What is a set?

A **set** is a collection of well-defined objects, so that it is possible to decide whether any given object is, or is not, in the set.
An object in a set is called an **element** or a **member** of the set.

Notation and Representation

Use capital letters $A, B, C, D, \ldots$ to represent sets and lower case letters $a, b, c, d, \ldots$ to represent elements of a set..
'$a \in A$' denotes 'a is a member or an element of set A'
'$a \notin A$' denotes 'a is not a member or an element of set A'.

Description of sets

(i) **Using a diagram.** A set can be described by drawing the objects in the set in a diagram.

(ii) **By Word Description.** A set can be described in words. For example a set containing the numbers 1, 2, 3, 4, 5 can be described as 'a set of natural numbers from 1 to 5.

(iii) **By Listing the Elements.** A set can be described by listing the elements inside a pair of braces or curly brackets $\{\cdots\}$. For example, a set consisting of numbers 1,2,3,4,5 is described as

$$\{1, 2, 3, 4, 5\}$$

(iv) **Sample Elements Description.** A set can be described by using a sample element to describe the set. For example, the set consisting of all odd positive integers can be described as

$$\{x : x \text{ is an odd positive integer}\}$$

or

$$\{x | x \text{ is an odd positive integer}\}$$

where : or | is read as 'such that'.

Example 1

If $\mathcal{U} = \{2, 4, 6, 8, \ldots, 20\}$ is the universal set, write down by listing the elements, the set $\{\text{multiples of } 3\}$.

Solution

$$\{\text{multiples of } 3\} = \{6, 12, 18\}.$$

Different Types of Sets

1. **Null or Empty Set.** The set containing no element is called the null set or the empty set. It is denoted by ϕ or $\{\ \}$.

2. **Singleton set.** A set containing only one member is called a singleton set.

3. **Finite and Infinite sets.** If the number of elements in a set is finite, the set is called a finite set. The number of elements in a finite set A is denoted by $n(A)$.
   A set containing an infinite number of elements is called an infinite set. For example, the following sets are infinite sets:
   (a) $\{x : x \text{ is a natural number}\} = \{1, 2, 3, 4, 5, \ldots\}$
   (b) $\{x | x \text{ is an integer}\} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

4. **Subsets.** A set B, consisting of some or all the elements of a set A, is called a subset of A. '$B \subseteq A$' denotes 'B is a subset of A'.
   Every set is a subset of itself, i.e. $A \subseteq A$ for any set A. The empty set is considered a subset of every set, i.e. $\phi \subseteq A$, for any set A. If A is a finite set containing n elements, then there are 2^n subsets that can be formed from set A.

5. **Equal sets.** Two sets A and B are equal (denoted $A = B$), if they contain identically the same elements (not necessarily in the same order). Thus $A = B$ means
   $$A \subseteq B \quad \text{and} \quad B \subseteq A.$$

6. **Universal Sets.** The set containing all elements under discussion in a particular problem, is called the universal set for that problem. The universal set is denoted by $\mathcal{U}$ or ϵ. Then all other sets in the problem are subsets of this universal set. Thus the universal set changes fro problem to problem.

7. **Complement.** If A is any set in a universal set $\mathcal{U}$, then the complement of A, denoted by A^c or A', is the set consisting of all the elements in $\mathcal{U}$ which are not in A.

8. **Venn Diagrams.** A Venn diagram consists of a rectangle, which represents the universal set in a particular problem, and circles inside the rectangle, representing subsets of the universal set.

Example 2
Obtain all the subsets of the set $\{a, b, c, d\}$.

Solution
There are $2^4 = 16$ subsets.
Subset with 0 element (1): ϕ
Subsets with 1 element (4): $\{a\}, \{b\}, \{c\}, \{d\}$
Subsets with 2 elements (6): $\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$
Subsets with 3 elements (4):
$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$
Subsets with 4 elements (1): $\{a, b, c, d\}$
Total $= 1 + 4 + 6 + 4 + 1 = 16$.

Operations on Sets
Union of Sets. If A and B are sets, define the union of A and B, denoted by $A \cup B$, as the set

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

If $A \subseteq B$, then $A \cup B = B$.

Intersection of Sets. If A and B are sets, define the intersection of A and B, denoted by $A \cap B$, as the set

$$A \cap B = \{x| : x \in A \text{ and } x \in B\}$$

Thus $A \cap B$ consists of common elements in A and B.
If $A \subseteq B$, then $A \cap B = A$.

Example 3
Let A be a set in a universal set $\mathcal{U}$. such that A^c is the complement of A. Draw the operation table for

$$(\{\phi, A, A^c, \mathcal{U}\}, \cap)$$

Solution
Operation Table

$\cap$	ϕ	A	A^c	$\mathcal{U}$

Solution
Operation Table

$\cap$	ϕ	A	A^c	$\mathcal{U}$
ϕ	ϕ	ϕ	ϕ	ϕ
A	ϕ	A	ϕ	A
A^c	ϕ	ϕ	A^c	A^c
$\mathcal{U}$	ϕ	A	A^c	$\mathcal{U}$

Properties of Binary Operations

1. **Closure.** Let S be a set. An operation $*$ on S is a **binary operation** if for every pair of elements a, b in S, $a * b$ is in S. Then say that set S is closed with respect to the binary operation $*$, or say that $*$ satisfies the closure property on set S.

2. **Commutative Property:** Let $(S, *)$ be a set S together with a binary operation $*$ on S is said to satisfy the commutative law or property, if for every pair a, b in S,
$$a * b = b * a$$

3. **Associative Property:** Let $(S, *)$ be a set S together with a binary operation $*$ on S. The binary operation $*$ on S is said to satisfy the associative law or property, if for every triple a, b, c in S,
$$(a * b) * c = a * (b * c)$$

4. **Identity Element:** Let $(S, *)$ be a set S together with a binary operation $*$ on S. If there is an element denoted generally by e, in S such that
$$e * a = a * e \text{ for all } a \in S$$
then e is called an identity element for $*$.

5. **Inverses:** Let $(S, *)$ be a set S together with a binary operation $*$ on S, having an identity e. If a and b are elements in S such that
$$a * b = b * a = e$$

then a is called the inverse of b, and b is called the inverse of a in $(S, *)$. Denote the inverse of a by a^{-1}. Thus

$$a * b = b * a = e \Rightarrow b = a^{-1} \text{ and } a = b^{-1}$$

6. **Distributive Laws:** Let $(S, *, \circ)$ be a set together with two binary operation $*$ and $\circ$ on S. If for every a, b, c in S

$$a * (b \circ c) = (a * b) \circ (a * c)$$

and

$$(b \circ c) * a = (b * a) \circ (c * a)$$

then we say that $*$ is distributive over $\circ$.

Notation for Sets of Numbers

$\mathbb{N}$ = the set of all counting or natural numbers
$\mathbb{I}$ or $\mathbb{Z}$ = the set of all integers
$\mathbb{Q}$ = the set of all rational numbers
$\mathbb{R}$ = the set of all real numbers
$\mathbb{C}$ = the set of all complex numbers.

Example 4

(a) If $\mathbb{N}$ is the set of all natural numbers, is $(\mathbb{N}, \div)$ closed? Give reason/counter example.

(b) If $\mathbb{R}$ is the set of all real numbers, is $(\mathbb{R}, -)$ (i) associative?, (ii) commutative?. [Give reason/counter-example].

Solution

(a) Counter-example:
$a = 3$, $b = 4$, $a \div b = \frac{3}{4} \notin \mathbb{N}$.
Therefore $(\mathbb{N}, \div)$ is not closed.

(b)(i) $a = 2$, $b = 3$, $c = 4$
$(a - b) - c = (2 - 3) - 4 = -1 - 4 = -5$
$a - (b - c) = 2 - (3 - 4) = 2 + 1 = 3$
Hence $(2 - 3) - 4 \neq 2 - (3 - 4)$
Therefore $(\mathbb{R}, -)$ is **NOT** associative.

(ii) $a = 5, b = 7$
$a - b = 5 - 7 = -2$
$b - a = 7 - 5 = 2$
Hence $5 - 7 \neq 7 - 5$
Therefore $(\mathbb{R}, -)$ is **NOT** commutative.

Remark: For a property not to hold, it is sufficient to give a counter-example.

Example 5
Consider the set $S = \{4, 8, 12, 16\}$ and a binary operation $\otimes$ on S defined by $a \otimes b$ is the remainder when ab is divided by 20 (i.e. multiplication modulo 20)

(a) Draw the Operation Table for $(S, \otimes)$.

(b) From the Operation Table, determine if it exists

(i) an identity

(ii) an inverse of each element in S.

Solution

(a) **Operation Table**

$\otimes$	4	8	12	16
4	16	12	8	4
8	12	4	16	8
12	8	16	4	12
16	4	8	12	16

(b)(i) From the last row and last column of the Operation Table, 16 is an identity.

(ii) From the Operation Table

$4 \otimes 4 = 16 \Rightarrow 4^{-1} = 4$
$8 \otimes 12 = 16 \Rightarrow 8^{-1} = 12, 12^{-1} = 8$
$16 \otimes 16 = 16 \Rightarrow 16^{-1} = 16.$

Example 6
If $\mathcal{P}(X)$ is the set of all subsets of a set X, then in $(\mathcal{P}(X), \cup, \cap)$,

(a) $\cup$ is distributive over $\cap$
i.e. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(b) $\cap$ is distributive over $\cup$
i.e. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
$a(b + c) = ab + ac$, for all a, b, c in $\mathbb{R}$.

Example 7
Multiplication is distributive over addition.

Practice Exercise 15

1. Write the set $\{x | -3 < x \leq 5, x \text{ is an integer}\}$ by listing the elements.
2. Write down all the subsets of the set $\{a, b, c\}$.
3. Given the universal set $\mathcal{U} = \{1, 2, 3, 4, 5\}$, $M = \{1, 2, 5\}$ and $N = \{1, 3, 5\}$, find (a) $(M \cup N)^c$, (b) M^c, (c) N^c, (d) $M^c \cap N^c$, (e) $M^c \cup N^c$.
4. In $(\mathbb{R}, *)$, where

$$x * y = x + y - xy \text{ for all } x, y \text{ in } \mathbb{R}$$

   (a) Is the operation $*$ associative?
   (b) Does $(\mathbb{R}, *)$ contain an identity?
   (c) What members of $\mathbb{R}$ have inverses?
5. If $S = \{1, 3, 5, 7\}$ and $*$ is multiplication modulo 8,
   (a) Draw the Operation Table for $(S, *)$;
   (b) From the Operation Table, determine if it exists,
      (i) an identity
      (ii) an inverse for each element in S.