Jeffrey Zheng
Math E-156

Week 2 Proof of the Week

Let X_1, X_2, … X_n be random variables from a distribution with $Var[X_i] = \sigma^2 < \infty$

$$E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = E\left[\frac{1}{n}\sum_{i=1}^{n}\left(X_i^2 - 2\bar{X}X_i + \bar{X}^2\right)\right] = E\left[\frac{1}{n}\left(\sum_{i=1}^{n}X_i^2 - 2\bar{X}\sum_{i=1}^{n}X_i + \bar{X}^2\sum_{i=1}^{n}1\right)\right]$$

$$= E\left[\frac{1}{n}\left(\sum_{i=1}^{n}X_i^2 - 2\bar{X}(n\bar{X}) + \bar{X}^2 n\right)\right] = \frac{1}{n}E\left[\sum_{i=1}^{n}X_i^2 - \bar{X}^2 n\right] = \frac{1}{n}E\left(\left[\sum_{i=1}^{n}X_i^2\right] - nE[\bar{X}^2]\right)$$

Where we can break up the expectation because of linearity

$Var[X_i] = E[X_i^2] - E[X_i]^2$ or $\sigma^2 = E[X_i^2] - \mu^2$, so $E[X_i^2] = \sigma^2 + \mu^2$

$$E\left(\left[\sum_{i=1}^{n}X_i^2\right]\right) = n(\sigma^2 + \mu^2)$$

$Var[\bar{X}] = E[\bar{X}^2] - E[\bar{X}]^2$ or $\frac{\sigma^2}{n} = E[\bar{X}^2] - \mu^2$, so $E[\bar{X}^2] = \frac{\sigma^2}{n} + \mu^2$

Combining those properties into the expectation equation

$$\frac{1}{n}E\left(\left[\sum_{i=1}^{n}X_i^2\right] - nE[\bar{X}^2]\right) = \frac{1}{n}\left[n(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n} + \mu^2\right)\right] = \frac{(n-1)\sigma^2}{n}$$

Similarly,

$$s^2 = E\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = \sigma^2$$

■