

When estimating an entropy $H[X]$, call the plug-in / nonparametric maximum likelihood estimator (i.e. the estimator you've been using) $\hat{H}[X]$. Call $\hat{\mathcal{X}}$ the *observed* alphabet of X . Then the Miller-Madow estimator of the entropy is

$$\hat{H}_{MM}[X] = \hat{H}[X] + \frac{|\hat{\mathcal{X}}| - 1}{2n} \quad (1)$$

where $|\hat{\mathcal{X}}|$ is the number of observed symbols and n was the number of samples used to estimate $\hat{H}[X]$.

Using the definition from line 368 of your code, we see that transfer entropy can be written as

$$TE_{X \rightarrow Y}^{(k)} = H(Y_t | Y_{t-k}^{t-1}) - H(Y_t | Y_{t-k}^{t-1}, X_{t-k}^{t-1}) \quad (2)$$

$$= H(Y_t, Y_{t-k}^{t-1}) - H(Y_{t-k}^{t-1}) - H(Y_t, Y_{t-k}^{t-1}, X_{t-k}^{t-1}) + H(Y_{t-k}^{t-1}, X_{t-k}^{t-1}), \quad (3)$$

where $X_{t-k}^{t-1} = (X_{t-k}, X_{t-(k-1)}, \dots, X_{t-1})$. We would apply the Miller-Madow estimator individually to each of the entropy terms. For example, for the first term, we have

$$\hat{H}_{MM}[Y_t, Y_{t-k}^{t-1}] = \hat{H}_{MM}[Y_{t-k}^t] = \hat{H}[Y_{t-k}^t] + \frac{|\widehat{\mathcal{Y}^{k+1}}| - 1}{2n}, \quad (4)$$

where $|\widehat{\mathcal{Y}^{k+1}}|$ is the number of $(k+1)$ -tuples we actually observe (of the 2^{k+1} possible tuples). Doing this for each term, the overall Miller-Madow estimator for the transfer entropy is

$$\widehat{TE}_{X \rightarrow Y}^{(k)} = \hat{H}_{MM}(Y_t | Y_{t-k}^{t-1}) - \hat{H}_{MM}(Y_t | Y_{t-k}^{t-1}, X_{t-k}^{t-1}) \quad (5)$$

$$= \hat{H}_{MM}(Y_t, Y_{t-k}^{t-1}) - \hat{H}_{MM}(Y_{t-k}^{t-1}) - \hat{H}_{MM}(Y_t, Y_{t-k}^{t-1}, X_{t-k}^{t-1}) + \hat{H}_{MM}(Y_{t-k}^{t-1}, X_{t-k}^{t-1}). \quad (6)$$

One possible problem with this estimator is that it can result in *negative* estimates of entropies. That usually occurs when \hat{H} is very small. I usually record both \hat{H} and \hat{H}_{MM} .