MATHEMATICS E-156, FALL 2014
Mathematical Foundations of Statistical Software
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Proof of the Week #10

Theorem:
A random variable X has the chi-square distribution with m degrees of freedom if

$$X \sim \chi^2_m \sim \text{Gamma}\left(\frac{m}{2}, \frac{1}{2}\right).$$

Using properties of the gamma distribution, prove the followingL

- If $X_1, X_2, \ldots, X_n$ are independent chi-square random variables with degrees of freedom $m_1, m_2, \ldots, m_n$, then $X = X_1 + X_2 + \cdots + X_n$ is chi-square with $m = m_1 + m_2 + \cdots + m_n$ degrees of freedom.
- If $Z \sim N(0,1)$, then Z^2 is chi-square with one degree of freedom.
- If $Z_1, Z_2, \ldots, Z_k$ are independent $N(0,1)$ random variables, then $X = Z_1^2 + Z_2^2 + \cdots + Z_k^2$ has a chi-square distribution with k degrees of freedom.

Proof: We know

$$X_i \sim \text{Gamma}\left(\frac{m_i}{2}, \frac{1}{2}\right)$$

so

$$X \sim \text{Gamma}\left(\frac{m}{2}, \frac{1}{2}\right) \sim \chi^2_m.$$

Now, from the definition,

$$\chi^2_1 \sim \text{Gamma}\left({}^1\!/\!_2, {}^1\!/\!_2\right)$$

with probability density function

$$f(x) = \frac{\left({}^1\!/\!_2\right)^{1/2}}{\Gamma\left({}^1\!/\!_2\right)} x^{-1/2} e^{-1/2x}.$$

Now, the distribution function of Z^2 is

$$F_{Z^2}(x) = P\left(Z^2 \leq x\right) = P\left(-\sqrt{x} \leq Z \leq \sqrt{x}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{x}}^{\sqrt{x}} e^{-z^2/2} dz.$$

Thus the density function is

$$f_{z^2}(x) = F'(x) = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{2\sqrt{x}} e^{-x/2} + \frac{1}{2\sqrt{x}} e^{-x/2}\right) = \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-1/2x}$$

and so $Z^2 \sim \chi^2_1$. The final part is a very, very direct consequence of the first two parts.