DEPT OF MATHS FUNAAB 2021-2022 MTS 101

FURTHER NOTE ON APPLICATIONS OF PARTIAL FRACTIONS

Some problems in Mathematics involving rational functions or rational expressions are tractable by resolving the rational functions or rational expressions involved in the problems into partial fractions. Also, some problems in Mathematics are tractable simply by transforming them into partial fraction problems. Within the scope of MTS 101, the following applications will be considered.

1. Summation of series.
2. Series expansion of rational functions or rational expressions.

Further applications of partial fractions to Integration of rational functions will be discussed in MTS 102 in the second semester.

SUMMATION OF SERIES

Problem 1 Find the sum of the first n terms of the series

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots$$

and hence obtain S_∞, the sum to infinity of the series.

Solution: The series can be written as $\sum_{r=1}^{n}\frac{1}{r(r+1)}$. Resolving into partial fractions, we have

$$\begin{aligned}
\sum_{r=1}^{n}\frac{1}{r(r+1)} &= \sum_{r=1}^{n}\left[\frac{1}{r}-\frac{1}{r+1}\right]\\
&= \frac{1}{1}-\frac{1}{2} \qquad [r=1]\\
&+ \frac{1}{2}-\frac{1}{3} \qquad [r=2]\\
&+ \frac{1}{3}-\frac{1}{4} \qquad [r=3]\\
&+ \bullet \quad \bullet\\
&+ \frac{1}{n-2}-\frac{1}{n-1} \qquad [r=n-2]\\
&+ \frac{1}{n-1}-\frac{1}{n} \qquad [r=n-1]\\
&+ \frac{1}{n}-\frac{1}{n+1} \qquad [r=n]\\
&= 1-\frac{1}{n+1} \quad \text{[after cancelling diagonally and adding]}\\
&= \frac{n}{n+1}.
\end{aligned}$$

Lastly,

$$\begin{aligned}
S_\infty &= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{r(r+1)} \\
&= \lim_{n\to\infty}\left[\frac{n}{n+1}\right] \\
&= \lim_{n\to\infty}\left[\frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}}\right] \\
&= \lim_{n\to\infty}\left[\frac{1}{1+\frac{1}{n}}\right] \\
&= 1.
\end{aligned}$$

Problem 2 Show that

$$\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)(2r+5)} = \frac{1}{60} - \frac{1}{4(2n+3)(2n+5)}$$

and deduce that

$$\sum_{r=1}^{\infty}\frac{1}{(2r+1)(2r+3)(2r+5)} = \frac{1}{60}.$$

Solution: Splitting into partial fractions, we have

$$\begin{aligned}
\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)(2r+5)} &= \sum_{r=1}^{n}\left[\frac{1}{8(2r+1)} - \frac{1}{4(2r+3)} + \frac{1}{8(2r+5)}\right] \\
&= \frac{1}{8}\sum_{r=1}^{n}\left[\frac{1}{2r+1} - \frac{2}{2r+3} + \frac{1}{2r+5}\right] \\
&= \frac{1}{8}\left[\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right. \\
&+ \frac{1}{5} - \frac{2}{7} + \frac{1}{9} \\
&+ \frac{1}{7} - \frac{2}{9} + \frac{1}{11} \\
&+ \frac{1}{9} - \frac{2}{11} + \frac{1}{13} \\
&+ \bullet \quad \bullet \quad \bullet \\
&+ \frac{1}{2n-5} - \frac{2}{2n-3} + \frac{1}{2n-1} \\
&+ \frac{1}{2n-1} - \frac{2}{2n+1} + \frac{1}{2n+3} \\
&+ \frac{1}{2n+1} - \frac{2}{2n+3} + \frac{1}{2n+5} \\
&= \frac{1}{3} - \frac{1}{5} + \frac{1}{2n+5} - \frac{1}{2n+3} \qquad \text{[after cancelling diagonally]} \\
&= \left.\frac{2}{15} - \frac{2}{(2n+3)(2n+5)}\right] \\
&= \frac{1}{60} - \frac{1}{4(2n+3)(2n+5)}.
\end{aligned}$$

Lastly,

$$\begin{aligned}
\sum_{r=1}^{\infty} \frac{1}{(2r+1)(2r+3)(2r+5)} &= \lim_{n\to\infty}\left[\frac{1}{60} - \frac{1}{4(2n+3)(2n+5)}\right] \\
&= \frac{1}{60} - \lim_{n\to\infty}\left[\frac{1}{4(2n+3)(2n+5)}\right] \\
&= \frac{1}{60} - 0 \\
&= \frac{1}{60}.
\end{aligned}$$

Problem 3 Show that

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+3)} = \frac{7}{36} - \frac{3n+7}{6(n+1)(n+2)(n+3)},$$

and deduce that $\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+3)} = \frac{7}{36}$.

Solution: Resolving into partial fractions, we have

$$\begin{aligned}
\sum_{r=1}^{n} \frac{1}{r(r+1)(r+3)} &= \sum_{r=1}^{n}\left[\frac{1}{3r} - \frac{1}{2(r+1)} + \frac{1}{6(r+3)}\right] \\
&= \frac{1}{3}\sum_{r=1}^{n}\left[\frac{1}{r} - \frac{3}{2(r+1)} + \frac{1}{2(r+3)}\right] \\
&= \frac{1}{3}\left[\frac{1}{1} - \frac{3}{4} + \frac{1}{8}\right. \\
&+ \frac{1}{2} - \frac{3}{6} + \frac{1}{10} \\
&+ \frac{1}{3} - \frac{3}{8} + \frac{1}{12} \\
&+ \frac{1}{4} - \frac{3}{10} + \frac{1}{14} \\
&+ \bullet \quad \bullet \quad \bullet \\
&+ \frac{1}{n-2} - \frac{3}{2(n-1)} + \frac{1}{2(n+1)} \\
&+ \frac{1}{n-1} - \frac{3}{2n} + \frac{1}{2(n+2)} \\
&+ \frac{1}{n} - \frac{3}{2(n+1)} + \frac{1}{2(n+3)} \\
&= \frac{1}{2} - \frac{3}{4} + \frac{1}{3} - \frac{2}{2(n+1)} + \frac{1}{2(n+2)} + \frac{1}{2(n+3)} \quad \text{[after cancelling diagonally]} \\
&= \left.\frac{7}{12} - \frac{3n+7}{2(n+1)(n+2)(n+3)}\right] \\
&= \frac{7}{36} - \frac{3n+7}{6(n+1)(n+2)(n+3)}.
\end{aligned}$$

Lastly,

$$\begin{aligned}
\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+3)} &= \lim_{n\to\infty}\left[\frac{7}{36} - \frac{3n+7}{6(n+1)(n+2)(n+3)}\right] \\
&= \frac{7}{36} - \lim_{n\to\infty}\left[\frac{3n+7}{6(n+1)(n+2)(n+3)}\right] \\
&= \frac{7}{36} - 0 \\
&= \frac{7}{36}.
\end{aligned}$$

Problem 4 Find the constants A and B such that

$$\frac{Ar}{(r+1)^2} + \frac{B(r+1)}{(r+2)^2} \equiv \frac{r^2+r-1}{(r+1)^2(r+2)^2}.$$

Hence find

$$\sum_{r=1}^{n} \frac{r^2+r-1}{(r+1)^2(r+2)^2} \quad \text{and} \quad \sum_{r=1}^{\infty} \frac{r^2+r-1}{(r+1)^2(r+2)^2}.$$

Solution: We are given the identity

$$\begin{aligned}
\frac{Ar}{(r+1)^2} + \frac{B(r+1)}{(r+2)^2} &\equiv \frac{r^2+r-1}{(r+1)^2(r+2)^2} \\
\Rightarrow \quad Ar(r+2)^2 + B(r+1)^3 &\equiv r^2+r-1.
\end{aligned}$$

Setting $r = -1$, we have

$$-A = 1 - 1 - 1$$

from which we obtain

$$A = 1.$$

Setting $r = -2$, we have

$$-B = 4 - 2 - 1$$

from which we obtain

$$B = -1.$$

Hence,

$$\frac{r}{(r+1)^2} - \frac{r+1}{(r+2)^2} \equiv \frac{r^2+r-1}{(r+1)^2(r+2)^2}.$$

Now,

$$\begin{aligned}
\sum_{r=1}^{n}\left[\frac{r^2+r-1}{(r+1)^2(r+2)^2}\right] &= \sum_{r=1}^{n}\left[\frac{r}{(r+1)^2}-\frac{r+1}{(r+2)^2}\right]\\
&= \frac{1}{4}-\frac{2}{9}\\
&+ \frac{2}{9}-\frac{3}{16}\\
&+ \frac{3}{16}-\frac{4}{36}\\
&+ \frac{4}{25}-\frac{5}{49}\\
&+ \bullet \qquad \bullet\\
&+ \frac{n-2}{(n+1)^2}-\frac{n-1}{n^2}\\
&+ \frac{n-1}{(n)^2}-\frac{n}{(n+1)^2}\\
&+ \frac{n}{(n+1)^2}-\frac{n+1}{(n+2)^2}\\
&= \frac{1}{4}-\frac{n+1}{(n+2)^2} \qquad \text{[after cancelling diagonally]}
\end{aligned}$$

Lastly,

$$\begin{aligned}
\sum_{r=1}^{\infty}\frac{r}{(r+1)^2}-\frac{r+1}{(r+2)^2} &= \lim_{n\to\infty}\left[\frac{1}{4}-\frac{n+1}{(n+2)^2}\right]\\
&= \frac{1}{4}-\lim_{n\to\infty}\left[\frac{n+1}{(n+2)^2}\right]\\
&= \frac{1}{4}-0\\
&= \frac{1}{4}.
\end{aligned}$$

SERIES EXPANSION OF RATIONAL FUNCTIONS OR RATIONAL EXPRESSIONS

Problem 1 Obtain the first three terms of the expansion of

$$\frac{3x-5}{(x+3)(2x-1)}$$

in ascending powers of x.

Solution: Resolving into partial fractions, we have

$$\frac{3x-5}{(x+3)(2x-1)}=\frac{2}{x+3}+\frac{1}{1-2x}.$$

Now,

$$
\begin{aligned}
\frac{2}{x+3} &= \frac{2}{3}\left(1+\frac{x}{3}\right)^{-1} \\
&= \frac{2}{3}\left(1-\frac{x}{3}+\left(\frac{x}{3}\right)^2-\cdots\right) \quad \text{[Binomial Expansion]} \\
&= \frac{2}{3}-\frac{2}{9}x+\frac{2}{27}x^2-\cdots \qquad (1)
\end{aligned}
$$

Also,

$$
\begin{aligned}
\frac{1}{1-2x} &= (1-2x)^{-1} \\
&= 1+2x+4x^2+\cdots \qquad (2)
\end{aligned}
$$

Adding (1) and (2), we get

$$
\begin{aligned}
\frac{3x-5}{(x+3)(2x-1)} &= \left(\frac{2}{3}+1\right)+\left(2-\frac{2}{9}\right)x+\left(4+\frac{2}{27}\right)x^2+\cdots \\
&= \frac{5}{3}+\frac{16}{9}x+\frac{110}{27}x^2+\cdots
\end{aligned}
$$

INTEGRATION OF RATIONAL FUNCTIONS

Problem 1 Evaluate:

$$\int_2^3 \frac{x^2dx}{x^2-1}.$$

Solution: Resolving into partial fractions, we have

$$\frac{x^2}{x^2-1}=\frac{1/2}{x-1}-\frac{1/2}{x+1}.$$

Hence,

$$
\begin{aligned}
\int_2^3 \frac{x^2dx}{x^2-1} &= \frac{1}{2}\int_2^3\left[\frac{1}{x-1}-\frac{1}{x+1}\right]dx \\
&= \frac{1}{2}\left|\ln(x-1)-\ln(x+1)\right|_2^3 \\
&= \frac{1}{2}[\ln 2-\ln 4-\ln 1+\ln 3] \\
&= \frac{1}{2}[\ln 2-2\ln 2+\ln 3] \\
&= \frac{1}{2}[\ln 3-\ln 2] \\
&= \frac{1}{2}\ln(3/2) \\
&= \frac{1}{2}\ln 1.5
\end{aligned}
$$

FURTHER PRACTICE PROBLEMS

1. Find the constants A, B and C if:

   (a) $x^2 + 3x + 1 \equiv Ax(x+1) + B(x-1) + C.$

   (b) $x^2 \equiv A(x-1)(x-2) + B(x-2)(x-3) + C(x-1)(x-3).$

   (c) $4x^2 + 3x + 2 \equiv A((x+B)^2 + C).$

2. Obtain the first four nonzero terms in the expansion of p(x) where

$$p(x) = \frac{1}{(2x+1)(2x+3)(2x+5)}.$$

3. Given that

$$f(x) = \frac{x+1}{(2x-1)(2x+1)(2x+3)}.$$

   Show that

$$16f(x) = \frac{3}{2x-1} - \frac{2}{2x+1} - \frac{1}{2x+3}.$$

   Hence or otherwise, show that the sum of the first n terms of the series

$$\frac{2}{1 \times 3 \times 5} + \frac{3}{3 \times 5 \times 7} + \frac{4}{5 \times 7 \times 9} + \cdots$$

   is

$$\frac{5}{24} - \frac{4n+5}{8(2n+1)(2n+3)}$$

   and deduce that the sum to infinity of the series is 5/24.

4. A function f(x) is given by

$$f(x) = \frac{1 + 2x + 3x^2}{(1-x)(1+x^2)}.$$

   Show that

$$f(x) = \frac{3}{1-x} - \frac{2}{1+x^2}.$$

   Hence show that if x is so small that powers of x higher than the third may be neglected, the expansion of f(x) in ascending powers of x is

$$1 + 3x + 5x^2 + 3x^3.$$

5. Let

$$f(x) = \frac{x+2}{(1-x)(1+x^2)}.$$

   (a) Resolve f(x) into partial fractions.

   (b) Find the first three nonzero terms in the expansion of f(x).

6. Resolve

$$\frac{1}{x(x+1)(x+2)}$$

   into partial fractions and hence find the sum of the first n terms of the series

$$\frac{1}{1\times 2\times 3} + \frac{1}{2\times 3\times 4} + \frac{1}{3\times 4\times 5} + \cdots$$

   Deduce the sum to infinity of the series.

7. Given that

$$f(x) = \frac{1+x}{(1-x)(1+x^2)},$$

   show that

$$f(x) = \frac{1}{1-x} + \frac{1}{2}\frac{2x}{1+x^2}.$$

   Hence evaluate

$$\int_0^{3/4} f(x)dx$$

8. Find the constants A, B, C in the identity

$$\frac{3x^2-kx}{(x-2k)(x^2+k^2)} \equiv \frac{A}{x-2k} + \frac{Bx+kC}{x^2+k^2}$$

   where k is a constant. Hence show that

$$\int_0^k \frac{3x^2-kx}{(x-2k)(x^2+k^2)}dx = \frac{1}{4}\pi - \frac{3}{2}\ln 2.$$

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