MATHEMATICS E-156, FALL 2014
Mathematical Foundations of Statistical Software
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Proof of the Week #9

Theorem:
A random variable X has the gamma distribution if its probability density function is

$$f(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}, x \geq 0$$

Prove the following:

- The moment generating function is $M(t) = \left(\frac{\lambda}{\lambda - t}\right)^r$.
- $$E[X] = \frac{r}{\lambda}$$
- $$\text{Var}[X] = \frac{r}{\lambda^2}$$
- If $X_1, X_2, \ldots, X_n$ are independent random variables with $X_i \sim \text{Gamma}(r_i, \lambda)$, then

$$X = X_1 + X_2 + \cdots + X_n \sim \text{Gamma}\left(r_1 + r_2 + \cdots + r_n, \lambda\right).$$

Proof: We know that the moment generating function is

$$\begin{aligned}
M(t) &= E[e^{tX}] \\
&= \int_0^\infty f(x) e^{tx} dx \\
&= \frac{\lambda^r}{\Gamma(r)} \int_0^\infty x^{r-1} e^{-(\lambda - t)x} dx \\
&\qquad \left(x = \frac{u}{\lambda - t}, dx = \frac{du}{\lambda - t}\right) \\
&= \frac{\lambda^r}{\Gamma(r)} \frac{1}{(\lambda - t)^r} \int_0^\infty u^{r-1} e^{-u} du \\
&= \frac{\lambda^r}{\Gamma(r)} \frac{1}{(\lambda - t)^r} \Gamma(r) \\
&= \frac{\lambda^r}{(\lambda - t)^r}.
\end{aligned}$$

This means that

$$M'(t) = \lambda^r \frac{d}{dt}(\lambda - t)^{-r} = \frac{r\lambda^r}{(\lambda - t)^{r+1}}$$

and

$$M''(t) = \frac{r(r+1)\lambda^r}{(\lambda - t)^{r+2}}.$$

Thus

$$E[X] = M'(0) = \frac{r\lambda^r}{\lambda^{r+1}} = \frac{r}{\lambda},$$

$$E[X^2] = M''(0) = \frac{r(r+1)\lambda^r}{\lambda^{r+2}} = \frac{(r+1)r}{\lambda^2}$$

and so

$$\mathrm{Var}[X] = E[X^2] - (E[X])^2 = \frac{r^2 + r}{\lambda^2} - \frac{r^2}{\lambda^2} = \frac{r}{\lambda^2}.$$

Now, for the final part,

$$M_X(t) = M_{X_1}(t) \cdots M_{X_n}(t) = \left(\frac{\lambda}{\lambda - t}\right)^{r_1} \cdots \left(\frac{\lambda}{\lambda - t}\right)^{r_n} = \left(\frac{\lambda}{\lambda}\right)^{r_1 + \cdots + r_n}$$

so

$$X_1 + \cdots + X_n \sim \mathrm{Gamma}\left(r_1 + \cdots + r_n, \lambda\right).$$