# Algebra and Trigonometry for Biological Sciences

MTS 105

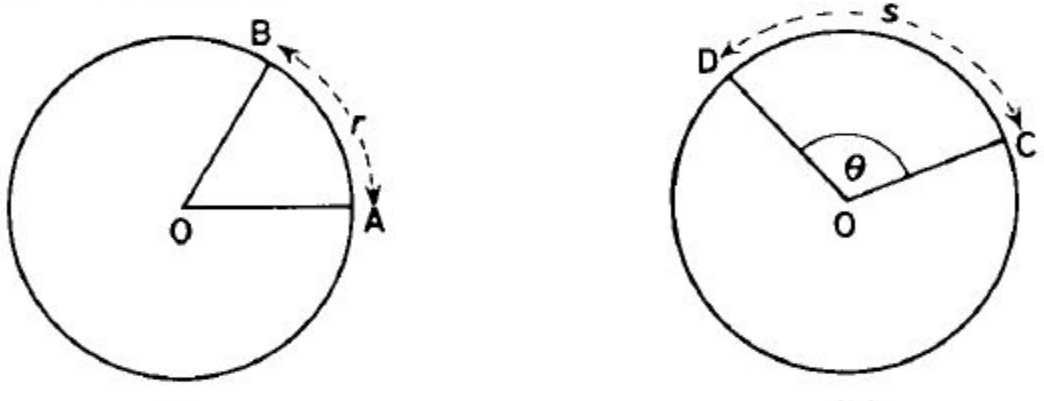
Department of Mathematics

**Dr. D. O. Adams**

# Elementary Trigonometry: Degree and Radian Measures

- Angles are generally measured in degrees (360° = one revolution) or radians. Let AB be an arc of a circle with centre O, equal in length to the radius r of the circle. Then <AOB is one radian.

$$\theta = \frac{s}{r} \quad radian \qquad (1)$$




(a) (b)

It follows from equation (1) that one complete revolution is equivalent to $\dfrac{2\pi r}{r} = 2\pi$ radians. Thus, we have the relationship between the two units:

$$2\pi \text{ radians} = 360^\circ$$

$$\pi \text{ radians} = 180^\circ$$

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^\circ = 57{\cdot}2958^\circ = 57^\circ\ 17'\ 45''$$

$$1^\circ = \frac{\pi}{180} \text{ radians}$$

*Example 1* Express the following angles in radians: (i) 37°, (ii) $-143^\circ\ 10'$.

(i) Note that $1^o = \dfrac{\pi}{180}$ *radians*. Then

$$37^\circ = \frac{37 \times \pi}{180} \text{ radian} = 0.6458 \text{ radian}$$

(ii) $$-143^\circ\ 10' = -143\tfrac{1}{6}^\circ = -143{\cdot}166^\circ$$

Therefore

$$-143^\circ\ 10' = -\frac{143{\cdot}166 \times \pi}{180} \text{ radians} = -2{\cdot}499 \text{ radians}$$

*Example 2* Express the following angles in degrees and minutes correct to the nearest minute: (i) 2·1 radians, (ii) $\pi/12$ radian.

(i) Since $1 radian = \left(\frac{180}{\pi}\right)^o$. Now

$$2{\cdot}1 \text{ radians} = \frac{2{\cdot}1 \times 180}{\pi} \text{ degrees} = 120{\cdot}316^\circ$$

$$= 120^\circ\ 19' \quad \text{(correct to the nearest minute)}$$

(ii)
$$\frac{\pi}{12} \text{radian} = \left(\frac{\pi}{12} \times \frac{180}{\pi}\right)^\circ = 15^\circ$$

Although an understanding of the relationship between the units concerned is desirable, in practice the conversions of the examples above are best carried out with the aid of tables or a calculator.

## Practice Exercise I

**1** Express the following angles in degrees and minutes correct to the nearest minute: (i) 3·2 radians, (ii) $-1{\cdot}58$ radians, (iii) $\pi/5$ radian.

**2** Express in radians (i) $235^\circ$, (ii) $14^\circ\ 4'$, (iii) $-128^\circ\ 10'$.

**3** Which is the larger of the following pairs? (i) $129^\circ$ or 2·16 radians, (ii) $19^\circ$ or $\frac{1}{3}$ radian.

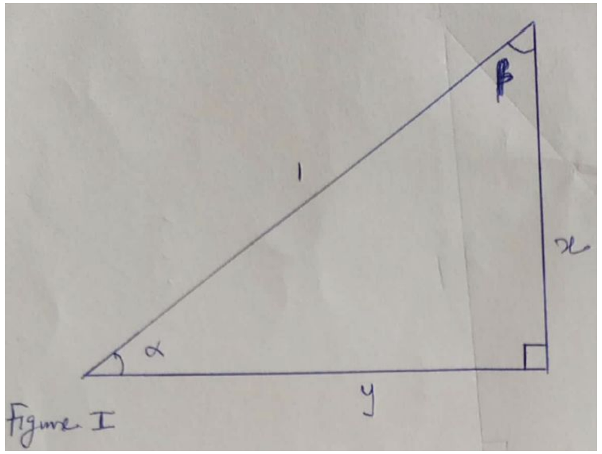
**4** Verify the correctness of the following useful equivalents:

(i) $\dfrac{\pi}{2}$ radians $= 90^\circ$ (ii) $\dfrac{\pi}{4}$ radian $= 45^\circ$ (iii) $\dfrac{\pi}{3}$ radians $= 60^\circ$

LEARNING MANAGEMENT SYSTEM
LMS

# Complementary Angles

Angles are complementary if their sum is $90^o$.




In Fig.I, $\alpha$ and $\beta$ are complementary angles. Then,

$$\alpha = 90^o - \beta \text{ and } \beta = 90^o - \alpha.$$

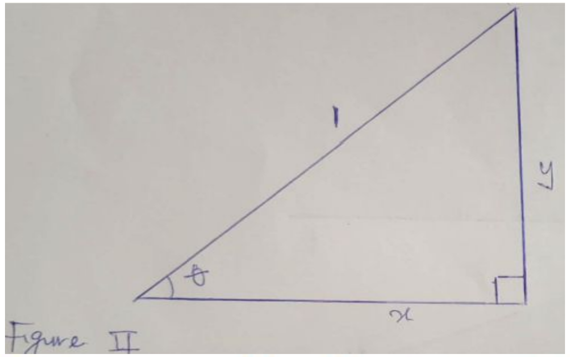
From Fig.I, $\sin\alpha = \frac{x}{1} = \cos\beta = \cos(90^o - \alpha)$,

$$\cos\alpha = \frac{y}{1} = \sin\beta = \sin(90^o - \alpha).$$

# Special Angle

The ratios for angles $0^o$, $30^o$, $45^o$, $60^o$, $90^o$, $180^o$, $270^o$ occur frequently especially in mechanics, and it is useful to remember their values (some are in surd form).

$\boxed{0^o, 90^o}$




In Fig.II, if $\theta$ decreases to $0^o$, then $y$ decreases to 0 and $x$ tends to 1. So,

$$\sin\theta^o = \cos 90^o = \frac{y}{1} = 0;$$

$$\tan\theta^o = \frac{y}{x} = \frac{0}{1} = 0;$$

$$\cos\theta^o = \sin 90^o = \frac{x}{1} = 1.$$

$\tan 90^o$ is not defined ($\tan 90^o$ would be $\frac{y}{0}$ which has no value).

$\boxed{30^o,\ 60^o}$

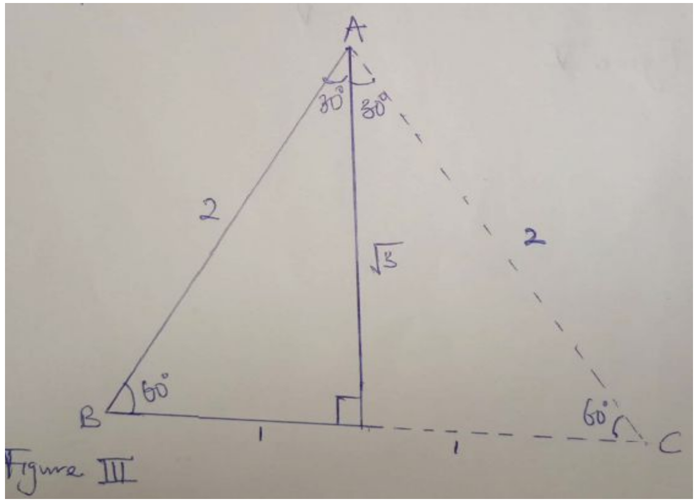


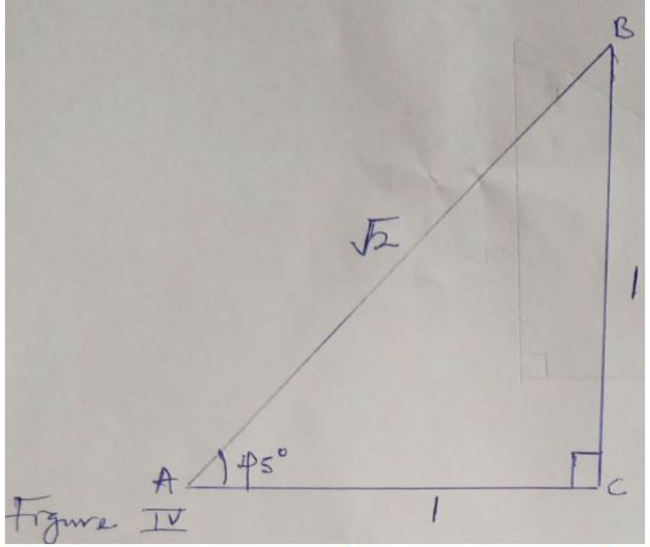

Fig.III, is an equilateral triangle $ABC$, of side 2 units. AD bisects angle A, and by symmetary is perpendicular to BC. Then $BD = 1$ and $AD = \sqrt{3}$ units.

$$\sin 30^o = \cos 60^o = \frac{1}{2};$$

$$\cos 30^o = \sin 60^o = \frac{\sqrt{3}}{2};$$

$$\tan 30^o = \frac{1}{\sqrt{3}}, \quad \tan 60^o = \sqrt{3}.$$

$45^o$




In Fig.IV, ABC is an isosceles right-angled triangle $\angle A = \angle B = 45^o$.

$AC = CB = 1$ unit and $AB = \sqrt{2}$ units.

$$\sin 45^o = \cos 45^o = \frac{1}{\sqrt{2}};$$

$$\tan 45^o = 1.$$

$180^o$

$$\sin 180^o = \sin(180-0)^o = \sin 0^o = 0;$$

$$\cos 180^o = \cos(180-0)^o = -\cos 0^o = -1;$$

$$\tan 180^o = \tan(180-0)^o = -tan0^o = 0.$$

$\boxed{270^o}$

$$\sin 270^o = \sin(360-90)^o = -\sin 90^o = -1;$$

$$\cos 270^o = \cos(360-90)^o = \cos 90^o = 0;$$

$\tan 270^o$ is undefined.

## Summary

Table I

| | $0^o$ | $30^o$ | $45^o$ | $60^o$ | $90^o$ | $180^o$ | $270^o$ |
|---|---|---|---|---|---|---|---|
| sin | 0 | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1 | 0 | -1 |
| cos | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 | -1 | 0 |
| tan | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | - | 0 | - |

Ratios for angles connected with these can be easily found.

**Example 3.** Find the values of (a) $\sin 120^o$ (b) $\cos 240^o$ (c) $\tan 315^o$.

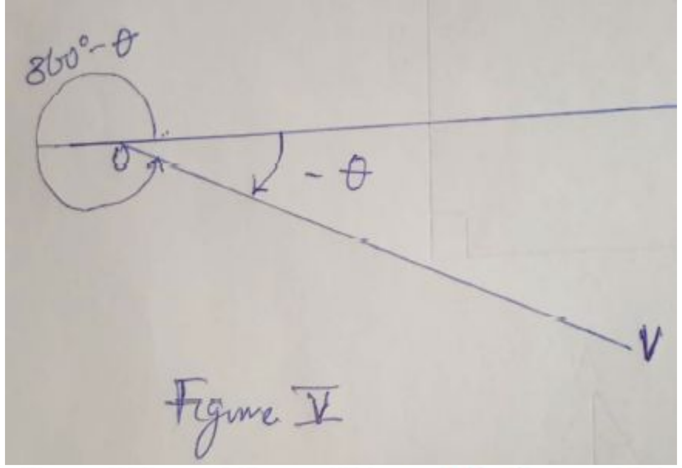
Solution.

(a) $\sin 120^o = \sin(180-60)^o = \sin 60^o = \frac{\sqrt{3}}{2}$.

(b) $\cos 240^o = \cos(180+60)^o = -\cos 60^o = -\frac{1}{2}$

(c) $\tan 315^o = \tan(360-45)^o = -tan45^o = -1$.

# Negative Angle




If the arm $OV$ rotates in a clockwise direction, it will describe a negative angle, $-\theta$. The positive angle is then $(360-\theta)^o$, see Fig.V. Hence,

$$\sin(-\theta) = \sin(360-\theta)^o = -\sin\theta$$

$$\cos(-\theta) = \cos(360-\theta)^o = \cos\theta$$

$$\tan(-\theta) = \tan(360-\theta)^o = -\tan\theta.$$

For example,

$$\tan(-210^o) = -\tan 210^o = -\tan(180+30)^o = -\tan 30^o = \frac{-1}{\sqrt{3}};$$

$$\cos(-120^o) = \cos 120^o = \cos(180-60)^o = -\cos 60^o = -\frac{1}{2}.$$

**Example 4.** Given that $\dfrac{4\tan 75^o}{1-\tan^2 75^o} = \dfrac{1}{\cos 150^o}$, find $\tan 75^o$ in surd form.

$$\frac{1}{\cos 150^o} = \frac{1}{-\cos 30^o} = \frac{1}{-\sqrt{3}/2} = -\frac{2}{\sqrt{3}}.$$

Hence,

$$\frac{4\tan 75^o}{1-\tan^2 75^o} = -\frac{2}{\sqrt{3}} \quad or \quad \frac{\tan 75^o}{1-\tan^2 75^o} = -\frac{1}{2\sqrt{3}}.$$

For ease of writing, let $\tan 75^o = t$.

Then, $\dfrac{t}{1-t^2} = -\dfrac{1}{2\sqrt{3}}$ which gives $2\sqrt{3}t = -1 + t^2$ or $t^2 - 2\sqrt{3}t - 1 = 0$.

This is a quadratic equation in $t$. Solving,

$$t = \frac{2\sqrt{3} \pm \sqrt{16}}{2} = \sqrt{3} \pm 2.$$

But $\sqrt{3} - 2$ is negative and $\tan 75^o$ must be positive. Hence, $t = \tan 75^o = 2 + \sqrt{3}$.

LEARNING MANAGEMENT SYSTEM LMS

# Practice Exercise II

1. Without using tables, find the values of the following, in a simplifie

   surd form where necessary:

   (a) $\sin 135^o$ (b) $\tan 240^o$ (c) $\cos 210^o$.

2. Find the value of $\sin 15^o$ given that

$$\sin 15^o = \sin 60^o \cos 45^o - \cos 60^o \sin 45^o.$$

3. Find the value of $\cos 15^o = \cos 60^o \cos 45^o + \sin 60^o \sin 45^o$.

# Trigonometric Identities

For an acute angle $\theta$, the trigonometric ratios are defined as follows

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x} \qquad \text{(i)}$$

$$\operatorname{cosec}\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y} \qquad \text{(ii)}$$

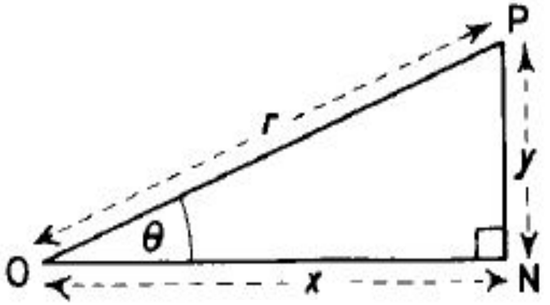



Fig.VI

Thus, we have immediately the following relationships between the six ratios:

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} \qquad \text{(iii)}$$

Also

$$\tan\theta = \frac{y}{x} = \frac{y}{r}\bigg/\frac{x}{r} = \frac{\sin\theta}{\cos\theta}$$

and

$$\cot\theta = \frac{x}{y} = \frac{x}{r}\bigg/\frac{y}{r} = \frac{\cos\theta}{\sin\theta} \qquad \text{(iv)}$$

Furthermore, $x^2 + y^2 = r^2$, therefore

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

that is

$$\cos^2\theta + \sin^2\theta = 1 \qquad \text{(v)}$$

This may be written in either of the forms

$$\sin^2\theta = 1 - \cos^2\theta \quad \text{or} \quad \cos^2\theta = 1 - \sin^2\theta \qquad \text{(vi)}$$

LEARNING MANAGEMENT SYSTEM LMS

In addition, we have

$$1 + \tan^2\theta = 1 + \frac{y^2}{x^2} = \frac{x^2 + y^2}{x^2} = \frac{r^2}{x^2}$$

therefore

$$1 + \tan^2\theta = \sec^2\theta \qquad \text{(vii)}$$

Also

$$1 + \cot^2\theta = 1 + \frac{x^2}{y^2} = \frac{y^2 + x^2}{y^2} = \frac{r^2}{y^2}$$

therefore

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta \qquad \text{(viii)}$$

The relationships (i) to (viii) which are true generally for any angle, enable us to calculate all the trigonometric ratios if one is known, and are of value in rewriting trigonometric expressions in alternative and simpler forms.

*Example 1* If $\sin\theta = 1/\sqrt{3}$ and $0^\circ \leqslant \theta \leqslant 90^\circ$, find the values of the other trigonometric ratios of the angle $\theta$.

From (vi),

$$\cos^2\theta = 1 - \sin^2\theta = 1 - \tfrac{1}{3} = \tfrac{2}{3}$$

$$\cos\theta = \sqrt{\tfrac{2}{3}}$$

By (iv),

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1}{\sqrt{2}}$$

and by (iii), $\sec\theta = \sqrt{\tfrac{3}{2}}$ $\quad \operatorname{cosec}\theta = \sqrt{3}$ $\quad \cot\theta = \sqrt{2}$

*Example 2* Show that $\sin^2\theta + (1+\cos\theta)^2 = 2(1+\cos\theta)$.

The left-hand side (LHS) $= \sin^2\theta + (1+\cos\theta)^2$

$$\begin{aligned}
&= \sin^2\theta + 1 + 2\cos\theta + \cos^2\theta \\
&= 2 + 2\cos\theta \\
&\quad (\text{since } \cos^2\theta + \sin^2\theta = 1) \\
&= 2(1+\cos\theta) \\
&= \text{right-hand side (RHS), as required.}
\end{aligned}$$

# Practice Exercise III

1 If $\cos\theta = \frac{12}{13}$ and $0° \leqslant \theta \leqslant 90°$, evaluate $\sin\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, $\operatorname{cosec}\theta$.

2 If $\tan\theta = \frac{3}{2}$ and $0° \leqslant \theta \leqslant 90°$, evaluate $\sin\theta$ and $\cos\theta$.

3 If $x = a\cos\theta$, simplify (i) $a^2 - x^2$, (ii) $\left(1 - \dfrac{x^2}{a^2}\right)^{5/2}$

4 If $x = a\tan\theta$, simplify (i) $\dfrac{1}{a^2 + x^2}$, (ii) $\sqrt{\left(1 + \dfrac{x^2}{a^2}\right)}$.

# Graphs of Trigonometric Functions

By drawing up tables of values from 0° to 360°, graphs of $y = \sin A$, $y = \cos A$ and $y = \tan A$ may be plotted. Values obtained with a calculator (correct to 3 decimal places — which is more than sufficient for plotting graphs), using 30° intervals, are shown below, with the respective graphs shown in Fig. VII.

(a) $y = \sin A$

| $A$ | 0 | 30° | 60° | 90° | 120° | 150° | 180° |
|---|---|---|---|---|---|---|---|
| $\sin A$ | 0 | 0.500 | 0.866 | 1.000 | 0.866 | 0.500 | 0 |

| A | 210° | 240° | 270° | 300° | 330° | 360° |
|---|---|---|---|---|---|---|
| $\sin A$ | −0.500 | −0.866 | −1.000 | −0.866 | −0.500 | 0 |

(b) $y = \cos A$

| $A$ | 0 | 30° | 60° | 90° | 120° | 150° | 180° |
|---|---|---|---|---|---|---|---|
| $\cos A$ | 1.000 | 0.866 | 0.500 | 0 | −0.500 | −0.866 | −1.000 |

| $A$ | 210° | 240° | 270° | 300° | 330° | 360° |
|---|---|---|---|---|---|---|
| $\cos A$ | −0.866 | −0.500 | 0 | 0.500 | 0.866 | 1.000 |

(c) $y = \tan A$

| A | 0 | 30° | 60° | 90° | 120° | 150° | 180° |
|---|---|---|---|---|---|---|---|
| $\tan A$ | 0 | 0.577 | 1.732 | $\infty$ | $-1.732$ | $-0.577$ | 0 |

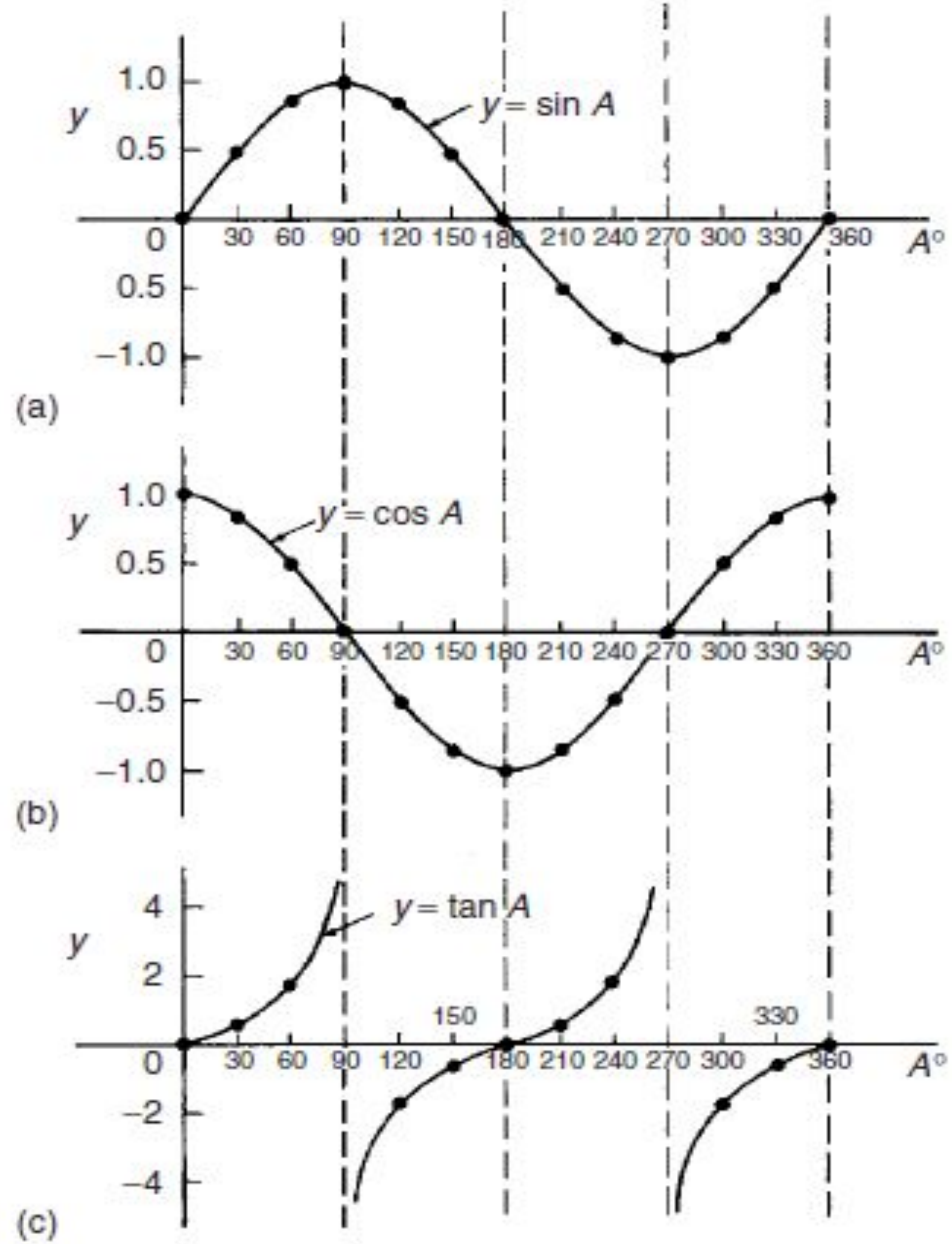
| $A$ | 210° | 240° | 270° | 300° | 330° | 360° |
|---|---|---|---|---|---|---|
| $\tan A$ | 0.577 | 1.732 | $\infty$ | $-1.732$ | $-0.577$ | 0 |

From Fig. VII it can be seen that:

(i) Sine and cosine graphs oscillate between peak values of $\pm 1$

(ii) The cosine curve is the same shape as the sine curve but displaced by 90°.

(iii) The sine and cosine curves are continuous and they repeat at intervals of 360°; the tangent curve appears to be discontinuous and repeats at intervals of 180°.



y
1.0
0.5
0
0.5
−1.0
30 60 90 120 150 180 210 240 270 300 330 360
A°
y = sin A
(a)
y
1.0
0.5
0
−0.5
−1.0
y = cos A
(b)
y
4
2
0
−2
−4
150
330
y = tan A
(c)


Fig. VII

## Graphs of sine and cosine waveforms

(i) A graph of y = sin A is shown by the broken line in Fig. VIII and is obtained by drawing up a table of values as shown for (a) above. A similar table may be produced for y = sin 2A.

| $A°$ | 0 | 30 | 45 | 60 | 90 | 120 |
|---|---|---|---|---|---|---|
| $2A$ | 0 | 60 | 90 | 120 | 180 | 240 |
| $\sin 2A$ | 0 | 0.866 | 1.0 | 0.866 | 0 | −0.866 |

| $A°$ | 135 | 150 | 180 | 210 | 225 | 240 |
|---|---|---|---|---|---|---|
| $2A$ | 270 | 300 | 360 | 420 | 450 | 480 |
| $\sin 2A$ | −1.0 | −0.866 | 0 | 0.866 | 1.0 | 0.866 |

| $A°$ | 270 | 300 | 315 | 330 | 360 |
|---|---|---|---|---|---|
| $2A$ | 540 | 600 | 630 | 660 | 720 |
| $\sin 2A$ | 0 | −0.866 | −1.0 | −0.866 | 0 |

(ii) A graph of y = cos A is shown by the broken line in Fig. IX and is obtained by drawing up a table of values as shown in (b) above. A similar table may be produced for y = cos 2A with the result as shown.

LEARNING MANAGEMENT SYSTEM LMS

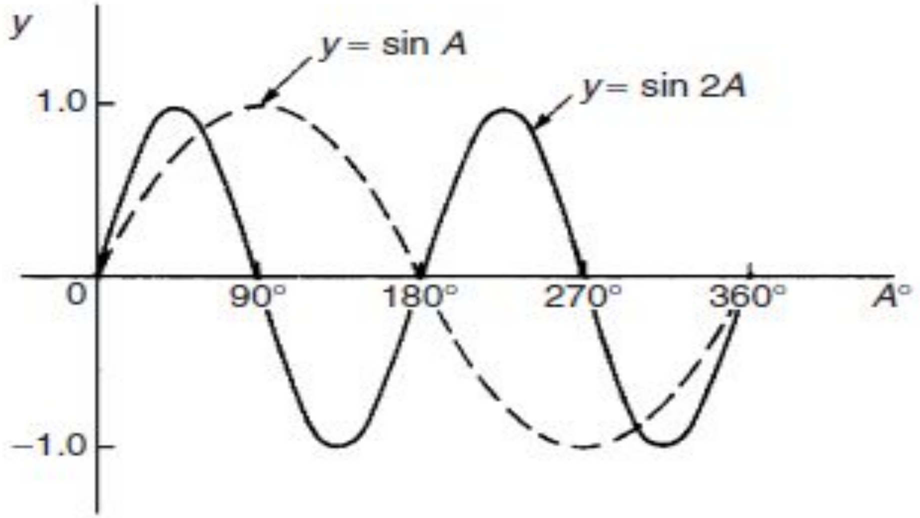



Fig. VIII

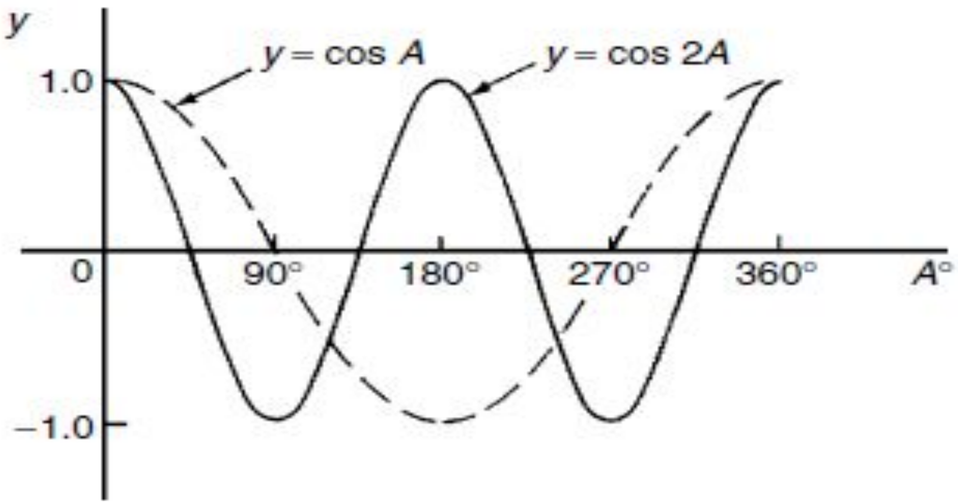



Fig. IX

# Practice Exercise IV

1. Sketch $y = \sin 3A$ between $A = 0°$ and $A = 360°$.

2. Sketch a graph of $y = \cos\frac{1}{2}A$ between $A = 0°$ and $A = 360°$.

# Inverse Trigonometric Functions

If $y = \sin x$, then $x$ is the angle whose sine is $y$. Inverse trigonometrical functions are denoted either by prefixing the function with 'arc' or by using $^{-1}$. Hence transposing $y = \sin x$ for $x$ gives $x = \arcsin y$ or $\sin^{-1} y$. Interchanging $x$ and $y$ gives the inverse $y = \arcsin x$ or $\sin^{-1} x$.

Similarly, $y = \arccos x$, $y = \arctan x$, $y = \operatorname{arcsec} x$, $y = \operatorname{arccosec} x$ and $y = \operatorname{arccot} x$ are all inverse trigonometric functions. The angle is always expressed in radians.

Inverse trigonometric functions are periodic so it is necessary to specify the smallest or principal value of the angle. For $y = \arcsin x$, $\arctan x$, $\operatorname{arccosec} x$ and $\operatorname{arccot} x$, the principal value is in the range $-\frac{\pi}{2} < y < \frac{\pi}{2}$. For $y = \arccos x$ and $\operatorname{arcsec} x$ the principal value is in the range $0 < y < \pi$.

For example:

Determine the principal values of

(a) arcsin 0.5 (b) arctan(−1)

(c) $\arccos\left(-\frac{\sqrt{3}}{2}\right)$

Solution:

Using a calculator,

(a) $\arcsin 0.5 \equiv \sin^{-1} 0.5 = 30° = \frac{\pi}{6}$ **rad** or **0.5236 rad**

(b) $\arctan(-1) \equiv \tan^{-1}(-1) = -45° = -\frac{\pi}{4}$ **rad** or **−0.7854 rad**

(c) $$\arccos\left(-\frac{\sqrt{3}}{2}\right) \equiv \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = 150°$$

$$= \frac{5\pi}{6} \text{ rad or } \mathbf{2.6180\ rad}$$

LEARNING MANAGEMENT SYSTEM LMS

# Practice Exercise V

Determine the principal values of the following inverse trigonometric functions:

1. $\arcsin(-1)$ $\left[-\frac{\pi}{2} \text{ or } -1.5708 \text{ rad}\right]$

2. $\arccos 0.5$ $\left[\frac{\pi}{3} \text{ or } 1.0472 \text{ rad}\right]$

3. $\arctan 1$ $\left[\frac{\pi}{4} \text{ or } 0.7854 \text{ rad}\right]$

# Compound Angles

The **compound angle formulae** for sines and cosines of the sum and difference of two angles $A$ and $B$ are:

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

(Note, $\sin(A+B)$ is **not** equal to $(\sin A + \sin B)$, and so on.)

The formulae stated above may be used to derive two further compound angle formulae:

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

The compound-angle formulae are true for all values of $A$ and $B$, and by substituting values of $A$ and $B$ into the formulae they may be shown to be true.

For example:

Expand and simplify the following expressions:

(a) $\sin(\pi + \alpha)$ (b) $-\cos(90^\circ + \beta)$

(c) $\sin(A - B) - \sin(A + B)$

Solution:

(a) $\sin(\pi + \alpha) = \sin\pi\cos\alpha + \cos\pi\sin\alpha$ (from the formula for $\sin(A + B)$)

$$= (0)(\cos\alpha) + (-1)\sin\alpha$$

$$= -\boldsymbol{\sin\alpha}$$

(b) $-\cos(90^\circ + \beta)$

$$= -[\cos 90^\circ \cos\beta - \sin 90^\circ \sin\beta]$$

$$= -[(0)(\cos\beta) - (1)\sin\beta] = \boldsymbol{\sin\beta}$$

(c) $\sin(A - B) - \sin(A + B)$

$$= [\sin A\cos B - \cos A\sin B]$$

$$- [\sin A\cos B + \cos A\sin B]$$

$$= \boldsymbol{-2\cos A\sin B}$$

LEARNING MANAGEMENT SYSTEM LMS

# Practice Exercise VI

1. If $\sin P = 0.8142$ and $\cos Q = 0.4432$ evaluate, correct to 3 decimal places: (a) $\sin(P - Q)$, (b) $\cos(P + Q)$ and (c) $\tan(P + Q)$, using the compound angle formulae

2. Show that

$$\tan\left(x + \frac{\pi}{4}\right)\tan\left(x - \frac{\pi}{4}\right) = -1$$

LEARNING MANAGEMENT SYSTEM LMS

References


1. Bunday, B. D. and Mulholland, H. (1983). Pure mathematics for advanced level. Second Edition. Butterworth and Co. (Publishers) Limited.
2. Talbert, J. F., Godman, A. and Ogum, G. E. O. (1984). Additional Mathematics for West Africa. Longman Group UK Limited, England.
3. Bird, J. (2003). Engineering Mathematics. Newnes. An imprint of Elsevier Science, Linacre House, Jordan Hill, Oxford OX2 8DP.