MATHEMATICS E-156, FALL 2014
Mathematical Foundations of Statistical Software

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**Proof of the Week #:** 5

**Theorem:** Proof of the week

For random variable $X$, define the moment generating function

$$M(t) = E[e^{tX}].$$

Prove that

- The $n$th derivative of $M(t)$, evaluated at $t = 0$, is equal to the $n$th moment $E[X^n]$.
- If $X_1$ and $X_2$ are independent random variables with moment generating functions $M_1(t)$ and $M_2(t)$, then

$$M_{X_1+X_2}(t) = M_1(t)M_2(t)$$

1.
$$e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + ...$$

Because expectation is linear the moment generating function may be rewritten as,

$$M(t) = E[e^{tX}] = 1 + tE[X] + \frac{t^2}{2!}E[X^2] + \frac{t^3}{3!}E[X^3] + ...$$

First derivative of $M(t)$

$$\frac{d}{dt}M(t) = 0 + E[X] + tE[X^2] + \frac{t^2}{2}E[x^3] + ...$$

$$\frac{d}{dt}M(0) = 0 + E[X] + 0 + ... = E[X]$$

Second derivative of $M(t)$

$$\frac{d^2}{dt^2}M(t) = 0 + 0 + E[X^2] + tE[X^3] + ...$$

$$\frac{d^2}{dt^2}M(0) = 0 + 0 + E[X^2] + 0 + ... = E[X^2]$$

Thus, by induction, we conclude that

$$M^n(t) = E[X^n]$$

2. Given that $X_1$ and $X_2$ are independent random variables,

$$E[f(X_1)g(X_2)] = E[f(X_1)]E[g(X_2)]$$

Thus, for moment generating function of $X_1 + X_2$,

$$M_{X_1+X_2}(t) = E[e^{t(X_1+X_2)}] = E[e^{tX_1}e^{tX_2}] = E[e^{tX_1}]E[e^{tX_2}] = M_{X_1}(t)M_{X_2}(t)$$