Proof 4-1

Larson Hogstrom - Math E156 - 2014

Poisson distribution as the limit of a binomial distribution

Random variable X_n has a binomial distribution with parameters n and p, expectation $\lambda = np$. So $p = \lambda/n$.

Its mass function is

$$P(X_n = x) = \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{x!(n-x)!}\frac{\lambda^x}{n^x}(1-\frac{\lambda}{n})^{n-x}.$$

Take the limit as $n \to \infty$ to get the mass function for a Poisson random variable X:

$$P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}.$$

Solution:

$$\lim_{n\to\infty} \frac{n!}{x!(n-x)!}\frac{\lambda^x}{n^x}(1-\frac{\lambda}{n})^{n-x}$$

$$\lim_{n\to\infty} \frac{n(n-1)(n-2)..(n-x+1)!}{x!}\frac{\lambda^x}{n^x}(1-\frac{\lambda}{n})^{n-x}$$

- $n(n-1)(n-2)...(n-x+1) = n^x + 0(n^{x-1})$
- $\lim_{n\to\infty} \frac{n(n-1)(n-2)...(n-x+1)}{n^x x!} = \frac{1}{x!}$

Now

$$\lim_{n\to\infty} \frac{\lambda^x}{x!}(1-\frac{\lambda}{n})^n(1-\frac{\lambda}{n})^{-x}$$

- $\lim_{n\to\infty}(1-\frac{\lambda}{n})^n = e^{-\lambda}$
- $\lim_{n\to\infty}(1-\frac{\lambda}{n})^{-x} = 1$

$$\lim_{n\to\infty} \frac{\lambda^x}{x!}e^{-\lambda} = Pois(\lambda)$$

$source : http : //en.wikipedia.org/wiki/Poisson_limit_theorem$