ALGEBRA

MTS 101/MTS DEPT

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MODULE TWO

- **Course outline:**

Fundamental of set

Set notation and terminologies

Operations on Set

Order of Sets

Applications

INDICES

For any positive integral powers, the index (or power) of a number indicates how many times that number must be multiplied by itself.

e.g. $x^3 = x \times x \times x$ and $x^m = x \times x \times x \times x \times \cdots$ (m times)

the letter x is called the **base** while m is the **power**, **index** or **exponent**

LAWS OF INDICES

If m, n are positive integers and x, y are real numbers then

1. $x^m \times x^n = x^{m+n}$

2. $x^m \div x^n = x^{m-n}$

3. $(x^m)^n = x^{m \times n}$

4. $(x \times y)^m = x^m \times y^m$

5. $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$

Laws of Indices

6. $x^{-m} = \frac{1}{x^{m}}$

7. $x^{m/n} = \sqrt[n]{x^{m}} = \left(\sqrt[n]{x}\right)^{m}$

8. $x^{-1/n} = \frac{1}{x^{1/n}} = \frac{1}{\sqrt[n]{x}}$

9. $x^{-m/n} = \frac{1}{x^{m/n}} = \frac{1}{\sqrt[n]{x^{m}}} = \frac{1}{\left(\sqrt[n]{x}\right)^{m}} = \left(\frac{1}{\sqrt[n]{x}}\right)^{m}$

Examples 1

Evaluate

i. $9^{1.5}$ ii. $10^{1/2} \div 10^{3/2}$ iii. $\left(1\frac{9}{16}\right)^{\frac{1}{2}}$ iv. $9^{-n/2} \times 3^{n+2} \times 81^{-1/4}$

Solution

i. $9^{1.5} = 9^{3/2} = \left(3^{2}\right)^{3/2} = 3^{2\times 3/2} = 3^{3} = 27$

ii. $10^{1/2} \div 10^{3/2} = 10^{1/2} \times \frac{1}{10^{3/2}} = 10^{1/2} \times 10^{-3/2} = 10^{1/2-3/2} = 10^{-1} = \frac{1}{10}$

iii. $\left(1\frac{9}{16}\right)^{\frac{1}{2}} = \left(\frac{25}{16}\right)^{\frac{1}{2}} = \frac{25^{\frac{1}{2}}}{16^{\frac{1}{2}}} = \frac{5}{4}$

iv. $9^{-n/2} \times 3^{n+2} \times 81^{-1/4} = 3^{2\times -n/2} \times 3^{n+2} \times 3^{4\times -1/4} = 3^{-n} \times 3^{n+2} \times 3^{-1}$

$3^{-n+n+2-1} = 3$

Exercises 1

Simplify the following: Type equation here.

1. $\frac{x^{1/3} \times 2y^{1/4} \times \sqrt{xy}}{(x^{10}y^{9})^{1/2}}$

2. $\frac{x^{-2/3} \times y^{-1/3}}{(x^{4}y^{2})^{-1/8}}$

3. $\frac{x^{3n+1}}{x^{2n+5/2}\sqrt{x^{2n-3}}}$

4. $\frac{-(x^{2}+1)^{1/2} + x^{2}(x^{2}+1)^{-1/2}}{x^{2}}$

LOGARITHMS

If $a \neq 1$ is a positive real number, then the logarithms of a real number y to the base a is the index to which a should be raised to obtain y.

i.e. if $a^x = y$ then $\log_a y = x$

i.e. $a^0 = 1 \leftrightarrow \log_a 1 = 0$, $a^1 = a \leftrightarrow \log_a a = 1$

The definition shows that logarithms is defined only for positive values

Properties of Logarithms

If $M.N$ are two numbers, then for any positive real number a (where $a \neq 1$

1. $\log_a MN = \log_a M + \log_a M$ i.e. the logarithm of a product is a sum of their logarithms

$\log_a (M + N) \neq \log_a M + \log_a N$ – *PLEASE NOTE*

2. $\log_a \frac{M}{N} = \log_a M - \log_a N$ i.e. the logarithm of a ratio in equal to the logarithm of the numerator minus the logarithm of the denominator.

$\mathbf{log}_a\, M - N \neq \mathbf{log}_a\, M - \mathbf{log}_a\, N$ − ***PLEASE NOTE***Type equation here.

3. $\log_a M^P = p \log_a P$ (where P is any real number)

4. $\log_a M = \frac{1}{\log_b a} \times \log_b M = \log_b M \, T \log_b a$

COMMON LOGARITHMS

Any number y can be expressed as $y = a + 10^b$ where $1 \leq a < 10,\ a \in \mathbb{R}$ and $b \in \mathbb{Z}$

b is called the characteristics of y and $\log_{10} a$ is called the Mantissa of $\log_{10} y$. The Mantissa is given in the common logarithms tables to 4 significant figures. The characteristics shows how many the decimal point of the number y has been moved and in what direction.

e.g.

i. $4326 = 10^3 \times 4.326$ (Characteristic = 3, Mantisaa = $\log_{10} 4326$)

$= 10^3 \times 10^{0.6361}$ (from tables, Mantissa = 0.6361)

$= 10^{3.06361} \leftrightarrow \log_{10} 4326 = 3.6361$ (definition of logarithms)

Similarly

ii. $432.6 = 10^2 \times 4.326 = 10^2 \times 10^{0.6361} = 10^{3.6361} \leftrightarrow \log_{10} 4326 = 3.6361$

iii. $2 = 1 \times 2 = 10^0 \times 2 = 10^0 \times 10^{0.3010} = 10^{0.3010} \leftrightarrow \log_{10} 2 = 0.3010$

iv. $0.04326 = 10^{-2} \times 4.326 = 10^{-2} \times 10^{0.6361} = 10^{-2.6361}$ (the negative sign shows that only the characterist is –ve)

Exercise 2

1. Evaluate the following to 3 significant figures:

i. $\left(\dfrac{8.621 \times \sqrt[3]{0.02734}}{\sqrt[5]{52.18 \times 0.724}}\right)^2$ ii. $\dfrac{0.3581 \times 0.028447}{0.009418 \times 3.291}$

2. Evaluate the following without using tables:

i. $3.05 \times \log_2 3 \times \log_3 4 \times \log_4 5 \times \log_5 6 \times \log_6 7 \times \log_7 8$

ii. $\dfrac{\log 81}{\log 9} - \dfrac{\log 49}{\log 343} + 5\log 2 - \log 32$ iii. $\dfrac{\frac{2}{3}\log x - \log y}{\frac{1}{3}\log 3 - 0.5\log y} + 3$

iv. $\log_{10} 105$ given that $\log_{10} 2 = 0.3010$, $\log_{10} 3 = 0.4771$ and $\log_{10} 7 = 0.8451$

Indicial and Logarithmic Equations

These are equations involving either logarithms or indices or both

Examples

Solve for x in the following equations

i. $2^{9x-3} = 8^{3-x^2}$ ii. $2^{2x+1} - 9(2^x) + 4 = 0$

iii. $\log_2 x + \log_4 2x + \log_{16} 3x = \frac{21}{4}$

iv. $\log_2(11 - x) = \log_2(x + 1) + 3$

Solution

i. $2^{9x-3} = 8^{3-x^2} \rightarrow 2^{3(3x-1)} = 2^{3(3-x^2)}$

applying logarithms on both sides we have =

$3(3x - 1) = 3(3 - x^2) \rightarrow 3(x^2 + 3x - 4) = 0 \rightarrow x^2 + 3x - 4 = 0 \rightarrow (x + 4)(x - 1) = 0$

$x = -4$ or 1

ii. $2^{2x+1} - 9(2^x) + 4 = 0 \longrightarrow 2(2^x) - 9(2^x) + 4 = 0$

Let $y = (2^x)$ $then$ $y^2 = 2^{2x}$ hence we have $2y^2 - 9y + 4 = 0$

Which imply that

$(2y - 1)(y - 4) = 0 \longrightarrow y = \frac{1}{2}$ or $y = 4$

Now if $y = \frac{1}{2}$ then $2^x = 2^{-1} \longrightarrow x = -1$

If $y = 4$ then $2^x = 2^2 \longrightarrow x = 2$

iii. $\log_2 x + \log_4 2x + \log_{16} 3x = \frac{21}{4} \longrightarrow \frac{\log_2 x}{\log_2 2} + \frac{\log_2 2x}{\log_2 4} + \frac{\log_2 3x}{\log_2 16} = \frac{21}{4}$

$\log_2 x + \frac{\log_2 2x}{2} + \frac{\log_2 3x}{4} = \frac{21}{4} \longrightarrow \frac{(4\log_2 x + 2\log_2 2x + \log_2 3x)}{4} = \frac{21}{4}$.

iv.

$\left(4\log_2 x + 2\log_2 2x + \log_2 3x\right) = 21 \Longrightarrow \log_2(x^4 \times 4x^2 \times 3x) = 21$

Hence $12x^7 = 2^{21} \Longrightarrow x^7 = \frac{2^{21}}{12}$

v. $\log_2(11 - x) = \log_2(x + 1) + 3 \Longrightarrow \log_2(11 - x) - \log_2(x + 1) = 3$

implying that $\log_2 \frac{11 - x}{x + 1} = 3$ hence $\frac{11 - x}{x + 1} = 2^3 = 8 \therefore 11 - x = 8x + 8 \Longrightarrow x = \frac{1}{3}$

SURDS OR RADICALS

A surd is an irrational root of a rational number. Therefore, surds are of the form $y = \sqrt[n]{x}$ where x is a rational number which is not a perfect nth of a rational number.

In $\sqrt[n]{x}$, n is called the **INDEX** or **ORDER** of y and x is called the **RADICAND.**

Invariably, surds of 2^{nd}, 3^{rd} 4^{th} and 5th orders are called quadratics, cubic and quintic surds respectively considering the index.

Examples: $3\sqrt{2}, 3 + \sqrt{2}, \sqrt{5}, \sqrt{5} + \sqrt{3}\sqrt{4}, 2 + \sqrt{2} - 3\sqrt{3}$ etc are all surds but $\sqrt[3]{27}, \sqrt[4]{16}, \sqrt{4}$ are not. $3\sqrt{2}, \sqrt{5}$ are monomials (one term) surds while $3 + \sqrt{7}, \sqrt{3} + 3\sqrt{5}$ are binomial (two terms) and $3\sqrt{2} - 5\sqrt{7}$ etc trinomial (three terms) surds.

Properties of Surds

If x, y are non-negative rational numbers and n is a positive integer then

1. $\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$

2. $\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}, y \neq 0$

3. $\sqrt[n]{x^n} = x$

Addition of surds is possible only when the simplified form of the surds contain the same surd. Such surds are called similar surds. Examples of this is $\sqrt{12} = 2\sqrt{3}$ *and* $\sqrt{75} = 5\sqrt{3}$ are similar.

A surd is in its simplest form if

i. the radicant contains no fractions

ii. the radicant contains no factors that are perfect nth power where n is the index of the radical.

iii. the index is the smallest possible positive integer

Rationalization of Surds

If the product of two surds is a rational number, then each is called a rationalizing factor, the process of multiplying them together is called rationalization.

The binomial quadratic surds $a \pm \sqrt{b}$ are called <u>conjugate pairs</u>. The product and sum of two conjugates pairs is a rational number. Note the identities

i. $(a + b\sqrt{d})(a - b\sqrt{d}) = a^2 - (b\sqrt{d})^2 = a^2 - b^2 d$ (is rational)

ii. $(a + b\sqrt{d})(a - b\sqrt{d}) = 2a$ (is rational)

iii. $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b$ (is rational)

iv. $\frac{1}{\sqrt{a}} = \frac{\sqrt{a}}{\sqrt{a}\sqrt{a}} = \frac{\sqrt{a}}{(\sqrt{a})^2} = \frac{\sqrt{a}}{a}$ (the denominator is rational)

v. $\frac{1}{a \pm b\sqrt{d}} = \frac{a \mp b\sqrt{d}}{a^2 - b^2 d}$ (denominator is rational)

Execises

1. Simplify the following

i. $\sqrt{\frac{25x^8y^6}{16}}$

ii. $\frac{\sqrt{5}-2}{\sqrt{2}+2}+\frac{\sqrt{5}+2}{\sqrt{5}-2}$

iii. Express $\frac{\sqrt{2}+5}{\sqrt{10}}$ in the form $a\sqrt{5}+b\sqrt{2}$

iv. $\frac{\sqrt{\frac{1}{4}}+\sqrt{\frac{1}{8}}+\sqrt{\frac{1}{2}}}{(1-\sqrt{2})(1+2\sqrt{2})}$

2. Express $\sqrt{32}+\frac{6}{\sqrt{2}}$ as single surd and hence find the value of $\frac{7}{\sqrt{2}}\left(\sqrt{32}+\frac{6}{\sqrt{2}}\right)$