Calculus and Trigonometry

MTS 102 / Department of Mathematics

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The maximum and minimum values of a function

Figure 11.3 shows the graph of a function $y = f(x)$. The point A is called a local maximum of this function. The value of the function at A exceeds its values in a certain neighbourhood of A. Similarly, C is a local maximum and B a local minimum. As can be seen, B is not the absolute minimum value of the function, the value at D for example being less than the value at B. This is the reason for the term 'local' minimum, although in much of the literature this word is omitted although generally implied.

Fig. 11.3

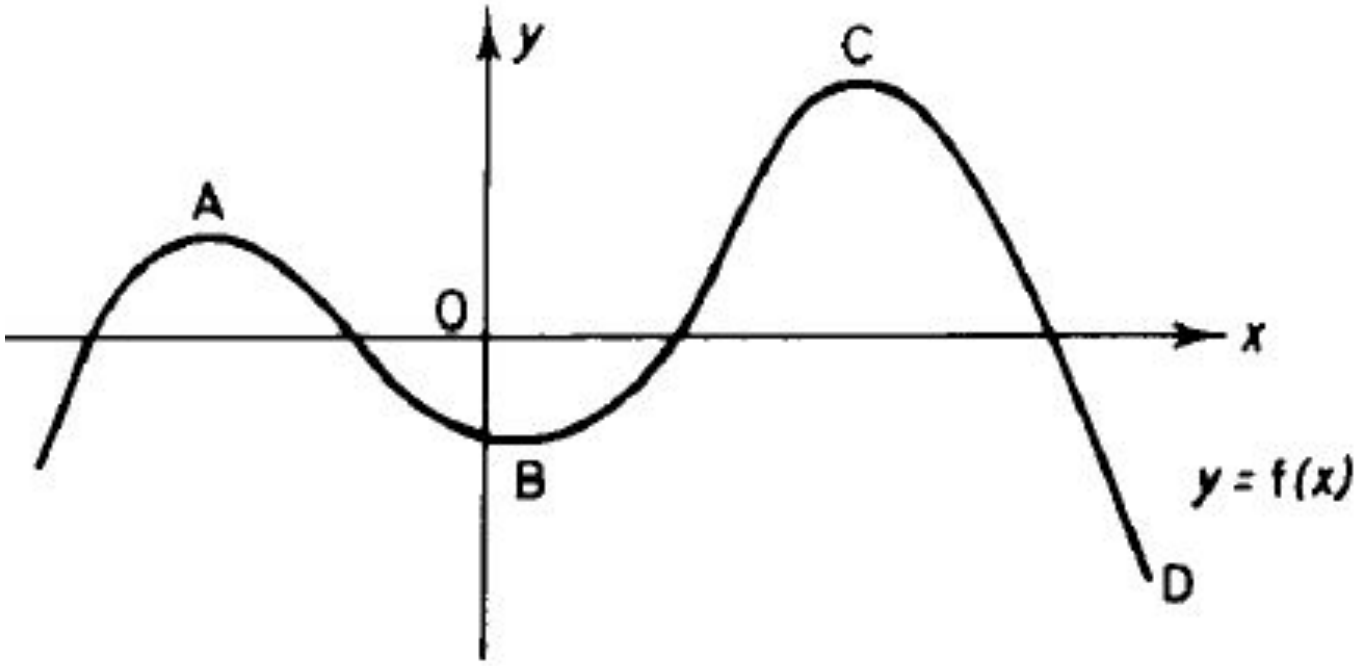

Figure 11.3

The positions of the points A, B, C may be determined by using the property that the derivative is zero (the tangent is parallel to the x-axis) at local maxima or minima.

To distinguish between local maxima and local minima, we shall examine the derivative in the neighbourhood of A and B respectively. Near A, the derivative is positive to the left of A, zero at A and negative to the right of A, i.e. it changes sign from positive to negative as x increases. Near B, the derivative is negative to the left of B, zero at B, and positive to the right of B, i.e. it changes sign from negative to positive as x increases. *Figure 11.4* shows the graph $y = \mathrm{f}(x)$ together with the sign (or zeros) of its derivative marked on it.

In those regions in which $\mathrm{f}'(x)$ is positive, we say that $\mathrm{f}(x)$ is an increasing function of x; in those regions in which $\mathrm{f}'(x)$ is negative, we say that $\mathrm{f}(x)$ is

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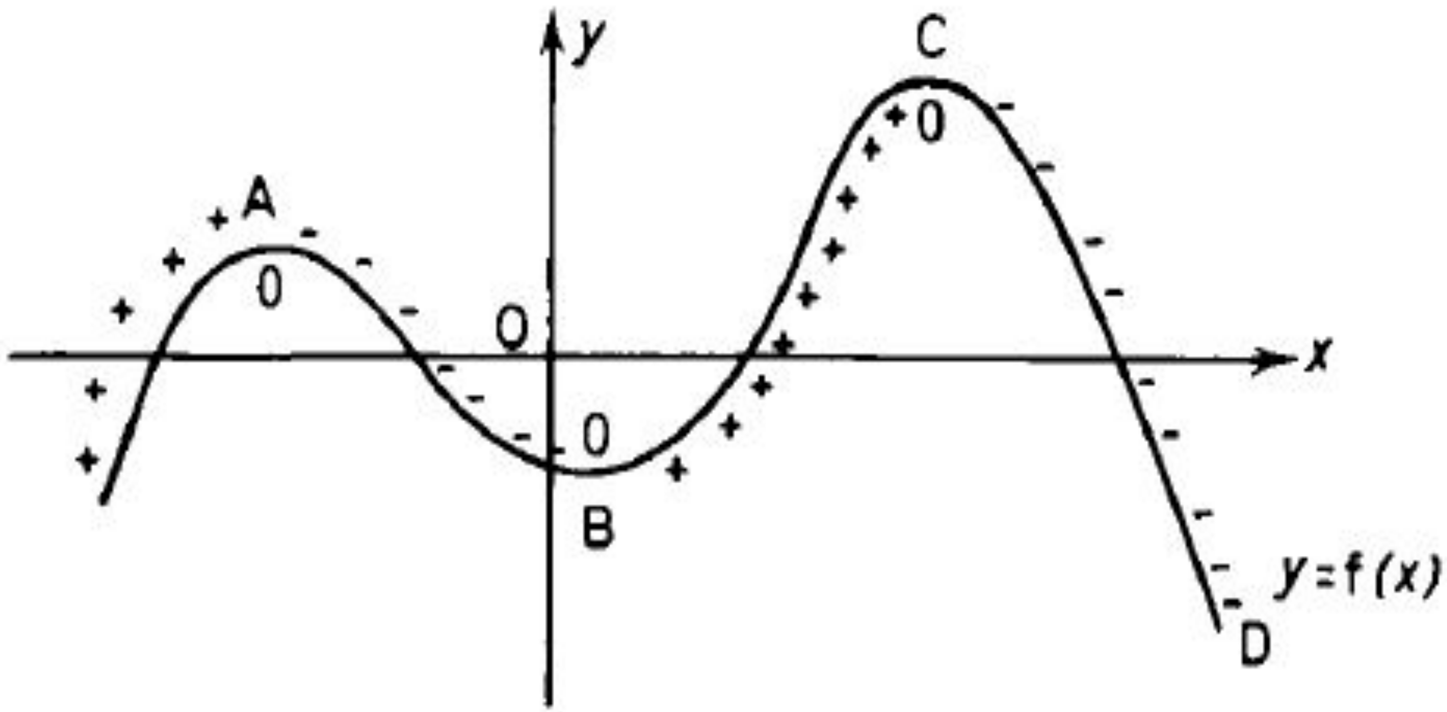

Figure 11.4

a decreasing function of x; and at points where $f'(x)$ is zero, we say that $f(x)$ is stationary. In the case of local maxima and minima (see Section 11.6 for other cases) the stationary points are sometimes called turning points. We can state the following rules for determining the stationary points which are also (local) maxima and minima, i.e. turning points, and for distinguishing between them.

RULE I At a turning point, $f'(x) = 0$.
RULE II At a local maximum, $f'(x)$ changes from positive to negative as x increases.
RULE III At a local minimum, $f'(x)$ changes from negative to positive as x increases.

In practice we evaluate (or at least examine the sign of) $f'(x)$ for values of x just less than and just greater than its value at the turning point.

Example 1 Find the nature of the turning points of the function $y = x^3 - 2x^2 + x + 4$.

$$\frac{dy}{dx} = 3x^2 - 4x + 1$$

At the turning points $dy/dx = 0$, that is

$$3x^2 - 4x + 1 = 0$$
$$(3x - 1)(x - 1) = 0$$
$$x = \tfrac{1}{3} \quad \text{or} \quad x = 1$$

Consider the value $x = \frac{1}{3}$. When $x = \frac{1}{4}$ (a convenient value just less than $\frac{1}{3}$)

$$\frac{dy}{dx} = 3 \times \tfrac{1}{16} - 4 \times \tfrac{1}{4} + 1 = \tfrac{3}{16}$$

When $x = \frac{1}{2}$ (a convenient value just greater than $\frac{1}{3}$)

$$\frac{dy}{dx} = 3 \times \tfrac{1}{4} - 4 \times \tfrac{1}{2} + 1 = -\tfrac{1}{4}$$

Thus, since dy/dx changes sign from positive to negative, when $x = \frac{1}{3}$, y is a maximum, the value of y being

$$\tfrac{1}{27} - \tfrac{2}{9} + \tfrac{1}{3} + 4 = \tfrac{112}{27}$$

For the value $x = 1$. When $x = 0{\cdot}9$

$$\frac{dy}{dx} = 3 \times 0{\cdot}81 - 4 \times 0{\cdot}9 + 1 = -0{\cdot}17 \quad \text{is negative}$$

When $x = 1{\cdot}1$

$$\frac{dy}{dx} = 3 \times 1{\cdot}21 - 4 \times 1{\cdot}1 + 1 = 0{\cdot}23 \quad \text{is positive}$$

Thus when $x = 1$, y is a minimum, the value of y being $1 - 2 + 1 + 4 = 4$.

A second procedure for distinguishing between maximum and minimum values may be obtained as follows. In the region of a maximum, $f'(x)$ changes sign from positive to negative as x increases. Thus $f'(x)$ is a decreasing function of x in the region so that $f''(x)$ is negative. Near a local minimum, $f'(x)$ is an increasing function so that $f''(x)$ is positive.

Hence at stationary points giving maximum values, $f''(x) < 0$; and at stationary points giving minimum values, $f''(x) > 0$.

If at a stationary point $f''(x) = 0$, *no conclusions* can be drawn using the above argument and we have to resort to our original criterion for distinguishing between maximum and minimum values.

For the example just considered.

$$f''(x) = \frac{d^2y}{dx^2} = 6x - 4$$

When $x = \frac{1}{3}$

$$\frac{d^2y}{dx^2} = 2 - 4 = -2 \quad \text{is negative}$$

When $x = 1$

$$\frac{d^2y}{dx^2} = 6 - 4 = 2 \quad \text{is positive}$$

Thus $x = \frac{1}{3}$ gives a maximum and $x = 1$ gives a minimum for y.

Example 2 Find the maximum and minimum values of $y = x^3 - 6x^2 + 9x$.

$$\frac{dy}{dx} = 3x^2 - 12x + 9$$

$$= 3(x^2 - 4x + 3) = 3(x - 1)(x - 3)$$

Therefore, $dy/dx = 0$ when $x = 1$ or $x = 3$ and these could give turning values.

$$\frac{d^2y}{dx^2} = 6x - 12$$

When $x = 1$

$$\frac{d^2y}{dx^2} = -6 < 0 \quad \text{giving a maximum value of 4 for } y$$

When $x = 3$

$$\frac{d^2y}{dx^2} = 18 - 12 = 6 > 0 \quad \text{giving a minimum value of 0 or } y$$

Exercise A

1 If $y = (x-1)(x+2)^2$, find the maximum and minimum values of y.

2 Find the maximum and minimum values of $y = x(x-1)^2$.

3 Find the maximum and minimum values of $y = \dfrac{x}{x^2+1}$.

4 Find the maximum and minimum values of $\sin t + \frac{1}{2}\cos 2t$.

11.6 Points of inflexion

Consider the function $y = x^3(x-4) = x^4 - 4x^3$.

$$\frac{dy}{dx} = 4x^3 - 12x^2 = 4x^2(x-3)$$

$$\frac{dy}{dx} = 0 \qquad \text{when} \quad x = 0 \quad \text{or} \quad x = 3$$

Near $x = 3$, dy/dx changes sign from negative to positive as x increases through the value 3. Thus $x = 3$ gives a minimum value of -27 for y. Near $x = 0$, dy/dx is negative for x just below zero, is zero when x is zero, and is negative again for x just greater than zero. Thus, although dy/dx is zero, since dy/dx does not change sign as x passes through this value, this point gives neither a maximum nor a minimum value for y.

Figure 11.9 shows the graph of the function $y = x^3(x-4)$. The sign of dy/dx is indicated on the graph.

At B the function has a minimum value and at the origin there is a point called a point of inflexion. At such a point, the graph of the function changes from being concave up to concave down or vice versa. (In our particular case, it is the former.) As the value of x increases through zero, the *derivative* changes from negative to zero, and then to negative again, i.e. at O the derivative has a maximum. This is true quite generally; at a point of inflexion the derivative has a maximum value or a minimum value. This latter condition enables us to give a criterion for finding points of inflexion, for at points at which dy/dx is a maximum or a minimum

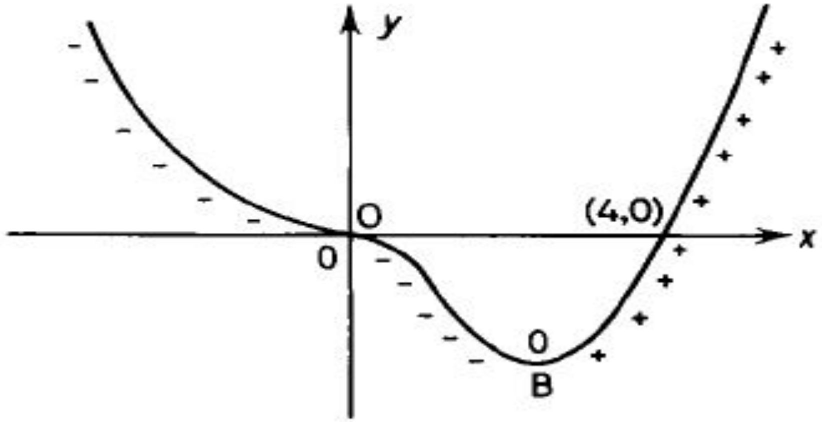

Figure 11.9

$$\frac{d^2y}{dx^2} = 0 \quad \text{and changes sign}$$

This is the case in our present example where

$$\frac{d^2y}{dx^2} = 12x^2 - 24x = 12x(x-2)$$

which vanishes and changes sign at the origin from positive to negative.

Although it is so in the example chosen, it is not necessary that dy/dx be zero at a point of inflexion (see *Figure 11.10*). In *Figure 11.10a*, the curve

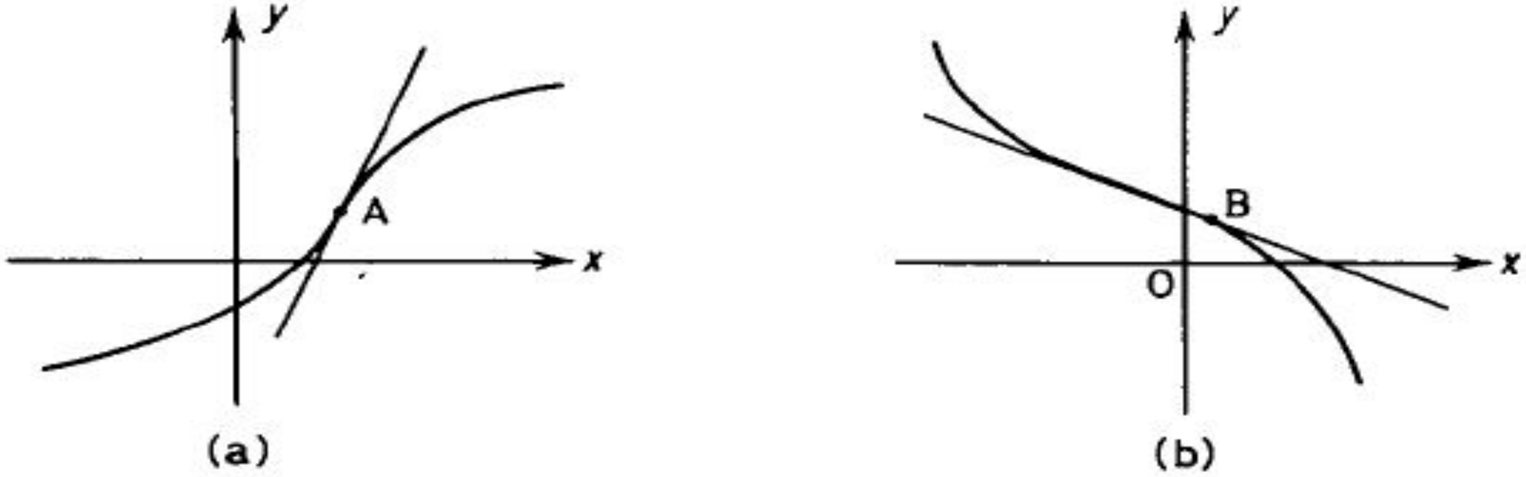

Figure 11.10

has a point of inflexion at A; dy/dx is a maximum. In *Figure 11.10b*, the curve has a point of inflexion at B; dy/dx is a minimum. In both cases, $d^2y/dx^2 = 0$ and changes sign but $dy/dx \neq 0$.

To sum up the results of this and the previous section:

If $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} < 0$, there is a maximum value.

If $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} > 0$, there is a minimum value.

If $\dfrac{d^2y}{dx^2} = 0$ *and changes sign*, there is a point of inflexion.

Example 1 Find the maximum and minimum values and the points of inflexion of $y = x^3 - 6x^2 + 9x + 1$.

$$\frac{dy}{dx} = 3x^2 - 12x + 9 = 3(x-1)(x-3)$$

therefore, dy/dx is zero when $x = 1$ or when $x = 3$

$$\frac{d^2y}{dx^2} = 6x - 12$$

When $x = 1$

$$\frac{d^2y}{dx^2} = -6 < 0$$

When $x = 3$

$$\frac{d^2y}{dx^2} = 6 > 0$$

$d^2y/dx^2 = 6x - 12$ is zero when $x = 2$ and changes sign. Thus

when $x = 1$ y has a maximum value of 5
when $x = 3$ y has a minimum value of 1

and there is just one point of inflexion at the point (2, 3).

Exercise B

1 Find the position of the point of inflexion of the curve $y = 2x^3 - 5x^2 - 4x + 1$.

2 Find the positions of the points of inflexion of the curve $y = 3x^4 - 4x^3 + 2$.

3 Find the positions of the turning values and the point of inflexion of the curve $y = x^3 - 2x^2 + x + 3$.

Reference

B. D. Bunday and H. Mulholland, First published 1967. Pure Mathematics for Advanced Level (Second Edition). Butterworth and Co (Publishers) Ltd, 1983.