Homework 1

Linear Algebra and Probability, Fall 2024

Due Wednesday, September 11, at 5:00 PM

You may work in groups, but you must write your solutions on your own. You are also expected to acknowledge the names of your collaborators on the top of your homework, as this is the appropriate academic practice.

Problems 6,7, and 10 all ask you to write proofs. These should be clearly written in complete sentences.

1. For each of the following statements, indicate whether it reflects a fixed mindset or a growth mindset.

   (a) I'm not good enough at math to succeed in this class.

   (b) Math is difficult and takes time to learn. I must be persistent in my study so that I can master the material of this course.

   (c) If I am confused and see others are getting it, that is because they are smarter than I am.

   (d) If I am confused, it is okay. Everyone in this class comes from different backgrounds and experiences. I can't really know if somebody else is really understanding it. I belong in this class, and I can ask questions to make sure I understand. I will seek out additional help in office hours.

   (e) I have had success in math before, but will need to continue to work hard to reach my full potential. If I haven't struggled with math yet, I haven't pushed myself hard enough.

   (f) I am good at math.

2. In each of the following examples, A is a set, and x is some object. In each case, determine whether $x \in A, x \subseteq A$, or if neither or both is true.

   (a) $x = 5, \quad A = \{1, 2, 5\}$

   (b) $x = \{5\}, \quad A = \{1, 2, 5\}$.

   (c) $x = \{3, 4\}, \quad A = \{1, 3, \{4\}\}$.

   (d) $x = \{\text{Apples, Carrots}\}, \quad A = \{\text{Apples, Oranges, Cherries, Carrots, Lettuce}\}$.

   (e) $x = \{1, 2, 3\}, \quad A = \{1, 2, 3, \{1\}, \{2\}, \{1, 2, 3\}\}$.

   (f) $x = \emptyset, \quad A = \{\emptyset\}$.

3. Let $A = \{1, 4, 3\}$. Write out the elements of the power set $\mathcal{P}(A)$.

4. Let A = $\{x \in \mathbb{Z} : |x| \leq 10 \quad \text{and} \quad x = y^2 \text{ for some } y \in \mathbb{Z}\}$ (Here, $|x|$ means absolute value of x)

   (a) Write out the elements of A.

   (b) What is the cardinality of the power set of A, $\mathcal{P}(A)$?

5. For each of the following pairs of sets A and B, determine the cardinality of $A \cup B$ and the cardinality of $A \cap B$.

   (a) $A = \{1, 2, 3\}, B = \{1, 2, 4\}$.

   (b) $A = \{4, 2, 8, 3\}, B = \{\{4\}, \{2\}, \{8\}, \{3\}\}$.

   (c) $A = \{3, 5, 4, 2\}, B = \{5, 7, \{3, 5\}, \{4, 2\}\}$.

6. Let

   $$A = \{(x, y) \in \mathbb{R}^2 : 3x+2y = 4\}, \qquad \text{and} \qquad B = \{(x, y) : 4y^2-9x^2+24x = 16\}.$$

   Prove that $A \subseteq B$, but $A \neq B$.

7. Prove that $\bigcap_{n=1}^{\infty}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$. Here, the open interval (a, b) is the set

   $$\{x \in \mathbb{R} : a < x < b\}.$$

   Hint: You need to show that

   $$\{0\} \subseteq \bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right) \qquad \text{and} \qquad \bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right) \subseteq \{0\}.$$

   For the second part, this means you need to show that if $x \in \bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right)$, then $x = 0$. Prove this by contrapositive.

8. Determine whether the following statements are true or false. If false, explain why.

   (a) If it is not spring, then it is not April.

   (b) If $x = 4$, then $5 = 5$.

   (c) If $5 = 7$, then pigs can fly.

   (d) Assuming x is a real number, $x = 1$ if and only if $x^2 = 1$.

   (e) Assuming x is an integer, x is odd if and only if $x + 1$ is even.

9. Assume x is an integer. Use proof by contrapositive to show that, if $7x+5$ is even, then x is odd.