MTS 101 - Algebra - 3 Units

Professor Agboola A.A.A.
Department of Mathematics, Federal University of Agriculture,
Abeokuta, Nigeria. (agboolaaaa@funaab.edu.ng)

A Virtual Lecture Presented To: Physical Sciences, Engineering and Technology Students

Mounted by: Department of Mathematics
Federal University of Agriculture,
Abeokuta, Nigeria

December 14 - 28, 2021

Abstract

Abstract

In this lecture note, we are going to study: Linear Equations in one Variable, Inequalities, Theory of Quadratic Equations and Cubic Equations, Equations Reducible to Quadratic Type and Simultaneous Equations. Several Examples and Practice Problems will provided.

1. Linear Equations in one Variable

Definition

A linear equation in one variable say x is an equation that can be reduced to the form $ax = b$ where $a, b \in \mathbb{Z}$, $a \neq 0$ and $x = b/a$.

Example

Solve for x given that

$$\frac{x+1}{3} - \frac{x-2}{4} = \frac{2x+3}{6}.$$

Solution: Multiply through by 12 the LCM of 3, 4 and 6 to have

$$\begin{aligned} 4(x+1) - 3(x-2) &= 2(2x+3) \\ \Rightarrow 4x + 4 - 3x + 6 &= 4x + 6 \\ \Rightarrow -3x &= -4 \\ \therefore x &= 4/3. \end{aligned}$$

Example

Solve the equation

$$\frac{x}{2x+1} = \frac{2x+3}{4x-3}.$$

Solution: Cross multiply to have

$$\begin{aligned} x(4x-3) &= (2x+1)(2x+3) \\ \Rightarrow 4x^2-3x &= 4x^2+8x+3 \\ \Rightarrow -11x &= 3 \\ \therefore x &= -\frac{3}{11}. \end{aligned}$$

Example

Solve the equation

$$\frac{2}{x^2-x-6} = \frac{3}{x^2-5x+6}.$$

Solution:

$$\begin{aligned}
\frac{2}{x^2 - x - 6} &= \frac{3}{x^2 - 5x + 6} \\
\Rightarrow \frac{2}{(x-3)(x+2)} &= \frac{3}{(x-2)(x-3)}
\end{aligned}$$

Multiply through by $(x-2)(x-3)(x+2)$ to have

$$\begin{aligned}
2(x-2) &= 3(x+2) \\
\Rightarrow 2x - 4 &= 3x + 6 \\
\Rightarrow -x &= 10 \\
\therefore x &= -10.
\end{aligned}$$

Practice Problems 1

1. Solve the following equations:
   (a) $\frac{3}{4}(3x-4) - \frac{4}{5}(4-5x) - \frac{5}{6}(5-6x) = \frac{9}{10}(9x-10)$.
   (b) $\frac{4}{x+3} - \frac{3}{x+2} = 0$.
   (c) $\frac{3}{2x-5} - \frac{4}{x-3} = 0$.

(d) $\frac{4x-3}{6x+1} = \frac{2x-1}{3x+4}$.

(e) $\frac{2x+3}{2x+5} - \frac{x-1}{x-2} = 0$.

(2) Solve the following equations:

(a) $\frac{3}{2x+3} - \frac{1}{2x+1} = \frac{1}{x+1}$.

(b) $\frac{4x-1}{x+4} - 2 = \frac{2x-1}{x+2}$.

(c) $\frac{2}{x+3} - \frac{x-6}{x^2-9}$.

(d) $\frac{4}{x^2+3x+2} - \frac{3}{x^2+5x+6} = 0$.

(e) $\frac{2}{x-2} = \frac{2x-1}{x^2+x-6} - \frac{3}{x+3}$.

(f) $\frac{3}{x-4} - \frac{x+2}{x^2-3x-4} = \frac{1}{2x+2}$.

Linear Inequalities in One Unknown

Definition

A linear inequality in one unknown x is either of the form

$$ax < b, ax > b, ax \leq x, ax \geq x$$

where $a, b \in \mathbb{Z}$ and $a \neq 0$.

Linear inequalities can be solved like linear equations but attention must be placed to the position of inequalities. Unlike linear equations with unique solutions, linear inequalities do not have unique solutions but ranges where solutions exist. Hence, number lines must always be drawn to display the ranges where solutions can be found. The following should be noted:

If $a \leq b$ and $c \in \mathbb{R}^+$, then

(i) $a \pm c \leq b \pm c$,

(ii) $ac \leq bc$,

(iii) $1/a \geq 1/b$, $a \div c \leq b \div c, c \neq 0$.

If $a \geq b$ and $c \in \mathbb{R}^+$, then

(i) $a \pm c \geq b \pm c$,

(ii) $ac \geq bc$,

(iii) $1/a \leq 1/b$, $a \div c \geq b \div c, c \neq 0$.

If $a \leq b$ and $c \in \mathbb{R}^-$, then

(i) $ac \geq bc$,

(ii) $a \div c \geq b \div c, c \neq 0$.

If $a \geq b$ and $c \in \mathbb{R}^-$, then

(i) $ac \leq bc$,

(ii) $a \div c \leq b \div c, c \neq 0$.

Example

Find the range of values of x for which

$$\frac{1}{2}(3x-1) - \frac{5}{3}(x-2) \leq \frac{3}{4}(2x+3).$$

Solution: Multiplying through by 12 the LCM of 2, 3 and 4, we have

$$\begin{aligned} 6(3x-1)-20(x-2) &\leq 9(2x+3) \\ \Rightarrow -20x &\leq 5 \\ \Rightarrow 20x &\geq -5 \\ \therefore x &\geq -1/4. \end{aligned}$$

$$\text{Number Line:} \quad -1/4\bullet \xrightarrow{x\geq -1/4} X$$

Example

Find the interval(s) in which x can lie given that

$$5-(4-x)\leq 3 \quad \text{and} \quad 5-\frac{1}{3}(x+6)<4.$$

Solution: From the 1st inequality we obtain $x \leq 2$ and from the 2nd inequality we obtain $x > -3$. Combining the two solutions to obtain $-3 \leq x \leq 2$ which is the required interval. The number line is

$$\text{Number Line:} \quad -3\circ \xrightarrow[\;]{-3<x\leq 2} \bullet 2$$

Modulus or Absolute Value of a Number x

Definition

The modulus or absolute value of a number x denoted by $\mid x \mid$ is the positive number having the same magnitude as x. $\mid x \mid$ is defined by

$$\mid x \mid = \begin{cases} 0 & \text{if} \quad x = 0, \\ x & \text{if} \quad x > 0, \\ -x & \text{if} \quad x < 0. \end{cases}$$

The following should be noted:

(i) If $\mid x \mid \leq k$, then $-k \leq x \leq k$.

(ii) If $\mid x \mid \geq k$, then $x \geq k$ or $x \leq -k$.

(iii) If $\mid ax + b \mid \leq k$, then $-k \leq ax + b \leq k$.

(iv) If $\mid ax + b \mid \geq k$, then $ax + b \geq k$ or $ax + b \leq -k$.

Example

Solve the following inequalities

(i) $\mid x - 1 \mid < 2$.

(ii) $\mid 1 - 3x \mid > 1$.

(iii) $\mid 2 - \frac{1}{3}(1 - x) \mid < 5$.

Solution: (i) $\mid x - 1 \mid < 2$ implies that $-2 < x - 1 < 2$ from which we obtain $-1 < x < 3$.

$$\text{Number Line:} \quad -1\circ \xrightarrow{-1<x<3}\!\!\!\!\!\xleftarrow{} \circ 3$$

(ii) $\mid 1 - 3x \mid > 1$ implies that $1 - 3x > 1$ or $1 - 3x < -1$ from which obtain $x < 0$ or $x > 2/3$.

$$\text{Number Line:} \quad \xleftarrow{x<0}\!\!\!\!\!\xleftarrow{} \circ 0 \qquad 2/3\circ \xrightarrow{x>2/3}\!\!\!\!\!\xrightarrow{} X$$

(iii) $\mid 2 - \frac{1}{3}(1 - x) \mid < 5$ implies that $-5 < 2 - \frac{1}{3}(1 - x) < 5$ that is $-15 < 6 - 1 + x < 15$ from which we obtain $-20 < x < 10$.

$$\text{Number Line:} \quad -20\circ \xrightarrow{-20<x<10}\!\!\!\!\!\xleftarrow{} \circ 10$$

Quadratic Inequalities

Definition

Quadratic inequalities are inequalities either of the form $ax^2 + bx + c < 0$, $ax^2 + bx + c > 0$, $ax^2 + bx + c \leq 0$ or $ax^2 + bx + c \geq 0$.

If α and β are the real roots of the equation $ax^2 + bx + c = 0$ with $\alpha < \beta$, then

(i) $\alpha < x < \beta$ are the solutions of the inequality $ax^2 + bx + c < 0$.

(ii) $x > \beta$ or $x < \alpha$ are the solutions of the inequality $ax^2 + bx + c > 0$.

(iii) $\alpha \leq x \leq \beta$ are the solutions of the inequality $ax^2 + bx + c \leq 0$.

(iv) $x \geq \beta$ or $x \leq \alpha$ are the solutions of the inequality $ax^2 + bx + c \geq 0$.

Example

Solve the following inequalities:

(i) $x^2 - 5x + 6 < 0$.

(ii) $6x^2 - 5x - 6 \geq 0$

Solution:

(i) $x^2 - 5x + 6 < 0$ implies that $(x-2)(x-3) < 0$. Since 2 and 3 are the roots of the equation $x^2 - 5x + 6 = 0$ with $2 < 3$, it follows that the solutions of the inequalities $x^2 - 5x + 6 < 0$ are $2 < x < 3$.

$$\text{Number Line:} \quad 2\circ \xrightarrow{2<x<3}\!\!\!\!\longleftarrow \circ 3$$

(ii) $6x^2 - 5x - 6 \geq 0$ implies that $(2x-3)(3x+2) \geq 0$. Since -2/3 and 3/2 are the roots of the equation $6x^2 - 5x - 6 = 0$ with $-2/3 < 3/2$, it follows that the solutions of the inequalities $6x^2 - 5x - 6 \geq 0$ are $x \leq -2/3$ or $x \geq 3/2$.

$$\text{Number Line:} \quad \longleftarrow\!\!\!\!\xleftarrow{x\leq -2/3} \bullet -2/3 \qquad 3/2\bullet \xrightarrow{x\geq 3/2}\!\!\!\!\longrightarrow X$$

Inequalities Reducible to Quadratic Inequalities

Example

Solve the following inequalities:

(i) $\frac{2x-1}{x+3} < \frac{2}{3}$.

(ii) $\frac{x-1}{x-2} > \frac{x-2}{x-3}$.

Solution:

(i) Multiplying through by $(x+3)^2$ which is a positive quantity we have

$$\begin{aligned} 3(2x-1)(x+3) &< 2(x+3)^2 \\ \Rightarrow 4x^2+3x-27 &< 0 \\ \Rightarrow (x+3)(4x-9) &< 0 \\ \therefore -3 &< x < 9/4. \end{aligned}$$

$$\text{Number Line:} \quad -3\circ \overset{-3<x<9/4}{\longrightarrow\!\!\longleftarrow} \circ 9/4$$

(ii)

$$\begin{aligned} \frac{x-1}{x-2} &> \frac{x-2}{x-3} \\ \frac{x-1}{x-2}-\frac{x-2}{x-3} &> 0 \\ \Rightarrow \frac{-1}{x^2-5x+6} &> 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow -(x^2 - 5x + 6) &> 0 \times (x^2 - 5x + 6)^2 \\ \Rightarrow x^2 - 5x + 6 &< 0 \\ \Rightarrow (x-2)(x-3) &< 0 \\ \therefore 2 &< x < 3. \end{aligned}$$

$$\text{Number Line:} \quad 2\circ \xrightarrow{2<x<3} \!\!\!\!\! \xleftarrow{} \circ 3$$

Example

Find the range of values of x for which

$$\frac{2x^2 - 3x - 5}{x^2 + 2x + 6} < \frac{1}{2}.$$

Solution: By completing the squares, we can express $x^2 + 2x + 6$ as $(x+1)^2 + 5$ which shows that $x^2 + 2x + 6$ is always positive. Hence we can multiply both sides of the given inequality by $x^2 + 2x + 6$ without affecting the position of the inequality. Hence we obtain

$$
\begin{aligned}
4x^2 - 6x - 10 &< x^2 + 2x + 6 \\
\Rightarrow 3x^2 - 8x - 16 &< 0 \\
\Rightarrow (3x+4)(x-4) &< 0 \\
\therefore -4/3 &< x < 4.
\end{aligned}
$$

$$
\text{Number Line:} \quad -4/3\circ \overset{-4/3<x<4}{\longrightarrow\!\!\longleftarrow} \circ 4
$$

Example

Find the range of values of x given that $\mid x-1 \mid > 3 \mid x-2 \mid$.

Solution: For $x < 2$, we have $\mid x-2 \mid < 0$. Dividing through by $\mid x-2 \mid$ we have

$$
\frac{\mid x-1 \mid}{\mid x-2 \mid} < 3
$$

$$\begin{aligned}\Rightarrow \left| \frac{x-1}{x-2} \right| &< 3\\ \Rightarrow -3 &< \frac{x-1}{x-2} < 3\end{aligned}$$

From the 1st inequality we have

$$\begin{aligned}-3(x-2)^2 &< (x-1)(x-2)\\ \Rightarrow 4x^2-15x+14 &> 0\\ \Rightarrow (4x-7)(x-2) &> 0\\ \therefore x &< 7/4 \quad \text{or} \quad x > 2.\end{aligned}$$

From the 2nd inequality we have

$$\begin{aligned}(x-1)(x-2) &< 3(x-2)^2\\ \Rightarrow 2x^2-9x+10 &> 0\\ \Rightarrow (2x-5)(x-2) &> 0\end{aligned}$$

$$\therefore x < 2 \quad \text{or} \quad x > 5/2.$$

The required solution is $\therefore 7/4 < x < 5/2$ with $x \neq 2$.

$$\text{Number Line:} \quad 7/4 \circ \overset{7/4<x<5/2}{\longrightarrow\!\longleftarrow} \circ 5/2$$

Applications of inequalities

Linear and quadratic inequalities have so many real life applications in Sciences, Engineering and Technology.

Example

A rectangular plot of breadth $2x$ m is located inside a right-angled isosceles triangular farm plot of base length 10m for keeping of seedlings.

(a) Show that the area of the rectangular plot is $2x(5-x)$ m^2.

(b) If the area of the rectangular plot is to lie between 8 m^2 and 12 m^2, find the range of values of x.

Solution:

(a) Let ABC be the isosceles triangular plot such that $B\hat{A}C = 90^o$, $AB = AC$ and $BC = 10$. Let $EFGH$ be the rectangular plot such that E is on AB, F and G on BC and H on AC such that $EH = FG = 2x$. Let $EF = HG = y$. It can be shown that $y = 5 - x$ and

$$\text{Area of rectangle } EFGH = y \times 2x = 2x(5 - x).$$

(b) If the area of the rectangular plot is to lie between 8 m^2 and 12 m^2, then we must have

$$\begin{aligned}
8 &< 2x(5-x) < 12 \\
\Rightarrow 8 &< 10x - 2x^2 < 12 \\
\Rightarrow x^2 - 5x + 4 &< 0, \quad x^2 - 5x + 6 > 0 \\
\Rightarrow (x-1)(x-4) &< 0, \quad (x-2)(x-3) > 0 \\
\therefore 1 &< x < 4, \quad x < 2 \quad \text{or} \quad x > 3.
\end{aligned}$$

The required range of values of x is $\therefore 1 < x < 2$ or $3 < x < 4$.

$$\text{Number Lines:} \quad 1\circ \xrightarrow{1<x<2}\!\!\!\!\longleftarrow \circ 2 \quad \text{or} \quad 3\circ \xrightarrow{3<x<4}\!\!\!\!\longleftarrow \circ 4$$

Practice Problems 2

(1) Find range of values of x for which:

(a) $-5 \leq \frac{1}{3}x - 1 \leq 1$.

(b) $1 \leq \frac{1}{4}(3x + 1)4 < 2$.

(c) $\frac{1}{3}(x + 1) - \frac{1}{4}(x - 1) \geq \frac{1}{2}$.

(d) $-\frac{1}{2} \leq \frac{1}{5}(3 - 2x) \leq 1$.

(2) Solve the following inequalities:

(a) $(3x - 1)(x + 1) \leq x(x + 3)$.

(b) $1 \leq \log_2(x^2 + x + 2) \leq 2$.

(c) $2\sqrt{x^2 - x + 2} < 3$.

(d) $(2x - 1)(x + 2) > (x + 1)(x - 2)$.

(3) Solve the following inequalities:

(a) $2\left(\frac{2}{x} - \frac{1}{3}\right) > -\frac{1}{6}$.

(b) $\frac{1}{3x} \leq \frac{1}{2x} + \frac{1}{4}$.

(c) $\frac{1}{5x} > \frac{4}{15x} - \frac{1}{20}$.

(d) $\frac{5}{3x} + \frac{3}{4} \geq \frac{1}{12}$.

(e) $\frac{2}{x} < \frac{5}{x} + 2$.

(4) For what values of x is:

(a) $\mid x-1 \mid = 3 \mid x+1 \mid$.

(b) $\mid \frac{1}{2}(x+1) - \frac{1}{3}(x-1) \leq 1$.

(c) $\mid \frac{1}{x+1} \mid = 1$.

(d) $\mid \frac{x-3}{x+1} \mid < 2$.

(e) $\frac{x^2-5x+6}{x-4} \geq 0$.

(5) (a) The dimensions of a rectangular farm plot are $(x-1)$ m and $(3x+2)$ m. What is the range of values of x, if the area of the rectangle must not be less than 12 m^2 or greater that 22 m^2 ?

(b) An experimental farm plot is to be in the shape of a right-angled triangle. If the two shorter sides of the triangle are x m and $(x+1)$ m, find the range of possible values of x if the hypotenuse is to be longer than 5 m but shorter than $\sqrt{85}$ m.

(c) The resistance, R units of force, to the motion of a planter planting soyabeans moving at velocity v ms^{-1} is given by

$$R = 0.1v + 0.5v^2.$$

If R varies between 1.8 and 7.6 units, find the range of values of v.

Theory of Quadratic Equations

Definition

Quadratic equations are equations of the second degree of the form

$$ax^2 + bx + c = 0 \quad \text{where} \quad a, b, c \in \mathbb{R} \quad \text{but} \quad a \neq 0 \tag{1}$$

Any quadratic equation must have two solutions/roots which may be real or complex.

Dividing through equation (1) by a, we have

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \tag{2}$$

$$\Rightarrow x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2 = 0$$

$$\Rightarrow \left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2} = 0 \tag{3}$$

$$
\begin{aligned}
\Rightarrow \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
\Rightarrow x + \frac{b}{2a} &= \sqrt{\frac{b^2 - 4ac}{4a^2}} \\
\Rightarrow x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
\therefore x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (4)
\end{aligned}
$$

If we set $D \equiv b^2 - 4ac$ where D is called the discriminant of equation (1), then equation (4) will become

$$
x = \frac{-b \pm \sqrt{D}}{2a} \qquad (5)
$$

If α and β are the roots of equation (1), then from equation (5), we can write

$$
\alpha = \frac{-b + \sqrt{D}}{2a} \qquad (6)
$$

$$\beta = \frac{-b-\sqrt{D}}{2a} \tag{7}$$

By adding equations (6) and (7) and also multiplying equations (6) and (7) we obtain

$$\alpha+\beta = \text{Sum of Roots } = -\frac{b}{a} \tag{8}$$

$$\alpha\times\beta = \text{Product of Roots } = \frac{c}{a} \tag{9}$$

It follows from equation (2) that any quadratic equation can be written in the form

$$x^2 - [\text{Sum of Roots}]x + \text{ Product of Roots} = 0 \tag{10}$$

Example

(a) Find the quadratic equation whose roots are 3/2 and -2/5.
(b) If α and β are the roots of the equation $2x^2-4x-3=0$, find an equation whose roots are:
    (i) α^2 and β^2.

(ii) $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.
(iii) α^3 and β^3.
(iv) $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$.
(v) $\frac{\alpha}{\beta^2}$ and $\frac{\beta}{\alpha^2}$.

Solution: (a)

$$\begin{aligned}\text{Sum of Roots} &= \frac{3}{2}+\frac{-2}{5}=\frac{11}{10}\\ \text{Product of Roots} &= \frac{3}{2}\times\frac{-2}{5}=-\frac{3}{5}\\ \therefore x^2-\frac{11}{10}x-\frac{3}{5} &= 0\\ \therefore 10x^2-11x-6 &= 0 \quad \text{is the required equation.}\end{aligned}$$

(b) $\alpha+\beta=2$ and $\alpha\beta=-\frac{3}{2}$.
(i)

$$\begin{aligned}\text{Sum of Roots} &= \alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=4+3=7\\ \text{Product of Roots} &= \alpha^2\times\beta^2=(\alpha\beta)^2=\frac{9}{4}\end{aligned}$$

$$\therefore x^2 - 7x + \frac{9}{4} = 0$$
$$\therefore 4x^2 - 28x + 9 = 0 \text{ is the required equation.}$$

(ii)

$$\begin{aligned}
\text{Sum of Roots} &= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = -\frac{14}{3} \\
\text{Product of Roots} &= \frac{\alpha}{\beta} \times \frac{\beta}{\alpha} = 1 \\
\therefore x^2 + \frac{14}{3}x + 1 &= 0 \\
\therefore 3x^2 + 14x + 3 &= 0 \text{ is the required equation.}
\end{aligned}$$

(iii)

$$\begin{aligned}
\text{Sum of Roots} &= \alpha^3 + \beta^3 = (\alpha + \beta)\left((\alpha + \beta)^2 - 3\alpha\beta\right) = 17 \\
\text{Product of Roots} &= \alpha^3 \times \beta^3 = (\alpha\beta)^3 = -\frac{27}{8}
\end{aligned}$$

$$\therefore x^2 - 17x - \frac{27}{8} = 0$$

$$\therefore 8x^2 - 136x - 27 = 0 \quad \text{is the required equation.}$$

(iv)

$$\text{Sum of Roots} = \frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3 + \beta^3}{\alpha\beta} = -\frac{34}{3}$$

$$\text{Product of Roots} = \frac{\alpha^2}{\beta} \times \frac{\beta^2}{\alpha} = \alpha\beta = -\frac{3}{2}$$

$$\therefore x^2 + \frac{34}{3}x - \frac{3}{2} = 0$$

$$\therefore 6x^2 + 68x - 9 = 0 \quad \text{is the required equation.}$$

(v)

$$\text{Sum of Roots} = \frac{\alpha}{\beta^2} + \frac{\beta}{\alpha^2} = \frac{\alpha^3 + \beta^3}{(\alpha\beta)^2} = \frac{68}{9}$$

$$\text{Product of Roots} = \frac{\alpha}{\beta^2} \times \frac{\beta}{\alpha^2} = \frac{1}{\alpha\beta} = -\frac{2}{3}$$

$$\therefore x^2 - \frac{68}{9}x - \frac{2}{3} = 0$$

$$\therefore 9x^2 - 68x - 6 = 0 \quad \text{is the required equation.}$$

Nature of Roots of Quadratic Equations

The nature and types of roots of any quadratic equation $ax^2 + bx + c = 0$ can be determined by the discriminant D of the equation given by

$$D = b^2 - 4ac \tag{11}$$

There are 3 possible cases:

(i) **Two Equal Roots:** For two equal roots,

$$D = b^2 - 4ac = 0 \quad \therefore \quad b^2 = 4ac \tag{12}$$

(ii) **Two Real Roots:** For two real roots,

$$D = b^2 - 4ac \geq 0 \quad \therefore \quad b^2 \geq 4ac \tag{13}$$

(iii) **Two Complex Roots:** For two complex roots,

$$D = b^2 - 4ac < 0 \quad \therefore \quad b^2 < 4ac \tag{14}$$

Example

(a) Find k if the equation $(7k+1)x^2 + (5k-1)x + k = 1$ has equal roots.

(b) For what values of k does the equation $x^2 - (k+4)x + 9 = 0$ have real roots?

(c) Show that the equation $a^2x^2 - ax + 1 = 0$ can never have real roots.

Solution: (a)
$(7k+1)x^2 + (5k-1)x + k = 1 \;\Rightarrow\; (7k+1)x^2 + (5k-1)x + k - 1 = 0$. For equal roots, we have

$$\begin{aligned}
(5k-1)^2 &= 4(k-1)(7k+1)\\
\Rightarrow 3k^2 - 14k - 5 &= 0\\
\Rightarrow (3k+1)(k-3) &= 0\\
\therefore k &= -\frac{1}{3} \quad \text{or} \quad k = 3.
\end{aligned}$$

(b) For real roots, we have

$$
\begin{aligned}
(-(k+4))^2 &\geq 36 \\
\Rightarrow k^2 + 8k - 20 &\geq 0 \\
\Rightarrow (k-2)(k+10) &\geq 0 \\
\therefore k &\leq -10 \quad \text{or} \quad k \geq 2.
\end{aligned}
$$

$$
\text{Number Line:} \quad \xleftarrow{x \leq -10} \bullet -10 \qquad 2\bullet \xrightarrow{x \geq 2} K
$$

(c) For real roots, we have

$$
\begin{aligned}
(-a)^2 &\geq 4a^2 \\
\Rightarrow 3a^2 &\leq 0 \\
\therefore a &\leq 0.
\end{aligned}
$$

This is a contradiction since a cannot be 0. Hence $a^2x^2 - ax + 1 = 0$ can never have real roots.

Maximum and Minimum Values of Quadratic Functions

Definition

A quadratic function is a function given by

$$y = f(x) = ax^2 + bx + c \quad \text{where} \quad 0 \neq a, b, c \in \mathbb{Z} \quad \text{are the coefficients} \tag{15}$$

By completing the squares, we have

$$\begin{aligned} y &= a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] \\ &= a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right] \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right] \\ &= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \end{aligned} \tag{16}$$

When $a < 0$, we have from equation (16)

$$y = y_{\max} = \frac{4ac - b^2}{4a} \text{ which occurs when } x = x_{\max} = -\frac{b}{2a} \tag{17}$$

When $a > 0$, we have from equation (16)

$$y = y_{\min} = \frac{4ac - b^2}{4a} \text{ which occurs when } x = x_{\min} = -\frac{b}{2a} \tag{18}$$

Example

Find the maximum or minimum values of the following quadratic functions and obtain the corresponding values of x for which they occur.

(a) $y = -4x^2 + 3x - 2$,

(b) $y = 5x^2 - 4x - 3$.

Solution: (a) $a = -4 < 0, b = 3, c = -2$

$$\begin{aligned} y_{\max} &= \frac{32 - 9}{-16} = -\frac{23}{16}. \\ x_{\max} &= -\frac{3}{-8} = \frac{3}{8}. \end{aligned}$$

(b) $a = 5 > 0, b = -4, c = -3$

$$\begin{aligned} y_{\min} &= \frac{-60-16}{20} = -\frac{19}{5}. \\ x_{\min} &= -\frac{-4}{10} = \frac{2}{5}. \end{aligned}$$

Example

Show that for real values of x, the function

$$\frac{x+2}{x^2+3x+6}$$

cannot be greater than $1/3$ nor less than $-1/5$. Find for what values of x, if any, it attains these values.

Solution: Let $\frac{x+2}{x^2+3x+6} = k$ so that $kx^2 + (3k-1)x + 2(3k-1) = 0$. For real roots, we must have

$$(3k-1)^2 \geq 8k(3k-1)$$

$$
\begin{aligned}
\Rightarrow 15k^2 - 2k - 1 &\leq 0 \\
\Rightarrow (3k-1)(5k+1) &\leq 0 \\
\therefore -\frac{1}{5} &\leq k \leq \frac{1}{3} \\
\therefore -\frac{1}{5} &\leq \frac{x+2}{x^2+3x+6} \leq \frac{1}{3}.
\end{aligned}
$$

This shows that the function cannot be greater than $1/3$ nor less than $-1/5$. When $k = 1/3$, we have

$$
\begin{aligned}
\frac{x+2}{x^2+3x+6} &= \frac{1}{3} \\
\Rightarrow x^2 &= 0 \\
\therefore x &= 0.
\end{aligned}
$$

When $k = -1/5$, we have

$$
\begin{aligned}
\frac{x+2}{x^2+3x+6} &= -\frac{1}{5} \\
\Rightarrow (x+4)^2 &= 0
\end{aligned}
$$

$$\therefore x = -4.$$

Example

Given that

$$\frac{1-2x}{x^2+2} = k,$$

form a quadratic equation in x. If x is always real, show that k must lie between two values and find these values. Hence state the maximum and minimum values of the expression

$$\frac{1-2x}{x^2+2}.$$

Solution: $\frac{1-2x}{x^2+2} = k \Rightarrow kx^2 + 2x + (2k-1) = 0$. For real roots, we have

$$\begin{aligned} 4 &\geq 4k(2k-1) \\ \Rightarrow 2k^2 - k - 1 &\leq 0 \\ \Rightarrow (k-1)(2k+1) &\leq 0 \end{aligned}$$

$$\begin{aligned} \therefore -\frac{1}{2} &\leq k \leq 1 \\ \therefore -\frac{1}{2} &\leq \frac{1-2x}{x^2+2} \leq 1. \end{aligned}$$

Hence the maximum value if the expression is 1 and the minimum value is -1/2.

Theory of Cubic Equations

Definition

A cubic equation is an equation of the form

$$ax^3 + bx^2 + cx + d = 0 \tag{19}$$

where $a, b, c, d \in \mathbb{Z}$ with $a \neq 0$.

Dividing through Equation (19) by a, we obtain

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0 \tag{20}$$

If α,β and γ are the roots of Equation (20), then we have

$$\begin{aligned} (x-\alpha)(x-\beta)(x-\gamma) &= 0 \\ \Rightarrow x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\alpha\gamma+\beta\gamma)x - \alpha\beta\gamma &= 0 \qquad (21) \end{aligned}$$

Comparing Equations (20) and (21), we obtain

$$\alpha+\beta+\gamma = -\frac{b}{a} \qquad (22)$$

$$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a} \qquad (23)$$

$$\alpha\beta\gamma = -\frac{d}{a} \qquad (24)$$

which give the relationships between the roots and coefficients of Equation (19).

Example

If p, q and r are the roots of the equation $2x^3 - 3x^2 - 4x - 2 = 0$, find the value of:

(i) $p+q+r$.

(ii) $pq+pr+qr$.

(iii) pqr.

Solution: Comparing the given equation with Equation (19), we have $a = 2, b = -3, c = -4, d = -2$ and $\therefore$

$$(i) \quad p + q + r = -\frac{b}{a} = \frac{3}{2}.$$

$$(ii) \quad pq + pr + qr = \frac{c}{a} = -2.$$

$$(iii) \quad pqr = -\frac{d}{a} = 1.$$

Example

Given that the sum of two roots of the equation $x^3 - 5x^2 - 8x + 12 = 0$ is 7, find the roots of the equation.

Solution: In this problem, $a = 1, b = -5, c = -8, d = 12$. Let p, q, r be the three roots of the equation such that $p + q = 7$ from which we obtain $q = 7 - p$. But then,

$$\begin{aligned} p + q + r &= 5 \qquad (1) \\ pq + pr + qr &= -8 \qquad (2) \end{aligned}$$

$$pqr = -12 \qquad (3)$$

From (2) we have

$$\begin{aligned} 7+r &= 5 \\ \therefore r &= -2. \end{aligned}$$

Using the value of r in (3), we obtain

$$\begin{aligned} pq &= 6 \\ \Rightarrow p(7-p) &= 6 \\ \Rightarrow p^2-7p+6 &= 0 \\ \Rightarrow (p-1)(p-6) &= 0 \\ \therefore p &= 1 \text{ or } p=6 \\ \therefore q &= 6 \text{ or } q=1. \end{aligned}$$

Hence the roots of the equation are $6, 1, -2$.

Example

Given that the roots of the equation $2x^3 + 3x^2 - 11x - 6 = 0$ are in arithmetic progression, solve the equation.

Solution Let $\alpha = p - d, \beta = p, \gamma = p + d$ be the roots of the equation. Then

$$\begin{aligned}
\alpha + \beta + \gamma &= p - d + p + p + d = 3p = -\frac{3}{2} \\
\therefore p &= -\frac{1}{2} \\
\alpha\beta + \alpha\gamma + \beta\gamma &= p(p-d) + (p-d)(p+d) + p(p+d) \\
&= -\frac{11}{2} \\
\Rightarrow p^2 - pd + p^2 - d^2 + p^2 + pd &= -\frac{11}{2} \\
\Rightarrow 3p^2 - d^2 &= -\frac{11}{2}
\end{aligned}$$

$$\Rightarrow d^2 = \frac{25}{4}$$

$$\therefore d = \pm\sqrt{\frac{25}{4}} = \pm\frac{5}{2}$$

$$\therefore x = -\frac{1}{2}, -3, 2 \text{ are the roots.}$$

Practice Problems 3

1. Find the quadratic equation whose roots are:
   (a) $\frac{1}{1+\sqrt{2}}$ and $\frac{1}{1-\sqrt{2}}$.
   (b) $\frac{1}{2}(1-\sqrt{5})$ and $\frac{1}{2}(1+\sqrt{5})$.
   (c) $\frac{2}{\sqrt{2}-\sqrt{3}}$ and $\frac{2}{\sqrt{2}+\sqrt{3}}$.
   (d) $10-\sqrt{10}$ and $10+\sqrt{10}$.
2. If α and β are the roots of the equation $3x^2+4x-5=0$, find the values of the following:
   (a) $\alpha-\beta$.
   (b) $\alpha^2-\beta^2$.
   (c) $\alpha^3-\beta^3$.
   (d) $\alpha^4+\beta^4$.

(e) $\alpha^4 - \beta^4$.

(3) If α and β are the roots of the equation $-5x^2 + 4x - 3 = 0$, obtain the quadratic equation whose roots are:

- (a) $2\alpha^2 + 3$ and $2\beta^2 + 3$.
- (b) $3\alpha^3 - 4$ and $3\beta^3 - 4$.
- (c) $4\alpha^4 + 5$ and $4\beta^4 + 5$.
- (d) $\frac{\alpha+1}{\beta+1}$ and $\frac{\beta+1}{\alpha+1}$

(4) (a) If one roots of the equation $x^2 - px + q = 0$ is square of the other, show that

$$p^3 - q(3p + 1) - q^2 = 0.$$

- (b) Form the quadratic equation whose sum of the roots is 5 and the sum of the squares of the roots is 53.
- (c) If the quadratic equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ $(a \neq b)$ have a common root, show that the solutions of $2x^2 + (a + b)x = (a + b)^2$ are $x = 1$ and $x = -\frac{1}{2}$.
- (d) The roots of the equation $x^2 + px + q = 0$ are α and β.
  - (i) Given that the roots differ by $2\sqrt{3}$ and that the sum of the reciprocals of the roots is 4, find the possible values of p and q.
  - (ii) Find an equation whose roots are $\alpha + \frac{2}{\beta}$ and $\beta + \frac{2}{\alpha}$, expressing the coefficients in terms of p and q.

(5) (a) Find the values of k if the equation $(k+3)x^2 - (11k+1)x + k = 2(k-5)$ has equal roots.

(b) Show that the equation $x^2 - 2kx + 3k^2 + m^2 = 0$ cannot have real roots if k and m are real.

(c) Find the range of values of k for which the equation $3x^2 - 3kx + (k^2 - k - 3) = 3$ has real roots.

(d) Find the range of values of k if $2x^2 + x + k$ is always positive.

(6) Find the maximum and minimum values of the following quadratic functions and the values of x for which they occur:

(a) $y = -\frac{3}{5}x^2 + \frac{3}{4}x - \frac{1}{2}$.

(b) $y = 10x^2 - 11x - 20$.

(c) $y = \frac{9}{10}x^2 - \frac{7}{8}x + \frac{5}{6}$.

(d) $y = -64x^2 + 30x - 77$.

(e) $y = 300x^2 - 200x - 100$.

(f) $y = \frac{3}{5} + 20x - \frac{5}{2}x^2$.

(7)

(a) If x is real, show that

$$\frac{(x-2)^2+16}{2(x+2)}$$

can take on any real value which does not lie between $-4(1+\sqrt{2})$ and $4(-1+\sqrt{2})$.

(b) Find the possible values of k if

$$\frac{x^2+3x-4}{5x-k}$$

may be capable of taking on all values when x is real.

(c) Show that

$$\frac{5}{x^2+3x+3}$$

is positive for all real values of x and find its greatest values.

(8)

(a) Given that the sum of two roots of the equation $4x^3-4x^2-5x+3=0$ is 2, obtain the roots of the equation.

(b) If the roots of the equation $x^3+3x^2-6x-8=0$ are in geometric progression, find the roots.

Some equations which are not quadratic in nature can be reduced or transformed into quadratic equations by some transformations or substitutions. The following examples are typical ones.

Example

Solve the equation $6x^4 - 13x^2 + 6 = 0$.

Solution: $6x^4 - 13x^2 + 6 = 0 \Rightarrow 6\left(x^2\right)^2 - 13x^2 + 6 = 0$. Putting $x^2 = k$, we have

$$\begin{aligned} 6k^2 - 13k + 6 &= 0 \\ \Rightarrow (3k-2)(2k-3) &= 0 \\ \therefore k &= 2/3 \text{ or } k = 3/2 \\ \therefore x &= \pm\sqrt{2/3} \text{ or } x = \pm\sqrt{3/2}. \end{aligned}$$

Example

Find the real values of x given that:

(a) $36x - 5\sqrt{x} - 1 = 0$,

(b) $\frac{36}{x^2} - \frac{5}{x} - 1 = 0$.

Solution: (a) $36x - 5\sqrt{x} - 1 = 0 \;\Rightarrow\; 36(\sqrt{x})^2 - 5\sqrt{x} - 1 = 0$. Putting $\sqrt{x} = k$, we have $36k^2 - 5k - 1 = 0$

$$\begin{aligned} \Rightarrow (9k+1)(4k-1) &= 0 \\ \therefore k &= -1/9 \text{ or } k = 1/4 \\ \therefore x &= 1/81 \text{ or } x = 1/16. \end{aligned}$$

(b) $\frac{36}{x^2} - \frac{5}{x} - 1 = 0 \;\Rightarrow\; 36\left(\frac{1}{x}\right)^2 - 5\left(\frac{1}{x}\right) - 1 = 0$. Putting $\frac{1}{x} = k$, we have

$$\begin{aligned} 36k^2 - 5k - 1 &= 0 \\ \Rightarrow (9k+1)(4k-1) &= 0 \\ \therefore k &= -1/9 \text{ or } k = 1/4 \\ \therefore x &= -9 \text{ or } x = 4. \end{aligned}$$

Example

Solve the following equations.

(a) $5^{2x} - 5^{1+x} + 6 = 0$.

(b) $x^4 + \frac{9}{x^2} - 4\left(x + \frac{3}{x}\right) - 6 = 0$.

Solution: (a)

$$\begin{aligned}
5^{2x} - 5^{1+x} + 6 &= 0 \\
\Rightarrow (5^x)^2 - 5 \times 5^x + 6 &= 0 \\
\Rightarrow k^2 - 5k + 6 &= 0 \quad [\text{where} \quad 5^x = k] \\
\Rightarrow (k-2)(k-3) &= 0
\end{aligned}$$

$$\begin{aligned}
\therefore k &= 2 \quad \text{or} \quad k = 3 \\
\therefore x &= \frac{\log 2}{\log 5} \quad \text{or} \quad x = \frac{\log 3}{\log 5}.
\end{aligned}$$

(b) Note that

$$
\begin{aligned}
\left(x+\frac{3}{x}\right)^2 &= x^2+\frac{9}{x^2}+6\\
\therefore x^2+\frac{9}{x^2} &= \left(x+\frac{3}{x}\right)^2-6\\
\therefore x^4+\frac{9}{x^2}-4\left(x+\frac{3}{x}\right)-6 &= 0\\
\Rightarrow \left(x+\frac{3}{x}\right)^2-4\left(x+\frac{3}{x}\right)-12 &= 0\\
\Rightarrow k^2-4k-12 &= 0 \quad \text{where} \quad x+\frac{3}{x}=k\\
\Rightarrow (k+2)(k-6) &= 0\\
\therefore k &= -2 \quad \text{or} \quad k=6.
\end{aligned}
$$

When $k=-2$, we have $x^2+2x+3=0$ which has no real solutions. When $k=6$, we have $x^2-6x+3=0$ which has real solutions $x=3\pm\sqrt{6}$.

Practice Problems 4

1. Solve the following equations:

$$(a) \qquad 6\sqrt{\frac{2x}{x-1}} + 5\sqrt{\frac{x-1}{2x}} = 13.$$

$$(b) \qquad \frac{x-1}{x+1} + \frac{x+2}{x-2} = 2 + \frac{2}{x+4}.$$

$$(c) \qquad x(x+1) + \frac{12}{x(x+1)} = 8.$$

2. Solve the following equations:

$$(a) \qquad x^4 - 2x^3 - 2x^2 + 2x + x = 0 \quad \left[\text{Hint: put} \quad k = x - \frac{1}{x}\right].$$

$$(b) \qquad \sqrt{\frac{p+x}{q+x}} - \sqrt{\frac{q+x}{p+x}} = \frac{3}{2}.$$

$$(c) \qquad \frac{x+7}{x+5} + \frac{x+9}{x+7} = \frac{x+6}{x+4} + \frac{x+10}{x+8}.$$

5. Simultaneous Equations

Simultaneous equations in two unknowns say x and y are of different types and forms. The two equations may be linear, one may be linear and the other quadratic and the two equations may be quadratic. When the two equations are linear, the solutions (x, y) may be found either by elimination method, substitution method or graphical method. When one of the equations is linear and the other is quadratic, or, when the two equations are quadratic, there are no known specific methods in obtaining the solutions (x, y). In these cases,the solutions can be found by ingenuity.

Example

Solve the simultaneous equations

$$\begin{aligned} 5y + 1 &= 2xy \qquad (1) \\ 3x + y - 7 &= xy \qquad (2) \end{aligned}$$

Solution: From (1) and (2) we obtain

$$\begin{aligned} y(2x-5) &= 1 \qquad (3) \\ y(x-1) &= 3x-7 \qquad (4) \end{aligned}$$

Dividing (3) by (4) to have

$$\begin{aligned} \frac{2x-5}{x-1} &= \frac{1}{3x-7} \\ \Rightarrow (2x-5)(3x-7) &= x-1 \\ \Rightarrow 6x^2-30x+36 &= 0 \\ \Rightarrow x^2-5x+6 &= 0 \\ \Rightarrow (x-2)(x-3) &= 0 \\ \therefore x &= 2 \quad \text{or} \quad x=3 \\ \therefore y &= -1 \quad \text{or} \quad y=1 \\ \therefore (x,y) &= (2,-1) \quad \text{or} \quad (3,1) \quad \text{are the solutions.} \end{aligned}$$

Example

Solve the simultaneous Equations

$$\begin{aligned} \frac{x}{3} + \frac{y}{4} &= 1 \qquad (1) \\ \frac{3}{x} - \frac{2}{y} &= \frac{7}{12} \qquad (2) \end{aligned}$$

Solution: From (1) and (2) we have

$$\begin{aligned} \frac{3}{x} &= \frac{4}{4-y} \qquad (3) \\ \frac{3}{x} &= \frac{24+7y}{12y} \qquad (4) \end{aligned}$$

Equating (3) and (4) we have

$$\begin{aligned} \frac{4}{4-y} &= \frac{24+7y}{12y} \\ \Rightarrow (4-y)(24+7y) &= 48y \\ \Rightarrow 7y^2 + 44y - 96 &= 0 \\ \Rightarrow (y+8)(7y-12) &= 0 \\ \therefore y &= -8 \text{ or } y = \frac{12}{7} \end{aligned}$$

$$\therefore x = 9 \text{ or } x = \frac{12}{7}$$

$$\therefore (x, y) = (9, -8) \text{ or } \left(\frac{12}{7}, \frac{12}{7}\right) \text{ are the solutions.}$$

Example

Solve the simultaneous equations

$$x^2 - y^2 = 24 \qquad (1)$$

$$\frac{1}{x+y} + \frac{3}{x-y} = \frac{11}{12} \qquad (2)$$

Solution: From (2) we have

$$\frac{x - y + 3x + 3y}{(x-y)(x+y)} = \frac{11}{12}$$

$$\Rightarrow \frac{4x + 2y}{x^2 - y^2} = \frac{11}{12}$$

$$
\begin{aligned}
\Rightarrow \frac{4x+2y}{24} &= \frac{11}{12} \\
\Rightarrow 2x+y &= 11 \\
\therefore y &= 11-2x \qquad (3)
\end{aligned}
$$

Using (3) in (1) we have

$$
\begin{aligned}
x^2-(11-2x)^2 &= 24 \\
\Rightarrow 3x^2-44x-121 &= 0 \\
\Rightarrow (x-5)(3x-29) &= 0 \\
\therefore x &= 5 \text{ or } x=\frac{29}{3} \\
\therefore y &= 1 \text{ or } y=-\frac{25}{3} \\
\therefore (x,y) &= (5,1) \text{ or } \left(\frac{29}{3},-\frac{25}{3}\right) \text{ are the solutions.}
\end{aligned}
$$

Example

Solve the simultaneous equation

$$\begin{aligned} x^2 + y^2 &= 13 \qquad (1) \\ x^2 - 3xy + 2y^2 &= 35 \qquad (2) \end{aligned}$$

Solution: Putting $y = kx$ in (1) and (2) we have

$$\begin{aligned} (1 + k^2)x^2 &= 13 \qquad (3) \\ (1 - 3k + 2k^2)x^2 &= 35 \qquad (4) \end{aligned}$$

Dividing (4) by (3) to have

$$\begin{aligned} \frac{1 - 3k + 2k^2}{1 + k^2} &= \frac{35}{13} \\ \Rightarrow 9k^2 + 39k + 22 &= 0 \\ \Rightarrow (3k + 2)(3k + 11) &= 0 \\ \therefore k &= -\frac{2}{3} \text{ or } k = -\frac{11}{3} \end{aligned}$$

$$\therefore y = -\frac{2}{3}x \text{ or } y = -\frac{11}{3}x$$

Put $y = -\frac{2}{3}x$ in (1) to have

$$\begin{aligned} x^2 + \frac{4}{9}x^2 &= 13 \\ \Rightarrow 13x^2 &= 117 \\ \Rightarrow x^2 &= 9 \\ \therefore x &= \pm\sqrt{9} = \pm 3 \\ \therefore y &= \mp 2 \\ \therefore (x, y) &= (-3, 2) \quad \text{or} \quad (3, -2) \quad \text{are the solutions.} \end{aligned}$$

Put $y = -\frac{11}{3}x$ in (1) to have

$$\begin{aligned} x^2 + \frac{121}{9}x^2 &= 13 \\ \Rightarrow 130x^2 &= 117 \\ \Rightarrow x^2 &= \frac{117}{130} = \frac{9}{10} \end{aligned}$$

$$\therefore x = \pm\sqrt{\frac{9}{10}} = \pm\frac{3}{10}\sqrt{10}$$

$$\therefore y = \mp\frac{11}{10}\sqrt{10}$$

$$\therefore (x, y) = \left(-\frac{3}{10}\sqrt{10}, \frac{11}{10}\sqrt{10}\right) \text{ or } \left(\frac{3}{10}\sqrt{10}, -\frac{11}{10}\sqrt{10}\right) \text{ are the solutions.}$$

Practice Problems 5

1. Solve the following simultaneous equations.

$$(a)\ \ x^2 - xy + 7y^2 = 27, x^2 - y^2 = 15.$$

$$(b)\ \ x^2 + y^2 = 5, \frac{1}{x^2} + \frac{1}{y^2} = \frac{5}{4}.$$

$$(c)\ \ \frac{x}{y} + \frac{y}{x} = \frac{17}{4}, x^2 - 4xy + y^2 = 1.$$

$$(d)\ \ x^2 + y^2 = 10, \frac{1}{x} + \frac{1}{y} = \frac{4}{3}.$$

(2) Solve the following simultaneous equations.

$$(a)\ \ x + 2y = 7, x^2 + 2y^2 = 17.$$
$$(b)\ \ xy = 2(x + y - 1), yz = 2(y + z - 6), zx = 2(z + x - 4).$$
$$(c)\ \ 2x^2 + 3xy + 4y^2 = 1, 4x^2 + 3xy + 2y^2 = 2.$$
$$(d)\ \ xy - 3x - 3y + 12 = 0, 2xy + 4x + 4y = 56.$$

(3)

(a) Given the simultaneous equations $x^2 - 6xy + 11y^2 = 3k^2$, $x^2 - 2xy - 3y^2 = 5k^2$, derive an equation in x and y only, and hence solve the equations for x and y in terms of k.

(b) If $\frac{x^2}{y} + \frac{y^2}{x} = 12$ and $\frac{1}{x} + \frac{1}{y} = \frac{1}{3}$, find the values of xy and hence solve the equations.

(c) Given that

$$\log_2(x - 5y + 4) = 0 \ \text{ and } \ \log_2(x + 1) - 1 = 2\log_2 y,$$

find the values of x and y.

Useful Textbooks

1. Intermediate Pure Mathematics by Blakey J.
2. Pure Mathematics for Advanced Level by Bunday B.D. and Mulholland H.
3. Additional Mathematics for West Africa by Talbert J.F., Godman A. and Ogum G.E.O.
4. Pure Mathematics by Tranter C.J.
5. Mathematical Basics Volumes 1 and 2 by Sikiru T.O.
6. Further Mathematics by Egbe E., Odili G.A. and Ugbebor O.O.
7. Pure Mathematics by Backhouse J.K., Houldsworth S.P.T and Cooper B.E.D.
8. Any available online textbooks and materials in Pure Mathematics.