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Math E-156

Week 6 Proof of the Week

Given a normal distribution $X \sim N(\mu, \sigma^2)$,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$M(t) = E[e^{tX}] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tX} e^{-\frac{(X-\mu)^2}{2\sigma^2}} dx$$

The density function $f(x)$ integrates to 1 independent of the value of μ because it's with respect to x

$$z = \frac{X-\mu}{\sigma}$$

$$X = z\sigma + \mu$$

$$dX = \sigma\, dz$$

Perform a u-substitution for X

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{z\sigma t} e^{\mu t} e^{-\frac{z^2}{2}} \sigma\, dz = e^{\mu t} e^{\frac{\sigma^2 t^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\sigma t)^2}{2}} dz$$

Because,

$$e^{\frac{\sigma^2 t^2}{2}} e^{-\frac{(z-\sigma t)^2}{2}} = e^{z\sigma t} e^{-\frac{z^2}{2}}$$

$$\frac{\sigma^2 t^2}{2} - \frac{z^2 - 2z\sigma t + \sigma^2 t^2}{2} = z\sigma t - \frac{z^2}{2}$$

$$= e^{\mu t} e^{\frac{\sigma^2 t^2}{2}} = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

Where,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\sigma t^2)}{2}} dz = 1$$

$$M(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

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$$M'(t) = (\mu + \sigma^2 t) e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$M'(0) = E[X] = \mu$$

■

$M''(t) = \sigma^2 e^{\mu t + \frac{\sigma^2 t^2}{2}} + (\mu + \sigma^2 t) e^{\mu t + \frac{\sigma^2 t^2}{2}}$

$M''(0) = \sigma^2 + \mu^2$

$Var[X] = \sigma^2 + \mu^2 - \mu^2 = 0$

■

If $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ with X_1 and X_2 independent,

$M_{X_1+X_2}(t) = e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}} e^{\mu_2 t + \frac{\sigma_2^2 t^2}{2}}$

$= e^{(\mu_1+\mu_2)t + \frac{(\sigma_1^2+\sigma_2^2)t^2}{2}}$

$X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

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