6 COMPLEX NUMBER

6.1 Introduction

The real number system had limitations that were later overcome by a series of improvements in both concepts and mechanics. In connection with, quadratic equations we encountered the concept of imaginary number and the device invented for handling it, the notation $i^2 = -1$ or $i = \sqrt{-1}$. The extension of real number system to include imaginary' numbers is called the complex number system. A particular classification of numbers is their division into real and imaginary numbers, which is the focus of this teaching. Most of the numbers you might have encountered are real. Examples include integers, rational and irrational numbers. Surds are also considered real numbers. Imaginary numbers on the other hand are not very common except in science and engineering, and together with real numbers form the basis of complex numbers.

6.2 Complex Number

A complex number is a number of the form $a + bi$, where a and b are real. . The letter a is called the real part and b is called the imaginary part of $a+bi$. If $a = 0$, the number ib is said to be a purely imaginary number and if $b = 0$, the number a is real. Hence, real numbers and pure imaginary numbers are special cases of complex numbers. The complex numbers are denoted by Z , i.e., $Z = a + bi$. In coordinate form, $Z = (a, b)$. Note : Every real number is a complex number with 0 as its imaginary part.

6.3 Properties of Complex Number

(i) The two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$ for example, if $x - 2 + 4yi = 3 + 12i$ Then $x - 2 = 3$ and $y = 3$.
(ii) If any complex number vanishes then its real and imaginary parts will separately vanish. For example, if $a + ib = 0$, then $a = -ib$. Squaring both sides, $a^2 = -b^2$, $a^2 + b^2 = 0$, which is possible only when $a = 0$, $b = 0$.

6.4 Basic Algebraic Operation on Complex Numbers

There are four algebraic operations on complex numbers.

(i) Addition

If $Z_1 = a_1 + b_1 i$ and $Z_2 = a_2 + b_2 i$, then

$$Z_1 + Z_2 = (a_1 + b_1 i) + (a_2 + b_2 i)$$

$$= (a_1 + a_2) + i(b_1 + b_2)$$

(ii) Subtraction

$$Z_1 - Z_2 = (a_1 + b_1 i) - (a_2 + b_2 i)$$
$$= (a_1 - a_2) + i(b_1 - b_2)$$

(iii) Multiplication

$$Z_1 Z_2 = (a_1 + b_1 i)(a_2 + b_2 i)$$
$$= (a_1 a_2 - b_1 b_2) + a_1 b_2 i + b_1 a_2 i$$
$$= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)$$

.

(iv) Division

$$\frac{z_1}{z_2} = \frac{a_1 + b_1 i}{a_2 + b_2 i}$$

.
Multiply numerator and denominator by the conjugate of the denominator i.e $a_2 - b_2 i$ in order to make the denominator real.

$$\frac{z_1}{z_2} = \frac{(a_1 + b_1 i)(a_2 - b_2 i)}{(a_2 + b_2 i)(a_2 - b_2 i)}$$
$$= \frac{(a_1 a_2 + b_1 b_2) + i(b_1 a_2 - a_1 b_2)}{a_2^2 + b_2^2}$$
$$= \frac{(a_1 a_2 + b_1 b_2)}{a_2^2 + b_2^2} + i\frac{(b_1 a_2 - a_1 b_2)}{a_2^2 + b_2^2}.$$

Generally result will be expressed in the form a + ib.

Example 1
Add and subtract the numbers $3 + 4i$ and $2 - 7i$.
Solution
Addition: $(3 + 4i) + (27i) = (3 + 2) + i(47) = 5 - 3i$
Subtraction: $(3 + 4i)(27i) = (32) + i(4 + 7) = 1 + 11i$

Example 2: Find the product of the complex numbers: $3 + 4i$ and $2 - 7i$.
Solution
$(3 + 4i)(2 - 7i) = 6 - 21i + 8i - 28i2$
$= 6 + 28 - 13i = 34 - 13i$.

Example 3
Divide $3 + 4i$ by $2 - 7i$.
Solution

$$\frac{3+4i}{2-7i} = \frac{(3+4i)(2+7i)}{(2-7i)(2+7i)}$$
$$= \frac{6+28i^2+i(21+8)}{4+49} = \frac{22+29i}{53}$$
$$= \frac{22}{53} + i\frac{29}{53}.$$

Example 4: Express $\frac{(2+i)(1-i)}{4-3i}$ in the form of $a + ib$.
Solution

$$\frac{(2+i)(1-i)}{4-3i} = \frac{(2+1)+i(1-2)}{4-3i} = \frac{3-i}{4-3i} = \frac{(3-i)(4+3i)}{(4+3i)(4-3i)}$$
$$= \frac{(12+3)+i(9-4)}{16+9} = \frac{15+i(5)}{25} = \frac{15}{25} + i\frac{5}{25} = \frac{3}{5} + i\frac{1}{5}.$$

Example 5: Separate into real and imaginary parts: $\frac{1+4i}{3+i}$
. Solution

$$\frac{1+4i}{3+i} = \frac{(1+4i)(3-i)}{(3+i)(3-i)}$$

$$= \frac{(3+4)+i(12-1)}{9+1} = \frac{7+11i}{10} = \frac{7}{10} + i\frac{11}{10}$$

. Thus, the real part is $a = \frac{7}{10}$ and imaginary part is $b = \frac{11}{10}$.

Extraction of square roots of a complex number

Example 6
Extract the square root of the complex numbers $21 - 20i$.
Solution
Let $a + ib = \sqrt{21 - 20i}$.
Squaring both sides we have

$$(a+ib)^2 = \sqrt{21-20i}$$

$$= a^2 - b^2 + 2abi = 21 - 20i$$

Comparing both sides, we have

$$a^2 - b^2 = 21 \qquad (1)$$

$$2ab = -20 \qquad (2)$$

From (2) $b = -10/a$. Put b in equation (1),

$$a^2 - 100/a^2 = 21$$

$$a^4 - 21a^2 - 100 = 0$$

$$(a^2 - 25)(a^2 + 4) = 0$$

$$a^2 = 25 \text{ or } a^2 = -4$$

$$a = +5 \text{ or } a = +2i.$$

But a is not imaginary, so the real value of a is $a = 5$ or $a = 5$. The corresponding value of b is $b = 2$ or $b = 2$. Hence the square roots of $21 - 20i$ are: $5 - 2i$ and $-5 + 2i$.

Factorization of a complex numbers

Example 7 Factorise: $a^2 + b^2$

Solution We have

$$a^2 + b^2 = a^2 - (-b^2)$$
$$= a^2 - (i^2 b^2), i^2 = -1$$
$$= (a)^2 - (ib)^2 = (a + ib)(a - ib)$$

6.5 Additive Inverse of a Complex Number

Let $Z = a + ib$ be a complex number, then the number $-(a + ib)$ is called the additive inverse of Z . It is denoted by $-Z$ i.e., $-Z = -a - ib$. Also $Z - Z = 0$

6.6 Multiplicative inverse of a complex number

Let $a + ib$ be a complex number, then $x + iy$ is said to be multiplicative inverse of $a + ib$ if

$$(x + iy)(a + ib) = 1$$

Or

$$(x + iy) = \frac{1}{a + ib}$$
$$= \frac{1}{a + ib} \times \frac{a - ib}{a - ib}$$
$$x + iy = \frac{a - ib}{a^2 + b^2}$$
$$x + iy = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}.$$

So

$$x = \frac{a}{a^2 + b^2}, y = -\frac{b}{a^2 + b^2}.$$

Hence multiplicative inverse of (a, b) $(\frac{a}{a^2+b^2}, -\frac{b}{a^2+b^2})$

6.7 Conjugate of a complex number

Two complex numbers are called the conjugates of each other if their real parts are equal and their imaginary parts differ only in sign. If $Z = a + bi$, the complex number $a-bi$ is called the conjugate of Z and is denoted by $\bar{Z}$. Therefore,

$$\bar{Z}=\overline{a+bi}=a-bi.$$

Theorem 1 :
If Z_1 and Z_2 are complex numbers, then
(i) $\overline{Z_1+Z_2}=\overline{Z_1}+\overline{Z_2}$
(ii) $\overline{Z_1-Z_2}=\overline{Z_1}-\overline{Z_2}$
(iii) $\overline{Z_1Z_2}=\overline{Z_1}\times\overline{Z_2}$
(i) $\overline{(\frac{Z_1}{Z_2})}=\frac{\overline{Z_1}}{\overline{Z_2}}$.
Proof
Exercise
Example 11: Find the conjugate of the complex number 2i(3 + 8i)

Example 11: Find the conjugate of the complex number $2i(-3+8i)$
Solution

$$2i(-3+8i)=-16-6i$$
$$\overline{2i(-3+8i)}=-16+6i$$

In another way we have,

$$\overline{2i(-3+8i)}=\overline{2i}(\overline{-3+8i})$$

By above theorem, we have

$$\overline{2i(-3+8i)}=\overline{2i}(\overline{-3+8i})$$

$$=-2i(+3-8i)$$
$$=-16-6i$$

6.8 Argand Plane

An Argand plane or Gauss Plane, named after the French mathematician Jean-Robert Argand and the great German Mathemation Carl Friedrich Gauss, is a geometrical plot of complex numbers on $x-y$ Cartesian plane, also known as complex plane. The x-axis (horizontal axis) represents the real parts and the y-axis (vertical axis) represents the imaginary parts of complex numbers. They are called real axis and imaginary axis respectively. Since a complex number $Z = a + ib$ can also be represented by an ordered pair (a, b), each point in the plane can be viewed as the graph of a complex number. The real part a of $a + ib$ taken as the x-coordinate of a point $P = P(a, b)$. Similarly, since the imaginary part b of $a + ib$ is taken as y - coordinate of P, the y-axis is called the imaginary axis.

Exercise: Read and check for the graphical representation of complex number or Argand plane

6.9 Modulus of a Complex Number

The Modulus or the absolute value of the complex number $Z = a + ib$ is denoted by r, $|Z|$ or $|a + ib|$ and is given by,

$$r = |Z| = |a + ib| = \sqrt{a^2 + b^2}.$$

Theorem 2

Let Z_1 and Z_2 be any two complex numbers, then
(i) $|Z_1 \times Z_2| = |Z_1| \times |Z_2|$
(ii) $|Z_1 \div Z_2| = |Z_1| \div |Z_2|$
(iii) $|Z_1 + Z_2| \leq |Z_1| + |Z_2|$
(iv) $|Z_1 - Z_2| \geq |Z_1| - |Z_2|$.

Proof

Exercise

6.10 Polar form of a complex number

The polar form of a complex number $Z = a + ib$ is given as

$$Z = r(\cos\theta + i\sin\theta)$$

or

$$Z = rCis\theta$$

where, r is called the absolute value or modulus of Z and θ the angle from the positive real axis to this line, as the argument or amplitude of Z and is denoted by arg Z i.e., $\theta = argZ$.

Given the complex numbers $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then we have the following,

$$Z_1Z_2 = r_1r_2\Big(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\Big)$$

and

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2}\Big(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\Big).$$

Also, the following is true for negative power of complex number

$$Z^{-n} = \frac{1}{r^n}\Big[\frac{1}{\cos n\theta + i\sin n\theta}\Big].$$

Example 12: Express the complex number $1 + i\sqrt{3}$ in polar form.
Solution
$a = 1, b = \sqrt{3}$. Therefore,

$$\begin{aligned} r &= \sqrt{a^2+b^2} \\ &= \sqrt{1^2 + (\sqrt{3})^2} = 2. \\ \tan\theta &= \frac{b}{a} = \sqrt{3}. \\ \theta &= 60^0 \end{aligned}$$

Thus,

$$1 + i\sqrt{3} = 2(\cos 60^0 + i\sin 60^0) = 2Cis60^0.$$

Example 13: Express $4(\cos 225^0 + i\sin 225^0)$ in rectangular form.
Solution
$\cos 225^0 = \frac{-1}{\sqrt{2}}$ and $\sin 225^0 = \frac{-1}{\sqrt{2}}$.
Thus,

$$\begin{aligned} 4(\cos 225^0 + i\sin 225^0) &= 4(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}) \\ &= -2\sqrt{2} - 2i\sqrt{2} \end{aligned}$$

Example 14: Find the magnitude (modulus) and argument of $(4+7i)(3-2i)$.

Solution
$(4+7i)(3-2i) == 26+13i$ (after expanding the LHS). Therefore,

$$r = \sqrt{26^2+13^2} = 29.1$$
$$\tan\theta = \frac{1}{2},$$
$$\theta = 26.57^0.$$

6.11 De Moivres Theorem and n^{th} root of unity
The theorem state that if $Z = r(\cos\theta + i\sin\theta)$ is a complex number and n is a positive integer, then

$$Z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).$$

This theorem helps to easily compute the power of a complex number such as $Z = a+2i$. Also, some of the application of this theorem include nth roots, and roots of unity. The solutions of the equation $Z^n = 1$, for positive values of integer n, are the n roots of the unity.
In polar form the equation $Z^n = 1$ can be written as

$$Z^n = \cos(0+2k\pi) + i\sin(0+2k\pi) = \exp^{i2k\pi}, \quad K = 0,1,2...$$

Using the De Moivre's theorem, we find the n^{th} roots of unity from the equation given below:

$$z = \left(\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)\right) = \exp^{\frac{i2\pi k}{n}}, \quad k = 0,1,2...n-1$$

Note the following
(1) All the n roots of nth roots unity are in Geometrical Progression
(2) Sum of the n roots of nth roots of unity is always equal to zero. Also the Sum of the n roots of nth roots of any complex number is always equal to zero
(3) Product of the n roots of nth roots unity is equal to $(-1)^{n-1}$.
(4) All the n roots of nth roots unity lie on the circumference of a circle whose centre is at the origin and radius equal to 1 and these roots divide the circle into n equal parts and form a polygon of n sides.

(5) The nth roots of unity are $1, w, w^2, ...w^{n-1}$ where $w = cis\frac{2\pi}{n}$
Example 15
Find the cube roots of unity.
Solution
We have to find $1^{\frac{1}{3}}$. Let $z = 1^{\frac{1}{3}}$ then $z^3 = 1$. In polar form, the equation $z^3 = 1$ can be written as

$$z^3 = \cos(0 + 2k\pi) + i\sin(0 + 2k\pi) = e^{i2k\pi},\ k = 0, 1, 2, ...$$

Therefore,

$$z = \cos(\frac{2k\pi}{3}) + i\sin(\frac{2k\pi}{3}) = e^{i\frac{2k\pi}{3}},\ k = 0, 1, 2, ...$$

By taking $k = 0, 1, 2$, we obtain,

$$\begin{aligned} k = 0: \quad z &= \cos 0 + i\sin 0 = 1, \\ k = 1: \quad z &= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} \\ &= \cos(\pi - \frac{\pi}{3}) + i\sin(\pi - \frac{\pi}{3}) \\ &= -\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} \\ &= -\frac{1}{2} + i\frac{\sqrt{3}}{2}. \end{aligned}$$

$$\begin{aligned} k = 2: \quad z &= \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} \\ &= \cos(\pi + \frac{\pi}{3}) + i\sin(\pi + \frac{\pi}{3}) \\ &= -\cos\frac{\pi}{3} - i\sin\frac{\pi}{3} \\ &= -\frac{1}{2} - i\frac{\sqrt{3}}{2}. \end{aligned}$$

Therefore, the cube roots of unity are

$$1, \frac{-i + i\sqrt{3}}{2}, \frac{-i - i\sqrt{3}}{2}.$$

Exercise

Obtain the fourth and fifth roots of unity.

Example 16 Compute Z^n given that $Z = 2 + 2i$.

Solution

We fist convert the complex number to polar form i.e

$$Z = 2 + 2i = 2\sqrt{2}(\cos 45^0 + \imath sin 45^0)$$

, and $r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$ and $\theta = \tan(a/b) = \tan 1 = 45^0$.

Then,

$$\begin{aligned} Z^6 = (2+2i)^6 &= 2\sqrt{2}[(\cos 45^0 + i\sin 45^0)]^6 \\ &= (2\sqrt{2})^6[(\cos 270^0 + i\sin 270^0)] \\ &= -512i \end{aligned}$$

Example 17 Find the values of the following

(1) $(1+i\sqrt{3})^3$ (ii) $(1-i)^8$ (iii) $(\frac{\sqrt{3}}{2} + \frac{i}{2})^5 - (\frac{\sqrt{3}}{2} - \frac{i}{2})^5$

Solution

(i)

$$(1+i\sqrt{3}) = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}).$$

Therefore,

$$\begin{aligned} (1+i\sqrt{3})^3 &= 8[(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})]^3 \\ &= 8[\cos\pi + i\sin\pi] = -8. \end{aligned}$$

(ii)

$$\begin{aligned} (1-i)^8 &= \left[\sqrt{2}(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}})\right]^8 \\ &= \left[\sqrt{2}(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4})\right]^8 \\ &= 16. \end{aligned}$$

(iii)

$$(\frac{\sqrt{3}}{2} + \frac{i}{2})^5 - (\frac{\sqrt{3}}{2} - \frac{i}{2})^5 = (\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})^5 - (\cos\frac{\pi}{6} - i\sin\frac{\pi}{6})^5$$

$$= \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} - \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$$
$$= i.$$

Exercises

(1)Find the magnitude (Modulus) of the following
(a) $(-2,-1)$ (b) $\frac{1+2i}{2-i}$ (c) $\frac{(3-5i)(1+i)}{4+2i}$
(2)Express of the following complex number in the polar
(a) $2+2\sqrt{3}i$ (b) $\frac{1+2i}{1-3i}$ (c) $\left(\frac{2+i}{3-i}\right)^2$
(3)Write each complex number in the form $a+bi$.
(a) $4Cis230^0$ (b) $2Cis(-30^0)$ (c) $12Cis420^0$
(4)Find the magnitude and the principle argument of
(a) $5-7i$ (b) $(5-7i)(8+5i)$ (c) $\frac{1+i}{1-i}$
(5)Find Z such that
(a) $|Z| = 8\sqrt{2}, argZ = \frac{\pi}{4}$ (b) $|Z| = 5, argZ = \frac{\pi}{2}$ (c) $|Z| = \frac{1}{3}, argZ = \frac{\pi}{3}$
(6)Find the conjugate and modulus of
(a) $\frac{1+i}{1-i}$ (b) $\frac{1+2i}{2-i}$ (c) $\frac{-2}{3} - \frac{4}{9}i$
(7) Find the additive and multiplicative inverse of the following
(a) $(-3,2)$ (b) $\frac{1+2i}{2-i}$ (c) $(5-7i)(8+5i)$
(8) If $Z = 2+3i$ Prove that $\overline{Z}Z = 13$.
(9) Write complex number (a) $-\sqrt{2}+\sqrt{6}i$ (b) $-+\sqrt{3}i$ in polar (trigonometric)form.
(10) Show that
$$|\frac{1+2i}{2-i}| = 1$$
(11)Factorise each of the following (a) $36a^2+100b^2$ (b) $2x^2+5y^2$ (c)$9a^2+64b^2$.
(12)Perform the division and write the result in standard form for each of the following. (a)$\frac{8-7i}{1-2i}$ (b) $\frac{1}{(4-5i)^2}$ (c)$\frac{2i}{2+i} + \frac{5}{2-i}$ (d) $\frac{(2-3i)(5i)}{2+3i}$
(13) Simplify the following complex numbers and write them in standard for.
(a)$4i^2 - 7i^7$ (b) $(\sqrt{-2})^8$ (c) $-5i^9$
(13)Use De Moivres Theorem to find the indicated power of the complex number. Express the result in standard form.
(a)$(i+3)^3$ (b)$(-1+i)^{10}$ (c)$\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)^{10}$ (d) $\left(5(\cos 20^0 + i\sin 20^0)\right)^3$.
(15) Find (a) the fifth roots of $16(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3})$ (b) fourth roots of 1.
(16) Find all the solutions of the following equations. (a) $x^4 - i = 0$ (b) $x^5 - 243 = 0$.