MATHEMATICS E-156, SPRING 2014
MATHEMATICAL FOUNDATIONS OF STATISTICAL SOFTWARE
Proof #1

Richard Cruse

The variance Var$[X]$ of a random variable X is defined as $E[(X - E[X])^2]$.

Given that $E[a_1X_1 + a_2X_2] = a_1E[X_1] + a_2E[X_2]$ in all cases and that $E[X_1X_2] = E[X_1]E[X_2]$ for independent random variables, prove that

1. Var$[X] = E[X]^2 - (E[X])^2$.

   Proof.

$$\begin{aligned} Var[X] = E[(X - E[X])^2] &= E[X^2 - 2E[X]X + E[X]^2] \\ &= E[X^2] - 2(E[X])^2 + E[X]^2 \\ &= E[X^2] - (E[X])^2 \end{aligned}$$

$\square$

2. Var$[aX + b] = a^2$ Var$[X]$.

   Proof.

$$\begin{aligned} Var[aX + b] &= E[(aX + b)^2] - (E[aX + b])^2 \\ &= E[a^2X^2 + 2abX + b^2] - (aE[X] + b)^2 \\ &= a^2E[X^2] + 2abE[X] + b^2 - a^2E[X]^2 - 2abE[X] - b^2 \\ &= a^2Var[X] \end{aligned}$$

$\square$

3. If X_1 and X_2 are independent, Var$[X_1 + X_2] =$ Var$[X_1]$+Var$[X_2]$

   Proof.

$$\begin{aligned} Var[X_1 + X_2] &= E[(X_1 + X_2)^2] - (E[X_1] + E[X_2])^2 \\ &= E[X_1^2] + E[2X_1X_2] + E[X_2^2] - E[X_1]^2 - 2E[X_1]E[X_2] - E[X_2]^2 \\ &= Var[X_1] + Var[X_2] \end{aligned}$$

$\square$