DEPT OF MATHS FUNAAB 2021-2022 MTS 101

FURTHER NOTE ON APPLICATION OF FACTOR THEOREM

Recall that any given polynomial $p(x)$ can be expressed as

$$\begin{aligned} p(x) &= q(x)d(x) + r(x) \quad \text{where} && (1)\\ q(x) &= \quad \text{the quotient} && (2)\\ d(x) &= \quad \text{the divisor} && (3)\\ r(x) &= \quad \text{the remainder.} && (4) \end{aligned}$$

Given that $d(x) = ax + b$, then equation (1) becomes

$$p(x) = q(x)(ax + b) + r(x). \qquad (5)$$

By setting $d(x) = ax + b = 0$, we obtain $x = -\frac{b}{a}$ and using this in equation (5), we obtain

$$\begin{aligned} p\left(-\frac{b}{a}\right) &= q(x) \times 0 + r\left(-\frac{b}{a}\right)\\ \therefore \quad p\left(-\frac{b}{a}\right) &= r\left(-\frac{b}{a}\right) && (6) \end{aligned}$$

which is the **Remainder's Theorem**.

Given that $d(x) = ax + b$ is a factor of $p(x)$, then equation (6) gives

$$p\left(-\frac{b}{a}\right) = 0 \qquad (7)$$

which is the **Factor Theorem**. It should be noted that if $(ax + b)$ and $(cx + d)$ are factors of $p(x)$, then their product given by $(ax + b)(cx + d)$ is also a factor of $p(x)$.

APPLICATIONS OF FACTOR THEOREM

The Factor Theorem has so many applications in Mathematics. For the purposes of MTS 101, only its application to factorization of symmetrical functions will be considered.

Definition 1: A function $f(a, b, c)$ in three variables a, b, c is called a symmetrical function if the value of the function remained unchanged when a is replaced by b, b is replaced by c and c is replaced by a simultaneously.

Example 1:

$$\begin{aligned} f(a, b, c) &= a + b + c, ab + bc + ca, a^2 + b^2 + c^2,\\ &\quad a^2bc + ab^2c + abc^2 + abc, a^3 + b^3 + c^3 \end{aligned}$$

are examples of symmetrical functions.

Definition 2: If $f(a,b,c)$ is a symmetrical function and $k \in \mathbb{Z}^+$ such that

$$f(ka,kb,kc) = k^n f(a,b,c). \tag{8}$$

The positive integer n is called the degree or order of $f(a,b,c)$.

Example 2: From Example 1, we have

$$\begin{aligned}
f(a,b,c) &= a+b+c \\
\Rightarrow f(ka,kb,kc) &= ka+kb+kc \\
&= k(a+b+c) \\
&= k^1 f(a,b,c) \quad \therefore \quad f(a,b,c) \text{is of order 1.} \\
f(a,b,c) &= ab+bc+ca \\
f(ka,kb,kc) &= (ka)(kb)+(kb)(kc)+(kc)(ka) \\
&= k^2ab+k^2bc+k^2ca \\
&= k^2(ab+bc+ca) \quad \therefore \quad f(a,b,c) \text{is of order 2.} \\
f(a,b,c) &= a^2+b^2+c^2 \\
\Rightarrow f(ka,kb,kc) &= k^2 f(a,b,c) \quad \therefore \quad f(a,b,c) \text{is of order 2.} \\
f(a,b,c) &= a^2bc+ab^2c+abc^2+abc \\
\Rightarrow f(ka,kb,kc) &= k^3 f(a,b,c) \quad \therefore \quad f(a,b,c) \text{is of order 3.} \\
f(a,b,c) &= a^3+b^3+c^3 \\
\Rightarrow f(ka,kb,kc) &= k^3 f(a,b,c) \quad \therefore \quad f(a,b,c) \text{is of order 3.}
\end{aligned}$$

Theorem 1: Let $f(a,b,c)$ be a symmetrical function.

(i) If a is a factor of $f(a,b,c)$ so also are b and c.

(ii) If $(a+b)$ is a factor of $f(a,b,c)$ so also are $(b+c)$ and $(c+a)$.

(iii) If $(a-b)$ is a factor of $f(a,b,c)$ so also are $(b-c)$ and $(c-a)$.

Example 3: Show that

$$f(a,b,c) = bc(b-c)+ca(c-a)+ab(a-b)$$

is a symmetrical function of order 3 and factorize $f(a,b,c)$ completely.

Solution: Obviously, $f(a,b,c)$ is a symmetrical function. For the order:

$$\begin{aligned}
f(a,b,c) &= bc(b-c)+ca(c-a)+ab(a-b) \\
\Rightarrow f(ka,kb,kc) &= (kb)(kc)((kb)-(kc))+(kc)(ka)((kc)-(ka))+(ka)(b)((ka)-(kb)) \\
&= k^3[bc(b-c)+ca(c-a)+ab(a-b)] \\
&= k^3 f(a,b,c)
\end{aligned}$$

showing that $f(a,b,c)$ is a symmetrical function of oder 3.

To factorize $f(a,b,c)$, let

$$f(a) = bc(b-c)+ca(c-a)+ab(a-b) \tag{9}$$

be a function of a. Then

$$\begin{aligned}
f(b) &= bc(b-c)+cb(c-b)+b^2(b-b) \\
&= b^2c-bc^2+bc^2-b^2c+0 \\
&= 0.
\end{aligned}$$

By Factor Theorem, $(a-b)$ is a factor of $f(a,b,c)$ and by Theorem 1(iii), $(b-c)$ and $(c-a)$ are factors of $f(a,b,c)$ and therefore since $f(a,b,c)$ is of oder 3 and the product $(a-b)(b-c)(c-a)$ is also of order 3, we must express $f(a,b,c)$ as

$$f(a,b,c) = \alpha(a-b)(b-c)(c-a) \tag{10}$$

where α is a constant to be determined. From equation (10), we have

$$\begin{aligned}
f(a,b,c) &= \alpha(a-b)(b-c)(c-a) \\
\Rightarrow bc(b-c)+ca(c-a)+ab(a-b) &\equiv \alpha(a-b)(b-c)(c-a) \\
\Rightarrow b^2c-bc^2+ac^2-a^2c+a^2b-ab^2 &\equiv -\alpha[b^2c-bc^2+ac^2-a^2c+a^2b-ab^2] \\
\Leftrightarrow \alpha &= -1 \\
\therefore \quad f(a,b,c) &= -(a-b)(b-c)(c-a) \\
&= (b-a)(b-c)(c-a) \\
&= (a-b)(c-b)(c-a) \\
&= (a-b)(b-c)(a-c).
\end{aligned}$$

Example 4: Show that

$$f(a,b,c) = a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$$

is a symmetrical function of order 5 and factorize $f(a, b, c)$ completely.

Solution: Obviously, $f(a, b, c)$ is a symmetrical function. For the order:

$$f(ka, kb, kc) = k^5 f(a, b, c)$$

which shows that $f(a, b, c)$ is a symmetrical function of order 5.

To factorize $f(a, b, c)$, let

$$f(a) = a^3(b^2 - c^2) + b^3(c^2 - a^2) + c^3(a^2 - b^2) \tag{11}$$

be a function of a. Then

$$\begin{aligned} f(b) &= b^3(b^2 - c^2) + b^3(c^2 - b^2) + c^3(b^2 - b^2) \\ &= 0. \end{aligned}$$

By Factor Theorem, $(a - b)$ is a factor of $f(a, b, c)$ and by Theorem 1(iii), $(b - c)$ and $(c - a)$ are factors of $f(a, b, c)$ and therefore since $f(a, b, c)$ is of oder 5 and the product $(a - b)(b - c)(c - a)$ is also of order 3 , we must express $f(a, b, c)$ as

$$f(a, b, c) = (a - b)(b - c)(c - a)(\alpha(a^2 + b^2 + c^2) + \beta(ab + bc + ca)) \tag{12}$$

where α and β are constants to be determined. From equation (12), we have

$$\begin{aligned} f(a, b, c) &= (a - b)(b - c)(c - a)(\alpha(a^2 + b^2 + c^2) + \beta(ab + bc + ca)) \\ \Rightarrow a^3(b^2 - c^2) + b^3(c^2 - a^2) + c^3(a^2 - b^2) &\equiv (a - b)(b - c)(c - a)(\alpha(a^2 + b^2 + c^2) + \beta(ab + bc + ca)) \\ \Leftrightarrow -\alpha &= 0, \alpha - \beta = 1 \\ \Leftrightarrow \alpha &= 0, \beta = -1 \\ \therefore \quad f(a, b, c) &= (a - b)(b - c)(c - a)(-(ab + bc + ca)) \\ &= -(a - b)(b - c)(c - a)(ab + bc + ca) \\ &= (b - a)(b - c)(c - a)(ab + bc + ca) \\ &= (a - b)(c - b)(c - a)(ab + bc + ca) \\ &= (a - b)(b - c)(a - c)(ab + bc + ca). \end{aligned}$$

Example 5: Show that

$$f(a, b, c) = a(b^3 - c^3) + b(c^3 - a^3) + c(a^3 - b^3)$$

is a symmetrical function of order 4 and factorize $f(a, b, c)$ completely.

Solution: Obviously, $f(a,b,c)$ is a symmetrical function. For the order:

$$f(ka,kb,kc) = k^4 f(a,b,c)$$

which shows that $f(a,b,c)$ is a symmetrical function of order 4.

To factorize $f(a,b,c)$, let

$$f(a) = a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3) \tag{13}$$

be a function of a. Then

$$\begin{aligned} f(b) &= b(b^3-c^3)+b(c^3-b^3)+c(b^3-b^3) \\ &= 0. \end{aligned}$$

By Factor Theorem, $(a-b)$ is a factor of $f(a,b,c)$ and by Theorem 1(iii), $(b-c)$ and $(c-a)$ are factors of $f(a,b,c)$ and therefore since $f(a,b,c)$ is of oder 4 and the product $(a-b)(b-c)(c-a)$ is also of order 3 , we must express $f(a,b,c)$ as

$$f(a,b,c) = (a-b)(b-c)(c-a)(\alpha(a+b+c)) \tag{14}$$

where α is a constant to be determined. From equation (13), we have

$$\begin{aligned} f(a,b,c) &= (a-b)(b-c)(c-a)(\alpha(a+b+c)) \\ \Rightarrow a(b^3-c^3)+b(c^3-a^3)+c(a^3-b^3) &\equiv (a-b)(b-c)(c-a)(\alpha(a+b+c)) \\ \Leftrightarrow -\alpha &= -1 \\ \therefore \quad \alpha &= 1 \\ \therefore \quad f(a,b,c) &= (a-b)(b-c)(c-a)(a+b+c). \end{aligned}$$

FURTHER PRACTICE PROBLEMS

1. Let $f(a,b,c)$ be a function given by

$$f(a,b,c) = a^3+b^3+c^3-3abc.$$

   (a) Show that $f(a,b,c)$ is a symmetrical function of order 3.

   (b) Show that $(a+b+c)$ is a factor of $f(a,b,c)$ and hence show that

$$f(a,b,c) = \frac{1}{2}(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right].$$

(c) Given that $a+b+c=4, a^2+b^2+c^2=10$ and $a^3+b^3+c^3=16$, find the value of abc.

(d) Eliminate a, b and c in the equations

$$a+b+c=k, x^2+y^2+z^2=l^2, x^3+y^3+z^3=m^3 \quad \text{and} \quad abc=n^3.$$

(e) If

$$a=k(y+z-x), b=k(z+x-y), c=k(x+y-z),$$

show that

$$a^3+b^3+c^3-3abc=4k^3[x^3+y^3+z^3-3xyz].$$

2. Let a $f(a,b,c)$ be a function given by

$$f(a,b,c)=(ab+bc+ca)^3-a^3b^3-b^3c^3-c^3a^3.$$

(a) Show that $f(a,b,c)$ is a symmetrical function of order 6.

(b) Show that a and $(a+b)$ is a factor of $f(a,b,c)$ and hence show that

$$f(a,b,c)=3abc(a+b)(b+c)(c+a).$$

3. (a) Let $f(a,b,c)$ be a function given by

$$f(a,b,c)=(b-c)^3+(c-a)^3+(a-b)^3.$$

i. Show that $f(a,b,c)$ is a symmetrical function of order 3.

ii. Show that

$$f(a,b,c)=-3\left[(b+c)^2(b-c)+(c+a)^2(c-a)+(a+b)^2(a-b)\right].$$

(b) Let $f(a,b,c)$ be a function given by

$$f(a,b,c)=(b+c)^3(b-c)+(c+a)^3(c-a)+(a+b)^3(a-b).$$

i. Show that $f(a,b,c)$ is a symmetrical function of order 4.

ii. Factorize $f(a,b,c)$ completely.

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