4.0 Simple Harmonic Motion

Definition 4.1.1 Simple Harmonic Motion or SHM is defined as a motion in which the restoring force is directly proportional to the displacement of the body from its mean position. the direction of this restoring force is always towards the mean position. The acceleration of a particle executing simple harmonic motion is given by $a(t) = w^2 x(t)$ where w is the angular velocity of the particle.

All the Simple Harmonic Motions are oscillatory and also periodic, but not all oscillatory motions are SHM.

The study of Simple Harmonic Motion is very useful and forms an important tool in understanding the characteristics of sound waves, light waves and alternating current.

4.1.2 Types of Simple Harmonic Motion:

- Linear SHM
- Angular SHM.

4.1.3 Linear Simple Harmonic Motion

When a particle moves to and fro about a fixed point (called equilibrium position) along with a straight line then its motion is called linear Simple Harmonic Motion. Example Spring-mass system.

$$\vec{F} \propto -\vec{x}$$

$$\vec{a} \propto -\vec{x}$$

$\vec{x}$ = displacement of particle from equilibrium

$\vec{F}$ = Restoring force

$\vec{a}$ = acceleration

4.1.4 Angular Simple Harmonic Motion

When a system oscillates angularly with respect to a fixed axis, then the motion is called angular simple harmonic motion.

$$T \propto \theta$$

or

$$\alpha \propto \theta$$

where

T = Torque

α = angular acceleration

θ = angular displacement.

4.1.5 Time period and Frequency of SHM.

the minimum time after which the particle keeps on repeating its motion is known as the period or the shortest time taken to complete one oscillation

$$T = \frac{2\pi}{w}$$

where Frequency $= \frac{1}{T}$ so that $w = 2\pi f = \frac{2\pi}{T}$.

4.1.6 Phase in SHM.

The phase of a vibrating particle at any instant is the state of the vibrating (oscillating) particle regarding its displacement and direction of vibration at that particular instant.

the expression is given as

$$x = A \sin(wt + \phi)$$

where $(wt + \phi)$ is the phase of the particle, the phase at time $t = 0$ is known as the initial phase

4.1.7 Phase Difference

the ~~difference~~ of ~~total~~ phase angles of two particles executing simple harmonic motion with respect to the mean position is known as the phase difference.

~~[illegible]~~

Two vibrating particles are said to be on the same phase, the phase difference between them is an even multiple of π.

$\nabla\phi = n\pi$ where $n = 0, 1, 2, 3 \dots$

Two vibrating particles are said to be in the opposite phases if the phase difference between them is an odd multiple of π

$\nabla\phi = (2n+1)\pi$ where $n = 0, 1, 2, 3 \dots$

Recall that $\vec{F} = -K\vec{x}$ where $-K$ is a +ve constant.

$\Rightarrow \vec{F} = m\vec{a}$, where $\vec{a}$ is the acceleration

~~Hence~~,

$\Rightarrow m\vec{a} = -K\vec{x}$

$\Rightarrow \vec{a} = -\left(\frac{K}{m}\right)\vec{x}$

put $\frac{K}{m} = w^2$

$\Rightarrow w = \sqrt{\frac{K}{m}}$

$\Rightarrow \vec{a} = -\left(\frac{K}{m}\right)\vec{m} = -w^2\vec{x}$

Since $\vec{a} = \frac{d^2x}{dt^2}$

therefore $\frac{d^2x}{dt} = -w^2\vec{x}$.

So $\frac{d^2x}{dt^2} + w^2x = 0$, which is the differential equation for linear simple harmonic motion.

Also

Recall that $T \propto -\theta$

$$T = -K\theta$$

$$T = Ia$$

where $a = -K\theta$

$$I\frac{d^2\theta}{dt^2} = -K\theta$$

$$\frac{d^2\theta}{dt^2} = \left(\frac{K}{I}\right)\theta = -w^2\theta$$

$\frac{d^2\theta}{dt^2} = -w^2\theta = 0$ is the differential equation of an angular Simple Harmonic Motion.

Conditions for Simple Harmonic Motion

Since $\vec{F} \propto -\vec{x}$

$$\vec{a} \propto -\vec{x}$$

Then $\vec{a} = -w^2x$

$$\vec{a} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

$$\vec{a} = v\frac{dv}{dx} = -w^2x.$$

$$\int_0^v v\,dv = \int_0^x -w^2 x\,dx$$

$$\frac{v^2}{2} = \frac{-w^2x^2}{2} + c \quad \text{———} (1)$$

At the point $v=0$, $[x=A)$ equation (1) becomes

$$\frac{v^2}{2} = \frac{-w^2A^2}{2} + c$$

$v=0$,

$$0 = \frac{-w^2A^2}{2} + c$$

$$c = \frac{w^2A^2}{2}$$

Substituting c into equation (1)

$$\frac{v^2}{2} = -\frac{w^2x^2}{2} + \frac{w^2A^2}{2}$$

$$\Rightarrow v^2 = -w^2x^2 + w^2A^2$$

$$\Rightarrow v^2 = w^2(A^2 - x^2)$$

$$v = \sqrt{w^2(A^2-x^2)}$$

$$v = w\sqrt{A^2-x^2} \quad \text{———} (2)$$

where v is the velocity of the particle executing simple harmonic motion from the definition of instantaneous velocity

$$v = \frac{dx}{dt} = w\sqrt{A^2-x^2}$$

$$\Rightarrow \int \frac{dx}{\sqrt{A^2 - x^2}} = \int_0^t w\,dt$$

$$\Rightarrow \sin^{-1}\left(\frac{x}{A}\right) = wt + \phi$$

So that

$$x = A\sin(wt + \phi) \quad \text{———} (3)$$

Equation (3) is the equation of position of a particle as a function of time.

4.2 Energy in Simple Harmonic Motion (SHM)

The system that executes SHM is called the harmonic oscillator.

Consider a particle of mass m, executing linear simple harmonic motion of angular frequency w and amplitude (A). the displacement $(\vec{x})$, velocity $(\vec{v})$ and acceleration $(\vec{a})$ at any time t are given by

$$x = A\sin(wt + \phi) \quad \text{[by differentiation]}$$

$$v = Aw\cos(wt+\phi) = w\sqrt{A^2 - x^2}$$

$$a = -w^2 A\sin(wt + \phi) = -w^2 x$$

The restoring force $(\vec{F})$ acting on the particle are given by

$$F = -Kx, \quad \text{where } K = mw^2.$$

4.2.1 Kinetic Energy of a Particle in SHM

Given that

$$K = \frac{1}{2}mv^2, \quad \text{Since } v^2 = A^2 w^2 \cos^2(wt+\phi)$$

$$K = \frac{1}{2} m\omega^2 A^2 \cos^2(\omega t + \phi)$$
$$= \frac{1}{2} m\omega^2 (A^2 - x^2)$$

therefore $K = \frac{1}{2} m\omega^2 A^2 \cos^2(\omega t + \phi) = \frac{1}{2} m\omega^2 (A^2 - x^2)$

4.2.2 Potential Energy of SHM.

the total work done by the restoring force in displacing the particle from $x = 0$ to $x = x$, When the particle has been displaced from x to $x + dx$, the work done by restoring force

$$dw = F dx = -Kx\, dx.$$
$$w = \int dw = \int_0^x -Kx\, dx = -\frac{Kx^2}{2}$$
$$= -\frac{m\omega^2 x^2}{2} \quad [K = m\omega^2]$$
$$w = \frac{-m\omega^2 A^2}{2} \sin^2(\omega t + \phi)$$. is the

potential Energy.

i.e $\frac{m\omega^2 x^2}{2} = \frac{m\omega^2 A^2}{2} \sin^2(\omega t + \phi)$

4.2.3 Total Mechanical Energy of particle Executing SHM.

$$E = K.E + P.E$$
$$= \frac{1}{2} m\omega^2 (A^2 - x^2) + \frac{1}{2} m\omega^2 x^2$$
$$\Rightarrow E = \frac{1}{2} m\omega^2 A^2.$$

Example 1 Consider a particle undergoing simple harmonic motion. The velocity of the particle at position x_1 is V_1 and velocity of the particle at position x_2 is V_2. Show that the ratio of time period and amplitude is

$$\frac{T}{A} = 2\pi \sqrt{\frac{x_2^2 - x_1^2}{V_1^2 x_2^2 - V_2^2 x_1^2}}$$

Solution

Using the equation

$$v = w\sqrt{A^2 - x^2} \Rightarrow v^2 = w^2(A^2 - x^2) \quad \text{--- (1)}$$

then

$$v_1^2 = w^2(A^2 - x_1^2) \quad \text{--- (2)}$$

and

$$v_2^2 = w^2(A^2 - x_2^2)$$

Substracting (2) from (1), then

$$v_1^2 - v_2^2 = w^2(A^2 - x_1^2) - w^2(A^2 - x_2^2)$$

$$= w^2(x_2^2 - x_1^2)$$

$$w = \sqrt{\frac{v_1^2 - v_2^2}{x_2^2 - x_1^2}} \Rightarrow T = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}} \quad \text{--- (3)}$$

Dividing (1) and (2), then

$$\frac{v_1^2}{v_2^2} = \frac{w^2(A^2 - x_1^2)}{w^2(A^2 - x_2^2)} \Rightarrow A = \sqrt{\frac{v_1^2 x_2^2 - V_2^2 x_1^2}{v_1^2 - V_2^2}} \quad \text{--- (4)}$$

Dividing (3) & (4), then