MTS 102 – CALCULUS AND TRIGONOMETRY (2022/2023 Session)

Module 5 – INTEGRATION AND ITS APPLICATIONS

Objectives:

At the end of this module, students should be able to

- see integration as reverse process of differentiation;
- find a function whose derivative we already know

Recommended Texts:

- **BD Bunday & H Mulholland (2004): Pure Mathematics for A' level (second edition)**, Heinemann Educational Books (Nigeria) Plc: Ibadan
- **MR Tuttuh-Adegun, S Sivasubramaniam & R Adegoke (1997): Further Mathematics Project (revised edition)**, NPS Educational Publishers Limited: Ibadan
- **CJ Tranter & CG Lambe (1975): Advanced Level Mathematics (Pure and Applied),** The English Universities Press Limited: Great Britain

3.1 Introduction

Previously, we have been considering the problem of finding the **differential coefficient** or **rate of change** of a given function. The **integral calculus** to which we now turn our attention is concerned with the inverse problem, namely, given the rate of change of a function, to find the function. In symbols, we require to find $f(x)$ where

$$\frac{df(x)}{dx} = g(x) \qquad \ldots \qquad (1)$$

and $g(x)$ is given . It is more usual to write

$$f(x) = \int g(x)dx$$

and we define integration as follows. The integral of a function $g(x)$ with respect to x is the function whose differential coefficient with respect to x is $g(x)$ and it is written $\int g(x)dx$. The justification for the choice of the symbol will be much appreciated as we process as integration will be seen as a process of summation. If we are required to find $\int 3x^2 dx$, then $\int 3x^2 dx = x^3$ because

$$\frac{d(x^3)}{dx} = 3x^2$$

Similarly $\int \sin x dx = -\cos x$; because $\dfrac{d(-\cos x)}{dx} = \sin x$

$$\int \frac{dx}{x}\,dx = \log_e x;\text{because } \frac{d(\log_e x)}{dx} = \frac{1}{x}$$

3.2 The arbitrary constant

If $f(x) = x^3$ then $\frac{df(x)}{dx} = 3x^2$ so x^3 is the integral of $3x^2$.

If $f(x) = x^3 + 5$ then $\frac{df(x)}{dx} = 3x^2$ and $x^3 + 5$ is the integral of $3x^2$.

In general, if $f(x) = x^3 + c$ then $\frac{df(x)}{dx} = 3x^2$. So $x^3 + c$ is integral of $3x^2$, $f(x) = x^3 + c$

is called the **general solution** of

$f(x) = x^3$

$f(x) = x^3 + 5$

$f(x) = x^3 + c$

are particular solution of $\frac{df(x)}{dx} = 3x^2$

The constant c , in the general solution is called an arbitrary constant of integration .The general

solution of $\frac{df(x)}{dx} = 3x^2$

$$f(x) = \frac{x^{n+1}}{n+1} + c, n \neq -1$$

Remarks: To find $f(x)$ given $g(x)$ means that we have to retrace the steps we made in the process of differentiation and then add constant. Unfortunately, there seems to be no general method for doing this, but a few of the more common integrals can be stated from our knowledge of differential coefficients. These results are known as **standard forms** and can be summarized in a tabular form for easy reference.

S/N	$f(x)$	$\int f(x)dx$
1	ax^n	$\frac{ax^{n+1}}{n+1}+c,(n\neq -1)$
2	$\cos x$	$\sin x+c$
3	$\sin x$	$-\cos x+c$
4	e^x	e^x+c
5	$\frac{1}{x}$	$\log_e x+c$
6	$\frac{1}{a^2+x^2}$	$\frac{1}{a}\arctan(\frac{x}{a})+c,$ a and c are constants
7	$\sec^2 x$	$\tan x+c$

Example 1: Integrate the following functions with respect to x:

(i) x^8 (ii) $\frac{1}{\sqrt{x^3}}$ (iii) $\frac{1}{25+x^2}$

Solution: (i) $\int x^8 dx=\frac{1}{9}x^9+c$ (ii) $\int x^{-\frac{3}{2}}dx=\frac{-2}{\sqrt{x}}+c$

(iii) $\int\frac{1}{25+x^2}dx=\frac{1}{5}\arctan(\frac{x}{5})+c$

3.3 Few useful rules of operation

Rule 1: The integral of a sum of a finite number of functions is the sum of their separate integrals (‘sum’ includes the addition of negative quantities, i.e. ‘differences’)
Rule 2: The addition of a constant to the variable makes no difference to the form of the result
Rule 3: Multiplying the variable by a constant makes no difference to the form of the result but we have to divide by the constant
Rule 4: The integral of a fraction whose numerator is the derivative of the denominator is the logarithm of the denominator
Rule 5: A constant factor may be brought outside the integral sign

Example 2: Integrate the following functions with respect to x:

(i) $(1+x)^2$ (ii) $\cos(x+3)$ (iii) e^{-2x} (iv) $\frac{4x^3-1}{x^4-x+2}$

Solution: (i) $\int(1+x)^2dx=\int(1+2x+x^2)dx=\int dx+2\int xdx+\int x^2dx=x+x^2+\frac{x^3}{3}+c$

(ii) $\int \cos(x+3)dx = \sin(x+3)+c$ (iii) $\int e^{-2x}dx = -\frac{1}{2}e^{-2x}+c$

(iv) $\int \frac{4x^3-1}{x^4-x+2}dx = \log_e(x^4-x+2)+c$

3.4 Some techniques of integration

3.4.1: **Integration by algebraic substitution (change of variable)**

A widely used device in integration is to reduce an integral that is not in standard form by making an appropriate algebraic substitution (to reduce to one that is in standard form)

Example 3: Evaluate (i) $\int x^2 \cos x^3 dx$ (ii) $\int x\sqrt{1+x^2}\,dx$

Solution: (i) Let $u = x^3, \frac{du}{dx} = 3x^2 \Rightarrow dx = \frac{du}{3x^2}$,

$$\int x^2 \cos x^3 dx = \int x^2 \cos u \frac{du}{3x^2} = \frac{1}{3}\int \cos u\,du = \frac{1}{3}\sin x^3 + c$$

(iii) Let $u = (1+x^2), \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$, $\int x\sqrt{1+x^2}\,dx = \frac{1}{2}\int u^{\frac{1}{2}}du = \frac{1}{3}(1+x^2)^{\frac{3}{2}} + c$

3.4.2: **Integration by trigonometric substitution**

Certain integral forms require trigonometric substitution if the integrands involves

(i) $\sqrt{a^2-x^2}$, we try $x = a\sin\theta$

(ii) $\sqrt{a^2+x^2}$, we try $x = a\tan\theta$

(iii) $\sqrt{x^2-a^2}$, we try $x = a\sec\theta$

Example 4: Evaluate $\int \frac{2}{x^2+5}dx$

Solution: Put $x = \sqrt{5}\tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{x}{\sqrt{5}}\right), dx = \sqrt{5}\sec^2\theta d\theta; x^2+5 = 5\sec^2\theta$

$$\int \frac{2}{x^2+5}dx = 2\int \frac{\sqrt{5}\sec^2\theta}{5\sec^2\theta}d\theta = \frac{2}{\sqrt{5}}\int d\theta = \frac{2}{\sqrt{5}}\theta + c = \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right) + c$$

Powers of sine and cosine

Example 5: Evaluate $\int \sin^2 x \cos x dx$

Solution: Put $u = \sin x$ $\frac{du}{dx} = \cos x \Rightarrow dx = \frac{du}{\cos x}$

$$\int \sin^2 x \cos x dx = \int u^2 \cos x \frac{du}{\cos x} = \frac{u^3}{3} + c = \frac{\sin^3 x}{3} + c$$

If the integrand is a product of a sine and/or a cosine of a multiple angle

$$\sin mx \cos nx = \frac{1}{2}\left(\sin(m+n)x + \sin(m-n)x\right)$$

$$\cos mx \cos nx = \frac{1}{2}\left(\cos(m+n)x + \cos(m-n)x\right)$$

$$\sin mx \sin nx = \frac{1}{2}\left(\cos(m-n)x + \cos(m+n)x\right)$$

Example 6: Evaluate $\int \sin 3x \cos x$

Solution: $\int \sin 3x \cos x = \int \frac{1}{2}(\sin 4x + \sin 2x)dx = \frac{-\cos 4x}{8} - \frac{\cos 2x}{4} + c$

3.4.3 **Integration by partial fraction**

If a rational expression is not in a standard integral form, it could be transformed into a standard form by splitting it into partial fractions.

Example 7: Evaluate (i) $\int \frac{4x-5}{(x+1)(x-2)} dx$ (ii) $\int \frac{2x^3 - 2x^2 - 2x - 7}{x^2 - x - 2} dx$

Solution: (i) Using partial fraction $\int \frac{4x-5}{(x+1)(x-2)} dx = \int (\frac{3}{x+1} + \frac{1}{x-2})dx = \int \frac{3}{x+1} dx + \int \frac{1}{x-2} dx$

$$= 3\log_e(x+1) + \log_e(x-2) + c$$

(ii) In this case the degree of the numerator is higher than the degree of the denominator. We decide to make the expression a proper algebraic fraction

$$\int \frac{2x^3 - 2x^2 - 2x - 7}{x^2 - x - 2} dx = \int (2x + \frac{3}{x+1} - \frac{1}{x-2})dx$$

$$= x^2 + 3\log_e(x+1) + \log_e(x-2) + c$$

3.4.4: **Integration by parts**

This is a method of integrating a product of two functions. From

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

on integrating both sides with respect to x, we have

$$uv = \int v\frac{du}{dx}dx + \int u\frac{dv}{dx}dx$$

whence

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Example 8: Evaluate $\int x\sin x dx$

Solution: Put $v = x$ and $du = \sin x dx$

$dv = dx; u = -\cos x$

$\int x\sin x dx = -x\cos x + \sin x + c$

3.5 Some applications of integration

At the end of this section, students should be able to:

(i) associate the area under a curve with an integral;
(ii) calculate area under a curve using definite integral;
(iii) apply integration to solve problems relating to dynamics;
(iv) find volume of solids generated by revolution using integrals

3.5.1 **The area under a curve**

The formula

$$\int_a^b f(x)dx$$

is interpreted as the area bounded by the curve $y = f(x)$, and the ordinates corresponding to $x = a, x = b$ and the $x - axis$.

Example 9: Calculate the area between the curve $y = 3x(x-4)$ and the axis of x.

Solution:

Hence the required area

$$= \int_0^4 y dx = \int_0^4 3x(x-4) dx = \int_0^4 (3x^2 - 12x) dx$$
$$= [x^3 - 6x^2]_0^4 = -32 units$$

Remarks: The negative sign is explained by the fact that

the curve lies below the axis of x under consideration .

3.5.2 Volumes of solids of revolution:

Another simple application of the integral calculus is the calculation of the volume of the solid formed by the rotation of the curve y = f(x) about the axis of x. The integral

$$\int_a^b \pi y^2 dx = \pi \int_a^b y^2 dx$$

is the required formula for the **volume of a solid of revolution.**

Example 10:

Find the volume of solid generated when the region bounded by $y = 2x$,

$y = 2x, the$ ordinate at x =2, x = 4 and the x- axis,

is revolved about the x-axis.

Solution: Let V be the volume of solid generated when the region is revolved about the x-axis.

$$V= V = \int_2^4 \pi y^2 dx = \pi \int_0^4 4x^2 = 4/3 \pi [(x)]_2^4 = \frac{224}{3}\pi \text{ units}^3$$

3.5.3. Centre of mass & Centroid

Finding the total mass M and the calculation of first moments (N_x and N_y) with respect to x and y, respectively. The total mass M can be considered as the limiting value as tends to zero of the sum of the approximate elementary masses $\rho dx, i.e., M = \int_0^l \rho dx$

where ρ is the variable density of the rod at a point of abscissa x and l is the length of the rod. In the same way the first moment δN_x of the element is approximately $x\rho\delta x$ so that $N_x = \int_0^l x\rho dx$

and the abscissa $\bar{x}$ of the centre of gravity or centre of mass of the rod is given by

$$\bar{x} = \frac{N_x}{M} = \frac{\int_0^l x\rho dx}{\int_0^l \rho dx} \text{ and respectively } \bar{y} = \frac{N_y}{M} = \frac{\int_0^l y\rho dy}{\int_0^l \rho dy}$$

and the point $(\bar{x}, \bar{y})$ is called the **centre of mass** or **centre of gravity.**

Now consider the centre of gravity of a lamina of uniform density bounded by the curve $y = f(x)$,

the axis of x and ordinates $x = a, x = b.$

The point $(\bar{x}, \bar{y})$ found by assuming uniform density of material over an area is usually called the **centroid** of the area; when the density is uniform and the centre of gravity coincide. In this case

$$\bar{x} = \int_a^b xydx \div \int_a^b ydx, \quad y = \frac{1}{2}\int_a^b y^2dx \div \int_a^b ydx$$