Lectures on Cosmological Correlations

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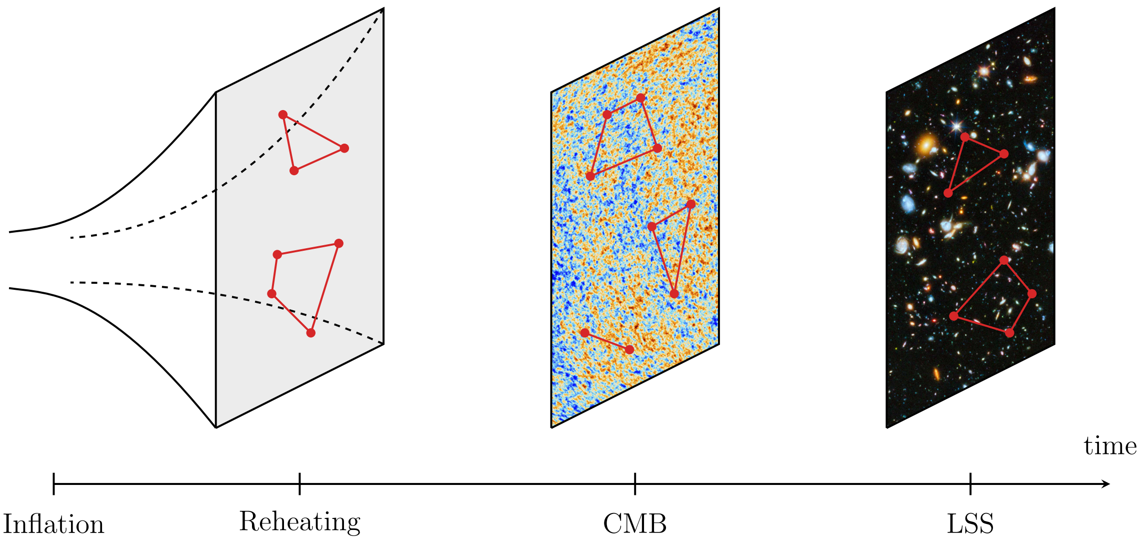


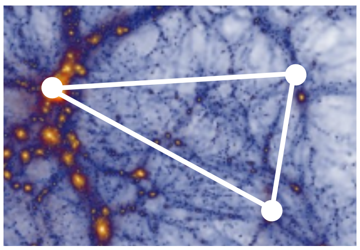
OUTLINE

Lecture notes and lecture scripts can be found at:
https://github.com/ddbaumann/cosmo-correlators

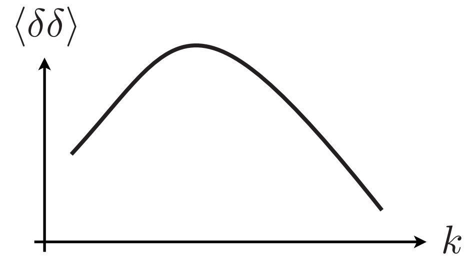
I. MOTIVATION

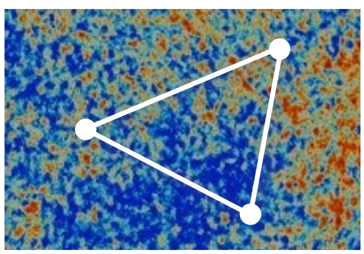
1.1. Cosmological Correlations

In cosmology, we measure **spatial correlations**:

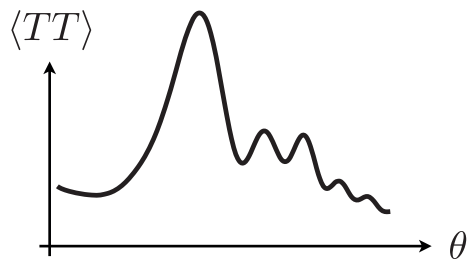


$$\langle \delta\rho(\mathbf{x}_1)\cdots\delta\rho(\mathbf{x}_N)\rangle$$

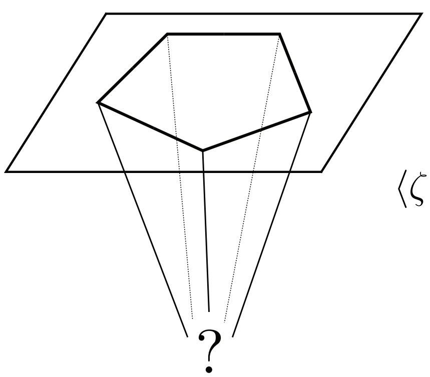




$$\langle \delta T(\theta_1)\cdots\delta T(\theta_N)\rangle$$



These correlations can be traced back to the origin of the hot Big Bang:



$$\langle \zeta(\mathbf{x}_1)\cdots\zeta(\mathbf{x}_N)\rangle$$

Where did the primordial correlations come from?

- Clue 1: The correlations span superhorizon scales.
- Clue 2: They are scale-invariant.

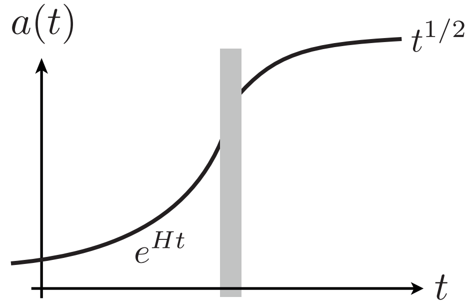
This suggests that the fluctuations were created **before the hot Big Bang**, during a phase of approximate time-translation invariance.

1.2. Inflation and De Sitter Space

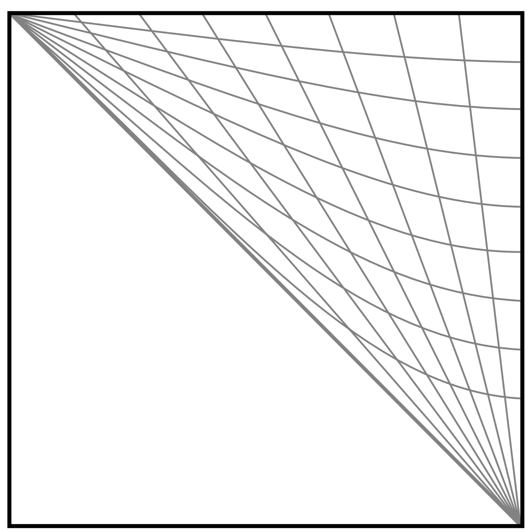
The observed correlations can be explained by an extended period of **accelerated expansion** (= inflation):

$$H(t) \equiv \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} \approx \text{const.}$$

$$\varepsilon(t) \equiv -\frac{\dot{H}}{H^2} \ll 1\,.$$



The spacetime during inflation is approximately **de Sitter space**.



In conformal time, $\mathrm{d}\eta = \mathrm{d}t/a(t)$, the de Sitter metric is

$$\mathrm{d}s^2 = \frac{-\mathrm{d}\eta^2 + \mathrm{d}\mathbf{x}^2}{(H\eta)^2}\,,$$

where $-\infty < \eta < 0$. The primordial correlations live on the **future boundary** of the de Sitter space at $\eta_* \approx 0$.

1.3. Quantum Fluctuations

Consider a massless scalar field during inflation:

$$\begin{aligned} S &= \frac{1}{2}\int \mathrm{d}\eta \mathrm{d}^3x\, a^2 \left(\dot\phi^2 - (\partial_i\phi)^2\right) \\ &= \frac{1}{2}\int \mathrm{d}\eta \mathrm{d}^3x \left(\dot u^2 - (\partial_i u)^2 + \frac{\ddot a}{a}u^2\right), \quad \text{where} \quad u(\eta,\mathbf{x}) \equiv a(\eta)\phi(\eta,\mathbf{x})\,. \end{aligned}$$

Classical dynamics

The classical equation of motion (for each Fourier mode) is

$$\boxed{\ddot u_{\mathbf{k}} + \left(k^2 - \frac{2}{\eta^2}\right)u_{\mathbf{k}} = 0}\,.$$

- At *early times* ($-k\eta \ll 1$), we have

$$\ddot u_{\mathbf{k}} + k^2 u_{\mathbf{k}} = 0 \qquad \Longrightarrow \qquad u_k \sim \frac{1}{\sqrt{2k}}e^{\pm ik\eta}\,.$$

- At *late times* ($-k\eta \to 0$), we have

$$\ddot u_{\mathbf{k}} - \frac{2}{\eta^2}u_{\mathbf{k}} = 0 \qquad \Longrightarrow \qquad u_k \sim c_1\eta^{-1} + c_2\eta^2 \to c_1\eta^{-1}\,.$$

The exact solution is

$$\boxed{u_{\mathbf{k}}(\eta) = a_{\mathbf{k}}^{\pm}\left(1 \pm \frac{i}{k\eta}\right)\frac{e^{\pm ik\eta}}{\sqrt{2k}}}\,.$$

Canonical quantization

- Promote classical fields $u, \pi = \dot u$ to **quantum operators** $\hat u, \hat\pi$, with

$$[\hat u(\eta,\mathbf{x}), \hat\pi(\eta,\mathbf{x}')] = i\delta_D(\mathbf{x}-\mathbf{x}') \quad \Longrightarrow \quad [\hat u_{\mathbf{k}}(\eta), \hat\pi_{\mathbf{k}'}(\eta)] = i(2\pi)^3\delta_D(\mathbf{k}+\mathbf{k}')\,.$$

- Define the **mode expansion**

$$\hat u_{\mathbf{k}}(\eta) = u_k^*(\eta)\,\hat a_{\mathbf{k}} + u_k(\eta)\,\hat a^\dagger_{-\mathbf{k}}\,,$$

where $[\hat a_{\mathbf{k}}, \hat a^\dagger_{\mathbf{k}'}] = (2\pi)^3\delta_D(\mathbf{k}-\mathbf{k}')$ and $\dot u_k\, u_k^* - u_k\, \dot u_k^* = i$ (Wronksian).

- Define the **vacuum** by (see **Exercise 2.1**)

$$\hat{a}_{\mathbf{k}}|0\rangle = 0 \quad \text{and} \quad u_k(\eta) = \left(1 + \frac{i}{k\eta}\right)\frac{e^{ik\eta}}{\sqrt{2k}} \quad \text{(BD)}\,.$$

- Compute the two-point function of **zero-point fluctuations**:

$$\begin{aligned}\langle 0|\hat{u}_{\mathbf{k}}(\eta)\hat{u}_{\mathbf{k}'}(\eta)|0\rangle &= \langle 0| \left(u_k^*(\eta)\,\hat{a}_{\mathbf{k}} + u_k(\eta)\,\hat{a}^\dagger_{-\mathbf{k}}\right)\left(u_{k'}^*(\eta)\,\hat{a}_{\mathbf{k}'} + u_{k'}(\eta)\,\hat{a}^\dagger_{-\mathbf{k}'}\right)|0\rangle \\ &= (2\pi)^3\delta_D(\mathbf{k}+\mathbf{k}')\,|u_k(\eta)|^2\,.\end{aligned}$$

- The **power spectrum** of ϕ is

$$P_\phi(k) = \frac{|u_k(\eta)|^2}{a^2(\eta)} = \frac{H^2}{2k^3}\left(1 + k^2\eta^2\right) \xrightarrow{k\eta\to 0} \boxed{\frac{H^2}{2k^3}}\,.$$

  This is the famous **scale-invariant** spectrum of a massless field in de Sitter.

Curvature perturbations

The initial conditions of the hot Big Bang are typically defined in terms of the comoving curvature perturbation:

$$\delta g_{ij} = a^2(\eta)e^{2\zeta(\eta,\mathbf{x})}\delta_{ij}\,,$$

which is related to the inflaton fluctuations (in spatially flat gauge) by

$$\zeta = -\frac{H}{\dot{\phi}}\delta\phi\,.$$

The power spectrum of ζ then is

$$\boxed{P_\zeta(k) = \left(\frac{H}{\dot{\phi}}\right)^2\frac{H^2}{2k^3}\bigg|_{k=aH} \equiv A_{\rm s}k^{n_{\rm s}-1}}\,.$$

Massive fields

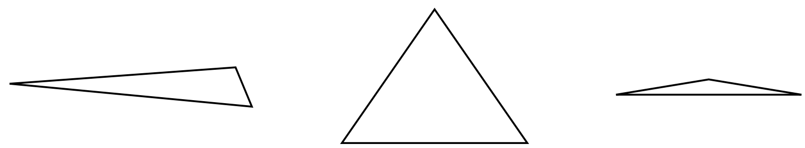
Massive fields are also produced during inflation, but don't survive until late times (see **Exercise 2.2**). Their imprints can, however, be found in the correlations of the light fields (= "cosmological collider physics")

1.4. Primordial Non-Gaussianity

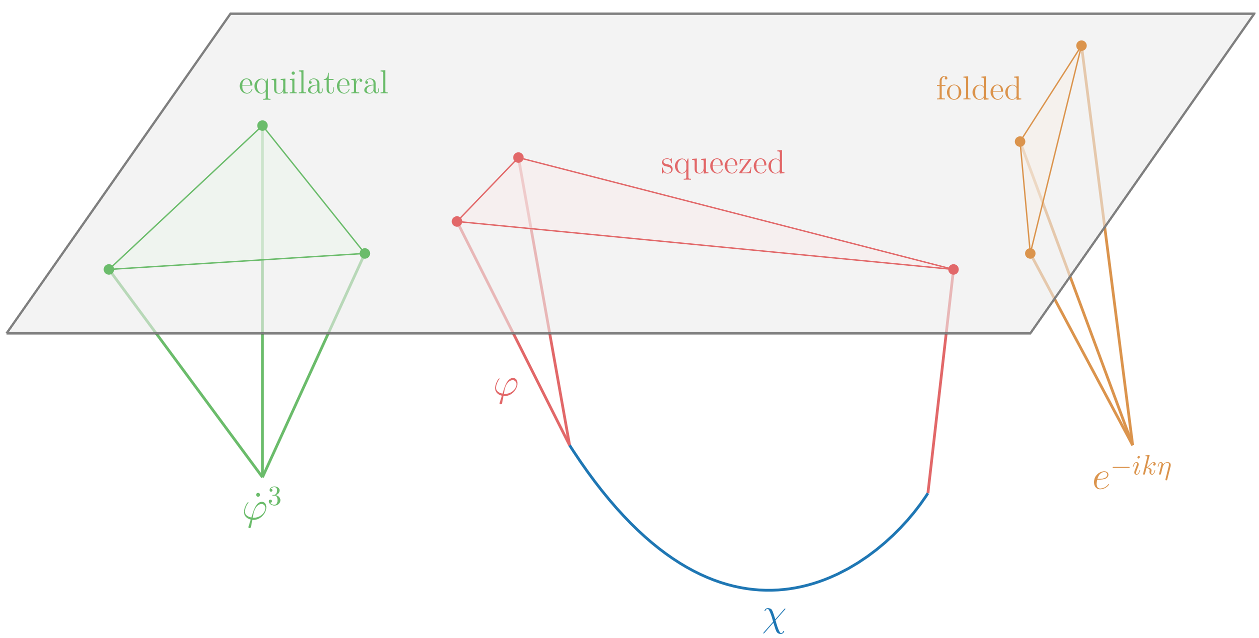
The main diagnostic of primordial non-Gaussianity is the **bispectrum**:

$$\langle \zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle = B_\zeta(k_1,k_2,k_3) \times (2\pi)^3 \delta_D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\,.$$

- amplitude: $f_{\rm NL} \equiv \dfrac{5}{18}\dfrac{B_\zeta(k,k,k)}{P_\zeta^2(k)}$
- shape:



• effect:	new particles	new interactions	excited states
• Planck constraints:	$\lvert f_{\rm NL}^{\rm loc}\rvert < 5$	$\lvert f_{\rm NL}^{\rm equil}\rvert < 40$	$\lvert f_{\rm NL}^{\rm flat}\rvert < 20$



Massive particles can be created by the inflationary expansion.
The **decay** of particles produces distinct correlations.

These correlations are **tracers** of the inflationary dynamics.

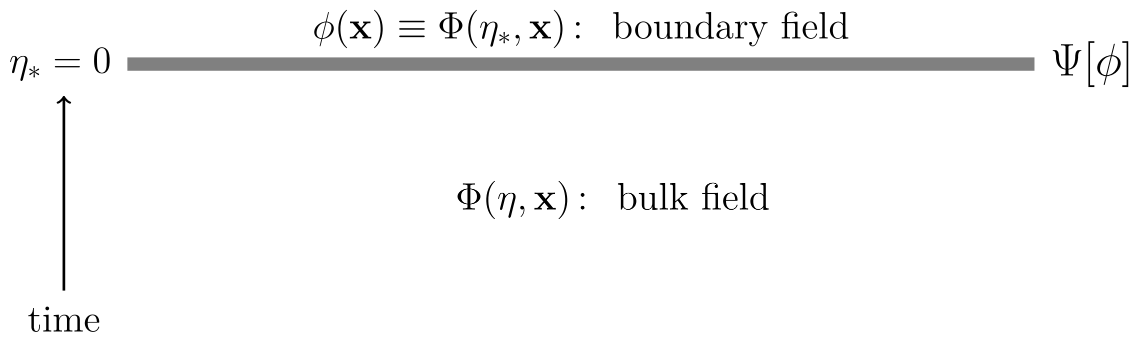
In the lectures notes, we describe three methods for computing these higher-point correlations:

- In-In Formalism
- Wavefunction Approach
- Cosmological Bootstrap

In the actual lectures, I will only have time for the latter two.

II. WAVEFUNCTION APPROACH

2.1. Wavefunction of the Universe



The "wavefunction of the universe" is

$$\Psi[\phi] \equiv \langle \phi(\mathbf{x}) | 0 \rangle = \int\limits_{\substack{\Phi(\eta_*)=\phi \\ \Phi(-\infty)=0}} \mathcal{D}\Phi \, e^{iS[\Phi]} \approx e^{iS[\Phi_{\rm cl}]} \, .$$

It defines boundary correlators

$$\langle \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_N) \rangle = \int \mathcal{D}\phi \, \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_N) \, |\Psi[\phi]|^2 \, .$$

The perturbative expansion of the wavefunction (in momentum space) is

$$\Psi[\phi] \; = \; \exp\left(- \sum_{N=2}^{\infty} \frac{1}{N!} \int {\rm d}^3 k_1 \cdots {\rm d}^3 k_N \, \Psi_N(\underline{\mathbf{k}}) \, \phi_{\mathbf{k}_1} \cdots \phi_{\mathbf{k}_N} \right) ,$$

where the "wavefunction coefficients" are

$$\Psi_N(\underline{\mathbf{k}}) = (2\pi)^3 \delta_D(\mathbf{k}_1 + \cdots + \mathbf{k}_N) \, \langle \underset{\substack{\uparrow \\ \text{dual operators: } \phi \to O, \; \gamma_{ij} \to T_{ij}}}{O_{\mathbf{k}_1}} \cdots O_{\mathbf{k}_N} \rangle' \, .$$

The relation between correlators and wavefunction coefficients is

$$\begin{aligned}
\langle \phi\phi \rangle &= \frac{1}{2{\rm Re}\langle OO \rangle} \, , \\
\langle \phi\phi\phi \rangle &= \frac{2{\rm Re}\langle OOO \rangle}{\prod_{n=1}^{3} 2{\rm Re}\langle O_n O_n \rangle} \, , \\
\langle \phi\phi\phi\phi \rangle &= \frac{\langle OOOO \rangle}{\langle OO \rangle^4} + \frac{\langle OOX \rangle^3}{\langle XX \rangle \langle OO \rangle^4} \, .
\end{aligned}$$

2.2. A Warmup

As a warmup, we apply the wavefunction approach to a **harmonic oscillator**:

$$S[\Phi] = \int \mathrm{d}t \left(\frac{1}{2}\dot{\Phi}^2 - \frac{1}{2}\omega^2\Phi^2 \right).$$

The classical solution is

$$\Phi_{\rm cl} = \phi e^{i\omega t}\,,$$

and the on-shell action becomes

$$\begin{aligned} S[\Phi_{\rm cl}] &= \int_{t_i}^{t_*} \mathrm{d}t \left[\frac{1}{2}\partial_t(\dot{\Phi}_{\rm cl}\Phi_{\rm cl}) - \frac{1}{2}\Phi_{\rm cl}\underbrace{(\ddot{\Phi}_{\rm cl} + \omega^2\Phi_{\rm cl})}_{=\,0} \right] \\ &= \frac{1}{2}\dot{\Phi}_{\rm cl}\Phi_{\rm cl}\Big|_{t=t_*} \\ &= \frac{i\omega}{2}\phi^2\,. \end{aligned}$$

The wavefunction then is

$$\Psi[\phi] \approx \exp(iS[\Phi_{\rm cl}]) = \exp\left(-\frac{\omega}{2}\phi^2\right),$$

which implies

$$\boxed{\langle \phi^2 \rangle = \frac{1}{2\omega}}\,.$$

In QFT, the same result applies for each Fourier mode:

$$\boxed{\langle \phi_{\mathbf{k}}\phi_{-\mathbf{k}} \rangle' = \frac{1}{2\omega_k}}\,,$$

where $\omega_k = \sqrt{k^2 + m^2}$.

To make this more interesting, consider a **time-dependent oscillator**:

$$S[\Phi] = \int \mathrm{d}t \left(\frac{1}{2} A(t) \dot{\Phi}^2 - \frac{1}{2} B(t) \Phi^2 \right).$$

The classical solution is

$$\Phi_{\text{cl}} = \phi K(t)\,, \quad \text{with} \quad \begin{array}{l} K(0) = 1 \\ K(-\infty) \sim e^{i\omega t} \end{array}$$

and the on-shell action becomes

$$\begin{aligned} S[\Phi_{\text{cl}}] &= \int_{t_i}^{t_*} \mathrm{d}t \left[\frac{1}{2} \partial_t (A \dot{\Phi}_{\text{cl}} \Phi_{\text{cl}}) - \frac{1}{2} \Phi_{\text{cl}} \underbrace{\left(\partial_t (A \dot{\Phi}_{\text{cl}}) + B \Phi_{\text{cl}} \right)}_{=0} \right] \\ &= \frac{1}{2} A \dot{\Phi}_{\text{cl}} \Phi_{\text{cl}} \Big|_{t=t_*} \\ &= \frac{1}{2} A \phi^2 \partial_t \log K \Big|_{t=t_*} . \end{aligned}$$

The wavefunction then is

$$\Psi[\phi] \approx \exp(iS[\Phi_{\text{cl}}]) = \exp\left(\frac{i}{2} (A \partial_t \log K)\Big|_* \phi^2 \right),$$

which implies

$$|\Psi[\phi]|^2 = \exp\left(-\mathrm{Im}(A \partial_t \log K)\big|_* \phi^2 \right) \quad \Longrightarrow \quad \boxed{\langle \phi^2 \rangle = \frac{1}{2\, \mathrm{Im}(A \partial_t \log K)\big|_*}} .$$

2.3. Free Fields in de Sitter

Consider a **massless field in de Sitter**:

$$
\begin{aligned}
S &= \int \mathrm{d}\eta \mathrm{d}^3x\, a^2(\eta) \left[(\Phi')^2 - (\nabla\Phi)^2\right] \\
&= \frac{1}{2}\int \mathrm{d}\eta \frac{\mathrm{d}^3k}{(2\pi)^3}\left[\frac{1}{(H\eta)^2}\Phi'_{\mathbf{k}}\Phi'_{-\mathbf{k}} - \frac{k^2}{(H\eta)^2}\Phi_{\mathbf{k}}\Phi_{-\mathbf{k}}\right],
\end{aligned}
$$

which is the same as the time-dependent oscillator.

The classical solution is

$$
\Phi_{\rm cl} = \phi K(\eta)\,, \quad \text{with} \quad \begin{array}{l} K(0)=1 \\ K(-\infty)\sim e^{ik\eta} \end{array}
$$

The function $K(\eta)$ is the *bulk-to-boundary propagator*.
For a massless field, we have

$$
\begin{aligned}
K(\eta) &= (1-ik\eta)e^{ik\eta}\,, \\
\log K(\eta) &= \log(1-ik\eta) + ik\eta\,,
\end{aligned}
$$

and hence

$$
\begin{aligned}
\mathrm{Im}(A\partial_\eta \log K)\big|_{\eta=\eta_*} &= \frac{1}{(H\eta_*)^2}\,\mathrm{Im}\left(\frac{-ik}{1-ik\eta_*} + ik\right) \\
&= \frac{1}{(H\eta_*)^2}\,\mathrm{Im}\left(\frac{k^2\eta + ik^3\eta_*^2}{1+k^2\eta_*^2}\right) \xrightarrow{\eta_*\to 0} \boxed{\frac{k^3}{H^2}}\,,
\end{aligned}
$$

The two-point function then is

$$
\boxed{\langle \phi_{\mathbf{k}}\phi_{-\mathbf{k}}\rangle' = \frac{H^2}{2k^3}}\,.
$$

The result for a massive field is derived in the lecture notes.

2.4. Anharmonic Oscillator

Consider the following **anharmonic oscillator**:

$$S[\Phi] = \int \mathrm{d}t \left(\frac{1}{2}\dot{\Phi}^2 - \frac{1}{2}\omega^2\Phi^2 - \frac{1}{3}\lambda\Phi^3\right).$$

The classical equation of motion is

$$\ddot{\Phi}_{\rm cl} + \omega^2\Phi_{\rm cl} = -\lambda\Phi_{\rm cl}^2\,.$$

A formal solution is

$$\Phi_{\rm cl}(t) = \phi K(t) + i\int \mathrm{d}t'\, G(t,t')\big(-\lambda\Phi_{\rm cl}^2(t')\big),$$

where

$$\begin{aligned} K(t) &= e^{i\omega t}\,, \\ G(t,t') &= \frac{1}{2\omega}\left(e^{-i\omega(t-t')}\theta(t-t') + e^{i\omega(t-t')}\theta(t'-t) - e^{i\omega(t+t')}\right). \end{aligned}$$

Computing the on-shell action is now a bit more subtle.

As before, we first write

$$\begin{aligned} S[\Phi] &= \int_{t_i}^{t_*} \mathrm{d}t \left[\frac{1}{2}\partial_t(\Phi\dot{\Phi}) - \frac{1}{2}\Phi(\ddot{\Phi}+\omega^2\Phi) - \frac{\lambda}{3}\,\Phi^3\right] \\ &= \frac{1}{2}\Phi\dot{\Phi}\bigg|_{t=t_*} + \int \mathrm{d}t \left[-\frac{1}{2}\Phi(\ddot{\Phi}+\omega^2\Phi) - \frac{\lambda}{3}\,\Phi^3\right]. \end{aligned}$$

Since

$$\begin{aligned} &\lim_{t\to 0} G(t,t') = 0\,, \\ &\lim_{t\to 0} \partial_t G(t,t') = -ie^{i\omega t'} \neq 0\,, \end{aligned}$$

the boundary term is

$$\begin{aligned} \frac{1}{2}\Phi_{\rm cl}\dot{\Phi}_{\rm cl}\bigg|_{t=t_*} &= \frac{1}{2}\phi\left(i\omega\phi - i\lambda\int \mathrm{d}t'\,\big(-ie^{i\omega t'}\big)\,\Phi_{\rm cl}^2(t')\right) \\ &= \frac{i\omega}{2}\phi^2 - \frac{\lambda}{2}\phi\int \mathrm{d}t'\, e^{i\omega t'}\,\Phi_{\rm cl}^2(t')\,. \end{aligned}$$

The action then becomes

$$
\begin{aligned}
S[\Phi_{\rm cl}] \;&=\; \frac{i\omega}{2}\phi^2 - \frac{\lambda}{2}\phi \int \mathrm{d}t\, e^{i\omega t}\, \Phi_{\rm cl}^2 \\
&\quad + \int \mathrm{d}t \left[-\frac{1}{2}\left(\phi e^{i\omega t} - i\lambda \int \mathrm{d}t'\, G(t,t')\, \Phi_{\rm cl}^2(t')\right)\Big(-\lambda \Phi_{\rm cl}^2(t)\Big) - \frac{\lambda}{3}\, \Phi_{\rm cl}^3 \right].
\end{aligned}
$$

The terms linear in ϕ cancel.

The final on-shell action is

$$
\boxed{S[\Phi_{\rm cl}] = \frac{i\omega}{2}\phi^2 - \frac{\lambda}{3}\int \mathrm{d}t\, \Phi_{\rm cl}^3(t) - \frac{i\lambda^2}{2}\int \mathrm{d}t\, \mathrm{d}t'\, G(t,t')\, \Phi_{\rm cl}^2(t')\, \Phi_{\rm cl}^2(t)}\;.
$$

To evaluate this, we write the classical solution as

$$
\Phi_{\rm cl}(t) = \Phi^{(0)}(t) + \lambda \Phi^{(1)}(t) + \lambda^2 \Phi^{(2)}(t) + \cdots .
$$

where

$$
\begin{aligned}
\Phi^{(0)}(t) &= \phi e^{i\omega t}\,, \\
\Phi^{(1)}(t) &= i \int \mathrm{d}t'\, G(t,t')\, \Big(-(\Phi^{(0)}(t'))^2\Big) \\
&= i \int \mathrm{d}t'\, G(t,t')\, \Big(-\phi^2 e^{2i\omega t'}\Big) = \frac{\phi^2}{3\omega^2}\, \left(e^{2i\omega t} - e^{i\omega t}\right) .
\end{aligned}
$$

With this, the wavefunction becomes

$$
\boxed{\Psi[\phi] \approx e^{iS[\Phi_{\rm cl}]} = \exp\left(-\frac{\omega}{2}\phi^2 - \frac{\lambda}{9\omega}\phi^3 + \frac{\lambda^2}{72\omega^3}\phi^4 + \cdots\right)}\;.
$$

From this, we can compute $\langle \phi^3 \rangle$, $\langle \phi^4 \rangle$, etc.

2.5. Interactions in Field Theory

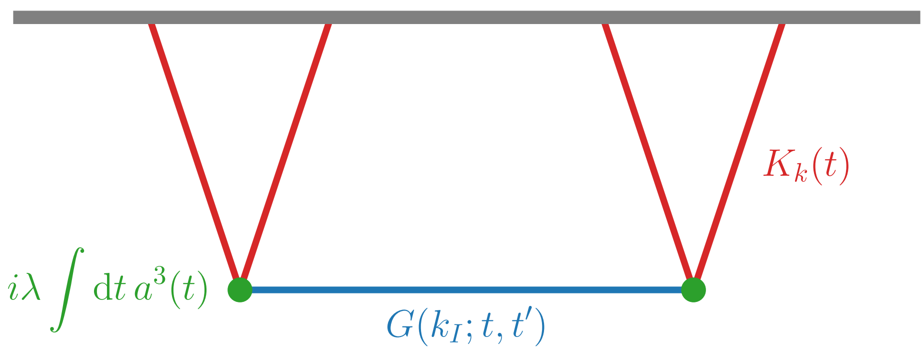
Back to field theory.

$$S[\Phi] = \int \mathrm{d}^4x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} m^2 \Phi^2 - \frac{1}{3} \lambda \Phi^3 \right).$$

The analysis is similar to that of the anharmonic oscillator ($\Rightarrow$ lecture notes).

In the interest of time, we jump directly to **Feynman rules** for wavefunction coefficients:

- bulk-to-boundary propagator K for every external line
- bulk-to-bulk propagator G for every internal line
- integrate each vertex over time.



Given a mode function $f_k(t)$, the bulk-to-boundary and bulk-to-bulk propagators are

$$K_k(t) = \frac{f_k(t)}{f_k(t_*)} ,$$

$$G(k; t, t') = \underbrace{f_k^*(t) f_k(t')\, \theta(t - t') + f_k^*(t') f_k(t)\, \theta(t' - t)}_{= G_F(k; t, t')} - \frac{f_k^*(t_*)}{f_k(t_*)} f_k(t) f_k(t') .$$

2.6. Correlators in Flat Space

Evaluate correlators at $t_* \equiv 0$. The flat-space mode function is

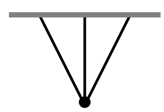
$$f_k(t) = \frac{1}{\sqrt{2k}} e^{ikt} .$$

Using this, the relevant propagators are

$$\begin{aligned} K_k(t) &= e^{ikt} , \\ G(k;t,t') &= \frac{1}{2k} \Big(e^{-ik(t-t')}\, \theta(t-t') + e^{ik(t-t')}\, \theta(t'-t) - e^{ik(t+t')} \Big) . \end{aligned}$$

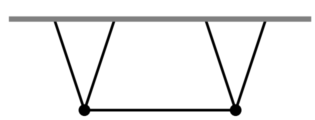
The three- and four-point wavefunction coefficients in Φ^3 theory are

$$\langle O_1 O_2 O_3 \rangle \equiv$$



$$\begin{aligned} &= i\lambda \int_{-\infty}^{0} \mathrm{d}t\, e^{i(k_1+k_2+k_3)t} \\ &= \frac{\lambda}{(k_1+k_2+k_3)} . \end{aligned}$$

$$\langle O_1 O_2 O_3 O_4 \rangle \ \equiv$$



$$\begin{aligned} &= \ -\lambda^2 \int_{-\infty}^{0} \mathrm{d}t'\, \mathrm{d}t''\, e^{ik_{12}t'} G(k_I; t', t'')\, e^{ik_{34}t''} , \quad k_{nm} \equiv k_n + k_m \\ &= \ \frac{\lambda^2}{(k_{12}+k_{34})(k_{12}+k_I)(k_{34}+k_I)} . \end{aligned}$$

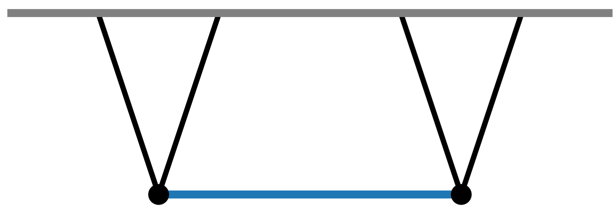
In the lecture notes, we show in detail how this reproduces the correct in-in correlators.

2.7. A Challenge

In general, correlators in de Sitter cannot be computed analytically (some exceptions are presented in the lecture notes).

Consider the four-point function of a **conformally coupled scalar** (with $m^2 = 2H^2$) mediated by the exchange of a **massive scalar**:

$$F \equiv \langle O_1 O_2 O_3 O_4 \rangle \equiv$$



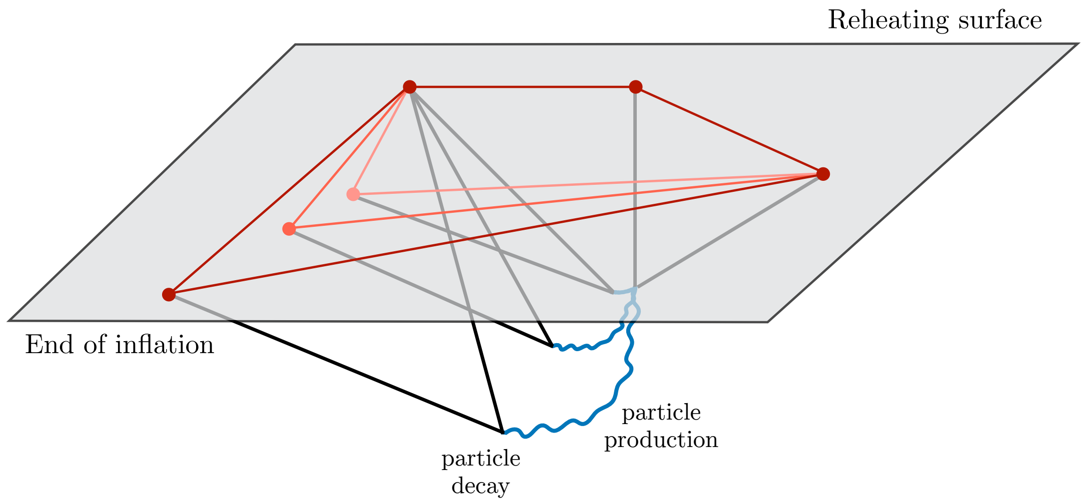
$$= -\lambda^2 \int \frac{\mathrm{d}\eta'}{\eta'^2} \int \frac{\mathrm{d}\eta''}{\eta''^2}\, e^{ik_{12}\eta'} e^{ik_{34}\eta''}\, G(k_I; \eta', \eta'')\,.$$

products of Hankel functions

The time integrals cannot be performed analytically.

Is there a way to obtain an analytic understanding of this (and other) correlators in de Sitter space?

III. COSMOLOGICAL BOOTSTRAP



3.1. Bootstrap Philosophy

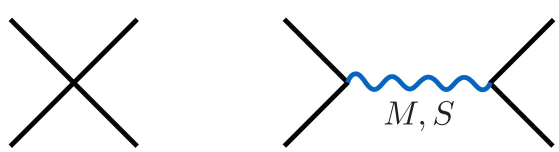
Principles		**Lagrangian**		**Observables**
Causality, unitarity, locality, symmetries, ...	$\Longrightarrow$	Fields, Equations of motion, Feynman diagrams, ...	$\Longrightarrow$	Cross sections, decay rates, cosmological correlators, ...

Bootstrap

S-matrix Bootstrap

$\Rightarrow$ scattering amplitudes are fixed by Lorentz invariance, locality and unitarity:

$$A(s,t) \;=\; \sum a_{nm} s^n t^m \;+\; \frac{g^2}{s-M^2} P_S\left(1+\frac{2t}{M^2}\right)$$



- No Lagrangian and Feynman diagrams are needed to derive this.
- Basic principles allow only a small menu of possibilities.

Cosmological Bootstrap

- What are the rules for consistent correlators?
- How are these rules implemented?

3.2. Symmetries

The metric of de Sitter is

$$\mathrm{d}s^2 = \frac{1}{H^2\eta^2}\left(-\mathrm{d}\eta^2 + \mathrm{d}\mathbf{x}^2\right).$$

The isometries of the metric are:

- P_i : translations
- J_{ij}: rotations
- D : dilatations $\qquad D = -\eta\partial_\eta - x^i\partial_i\,,$
- K_i: SCTs $\qquad K_i = 2x_i\eta\partial_\eta + \left(2x_ix^j\partial_j + (\eta^2 - x^2)\partial_i\right).$

We would like to see how these symmetries act on the boundary fields. Consider a massive scalar field in de Sitter space:

$$\Phi'' - \frac{2}{\eta}\Phi' - \nabla^2\Phi + \frac{m^2}{H^2}\frac{\Phi}{\eta^2} = 0\,.$$

At late times, the solution is

$$\Phi(\mathbf{x},\eta\to 0) = \underset{\substack{\uparrow \\ \text{dominates at late times } (\eta\to 0)\text{, for } m<\frac{3}{2}H}}{\phi(\mathbf{x})\,\eta^{\Delta}} + \bar\phi(\mathbf{x})\,\eta^{3-\Delta}\,, \qquad \Delta \equiv \frac{3}{2} - \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}\,. \qquad (\star)$$

Using $\eta\partial_\eta \mapsto \Delta$, the boundary field profiles $\phi(\mathbf{x})$ transform as

$$\begin{aligned} D\phi &= -(\Delta + x^i\partial_i)\phi\,, \\ K_i\phi &= \left[2x_i\Delta + \left(2x_ix^j\partial_j - x^2\partial_i\right)\right]\phi\,, \end{aligned}$$

like a primary operator of conformal weight Δ in a **CFT**.

The dual operator O transform in the same way, with $\Delta \mapsto 3-\Delta \equiv \bar\Delta$.

In cosmology, we work in **Fourier space**:

$$\begin{aligned} DO &= \left[\bar\Delta + k^i\partial_{k^i}\right]O\,, \\ K_iO &= \left[2\bar\Delta\partial_{k^i} - k_i\partial_{k^j}\partial_{k^j} + 2k^j\partial_{k^j}\partial_{k^i}\right]O\,. \end{aligned}$$

This leads to the following **Ward identity** for the wavefunction coefficients:

$$\left[-3+\sum_{a=1}^{N} D_a\right]\langle O_1\cdots O_N\rangle'(\underline{\mathbf{k}}) = 0\,,$$
$$\sum_{a=1}^{N} K_a^i\langle O_1\cdots O_N\rangle'(\underline{\mathbf{k}}) = 0\,.$$

Dilatation symmetry fixes the overall momentum scaling:

$$\langle O_1\cdots O_N\rangle' \propto k^{\Delta_t-3(N-1)}\,,\quad \text{where}\quad \Delta_t \equiv \sum_{a=1}^{N}\Delta_a\,.$$

All the juice is then in the SCT Ward identity.

Ex: Show that the **two-point function** of two scalar operators is

$$\langle O_1 O_2\rangle' \propto k_1^{2\Delta_1-3}\,\delta_{\Delta_1,\Delta_2}\,.$$

Ex: Consider the **three-point function** of two conformally coupled scalars Φ and a general scalar χ. Show that dilatation invariance implies

$$\langle O_1 O_2 X_3\rangle' = k_3^{\Delta-2}\hat{G}(u)\,,$$

where $u \equiv k_3/(k_1+k_2)$, and that special conformal invariance requires that

$$\boxed{\left[u^2(1-u^2)\partial_u^2 - 2u^3\partial_u \;+\; \frac{m^2}{H^2}-2\right]\hat{G}(u) = 0}\,.$$

This differential equation has a solution in terms of ${}_2F_1(u)$.

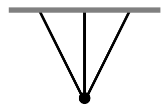
Four-point functions are less constrained by conformal symmetry alone. We need additional input to bootstrap them.

3.3. Singularities

Cosmological correlators have an interesting singularity structure.

We can see this in our results for the **flat-space wavefunction**:

$$\langle O_1 O_2 O_3 \rangle \equiv$$

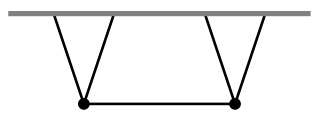


$$= i\lambda \int_{-\infty}^{0} \mathrm{d}t \, e^{iKt} = \frac{\lambda}{K} \,, \qquad \text{where} \quad K \equiv k_1 + k_2 + k_3 \,.$$

$\uparrow$

total energy singularity

$$\langle O_1 O_2 O_3 O_4 \rangle \equiv$$



$$= \frac{\lambda^2}{E E_L E_R} \,, \qquad \text{where} \quad \begin{aligned} E &\equiv k_{12} + k_{34} \,, \\ E_L &\equiv k_{12} + k_I \,, \\ E_R &\equiv k_{34} + k_I \,. \end{aligned}$$

$\uparrow$

partial energy singularity

The residues of these singularities are **scattering amplitudes**:

$$\begin{aligned} \lim_{E \to 0} \frac{\lambda^2}{E E_L E_R} &= \frac{1}{E} \frac{\lambda^2}{(k_{12} + k_I)(-k_{12} + k_I)} = \frac{1}{E} \frac{\lambda^2}{s} = \frac{A_4}{E} \,, \\ \lim_{E_L \to 0} \frac{\lambda^2}{E E_L E_R} &= \frac{\lambda}{E_L} \frac{\lambda}{(k_{34} + k_I)(k_{34} - k_I)} = \frac{A_3 \times \tilde{\psi}_3}{E_L} \,, \end{aligned}$$

where $\tilde{\psi}_3$ is the *shifted wavefunction coefficient*:

$$\tilde{\psi}_3 \equiv \frac{1}{2k_I} \Big[\psi_3(k_{34}, -k_I) + \psi_3(k_{34}, k_I) \Big] \,.$$

The same holds for de Sitter correlators:

- Correlators have singularities at points of would-be energy conservation.
- The residues of these singularities are scattering amplitudes.

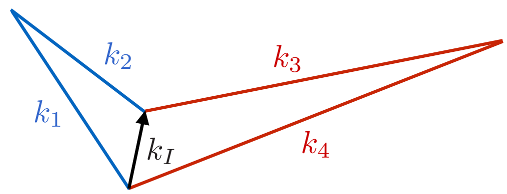
We will use these singularities as an input for the bootstrap.

3.4. Massive Exchange

We are now ready to return to the challenge of massive particle exchange:

The four-point function of conformally coupled scalars (in the s-channel) can be written as

$$F \equiv \langle O_1 O_2 O_3 O_4 \rangle' =$$



$$= \frac{1}{k_I} \hat{F}(u, v)\,,$$

where we have introduced

$$u = \frac{k_I}{k_1 + k_2}\,, \qquad v = \frac{k_I}{k_3 + k_4}\,.$$

This ansatz automatically solves the dilatation Ward identity.

After some work, the **conformal Ward identity** can be written as

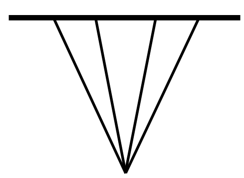
$$\boxed{(\nabla_u - \nabla_v)\hat{F} = 0}\,,$$

where $\Delta_u \equiv u^2(1-u^2)\partial_u^2 - 2u^3\partial_u$.
We will classify solution by their singularities.

Contact solutions

The simplest solutions correspond to contact interactions:

$$\hat{F}_c \equiv$$



$$= \sum_n \frac{c_n(u,v)}{E^{2n+1}}\,,$$

$$\uparrow$$

$$\Phi^4, (\partial_\mu \Phi)^4, \cdots$$

which have poles at vanishing total energy

$$E \equiv \sum_n k_n = \frac{u+v}{uv}\, k_I\,.$$

Note that $\hat{F}_{c,n} = \Delta_u^n \hat{F}_{c,0}$, where $\hat{F}_{c,0} \equiv uv/(u+v)$.

Exchange solutions

For tree exchange, we try

$$\begin{aligned}(\Delta_u + M^2)\hat{F}_e &= \hat{C} \\ (\Delta_v + M^2)\hat{F}_e &= \hat{C}\end{aligned} \quad \Leftrightarrow \quad (s+m^2)A_e = A_c(s,t)$$

where $\hat{C} = M^2 \hat{F}_c$ is a contact solution and $M^2 \equiv m^2/H^2 - 2$.

Using the simplest contact interaction as a source, we have

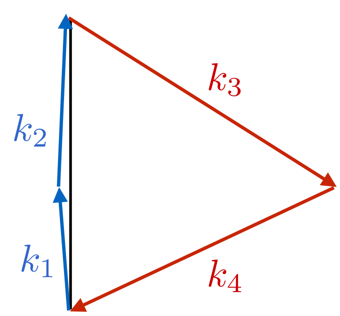
$$\boxed{\left[u^2(1-u^2)\partial_u^2 - 2u^3\partial_u + \left(\mu^2 + \frac{1}{4}\right)\right]\hat{F} = \frac{uv}{u+v}}\,, \qquad (\star)$$

where $\mu \equiv \sqrt{m^2/H^2 - 9/4}$.

Singularities

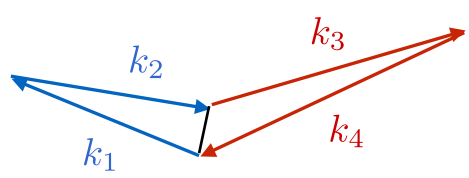
The equation has a number of interesting singularities:

- Factorization limit: $\lim\limits_{u,v\to -1} \hat{F} \propto A_3 \log(1+u) \times A_3 \log(1+v)$
- Folded limit: $\lim\limits_{u\to +1} \hat{F} \propto \log(1-u)$



This singularity should be absent in the standard vacuum.

- Collapsed limit: $\lim\limits_{u\to 0} \hat{F} \propto u^{\frac{1}{2}+i\mu}$



This non-analyticity corresponds to spontaneous particle production.

- Flat-space limit: $\lim\limits_{u\to -v} \hat{F} = A_4\,(u+v)\log(u+v)$

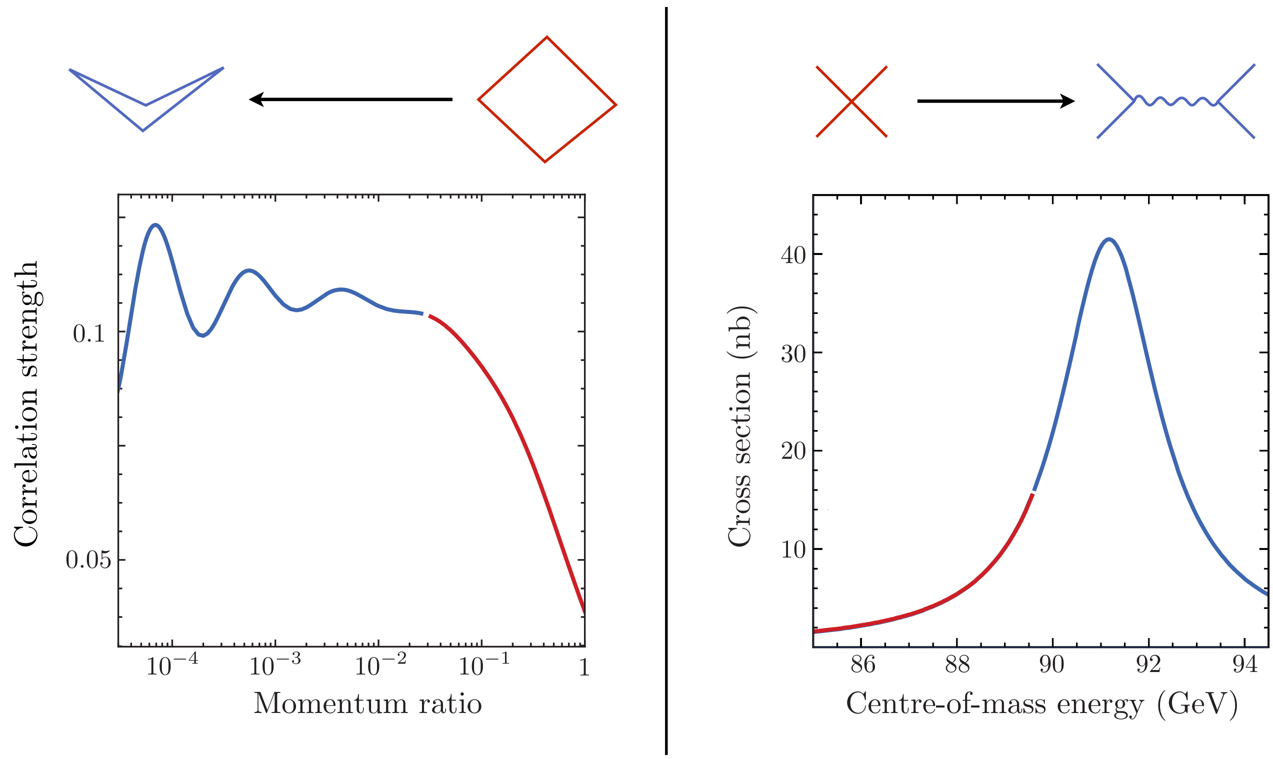
Factorization and folded limits uniquely fix the solution.
The remaining singularities become consistency checks.

Cosmological collider

The full solution can be written as (see lecture notes for explicit expressions)

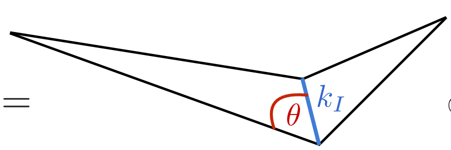
$$\boxed{\hat{F}(u,v) \;=\; \hat{F}_{\rm EFT}(u,v) \;+\; \hat{F}_{\rm pp}(u,v)}\,.$$

We are **forced** to have **particle production**.
This leads to **oscillations** in the collapsed limit:



These oscillations are the analog of a resonance in collider physics:

- The frequency of the oscillations depends on the **mass** of the particles.
- The angular dependence of the signal depends on the **spin** of the particles.

$F_S \;=$ 

$\propto \sin[M\log k_I]\, P_S(\cos\theta)$

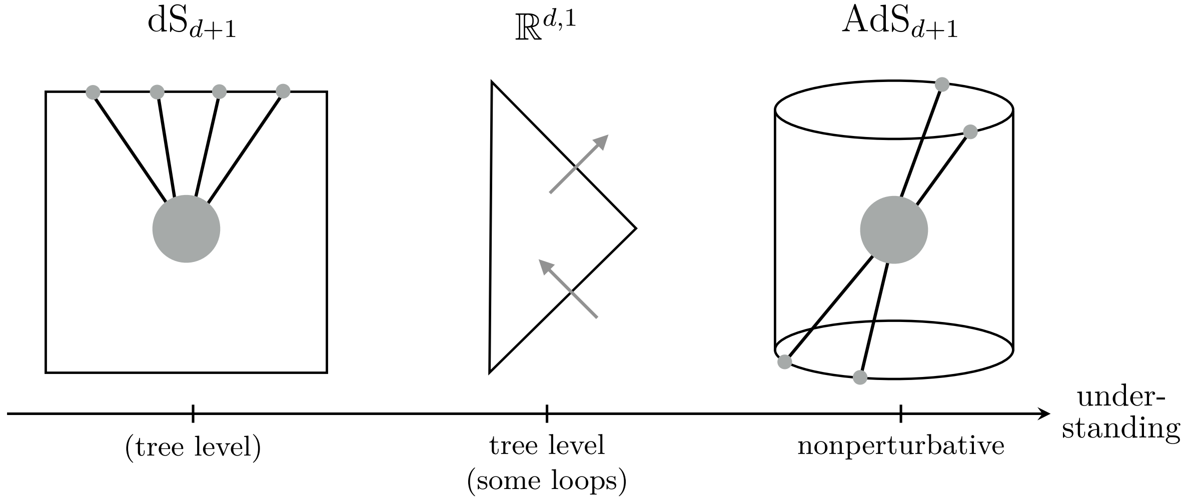
This is the analog of

$$A = \frac{g^2}{s - M^2} P_S(\cos\theta)$$

IV. OUTLOOK

There are many interesting things I didn't have time to describe in the lectures. Some more can be found in the lecture notes.

The bootstrap approach is only the beginning of a systematic exploration of cosmological correlators:



Many open questions remain:

• **Beyond Feynman** So far, the cosmological bootstrap has only been applied to individual Feynman diagrams (which by themselves are unphysical). However, in the S-matrix bootstrap, the real magic is found for on-shell diagrams. What is the on-shell formulation of the cosmological bootstrap? What magic will it reveal?

• **Beyond Perturbation Theory** So far, the bootstrap constraints (singularities, unitarity cuts, etc.) are only formulated in perturbation theory and implemented only at tree level. In contrast, we understand the conformal bootstrap nonperturbatively. Recently, some preliminary progress was made to understand unitarity constraints in de Sitter nonperturbatively, but much work remains. A nonperturbative bootstrap is required if we want to understand the UV completion of cosmological correlators.

• **Beyond Spacetime** In quantum gravity, spacetime is an emergent concept. In cosmology, the notation of "time" breaks down at the Big Bang. What replaces it? The cosmological bootstrap provides a modest form of emergent time, but there should be a much more radical approach where space and time are outputs, not inputs. For scattering amplitudes, such a reformulation was achieved in the form of the *amplituhedron*. What is the analogous object in cosmology? How does the Big Bang arise from it?