac+b\,d)+(bc-a\,d)i}{c^2+d^2}$$

$$= \frac{ac+bd}{c^2+d^2} + \frac{(bc-a\,d)i}{c^2+d^2}$$

Example 3.

(a.) Express $\frac{3+5i}{4+7i}$ in the form a + bi.

(b.) Express $\frac{\sqrt{3}+2i}{\sqrt{3}-2i}$ in the form x + yi

(a.) $$\frac{3+5i}{4+7i} = \frac{3+5i}{4+7i} \times \frac{4-7i}{4-7i}$$

$$= \frac{12-21i+20i-35i^2}{16-28i+28i-49i^2}$$

$$= \frac{12-i-35(-1)}{16-49(-1)} = \frac{12-i+35}{16+49} = \frac{47-i}{65} = \frac{47}{65} - \frac{1}{65}i$$

(b.) $\frac{\sqrt{3}+2i}{\sqrt{3}-2i} = \frac{\sqrt{3}+2i}{\sqrt{3}-2i} \times \frac{\sqrt{3}+2i}{\sqrt{3}+2i}$

$$= \frac{3+2\sqrt{3}i+2\sqrt{3}i+4i^2}{3+2\sqrt{3}i-2\sqrt{3}i-4i^2}$$

$$= \frac{3+4\sqrt{3}i+4(-1)}{3-4i^2}$$

$$= \frac{3+4\sqrt{3}i-1}{3-4(-1)}$$

$$= \frac{3-4+4\sqrt{3}i}{3+4}$$

$$= \frac{-1+4\sqrt{3}i}{7} = \frac{-1}{7} + \frac{4\sqrt{3}i}{7}$$

6▪3 Absolute value of a Complex Number

This is the non-negative square root of the real number z denoted by $|z|$, i.e. $|z| = \sqrt{z\bar{z}}$ where $\bar{z}$ = the complex conjugate of z .

If z = a + bi, $\bar{z}$ = a - bi

$|z| = \sqrt{z\bar{z}}$

$= \sqrt{(a+bi)(a-bi)}$

$= \sqrt{a^2-abi+abi-b^2i^2}$

$|z| = \sqrt{a^2-b^2i^2}$

$= \sqrt{a^2-b^2\,(-1)}$

$|z| = \sqrt{a^2+b^2}$

Example 4. Find the absolute value of $\frac{4}{5}-\frac{2}{5}i$.

Solution.

Let $z = \frac{4}{5}-\frac{2}{5}i$, $\bar{z} = \frac{4}{5}+\frac{2}{5}i$,

the absolute value of z, $|z| = \sqrt{z\bar{z}}$

$$|z| = \sqrt{(\frac{4}{5}-\frac{2}{5}i)(\frac{4}{5}+\frac{2}{5}i)}$$

$$= \sqrt{\frac{16}{25}+\frac{8}{25}i-\frac{8}{25}i-\frac{4}{25}i^2}$$

$$= \sqrt{\frac{16}{25}-\frac{4}{25}(-1)}$$

$$= \sqrt{\frac{16}{25}+\frac{4}{25}}$$

$$|z| = \sqrt{\frac{20}{25}} = \frac{\sqrt{20}}{\sqrt{25}} = \frac{\sqrt{4\times 5}}{5} = \frac{2\sqrt{5}}{5}.$$

Exercise:

1. If z = x + iy, Evaluate $(z)^2 + (\bar{z})^2 - \bar{z}z$.
2. Express the solution of the following equations in the form a + bi

   (a) $x^2 + 16 = 0$ (Ans.) $x = \pm 4i$

   (b.) $x^2 + 2x + 5 = 0$ (Ans.) $x = -1 \pm 2i$

   (c.) $x^2 + 4x + 40 = 0$

   (d.) Show that $1 + i - 3i^2 + i^7 = 4$

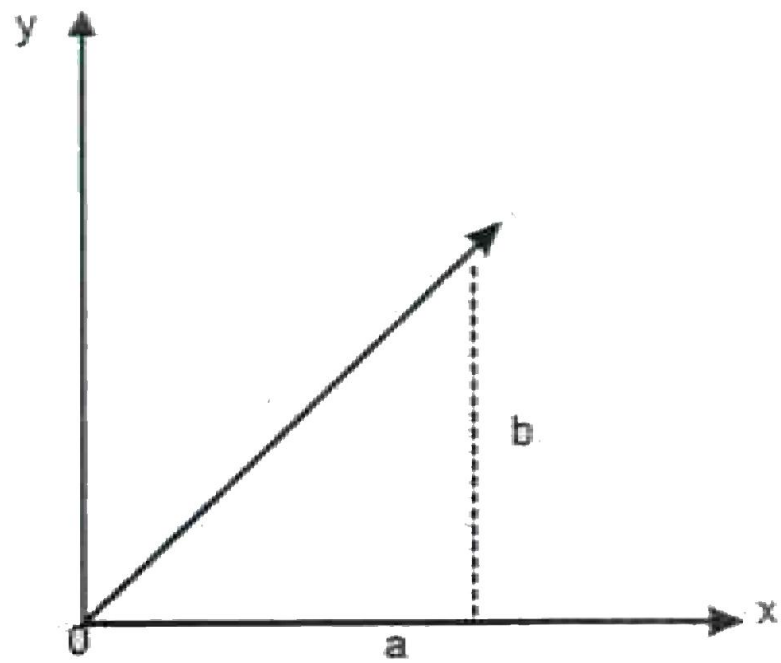
3. Evaluate in the form a + ib:

   (i) $\frac{1+i}{2-i}$ (ii) $\frac{3-4i}{5+2i}$

6▪4 Graphical Representation of a Complex Number (Argand diagram)

A complex number, $z = a + bi$ can be define as $z = a + bi = (a, b)$ where all the properties satisfied by $a + bi$ are also satisfied by (a, b) for example, $(a, b) + (c, d) = (a + c , b + d)$.

Complex numbers do not have the ordering property as they cannot be represented by points on a line but are represented by points on a plane [rectangular or polar].



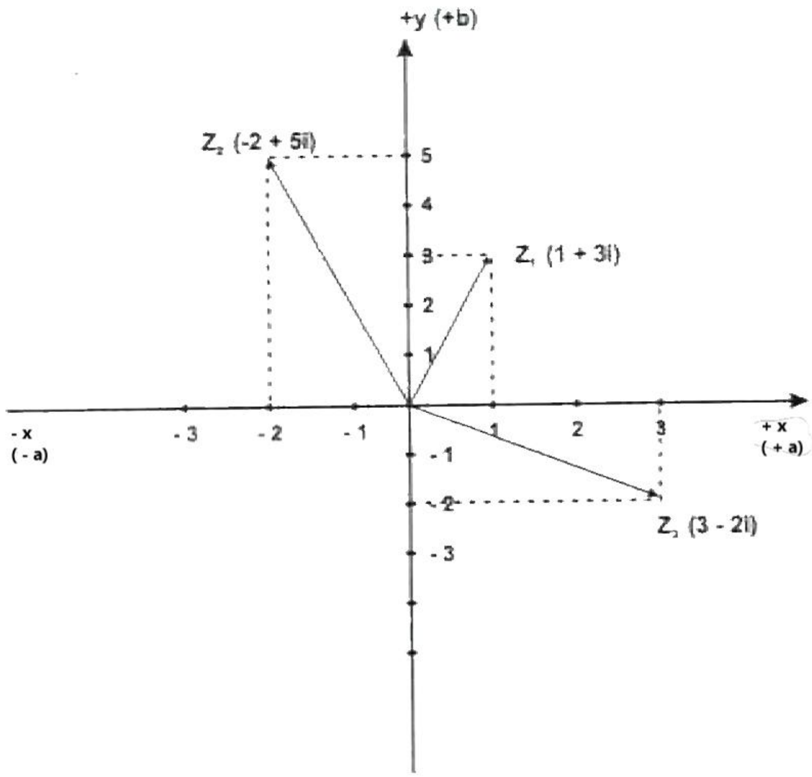
The graph above is called Argand diagram, where the y-axis is called the imaginary axis and the x-axis is the real axis.

Example 6.

Draw an argand diagram to represent the following:

(i) $z_1 = 1 + 3i$, (ii) $z_2 = -2 + 5i$ (iii) $z_3 = 3 - 2i$

Solution.



6▪5 De Moivre's Theorem

If n is a rational number, then it holds that [r (Cos Θ + i sin Θ)]^n = r^n (cos n Θ + i sin n Θ)

This statement is known as De Moivre's theorem.

Example 7.

Express $(\frac{1}{2} - \frac{\sqrt{3}}{\sqrt{2}})^3$ in the form r (Cos Θ + i sin Θ).

Solution

Let z = $\frac{1}{2} - i\frac{\sqrt{3}}{2}$

r = $\sqrt{(\frac{1}{2})^2 + (-\frac{\sqrt{3}}{2})^2}$

= $\sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{\frac{4}{4}} = \sqrt{1}$ = 1

$\Theta = \tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)$

$= \tan^{-1}(\sqrt{3})$

$= -60^{o}$

But $\alpha = 360° - 60° = 300°$ (since z falls in the fourth quadrant)

z = 1(cos 300 + i sin 300).

$= \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

By de Moivre's theorem

$[r(\cos\Theta + i\sin\Theta)]^n = r^n(\cos n\Theta + i\sin n\Theta)$

$[1(\cos 300 + i\sin 300]^3 = 1^3[(\cos(300 \times 3) + i\sin(300 \times 3)]$

= cos 900 + i sin 900.

900 can be reduced as

900-2(360) = 900 - 720 = 180°.

$\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^3 = 1(\cos 180^{o} + i\sin 180°)$.

Example 8

Express $\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^{99}$ in the form z = x + iy

Solution. Let

$Z = \frac{\sqrt{3}}{2} - \frac{i}{2} = \frac{\sqrt{3}}{2} - \frac{1}{2}i$

$r = \sqrt{(\frac{\sqrt{3}}{2})^2 + (-\frac{1}{2})^2} = = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{\frac{4}{4}} = \sqrt{1} = 1$

$\Theta = \tan^{-1}\left(-\frac{1/2}{\sqrt{3}}\right)$

$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= -30°$

$\alpha = 360° - 30° = 330^o$

$\frac{\sqrt{3}}{2} - \frac{1}{2} = 1(\cos 330° + i \sin 330^o)$

$z = \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^{99} = [1(\cos 330° + i \sin 330^o)]^{99}$

$= 1^{99}$ [cos(330 x 99)+ i sin(330 x 99)]

= cos 32670 + i sin 32670.

32670 can be reduced to

$32670 - (360 \times 90) = 32670 - 32400 = 270^o$

$z = \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^{99}$

$= \cos 270 + i \sin 270^o$

$= 0 - i = 0 + (-1)i.$

6.6 Root of Complex Numbers

Every complex number in the form r(cos Θ + i sin Θ) [r ≠ 0] has exactly *n* distinct *n*th root. This roots all have the same absolute values or modulus, the positive value $r^{1/n}$, the angles may be taken respectively as

$\frac{\Theta + k \cdot 360}{n}$, k = 0, 1, ... , n - 1

∴ $[r(\cos\Theta + i\sin\Theta)]^{1/n} = r^{1/n}\left[\cos\left(\frac{\Theta + k\cdot 360}{n}\right) + i\sin\left(\frac{\Theta + k\cdot 360}{n}\right)\right]$, k = 0 , 1 , 2 , ... , n - 1.

Example 9

Express $\sqrt{3}$-i in the form r(cos Θ + i sin Θ) where - π < Θ < Θ.

Using De Moivre's theorem, express the square root of this number in the same form.

Solution

Let z = $\sqrt{3}$-i

$$r = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3+1} = \sqrt{4} = 2$$

$$\Theta = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = 30^0$$

$$\alpha = 360° - 30^0 = 330^0$$

but the polar form of a complex number is r(cos Θ + i sin Θ)

z = 2(cos 330° + i sin 330°)

$z = \sqrt{(3-i)} = \left(\sqrt{3}-i\right)^{1/2}$

$\sqrt{3} - i$ = 2(cos 330° + i sin 330°)

$\left(\sqrt{3}-i\right)^{1/2} = [2(\cos 330° + i\sin 330°)]^{1/2}$

$\sqrt{2} = 2^{1/2}\left[\cos\left(\frac{330 + k\cdot 360}{2}\right) + i\sin\left(\frac{330 + k\cdot 360}{2}\right)\right]$, where k = 0, 1.

When k = 0

$$\sqrt{z_0} = 2^{1/2}\left[\cos\left(\frac{330 + 0\cdot 360}{2}\right) + i\sin\left(\frac{330 + 0\cdot 360}{2}\right)\right]$$

= $2^{1/2}$ [cos $\left(\frac{330}{2}\right)$+ i sin $\left(\frac{330}{2}\right)$]

= $2^{1/2}$ [cos 165°+ i sin 165°]

= 1.4142(cos 165° + i sin 165°).

When k = 1,

$\sqrt{z_1}$ = $2^{1/2}$ [cos $\left(\frac{330 + 1 \cdot 360}{2}\right)$+ i sin $\left(\frac{330 + 1 \cdot 360}{2}\right)$]

= $2^{1/2}$ [cos 345^o + i sin 345^o]

= 1.4142[cos 345^o + i sin 345^o].

Example 10

Find the 3 cubic roots of 1 + i$\sqrt{3}$ and exhibit 1 + $\sqrt{3}i$ and its cubic roots on an argand diagram.

Solution.

Let z = 1 + i$\sqrt{3}$

r = $\sqrt{(1)^2 + (\sqrt{3})^2}$

=$\sqrt{1+3}$ = $\sqrt{4}$

ϴ = $\tan^{-1}\sqrt{3}$ = 60°

but the polar form of a complex number is r(cos ϴ + i sin ϴ)

∴ z = 2(cos 60° + i sin 60°)

(ii) $\sqrt[3]{z}$ = $\sqrt[3]{(1+i\sqrt{3})}$ = $(1+i\sqrt{3})^{1/3}$

= [2(cos 60 + i sin 60)]$^{1/3}$

$Z^{1/3} = 2^{1/3}\ [\cos\left(\frac{60 + k \cdot 360}{3}\right) + i \sin\left(\frac{60 + k \cdot 360}{3}\right)]$, where k = 0, 1,2

When k = 0

$Z_0^{1/3} = 2^{1/3}\ [\cos\left(\frac{60 + 0 \cdot 360}{3}\right) + i \sin\left(\frac{60 + 0 \cdot 360}{3}\right)]$

$= 2^{1/3}\ [\cos 20° + i \sin 20°]$

= 1.184 + i0.431

When k = 1.

$Z_1^{1/3} = 2^{1/3}\ [\cos\left(\frac{60 + 1 \cdot 360}{3}\right) + i \sin\left(\frac{60 + 1 \cdot 360}{3}\right)]$

$=2^{1/3}\ [\cos 140^o + i \sin 140^o]$

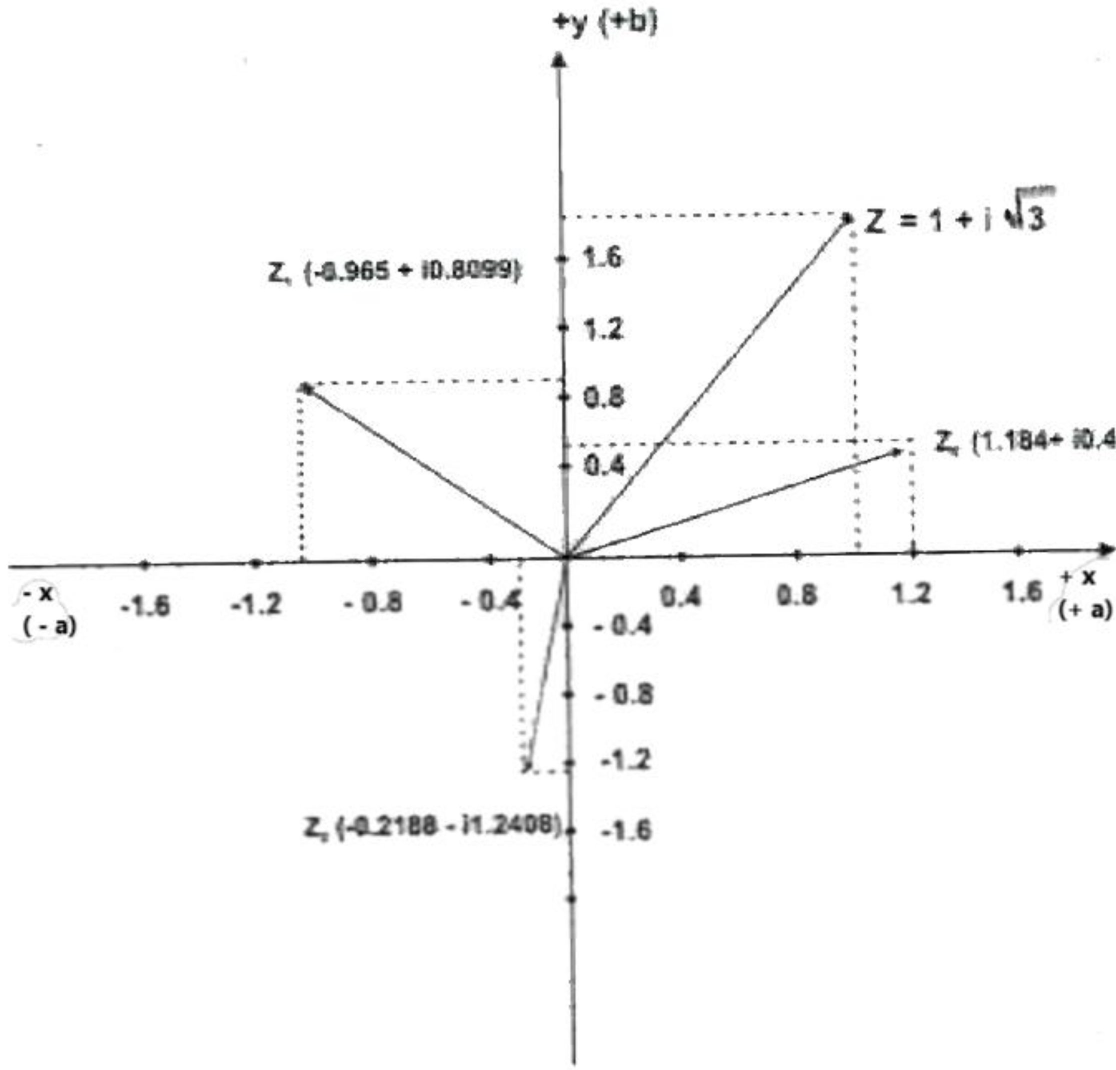
= -0.965+ i0.8099

When k = 2,

$Z_2^{1/3} = 2^{1/3}\ [\cos\left(\frac{60 + 2 \cdot 360}{3}\right) + i \sin\left(\frac{60 + 2 \cdot 360}{3}\right)]$

$= 2^{1/3}\ [\cos 260° + i \sin 260^o]$

=-0.2188 - i1.2408



References:

T. O. Sikiru (2018). Mathematical Basics: Algebra, Trigonometry and Complex Numbers, Volume 1. His Lineage Publishing House, Ibadan, Nigera.

B. D. Bunday (1967). Pure Mathematics for Advanced Level, Second Edition. Heinemana Educational Books (Nigeria) Plc, Nigeria.