MATHEMATICS E-156, SPRING 2014
MATHEMATICAL FOUNDATIONS OF STATISTICAL SOFTWARE
Proof list - 2014

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Last modified: December 22, 2013

## Proofs

1. Given that $E[X_1 + X_2] = E[X_1] + E[X_2]$ in all cases and that $E[X_1 X_2] = E[X_1]E[X_2]$ for independent random variables, prove that
   - $\mathrm{Var}[X] = E[X]^2 - (E[X])^2$.
   - $\mathrm{Var}[aX + b] = a^2\ \mathrm{Var}[X]$.
   - If $X_1$ and $X_2$ are independent, $\mathrm{Var}[X_1 + X_2] = \mathrm{Var}[X_1] + \mathrm{Var}[X_2]$
     (This is all done in section A.2)

2. Let $X_1, X_2, \cdots X_n$ be independent random variables from a distribution with $\mathrm{Var}[X_i] = \sigma^2 < \infty$.

   Prove that

   $$E[\frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2] = \frac{n-1}{n}\sigma^2.$$

   This is theorem 6.2 on page 149 but can be done much earlier.

3. Prove that the sum of $n$ independent Bernoulli random variables, each with parameter $p$, is a binomial random variable $Y \sim \mathrm{Binom}(n, p)$, and that

   $$E[Y] = np,\ \mathrm{Var}\ Y = np(1-p).$$

4. Prove that the Poisson distribution with parameter $\lambda$ has mean and variance both equal to $\lambda$ and that if $X_1$ and $X_2$ are independent Poisson random variables with parameters $\lambda_1$ and $\lambda_2$ respectively, then $X_1 + X_2$ is Poisson with parameter $\lambda_1 + \lambda_2$.

   (This is theorems B.5 and B.6 on page 380)

5. For random variable $X$, define the moment generating function

$$M(t) = E[e^{tX}].$$

Prove that

- The $n$th derivative of $M(t)$, evaluated at $t = 0$, is equal to the $n$ th moment $E[X^n]$.
- If $X_1$ and $X_2$ are independent random variables with moment generating functions $M_1(t)$ and $M_2(t)$, then

$$M_{X_1+X_2}(t) = M_1(t)M_2(t)$$

(This is done on pp. 370-372)

6. A normal random variable $X \sim N(\mu, \sigma^2)$ has density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Prove the following:

- The moment generating function of $X$ is

$$M(t) = e^{\mu t + \sigma^2 t^2/2}.$$

- $E[X] = \mu$.
- $\mathrm{Var}[X] = \sigma^2$.
- If $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma^2)$, and $X_1$ and $X_2$ are independent, then $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

(This is partially done on pp. 368-369)

7. Suppose that $X_1, X_2, \cdots X_n$ are independent random variables, all with expectation 0, variance 1, and the same moment generating function $M(t)$.

   Define $Z_n = \frac{1}{\sqrt{n}}(X_1 + X_2 + \cdots + X_n)$,
   and call its moment generating function $H_n(t)$.

   Using the fact that if $\lim_{n\to\infty} nr_n = 0$ then

$$\lim_{n\to\infty}(1 + \frac{x}{n} + r_n)^n = e^x$$

   prove that

$$\lim_{n\to\infty} H_n(t) = e^{\frac{t^2}{2}}$$

   This is a special case of the proof on page 110 of Haigh)

8. Define the gamma function for $r > 0$ by

$$\Gamma(r) = \int_0^\infty x^{r-1}e^{-x}dx.$$

Prove the following:

- $$\Gamma(n) = (n-1)!$$

- $$\Gamma(\frac{1}{2}) = \sqrt{\pi}.$$

- $$f(x) = \frac{\lambda^r}{\Gamma(r)}x^{r-1}e^{-\lambda x}, x \geq 0$$

  is a probability density function.

(This is done on pp. 383-384, though partially left as an exercise)

9. A random variable $X$ has the gamma distribution if its probability density function is

$$f(x) = \frac{\lambda^r}{\Gamma(r)}x^{r-1}e^{-\lambda x}, x \geq 0$$

The gamma distribution with $r = 1$ is called the exponential distribution.

Prove the following:

- The moment generating function is

$$M(t) = (\frac{\lambda}{\lambda - t})^r.$$

- $$E[X] = \frac{r}{\lambda}$$

- $$\mathrm{Var}[X] = \frac{r}{\lambda^2}$$

- If $X_1, X_2, \cdots X_n$ are independent random variables with $X_i \sim \mathrm{Gamma}(r_i, \lambda)$, then

$$X_1 + X_2 + \cdots + X_n \sim \mathrm{Gamma}(r_1 + r_2 + \cdots + r_n, \lambda).$$

(This is done on pp. 384-385, though partially left as an exercise)

10. A random variable $X$ has the chi-square distribution with $m$ degrees of freedom if
$$X \sim \text{Gamma}(\frac{m}{2}, \frac{1}{2})$$
. Using properties of the gamma distribution, prove the following:

   - If $X_1, X_2, \cdots X_n$ are independent chi-square random variables with degrees of freedom $m_1, m_2, \cdots m_n$, then $X_1 + X_2 + \cdots + X_n$ is chi-square with $m_1 + m_2 + \cdots + m_n$ degrees of freedom.
   - If $Z \sim N(0,1)$, then $Z^2$ is chi-square with one degree of freedom.
   - If $Z_1, Z_2, \cdots Z_k$ are independent $N(0,1)$ random variables, then $Z_1^2 + Z_2^2 + \cdots Z_k^2$ has a chi-square distribution with $k$ degrees of freedom.

   (This is done on pp. 386, though partially left as an exercise)

11. Let $X_1, X_2, \cdots X_n$ be a random sample from $N(\mu, \sigma^2)$, with sample mean $\overline{X}$ and sample variance $S^2$

   Prove the following:

   - $\overline{X}$ and $S^2$ are independent random variables.
   - $(n-1)S^2/\sigma^2$ has a chi-square distribution with $n-1$ degrees of freedom.

   (This is done on p 387, though the first part is left out)

12. Let $Z \sim N(0,1)$ and let $W$ denote a chi-square distribution with $k$ degrees of freedom, independent of $Z$. Let $T$ be the ratio
$$T = \frac{Z}{\sqrt{W/k}}$$

   Prove the following:

   - $T$ has a $t$ distribution: i.e. its probability density function is
$$f(x) = \frac{\Gamma((k+1)/2)}{\Gamma(k/2)\sqrt{k\pi}}(1 + \frac{x^2}{k})^{-(k+1)/2}.$$
   - $\text{Var}[T] = k/(k-2)$ if $k > 2$.
   - If $X_1, X_2, \cdots X_n$ are a random sample from $N(\mu, \sigma^2)$, then
$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$
     has a $t$ distribution with $n-1$ degrees of freedom.

   (This is done on pp. 388-389, though partially left as an exercise)

13. Suppose that random variable $X$ has a beta distribution, i.e. its probability density function is

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, 0 \leq x \leq 1$$

Prove the following:

- $$\int_0^1 f(x)dx = 1.$$
- $$E[X] = \frac{\alpha}{\alpha+\beta}$$
- $$\mathrm{Var}[X] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

(This is done on pp. 390-391, though partially left as an exercise)