Notes on Multi-Wavenumber Dynamics in Stratified Turbulence

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1 Harmonic Contributions to Stratified Turbulence

In intermediate-Fr regions, we need to adjust the assumptions of (a) frequency scale separation and (b) asymptotic breakdown of flow fields used in the model of [1]. In particular, their derivation makes an assumption that critical fluctuation fields are monochromatic—that is, with only one active Fourier mode—which we will see neglects order $O(\sqrt{Fr})$ contributions to the fluctuation fields and order $O(Fr)$ contributions to the large-scale fields.

By leveraging the scaling discovered by [1] but lifting the assumption of an infinite scale separation, we arrive here at a modified system still largely solvable by the two-timescale approach of the latter; algorithmically, the primary modification is the introduction of a second Fourier mode slaved to the first.

1.1 The Closed System in 3D

Define $\varepsilon := \sqrt{Fr}$, as above. Supposing a velocity field $(u_\perp, w) = (u, w)$ in 3D, a pressure field p, and a buoyancy field b, we decompose and rescale our system as follows:

$$\begin{aligned} u &:= \overline{u} + \varepsilon u', \qquad w := \overline{w} + \varepsilon^{-1} w', \\ p &:= \overline{p} + \varepsilon p', \qquad b := \overline{b} + \varepsilon b'. \end{aligned} \tag{1}$$

Here, the averaging operation $a \mapsto \overline{a}$ is assumed only to be a linear projection, so that $\overline{\overline{a}} = \overline{a}$ and $\overline{(a - \overline{a})} = 0$; we universally use the notation $a' := a - \overline{a}$. In practice, we will take it to be a sharp low-pass filter along the horizontal directions and, optionally, locally in time.

We further break down (and inhomogeneously scale) derivatives along these subspaces:

$$\nabla_\perp = \overline{\nabla}_\perp + \varepsilon^{-2}\nabla'_\perp, \qquad \partial_t = \overline{\partial}_t + \varepsilon^{-2}\partial'_t,$$

so that $\overline{\nabla}_\perp$ and $\overline{\partial}_t$ are the restrictions of each derivative to the image of $a \mapsto \overline{a}$. If we wish not to average locally over time in this projection, we can view the ∂_t breakdown simply as a heterogeneous scaling of flows in the image of $a \mapsto \overline{a}$ and in its complement.

Now, the scaling (1) is informed by the leading-order system derived in [1]; critically, note that vertical velocity is dominated by small-scale effects, though all other parameters are otherwise.

Consider the exact Boussinesq system [1, 2.1–2.4]:

$$
\begin{aligned}
\partial_t u + u \cdot \nabla_\perp u + w \partial_z u &= -\nabla_\perp p + \frac{1}{Re_b}\left[\varepsilon^4 \nabla_\perp^2 + \partial_z^2\right] u + f, \\
\varepsilon^2 \left[\partial_t w + u \cdot \nabla_\perp w + w \partial_z w\right] &= \frac{1}{\varepsilon^2}\left(-\partial_z p + b\right) + \frac{\varepsilon^2}{Re_b}\left[\varepsilon^4 \nabla_\perp^2 + \partial_z^2\right] w, \\
\partial_t b + u \cdot \nabla_\perp b + w \partial_z b + w &= \frac{1}{Pr\, Re_b}\left[\varepsilon^4 \nabla_\perp^2 + \partial_z^2\right] b, \\
\nabla_\perp \cdot u + \partial_z w &= 0,
\end{aligned}
\tag{2}
$$

introducing the buoyancy Reynolds number $Re_b = Re Fr^2$ and identifying the aspect ratio α with the Froude number itself. Before splitting the equations along smooth and fluctuating subspaces, we make an *intermediate separation of scales* (ISoS) assumption; for any quantities A, B, we have

$$
\overline{A} \cdot \overline{B} = \overline{\overline{A} \cdot \overline{B}}, \qquad \overline{\overline{A} \cdot B'} = 0. \tag{3}
$$

In the case of a sharp low-pass filter, this corresponds to a factor-of-two region devoid of energy between the upper end of the low band and the lower end of the high band. That is, if k_1 is the highest energy-containing wavenumber in the image of $a \mapsto \overline{a}$, then the lowest energy-containing wavenumber in its complement is $> 2k_1$. Note that this is easily satisfied in the case of ocean circulation, where it is believed that $Fr < 0.001$ [1].

Define $\Delta' := (\nabla'_\perp)^2 + \partial_z^2$. In splitting (2) along our smooth and fluctuating subspaces, first note that the continuity condition holds directly:

$$
\overline{\nabla}_\perp \cdot \overline{u} + \partial_z \overline{w} = 0, \qquad \nabla'_\perp \cdot u' + \partial_z w' = 0.
$$

Then for any fluctuating field ψ', we have

$$
\begin{aligned}
u' \cdot \nabla'_\perp \psi' + w' \partial_z \psi' &= u' \cdot \nabla'_\perp \psi' + \partial_z (w' \psi') - \psi' \partial_z w' \\
&= u' \cdot \nabla'_\perp \psi' + \partial_z (w' \psi') + \psi' \nabla'_\perp \cdot u' \\
&= \partial_z (w' \psi') + \nabla'_\perp \cdot (u' \psi').
\end{aligned}
\tag{4}
$$

In particular, the contribution of ψ' to the Reynolds stress reduces to $\partial_z \overline{w' \psi'}$, with the remaining terms directly affecting only the fluctuating equations.

Applying this to the system (2)—along with the scaling (1) and the ISoS assumption (3)—we arrive at the following closed system:

$$
\begin{aligned}
\overline{\partial}_t \overline{u} + \overline{u} \cdot \overline{\nabla}_\perp \overline{u} + \overline{w} \partial_z \overline{u} + \partial_z \overline{w' u'} &= -\overline{\nabla}_\perp \overline{p} + \frac{1}{Re_b} \partial_z^2 \overline{u} + \frac{\varepsilon^2}{Re_b} \overline{\nabla}_\perp^2 \overline{u} + \overline{f}, \\
-\partial_z \overline{p} + \overline{b} - \varepsilon^2 \partial_z \overline{w' w'} &= \varepsilon^4 \left(\overline{\partial}_t \overline{w} + \overline{u} \cdot \overline{\nabla}_\perp \overline{w} - \frac{1}{Re_b} \partial_z^2 \overline{w} \right) - \frac{\varepsilon^6}{Re_b} \overline{\nabla}_\perp^2 \overline{w}, \\
\overline{\partial}_t \overline{b} + \overline{u} \cdot \overline{\nabla}_\perp \overline{b} + \overline{w} \partial_z \overline{b} + \partial_z \overline{w' b'} + \overline{w} &= \frac{1}{Pr\, Re_b} \partial_z^2 \overline{b} + \frac{\varepsilon^2}{Pr\, Re_b} \overline{\nabla}_\perp^2 \overline{b}, \\
\overline{\nabla}_\perp \cdot \overline{u} + \partial_z \overline{w} &= 0.
\end{aligned}
$$

The fluctuating fields are derived similarly; note that the identity (4) gives a fluctuating quadratic term of

$$
\left(u' \cdot \nabla'_\perp \psi' + w' \partial_z \psi' \right)' = \partial_z (w' \psi')' + \nabla'_\perp \cdot (u' \psi').
$$

Then we derive

$$
\begin{aligned}
\partial'_t u' + \overline{u} \cdot \nabla'_\perp u' + w' \partial_z \overline{u} &= -\nabla'_\perp p' + f' - \varepsilon \partial_z (w' u')' - \varepsilon \nabla'_\perp (u' u')' \\
&\quad + \varepsilon^2 \left(\frac{1}{Re_b} \Delta' u' - u' \cdot \overline{\nabla}_\perp \overline{u} - \overline{w} \partial_z u' \right), \\
\partial'_t w' + \overline{u} \cdot \nabla'_\perp w' &= -\partial_z p' + b' - \varepsilon \partial_z (w' w')' - \varepsilon \nabla'_\perp \cdot (w' u')' \\
&\quad + \varepsilon^2 \left(\frac{1}{Re_b} \Delta' w' - \overline{w} \partial_z w' - w' \partial_z \overline{w} \right) - \varepsilon^4 u' \cdot \overline{\nabla}_\perp \overline{w}, \\
\partial'_t b' + \overline{u} \cdot \nabla'_\perp b' + w' \partial_z \overline{b} + w' &= -\varepsilon \partial_z (w' b')' - \varepsilon \nabla'_\perp \cdot (b' u')' \\
&\quad + \varepsilon^2 \left(\frac{1}{Re_b} \Delta' b' - u' \cdot \overline{\nabla}_\perp \overline{b} - \overline{w} \partial_z b' \right), \\
\nabla'_\perp \cdot u' + \partial_z w' &= 0.
\end{aligned}
$$

At this point, we look to retain terms of absolute order $\varepsilon^2 = Fr$ in the scaling used above; in light of (1), this means $O(\varepsilon^2)$ contributions in each smooth equation, $O(\varepsilon^3)$ contributions in the equation for w', and $O(\varepsilon)$ contributions in each other fluctuating equation.

The only changes to the smooth equations are (a) the introduction of weak horizontal diffusion of all quantities and (b) a pressure offset of $p_b := -Fr\,\overline{w'w'}$. The fluctuating equations change more substantially: in addition to the (still linear, so largely inconsequential) advection-diffusion terms added to the equation for w', we retain weakly nonlinear convection in each of our three quantities. Note that the now-neglected terms can largely be accounted for without changing the form of the algorithm, with the possible exception of the $\overline{w}$-momentum contribution in the smooth pressure equation—given the small ε^4 scaling in front of this, it could likely be calculated explicitly with minimal error.

All in all, we find

$$\boxed{\begin{aligned} \overline{\partial}_t\overline{u} + \overline{u}\cdot\overline{\nabla}_\perp\overline{u} + \overline{w}\partial_z\overline{u} + \partial_z\overline{w'u'} &= -\overline{\nabla}_\perp\overline{p} + \tfrac{1}{Re_b}\partial_z^2\overline{u} + \tfrac{\varepsilon^2}{Re_b}\overline{\nabla}_\perp^2\overline{u} + \overline{f},\\ \partial_z\overline{p} &= \overline{b} - \varepsilon^2\partial_z\overline{w'w'},\\ \overline{\partial}_t\overline{b} + \overline{u}\cdot\overline{\nabla}_\perp\overline{b} + \overline{w}\partial_z\overline{b} + \partial_z\overline{w'b'} + \overline{w} &= \tfrac{1}{Pr\,Re_b}\partial_z^2\overline{b} + \tfrac{\varepsilon^2}{Pr\,Re_b}\overline{\nabla}_\perp^2\overline{b},\\ \overline{\nabla}_\perp\cdot\overline{u} + \partial_z\overline{w} &= 0,\end{aligned}}$$

and

$$\boxed{\begin{aligned} \partial_t'u' + \overline{u}\cdot\nabla_\perp'u' + w'\partial_z\overline{u} &= -\nabla_\perp'p' + f' - \varepsilon\partial_z(w'u')' - \varepsilon\nabla_\perp'(u'u')',\\ \partial_t'w' + \overline{u}\cdot\nabla_\perp'w' + \partial_z p' - b' &= -\varepsilon\partial_z(w'w')' - \varepsilon\nabla_\perp'\cdot(w'u')' + \tfrac{\varepsilon^2}{Re_b}\Delta'w' - \varepsilon^2\partial_z(\overline{w}w'),\\ \partial_t'b' + \overline{u}\cdot\nabla_\perp'b' + w'\partial_z\overline{b} &= -w' - \varepsilon\partial_z(w'b')' - \varepsilon\nabla_\perp'\cdot(b'u')',\\ \nabla_\perp'\cdot u' + \partial_z w' &= 0.\end{aligned}}$$

1.2 2D System About a Single Large Gridpoint

Before delving into algorithmic details, we reduce the above system in the 2D toy example introduced in [1]. In our language, this means that $a \mapsto \overline{a}$ is the *streamwise average* of the flow, so our ISoS assumption is exactly true. As such, $\overline{\nabla}_\perp \equiv 0$, implying from the smooth continuity equation that $\overline{w} \equiv 0$. This gives the following two smooth equations:

$$\begin{aligned} \overline{\partial}_t\overline{u} + \partial_z\overline{w'u'} &= \tfrac{1}{Re_b}\partial_z^2\overline{u} + \overline{f},\\ \overline{\partial}_t\overline{b} + \partial_z\overline{w'b'} &= \tfrac{1}{Pr\,Re_b}\partial_z^2\overline{b},\end{aligned}$$

identical to those derived in [1]. Using the continuity condition, we further introduce the streamfunction ψ', with $u' = \partial_z\psi'$ and $w' = -\partial_x'\psi'$, and the fluctuating Laplacian $\Delta' = \partial_z^2 + (\partial_x')^2$.

Now, we differentiate the equations for u' and w' with respect to x and z, respectively, incorporating the previously neglected viscous term $\varepsilon^2\Delta'u'$ for later mathematical simplicity:

$$\partial_t'\partial_z u' + \partial_z(\overline{u}\partial_x'u') + \partial_z(w'\partial_z\overline{u}) = -\partial_x'\partial_z p' + \partial_z f' - \varepsilon\partial_z^2(w'u')' - \varepsilon\partial_x'\partial_z(u'u')' + \tfrac{\varepsilon^2}{Re_b}\Delta'\partial_z u',$$

$$\partial_t'\partial_x'w' + \overline{u}(\partial_x')^2w' + \partial_z\partial_x'p' - \partial_x'b' = -\varepsilon\partial_z\partial_x'(w'w')' - \varepsilon(\partial_x')^2(w'u')' + \tfrac{\varepsilon^2}{Re_b}\Delta'\partial_x'w' - \varepsilon^2\partial_z\partial_x'(\overline{w}w').$$

To obtain a single equation for ψ, we would like to subtract the w' equation above from that of u'; for this, note that the quadratic terms reduce as follows:

$$\begin{aligned} -\varepsilon\partial_z^2(w'u')' - \varepsilon\partial_x'\partial_z(u'u')' &+ \varepsilon\partial_z\partial_x'(w'w')' + \varepsilon(\partial_x')^2(w'u')'\\ &= \varepsilon\partial_z^2(\partial_x'\phi'\partial_z\phi')' - \varepsilon\partial_x'\partial_z(\partial_z\phi'\partial_z\phi')' + \varepsilon\partial_z\partial_x'(\partial_x'\phi'\partial_x'\phi')' - \varepsilon(\partial_x')^2(\partial_x'\phi'\partial_z\phi')'\\ &= \varepsilon\partial_z^3\phi'\partial_x'\phi' - \varepsilon\partial_z^2\partial_x'\phi'\partial_z\phi' + \varepsilon(\partial_x')^2\partial_z\phi'\partial_x'\phi' - \varepsilon(\partial_x')^3\phi'\partial_z\phi'\\ &= \varepsilon\partial_x'\phi'\Delta'\partial_z\phi' - \varepsilon\partial_z\phi'\Delta'\partial_x'\phi'.\end{aligned}$$

With this in mind, the fluctuating equations become

$$(\partial_t' + \overline{u}\partial_x')\Delta'\phi' - \partial_z^2\overline{u}\partial_x'\phi' + \partial_x'b' = \tfrac{\varepsilon^2}{Re_b}(\Delta')^2\phi' + \varepsilon(\partial_x'\phi'\Delta'\partial_z\phi' - \partial_z\phi'\Delta'\partial_x'\phi')'.$$

Note that retaining the $O(\varepsilon^2)$ diffusive terms of the u' equation affects the overall (non-dimensionalised) system only at order $O(\varepsilon^3)$. Similarly, we find

$$(\partial_t' + \overline{u}\partial_x')b' - \partial_x'\phi' - \partial_z\overline{b}\partial_x'\phi' = \tfrac{\varepsilon^2}{Pr\,Re_b}\Delta'b' + \varepsilon(\partial_x'\phi'\partial_z b' - \partial_z\phi'\partial_x b'),$$

including $O(\varepsilon^2)$ diffusive terms as before.

To apply the two-timescale approach of [1], we identify $\partial_t' := \partial_\tau$ as a distinct, short-time variable, and introduce the following, modified ansatz for ϕ':

$$\phi'(x,z,t,\tau) = \sum_{n\neq 0} A_n(t)\hat{\phi}_n(z,t)e^{ink(t)x+\sigma_n(t)\tau},$$

with the restriction that $A_{-n} = A_n^*$, $\hat{\phi}_{-n} = \hat{\phi}_n^*$, and $\sigma_{-n} = \sigma_n^*$, and similarly

$$b'(x,z,t,\tau) = \sum_{n\neq 0} A_n(t)\hat{b}_n(z,t)e^{ink(t)x+\sigma_n(t)\tau}.$$

With this in mind, the quadratic terms in the equation for ϕ' can be computed as follows:

$$\begin{aligned}(\partial_x'\phi'\Delta'\partial_z\phi' - \partial_z\phi'\Delta'\partial_x'\phi')' &=: \sum_{m,n\neq 0} R^\phi_{m,n} \\ &= \sum_{m,n\neq 0} A_m A_n e^{i(m+n)kx+(\sigma_n+\sigma_m)\tau}\cdot \\ &\qquad \left[imk\hat{\phi}_m(\partial_z^2 - n^2k^2)\partial_z\hat{\phi}_n - ink\partial_z\hat{\phi}_m(\partial_z^2 - n^2k^2)\hat{\phi}_n\right],\end{aligned}$$

and similarly for those in the equation for b':

$$\begin{aligned}(\partial_x'\phi'\partial_z b' - \partial_z\phi'\partial_x b')' &=: \sum_{m,n\neq 0} R^b_{m,n} \\ &= \sum_{m,n\neq 0} A_m A_n e^{i(m+n)kx+(\sigma_n+\sigma_m)\tau}\left[imk\hat{\phi}_m\partial_z\hat{b}_n - ink\hat{b}_n\partial_z\hat{\phi}_m\right].\end{aligned}$$

Using the technique of [1] to identify a leading-order critical wavenumber k, this gives a wavenumber $2k$ contribution to both fields of order $\varepsilon|A_1|^2$:

$$(2\sigma_1 + 2ik\overline{u})\Delta'\hat{\phi}_2 - 2ik(\partial_z^2\overline{u})\hat{\phi}_2 + 2ik\hat{b}_2 = \tfrac{\varepsilon^2}{Re_b}(\Delta')^2\hat{\phi}_2 + \varepsilon A_1^2\left[ik\hat{\phi}_1(\partial_z^2 - k^2)\partial_z\hat{\phi}_1 - ik\partial_z\hat{\phi}_1(\partial_z^2 - k^2)\hat{\phi}_1\right],$$

$$(2\sigma_1 + 2ik\overline{u})\hat{b}_2 - 2ik\hat{\phi}_2 - 2ik(\partial_z\overline{b})\hat{\phi}_2 = \tfrac{\varepsilon^2}{PrRe_b}\Delta'\hat{b}_2 + \varepsilon A_1^2\left[ik\hat{\phi}_1\partial_z\hat{b}_1 - ik\hat{b}_1\partial_z\hat{\phi}_1\right],$$

simply normalising $A_2 \equiv 1$.

In particular, this gives a sparse linear system to solve for $\hat{\phi}_2$ and $\hat{b}_2$ at each timestep, and similar analysis identifies higher-wavenumber contributions as well. Note that the n^{th} mode found in this manner has a magnitude of $O(\varepsilon^{n-1}|A_1|^n)$, suggesting that high frequency modes are relevant *only* as A_1 becomes large.

An example calculation is shown in Figure 1, using only wavenumbers k and $2k$. The correction provided by our "multi-timescale multi-frequency" (MTMF) approach is only minor, as we would predict, though it appears to reduce the asymmetry in the boundary between layers of flow. Notably, large-scale discrepancies between DNS and the "multi-timescale quasi-linear" (MTQL) approach of [1] are *not* corrected; these may require resolving feedback of higher modes on the first, for instance, or may require the techniques of the following section.

2 Considering Multiple Independent Modes

2.1 Multiple Independent Modes in Ferraro's Toy Model

Consider the following toy model of [2]:

$$\partial_t U_0 + U_0\partial_x U_0 = F - \nu U_0 + D\partial_z^2 U_0 - \overline{(\partial_\chi\eta_0)^2},$$

$$\partial_\tau\eta_0 = -\eta_0 - U_0\partial_\chi^2\eta_0 - \partial_\chi^4\eta_0 + \partial_z^2\eta_0.$$

Here, U_0 represents the large-scale, steady flow, and η_0 represents small-scale turbulence. Our goal is to rewrite the equation for η_0 as an eigenvalue problem, allowing us to avoid discretising the whole system on the scale of η_0.

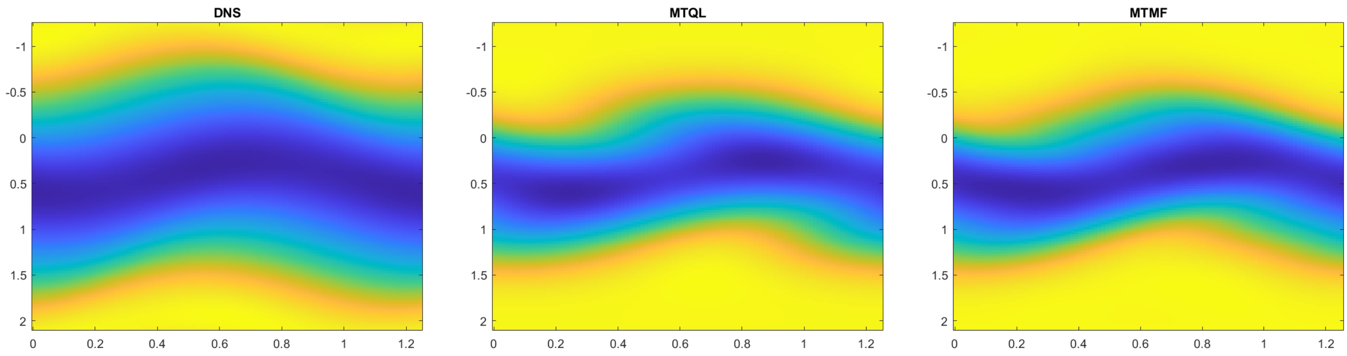

Figure 1: Plots of u in the two-dimensional toy model of Section 1.2, comparing the "multi-timescale quasi-linear" (MTQL) approach of [1] and our own "multi-timescale multi-frequency" (MTMF) against DNS, with a force $F \propto \cos(3z)$. The MTMF approach shown here resolves only wavenumber k and $2k$ components. The correction to MTQL is slight, as expected (i.e., of order $O(\mathrm{Fr})$), though it appears to reduce the asymmetry in the boundary between layers of flow.

For this, first write η_0 as follows:

$$\eta_0 = \sum\nolimits_j \eta_0^j = \sum\nolimits_j A_j(x,t)\hat{\eta}_0^j(x,z,t)e^{\sigma_j(x,t,k)+ik_j(x,t)\chi} + \text{c.c.},$$

noting that

$$\overline{(\partial_\chi \eta_0^j \partial_\chi \eta_0^\ell)} = 0$$

if $\ell \neq j$. We can rephrase our system in terms of these discretised variables:

$$\partial_t U_0 + U_0\partial_x U_0 = F - \nu U_0 + D\partial_z^2 U_0 - \sum k_j^2|A_j|^2|\hat{\eta}_0^j|^2,$$

$$\sigma_j\hat{\eta}_0^j = (-1 + k_j^2 U_0 - k_j^4 + \partial_z^2)\hat{\eta}_0^j =: L_j\hat{\eta}_0^j.$$

As we expect, we don't have a ready expression for $|A_j|^2$, but we can try (as a first step) to apply the Fredholm alternative condition used in [2]. That is, defining $\mathcal{L}_j := L_j - \sigma_j$, we know that $\mathcal{L}_j\hat{\eta}_0^j \equiv 0$ uniformly, so

$$\partial_t(\mathcal{L}_j\hat{\eta}_0^j) = \partial_k(\mathcal{L}_j\hat{\eta}_0^j) = 0.$$

Expanding the time derivative (though the k derivative gives similar results), a solution to our system above also solves the boundary value problem

$$\mathcal{L}_j\partial_t\hat{\eta}_0^j = -(\partial_t\mathcal{L}_j)\hat{\eta}_0^j.$$

From the Fredholm alternative, this is achievable if and only if $(\partial_t\mathcal{L}_j)\hat{\eta}_0^j$ is orthogonal (in z) to the left kernel of $\mathcal{L}_j$; noting that $\mathcal{L}_j$ is self-adjoint, we recover the same condition as Ferraro for each j:

$$\langle(\partial_t\mathcal{L}_j)\hat{\eta}_0^j|\hat{\eta}_0^j\rangle = 0.$$

Explicitly, we have

$$\partial_t\mathcal{L}_j = k_j^2\partial_t U_0 - \partial_t\sigma_j = k_j^2\left(F - \nu U_0 + D\partial_z^2 U_0 - U_0\partial_x U_0 - \sum k_\ell^2|A_\ell|^2|\hat{\eta}_0^\ell|^2\right) - \partial_t\sigma_j.$$

Writing

$$T(U_0) := F - \nu U_0 + D\partial_z^2 U_0 - U_0\partial_x U_0$$

for the advection-diffusion operator, we recover the final condition

$$k_j^{-2}(\partial_t\sigma_j)\int dz\ |\hat{\eta}_0^j|^2 = \int dz\ T(U_0)|\hat{\eta}_0^j|^2 - \sum k_\ell^2|A_\ell|^2\int dz\ |\hat{\eta}_0^\ell|^2|\hat{\eta}_0^j|^2.$$

Normalising $\hat{\eta}_0^j$ such that $\int dz\ |\hat{\eta}_0^j|^2 = k_j^{-2}$, this gives a diagonally dominant (and thus non-singular) linear system for $|A_j|^2$ for each (x,t).

References

[1] Gregory P. Chini, Guillaume Michel, Keith Julien, Cesar B. Rocha, and Colm-cille P. Caulfield. Exploiting self-organized criticality in strongly stratified turbulence. *Journal of Fluid Mechanics*, 933, 2022.

[2] Alessia Ferraro. Exploiting marginal stability in slow-fast quasilinear dynamical systems. 2022.