MATHEMATICS E-156, SPRING 2014
MATHEMATICAL FOUNDATIONS OF STATISTICAL SOFTWARE

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Proofs for Module #8 (Student t distribution)

1. Define the gamma function for $r > 0$:

$$\Gamma(r) = \int_0^\infty x^{r-1} e^{-x} dx \tag{1}$$

   (a) Prove that:

$$\Gamma(r+1) = r\Gamma(r) \text{ if } r > 0 \tag{2}$$

Proof. Substituting $(r+1)$ into the definition of the gamma function (1):

$$\Gamma(r+1) = \int_0^\infty x^r e^{-x} dx \tag{3}$$

Choose the following u and v for integration by parts:

$$u = x^r \qquad\qquad v = e^{-x}$$
$$du = rx^{r-1} \qquad\qquad dv = -e^{-x}$$

Recalling the definition of integration by parts:

$$uv - \int v du = \int u dv \tag{4}$$

and substituting our selected u and v into (4), evaluating from 0 to ∞ as in (1) yields:

$$x^r e^{-x}\Big|_0^\infty - \int_{x=0}^\infty e^{-x} r x^{r-1} = \int_{x=0}^\infty x^r(-e^{-x})$$

Recognizing (3) in the right-hand side, and evaluating the integral on the left, yields:

$$0 - r\int_{x=0}^\infty x^{r-1} e^{-x} dx = -\Gamma(r+1)$$

Similarly, recognizing (1) on the left gives the desired result:

$$r\Gamma(r) = \Gamma(r+1)$$

□

(b) Prove that: for integer $n > 0, \Gamma(n+1) = n!$

Proof. By induction: for the base case, evaluate the gamma function for $n = 1$ with definition (1):

$$\begin{aligned}\Gamma(1) &= \int_0^\infty x^0 e^{-x} dx \\ &= \int_0^\infty e^{-x} dx \\ &= -e^{-x} dx \Big|_0^\infty \\ &= 0 - -1 = 1\end{aligned}$$

and also consider $n = 2$; from (2) above:

$$\Gamma(1+1) = 1 \cdot \Gamma(1) = 1 \cdot 1 = 1$$

So for $n = 1$, $\Gamma(n+1) = n!$, as $\Gamma(1+1) = 1! = 1$, and we have shown that the base case holds. Though it isn't necessary, for fun we also consider $n = 3$; from (2) above:

$$\Gamma(2+1) = 2 \cdot \Gamma(1) = 2 \cdot 1 = 2$$

So, as expected, here also, $\Gamma(n+1) = n!$, as $\Gamma(2+1) = 2! = 2$.

For the induction step, assume that $\Gamma(n) = (n-1)!$ and show that $\Gamma(n+1) = n!$.
From (2):

$$\Gamma(n+1) = n\Gamma(n)$$

From the induction assumption, we can substitute $(n-1)!$ for $\Gamma(n)$:

$$\Gamma(n+1) = n(n-1)!$$

Therefore, by the definition of the factorial:

$$\Gamma(n+1) = n!$$

$\square$

(c) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Proof. By definition (1):

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{-\frac{1}{2}} e^{-x} dx$$

Choosing $x = \frac{u^2}{2}$ and $dx = udu$ yields:

$$\int_0^\infty \left(\frac{u^2}{2}\right)^{-\frac{1}{2}} e^{-\left(\frac{u^2}{2}\right)} udu$$

By algebra:

$$\int_0^\infty \left(\frac{\sqrt{2}}{u}\right) e^{-\left(\frac{u^2}{2}\right)} u du$$

Cancelling u's and dividing by 2 to compensate for doubling the value of this even function by integrating from $-\infty$ to ∞ rather than from 0 to ∞ yields:

$$\frac{1}{\sqrt{2}} \int_{-\infty}^\infty e^{-\left(\frac{u^2}{2}\right)} du$$

Closely following the discussion of the Gaussian integral on:

`http://mathworld.wolfram.com/GaussianIntegral.html`

and also referring to:

`http://en.wikipedia.org/wiki/Gaussian_integral`

and choosing $z = \frac{1}{\sqrt{2}} u$, $-z^2 = -\frac{u^2}{2}$, $\sqrt{2} dz = du$:

$$\frac{1}{\sqrt{2}} \int_{-\infty}^\infty e^{-z^2} \sqrt{2} dz$$

Cancelling $\sqrt{2}$s and using the fact that the square root of the square of a number is the same number:

$$\int_{-\infty}^\infty e^{-z^2} dz = \sqrt{\left(\int_{-\infty}^\infty e^{-z^2} dz\right)\left(\int_{-\infty}^\infty e^{-z^2} dz\right)}$$

Noting that the variable in the integral is a placeholder which integrates out, rename one from z to w:

$$\int_{-\infty}^\infty e^{-z^2} dz = \sqrt{\left(\int_{-\infty}^\infty e^{-w^2} dw\right)\left(\int_{-\infty}^\infty e^{-z^2} dz\right)}$$

Regrouping integral signs:

$$\int_{-\infty}^\infty e^{-z^2} dz = \sqrt{\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(w^2+z^2)} dw dz}$$

Switching to polar coordinates gives:

$$\int_{-\infty}^\infty e^{-z^2} dz = \sqrt{\int_{\theta=0}^{2\pi} \int_{r=0}^\infty e^{-r^2} r dr d\theta}$$

Integrating:

$$\int_{-\infty}^\infty e^{-z^2} dz = \sqrt{2\pi \left(-\frac{1}{2} e^{-r^2}\right)\Big|_0^\infty}$$

$$\int_{-\infty}^\infty e^{-z^2} dz = \sqrt{\pi}$$

$\square$

2. A random variable X has the gamma distribution if its probability density function is

$$f(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}, x \geq 0 \tag{5}$$

Prove the following:

(a) The moment generating function is $M(t) = (\frac{\lambda}{\lambda - t})^r$.

Proof. By definition:

$$M(t) = E[e^t X] = \int_0^\infty f(x) e^{tx} dx$$

Plugging in the gamma function probability density function (5) gives:

$$\frac{\lambda^r}{\Gamma(r)} \int_0^\infty x^{r-1} e^{-(\lambda - t)x} dx$$

Substitute in $x = \frac{u}{\lambda - t}$, $dx = \frac{du}{\lambda - t}$ to get:

$$M(t) = \frac{\lambda^r}{\Gamma(r)} \frac{1}{(\lambda - t)^r} \int_0^\infty u^{(r-1)} e^{-u} du$$

Recognizing the integral as Γ, and canceling with the Γ in the denominator, gives the desired result:

$$M(t) = \frac{\lambda^r}{(\lambda - t)^r}$$

□