MATHEMATICS E-156, FALL 2014
Mathematical Foundations of Statistical Software
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Proof of the Week 6:

Theorem:
A normal random variable $X \sim N(\mu, \sigma^2)$ has density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Prove the following:

- The moment generating function of X is
$$M(t) = e^{\mu t + \sigma^2 t^2/2}.$$
- $E[X] = \mu$.
- $\text{Var}[X] = \sigma^2$.
- If $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma^2)$, and X_1 and X_2 are independent, then $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

(This is partially done on pp. 368-369)
Proof: Firstly,

$$M(t) = E[e^{tX}] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Now,

$$\left(x - \left(\mu + \sigma^2 t^2\right)\right)^2 = x^2 - 2x\mu - 2x\sigma^2 t + \mu^2 + 2\mu\sigma^2 t + \sigma^4 t^2$$

so

$$\frac{\left(x - (\mu + \sigma^2 t)\right)^2}{2\sigma^2} = \frac{x^2 - 2x\mu - 2x\sigma^2 t + \mu^2 + 2\mu\sigma^2 t + \sigma^4 t^2}{2\sigma^2} = -tx + \frac{(x-\mu)^2}{2\sigma^2} + \mu t + \frac{\sigma^2 t^2}{2}.$$

Thus

$$\begin{aligned}
M(t) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\left(x-\left(\mu+\sigma^2 t\right)\right)^2}{2\sigma^2}} e^{\mu t + \frac{\sigma^2 t^2}{2}} dx \\
&= e^{\mu t + \frac{\sigma^2 t^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\left(x-\left(\mu+\sigma^2 t\right)\right)^2}{2\sigma^2}} dx \\
&= e^{\mu t + \frac{\sigma^2 t^2}{2}} \int_{-\infty}^{\infty} N(\mu + \sigma^2 t, \sigma^2) dx \\
&= e^{\mu t + \frac{\sigma^2 t^2}{2}}.
\end{aligned}$$

Now,

$$M(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

so

$$M'(t) = \left(\mu + \sigma^2 t\right) e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

and

$$M''(t) = \sigma^2 e^{\mu t + \frac{\sigma^2 t^2}{2}} + \left(\mu + \sigma^2 t\right)^2 e^{\mu t + \frac{\sigma^2 t^2}{2}} = \left(\left(\sigma^2 + \mu^2\right) + 2\mu\sigma^2 t + \sigma^4 t^2\right) e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

so $E[X] = M'(0) = \mu$ and

$$\mathrm{Var}[X] = M''(0) - M'(0)^2 = \sigma^2 + \mu^2 - \mu^2.$$

Finally, if $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma^2)$, and X_1 and X_2 are independent,

$$\begin{aligned}
M_{X_1+X_2}(t) &= E[e^{t(X_1+X_2)}] \\
&= E[e^{tX_1} e^{tX_2}] \\
&= E[e^{tX_1}]E[e^{tX_2}] \\
&= M_{X_1}(t) M_{X_2}(t) \\
&= e^{\mu_1 t + \sigma_1^2 \frac{t^2}{2}} e^{\mu_2 t + \sigma_2^2 \frac{t^2}{2}} \\
&= e^{(\mu_1+\mu_2)t + \left(\sigma_1^2 + \sigma_2^2\right)\frac{t^2}{2}} \\
&= M_{N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)}.
\end{aligned}$$

so $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.