ALGEBRA

MTS101 / MODULE 3

DR RAJI, M. T

POLYNOMIAL

A real polynomial of degree n is an expression of the form

$$y = f(x) = a_n x^n + a_{n-1} x^{n-1} \dots + a_1 x + a_0, a_n \neq 0$$

Where x is a variable real number and the a_n are real numbers.

The numbers $a_0, a_1, \dots, a_n$ are called coefficients of the polynomial.

The monomials $a_n x^n$ are called the term of the polynomial.

$a_n x^n$, is called the leading term and a_n the leading co-efficient (i.e. the co-efficient of the highest power of x).

a_0 is called the constant term

LEARNING MANAGEMENT SYSTEM LMS

Polynomial (cont'd)

Polynomial	Degree	Name
a	0	Constant
$ax + b, a \neq 0$	1	Linear
$ax^2 + bx + c, a \neq 0$	2	Quadratic
$ax^3 + bx^2 + cx + d, a \neq 0$	3	Cubic

Equality of Polynomials

Two polynomials are equal if they have the same degree and all the corresponding co-efficient are equal.

Addition, Subtraction and Multiplication of Polynomials

Define **addition and subtraction of polynomials:** as addition and subtraction of corresponding or like terms. Use distributive law of multiplication of numbers over addition to define multiplication of polynomials.

Polynomial (cont'd)

- **Example 1:** Simplify and arrange your results in descending powers of x

a.

$$3(4x^3 - 7x^2 + 6x - 2) - 2(3x^3 - 5x^2 - 2x + 5)$$

$$\begin{array}{r} 12x^3 - 21x^2 + 18x - 6 \\ -6x^3 + 10x^2 + 4x - 10 \\ \hline 6x^3 - 11x^2 + 22x - 16 \\ \hline \end{array}$$

b.

$$ax^3 - 3bx^2 + 5cx - 4dx^2 - bx^3 + 2ax - 7$$

$$= (a - b)x^3 - (3b + 4d)x^2 + (5c + 2a)x - 7$$

c.

$$(ax^2 + bx + c)(bx^2 - cx - a)$$

$$= abx^4 - acx^3 - a^2x^2 + b^2x^3 - bcx^2 - abx + bcx^2 - c^2x - ac$$

$$= abx^4 + (b^2 - ac)x^3 - (a^2 - bc)x^2 - (c^2 + ab)x - ac$$

Polynomial (cont'd)

- **Division of polynomials**
- Use a division algorithm, similar to the method of long division in numbers, to divide a polynomial $f(x)$ of degree n by a polynomial g(x) of degree m ≤ n, and obtain a quotient $g(x)$ and a remainder $r(x)$ of degree less than m, so that
- $f(x) = g(x)q(x) + r(x)$
- Where $q(x)$ and $r(x)$ are uniquely determined polynomials. The polynomial $f(x)$ is called the **dividend** and $g(x)$ is called the **divisor.**
- **Example:** Obtain the quotient and remainder when $x^3 + 2x^2 - 5x + 1$ is divided by $3x^2 - 2x + 1$

- **Solution**

$$\begin{array}{r l} & \frac{1}{3}x + \frac{8}{9} \\ 3x^2-2x+1 & \overline{\big)\, x^3 + 2x^2 - 5x + 1} \\ & -(x^3 - \frac{2}{3}x^2 + \frac{1}{3}x) \\ \hline & \frac{8}{3}x^2 - \frac{16}{3}x + 1 \\ & -\left(\frac{8}{3}x^2 - \frac{16}{3}x + \frac{8}{9}\right) \\ \hline & -\frac{32}{9}x + \frac{1}{9} \end{array}$$

- Quotient $= \frac{1}{3}x + \frac{8}{9}$; Remainder $= -\frac{32}{9}x + \frac{1}{9}$
- Now $x^3 + 2x^2 - 5x + 1 = (3x^2 - 2x + 1)\left(\frac{1}{3}x + \frac{8}{9}\right) - \frac{32}{9}x + \frac{1}{9}$

Polynomial (cont'd)

- **Linear Equation**
- $ax + b = 0, a \neq 0$, is a linear equation in a variable x, whose solution is $x = -\frac{b}{a}$.
- **Example:** Solve for x
- a) $4 + 5 = 2x + x$
- $9 = x$
- $\therefore x = 3$
- b) $\frac{x+1}{x-2} - 4 = 0$
- $multiply\ by\ (x-2)$
- $x + 1 - 4(x - 2) = 0$
- $-3x + 9 = 0$
- $x = 3.$

LEARNING MANAGEMENT SYSTEM LMS

Polynomial (cont'd)

- c) $3^{x+1} = 9^{x-2}$
- $\Rightarrow 3^{x+1} = 3^{2x-4}$
- $\Rightarrow x + 1 = 2x - 4$
- $\therefore x = 5$

d) $\log_2(2x - 1) - \log_2(2 - x) = 3$

- $\Rightarrow \log_2 \frac{2x-1}{2-x} = 3$
- $\frac{2x-1}{2-x} = 2^3 = 8$
- $\Rightarrow 2x - 1 = 16 - 8x$
- $\Rightarrow 10x = 17$
- $\therefore x = 1.7$

LEARNING MANAGEMENT SYSTEM LMS

Simultaneous Linear Equations in Two Variables

- **Example:** Solve simultaneously for x and y
- $10x - 4y = 35 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (a)$
- $4x + 9y = -39 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. (b)$
- **Method I: (Method of Substitution)**
- From (a): $x = \frac{1}{10}(4y + 35)$
- Substitute of x in (b)
- $\frac{4}{10}(4y + 35) + 9y = -39$
- $\times 5: \quad 2(4y + 35) + 45y = -195$
- $53y = -265 \longrightarrow \quad y = \frac{-265}{53} = -5$
- Then, $\quad x = \frac{1}{10}(4y + 35) \quad = \frac{1}{10}(-20 + 35) \quad = 1.5$
- Solution is: $x = 1.5, \quad y = -5$

Quadratic Equations

- A quadratic equation in x is of the form $ax^2 + bx + c = 0, \; a \neq 0$
- Methods of Solving
- a) Factorization Method
- b) Completing the Square Method
- c) General Formula

LEARNING MANAGEMENT SYSTEM LMS

Solution by factorization

- Use the fact that $ab = 0$, then either $a = 0$ or $b = 0$. Factorize the quadratic expression $ax^2 + bx + c$ with rational coefficients a, b and c, if the discriminant $D = b^2 - 4ac$ is a perfect square
- **Example:** solve for x

$$5x^2 + 4x - 1 = 0$$
$$a = 5, b = 4, c = -1$$
$$D = b^2 - 4ac = 16 - 4(5)(-1) = 36 = 6^2$$
$$5x(x + 1) - (x + 1) = 0$$
$$(x + 1)(5x - 1) = 0$$
$$\Rightarrow x + 1 = 0 \ or\ 5x - 1 = 0$$
$$\Rightarrow x = -1\ 0r \frac{1}{5}$$

LEARNING MANAGEMENT SYSTEM LMS

Solution by completing the square quadratic formula.

- $ax^2 + bx + c$
- $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- **Proof:** $x^2 + \frac{b}{a}x = -\frac{c}{a}$
- Complete the square on the Left Hand Side
- $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right) \Rightarrow \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$
- Square root of both sides
- $\sqrt{\left(x + \frac{b}{2a}\right)^2} = \sqrt{\frac{b^2 - 4ac}{4a^2}} \Rightarrow x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$
- $x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- **Note:** The quadratic formula can solve any quadratic equation

Nature of the roots of a quadratic equation

- Consider the quadratic equation $ax^2 + bx + c = 0, a \neq 0,$ whose discriminant is $D = b^2 - 4ac$. A solution is called a root of the quadratic equation. There are three types of roots.
- If D > 0, it square roots will be real number and we shall obtain two real distinct roots of the equation
- If D = 0, so its square roots and both roots of the equation will be real and equal to $-{}^{b}/_{2a}$
- If D < 0, its square root is that of a negative number. Such a square root is that of a negative. Such a square root cannot be a real number. It is a complex number.

- **Example 1:** Find the values of a for which the equation $(3a+1)x^2+(a+2)x+1=0$ has equal roots.
- The discriminant $D=(a+2)^2-4(3a+1)$
- $=a^2+4a+4-12a-4 \ =a^2-8a$
- For equal roots, D = 0; therefore $a^2-8a=0$
- $a(a-8)=0$
- $a=0 \ or \ a=8$
- If a and b are real numbers, prove that the roots of the equation
- $ax^2+(a+b)x+b=0$ are real.
- Solution
- $D=(a+b)^2-4ab$
- $(a-b)^2 \geq 0,$ for any real numbers a and b
- Hence, the roots of the equation are real.

Sum and Product of Roots of a Quadratic Equation

- Theorem 1: Let α and β be the roots of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$.
- Then
- Sum of roots: $\alpha + \beta = -\frac{b}{a}$
- Product of roots: $\alpha\beta = \frac{c}{a}$
- **Useful Identities** $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
- $\alpha^2\beta^2 = (\alpha\beta)^2,\ \alpha^3\beta^3 = (\alpha\beta)^3,\ \alpha^4\beta^4 = (\alpha\beta)^4$
- $(\alpha - \beta) = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$
- $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
- $\alpha^4 + \beta^4 = [(\alpha + \beta)^2 - 2\alpha\beta]^2 - 2(\alpha\beta)^2$

- If α and β are the roots of a quadratic equation in x, then the quadratic equation is
- $(x-\alpha)(x-\beta)=0$ i.e. $x^2-(\alpha+\beta)x+\alpha\beta=0$
- **Example**: If α and β are the roots of the equation
- $3x^2-4x-5=0$, find the equation
- $\alpha+\beta=\frac{4}{3}$, $\alpha\beta=\frac{-5}{3}$
- $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\left(\frac{4}{3}\right)^2-2\left(-\frac{5}{3}\right)$
- $=\frac{16}{9}+\frac{30}{9}=\frac{46}{9}$
- $\alpha^2\beta^2=(\alpha\beta)^2=\left(-\frac{5}{3}\right)^2=\frac{25}{9}$
- Equation in x with roots α^2 and β^2 is
- $x^2-(\alpha^2+\beta^2)x+\alpha^2\beta^2=0$
- i.e. $x^2-\frac{46}{9}x+\frac{25}{9}=0$

Remainder and Factor Theorems

- **Theorem 1. (Remainder Theorem)** If a polynomial $f(x)$ of degree $\geq$ **1** is divided by $(ax + b)$, then the remainder is $f\left(-\frac{b}{a}\right)$
- **Proof.** $f(x) = (ax + b)q(x) + R$
- $\Longrightarrow f\left(-\frac{b}{a}\right) = +R$
- **Note:** The Remainder theorem is used to find the remainder without performing the long division.
- **Theorem 2. (Factor Theorem)** If $(ax+b)$ is a factor of a polynomial $f(x)$, then $f\left(-\frac{b}{a}\right) = 0$, and conversely.

Example 1: Use the Remainder Theorem to find the remainder, when $f(x) = x^3 + 5x - 3$ is divided by $3x + 1$

- **Solution**:
- $3x + 1 = 0 \implies x = -\frac{1}{3}$
- By Remainder Theorem,
- The Remainder $= f\left(-\frac{1}{3}\right) = \left(-\frac{1}{3}\right)^3 + 5\left(-\frac{1}{3}\right) - 3 = -\frac{127}{27}$
- **Example 2:** Determine whether or not, $2x - 1$ is a factor of $g(x) = 2x^3 + x^2 + x - 1$
- **Solution:**
- $2x - 1 = 0 \implies x = \frac{1}{2}$
- $g\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right) - 1 = 0$
- By the Factor Theorem, it follows that $2x - 1$ is a factor of $g(x)$

- **Example 3:** Find a and b if $h(x) = 3x^4 + ax^3 + 12x^2 + bx + 4$
- is divisible by $(x - 1)$, and
- leaves remainder 18, when divided by $(x + 2)$
- **Solution**: By condition (i), and the Factor Theorem
- $h(1) = 0$
- $\Rightarrow 3 + a + 122 + b + 4 = 0$
- $\Rightarrow a + b = -19$ .... (1)
- By condition (ii), and the Remainder Theorem,
- $h(-2) = 3(-2)^4 + a(-2)^3 + 12(-2)^2 + b(-2) + 4$
- $\Rightarrow 4a + b = 41$ … … … … … … … … … … … … … … … … .(2)
- $(2) \rightarrow (1): 3a = 60$
- $a = 20, b = -39$

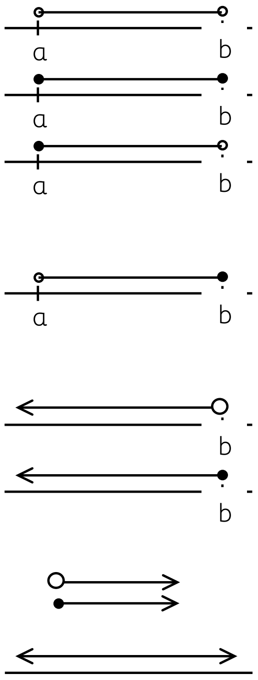
- **Theorem 4 (Rational Root Theorem).** Let $f(x)$ be a polynomial with integral coefficients and let $^p/_q$ be a rational number in its lowest terms. Then $x = {}^p/_q$ is a root of $f(x)$, (or $qx - p$ is a factor of $f(x)$, if (i) p is a factor of the constant term of $f(x)$, and (ii) q is a factor of the leading co-efficient of $f(x)$.
- **Example**: factorize $f(x) = x^3 + 2x^2 - 5x - 6$.
- Hence, solve the equation $f(x) = 0$ for x
- **Solution**: By the Rational Root Theorem, the values of x with testing are $\pm 1, \pm 2, \pm 3, \pm 6$
- $f(-1) = -1 + 2 + 5 - 6 = 0$
- $\Longrightarrow x + 1$ is a factor of $f(x)$
- $f(2) = 8 + 8 - 10 - 6 = 0 \quad \Longrightarrow x - 2$ is a factor of $f(x)$
- $f(-3) = -27 + 18 + 15 - 6 = 0$
- $\Longrightarrow x + 3$ is a factor of $f(x)$
- So, $f(x) = (x + 1)(x - 2)(x + 3)$

LEARNING MANAGEMENT SYSTEM LMS

Inequality

Inequalities of numbers can be represented using the interval notation, and geometrically on the real number line

1.	Inequality	Interval	Number line
	$a < x < b$	(a, b) Open interval	b a
	$a < x < b$	$[a, b]$ Closed interval	b a
	$a \leq x < b$	$[a, b)$ Interval closed on the left and open on the right	b a
	$a < x \leq b$	$(a, b]$ Interval open on the left and closed on the right	b a
	$x < b$	$(\quad \infty, b)$	b
	$x < b$	$(\quad \infty, b]$	b
	$x > a$ $x > a$	$(a - \infty)$ $[a, \infty)$	
	$\infty < x < \infty$	$(\quad \infty, \infty)$	



- **Properties of Inequalities**
- 1. Let a, b, c and d be any real numbers
- 2. $a < b$ and $b < c \Rightarrow a < c$, (Transitive law)
- 3. $a < b \Rightarrow a + c < b + c; a - c < b - c$

  $a > b \Rightarrow a + c > b + c; a - c > b - c$
- 4. $c > 0 \Rightarrow \frac{1}{c} > 0$

  $c < 0 \Rightarrow \frac{1}{c} < 0$
- 5. If $c > 0$

  $a < b \Rightarrow ac < bc, \frac{a}{c} < \frac{b}{c}$

  $a > b \Rightarrow ac > bc, \frac{a}{c} > \frac{b}{c}$

6. If $c < 0$

$$a < b \Rightarrow ac > bc, \frac{a}{c} > \frac{b}{c}$$

$$a > b \Rightarrow ac < bc, \frac{a}{c} < \frac{b}{c}$$

7. If $c \neq 0, c^2 > 0$

8. $ab > 0$ or $\frac{a}{b} > 0$

$$\Rightarrow (a > 0 \text{ and } b > 0) \text{ or } (a < 0 \text{ and } b < 0)$$

$ab < 0$ or $\frac{a}{b} < 0$

- $\Rightarrow$ either $(a > 0$ and $b < 0)$ or $(a < 0$ and $b > 0)$

9 $a < b$ and $c < d \Rightarrow a + c < b + d$

$a > b$ and $c > d \Rightarrow a + c > b + d$

10. If a, b, c and d are all positive

$a > b$ and $c > d \Longrightarrow ac > bd$

$a < b$ and $c < d \Longrightarrow ac < bd$

Hence $a > b \Longrightarrow a^2 > b^2$

and $a < b \Longrightarrow a^2 < b^2$

- **Absolute Values**

The modulus or the absolute value of a real number x denoted by $|x|$, is the positive number 3 which has the same magnitude as x. For example $|3| = 3 = |-3|$

Properties of absolute values

1. $|-x| = |x|, |xy| = |x|.|y|, \left|\frac{x}{y}\right| = \left|\frac{x}{y}\right|$

2. $|x| \geq 0$ for all real numbers .

$|x| = 0$ if and only if $x = 0$

3. $x > 0 \Longrightarrow |x| = x$; $x < 0 = |x| = -x.$

4. $|x| = a \Longrightarrow (x = a \text{ or } x = -a)$

5. $|x| < a$ means $-a < x < a$

6. $|x| > a$ means $(x > a \text{ or } x < -a)$

7. $|x - y| =$ distance on the real number line from x to y

8. $|x - b| < a$ means $-a < x - b < a$

9. $|x - b| > a$ means $(x - b > a \text{ or } x - b < -a)$

10. $|x + y| \leq |x| + |y|$ (Triangle inequality)

Linear Inequalities

- **Example**: Solve for x.

Solution: $\frac{3(1-4x)}{2} \leq \frac{3x-1}{3}$

Multiply by 6, the LCM of 2 and 3:

$$9(1-4x) \leq 2(3x-1);\ \text{by property 4}$$

$$9 - 36x < -6x - 2$$

$$-42x \leq -11$$

$$\Rightarrow x > -\frac{11}{42},\ \text{by property 4.}$$

$$\Rightarrow \left[\frac{11}{42}, \infty\right)$$

Exercise: Solve simultaneously

$$2 - \frac{x}{2} > 0 \text{ and } 6 - 3x < x - 2$$

Solution: $2 - \frac{x}{2} > 0$ and $6 - 3x < x - 2$

$$\Rightarrow 4 - x > 0 \text{ and } -4x < -8$$

$$\Rightarrow x < 4 \text{ and } x > 2$$

- **Prove of the triangle inequality**

Proof: $\left|x + y \le |x| + |y|\right|$

$$|x| \ge x, \qquad |y| \ge y$$

$$\Rightarrow |x| + |y| \ge x = y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$|x| \ge -x, \qquad |y| \ge -y$$

$$-|x| \le x, \qquad -|y| \le x = y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . (2)$$

and (2) imply that

$$-(|x| + |y|) \le x + y \le |x| + |y|$$

$$\Rightarrow |x + y| \le |x| + |y|$$

Partial Fractions

- Consider a rational function $\frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomials. If (degree of $f(x)$) < (degree of $g(x)$) then $\frac{f(x)}{g(x)}$ is called a **proper** rational function.
- If (degree of $f(x)$) $\geq$ (degree of $g(x)$) then $\frac{f(x)}{g(x)}$ is called an **improper** rational function.
- **Theorem 5.** Any rational function can be expressed as a sum of a polynomial (possibly the zero polynomial) and a proper rational function
- **Theorem 6.** If the polynomials $g(x)$ and $h(x)$ do not have any common factor of degree ≥ 1, then
- $\frac{f(x)}{g(x)h(x)} = q(x) + \frac{r(x)}{g(x)} + \frac{s(x)}{h(x)}$
- Where $\frac{r(x)}{g(x)}$and $\frac{s(x)}{h(x)}$ are proper rational functions, and $q(x)$is a polynomial.

LEARNING MANAGEMENT SYSTEM LMS

- **Theorem 7. (Distinct linear factors in the denominator).**

$$\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} \qquad a \neq b$$

- where

$$A = \frac{pa+q}{a-b}, \qquad B = \frac{pb+q}{b-a}$$

- **Proof.**

$$\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

$$= \frac{A(x-b)+B(x-a)}{(x-a)(x-b)}$$

$$\Longrightarrow px+q \equiv A(x-b)+B(x-a)$$

Put $x = a$: $\quad pa+q = A(x-b) \Longrightarrow A = \frac{pa+q}{a-b}$

Put $x = b$: $\quad pb+q = B(b-a) \Longrightarrow B = \frac{pb+q}{b-a}$

Theorem 8. (Repeated linear factor in the denominator)

$$\frac{px+q}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$$

Where $A = p, \quad B = pa + q$

Proof. $\frac{px+q}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2} = \frac{A(x-a)+B}{(x-a)^2}$

$$\Rightarrow px + q \equiv A(x-a) + B$$

Compare coefficients

$$px = Ax\ ; \qquad p = A$$

$$q = -Aa + B; q = -ap + B\ ; \qquad q + ap = B$$

Remark. The rule in Theorem 8 above can be generalized to the case

$$\frac{f(x)}{(x-a)^k} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_k}{(x-a)^k}$$

Where def $f(x) < k$ and the A_i's are constants.

For example:

$$\frac{px^2+qx+r}{(x-a)^3}=\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{(x-a)^3}$$

Where $A = p, \quad B = 2ap + q, \quad C = pa^2 + qa + r$

Example: Resolve into partial fractions

$$\frac{6r+1}{(4r^2-1)(2r+3)}$$

Solution:

$$\frac{6r+1}{(4r^2-1)(2r+3)}=\frac{6r+1}{(2r-1)(2r+1)(2r+3)}$$
$$=\frac{A}{2r-1}+\frac{B}{2r+1}+\frac{C}{2r+3}$$

Method [by substituting convenient values of r, by cover-up rule, by comparing coefficients of power of r]

- $$6r + 1 \equiv A(2r+1)(2r+3) + B(2r-1)(2r+3) + C(2r-1)(2r+1)$$

Coefficients of r^2: $\quad 0 = 4A + 4B + 4C \qquad (1)$

Coefficients of r: $\quad 6 = 8A + 4B \qquad (2)$

Constant: $\quad 1 = 3A - 3B - C \qquad (1)$

Solve equations (1) – (3) simultaneously

Then $\quad A = \frac{1}{2}, \quad B = \frac{1}{2}$ and $C = -1$

Hence

$$\frac{6r+1}{(4r^2-1)(2r+3)} = \frac{1}{2(2r-1)} + \frac{1}{2(2r+1)} - \frac{1}{2r+3}$$

Theorem 9.

(Linear and Irreducible quadratic factor in the denominator)

$$\frac{f(x)}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$$

Where deg $(f(x)) < 3$, and $A = \frac{f(x)}{\ldots}$

- **(Distinct irreducible quadratic factors in the denominator)**

If $\deg(f(x)) < 4$, $a \neq c$ or, $b \neq d$ then

$$\frac{f(x)}{(x^2+ax+b)(x^2+cx+d)} = \frac{Ax+B}{x^2+ax+b} + \frac{Cx+D}{x^2+cx+d}$$

(Repeated irreducible quadratic factors in the denominator)

If $\deg(f(x)) < k$, then

$$\frac{f(x)}{(x^2+ax+b)^k} = \frac{A_1x+B_1}{x^2+ax+b} + \frac{A_2x+B_2}{(x^2+ax+b)^2} + \cdots + \frac{A_kx+B_k}{(x^2+ax+b)^k}$$

Example. Split into partial fractions

$$\frac{3x^2}{1+x^3}$$

Solution

$$\frac{3x^2}{1+x^3} = \frac{3x^2}{(1+x)(1-x+x^2)} = \frac{A}{1+x} + \frac{B}{1-x+x^2}$$

- $$3x^2 \equiv A(1 - x + x^2) + (Bx + C)(1 + x)$$

By comparing coefficients of power of x

Coefficients of x^2 $\quad 3 = A + B$ ……………………….. (1)

Coefficients of x $\quad 0 = -A + B + C$ ……………………….. (2)

Constant` $\quad 0 = A + C$ ……………………….. (3)

Solve equations (1) – (3) simultaneously. Then $A = 1, \quad B = 2, \quad C = -1$

Hence, $\quad \frac{3x^2}{1+x^3} = \frac{1}{1+x} + \frac{2x+1}{1-x+x^2}$

Exercise: 1. Express as a sum of partial fractions

$$\frac{x+3}{(2x-1)^2(x+2)}$$

2. Resolve into partial fractions

$$\frac{x^2+1}{(x-1)^2(x^2+2x+2)}$$

3. Split into partial fractions $\dfrac{x^2-4x+5}{(3x-1\_^{3}(x+3)}$