Proof 1 — Skylar Sutherland — Math E-156

Given that $E[X_1 + X_2] = E[X_1] + E[X_2]$ in all cases and that $E[X_1X_2] = E[X_1]E[X_2]$ for independent random variables, prove that

- $\text{Var}[X] = E[X]^2 - (E[X])^2$.
- $\text{Var}[aX + b] = a^2 \text{Var}[X]$.
- If X_1 and X_2 are independent, $\text{Var}[X_1 + X_2] = \text{Var}[X_1] + \text{Var}[X_2]$.

Proof: Firstly, we point out that by the definition of expectation it is clear that $E[aX] = aE[X]$, and $E[a] = a$ (this was not explicitly stated in the assumptions). Then we have

$$\begin{aligned}
\text{Var}[X] &= E[(X - E[X])^2] \\
&= E[X^2 - 2E[X]X + (E[X])^2] \\
&= E[X^2] - E[2E[X]X] + E[E[X]^2] \\
&= E[X^2] - 2(E[X])^2 + (E[X])^2 \\
&= E[X^2] - (E[X])^2.
\end{aligned}$$

This means nthat

$$\begin{aligned}
\text{Var}[aX + b] &= E[(aX + b)^2] - (E[aX + b])^2 \\
&= E[a^2X^2 + 2abX + b^2] - (E[aX + b])^2 \\
&= a^2E[X^2] + 2abE[X] + b^2 - (aE[X] + b)^2 \\
&= a^2E[X^2] + 2abE[X] + b^2 - a^2(E[X])^2 - 2abE[X] - b^2 \\
&= a^2(E[X^2] - (E[X])^2) \\
&= a^2 \text{Var}[X].
\end{aligned}$$

And finally, for independent X_1 and X_2, since $E[X_1X_2] = E[X_1]E[X_2]$, we have

$$\begin{aligned}
\text{Var}[X_1 + X_2] &= E[(X_1 + X_2)^2] - (E[X_1] + E[X_2])^2 \\
&= E[X_1^2 + 2X_1X_2 + X_2^2] - (E[X_1])^2 - 2E[X_1]E[X_2] - (E[X_2])^2 \\
&= E[X_1^2] + 2E[X_1X_2] + E[X_2]^2 - (E[X_1)^2 - 2E[X_1]E[X_2] - (E[X_2)^2 \\
&= \text{Var}[X_1] + \text{Var}[X_2] + 2(E[X_1X_2] - E[X_1]E[X_2]) \\
&= \text{Var}[X_1] + \text{Var}[X_2].
\end{aligned}$$