MATHEMATICS E-156, FALL 2014
Mathematical Foundations of Statistical Software

Date: 13 Feb 2014
Name: Robert Shear
Proof of the Week #3:

Theorem: The sum of n independent Bernoulli random variables, each with parameter p, is a binomial random variable $Y \sim \text{Binom}(n,p)$, and

$$\text{E}[Y] = np, \text{Var}[Y] = np(1-p)$$

.

Proof: It is convenient to demonstrate the second part of the theorem first.

A Bernoulli random variable with parameter p, $X \sim \text{Bern}(p)$, is given by $X : S \to \{0, 1\}$ and $P(X = 1) = p$. Therefore,

$$\text{E}[X] = (1-p) \cdot 0 + p \cdot 1 = p.$$

Expectation is linear, so we may write

$$\text{E}[Y] = \sum_{i=1}^{n} \text{E}[X_i] = \sum_{i=1}^{n} p = np.$$

The variance of X may be expressed as the expectation of the squares minus the square of the expectation:

$$\begin{aligned}\text{Var}[X] &= \text{E}[X^2] - \text{E}[X]^2 \\ &= (1-p) \cdot 0^2 + p \cdot 1^2 - p^2 \\ &= p - p^2 \\ &= p\,(1-p)\,.\end{aligned}$$

By hypothesis, the X_i are independent and so the sum of the variances is the variance of the sums. Thus,

$$\begin{aligned}\text{Var}[Y] &= \sum_{i=1}^{n} \text{Var}[X_i] \\ &= \sum_{i=1}^{n} p\,(1-p) \\ &= np\,(1-p)\,.\end{aligned}$$

To say that $Y \sim \text{Binom}(n, p)$ is to say that the probability density function of $Y = \sum_{i=1}^{n} X_i$ is given by

$$P(Y = y) = \binom{n}{y} p^y \left(1 - p\right)^{n-y}.$$

If $Y = y$ then exactly y of the X's have the value 1. The probability of this occuring is p^y. There remain $n - y$ instances of X which must take the value 0. The probability of this occuring is $\left(1 - p\right)^{n-y}$. The term $p^y \left(1 - p\right)^{n-y}$ can occur exactly $\binom{n}{y}$ ways, so $P(Y = y) = \binom{n}{y} p^y \left(1 - p\right)^{n-y}$.

Note that by the binomial theorem,

$$\sum_{y=0}^{n} \binom{n}{y} p^y \left(1 - p\right)^{n-y} = (p + 1 - p)^n = 1$$

confirming that $\text{Binom}(n, p)$ is indeed a pdf.

$\square$