MTS 104 – MECHANICS (Part 2).

By

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(Venue: MP 02, Thur (11 am - 1 pm), 2022/2023)

Contents of Part 2

Texts:

- CJ Tranter & CG Lambe (1973): Advanced level Mathematics (Pure and Applied), The English Universities Press Limited, Warwick Lane, LONDON.
- MR Tuttuh-Adegun; S Sivasubramaniam & R Adegoke (1997): Further Mathematics Project. NPS Educational Publishers Limited, Ibadan, NIGERIA
- Any other Mathematics Textbooks

Section 2: Resultants of any number of forces acting on a particle and reduction of coplanar forces

2.1 Preamble:

Mechanics is defined as an area of study in physical sciences which deals with the effects of forces on a body. Statics and dynamics are two parts of the study of mechanics, in statics we discuss bodies which are at rest, in dynamics, bodies which are in motion. In this section, we shall be concerned with the study of bodies which remain at rest under the action of the given forces.

2.1.1 Force: This is defined as the action which tends to change the state of rest or uniform motion of a body in a straight line. The effect of force can be seen or felt but the force itself is invisible. The effect of a force on a body depends on:

- the magnitude of the force
- the direction in which the force acts
- the point of application of the force

Force is a vector quantity since both its magnitude and direction are taken into consideration. An absolute measure of force or unit of force is $kgms^{-2}$. The adopted unit of force commonly used is the Newton, N are is force which gives to a mass of $1kg$ an acceleration of 1 metre per second.

Weight: of a body is a force exerted on the body, due to the effect of gravity on the body

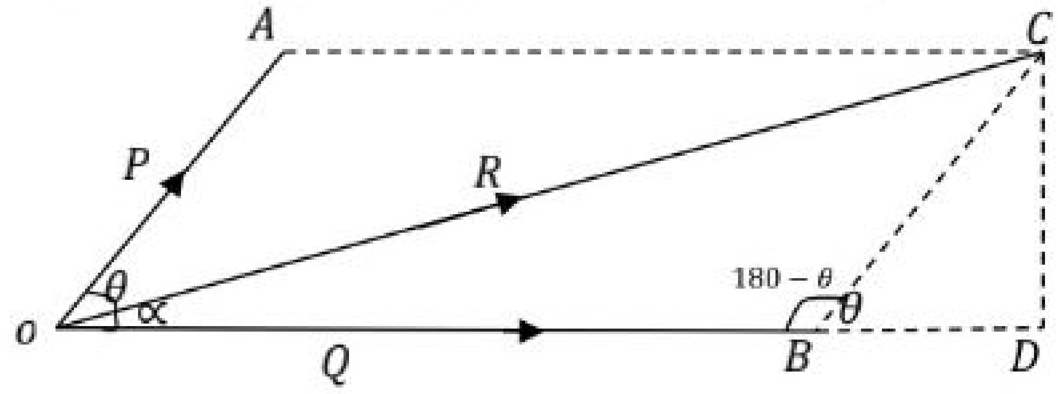
Particle: is the smallest part of the body

Rigid body: is a body in which the relative movements between it is negligible

2.1.2 Resultant of two forces acting at a point

Since a force is a vector quantity, two forces acting on a particle are equivalent to a single force acting on the particle which is the vector sum of the two forces. It could be combined according to the parallelogram law of vector combination.

Parallelogram Law: If two forces acting at a point are represented in magnitude and direction by the two adjacent side of a parallelogram, then the resultant of the two forces, is represented in magnitude and direction by the diagonal of the parallelogram, drawn from the point of action of the two forces



$Fig. 1.$

We can calculate R by using the cosine rule:

$$R^2 = P^2 + Q^2 - 2PQcos(180 - \theta)$$

If $\propto$ is the inclinations of R to Q, then

$$tan \propto = \frac{CD}{OD} = \frac{Psin\theta}{Q + pcos\theta}$$

Example 1: Forces P and Q are on a point O. The magnitude of P is $8N$ and that of Q is $12N$. The angle between P and Q at their point of action is 40^0. Find the resultant Force.

Solution: By the analytical method,

$$R^2 = 8^2 + 12^2 + 2(8)(12)cos40^0 = 64 + 144 + 192\cos 40^0 = 208 + 147.080 = 355.1$$

$$R^2 = \sqrt{355.1} = 18.84N$$

$$\tan \propto \ = \ \frac{CD}{OD} = \ \frac{8sin40^0}{12 + 8cos40^0} = \frac{5.142}{18.13} = 0.2836$$

$$\propto \ = \ \tan^{-1}(0.2836) = 15.83$$

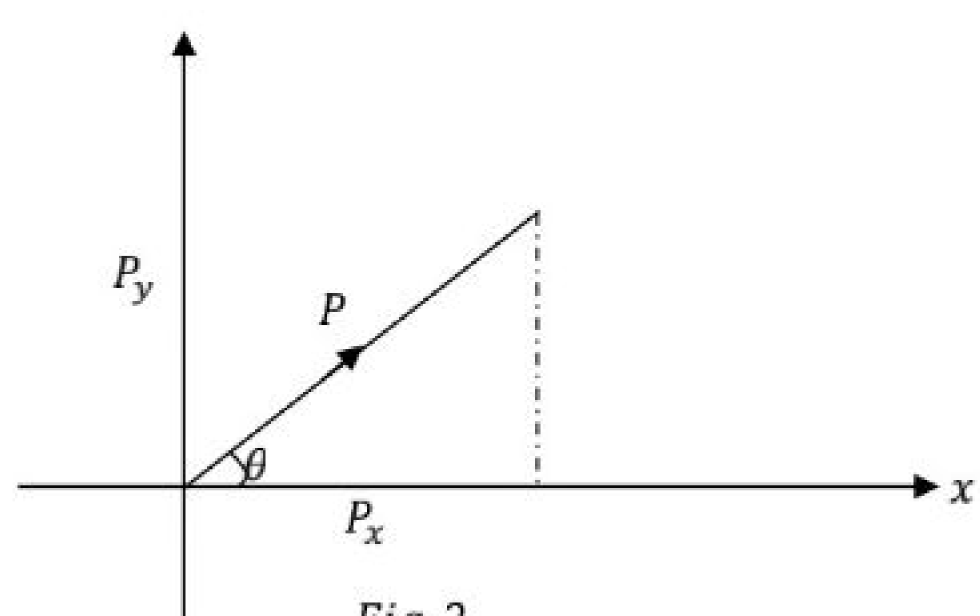
Also, a single force acting on a particle can be replaced by component forces in two directions which have the same effect as the single force. In most cases, we resolve a given force along the rectangular axes. We refer to the component along the $y - axis$ as the vertical component and the component along the $x - axis$ as the horizontal component. Consider figure 2, a force P is at θ^0 to the $x - axis$, let P_x be the horizontal component of P_y and P_i and P_j the vertical component of P. If i and j are unit vectors along x and y axes respectively, then we can write;

$P = P_{xi} + P_{yj}$

Also,

$$\frac{P_x}{P} = \cos 0 \ \Rightarrow P_x = Pcos\theta$$

$$\frac{P_y}{P} = \sin 0 \ \Rightarrow P_y = Psin\theta$$

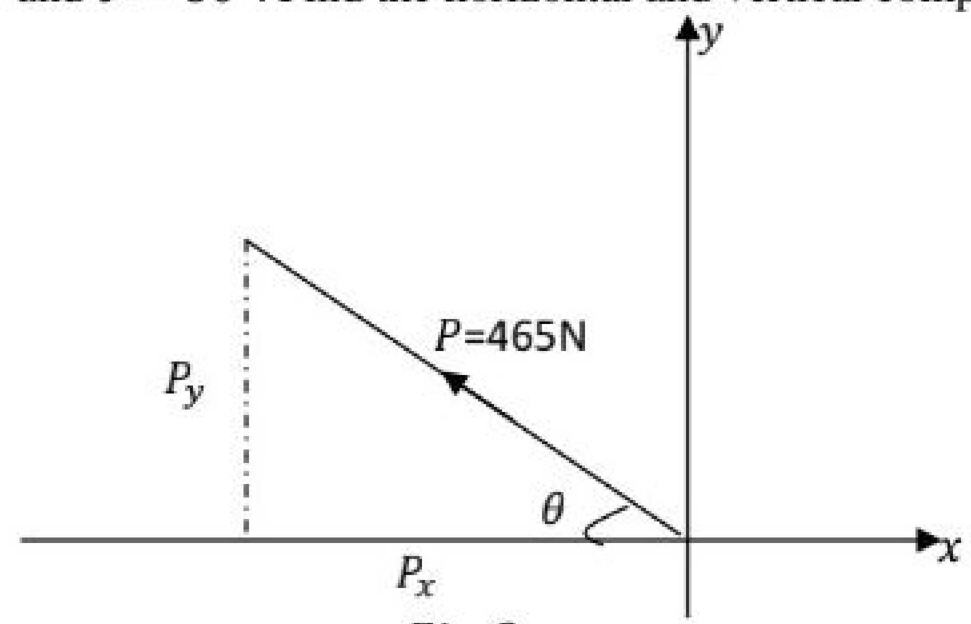


$Fig. 2.$

$$\tan\theta = \frac{P_y}{P_x} \Rightarrow \theta = \tan^{-1}\left(\frac{P_y}{p_x}\right)$$

However, the direction of inclination of the forces under consideration should be taken into account when finding components of a force.

Example 2: In figure 3, P is a force of $465N$ and $\theta = 30^0$. Find the horizontal and vertical components of P.



$Fig. 3.$

<u>Solution:</u>

$P_x = PCos\theta = -465\ cos30^0 = -402.7N$

$P_y = Psin\theta = \ 465\ sin30^0 = 232.5N$

2. 1. 3 <u>Resultant of Several Concurrent Forces</u>

In the earlier section, we have been able to find the method of two concurrent forces using the ||grm of vector addition. We can as well compute the resultant of a given number of forces by first reducing each force into horizontal and vertical components.

If R is the resultant and , the angle which the resultant formed with the horizontal, then:

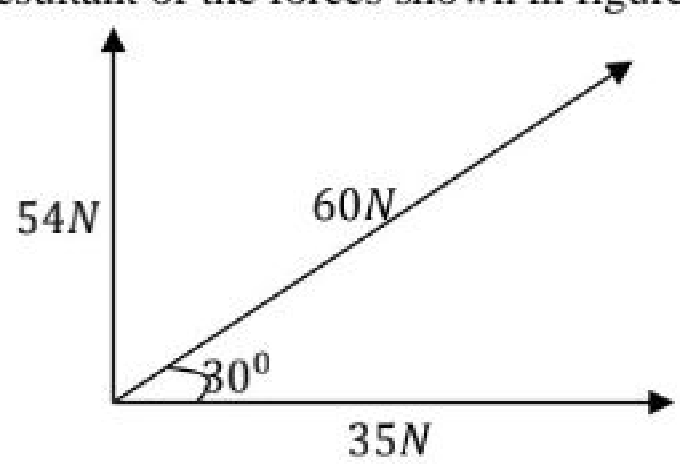
$$R = \sqrt{\left(\sum p_x\right)^2 + \left(\sum p_y\right)^2}$$

$$\tan\theta = \frac{\sum p_y}{\sum p_x}$$

Therefore,

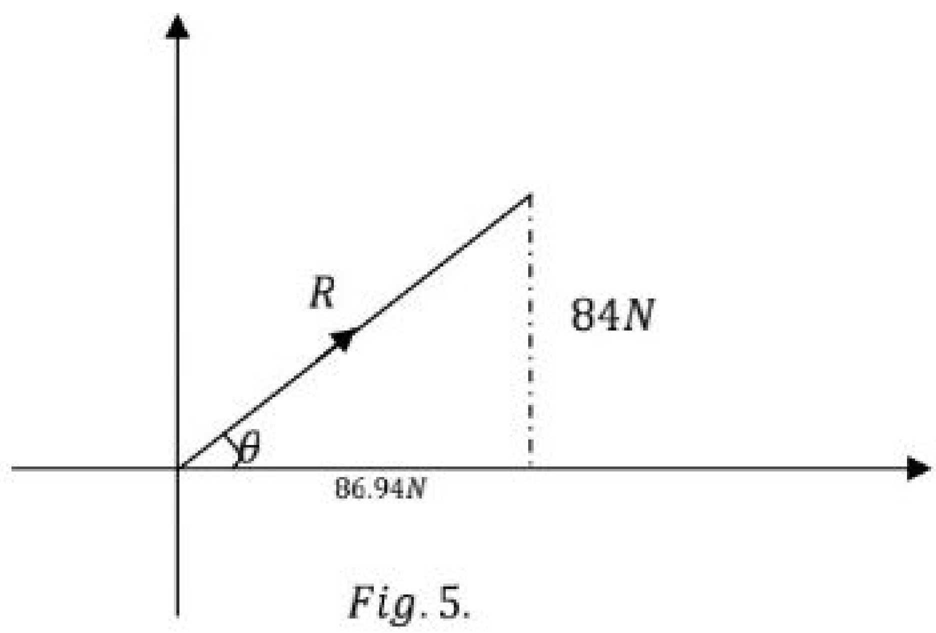
$$\theta = \tan^{-1}\frac{\sum p_y}{\sum p_x}$$

<u>Example 3</u>: Find the resultant of the forces shown in figure 4 below:



$Fig. 4.$

Solution:



$Fig. 5.$

Force	Vertical Component	Horizontal Component
$60N$	$60sin30^0 = 60 \times 0.5 = 30N$	$60cos30^0 = 60 \times 8660 = 57.96N$
$50N$	$54sin90^0 = 60 \times 1 = 54N$	$54cos90^0 = 54 \times 0 = 0$
$35N$	$36sin0^0 = 35 \times 0 = 0$	$35cos\ 0 = 35 \times 1 = 35N$
Total	$= 84N$	$= 86.96N$

Solution:

Let R be the magnitude of the resultant, then

$$R^2 = 84^2 + 86.96^2$$

$$R^2 = 7056 + 7562 = 14618$$

$$R = 120.9N$$

Let $\boldsymbol{\theta}$ be the inclination of the resultant to the horizontal, then

$$\tan\theta = \frac{84}{86.96} = 0.9660$$

$$\Rightarrow \theta = 44^0$$

Exercises:

1. A vehicle which has broken down is pushed by three men with forces $220N$ in the direction 030^0, $315N$ in the direction 090^0, $240N$ in the direction 120^0 respectively. Calculate the magnitude to the nearest N and the direction correct to the nearest degree, of the resultant of the three forces.
2. Forces at 8 and 5 Newton act on a particle in directions NE and $N30^0W$ respectively. Find the components of their resultant in direction N. and E.
3. The resultants of two forces P and Q acting on a particle is equal to P in magnitude; that of forces $2P$ and Q acting in the same directions as before is also equal to P. Find the magnitude of Q and prove that the direction of Q makes an angle of 150^0 with P.

2.2 Equilibrium of Coplanar forces and Couples

If a body remains at rest under the action given forces, we say that the body is in a state of *equilibrium*. In this section, we shall deal with bodies which remain at rest under the action of forces which have the tendency to cause *translation*. This state of equilibrium is called *Translational Equilibrium.*

An immediate question is; under what conditions will a body have a translational equilibrium? This is what this section is set to achieve.

First condition of equilibrium

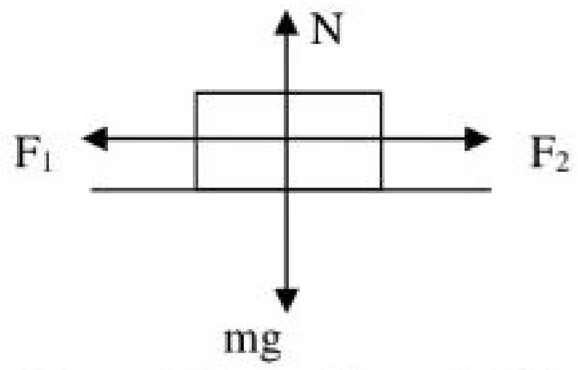


Fig. 6

Consider a block placed on a table as shown in Fig. 6. Forces F_1 and F_2 are applied to the block. If the magnitudes of F_1 and F_2 are equal, then the two forces balance each other. The block remains in translational equilibrium.

If $F_1 = -F_2$ then $F_1 + F_2 = 0$, i.e., sum of the horizontal components of the forces is equal to zero. In the same vein, upward force N balances the downward force mg on the block. We write $N = -mg$

$\therefore N + mg = 0$

Hence the sum of the vertical components of the forces is also equal to zero. In general, then the body is in translational equilibrium if the following conditions are satisfied:

$$R = 0, \sum F_x = 0 \text{ and } \sum F_y = 0$$

2.2.1 Triangle of forces. Lami's theorem

You will recall that if a system is in equilibrium under the action of a given number of forces, then resultant is equal to zero.

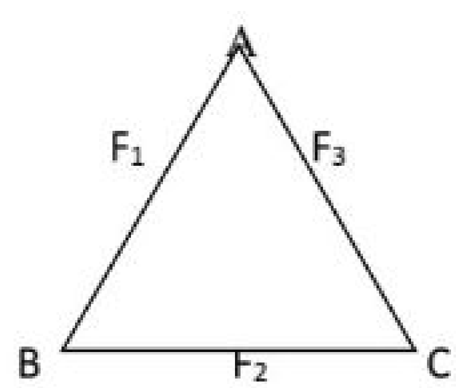


Fig. 7

Let us consider three coplanar forces F_1, F_2 and F_3 represented by the Triangle ABC. Hence *if a body is in equilibrium when acted upon by three coplanar forces, these forces can be represented by the sides of triangle taken in order.* The triangle thus drawn representing the three coplanar forces is called a *triangle of forces.*

Lami' s theorem

Lami's theorem is another form of the theorem of the triangle of forces, namely, *if three forces acting at a point are in equilibrium, then each force is proportional to the sine of the angle between the lines of action of the other two forces.*

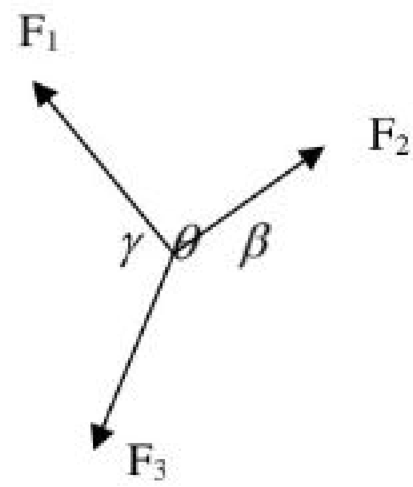


Fig. 8

Consider forces F_1, F_2 and F_3 acting at a point O as shown above in Fig. 8. Let the angle between F_1 and F_2 be θ, and the angle between F_2 and F_3 be β, while the angle between F_1 and F_3 be γ. By Lami's theorem

$$F_1 \propto \sin \beta \Rightarrow \frac{F_1}{\sin \beta} = k \text{ (constant)} \qquad \ldots \qquad (1)$$

$$F_2 \propto \sin \gamma \Rightarrow \frac{F_2}{\sin \gamma} = k \text{ (constant)} \qquad \ldots \qquad (2)$$

$$F_3 \propto \sin \theta \Rightarrow \frac{F_3}{\sin \theta} = k \text{ (constant)} \qquad \ldots \qquad (3)$$

$$\therefore \frac{F_1}{\sin \beta} = \frac{F_2}{\sin \gamma} = \frac{F_3}{\sin \theta} \qquad \ldots \qquad (4)$$

Equation (4) above is very useful in solving problems relating to three forces in equilibrium.

Example 4:

A body of mass 6.5 kg is supported by two strings. One of the strings is inclined at an angle of 30° and the other 40° to the horizontal. Find the tension in each string, if the system is in equilibrium (Take g = $10 m/s^2$)

Solution: **By Lami's theorem**

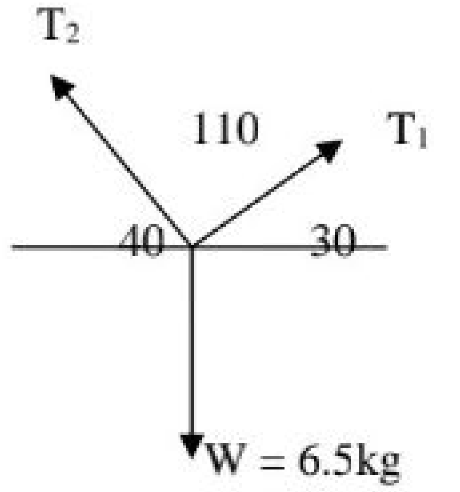


Fig. 9.

From Lami's theorem

$$\frac{T_1}{\sin 130} = \frac{W}{\sin 110}$$

$$\therefore T_1 = \frac{W \sin 130}{\sin 110} = \frac{65 \sin 50}{\sin 70} = 52.98N$$

$$\frac{T_2}{\sin 120} = \frac{W}{\sin 110}$$

$$\therefore T_2 = \frac{W \sin 120}{\sin 110} = \frac{65 \sin 60}{\sin 70} = 59.9N$$

Solution by resolution into components

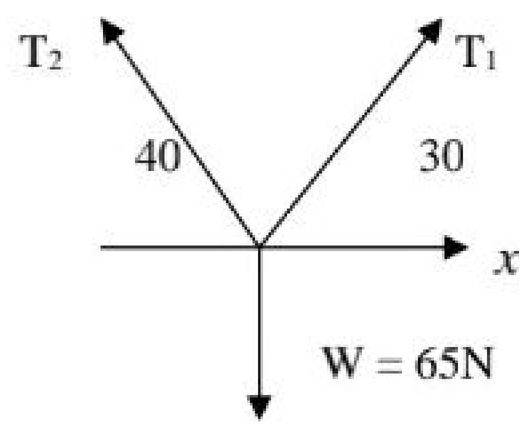


Fig. 10.

$$\sum F_x = T_1 \cos 30 - T_2 \cos 40$$

$$\sum F_y = T_1 \sin 30 + T_2 \sin 40 - 65$$

Since the system is in equilibrium;

$$\sum F_x = 0 \text{ and } \sum F_y = 0$$

$$\therefore T_1 \cos 30^\circ - T_2 \cos 40 = 0 \ldots (1)$$

$$T_1 \sin 30^\circ + T_2 \sin 40^\circ - 65 = 0 \ldots (2)$$

From (1) $T_1 = \dfrac{T_2 \cos 40}{\cos 30} = 0.8845T_2$ ... (3)

Substituting the value of T_1 in (3) into (2)

$$\begin{aligned}
&(0.8845T_2)\sin 30 + T_2 \sin 40 - 65 = 0 \\
&0.4423T_2 + 0.6428T_2 - 65 = 0 \\
&1.085T_2 - 65 = 0 \qquad \ldots (4) \\
&T_2 = 59.9N
\end{aligned}$$

Substituting the value of T_2 in (4) into (3)

$T_1 = 0.8845 \times 59.9 = 52.98N$

2. 2.3 Polygon of forces

The statement we made earlier about a system of three coplanar forces in equilibrium can be generalized for a system of forces that are more than three. If a system is in equilibrium under the action of three or more coplanar forces, then the forces can be represented in magnitude and direction by the sides of a polygon taken in order.

The polygon which is drawn to represent the system of forces is called *polygon of forces (See Fig.11)* . For any number of forces keeping a body in equilibrium, the first condition of equilibrium holds. Thus if a body is in equilibrium under the action of any number of forces then $\sum F_x = 0, \sum F_y = 0$

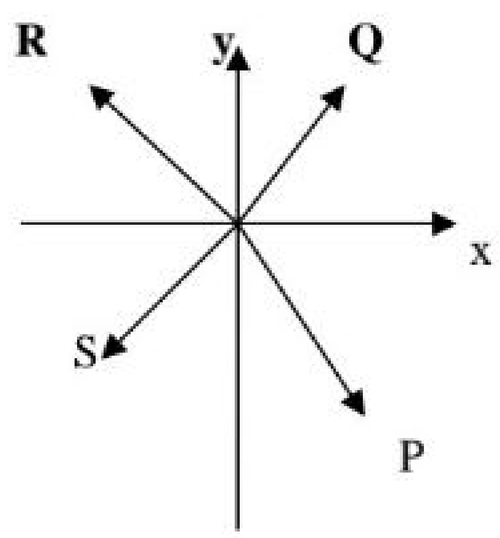
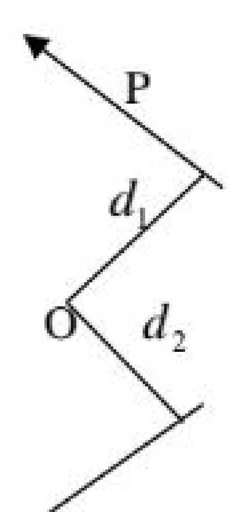


Fig. 11

Rotational Equilibrium

2.2 .4 Moment of force

Consider a flat sheet which is acted upon by two forces P and Q in the opposite sense, as shown in Fig. 12 about an axis through O, perpendicular to the plane of the sheet.



Q Fig. 12

The force P has the tendency to rotate the flat sheet about the point O in the counter-clockwise sense while the force Q has the tendency to rotate the body about the point O in the clockwise sense. The turning effect of the force P depends on d_1 which is the perpendicular distance from O to the line of action of the force P. This perpendicular distance is usually called *the force arm.* The greater the force arm, the greater the turning effect of the force P. Similarly, the turning effect of the force Q depends on d_2.

The *moment of a force* about the given point is a measure of the turning effect of the force on a body about this axis. The units of a moment are units of force and distance and we may speak of a moment of 10mN. Mathematically, the moment of a force about a reference point is defined as the product of the force and the force arm, i.e., $|M_1| = |F_1| x d_1$, where M_1 denote the magnitude of the moments of force P and force arm d_1.

Principle of moments

Let R be the resultant of two coplanar forces P and Q. We shall the moments of each of the two forces about a point A in the plane of the two forces. Let the moments of P, Q and R about point A be M_P, M_Q and M_A respectively. Then

$$M_P + M_Q = M_R$$

The above mathematical statement can be extended to any number of coplanar forces. As discussed earlier that if a system of coplanar forces acting at a point are in equilibrium, then their resultant is zero. It follows immediately, that if a system is in equilibrium under the action of any number of coplanar forces, then the sum of their moments about any point in the plane of the coplanar forces is zero. This is known as the *principle of moments*. Hence the necessary and sufficient condition for a system to be in equilibrium is that $\sum F_x = 0, \sum F_y = 0,$ and $\sum M_A = 0$, where $\sum M_A$ denote the sum of all the moments at a point A.

Couple

Two equal but opposite parallel forces constitute a couple.

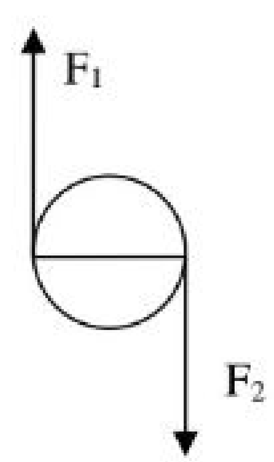


Fig. 13

Examples of couples are the two forces required to turn a tap and the force exerted at the end of a spanner to turn a nut together with the equal and opposite force exerted by the nut on the spanner. A couple will cause rotation but will not move things about. Let Γ be the moment of the couple, then

$$\Gamma = F_1 x d = F_2 x d$$

where d is the perpendicular distance between the two forces constituting the couple. If the resultant is defined in magnitude and direction as the vector sum of forces, the resultant of a couple is evidently zero. Its moment is not, however, zero.

2. 3 Friction

The opposing force between two bodies in contact is known as *frictional force*. We can therefore define *friction* as that force which tends to oppose relative sliding motion of two surfaces in contact. The mathematical treatment of friction is based on certain assumptions which are embodied in the so-called ***laws of friction*** which are found to be in close agreement with experience. These laws are set out below:

1. When two bodies are in contact the *direction* of friction is opposite to the direction in which motion will occur.
2. If the bodies are in equilibrium the force of friction is just sufficient to prevent motion and may be determined by applying the conditions for equilibrium to all the forces acting on each body.
3. *Limiting friction* is the frictional force which is being exerted when equilibrium is on the point of being broken
4. The *ratio* of limiting friction to the normal reaction depends on the nature of the surfaces in contact. This ratio is called the *coefficient of friction* and is denoted by the Greek letter μ (mu). Thus, if R be the normal reaction, the limiting friction is μR.
5. The amount of limiting friction is independent of the area of contact between the surfaces.
6. When motion takes place the direction of friction is opposite to that of relative motion and independent of velocity.

The *angle of friction* is defined as λ (lambda), where $\mu = \tan\lambda$. Thus if R be the normal reaction and the friction is limiting and equal to μR in a direction perpendicular to R, the resultant force at the point of contact is $R\sqrt{1+\mu^2}$ in a direction inclined to the normal reaction at an angle whose tangent is μ. Therefore, the resultant is $R\sqrt{(1+\tan^2\lambda)} = R\sec\lambda$, inclined at an angle λ to the normal.

2.3.1 Equilibrium on an inclined plane

If a body of weight W is placed on a rough plane inclined at an angle α to the horizontal it will slide down a line of greatest slope unless it is prevented by friction, which must therefore act upwards along the line. Let R be the normal reaction and F the force of friction (Fig. 14). The body is in equilibrium under the action of the forces W, R, and F, so that, since the weight acts in a direction making an angle 90-α with the line of greatest slope,

$$R = W\cos\alpha, F = W\sin\alpha, \frac{F}{R} = \tan\alpha$$

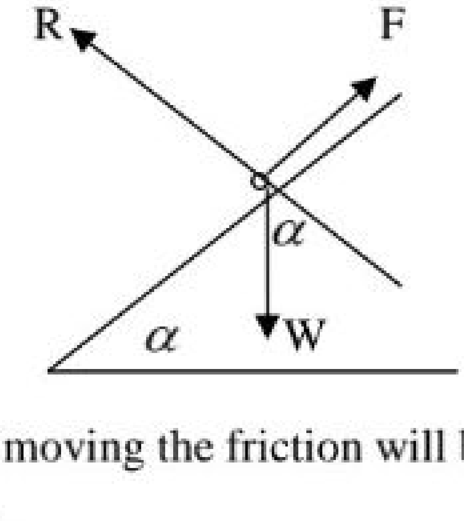


Fig. 14

If, however, the body is on the point of moving the friction will be limiting and $F = \mu R = R\tan\lambda$, so that $\tan\alpha = \mu = \tan\lambda$, that is, $\lambda = \alpha$.

Example 5.

A block of mass 2kg rests on a rough horizontal table. If the coefficient of friction between the block and the table is 0.43. Find the minimum horizontal force P that will make the block to slide on the table. (Take $g = 9.8\ ms^{-2}$).

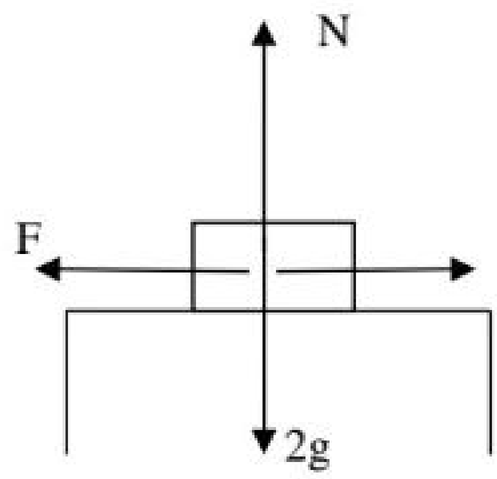


Fig. 15

Solution:

The block is at the point of sliding when friction is limiting. We say the block is at limiting equilibrium.

At limiting equilibrium P = F, but F = N $\therefore P = \mu N$

N = 2g, $\mu = 0.43, \therefore P = 0.43x2x9.8 = 8.428N$

Exercises 2:

1. A log of mass16kg rests on a horizontal floor. The coefficient of friction between the log and the floor is 0.62. If the minimum force required just to move the log along the horizontal floor is 58N, find the limiting friction.
2. A big stone is of mass 13kg. A boy whose weight is 59N sits on the stone. Find the minimum horizontal force P required to move the stone on the ground if the coefficient of friction is 0.25.

Due date is next TUESDAY ---, 2023 at Noon. Class Rep. should collect and submit to Mr. --------------- (at the Examination Venue)