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Math E-156

Week 4 Proof of the Week

The Poisson distribution is the limit of a binomial distribution as $n \to \infty$

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$E[X] = \sum_{j=0}^{\infty} \frac{je^{-\lambda}\lambda^j}{j!}$$

$$= \sum_{j=1}^{\infty} \frac{je^{-\lambda}\lambda^j}{j!}$$

The sum when $j = 0$ is 0 so we can reindex to start at 1

$$= e^{-\lambda} \sum_{j=1}^{\infty} \frac{j\lambda\lambda^{j-1}}{j(j-1)!}$$

Factor out some things to cancel out the numerator and pull out a λ

$$= \lambda e^{-\lambda} \sum_{j=1}^{\infty} \frac{\lambda^{j-1}}{(j-1)!}$$

$$= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

Where we let $k = j - 1$

$$= \lambda e^{-\lambda} e^{\lambda} = \lambda$$

Recognizing $\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda}$ as the Taylor Series expansion for the exponential function

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$$Var[X] = E[X^2] - E[X]^2$$

$$E[X^2] = \sum_{j=0}^{\infty} \frac{j^2 e^{-\lambda} \lambda^j}{j!} = \sum_{j=1}^{\infty} \frac{j^2 e^{-\lambda} \lambda^j}{j!} = e^{-\lambda} \sum_{j=1}^{\infty} \frac{j^2 \lambda \lambda^{j-1}}{j(j-1)!} = \lambda e^{-\lambda} \sum_{j=1}^{\infty} \frac{j \lambda^{j-1}}{(j-1)!}$$

$$= \lambda e^{-\lambda} \sum_{j=1}^{\infty} \frac{(j-1+1)\lambda^{j-1}}{(j-1)!} = \lambda e^{-\lambda} \left(\sum_{j=1}^{\infty} \frac{(j-1)\lambda^{j-1}}{(j-1)!} + \sum_{j=1}^{\infty} \frac{\lambda^{j-1}}{(j-1)!} \right)$$

Splitting the summation up so we can again set up our Taylor expansion

$$= \lambda e^{-\lambda} \left(\lambda \sum_{j=2}^{\infty} \frac{\lambda^{j-2}}{(j-2)!} + \sum_{j=1}^{\infty} \frac{\lambda^{j-1}}{(j-1)!} \right) = \lambda e^{-\lambda} \left(\lambda \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} + \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right) = \lambda e^{-\lambda} \left(\lambda e^{\lambda} + e^{\lambda} \right) = \lambda^2 + \lambda$$

Using the $i = j - 2$ and $k = j - 1$ substitutions and Taylor expansion

$$Var[X] = \left(\lambda^2 + \lambda\right) - \lambda^2 = \lambda$$

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If X_1 and X_2 are independent Poisson random variables with parameters λ_1 and λ_2 respectively,

$$P(X_1 + X_2 = m) = \sum_{j=0}^{m} P(X_1 = j, X_2 = m - j)$$

$$= \sum_{j=0}^{m} P(X_1 = j) P(X_2 = m - j)$$

Using the independence assumption

$$= \sum_{j=0}^{m} \frac{e^{-\lambda_1} \lambda_1^j}{j!} \frac{e^{-\lambda_2} \lambda_2^{m-j}}{(m-j)!}$$

$$= e^{-(\lambda_1 + \lambda_2)} \sum_{j=0}^{m} \frac{\lambda_1^j}{j!} \frac{\lambda_2^{m-j}}{(m-j)!}$$

$$= \frac{e^{-(\lambda_1 + \lambda_2)}}{m!} \sum_{j=0}^{m} \frac{m!}{j!\,(m-j)!} \lambda_1^j \lambda_2^{m-j} = \frac{e^{-(\lambda_1 + \lambda_2)}}{m!} (\lambda_1 + \lambda_2)^m$$

Inserting an $\frac{m!}{m!}$ to make the summation coincide with the Binomial Theorem

Thus, $X_1 + X_2$ is Poisson with parameter $\lambda_1 + \lambda_2$.

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