$$t = \frac{2\sqrt{3} \pm \sqrt{12+4}}{2}$$

$$= \frac{2\sqrt{3} \pm \sqrt{16}}{2} = \frac{2\sqrt{3} \pm 4}{2}$$

$$= \sqrt{3} \pm 2$$

$$t = \sqrt{3} + 2 \quad \text{or} \quad \sqrt{3} - 2$$

But $\sqrt{3}-2$ is relative and tan $75^0$ must be positive, hence $t = \tan 75^0 = \sqrt{3} + 2$

01/08/2023.

## Real Valued Function of Real Variables

**What is a function?**

A function is a rule that establishes the relationships between two or more variables.

**What is a variable?**

A variable is a letter than stands in for numbers.

e.g $2x+1=0$.

The variable in the above case is $x$.

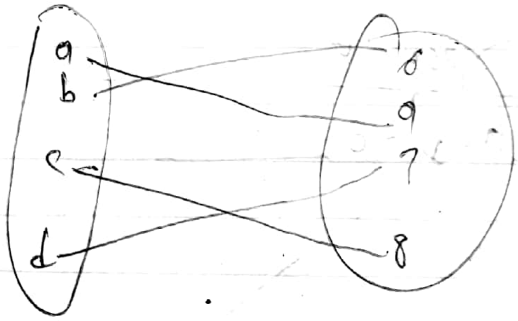
e.g If X and Y are arbitrary sets, ~~for every~~ $x \in Y$, there is assigned a unique element $f(x) = y \in Y$, then $f$ is called a function.

This is written as $f: X \to Y$, where the set $X$ is the domain of the function and $Y$ is the codomain of the function. The value of ~~set~~ $X$ is written as $f(x)$ and is called range or range of of a function. If the set X is actually a subset of $\mathbb{R}^n$, then we say that the function $f: \mathbb{R}^n \to Y$ is a function of several variables.

If the set $Y$ is $\mathbb{R}$ (set of real numbers), then we say say that $f: X \to \mathbb{R}$. The special cases of real values of single variable is a primary topic of discussion in the Calculus I and graphical methods do enhance intuition in attitude of surd function

**Definition**

A function whose range is a set of real number is called real value function. In other words, it is a function that assigns a real number to each member of each domain




Set D    Set $Y \in \mathbb{R}$

In the above example, ~~examp~~ every element in the set D is related to exactly one element in the set $Y \in \mathbb{R}$. The range $= \{8, 6, 7, 9\}$

The following are said to be real value functions since their range is a set of real numbers or some subsets of real numbers.

(i) $f(x) = 2$

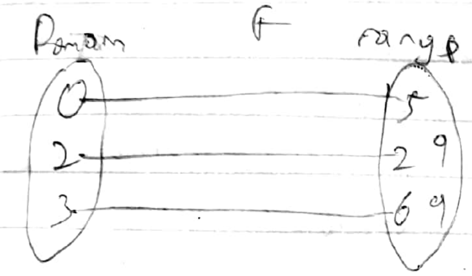
(ii) $f(x) = 1/x$, the range $\mathbb{R}$

(iii) $f(x) = e^x$

(iv) $f(x) = \frac{1}{x^2}$

Now given the domain ~~of~~ a function, we find the range of the function $g(t) = 6t^2 + 5$ at $t = 0$ to $t = 3$.

$g(t) = 6t^2 + 5$
at $t = 0$, $g(t) = 6(0)^2 + 5 = 5$
at $t = 1$, $g(t) = 6(1)^2 + 5 = 11$
at $t = 2$, $g(t) = 6(2)^2 + 5 = 29$
at $t = 3$, $g(t) = 6(3)^2 + 5 = 59$




Therefore, the domain $= \{0, 2, 3\}$
and the range $= \{5, 29, 69\}$

## Types of Functions

**(1) One-to-one function (Injective):**
This is a type of function that each element in the domain has a distinct imaged in the co-domain.
In another way, a function f from XY is said to be one-to-one or injective if $f(x_1) = f(x_2)$ implies that $x_1 = x_2$

~~Rectorally~~,
For instance, let $f: R \to R$ ($R$ is a set of real numbers) be a function defined by $f(x) = 3x+1$. Is f a one-to-one function?
$f(x) = f(y) = 3x + 1 = 3y + 1$
$\Rightarrow 3x + 1 - 3y - 1 = 0$
$\Rightarrow 3x = 3y$

For any $x, y \in R$,
$\Rightarrow x = y$

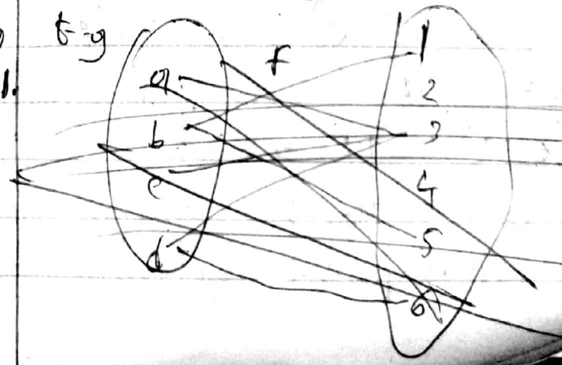
$\therefore \Rightarrow f(x) = f(y) = x = y$

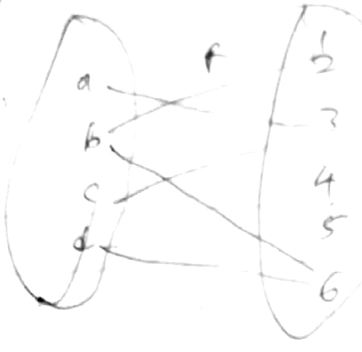
**(2) Unto function (Surjective):** This is a function that every element in the co-domain has at least one pre-image in the domain.

**Example:** Let f map R to R ($R$ is a set of real numbers) be a function defined by $f(x) = 3x + 1$.
If f a surjective function.
Let $y \in R$ be arbitrarily.
$\Rightarrow f(x) = f(y) \Rightarrow x = y$
$3x + 1 = y$
$x = \frac{y-1}{3}$
$f(x) = 3x + 1 = 3\left(\frac{y-1}{3}\right) + 1$
$= y$.
Hence, f is surjective.

**(3) Bijective function:** A function is said to be bijective when it is both one-to-one and unto (injective and surjective).

**(4) Many-to-one function:** A function is said to be many-to-one if there are at least two elements in the domain whose images are the same.
E.g






**Example:** A function $f: R \to R$ defined by $f(x) = x^2 + 1$

$f(1) = 1^2 + 1 = 2$
$f(-1) = (-1)^2 + 1 = 2$
$f(2) = 2^2 + 1 = 5$
$f(-2) = (-2)^2 + 1 = 5$

$f(1)$ and $f(-1)$ will have same image. Same for $f(2)$ and $f(-2)$. These are examples of many to one mapping.

1. If $X, Y$ be any two sets, which of the following relations $f: X \to Y$ is a function:
   when $f$ is a function, determine whether it is injective, surjective or bijective.

(i) $X = \{2, 6, 8, 12\}$, $Y = \{5, 12, 7, 10\}$, $f = \{(2,a), (6,b), (12,6)\}$

(ii) $X = \{a, b, c, d\}$, $Y = \{5, 12, 7, 10\}$ $f = \{(a,7), (b,12), (c,10), (d,5)\}$

(iii) $X = \{6, 4, 7, 11\}$, $Y = \{a, b, c\}$, $f = \{(6,a), (4,c), (7,b), (11,c)\}$

(iv) $X = Y = R = \{$the set of real numbers$\}$ $f(x) = \{x, 3x+5 : x \in X\}$

(v) $X = \{$set of integer$\}$
$Y = \{$set of positive integers$\}$

$f = \{(n, x^2 : n \in X)\}$

2. Let $f: R \to R$, $g: R \to R$ be defined by $f(x) = x^2$, if $x \geq 0$ or $2$, if $x < 0$, find

(i) $g(f(0))$ (ii) $g(f(-1))$ (iii) $g(f(x) : x \geq 0)$ (iv) $g(f(x) : x, 0)$

3. Let $x = \{(-1, 1)\}$ and let $f: X \to R$, $g: X \to R$, $h: X \to R$ be defined by $f(x) = \sin x$, $g(x) = \sin x$, $h(x) = \log x$. Examine whether:

(i) $f$ is injective, surjective
(ii) $g$ is injective, surjective
(iii) $h$ is injective, surjective

## Discontinuity

A function that fails any of the conditions of continuity is said to be discontinuous. Discontinuous function can either be removable or not removable

**Example 2:** Determine the nature of discontinuity of the function

$$f(x) = \frac{x-2}{x^2-4}$$

This function has 2 discontinuities as $x^2 - 4 = 0$ implies $x = 2$ or $x = -2$. However $f(x) = \frac{x-2}{x^2-4}$ is equal to $\frac{(x-2)}{(x-2)(x+2)}$

However, $f(x) =$

With $f(x) = \frac{1}{x+2}$, we have recovered one of the discontinuities, that is $x=2$ and we are left with $x=-2$, which is not removeable

## Exercise

(1) Determine whether there are any discontinuities in the following and if so, where did they occur

(i) $f(x) = \frac{x\pi}{x^3}$

(ii) $f(x) = x^2 - 3x + 2$

(iii) $f(x) = \frac{25x^2}{24-8x}$

(iv) $f(x) = (x^2+4)^{-1}$

(v) $f(x) = \frac{x^3 - 3x^2 + x + 1}{x^2+6}$

(vi) $f(x) = \frac{1}{x-1}$

04/09/2023

## DIFFERENTIAL CALCULUS

Differential calculus is a branch of pure mathematics established by Isaac Newton and Leibintz in the course of solving mechanical/celestial problems.

Differential calculus deals with determining the rate of a variable with respect to another variable.

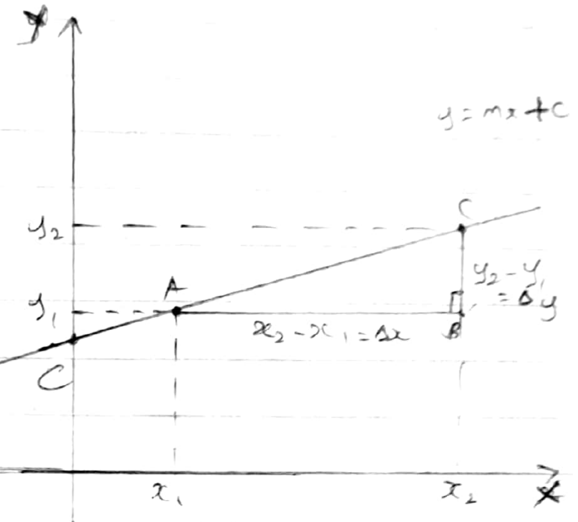
The first variable might be $y$ while the second $x$.

We have:

- Dependent variables
- Independent variables.

## Differentiation Using the first Principle:

This is the principle used by Isaac Newton and Leibntz.



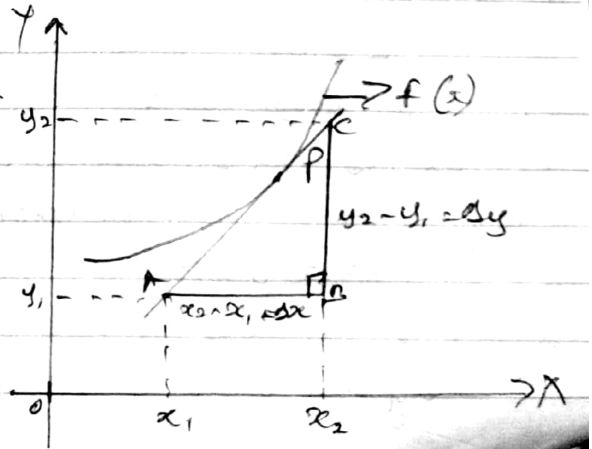

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Gradient of a line is the tangent on the angle that the line makes with the positive direction of the axis.

Slope = $\tan\theta$

Suppose $y$ is a function of $x$, i.e $y = f(x)$. ~~Suppose~~

Suppose the graph of $y$ is sketched below



To determine the slope of the line at point P, draw a tangent to the curve at point P

$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, the value we got is called the derivative of with with respect to $x$

OR

the differential coefficient of $y$ with respect to $x$.

This is denoted by :

(1) $\frac{dy}{dx}$ (2) $\frac{d}{dx}(y)$

(3) $y'(x)$ or $f'(x)$

Note: $\frac{d}{dx}$ is called a differential operator. When it comes across a function, it finds the derivative.

Therefore, $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

When the differential coefficient (derivative) of $y$ with respect to $x$ is obtained/gotten, then we say that we have gotten it using the first principle.

Examples

(1) If $y = x - 5$, find the derivative of $y$ with respect to $x$ using the principle.

Solution

$y$ is a function of $x$

Let $\delta x$ and $\delta y$ be small changes in $x$ and $y$ respectively, therefore, we have:

$y + \delta y = (x + \delta x) - 5$

$\cancel{x} - \cancel{5} + \delta y = \cancel{x} + \delta x - \cancel{5}$

$\delta y = \delta x$.

Divide both sides by $\delta x$

$\frac{\delta y}{\delta x} = \frac{\delta x}{\delta x} = 1$

$\therefore \frac{\delta y}{\delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

$= \lim_{\Delta x \to 0} 1 = 1$

$\therefore \frac{dy}{dx} = 1$

(2) If $y = x^2 - 2$

Let $\delta y$ and $\delta x$ be small changes in $y$ and $x$ respective so that

$y + \delta y = (x + \delta x)^2 - 2$

$\cancel{x^2} - \cancel{2} + \delta y = \cancel{x^2} + 2x\delta x + (\delta x)^2 - \cancel{2}$

$\delta y = 2x\delta x + (\delta x)^2$

Divide both sides by $\delta x$

$\frac{\delta y}{\delta x} = \frac{2x\delta x}{\delta x} + \frac{(\delta x)^2}{\delta x}$

$\frac{\delta y}{\delta x} = 2x + \delta x$

$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0} (2x + \delta x)$

$= 2x$

(3) Find, using the first principle, the derivative of $y$ with respect to $x$ of $y = x^3$

Solution

Let $\delta y$ and $\delta x$ be small changes in $y$ and $x$ respectively. Therefore:

$y + \delta y = (x + \delta x)^3$

$x^3 + \delta y = x^3 + 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3$

$\Rightarrow \delta y = 3x^2\delta x + 3x(\delta x)^2 + (\delta x)^3$

Divide through by $\delta x$

$\frac{\delta y}{\delta x} = 3x^2 + 3x\delta x + \delta x^2$

$\frac{dy}{dx} = \lim_{\Delta x \to 0} = \frac{\delta y}{\delta x}$

$= 3x^2$

④ If $y = \frac{1}{x}$, find $\frac{dy}{dx}$ using the first principle.

Solution

$y = \frac{1}{x}$

Let $\delta y$ and $\delta x$ be small changes in $y$ and $x$ respectively, we have:

$$y + \delta y = \frac{1}{x + \delta x}$$

$$\delta y = \frac{1}{x + \delta x} - \frac{1}{x}$$

$$\delta y = \frac{x - (x + \delta x)}{x(x + \delta x)}$$

$$= \frac{x - x - \delta x}{x(x + \delta x)}$$

$$\delta y = \frac{-\delta x}{x(x + \delta x)}$$

$$\cancel{\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\delta y}{}}$$

Divide both sides by $\delta x$

$$\frac{\delta y}{\delta x} = \frac{-1}{x(x + \delta x)}$$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{-1}{x(x + \delta x)}$$

$$= \frac{-1}{x^2} = -x^{-2}$$

~~Classwork~~

⑤ If $y = C$, where $C$ is a scalar constant, show, using the first principles that $\frac{dy}{dx} = 0$

Solution

Let $\delta y$ be a small increment in $y$. Then

$$y + \delta y = C$$

$$C + \delta y = C$$

$$\Rightarrow \delta y = 0$$

Divide both sides by $\delta x$

$$\frac{\delta y}{\delta x} = \frac{0}{\delta x} = 0$$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\delta y}{\delta x}$$

$$= 0$$

$\frac{1}{2}x^{-\frac{1}{2}}$

$\frac{1}{2x^{\frac{1}{2}}}$

Classwork

Using the first principle, find the derivative of $y$ WRT $x$ given that:

① $y = \sqrt{x}$ ② $y = \frac{1}{x-1}$

③ $y = \sin x$

Solution.

① $y = \sqrt{x}$

$y = x^{\frac{1}{2}}$

Let $\delta y$ and $\delta x$ be small changes in $y$ and $x$ respectively

$$y + \delta y = \sqrt{(x + \delta x)}$$

$$\delta y = \sqrt{(x + \delta x)} - \sqrt{x}$$

$$\delta y = \frac{\left[\sqrt{(x+\delta x)} - \sqrt{x}\right]}{1}$$

Multiply numerator and denominator by conjugate of $\left[\sqrt{(x+\delta x)} - \sqrt{x}\right]$

$$\delta y = \frac{\left(\sqrt{(x+\delta x)} - \sqrt{x}\right)}{1} \times \frac{\sqrt{(x+\delta x)} + \sqrt{x}}{\sqrt{(x+\delta x)} + \sqrt{x}}$$

$$\delta y = \frac{(x + \delta x) - x}{\sqrt{x + \delta x} + \sqrt{x}} = \frac{x - x + \delta x}{\sqrt{x + \delta x} + \sqrt{x}}$$

$$\delta y = \frac{\delta x}{\sqrt{x + \delta x} + x}$$

Divide both sides by $\delta x$

$$\frac{\delta y}{\delta x} = \frac{1}{\sqrt{x + \delta x} + x}$$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \delta x} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

(2) $y = \frac{1}{x-1}$

Let $\delta y$ and $\delta x$ be small changes in $y$ and $x$ respectively.

We have:

$$y + \delta y = \frac{1}{(x+\delta x)-1}$$

$$\delta y = \frac{1}{x+\delta x-1} - \frac{1}{x-1}$$

$$\delta y = \frac{(x-1)-(x+\delta x-1)}{(x+\delta x-1)(x-1)}$$

$$\delta y = \frac{\cancel{x}-\cancel{1}-\cancel{x}-\delta x+\cancel{1}}{(x+\delta x-1)(x-1)}$$

$$\delta y = \frac{-\delta x}{(x+\delta x-1)(x-1)}$$

Divide both sides by $\delta x$

$$\frac{\delta y}{\delta x} = \frac{-1}{(x+\delta x-1)(x-1)}$$

$$\therefore \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{-1}{(x+\delta x-1)(x-1)}$$

$$= \frac{-1}{(x-1)(x-1)} = \cancel{\frac{-1}{2(x-1)}}$$

$$= \cancel{\frac{-1}{2x+2}} = \frac{-1}{(x-1)^2}$$

(3) $y = \sin x$

Let $\delta y$ and $\delta x$ be little changes in $y$ and $x$ respectively.

$$y + \delta y = \sin(x+\delta x)$$

$$\delta y = \sin(x+\delta x) - \sin x$$

Using trigonometric identity

$$\delta y = [\sin x \cos(\delta x) + \cos x \sin(\delta x)] - \sin x$$

$$\delta y = \sin x \cos(\delta x) + \cos x \sin(\delta x) - \sin x$$

$$\delta y = \sin x \cos(\delta x) - \sin x + \cos x \sin(\delta x)$$

$$\delta y = \sin x[\cos(\delta x) - 1] + \cos x \sin(\delta x)$$

$$y = f(x) = x^2$$

$$\frac{f(x+h) - f(x)}{h}$$

$$= \frac{(x+h)^2 - x^2}{h}$$

$$= \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= 2x + h$$

$$\frac{dy}{dx} = f'(x)$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} (2x + h)$$

$$= 2x$$

## Techniques of Differentiation

If $y = f(x)$ ($y$ is a function of $x$), then $\frac{dy}{dx} = \frac{d}{dx}(y) = f'(x)$ can be obtained using different techniques, depending on the nature of $f(x)$.

① **Product rule**

If $y = UV$

where $U$ and $V$ are functions of $x$, then $\frac{dy}{dx} = V\frac{du}{dx} + U\frac{dv}{dx}$

Let $\delta y$, $\delta u$ and $\delta v$ be small increments in $y$, $u$ and $v$ respectively, we have

$$y + \delta y = (u + \delta u)(v + \delta v)$$

$$\Rightarrow \delta y = (u + \delta u)(v + \delta v) - uv$$

$$= uv + u\delta v + v\delta u - \delta u \delta v - uv$$

$$\delta y = u\delta v + v\delta u + \delta u \delta v$$

Divide through by $\delta x$

$$\frac{\delta y}{\delta x} = U\frac{\delta v}{\delta x} + V\frac{\delta u}{\delta x} + \frac{\delta u \delta v}{\delta x}$$

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$= U\frac{dv}{dx} + V\frac{du}{dx}$$

**Example:**

If $y = 2x^2(x^3 + 1)$

Put $U = 2x^2$

$V = x^3 + 1$

$$\frac{dy}{dx} = U\frac{dv}{dx} + V\frac{du}{dx}$$

$$= (2x^2)(3x^2) + (x^3+1)(4x)$$

$$= 6x^4 + 4x^4 + 4x$$

$$= 10x^4 + 4x$$

## Derivatives of some elementary functions: Circular functions

1. If $y = e^x$, then $\frac{dy}{dx} = e^x$.
2. If $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$
3. If $y = \sin x$, then $\frac{dy}{dx} = \cos x$
4. If $y = \cos x$, then $\frac{dy}{dx} = -\sin x$
5. If $y = \tan x$, then $\frac{dy}{dx} = \sec^2 x$

These are called hyperbolic functions

1. $\sinh x = \frac{1}{2}(e^x - e^{-x})$
2. $\cosh x = \frac{1}{2}(e^x + e^{-x})$
3. $\tanh x = \frac{\sinh x}{\cosh x} = \frac{(e^x - e^{-x})}{(e^x + e^{-x})}$

11/02/2023

From $\tanh x = \frac{\sinh x}{\cos x}$

$= \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

Divide num and den by $e^{-x}$

$= \frac{\cancel{e^{x} - e^{-x}}}{\cancel{e^{-x}}}$

$= \frac{e^{2x} - 1}{e^{2x} + 1}$

Example(s)

① Find $\frac{dy}{dx}$ of $y = e^{x}\ln x$

Let $u = e^{x}$, $v = \ln x$

$\frac{du}{dx} = e^{x}$, $\frac{dv}{dx} = \frac{1}{x}$

$\therefore \frac{dy}{dv} = V\frac{du}{dx} + U\frac{dv}{dx}$

$= \ln x(e^{x}) + e^{x}\left(\frac{1}{x}\right)$

$= e^{x}\left(\ln x + \frac{1}{x}\right)$

## Techniques of Differentiation

② Quotient Technique/Rule:

If $y = \frac{u}{v}$ where $v \neq 0$, then

$$\frac{dy}{dx} = \frac{V\frac{du}{dx} - U\frac{dv}{dx}}{V^2}$$

Using the first principles, let $\delta y$, $\delta u$ and $\delta v$ be small changes in y, u and v respectively. Therefore, we have:

$\delta y = \frac{u + \delta u}{v + \delta v}$

$\delta y = \frac{u + \delta u}{v + \delta v} - \frac{u}{v}$

$\delta y = \frac{v(u + \delta u) - u(v + \delta v)}{v(v + \delta v)}$

$\delta y = \frac{uv + v\delta u - uv - u\delta v}{v(v + \delta v)}$

$\delta y = \frac{v\delta u - u\delta v}{v(v + \delta v)}$

Divide both sides by $\delta x$

$\cancel{\frac{\delta y}{\delta x} = \frac{v\delta u - u\delta v}{v}}$

$\frac{\delta y}{\delta x} = \frac{v\frac{\delta u}{\delta x} - U\frac{\delta v}{\delta x}}{v(v + \delta v)}$

$\therefore \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$

$=$

Example: ① Find $\frac{dy}{dx}$ if $y = \frac{x+1}{x-1}$

Let $u = x + 1$, $v = x - 1$

$\frac{du}{dx} = 1$, $\frac{dv}{dx} = 1$

$\frac{dy}{dx} = \frac{V\frac{du}{dx} - U\frac{dv}{dx}}{V^2}$

$= \frac{1(x-1) - 1(x+1)}{(x-1)^2}$

$= \frac{x - 1 - x - 1}{(x-1)^2} = \frac{-2}{(x-1)^2}$

② From the definition of $\sinh x$ and $\cosh x$, show that $\cosh^2 x - \sinh^2 x = 1$ and deduce that: (i) $1 - \tanh^2 x = \operatorname{sech}^2 x$

(ii) $\coth^2 x - 1 = \operatorname{cosech}^2 x$

Solution

Put the LHS $= \cosh^2 x - \sinh^2 x$

$$= \frac{1}{4}(e^x+e^{-x})^2 - \frac{1}{4}(e^x - e^{-x})^2$$

$$= \frac{1}{4}\left[e^{x}+e^{-x}-e^{x}+e^{-x}\right]$$

$$\cancel{= \frac{1}{4}(2e^{x}) \frac{1}{4}}$$

$$= \frac{1}{4}\left[(e^x+e^{-x}+e^x-e^{-x})(e^x+e^{-x}-e^x+e^{-x})\right]$$

$$= \frac{1}{4}\left[(2e^{x})(2e^{-x})\right]$$

$$\cancel{= \frac{1}{4}[e^{x} e^{-x}]} = \frac{1}{4} \times 4 \times e^{x} \times e^{-x}$$

$$= 1 = RHS$$

(i) Dividing both sides by $\cosh^2 x$, then we obtain

$$\frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

(ii) Dividing both sides by $\sinh^2 x$, then we obtain

$$\frac{\cosh^2 x}{\sinh^2 x} - \frac{\sinh^2 x}{\sinh^2 x} = \frac{1}{\sinh^2 x}$$

$$\coth^2 x - 1 = \operatorname{cosech}^2 x$$

~~If $y = \sinh x \cosh x$, find $\frac{dy}{dx}$~~

U.

Find the derivation of the following WRT to $x$:

(1) $\sinh x$ (2) $\cosh x$ (3) $\tanh x$

Solution

(1) By definition, $y = \sinh x = \frac{1}{2}(e^x - e^{-x})$

$$\therefore \frac{dy}{dx} = \frac{1}{2}(e^x + e^{-x})$$

$$= \cosh x$$

(2) $y = \cosh x$

By definition, $\cosh x = \frac{1}{2}(e^x + e^{-x})$

$$\therefore \frac{dy}{dx} = \frac{1}{2}(e^x - e^{-x})$$

$$= \sinh x$$

(3) $y = \tanh x$

By definition, $\tanh x$

$$= \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{u}{v}$$

$$\frac{du}{dx} = e^x + e^{-x} \qquad \frac{dv}{dx} = e^x - e^{-x}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(e^x+e^{-x})(e^x+e^{-x}) - (e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2}$$

$$= \frac{(e^x+e^{-x})^2 - (e^x-e^{-x})^2}{(e^x+e^{-x})^2}$$

$$\cancel{= \frac{(e^x-e^{-x})^2}{(e^x+e^{-x})^2}}$$

$$= \frac{(e^x+e^{-x}+e^x-e^{-x})(e^x+e^{-x}-e^x+e^{-x})}{(e^x+e^{-x})^2}$$

$$= \frac{(2e^x)(2e^{-x})}{(e^x+e^{-x})^2} = \frac{4}{(e^x+e^{-x})^2}$$

$$= \frac{1}{\left[\frac{1}{2}(e^x+e^{-x})\right]^2}$$

$$= \frac{1}{\cosh^2 x}$$

$$= \operatorname{sech}^2 x$$

Techniques of Differentiation (Contd)

(3) Function of a Function (Chain Rule)

~~Derivation of function of a function~~

This is the most important of all the techniques of differentiation. It is also known as Derivative of Composite Functions

If $y = f(u)$ and $u = f(x)$,
then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Suppose we move from y to u, from u to v, from v to s, s to t and then, t to x

$y \to u \to v \to s \to t \to x$

with this, we have:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{ds} \times \frac{ds}{dt} \times \frac{dt}{dx}$$

Example: Differentiate the ffg ~~with~~ WRT to x:

1) $y = e^{-x}$ 2) $y = f(x \ln x)$
3) $y = \sinh^2 x$

Solutions

1) put $u = -x$, $\frac{du}{dx} = -1$

i.e $y = e^u$, $\frac{dy}{du} = e^u = e^{-x}$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= e^{-x} \times -1$

$= -e^{-x}$

2) Let $m = x \ln x$

$\therefore y = f(m)$

$\frac{dy}{dm} = f'(m)$, $\frac{dm}{dx} = 1 + \ln x$

$\frac{dy}{dx} = \frac{dy}{dm} \times \frac{dm}{dx}$

$= f'(x \ln x)(1 + \ln x)$

$= (1 + \ln x) \cdot f'(x \ln x)$

$y = \sinh^2 x$

Let $q = \sinh x$

$y = q^2$

$\frac{dy}{dq} = 2q$, $\frac{dq}{dx} = \cosh x$

$\frac{dy}{dx} = \frac{dy}{dq} \times \frac{dq}{dx}$

$= 2q \times \cosh x$

$= 2 \sinh x \cosh x$

Note that:

1) $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

2) $\frac{d}{dx}(y) = \frac{d}{du}(y) \cdot \frac{du}{dx}$

Differentiation of Implicit Function

If $y = f(x)$, then we say that we have expressed y explicitly in terms of x.

Example: Find $\frac{dy}{dx}$ if $x^2 + xy^2 + 1 = 0$

Solution

Differentiate both sides WRT x.

$\frac{d}{dx}(x^2 + xy^2 + 1) = \frac{d}{dx} 0$

$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(xy^2) + \frac{d}{dx}(1) = \frac{d}{dx}(0)$

$\Rightarrow \cancel{2x + [y^2 + 2xy] + 0 = 0}$

$\Rightarrow \cancel{2x + y^2 +}$

$\Rightarrow 2x + y^2 + x\frac{d}{dx}(y^2) + 0 = 0$

$\Rightarrow 2x + y^2 + x\frac{d}{dy}(y^2)\frac{dy}{dx} = 0$

$\Rightarrow 2x + y^2 + 2xy\frac{dy}{dx} = 0$

$\Rightarrow \therefore \frac{dy}{dx} = \frac{-y^2 \cancel{-2x}}{2xy}$

$\frac{dy}{dx} = \frac{-(y^2 + 2x)}{2xy}$

## Classwork

Find $\frac{dy}{dx}$ given that $x^y = y^x$

log both sides

$\log x^y = \log y^x$

$y \log x = x \log y$

$y \ln x = x \ln y$

Applying product rule to both sides

$\ln x + \left(y + \frac{1}{x}\right) = \ln y + \frac{1}{y}\frac{dy}{dx}$

$\ln x + \frac{xy + 1}{x} = \ln y + -y^{-2}$

18/09/2023

## Application of Differential Calculus

* Tangents/Normals to curves
* Increasing/Decreasing functions
* Approximation
* Rate of change
* Rectilinear motion
* Maximum & minimum values
* Curve sketching

### Tangents/Normals to Curves

The derivative of y w.r.t x at $x = x_1$ is denoted by:

$$\left.\frac{dy}{dx}\right|_{x=x_1}$$

If m is the gradient of the tangent to a curve at the point $x = x_1$, then $m = \left.\frac{dy}{dx}\right|_{x=x_1}$

**Recall:-**

The equation of the line of gradient, m through $(x_1, y_1)$ is

$$\boxed{y - x_1 = m(x - x_1)} \quad \text{——} \; (1)$$

The straight line perpendicular to the tangent at the point of contact on the tangent to the curve is called the normal to the curve.

If m' is the gradient of the normal, then $m' = \frac{-1}{m}$

~~The~~

The equation of the normal at the point $(x_1, y_1)$ is:

$$\boxed{y - y_1 = \frac{-1}{m}(x - x_1)} \quad (2)$$

## Increasing/Decreasing Functions

### Increasing functions

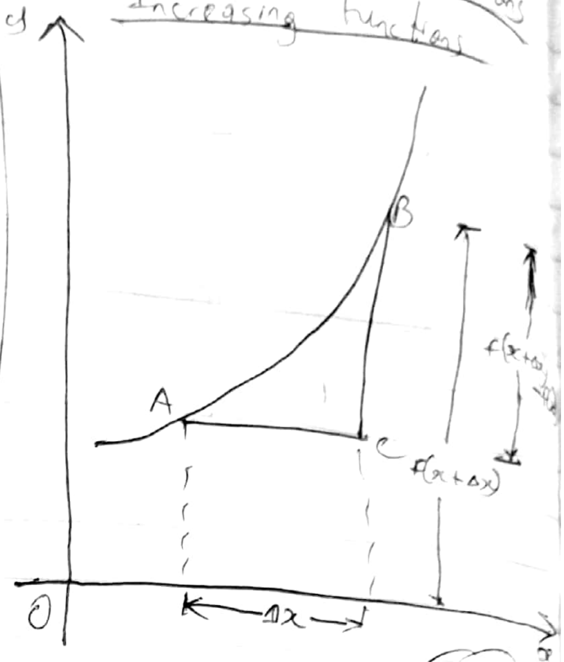



Fig. 1

Fig. 1 shows part of the curve $y = f(x)$. As $x$ increases, $y$ also increases.

Since $f(x + \Delta x) - f(x) \geq 0$ and as $x > 0$

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} > 0$$

Hence

$$\lim_{\Delta x \to 0} \left\{ \frac{f(x + \Delta x) - f(x)}{\Delta x} \right\} \geq 0$$

By definition

$$\lim_{\Delta x \to 0} \left\{ \frac{f(x + \Delta x) - f(x)}{\Delta x} \right\} = \frac{dy}{dx}$$

Hence if $y = f(x)$ is an increasing function at a given interval, then

$$\boxed{\frac{dy}{dx} \geq 0}$$

~~De~~

**Examples:**

(1) Find the equation of the tangent and the normal to the curve $y = 2x^3 - x^2 + 3x + 1$ at the point $x = 1$

Solution

$$y = 2x^3 - x^2 + 3x + 1$$

$$\frac{dy}{dx} = 6x^2 - 2x + 3$$

$$m = \left.\frac{dy}{dx}\right|_{x=1} = 6(1)^2 - 2(1) + 3$$

$$= 6 - 2 + 3$$

$$= 7$$

If $m$ is the gradient of the tangent at $x = 1$, then $m = 7$

~~Howe~~

However, when $x = 1$, $y = 5$

Then, the equation of the tangent at $x = 1$ is:

$$y - 5 = 7(x - 1)$$

$$y = 7x - 2$$

Hence, the equation of the normal at the point $x = 1$ is

$$y - 5 = \frac{-1}{7}(x - 1)$$

$$7y + x - 36 = 0$$

# Decreasing Functions

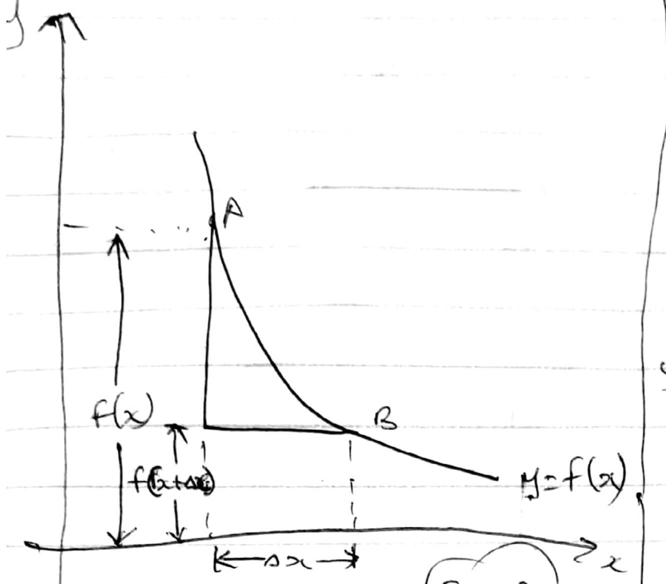



Fig. 2

## Examples:

(2) Find the range of the values of $x$ for which each of the following is increasing or decreasing

(a) $\frac{x^3}{3} - \frac{x^2}{2} - 2x$

(b) $\frac{x^2}{2} - 5x + 1$

### Solution

(1) Let $y = \frac{x^3}{3} - \frac{x^2}{2} - 2x$

$\frac{dy}{dx} = x^2 - x - 2$

$= (x+1)(x-2)$

$\therefore y = \frac{x^3}{3} - \frac{x^2}{2} - 2x$ is increasing since $\frac{dy}{dx} \geq 0$

$\frac{dy}{dx} = (x+1)(x-2) \geq 0$

$\frac{dy}{dx} \geq 0$ when $\underline{x < -1 \text{ and } x \geq 2}$

(b) Let $y = \frac{x^2}{2} - 5x + 1$

$\frac{dy}{dx} = x - 5$

$y$ is decreasing when $\frac{dy}{dx} < 0$

$x - 5 < 0$

$\Rightarrow x < 5$

# Approximations

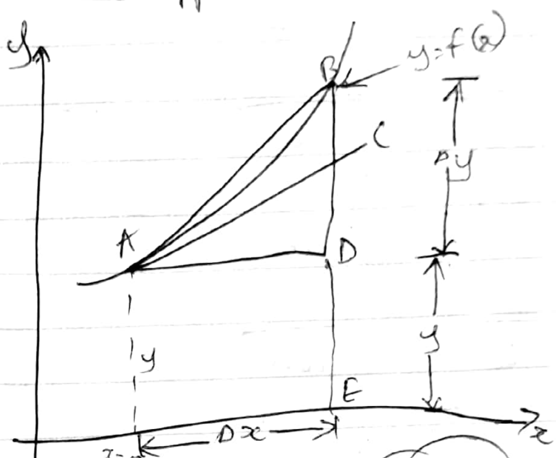



Fig. 3

Fig. 3 shows part of the curve $y = f(x)$

Gradient of the tangent $AB = \frac{\Delta y}{\Delta x}$

Gradient of the tangent at $A$

$= \left.\frac{dy}{dx}\right|_{x=a}$

If $\Delta x \to 0$, $CD = BD$

$$\boxed{\frac{\Delta y}{\Delta x} \simeq \left.\frac{dy}{dx}\right|_{x=a}}$$

Thus, $\frac{\Delta y}{\Delta x} \simeq \frac{dy}{dx}$

Hence, $\Delta y \simeq \frac{dy}{dx} \times \Delta x$

**Example 3:**

If a side of a square increases by 0.5%, find the approximate % increase in the area.

Solution

Let A be the area of the square, then $A = x^2$

$$\frac{dA}{dx} = 2x$$

$$\Delta A = \frac{dA}{dx} \cdot \Delta x$$

$$\approx 2x\,\Delta x$$

$$\frac{\Delta A}{A} = \frac{2x\,\Delta x}{x^2} = \frac{2\Delta x}{x}$$

Put $\frac{\Delta x}{x} = \frac{0.5}{100} = \frac{5}{1000}$

$$\frac{\Delta A}{A} = 2 \times \frac{5}{1000}$$

$$\frac{\Delta A}{A} = \frac{1}{100}$$

$\therefore$ % increase in A =

$$\frac{\Delta A}{A} \times \frac{100}{1}$$

$$= \frac{1}{100} \times 100\,\%$$

$$= 1\,\%$$

Hence, % increase in area of the square is 1 %

20/01/2023

## Rate of Change

If $y = f(x)$, $\frac{dy}{dx}$ can sometimes be introduced as the rate at which y is changing with respect to x. If y increases as x increases, $\frac{dy}{dx} > 0$, while if y decreases as x increases, $\frac{dy}{dx} < 0$.

**Example 4:** The radius of a circle is increasing at the rate of 0.01 cm/s. Find the rate at which the area is increasing when the radius of the circle is 5cm.

Solution

Let A be the area of the circle of radius, r.

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

By the chain rule, $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

From the info given, $\frac{dr}{dt} = 0.01$cm/s

$$\frac{dA}{dt} = 2\pi r \times 0.01$$

$$= 0.02\pi(5)$$

$$= 0.3142\,cm^2/s$$

**Assignment** ~~Example 5~~: Water is leaking from a hemispherical bowl of radius 20cm at the rate of 0.5cm³/s. Find the rate at which the surface area of the water is decreasing when the water level is halfway from the top.

## Rectilinear Motion

This is the motion of a particle along a straight line. It is specified by the equation $x = f(t)$ where x is the distance of the particle from an initial point O and t is the time.

If a direction is associated with the distance, we have a displacement. The velocity of a particle P is therefore the time rate of change of displacement. If V = velocity, $V = \frac{dx}{dt}$. If the particle moves away from O, the initial point, $\frac{dx}{dt} > 0$. If the particle moves towards O, $\frac{dx}{dt} < 0$. If the particle is momentarily at rest, $\frac{dx}{dt} = 0$.

Example 5: The motion of a particle starting from O is described by the equation $x = \frac{t^3}{3} - \frac{7}{2}t^2 + 10t$, how far is the particle from O when the particle is momentarily at rest.

Solution.

$$\frac{dx}{dt} = 3t^2 - 14t + 10 = 0$$

$$t^2 - 7t + 10 = 0$$

$$\cancel{t = \frac{-b \pm \sqrt{4ac}}{2a}}$$

$$\cancel{= \frac{14 \pm \sqrt{14 \times 3 \times 10}}{2 \times 14}}$$

$$\cancel{= \frac{14 \pm \sqrt{120}}{-28}}$$

$$\cancel{t = 3.}$$

$$t^2 - 5t - 2t + 10 = 0$$

$$t(t-5) - 2(t-5) = 0$$

$t = 2$ or $t = 5$

when $t = 2$

$$x = \frac{2^3}{3} - \frac{7}{2}(2)^2 + 10(2)$$

$$= \frac{8}{3} - \frac{28}{2} + 20$$

$$= \frac{16 - 84 + 120}{6}$$

$$= \frac{52}{6} = 8\tfrac{2}{3}$$

when $t = 5$

$$x = \frac{5^3}{3} - \frac{7}{2}5^2 + 10(5)$$

$$= \frac{125}{3} - \frac{175}{2} + 50$$

$$= \frac{250 - 525 + 300}{6}$$

$$= \frac{25}{6} = 4\tfrac{1}{6}$$

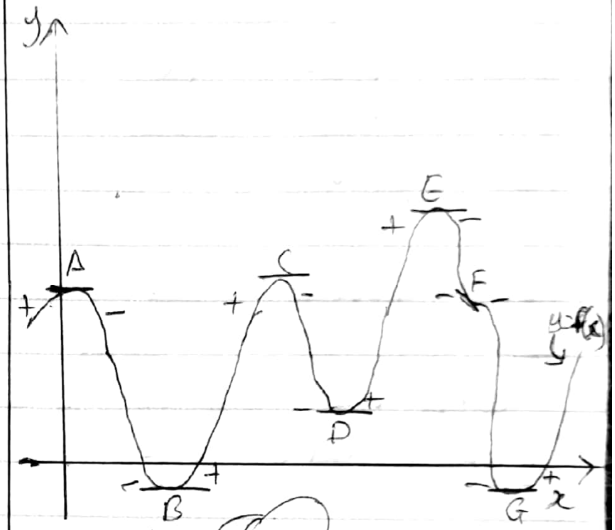



fig. 4

Figure 4 shows the graph of $y = f(x)$.

We consider the slope of the tangents at the points marked A, B, C, D, E, F and G. We also consider the slopes at the immediate neighbourhood of A, B, C, D, E, F and G as well.

~~Solution to Assignment~~

~~Let V be the volume of the bowl~~

~~$V = \frac{4}{6}\pi r^3 = \frac{2}{3}\pi r^3$~~

~~$\frac{dV}{dt} = \frac{2}{3}\pi r^2$~~

~~By chain rule~~ $\cancel{\frac{dV}{dt} = \frac{dv}{dr} \times \frac{dr}{dt}}$

$\cancel{\frac{dV}{dt}}$ = Let

## Solution

Let A be the area of the square, then $A = x^2$

$$\frac{dA}{dx} = 2x$$

$$\Delta A = \frac{dA}{dx} \cdot \Delta x$$

$$= 2x \Delta x$$

$$\frac{\Delta A}{A} = \frac{2x\Delta x}{x^2} = \frac{2\Delta x}{x}$$

But $\frac{\Delta x}{x} = \frac{0.5}{100} = \frac{5}{1000}$

$$\frac{\Delta A}{A} = 2 \times \frac{5}{1000}$$

$$\frac{\Delta A}{A} = \frac{1}{100}$$

$\therefore$ % increase in A =

$$\frac{\Delta A}{A} \times \frac{100}{1}$$

$$= \frac{1}{100} \times 100\%$$

$$= 1\%$$

Hence, % increase in area of the square is 1%

22/01/2023

## Rate of Change

If $y = f(x)$, $\frac{dy}{dx}$ can sometimes be introduced as the rate of which y is changing with respect to x. If y increases as x increases, $\frac{dy}{dx} > 0$, while if y decreases as x increases, $\frac{dy}{dx} < 0$.

cm/s. Find the rate at which the area is increasing when the radius of the circle is 5cm.

## Solution

Let A be the area of the circle of radius, r.

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

By the chain rule, $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

From the info given, $\frac{dr}{dt} = 0.01 cm/s$

$$\frac{dA}{dt} = 2\pi r \times 0.01$$

$$= 0.02\pi(5)$$

$$= 0.3142 cm^2/s$$

$\boxed{\text{Assignment}}$ ~~Example 5~~: Water is leaking hemispherical bowl of rad. at the rate of $0.5 cm^3/s$. Fi which the surface area is decreasing when the w halfway from the top.

## Rectilinear M

This is the m along a straight by the equation the distance initial point

If a d the distanc

The veloc

the t

If

Let $V$ be the volume of the water in the bowl, the volume of the water, when $h = r$ at full stage and height, $h$

$= \frac{1}{3}\pi h^2(3R-h)$ [volume of spherical segment]

At halfway, $h = 10cm$ and $R = 20cm$

Using $\frac{1}{3}\pi h^2(3R-h)$

$V = \frac{1}{3} \times \pi \times (10\,cm)^2 \times [(3 \times 20cm) - 10cm]$

$= \frac{1}{3} \times \pi \times 100cm^2 \times (60cm - 10cm)$

$= \frac{1}{3} \times \pi \times 100cm^2 \times 50cm$

$V = \frac{1}{3} \times 5000\pi\, cm^3$

$\frac{dv}{dt} = \frac{1}{3} \times 5000\pi \times \frac{dh}{dt}$

Given that $\frac{dh}{dt} = -0.5 cm^3/sec$

(since it is decreasing)

$\frac{dV}{dt} = \left(\frac{1}{3} \times 5000\pi\ cm^3/sec\right) \times -0.5 cm^3/sec$

$= \cancel{-833\pi\, cm^3}$ $-833.3 cm^3/sec$

~~The~~ $\cancel{\frac{dA}{dt}} =$

$A = 2\pi Rh$

At halfway, $h = 10cm$

$\therefore A = 2\pi \times 20cm \times 10cm$

$= 400\pi cm^2$

$\cancel{\frac{dA}{dt}} =$

$V = \frac{2}{3}\pi r^3$

$= \frac{2}{3} \times \pi \times \cancel{20cm^3}$

Let $A$ = heigh of full bowl = $r = 20cm$

When level is halfway, $h = 10cm$

Volume of water at height $h$ is calculated using formula for volume of a spherical segment:

$= \frac{1}{3}\pi h^2(3R - h)$

Given $R = 20cm$ and $h = 10cm$

$= \frac{1}{3} \times \pi \times (10\,cm)^2 \times [(3 \times 20cm) - 10cm]$

$= \frac{1}{3} \times \pi \times 100cm^2 \times 50cm$

$= \frac{1}{3}\pi\ 5000\pi m^3$

$\frac{dv}{dh} = \frac{2}{3}\pi h(3R-h) - \frac{1}{3}\pi h^2$

$= \frac{2}{3}\pi(10)(60-10) - \frac{1}{3}\pi(10)^2$

$= \frac{2}{3}\pi\ 5000 - \frac{1}{3}\pi\ 100$

$= \frac{10000\pi}{3} - \frac{100\pi}{3}$

$= \frac{10000\pi - 100\pi}{3}$

$= \frac{9900\pi}{3} = 3300\pi$

## Stationary Points.

Stationary points fall into three major categories:

(1) Those in which $\frac{dy}{dx}$ changes sign from positive through 0 to negative.

(2) Those in which $\frac{dy}{dx}$ changes sign from negative through 0 to positive. These are called minimum points.

(3) Those in which the sign of $\frac{dy}{dx}$ did not change in the immediate neighbourhood of the stationary points. These are called points of inflection.

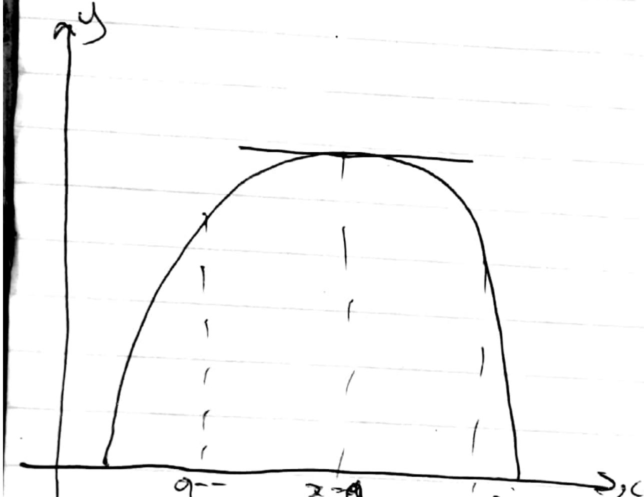



fig. 5

There is a maximum at the point where $x = a$, $\frac{dy}{dx} = 0$

$\therefore f'(a) = 0$

At $x = a^-$, $\frac{dy}{dx} > 0$

$f'(a^-) > 0$

At $x = a^+$, $\frac{dy}{dx} < 0$

$f'(a^+) < 0$

Hence, for the equation of a maximum at $x = a$, three conditions must be ~~involved~~ satisfied:

(i) $f'(a) = 0$

(ii) $f'(a^-) > 0$

(iii) $f'(a^+) < 0$

## Minimum points

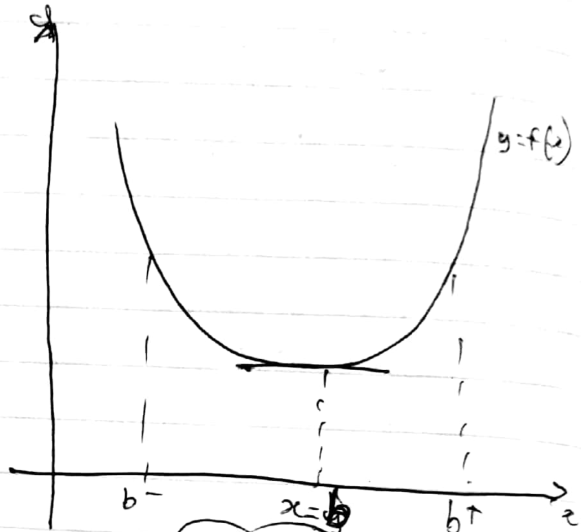



fig. 6

There is a minimum at $x = b$

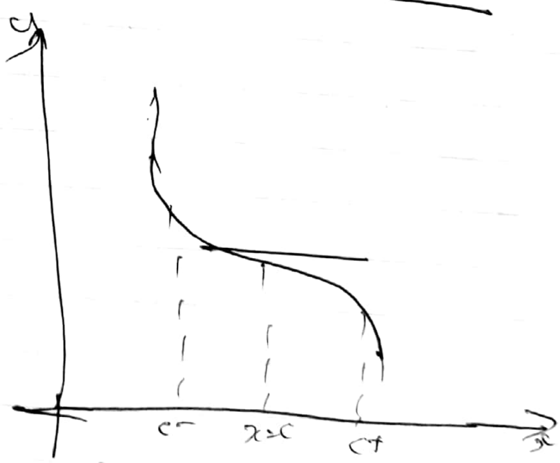
So, a point on a curve is a minimum at $x = b$

(i) $f'(b) = 0$

(ii) $f'(b^-) < 0$

(iii) $f'(b^+) > 0$

## Point of Inflexion



The point $x = c$ is a point of inflexion if:

(i) $f'(c) = 0$
(ii) $f'(c^-) < 0$
(iii) $f'(c^+) < 0$

Example 6: Find the stationary point of the following curves whose equations are:

(a) $y = \frac{x^3}{3} + x^2 - 3x + 4$

(b) $\frac{x^4}{4} + \frac{4}{3}x^3 - 2x^2 - 16x + 1$

Solution

(a) $\frac{dy}{dx} = x^2 + 2x - 3$

Let $\frac{dy}{dx} = 0$

$x^2 + 2x - 3 = 0$

~~$x^2 - 2x - x - 3 = 0$~~

~~$x(x-2) - 1(x-2)$~~

$x^2 - x + 3x - 3 = 0$

$x(x-1) + 3(x-1) = 0$

$x = -3$ or $x = 1$

(b) $\frac{dy}{dx} = x^3 + 4x^2 - 4x - 16$

Let $\frac{dy}{dx} = 0$

$x^3 + 4x^2 - 4x - 16 = 0$

$(x-2)(x+2)(x-4) = 0$

$x = 2, x = -2, x = 4$

Example 7:

Find the turning points on the curve $y = \frac{x^4}{2} + \frac{5}{3}x^3 - 2x^2 - 3x + 1$ and distinguish between them.

$\frac{dy}{dx} = 2x^3 + 5x^2 - 4x - 3$

At stationary points, $\frac{dy}{dx} = 0$

$2x^3 + 5x^2 - 4x - 3$
$= (2x+1)(x-1)(x+3) = 0$

$x = -\frac{1}{2}, x = 1, x = -3$

Let $x = -\frac{1}{2} = -0.5$

$a^- = -0.6$

$a^+ = -0.4$

Then $f'(a) = f'(0.5) = 0$

$f'(a^-) = f'(-0.6)$

$f'(a^-) = (-1.2+1)(-0.6-1)(-0.6+$

$> 0$

$f'(a^-) > 0$

$f'(a^+) = (-0.6+1)(-0.4-1)(-0.4+3)$
$< 0$

$\therefore f'(a^+) = 0$

Hence, there is a maximum point at $x = -\frac{1}{2}$

At $x = 1$

$a = 1, a^- = 0.9, a^+ = 1.1$

$f'(a^-) = (1.8+1)(0.9-1)(0.9+3)$

$f'(a^-) < 0$

$f'(a^+) = (2.2+1)(1.1-1)(1.1-3)$

$f(a^+) > 0$

At $x = -3$

$a = 3, a^- = 2.9, a^+ = 3.1$

$f'(a^-) = (6.2+1)(3.1+1)(-3.1+3)$

$f'(a^+) = (-5.8+1)(-2.9+1)(-2.9+3)$

~~The Second Derivative Test For Stationary Points~~

$\frac{dy}{dx}$ changes from negative through zero

## The Second Derivative Test for Stationary Points

The necessary and sufficient conditions for maximum points

(1) $\frac{dy}{dx} = 0$

(2) $\left.\frac{d^2y}{dx^2}\right|_{x=a} < 0$

There is a minimum point at $x = b$ on the curve $y = f(x)$ if

(i) $\frac{dy}{dx} = 0$

(ii) $\left.\frac{d^2y}{dx^2}\right|_{x=b} > 0$

The curve $y = f(x)$ has a point of inflexion at $x = c$, if

~~f'(c)~~ (i) $\frac{dy}{dx} = 0$ and

(ii) $\left.\frac{d^2y}{dx^2}\right|_{x=c^-} < 0$

$\left.\frac{d^2y}{dx^2}\right|_{x=c^+} > 0$

OR

(i) $\frac{dy}{dx} = 0$ and

(ii) $\left.\frac{d^2y}{dx^2}\right|_{x=c^-} < 0$

$\left.\frac{d^2y}{dx^2}\right|_{x=c^+} > 0$

## Assignment 1

Find the stationary points and determine whether the stationary points ~~are~~ maximum points, minimum points or points of inflexion in each of the following:

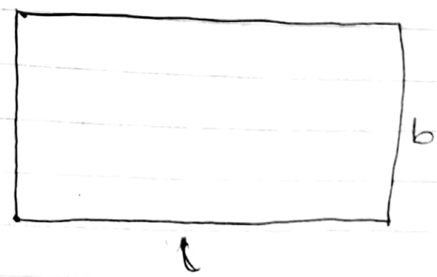
(a) $y = x(x^2 - 9)$

(b) $y = x^3 - 9x^2 - 27x + 27$

## Maximum and Minimum Value Problems

The concept of finding maximum and minimum points on a curve can be ~~tried out~~ carried over to other problems not ~~represented~~ connected in curves.

Example 9: 100m of wire is available for fencing a rectangular piece of land. Find the dimension of the land which maximize the area. Hence, determine the maximum area of the fence.



Let $x$ be the length of a side, then length of an adjacent side $= \left(\frac{100}{2}\right) - x$

$= 50 - x$

$A = x(50-x) = 16$

$A = 50x - x^2 \cancel{=16}$

$\frac{dA}{dx} = 50 - 2x = 0$

$\therefore x = 25$

$\frac{d^2y}{dx} = -2 < 0$

$l = x = 25$ , $b = 25$

$\therefore$ The maximum area of the fence

$= 25 \times 25 = 625m$

Assignment 2