MATHEMATICS E-156, FALL 2014
Mathematical Foundations of Statistical Software
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Proof of the Week #8

Theorem: Define the gamma function for $r > 0$ by

$$\Gamma(r) = \int_0^\infty x^{r-1} e^{-x} dx.$$

Prove the following:

- $$\Gamma(n) = (n-1)!$$
- $$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$
- $$f(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}, x \geq 0$$

Proof: We have that if $r > 0$,

$$\begin{aligned}
\Gamma(r+1) &= \int_0^\infty x^r e^{-x} dx \\
&\qquad \left(u = x^r, v = e^{-x}, du = rx^{r-1}, dv = -e^{-x}\right) \\
&= x^r e^{-x}\Big|_0^\infty + r \int_0^\infty x^{r-1} e^{-x} dx \\
&= r\Gamma(r).
\end{aligned}$$

This means that for integer $n > 1$,

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n(n-1)\ldots 2\Gamma(1)$$

Since

$$\Gamma(1) = \int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = 1,$$

this means $\Gamma(n+1) = n!$. Now,

$$
\begin{aligned}
\Gamma\left({}^1\!/\!_2\right) &= \int_0^\infty x^{-1/2} e^{-x} dx \\
&\qquad\qquad (x = {}^{u^2}\!/\!_2, dx = udu) \\
&= \int_0^\infty \frac{\sqrt{2}}{u} e^{-u^2/2} udu \\
&= \frac{1}{2} \int_{-\infty}^\infty \frac{\sqrt{2}}{u} e^{-u^2/2} udu \\
&= \frac{1}{\sqrt{2}} \int_\infty^\infty e^{-u^2/2} du
\end{aligned}
$$

which equals $\frac{1}{\sqrt{2}}\sqrt{2\pi} = \sqrt{\pi}$, as previously shown. Finally, we have that

$$
\begin{aligned}
\int_0^\infty \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} dx &= \frac{\lambda^r}{\Gamma(r)} \int_0^\infty x^{r-1} e^{-\lambda x} dx \\
&\qquad\qquad (u = \lambda x, du = \lambda dx) \\
&= \frac{\lambda^r}{\Gamma(r)} \int_0^\infty \frac{1}{\lambda^{r-1}} u^{r-1} e^{-u} \frac{1}{\lambda} du \\
&= \frac{\lambda^r}{\Gamma(r)} \frac{\Gamma(r)}{\lambda^r} \\
&= 1
\end{aligned}
$$

so f is a probability density function.