III. A NEW TWIST ON TIME

with Arkani-Hamed, Hillman, Joyce, Lee and Pimentel

3.1. Motivation

An essential feature of cosmology is that the Universe evolves in time.
However, there are reasons to believe that time is not a fundamental concept:

- Quantum gravity: cannot measure distances (and times) below the Planck scale
- Cosmology: classical spacetime breaks down at the Big Bang singularity

A deeper formulation of the physical laws—and one that has a chance of describing the Big Bang itself—therefore shouldn't involve time as an input, but must rather contain it as a derivable output.

- Scattering amplitudes: $\Rightarrow$ *amplituhedron*, *positive geometries*.
- Cosmology: If the notation of "time" breaks down at the Big Bang: What replaces it?

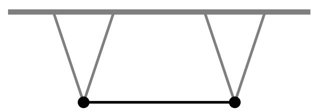
3.2. Correlators as Twisted Integrals

Consider conformal scalars in a power-law FRW background

$$a(\eta) = \left(\frac{\eta}{\eta_0}\right)^{-(1+\varepsilon)}$$

- The four-point function of conformal scalars in de Sitter space is

$$\psi \equiv$$



$$\begin{aligned} &= \int_{k_{12}}^{\infty} \mathrm{d}x_1 \int_{k_{34}}^{\infty} \mathrm{d}x_2 \, \frac{1}{(x_1+x_2)(x_1+k_I)(x_2+k_I)} \\ &= \int_0^{\infty} \mathrm{d}x_1 \mathrm{d}x_2 \, \frac{1}{(X_1+X_2+x_1+x_2)(X_1+x_1+1)(X_2+x_2+1)} \,, \end{aligned}$$

where $X_1 \equiv k_{12}/k_I$ and $X_2 \equiv k_{34}/k_I$.

Note: For conformal scalars, only to total energy entering a vertex matters, so we can work with truncated graphs. The exchange diagram above is then called a **two-site chain**.

- In a power-law FRW background, we get

$$\boxed{\psi = \int_0^{\infty} \mathrm{d}x_1 \mathrm{d}x_2 \, \frac{(x_1 x_2)^{\varepsilon}}{(X_1+X_2+x_1+x_2)(X_1+x_1+1)(X_2+x_2+1)}} \,,$$

where $(x_1 x_2)^{\varepsilon}$ is called a **twist**.

- All tree-level correlators can be written as such twisted integrals:

$$\psi(X_n) = \int_0^{\infty} \mathrm{d}x_1 \ldots \mathrm{d}x_n \, (x_1 \ldots x_n)^{\varepsilon} \, \psi_{\text{flat}}(X_n + x_n) \,,$$

Note that this is a purely boundary-centric definition of the wavefunction.

- Similar integrals arise for loop amplitudes in dimensional regularization.

3.3. Case Study: Two-Site Chain

Introduce a **family of integrals** with the same singularities:

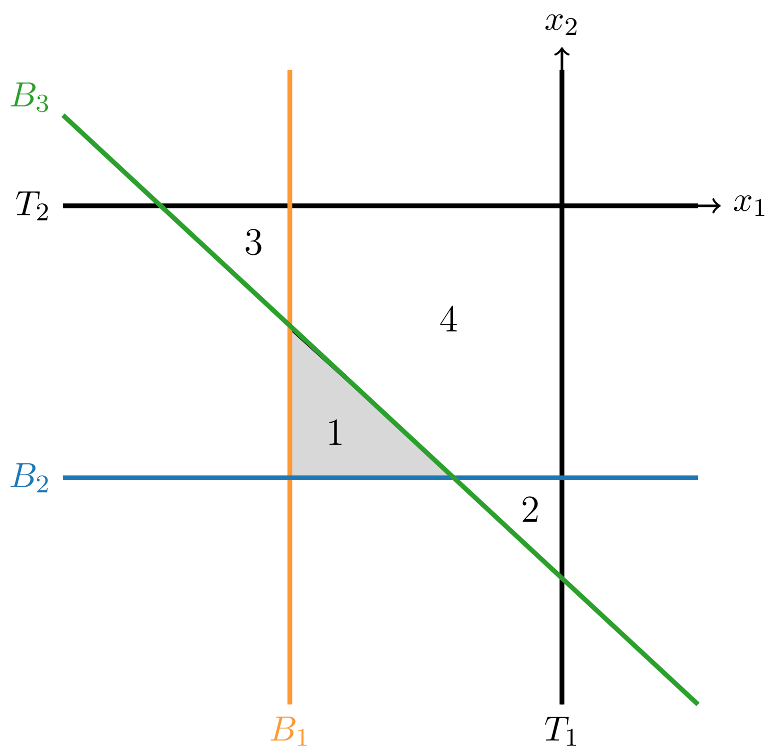
$$I_N = \int (x_1 x_2)^\varepsilon \, \Omega_N \,, \quad \Omega_N \equiv \frac{2}{T_1^{m_1} T_2^{m_2} B_1^{n_1} B_2^{n_2} B_3^{n_3}} \, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \,,$$

where we defined

$$\begin{aligned} T_1 &\equiv x_1 \,, \quad B_1 \equiv X_1 + x_1 + 1 \,, \\ T_2 &\equiv x_2 \,, \quad B_2 \equiv X_2 + x_2 + 1 \,, \quad B_3 \equiv X_1 + X_2 + x_1 + x_2 \,. \end{aligned}$$

- Not all integrals are independent because of IBP identities.
- How many independent **master integrals** are there?

Number of master integrals = number of bounded regions defined by the singularities of the integrand.



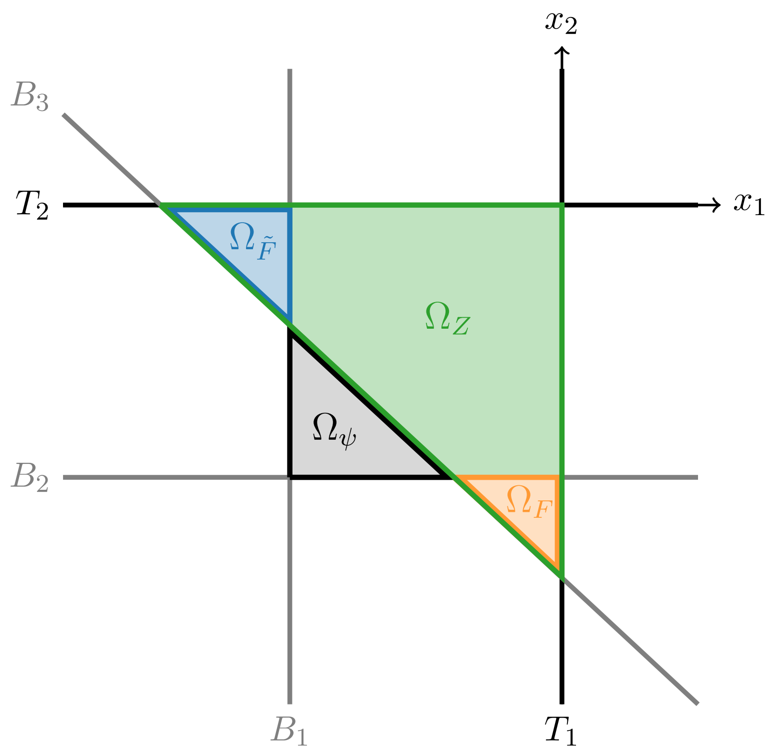
- Define a vector of master integrals

$$\vec{\mathcal{I}} \equiv \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \int (x_1 x_2)^\varepsilon \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{bmatrix}$$

- Taking $\partial_{X_{1,2}}$ leads to coupled differential equations:

$$\frac{\partial}{\partial X_i} I_a = \sum_{b=1}^{4} A_{i,ab} I_b \quad \Leftrightarrow \quad \mathrm{d}\vec{\mathcal{I}} = A\,\vec{\mathcal{I}}\,, \quad \text{where} \quad \mathrm{d} \equiv \mathrm{d}X_i \partial_{X_i}\,.$$

- The form of the equation depends on the choice of basis integrals.
  A preferred basis is in terms of the **canonical forms** of each bounded region:



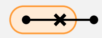
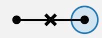
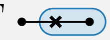
  The equation then takes the form:

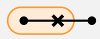
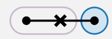
$$\mathrm{d}\vec{\mathcal{I}} = \varepsilon\, A\,\vec{\mathcal{I}}\,, \quad \text{where} \quad A \equiv \sum_i \alpha_i\, \mathrm{d}\log \underset{\substack{\uparrow \\ \textbf{letters}}}{\Phi_i(X)}$$

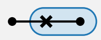
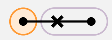
- The **letters** of the equation are

$$\begin{aligned} \Phi_1 &\equiv X_1 + 1\,, & \Phi_4 &\equiv X_1 - 1\,,\\ \Phi_2 &\equiv X_2 + 1\,, & \Phi_5 &\equiv X_2 - 1\,.\\ \Phi_3 &\equiv X_1 + X_2\,, \end{aligned}$$

• Explicitly, we get

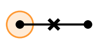
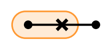
$\mathrm{d}\psi \;=\; \varepsilon\Big[(\psi - F)$  $+\; F$  $+\; (\psi - \tilde{F})$  $+\; \tilde{F}$  $\Big]$

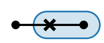
$\mathrm{d}F \;=\; \varepsilon\Big[F$  $+\; (F - Z)$  $+\; Z$  $\Big]$

$\mathrm{d}\tilde{F} \;=\; \varepsilon\Big[\tilde{F}$  $+\; (\tilde{F} - Z)$  $+\; Z$  $\Big]$

$\mathrm{d}Z \;=\; 2\varepsilon\, Z$ 

where we have introduced

 $\equiv \mathrm{d}\log(X_1 + 1)\,,$  $\equiv \mathrm{d}\log(X_1 - 1)\,,$

 $\equiv \mathrm{d}\log(X_2 + 1)\,,$  $\equiv \mathrm{d}\log(X_2 - 1)\,,$

 $\equiv \mathrm{d}\log(X_1 + X_2)\,.$

This graphical representation will play an important role below.

• The wavefunction satisfies a second-order inhomogeneous equation:

$$\Big[(X_1^2 - 1)\partial_{X_1}^2 + 2(1-\varepsilon)X_1\partial_{X_1} - \varepsilon(1-\varepsilon)\Big]\,\psi = g\left(\frac{1}{X_1 + X_2}\right)^{1-2\varepsilon}$$

This reflects **local evolution** in the bulk spacetime.

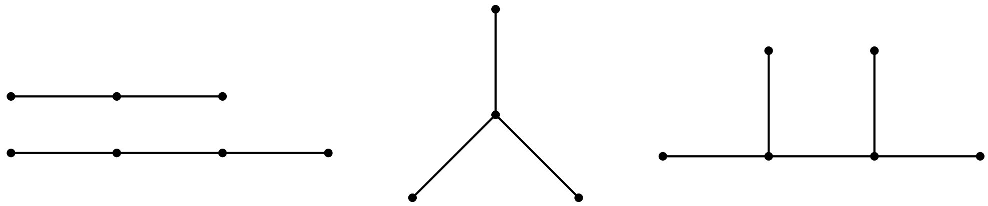
• The solution contains **two types of functions**:

$$\begin{aligned}\psi \;=\;& c_1(\varepsilon)\,(1+X_1)^{\varepsilon}(1+X_2)^{\varepsilon}\\ & c_2(\varepsilon)\,(X_1+X_2)^{2\varepsilon}\left(1 - {}_2F_1\left[\begin{matrix}1\,,\,\varepsilon\\ 1-\varepsilon\end{matrix}\,\middle|\,\frac{1-X_2}{1+X_1}\right] - (X_1 \leftrightarrow X_2)\right)\end{aligned}$$

In the de Sitter limit ($\varepsilon \to 0$), these become the familiar logs and dilogs.

3.4. More Complex Examples

We are also interested in more complicated graphs like



In that case, the integrals are higher-dimensional and harder to visualize.

For example, the wavefunction coefficient for the **three-site chain** is

$$\psi = \int_0^\infty (x_1x_2x_3)^\varepsilon\, \Omega_\psi\,, \quad \text{where} \quad \Omega_\psi \equiv \frac{4YY'}{B_1B_2B_3B_4}\left(\frac{1}{B_5}+\frac{1}{B_6}\right)\mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3\,.$$

where

$$\begin{aligned}
T_1 &\equiv x_1\,, & B_1 &\equiv X_1+x_1+Y\,, & B_4 &\equiv X_1+X_2+X_3+x_1+x_2+x_3\,,\\
T_2 &\equiv x_2\,, & B_2 &\equiv X_2+x_2+Y+Y'\,, & B_5 &\equiv X_1+X_2+x_1+x_2+Y'\,,\\
T_3 &\equiv x_3\,, & B_3 &\equiv X_3+x_3+Y'\,, & B_6 &\equiv X_2+X_3+x_2+x_3+Y\,.
\end{aligned}$$

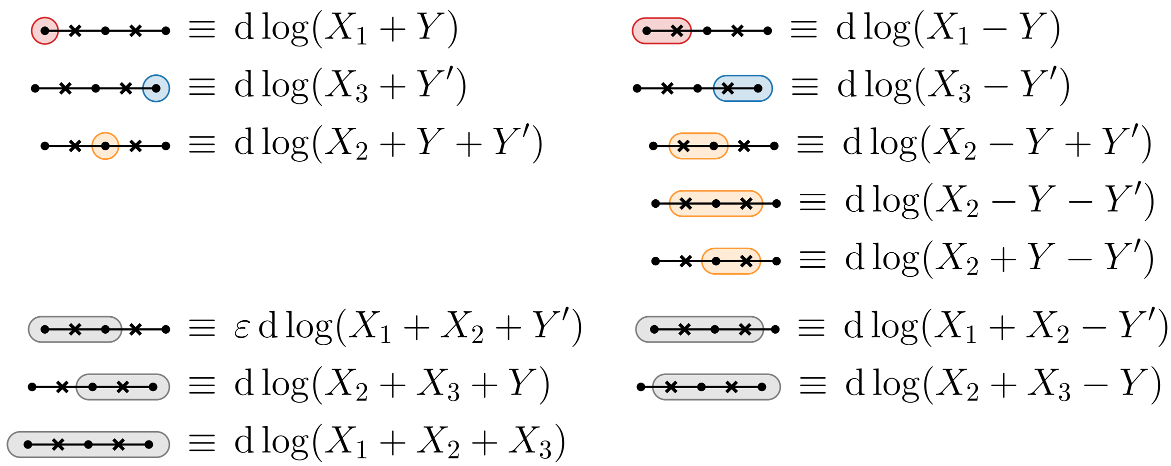
We need a different approach to tackle this.

We developed a systematic algorithm to derive differential equations for arbitrary tree graphs. Although this approach is completely systematic, for sufficiently complicated graphs it can become rather tedious and the results typically aren't very illuminating. However, closer inspection of the results has revealed something remarkable. The structure of the differential equations follows from very simple graphical rules.

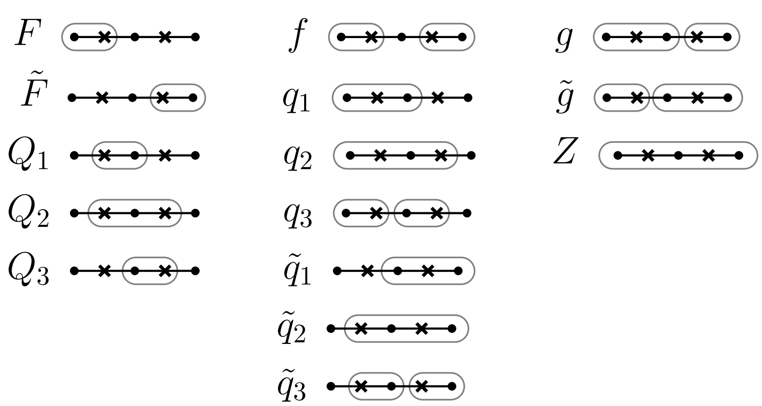
3.5. Time Evolution as Boundary Flow

Simple graphical rules govern the structure of the differential equations for cosmological correlators. I will illustrate these rules for the three-site chain, but they hold for all trees!

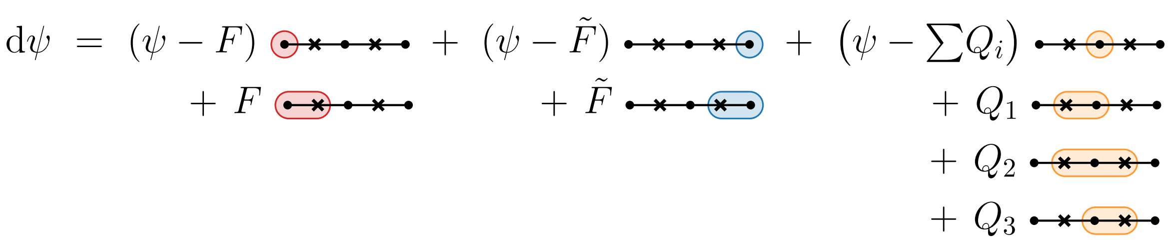
Letters = connected "tubings" of a "marked graph":



Sources = disconnected tubings (enclosing at least one cross):



The differential of the wavefunction is



When we take derivatives, letters get **activated**.
This satisfies **simple rules**.

The differential of Q_1 is

$$
\begin{aligned}
\mathrm{d}Q_1 \;=\; Q_1 \; + \; (Q_1 - q_1) \; + \; (Q_1 - \tilde{q}_3) \\
+ \; q_1 \; + \; (\tilde{q}_3 + \tilde{q}_2) \\
- \; \tilde{q}_2
\end{aligned}
$$

- **Nucleation**: $$\begin{aligned} \partial_{X_3} Q_1 \subset (Q_1 - \tilde{q}_3) \\ + \; \tilde{q}_3 \end{aligned}$$

- **Activation**: $$\partial_{X_2} Q_1 = Q_1$$

- **Merger**: $$\begin{aligned} \partial_{X_1} Q_1 = (Q_1 - q_1) \\ + \; q_1 \end{aligned}$$

- **Absorption**: $$\begin{aligned} \partial_{X_3} Q_1 = (Q_1 - \tilde{q}_3) \\ + \; (\tilde{q}_3 + \tilde{q}_2) \\ - \; \tilde{q}_2 \end{aligned}$$

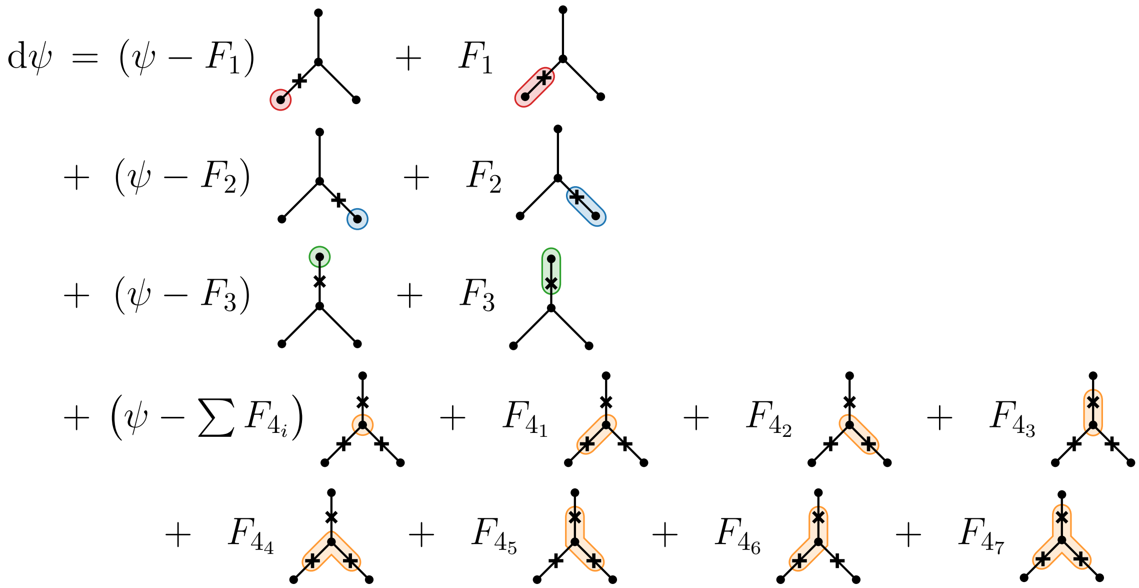
Remarkably, this is it!
The differentials of all other functions satisfy the same rules. (Try it!)

Exercise: Use the above rules to explain the following examples:

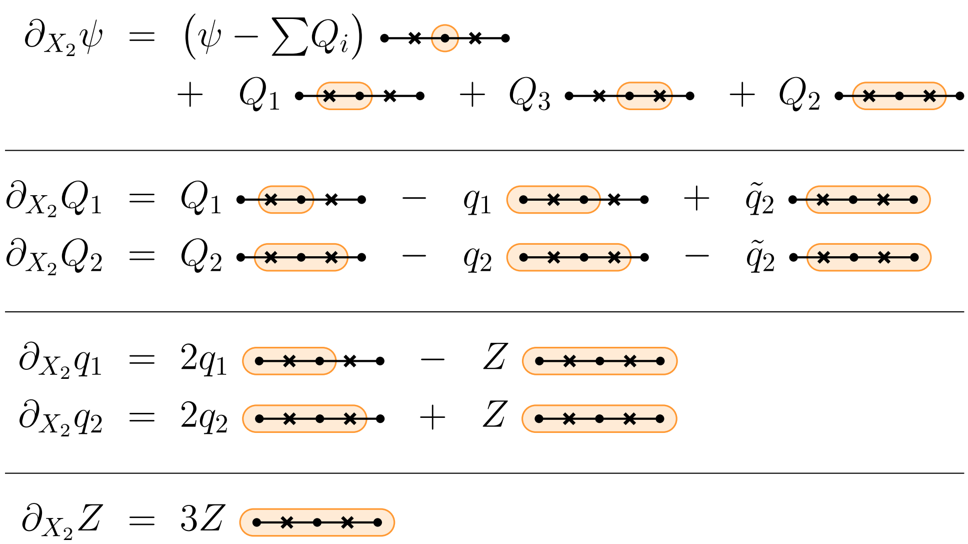
$$
\begin{aligned}
\mathrm{d}F \;=\; F \; + \; \big(F - f\big) \; + \; \big(F - \textstyle\sum q_i\big) \\
+ \; f \qquad\qquad + \; q_1 \\
+ \; q_2 \\
+ \; q_3
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}f \;=\; f \; + \; f \; + \; \big(f - g - \tilde{g} - Z\big) \\
+ \; g \\
+ \; \tilde{g} \\
+ \; Z
\end{aligned}
$$

• The rules stated above hold for arbitrary tree graphs!
In the paper, we present larger chains and the four-site star.



• Letters grow by merger and absorption:

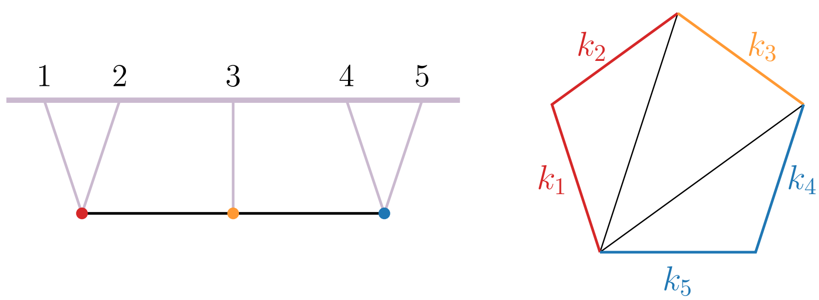


This is a new boundary manifestation of time evolution in cosmology!

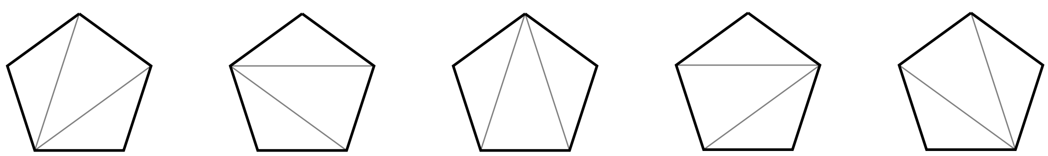
3.6. Beyond Single Graphs

So far, our treatment only applied to individual (off-shell) Feynman diagrams. However, Feynman diagrams are unphysical (especially for gauge theories and gravity) and only their sum has an invariant meaning. It is therefore important to find an on-shell formulation of cosmological correlators that doesn't rely on this unphysical separation into Feynman diagrams. In this work, we make a small step in this direction.

Each Feynman graph corresponds to a triangulation of a kinematic polygon:



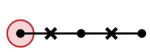
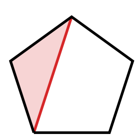
For colored scalars, there are 5 distinct triangulations:

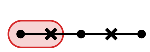
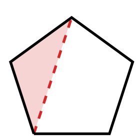


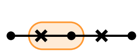
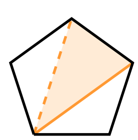
I II III IV V

- The differential equations produce these triangulations dynamically.
- Shared subpolygons becomes shared functions.

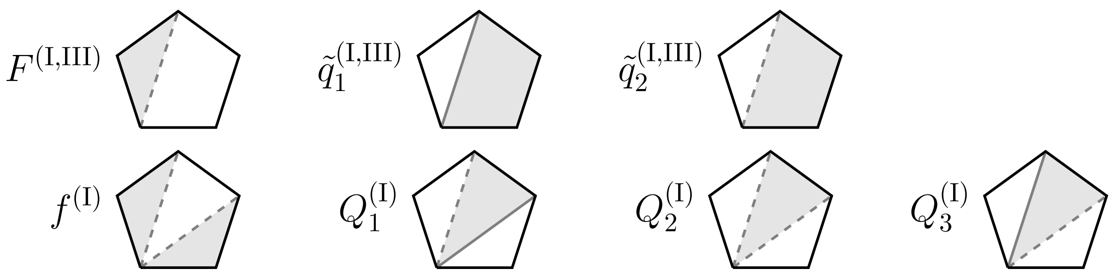
Letters = "activated subpolygons" with marked internal edges:

 $\Leftrightarrow$  $\equiv \mathrm{d}\log(k_{12} + y_{12})$

 $\Leftrightarrow$  $\equiv \mathrm{d}\log(k_{12} - y_{12})$

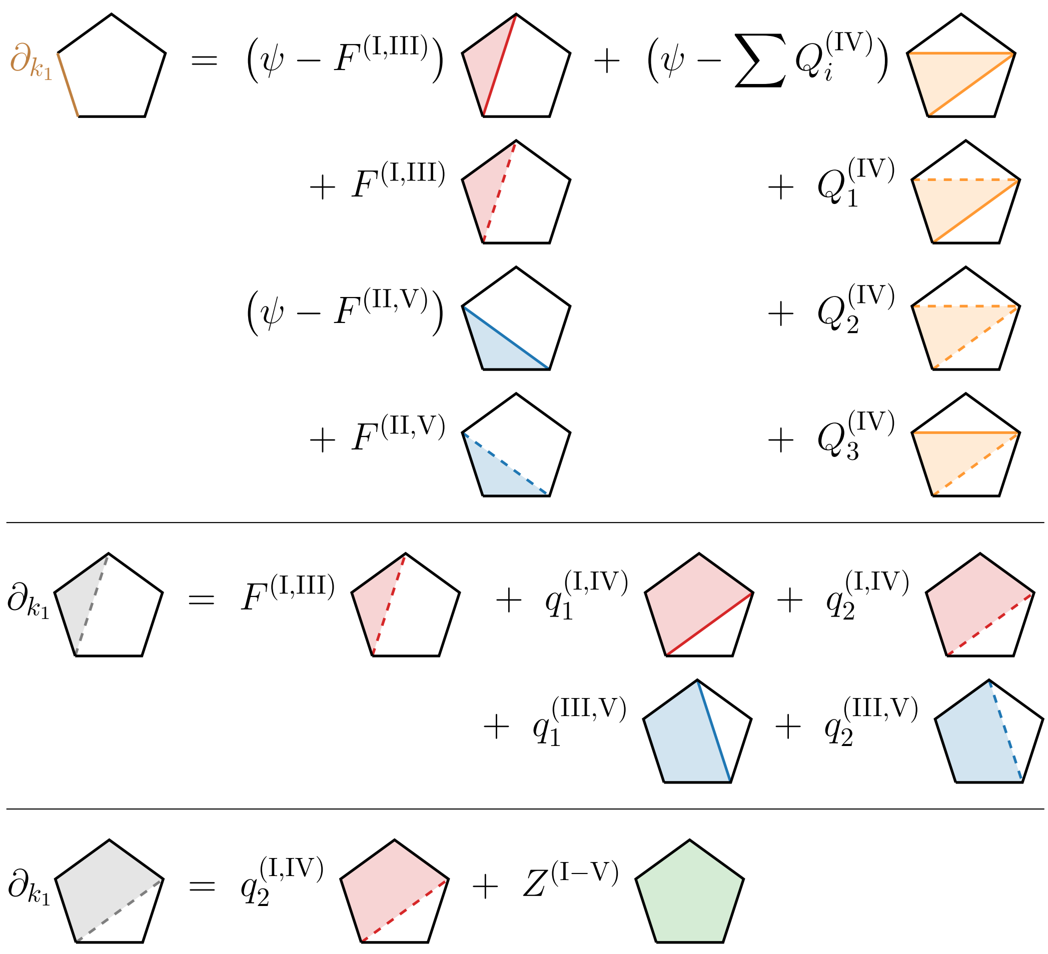
 $\Leftrightarrow$  $\equiv \mathrm{d}\log(k_3 - y_{12} + y_{45})$

Sources = (disconnected) subpolygons (with at least one marked edge):



The functions in the first line are shared between channels I and III.

Evolution = growing subpolygons (letters):



No separation into individual channels!

3.7. Outlook and Speculations

We discovered a new boundary description of cosmological time evolution:

- Time evolution is encoded in twisted integrals of the flat-space wavefunction.
- These integrals satisfy differential equations whose structure is predicted by simple rules.

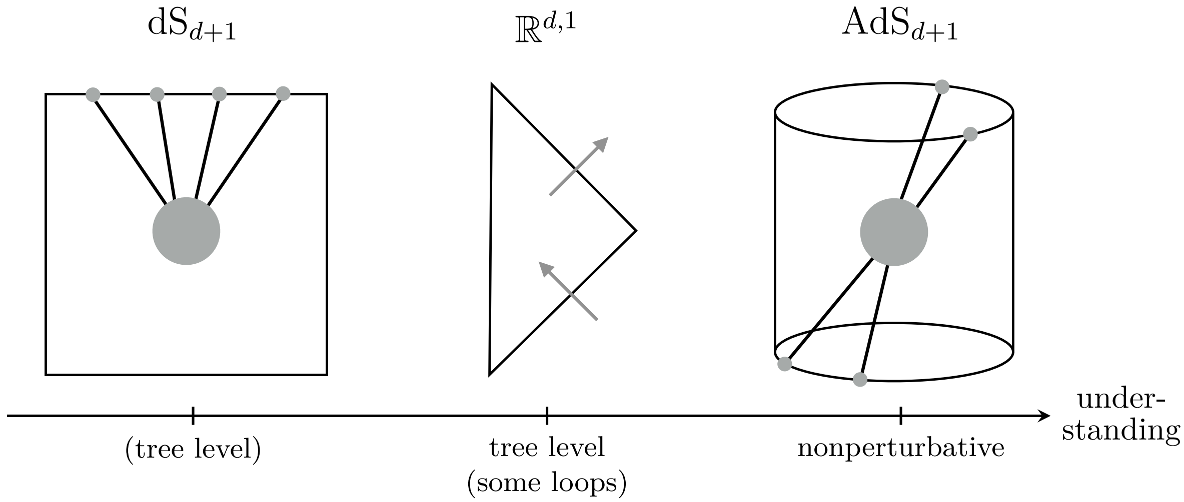
Open questions:

- **Massive particles**: So far, only conformal scalars. What about massive particles?
- **Loops**: So far, only trees. What about loops?
- **Positive geometry**: Is there a positive geometry underlying the structures that we found?
- **Emergent time**: Can the structures we found be derived from deeper principles?

IV. OUTLOOK

There are many interesting things I didn't have time to describe in the lectures. Some more can be found in the lecture notes.

Despite enormous progress in recent years, we are only at the beginning of a systematic exploration of cosmological correlators:



Many fundamental questions still remain unanswered:

• **Beyond Feynman** So far, the cosmological bootstrap has only been applied to individual Feynman diagrams (which by themselves are unphysical). However, in the S-matrix bootstrap, the real magic is found for on-shell diagrams. What is the on-shell formulation of the cosmological bootstrap? What magic will it reveal?

• **Beyond Perturbation Theory** So far, the bootstrap constraints (singularities, unitarity cuts, etc.) are only formulated in perturbation theory and implemented only at tree level. In contrast, we understand the conformal bootstrap nonperturbatively. Recently, some preliminary progress was made to understand unitarity constraints in de Sitter nonperturbatively, but much work remains. A nonperturbative bootstrap is required if we want to understand the UV completion of cosmological correlators.

• **Beyond Spacetime** In quantum gravity, spacetime is an emergent concept. In cosmology, the notation of "time" breaks down at the Big Bang. What replaces it? The cosmological bootstrap provides a modest form of emergent time, but there should be a much more radical approach where space and time are outputs, not inputs. For scattering amplitudes, such a reformulation was achieved in the form of the *amplituhedron*. What is the analogous object in cosmology? How does the Big Bang arise from it?