DEPT OF MATHS FUNAAB 2021-2022 MTS 101
FURTHER NOTE ON MATHEMATICAL INDUCTION

Introduction

The Principle of Mathematical Induction is based on the unique property of the set $\mathbb{Z}^+$ of positive integers. By definition,

$$\mathbb{Z}^+ = \{x \in \mathbb{Z} : x > 0\} = \{x \in \mathbb{Z} : x \geq 1\} = \{1, 2, 3, \cdots, \}. \tag{1}$$

To every positive integer $n \in \mathbb{Z}^+$, there exists the next positive integer $n+1$. The members of $\mathbb{Z}^+$ are well ordered as shown below

$$1 < 2 < 3 < 4 < 5 < 6 < \cdots < \cdots \tag{2}$$

Every nonempty subset $X \subset \mathbb{Z}^+$ contains an integer $m \in X$ such that $m \leq n$, $\forall n \in X$, that is, X contains a least (or smallest) element. This is the Well-Ordering Principle stated below.

Principle 1. [**The Well-Ordering Principle**] *Any nonempty subset of $\mathbb{Z}^+$ contains a smallest element.*

This principle is the basis of a proof technique known as mathematical induction. This technique will often help to prove a general mathematical statement involving positive integers when certain instances of that statement suggest a general pattern.

Principle 2. [**The Principle of Mathematical Induction**] *Let $P(n)$ be an open mathematical statement (or set of such open statements) that involves one or more occurrences of the variable n, which represents a positive integer. Suppose we can prove that:*

(i) $P(1)$ is true,

(ii) if $P(k)$ is true for some particular positive integer k, then $P(k+1)$ is also true.

Then we can conclude that $P(n)$ is true for all positive integers.

Further Examples

Further Example 1. Prove by induction that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \sum_{r=1}^{n} r^3 = \left[\frac{1}{2}n(n+1)\right]^2, \quad \forall n \in \mathbb{Z}^+.$$

Solution: Let $P(n)$ be the statement $S_n = \left[\frac{1}{2}n(n+1)\right]^2$. Then $P(1)$ is the statement

$$S_1 = \left[\frac{1}{2} \times 1 \times 2\right]^2 = 1^2 = 1$$

which is true since for $r = 1$, we have $1^3 = 1$. $P(2)$ is the statement

$$S_2 = \left[\frac{1}{2} \times 2 \times 3\right]^2 = 3^2 = 9$$

which is true since for $r = 2$, we have $1^3 + 2^3 = 1 + 8 = 9$. Assuming that the statement is true for $n = k$. Then $P(k)$ is the statement $S_k = \left[\frac{1}{2}k(k+1)\right]^2$. We next prove that the statement is true for $n = k + 1$. $P(k+1)$ is the statement

$$\begin{aligned}
S_{k+1} &= S_k + (k+1)^3 \\
&= \left[\frac{1}{2}k(k+1)\right]^2 + (k+1)^3 \\
&= \left[\frac{1}{2}(k+1)\right]^2 (k^2 + 4(k+1)) \\
&= \left[\frac{1}{2}(k+1)\right]^2 (k^2 + 4k + 4) \\
&= \left[\frac{1}{2}(k+1)\right]^2 (k+2)^2 \\
&= \left[\frac{1}{2}(k+1)(k+2)\right]^2 .
\end{aligned}$$

This shows that $P(k+1)$ is true. Hence by the Principle of Induction, the statement is true $\forall n \in \mathbb{Z}^+$.

Further Example 2. Prove by induction that

$$1^4 + 2^4 + 3^4 + \cdots + n^4 = \sum_{r=1}^{n} r^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2 + 3n - 1), \quad \forall n \in \mathbb{Z}^+.$$

Solution: Let $P(n)$ be the statement $S_n = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$. Then $P(1)$ is the statement

$$S_1 = \frac{1}{30} \times 1 \times 2 \times 3 \times 5 = 1$$

which is true since for $r = 1$, we have $1^4 = 1$. $P(2)$ is the statement

$$S_2 = \frac{1}{30} \times 2 \times 3 \times 5 \times 17 = 17$$

which is true since for $r = 2$, we have $1^4 + 2^4 = 1 + 16 = 17$. Assuming that the statement is true for $n = k$. Then $P(k)$ is the statement $S_k = \frac{1}{30}k(k+1)(2k+1)(3k^2+3k-1)$. We next prove that the statement is true for $n = k+1$. $P(k+1)$ is the statement

$$\begin{aligned}
S_{k+1} &= S_k + (k+1)^4 \\
&= \frac{1}{30}k(k+1)(2k+1)(3k^2+3k-1) + (k+1)^4 \\
&= \frac{1}{30}(k+1)\left[k(2k+1)(3k^2+3k-1) + 30(k+1)^3\right] \\
&= \frac{1}{30}(k+1)(6k^4+39k^3+91k^2+89k+30) \quad \text{[using binomial expansion]} \\
&= \frac{1}{30}(k+1)(k+2)(2k+3)(3k^2+9k+5) \quad \text{[using factor theorem]}
\end{aligned}$$

This shows that $P(k+1)$ is true. Hence by the Principle of Induction, the statement is true $\forall n \in \mathbb{Z}^+$.

Further Example 3. Prove by induction that

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \sum_{r=1}^{n}\left[\frac{1}{(2r-1)(2r+1)}\right] = \frac{n}{2n+1}, \ \forall n \in \mathbb{Z}^+.$$

Solution: Let $P(n)$ be the statement $S_n = \frac{n}{2n+1}$. Then $P(1)$ is the statement

$$S_1 = \frac{1}{2+1} = \frac{1}{3}$$

which is true since for $r = 1$, we have $\frac{1}{1\times 3} = \frac{1}{3}$. $P(2)$ is the statement

$$S_2 = \frac{2}{4+1} = \frac{2}{5}$$

which is true since for $r = 2$, we have $\frac{1}{1\times 3} + \frac{1}{3\times 5} = \frac{5+1}{15} = \frac{6}{15} = \frac{2}{5}$. Assuming that the statement is true for $n = k$. Then $P(k)$ is the statement $S_k = \frac{k}{2k+1}$. We next prove that the statement is true for $n = k+1$. $P(k+1)$ is the statement

$$\begin{aligned}
S_{k+1} &= S_k + \frac{1}{(2k+1)(2k+3)} \\
&= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \\
&= \frac{k(2k+3)+1}{(2k+1)(2k+3)} \\
&= \frac{2k^2+3k+1}{(2k+1)(2k+3)} \\
&= \frac{(k+1)(2k+1)}{(2k+1)(2k+3)} \\
&= \frac{k+1}{2k+3}.
\end{aligned}$$

This shows that $P(k+1)$ is true. Hence by the Principle of Induction, the statement is true $\forall n \in \mathbb{Z}^+$.

Alternative Solution

This problem could also be solved using partial fractions.

$$\begin{aligned}
\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} &= \sum_{r=1}^{n} \left[\frac{1/2}{2r-1} - \frac{1/2}{2r+1}\right] \\
&= \frac{1}{2}\sum_{r=1}^{n} \left[\frac{1}{2r-1} - \frac{1}{2r+1}\right] \\
&= \frac{1}{2}\left[\frac{1}{1} - \frac{1}{3} \right. \qquad [r=1] \\
&+ \quad \frac{1}{3} - \frac{1}{5} \qquad [r=2] \\
&+ \quad \frac{1}{5} - \frac{1}{7} \qquad [r=3] \\
&+ \quad \Updownarrow \quad \Updownarrow \quad \Updownarrow \\
&+ \quad \frac{1}{2n-5} - \frac{1}{2n-3} \qquad [r=n-2] \\
&+ \quad \frac{1}{2n-3} - \frac{1}{2n-1} \qquad [r=n-1] \\
&+ \quad \left. \frac{1}{2n-1} - \frac{1}{2n+1}\right] \qquad [r=n] \\
&= \frac{1}{2}\left[\frac{1}{1} - \frac{1}{2n+1}\right] \quad \text{[after canceling diagonally and adding]} \\
&= \frac{1}{2} \times \frac{2n}{2n+1} \\
&= \frac{n}{2n+1}.
\end{aligned}$$

The systematic way of summing from $r=1$ to $r=n$ should be noted and studied carefully. The common factor in the partial fractions (in this problem $1/2$) should be brought out. We then put $r = 1, 2, 3 \cdots, n-3, n-2, n-1, n$ in the partial fractions and then add the results as thus

$$\frac{1}{2}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{1}{2n-5} - \frac{1}{2n-3} + \frac{1}{2n-3} - \frac{1}{2n-1} + \frac{1}{2n-1} - \frac{1}{2n+1}\right].$$

The surviving terms are $\frac{1}{1}$ and $-\frac{1}{2n+1}$ and upon addition gives $\frac{2n}{2n+1}$ and multiplying this by the factor $1/2$ gives the required result $\frac{n}{2n+1}$.

Further Example 4. Prove by induction that

$$\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \cdots + \frac{1}{n(n+1)(n+2)} = \sum_{r=1}^{n}\left[\frac{1}{r(r+1)(r+2)}\right] = \frac{n(n+3)}{4(n+1)(n+2)}, \ \forall n \in \mathbb{Z}^+.$$

Solution: Let $P(n)$ be the statement $S_n = \frac{n(n+3)}{4(n+1)(n+2)}$. Then $P(1)$ is the statement

$$S_1 = \frac{1 \times 4}{4 \times 2 \times 3} = \frac{1}{6}$$

which is true since for $r=1$, we have $\frac{1}{1\times2\times3}=\frac{1}{6}$. $P(2)$ is the statement

$$S_2=\frac{2\times5}{4\times3\times4}=\frac{5}{24}$$

which is true since for $r=2$, we have $\frac{1}{1\times2\times3}+\frac{1}{2\times3\times4}=\frac{1}{6}+\frac{1}{24}=\frac{5}{24}$. Assuming that the statement is true for $n=k$. Then $P(k)$ is the statement $S_k=\frac{k(k+3)}{4(k+1)(k+2)}$. We next prove that the statement is true for $n=k+1$. $P(k+1)$ is the statement

$$\begin{aligned}
S_{k+1} &= S_k+\frac{1}{(k+1)(k+2)(k+3)}\\
&= \frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}\\
&= \frac{k(k+3)^2+4}{4(k+1)(k+2)(k+3)}\\
&= \frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}\\
&= \frac{(k+1)(k+1)(k+4)}{4(k+1)(k+2)(k+3)}\\
&= \frac{(k+1)(k+4)}{4(k+2)(k+3)}.
\end{aligned}$$

This shows that $P(k+1)$ is true. Hence by the Principle of Induction, the statement is true $\forall n\in\mathbb{Z}^+$.

Alternative Solution

This problem could also be solved using partial fractions.

$$\begin{aligned}
\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)} &= \sum_{r=1}^{n}\left[\frac{1/2}{r}-\frac{1}{r+1}+\frac{1/2}{r+2}\right]\\
&= \frac{1}{2}\sum_{r=1}^{n}\left[\frac{1}{r}-\frac{2}{r+1}+\frac{1}{r+2}\right]\\
&= \frac{1}{2}\Bigg[\frac{1}{1}-\frac{2}{2}+\frac{1}{3} \qquad [r=1]\\
&+ \frac{1}{2}-\frac{2}{3}+\frac{1}{4} \qquad [r=2]\\
&+ \frac{1}{3}-\frac{2}{4}+\frac{1}{5} \qquad [r=3]\\
&+ \frac{1}{4}-\frac{2}{5}+\frac{1}{6} \qquad [r=4]\\
&+ \quad \Updownarrow \quad \Updownarrow \quad \Updownarrow\\
&+ \frac{1}{n-4}-\frac{2}{n-3}+\frac{1}{n-2} \qquad [r=n-4]
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{n-3} - \frac{2}{n-2} + \frac{1}{n-1} \qquad\qquad [r = n-3] \\
&+ \quad \frac{1}{n-2} - \frac{2}{n-1} + \frac{1}{n} \qquad\qquad [r = n-2] \\
&+ \quad \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1} \qquad\qquad [r = n-1] \\
&+ \quad \left.\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right] \qquad\qquad [r = n] \\
&= \quad \frac{1}{2}\left[\frac{1}{1} - \frac{2}{2} + \frac{1}{2} + \frac{1}{n+1} - \frac{2}{n+1} + \frac{1}{n+2}\right] \\
&= \quad \frac{1}{2}\left[\frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}\right] \\
&= \quad \frac{1}{2}\left[\frac{(n+1)(n+2) + 2(n+1) - 2(n+2)}{2(n+1)(n+2)}\right] \\
&= \quad \frac{1}{2}\left[\frac{n^2+3n}{2(n+1)(n+2)}\right] \\
&= \quad \frac{n(n+3)}{4(n+1)(n+2)}.
\end{aligned}
$$

Further Example 5. Prove by induction that

$$\left(x^{2n-1} + y^{2n-1}\right) \quad \text{is divisible by} \quad (x+y) \quad \forall n \in \mathbb{Z}^+.$$

Solution: Let $P(n)$ be the statement that $(x^{2n-1} + y^{2n-1})$ is divisible by $(x+y)$. $P(1)$ is the statement

$$x^{2-1} + y^{2-1} = x^1 + y^1 = x + y$$

which is true. $P(2)$ is the statement

$$x^{4-1} + y^{4-1} = x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

which is true. Assuming that the statement is true for $n = k$. Then $P(k)$ will be the statement that $\left(x^{2k-1} + y^{2k-1}\right)$ is divisible by $(x+y)$. That is $\left(x^{2k-1} + y^{2k-1}\right) = N(x,y)(x+y)$ where $N(x,y)$ is a polynomial in x, y. Thus, we have $x^{2k-1} = (x+y)N(x,y) - y^{2k-1}$. We now prove that the statement is true for $n = k+1$. The statement $P(k+1)$ is

$$\begin{aligned}
x^{2(k+1)-1} + y^{2(k+1)-1} &= x^{2k+2-1} - y^{2k+2-1} \\
&= x^2 x^{2k-1} + y^{2k+1} \\
&= x^2\left[(x+y)N(x,y) - y^{2k-1}\right] + y^{2k+1} \\
&= x^2(x+y)N(x,y) - x^2y^{2k-1} + y^{2k+1} \\
&= x^2(x+y)N(x,y) - x^2y^{2k-1} + y^{2k+2-2+1} \\
&= x^2(x+y)N(x,y) - x^2y^{2k-1} + y^2 \times y^{2k-1} \\
&= x^2(x+y)N(x,y) - (x^2 - y^2)y^{2k-1} \\
&= x^2(x+y)N(x,y) - (x-y)(x+y)y^{2k-1} \\
&= (x+y)\left[x^2N(x,y) - (x-y)y^{2k-1}\right]
\end{aligned}$$

which is true. Hence by the Principle of Induction, the statement is true $\forall n \in \mathbb{Z}^+$.

Further Example 6. Use the Principle of Mathematical Induction to show that

$$4 \times 10^{2n} + 9 \times 10^{2n-1} + 5 \quad \text{is a multiple of } \; 99 \;\; \forall n \in \mathbb{Z}^+.$$

Solution: Let $P(n)$ be the statement $4 \times 10^{2n} + 9 \times 10^{2n-1} + 5$ is a multiple of 99. $P(1)$ is the statement

$$4 \times 100 + 9 \times 10 + 5 = 495 = 5 \times 99$$

which is true. $P(2)$ is the statement

$$4 \times 10000 + 9 \times 1000 + 5 = 49005 = 495 \times 99$$

which is true. Assuming that the statement is true for $n = k$. $P(k)$ is the statement

$$\begin{aligned}
4 \times 10^{2k} + 9 \times 10^{2k-1} + 5 &= 99N \quad [\text{where } N \text{ is an integer}] \\
\Rightarrow 4 \times 10^{2k} &= 99N - 9 \times 10^{2k-1} - 5.
\end{aligned}$$

We next prove that the statement is true for $n = k$. The statement $P(k+1)$ is given by

$$\begin{aligned}
4 \times 10^{2k+2} + 9 \times 10^{2k+1} + 5 &= 4 \times 10^2 \times 10^{2k} + 9 \times 10^{2k+1} + 5 \\
&= 10^2\left[99N - 9 \times 10^{2k-1} - 5\right] + 9 \times 10^{2k+1} + 5 \\
&= 99 \times 10^2 N - 9 \times 10^{2k+1} - 500 + 9 \times 10^{2k+1} + 5 \\
&= 99 \times 100N - 495 \\
&= 99(100N - 5)
\end{aligned}$$

which is true. Thus, by the Principle of Induction, the statement is true $\forall n \in \mathbb{Z}^+$

Further Example 7. Use the Principle of Mathematical Induction to prove that

$$\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15} \quad \text{is an integer.}$$

Solution: Let $P(n)$ be the statement $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is an integer. The statement $P(1)$ is

$$\frac{1}{5} + \frac{1}{3} + \frac{7}{15} = \frac{3+5+7}{15} = \frac{15}{15} = 1 \quad \text{which is true.}$$

The statement $P(2)$ is

$$\frac{32}{5} + \frac{8}{3} + \frac{14}{15} = \frac{96+40+14}{15} = \frac{150}{15} = 10 \quad \text{which is true.}$$

Assuming that the statement is true for $n = k$. Then the statement $P(k)$ is

$$\frac{k^5}{5} + \frac{k^3}{3} + \frac{7k}{15} \quad \text{is an integer}$$

We now prove that the statement is is true for $n = k+1$. The statement $P(k+1)$ is

$$\begin{aligned}
\frac{(k+1)^5}{5} + \frac{(k+1)^3}{3} + \frac{7(k+1)}{15} &= \frac{1}{5}\left[k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1\right] \\
&\quad + \frac{1}{3}\left[k^3 + 3k^2 + 3k + 1\right] + \frac{7k}{15} + \frac{7}{15} \quad \text{[using binomial expansions]} \\
&= \left[\frac{k^5}{5} + \frac{k^3}{3} + \frac{7k}{15}\right] + \left[k^4 + 2k^3 + 3k^2 + 2k\right] + \left[\frac{1}{5} + \frac{1}{3} + \frac{7}{15}\right] \\
&= \text{[an integer]} \quad + \quad \text{[an integer]} \quad + \quad \text{[an integer]} \\
&= \text{an integer.}
\end{aligned}$$

We have just shown that the statement is true for $n = k+1$. Hence by the Principle of Induction, the statement is true $\forall n \in \mathbb{Z}^+$.

Further Example 8. Use the Principle of Mathematical Induction to prove that

$$(xy)^n = x^n y^n \quad \forall n \in \mathbb{Z}^+.$$

Solution: Let $P(n)$ be the statement that $(xy)^n = x^n y^n$. $P(1)$ is the statement

$$x^1 y^1 = xy = (xy)^1 \quad \text{which is true.}$$

$P(2)$ is the statement

$$x^2 y^2 = (xx)(yy) = (xy)(xy) = (xy)^2 \quad \text{which is true.}$$

Assuming that the statement is true for $n = k$. Then the statement $P(k)$ is $(xy)^k = x^k y^k$. We now show that the statement is is true for $n = k + 1$. The statement $P(k + 1)$ is

$$\begin{aligned}(xy)^{k+1} &= (xy)^k(xy) \\ &= (x^k y^k)(xy) \\ &= (xx^k)(y^k y) \\ &= x^{k+1}y^{k+1}.\end{aligned}$$

We have just shown that the statement is true for $n = k+1$. Hence by the Principle of Induction, the statement is true $\forall n \in \mathbb{Z}^+$.

Further Practice Problems

1. For all positive integer n, use the principle of mathematical induction to prove the following:

   (a) The total number of distinct subsets of a finite set of n elements is 2^n.

   (b) $\sum_{r=1}^{n}(2r-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$.

   (c) $\sum_{r=1}^{n} r(r+2) = \frac{1}{6}n(n+1)(2n+7)$.

   (d) $\sum_{r=1}^{n} 2^{r-1} = \sum_{r=0}^{n-1} 2^r = 2^n - 1$.

   (e) $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2 = \left(\sum_{r=1}^{n} r\right)^2$.

   (f) $\sum_{r=1}^{n}(2r-1)^3 = n^2(2n^2-1)$.

   (g) $\sum_{r=1}^{n} r(2^r) = 2 + (n-1)2^{n+1}$.

   (h) $\sum_{r=1}^{n} 2 \times 3^{r-1} = 3^n - 1$.

   (i) $\sum_{r=1}^{n} r \times r! = (n+1)! - 1$.

2. (a) For $n \in \mathbb{Z}^+$, if $n > 10$, show that

   $$n - 2 < \frac{1}{12}n^2 - n.$$

   (b) Prove the following:

      i. $n > 3 \Rightarrow 2^n < n!$.

      ii. $n > 4 \Rightarrow n^2 < 2^n$.

      iii. $n > 9 \Rightarrow n^3 < 2^n$.

   (c) For $n \in \mathbb{Z}^+$, use mathematical induction to show that:

      i. $\sum_{r=1}^{n} r^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$.

ii. $\sum_{r=1}^{n} \frac{1}{n+r} = \sum_{r=1}^{2n} (-1)^{r-1} r^{-1}$.

iii. $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$.

iv. $\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}$.

v. $\sum_{r=1}^{n} \frac{r+3}{(r-1)r(r+1)} = \frac{3}{2} - \frac{n+2}{n(n+1)}$.

Deduce the sum to infinity for (iii), (iv) and (v).

3. (a) For all $n \in \mathbb{Z}^+$, let p(n) be the open statement: $n^2 + n + 41$ is prime.

   i. Show that p(n) is true for all $1 \leq n \leq 9$.

   ii. Does the truth of p(k) imply that of p(k+1) for all $k \in \mathbb{Z}^+$?

   (b) For $n \in \mathbb{Z}^+$ define the sum S_n by the formula

$$S_n = \sum_{r=1}^{n} \frac{n}{(n+1)!}.$$

   i. Verify that $S_1 = 1/2, S_2 = 5/6$ and $S_3 = 23/24$.

   ii. Compute S_4, S_5 and S_6.

   iii. On the basis of your results in (i) and (ii), conjecture a formula for the sum of the terms in S_n.

   iv. Verify your conjecture in part (iii) for all $n \in \mathbb{Z}^+$ by the principle of mathematical induction.

4. (a) For any $n \in \mathbb{Z}, n \geq 0$, show that:

   i. $2^{2n+1} + 1$ is divisible by 3.

   ii. $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

   iii. $\frac{n^7}{7} + \frac{n^3}{3} + \frac{11n}{21}$ is an integer.

   (b) i. If $n \in \mathbb{Z}^+$ and $n \geq 2$, show that $2^n <^{2n} C_n < 4^n$.

   ii. If $n \in \mathbb{Z}^+$, show that 57 divides $7^{n+2} + 8^{2n+1}$.

   iii. If $n \in \mathbb{Z}^+$, show that 5 divides $n^5 - n$.

   iv. If $n \in \mathbb{Z}^+$, show that 6 divides $n^3 + 5n$.

   v. If $n \in \mathbb{Z}^+$, show that 9 divides $5^{2n} + 3n - 1$.

5. Prove by induction that $\forall n \in \mathbb{Z}^+$,

   (a) $x^n - y^n$ is divisible by $x - y$.

   (b) $(1+x)^n \geq 1 + nx$.

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