MATHEMATICS E-156, FALL 2014
Mathematical Foundations of Statistical Software
Date: 2/19/2014
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Proof of the Week 5:

Theorem:
For random variable X, define the moment generating function

$$M(t) = E[e^{tX}].$$

Prove that

- The nth derivative of $M(t)$, evaluated at $t = 0$, is equal to the n th moment $E[X^n]$.
- If X_1 and X_2 are independent random variables with moment generating functions $M_1(t)$ and $M_2(t)$, then

$$M_{X_1+X_2}(t) = M_1(t)M_2(t)$$

(This is done on pp. 370-372)
Proof:

$$e^{tX} = \sum_{k=0}^{\infty} \frac{t^k X^k}{k!},$$

so

$$M(t) = \sum_{k=0}^{\infty} \frac{t^k E[X^k]}{k!},$$

Since

$$\frac{d^{(n)}}{dt^(n)} \frac{t^k E[X^k]}{k!} = \frac{t^{k-n} E[X^k]}{(k-n)!}$$

if $k \geq n$ and 0 if $k < n$, we have that

$$M^{(n)}(t) = \sum_{k=0}^{\infty} \frac{d^{(n)}}{dt^{(n)}} \frac{t^k E[X^k]}{k!} = \sum_{k=n}^{\infty} \frac{t^{k-n} E[X^k]}{(k-n)!} = E[X^n] + t \sum_{j=1}^{\infty} \frac{E[X^{j+n}]}{j!}$$

so $M^{(n)}(0) = E[X^n]$. Now, if X_1 and X_2 are independent, we know

$$E[f(X_1)g(X_1)] = E[f(X_1)]E[g(X_2)].$$

Thus

$$M_{X_1+X_2}(t) = E[e^{t(X_1+X_2)}] = E[e^{tX_1}e^{tX_2}] = E[e^{tX_1}]E[e^{tX_2}] = M_{X_1}(t)M_{X_2}(t).$$