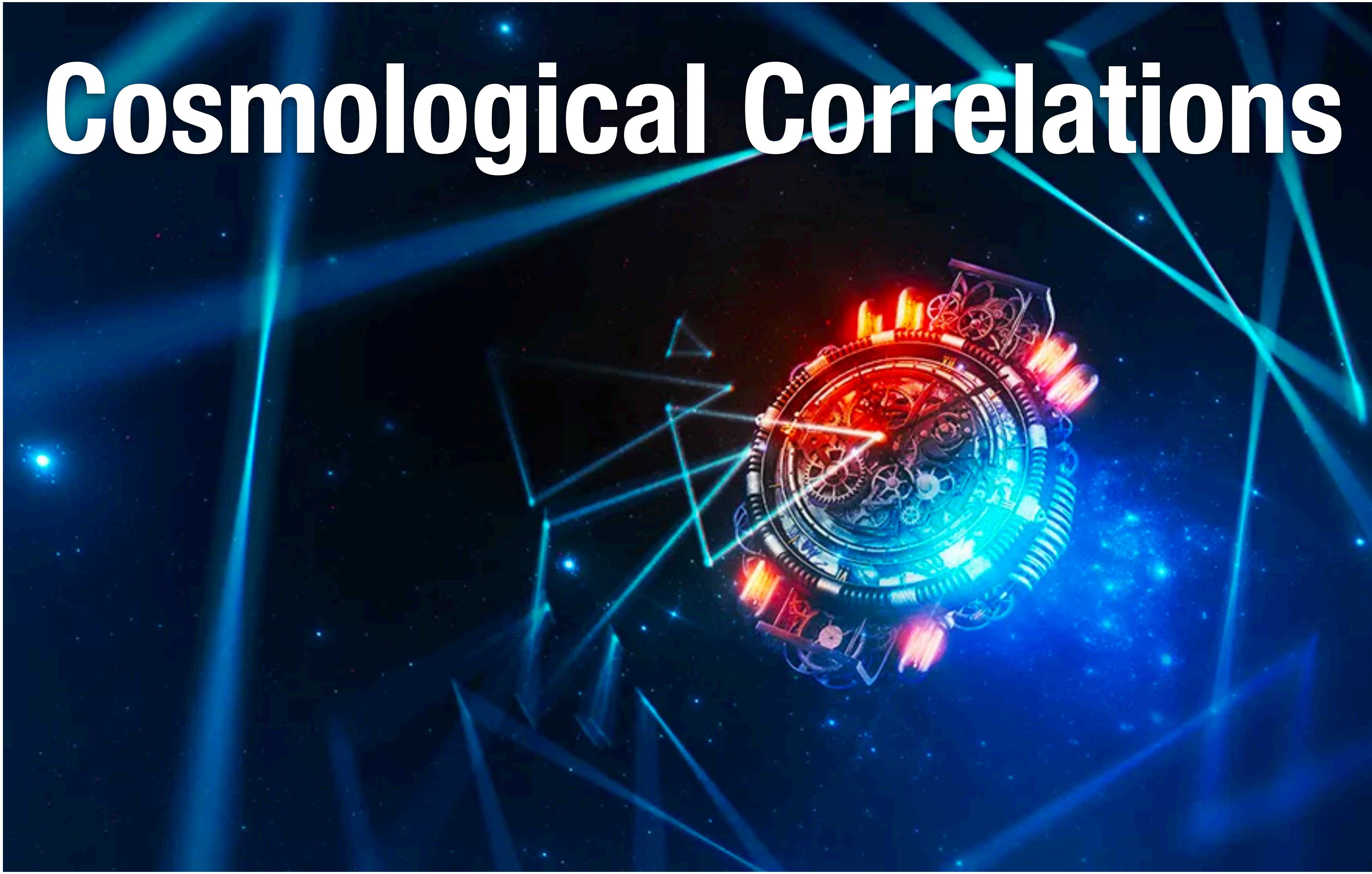
Cosmological Correlations



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Kavli Asian Winter School
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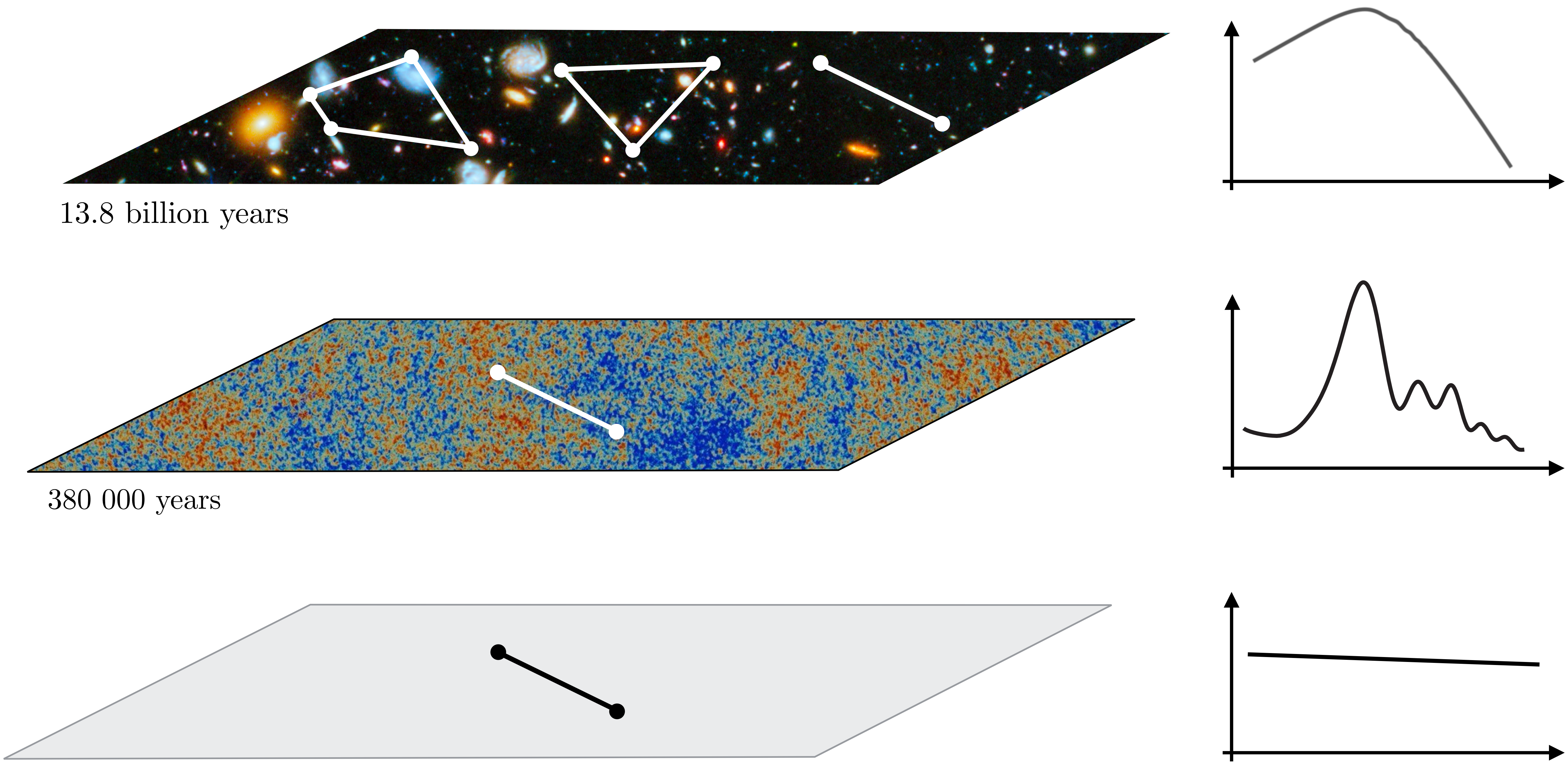
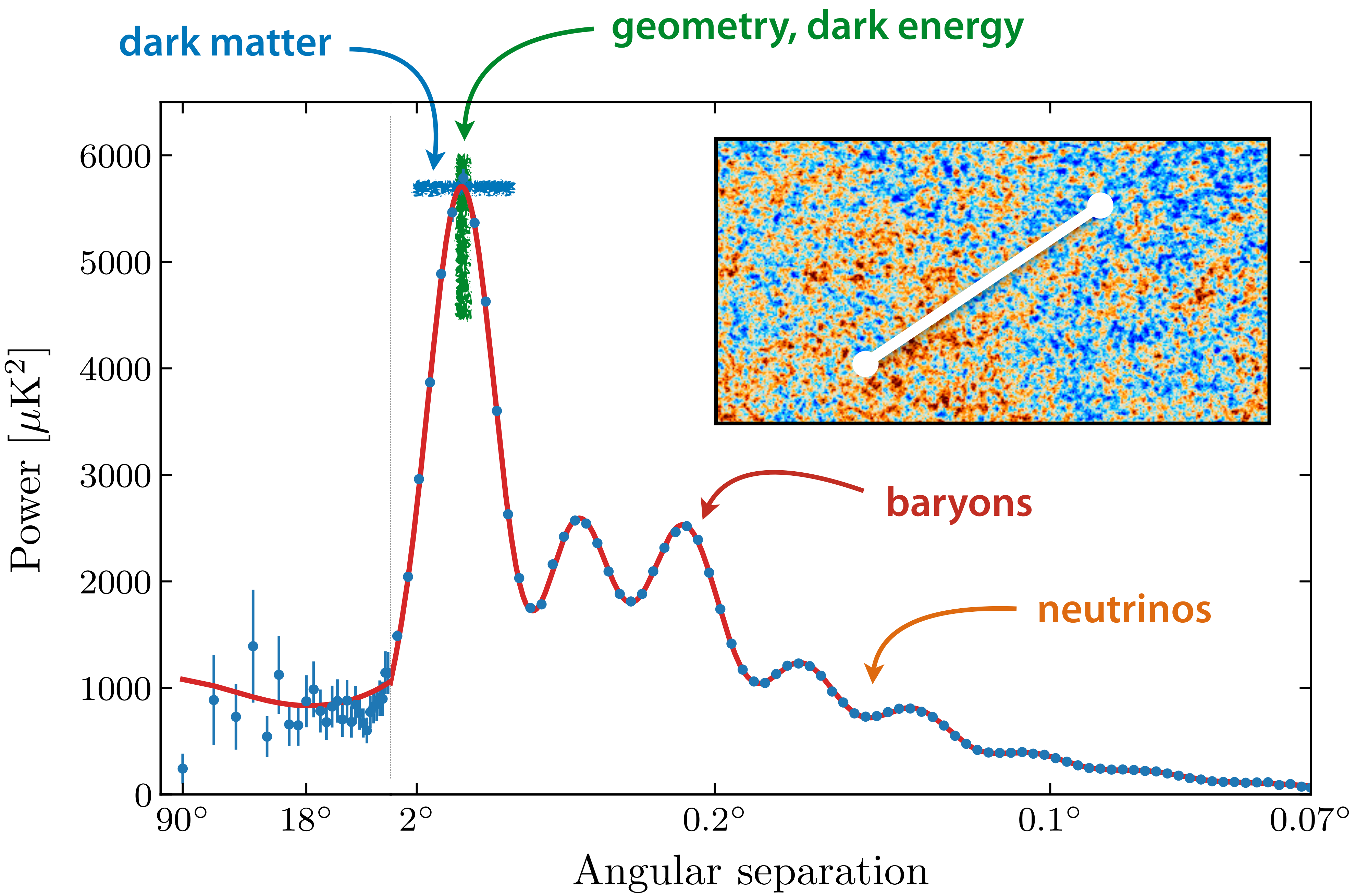
Motivation

By measuring cosmological correlations, we learn both about the **evolution** of the universe and its **initial conditions**:



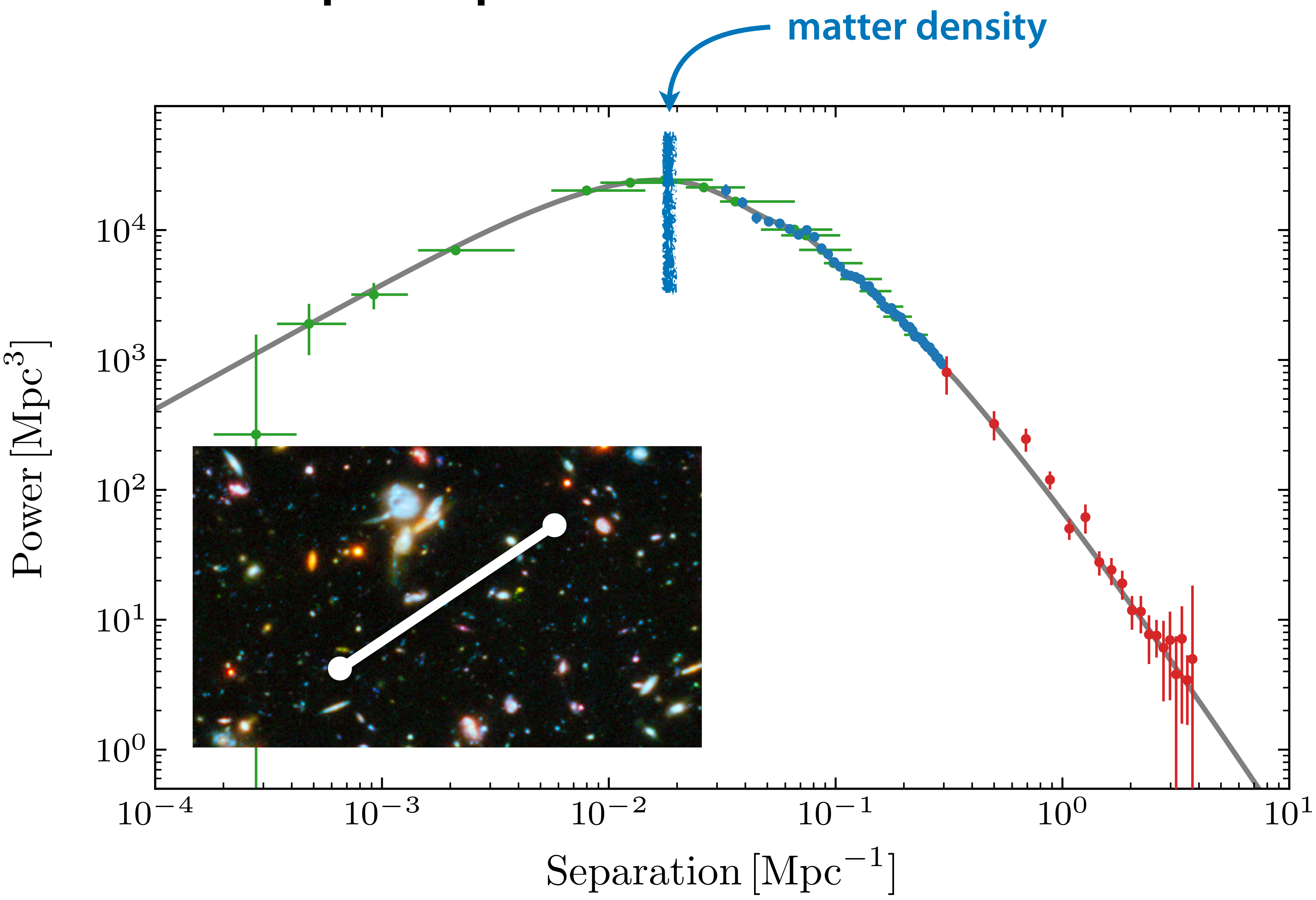
Motivation

The correlations in the CMB temperature anisotropies have revealed a great deal about the **geometry** and **composition** of the universe:



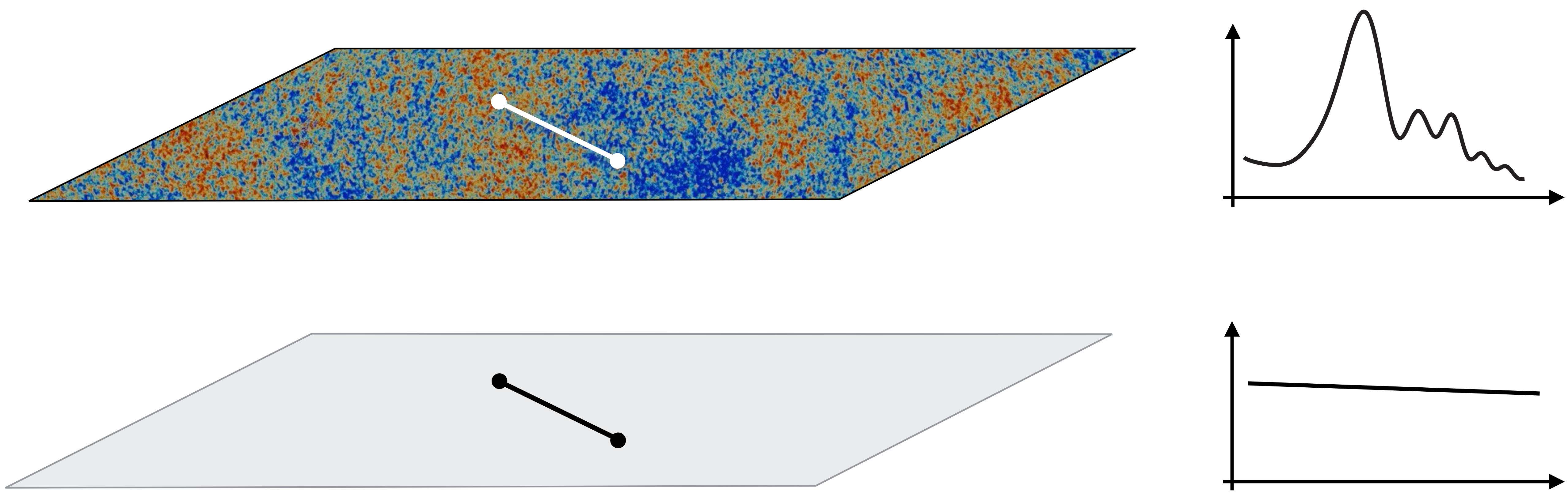
Motivation

Correlations are also observed in the large-scale structure of the universe. The observed **matter power spectrum** is



Motivation

Under relatively mild assumptions, the observed correlations can be traced back to primordial correlations on the **reheating surface**:

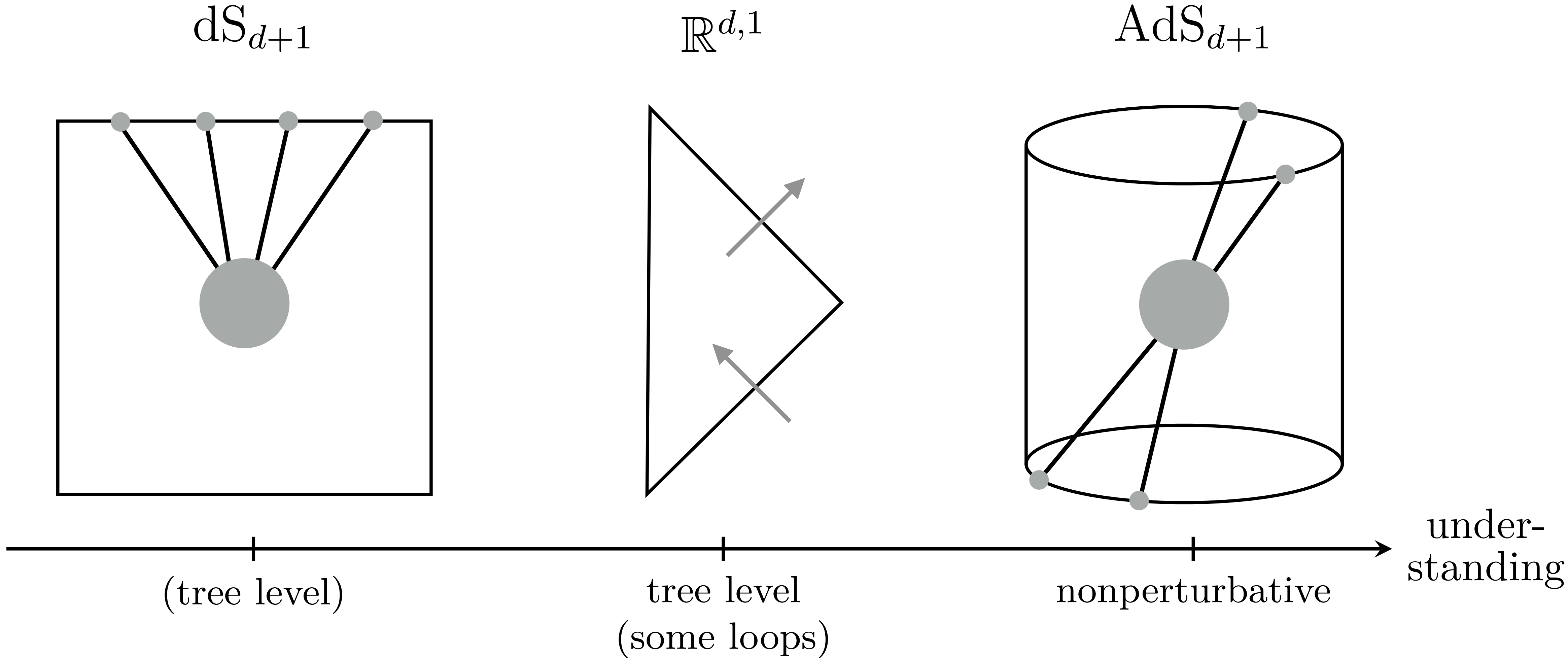


We have learned two interesting facts about these initial conditions:

1) The fluctuations were created **before the hot Big Bang**.
2) During a phase of time-translation invariance (= **inflation**).

Motivation

The study of cosmological correlators is also of **conceptual interest**:



Our understanding of quantum field theory in de Sitter space (cosmology) is still rather underdeveloped. **Opportunity for you to make progress!**

OUTLINE:

Cosmological Correlations	Wavefunction Approach	Cosmological Bootstrap
(Lecture 1)	(Lectures 2+3)	(Lectures 3+4)

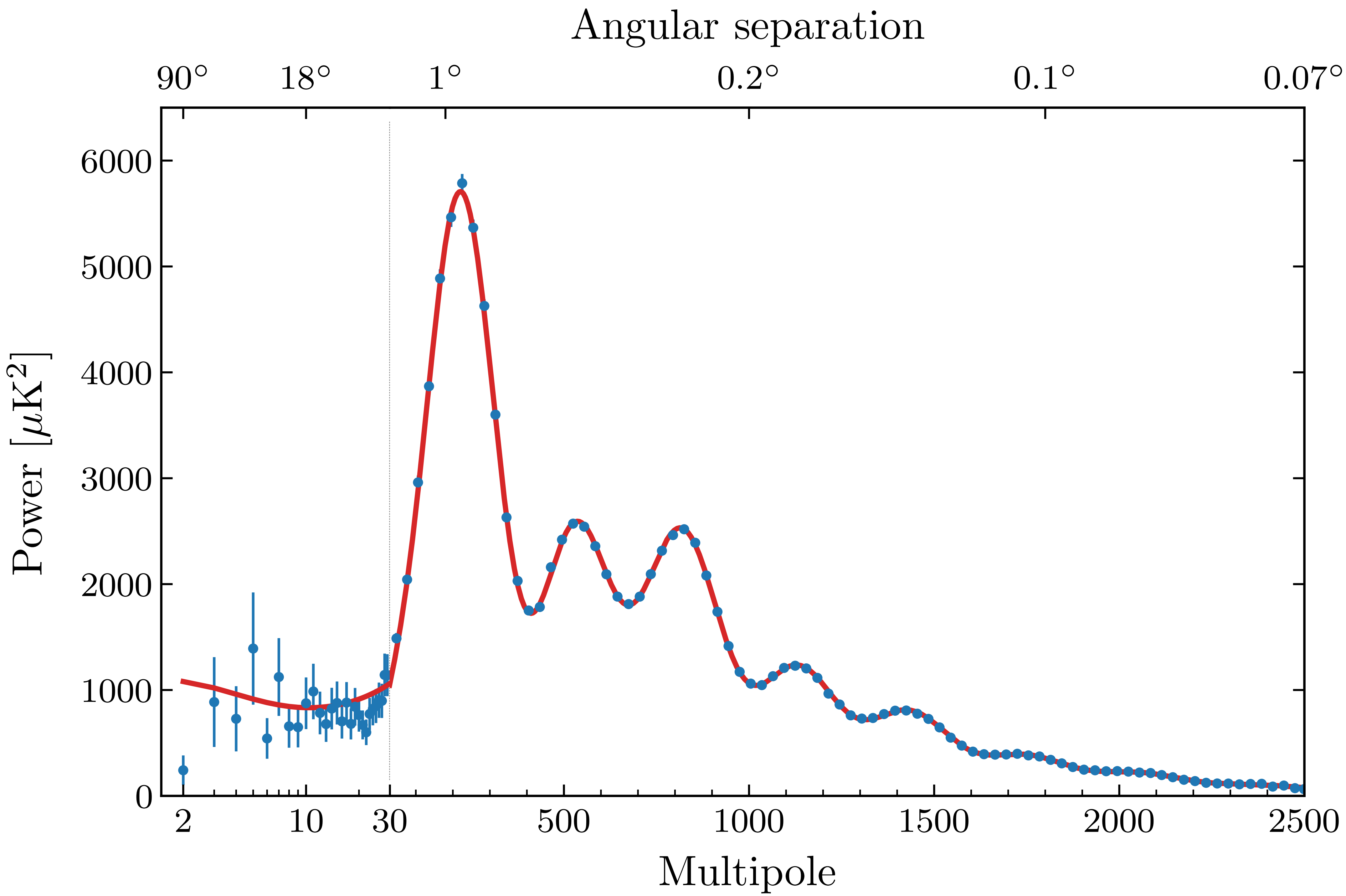
LECTURE NOTES:

https://github.com/ddbaumann/cosmo-correlators

written together with Austin Joyce.

Cosmological Correlations

CMB Power Spectrum



Large scales

Small scales

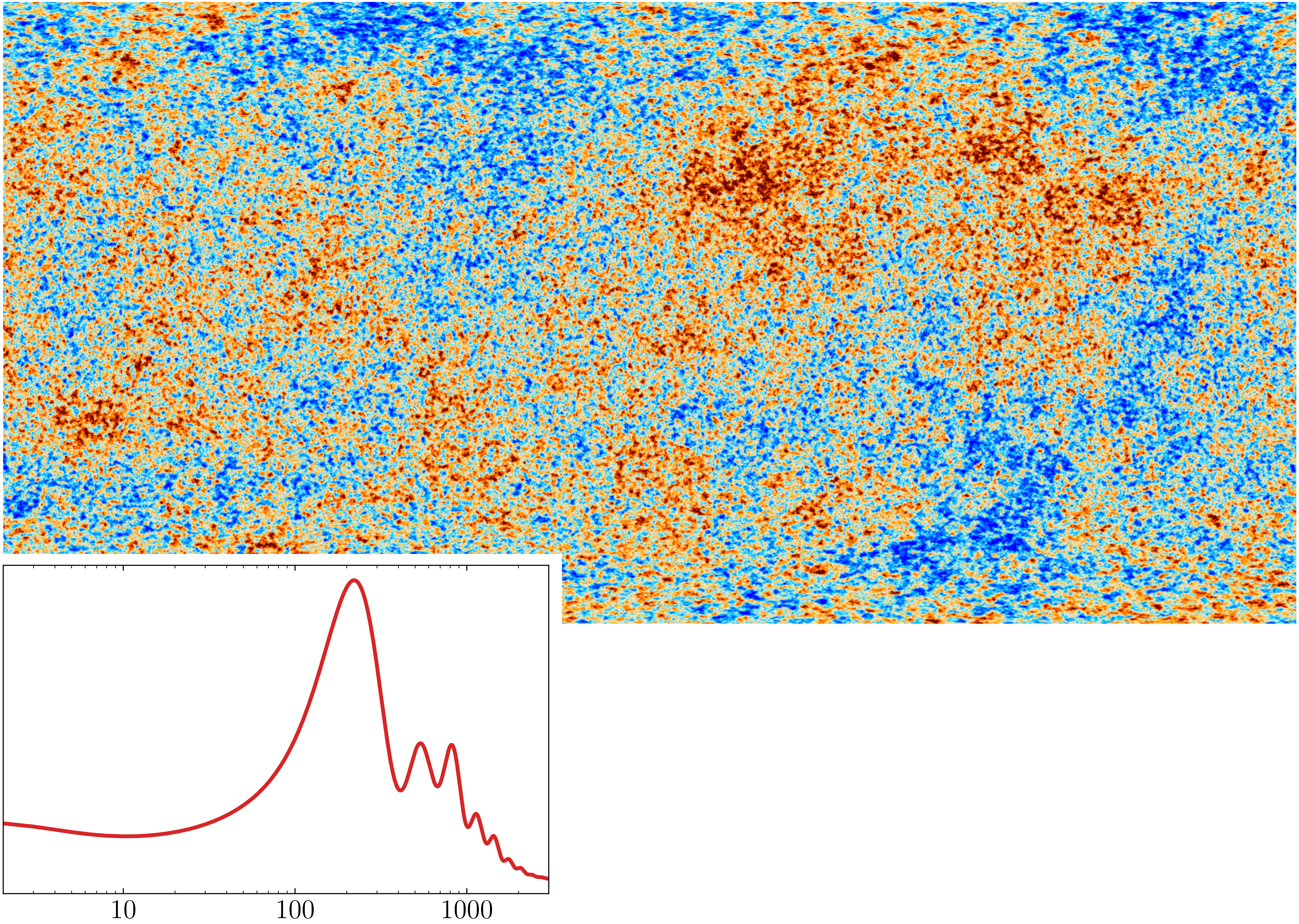
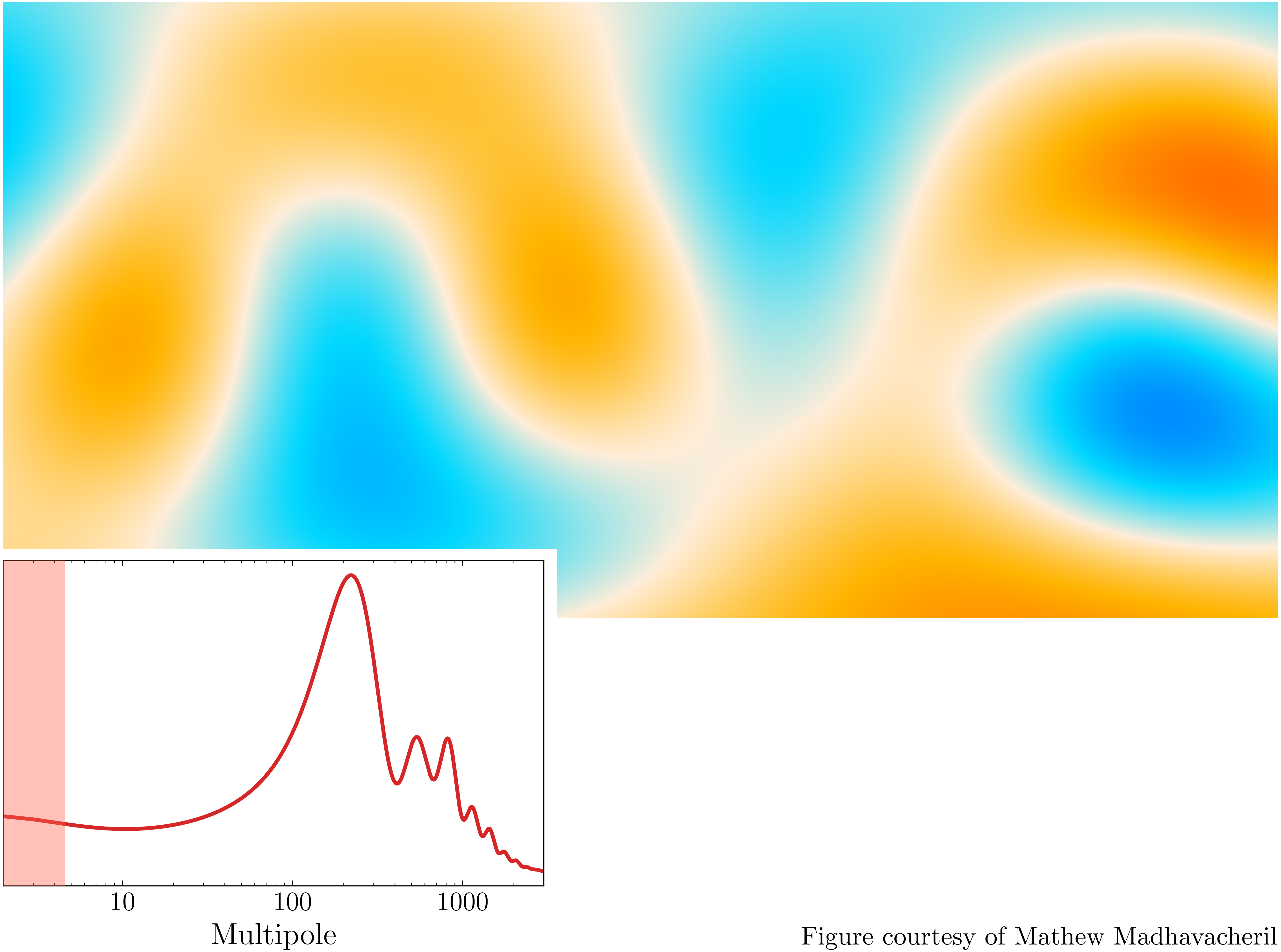


Figure courtesy of Mathew Madhavacheril



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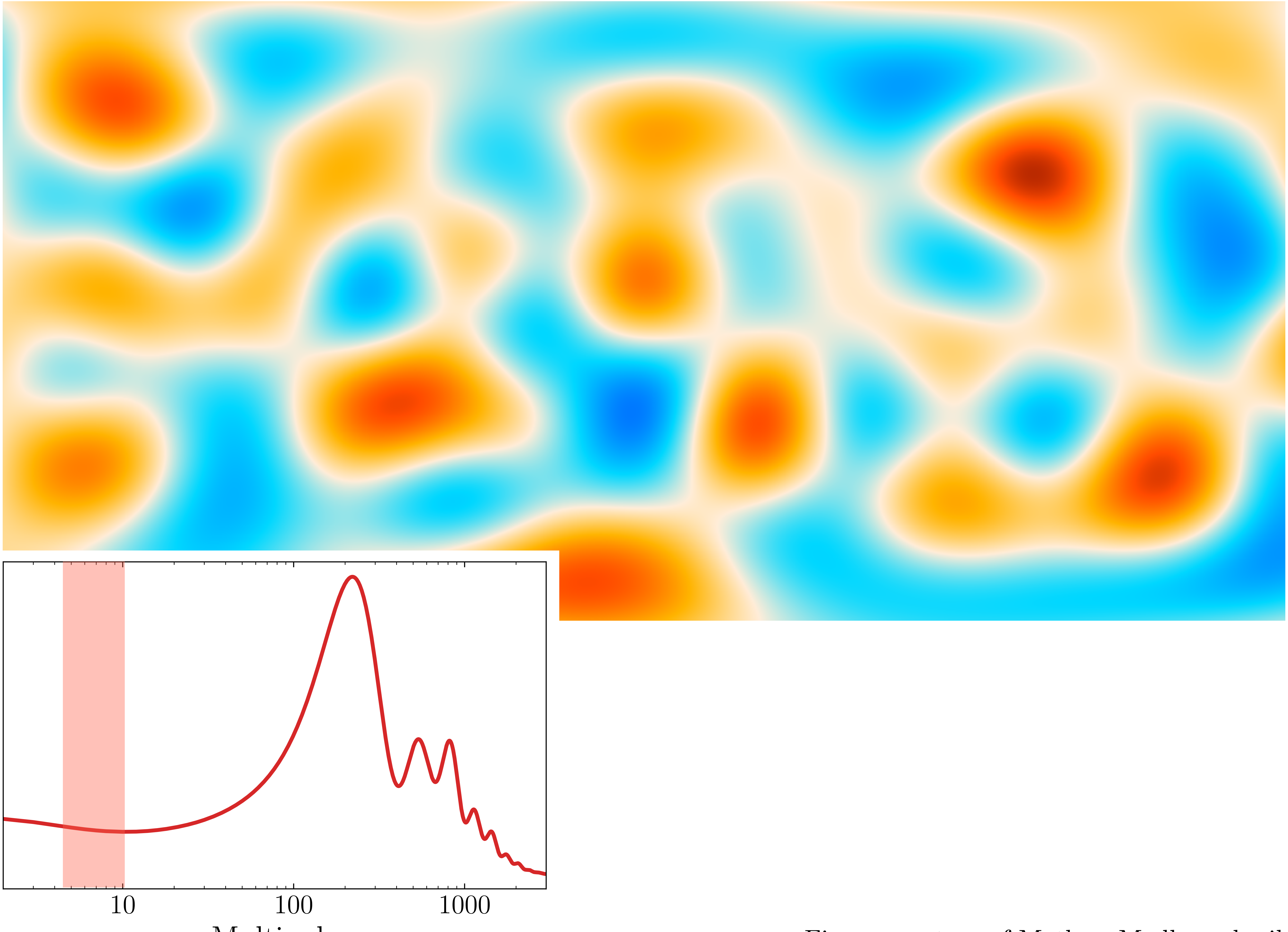

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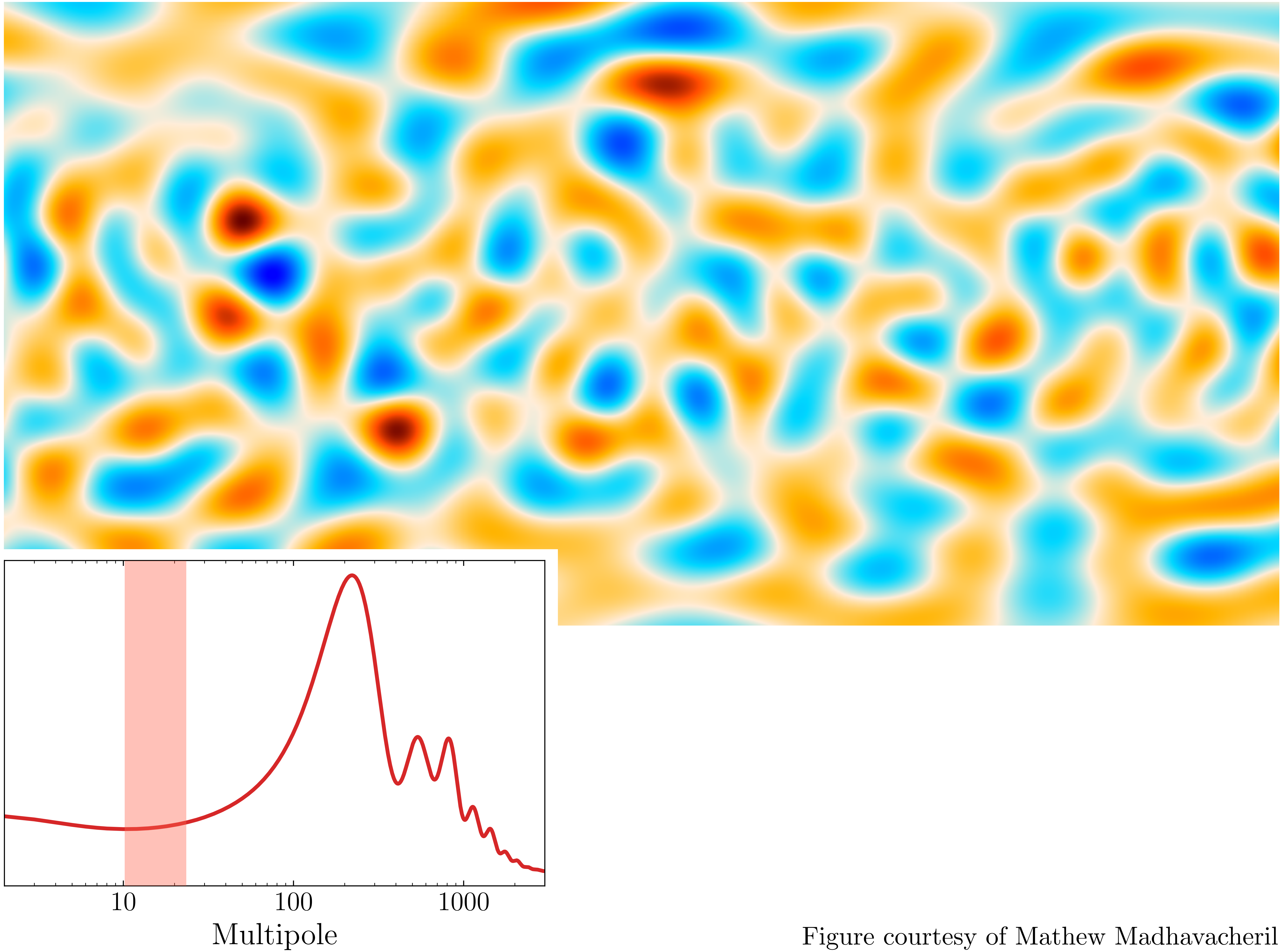

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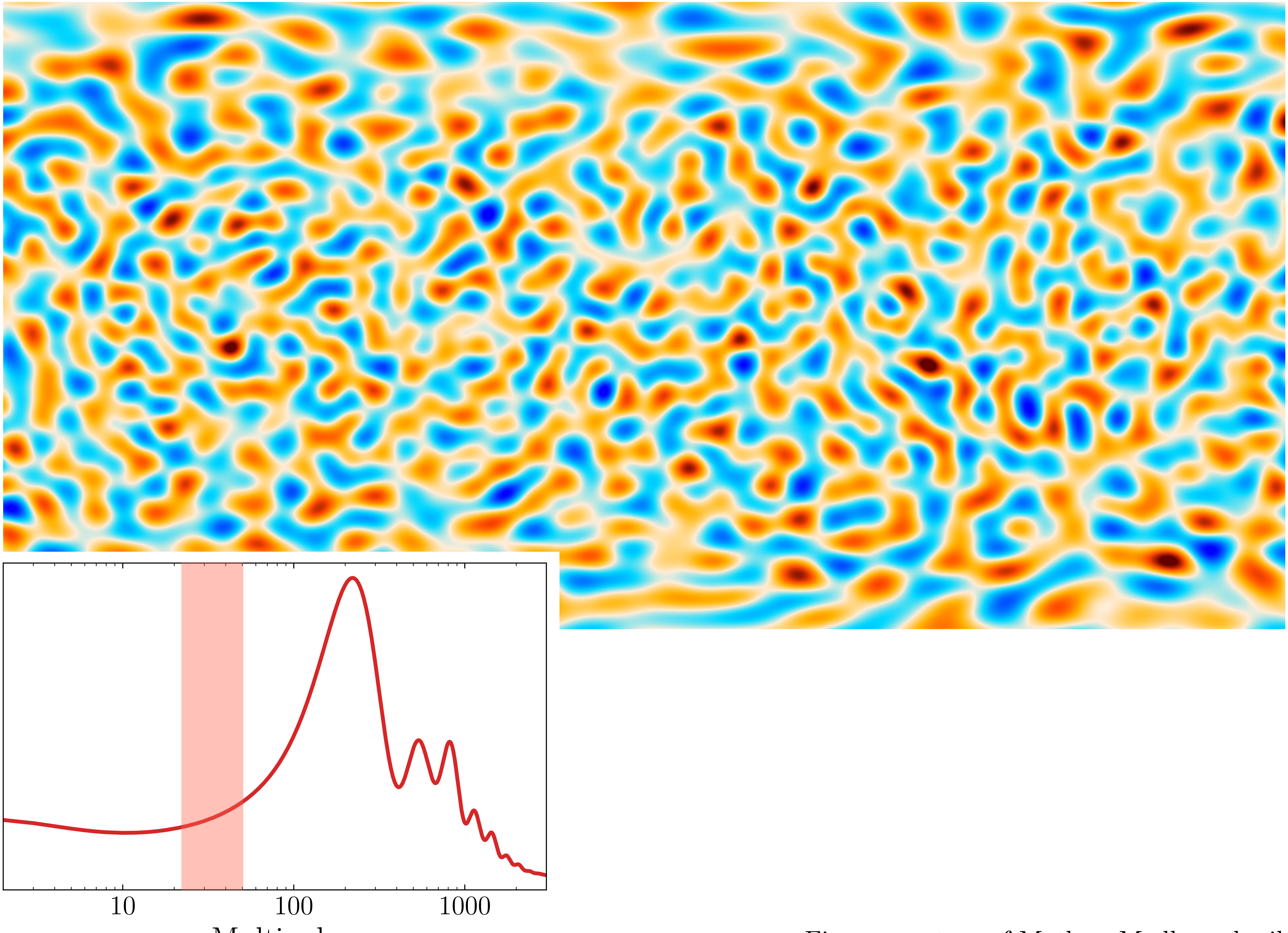

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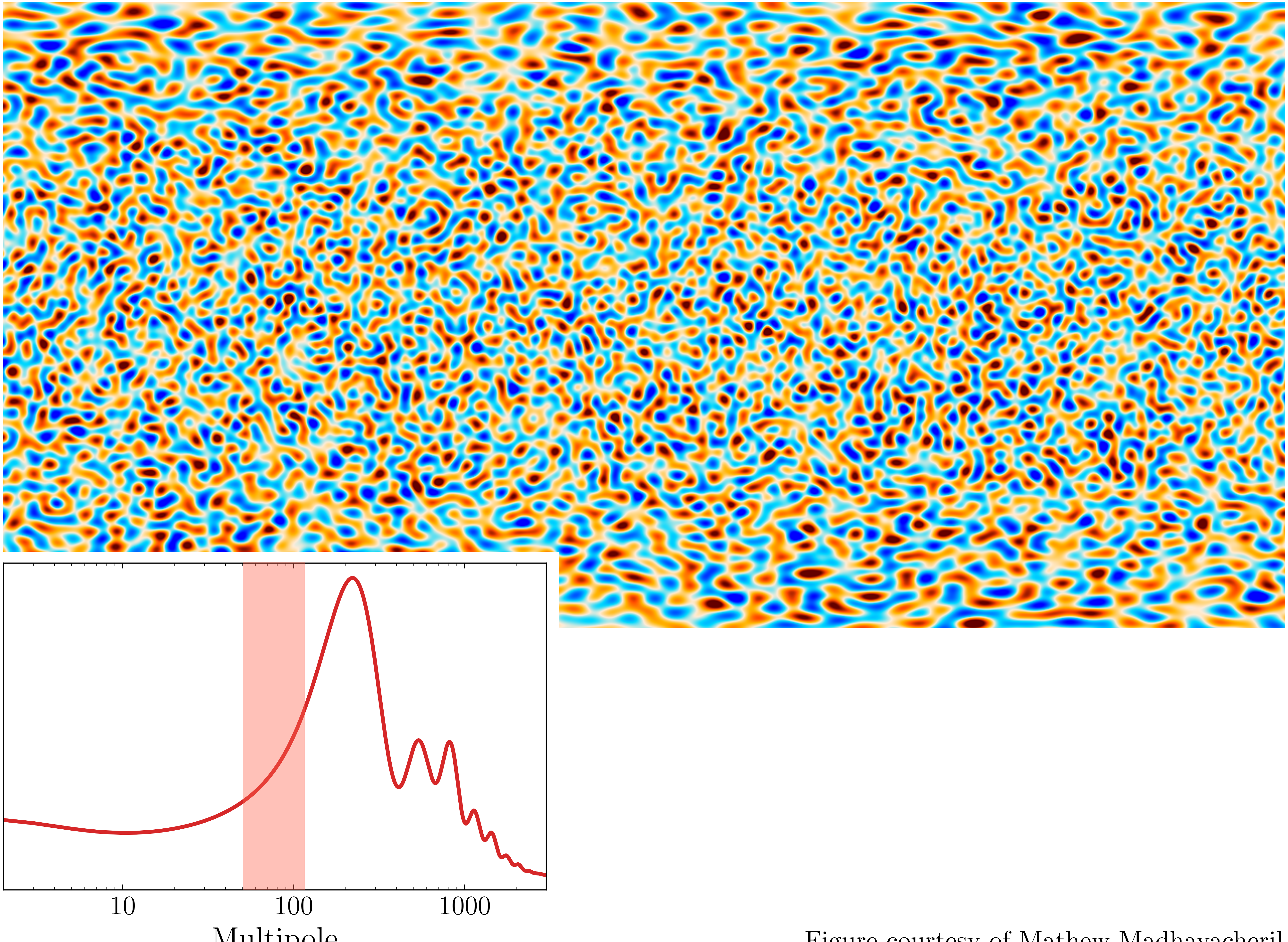

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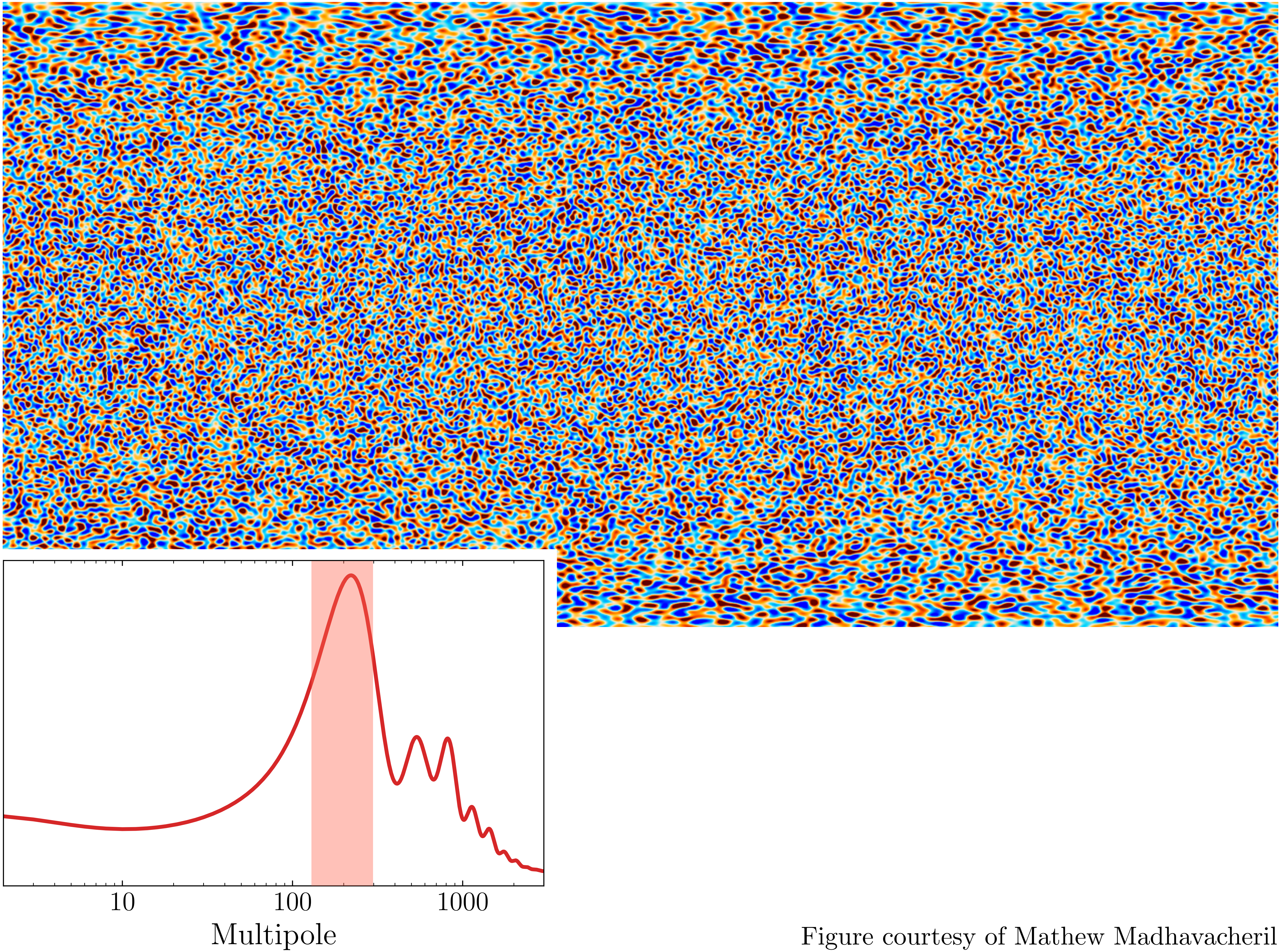

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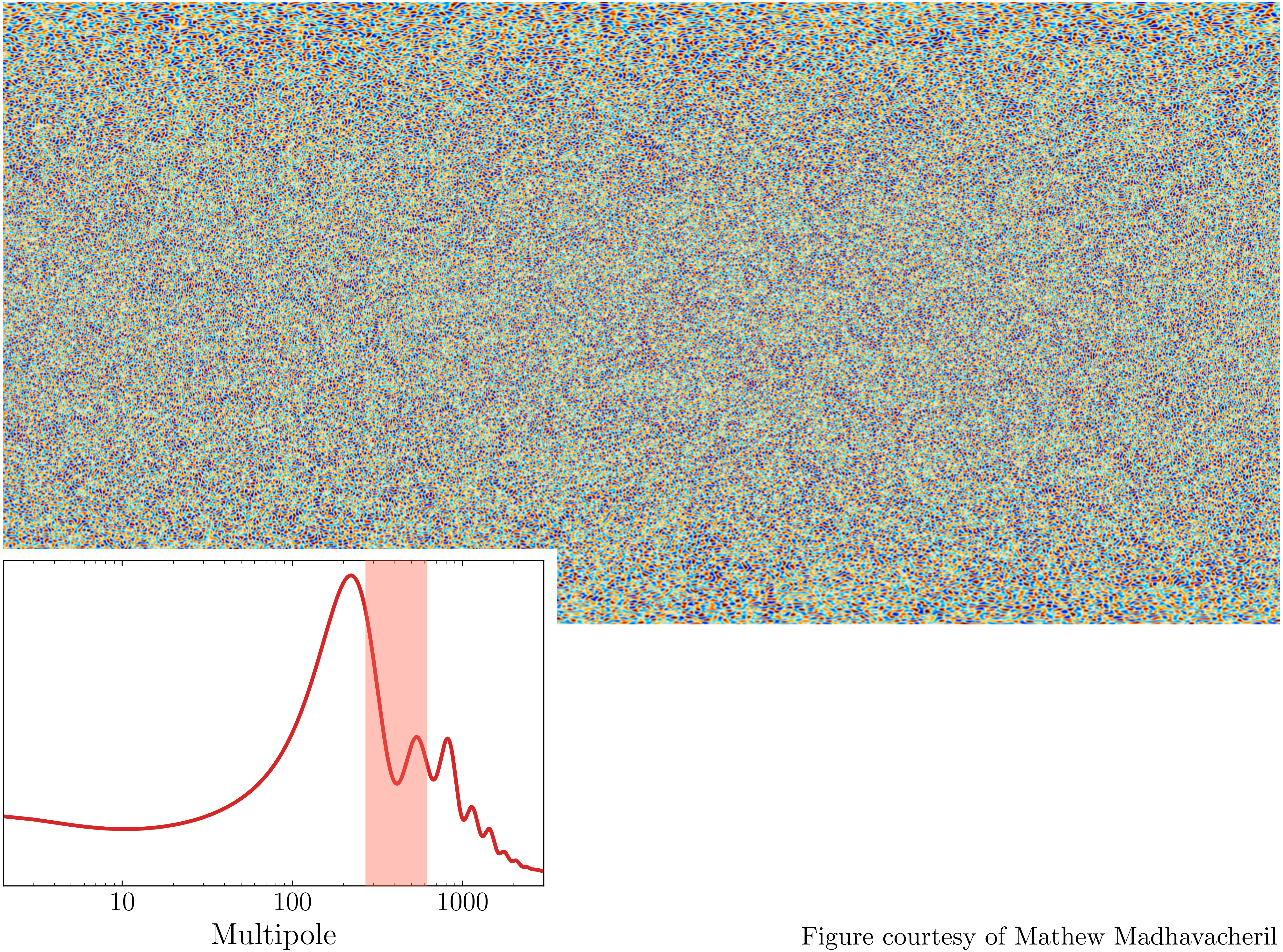

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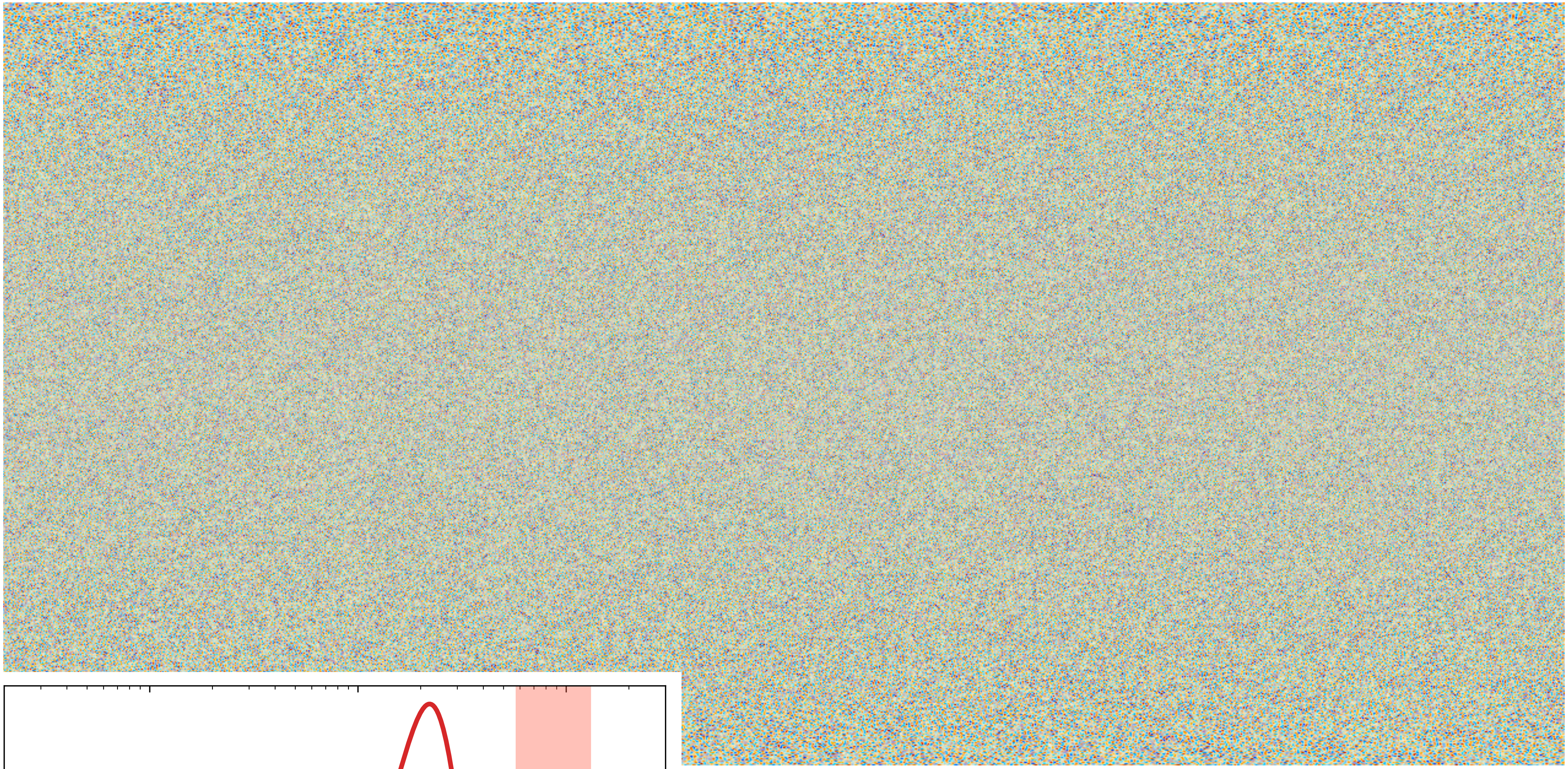

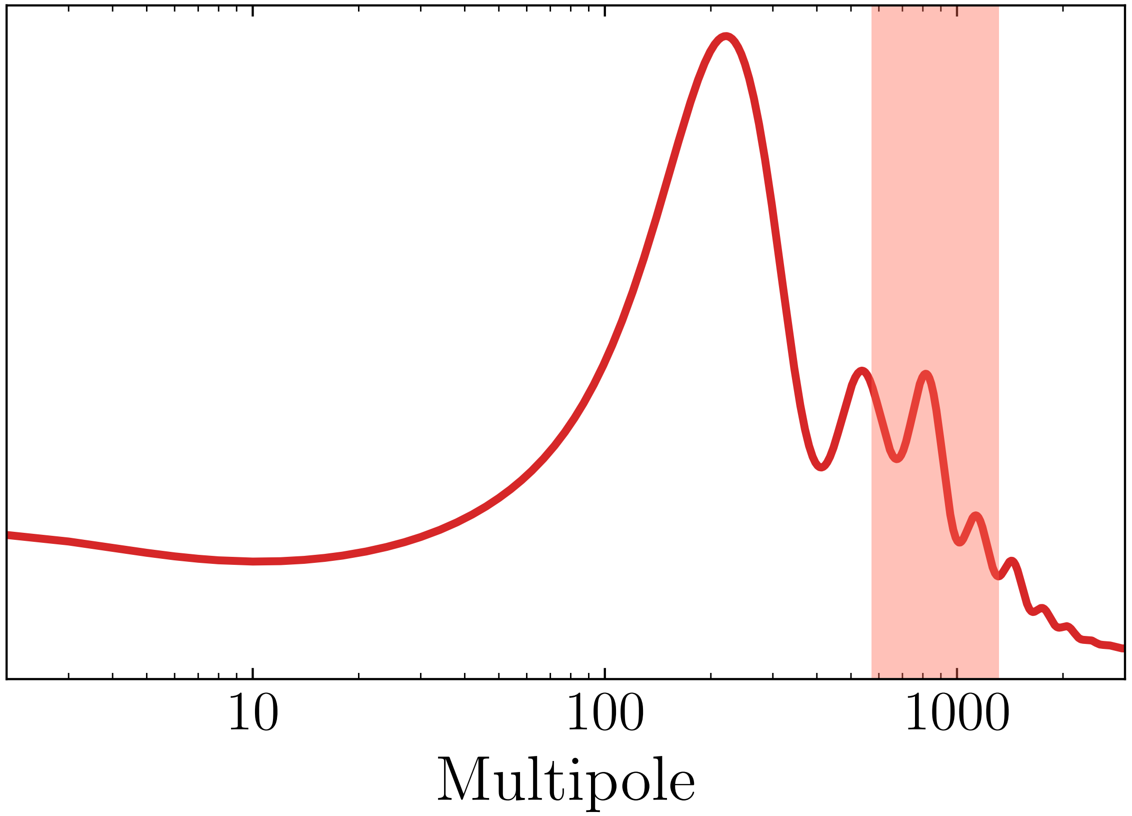

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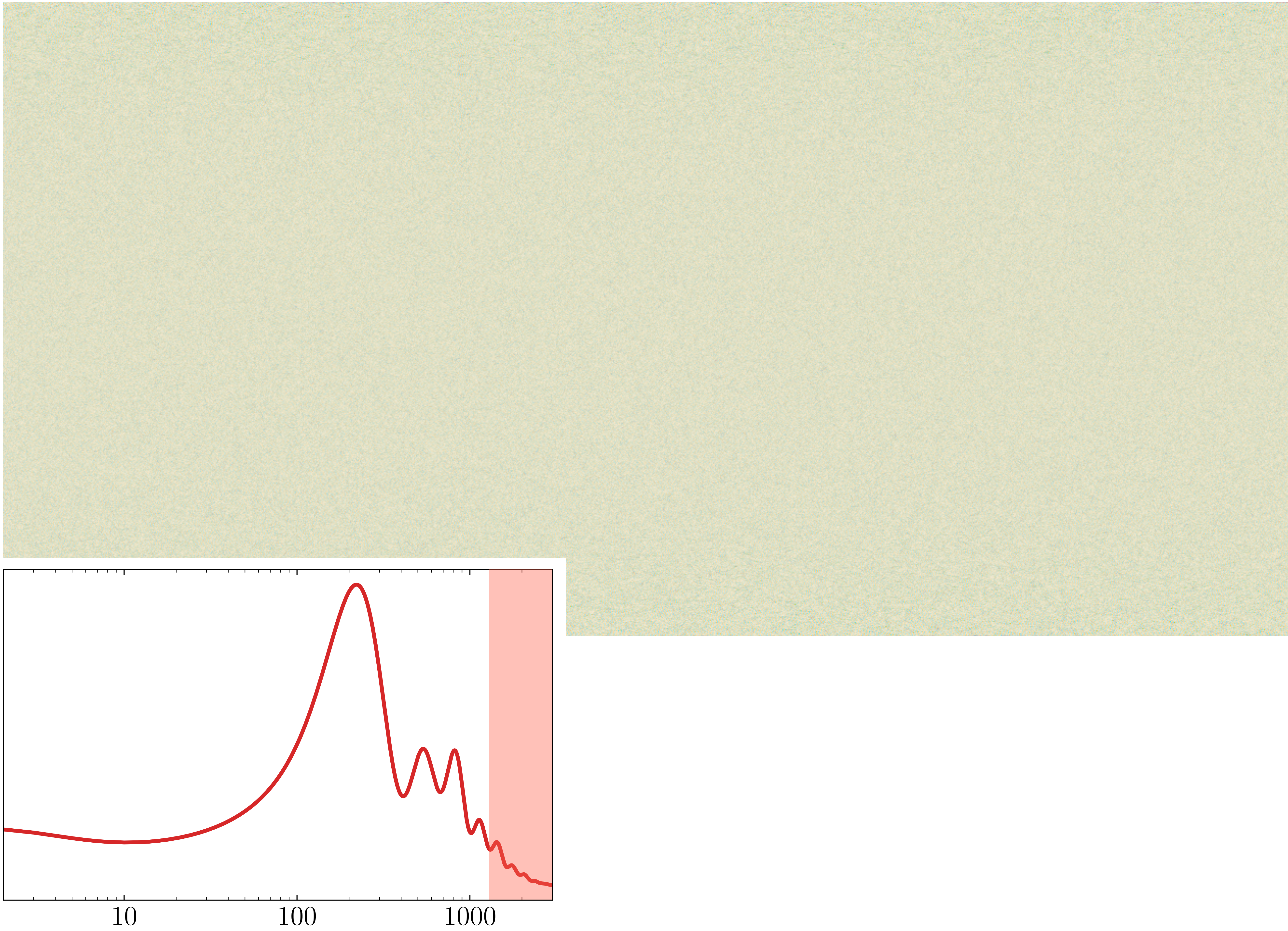

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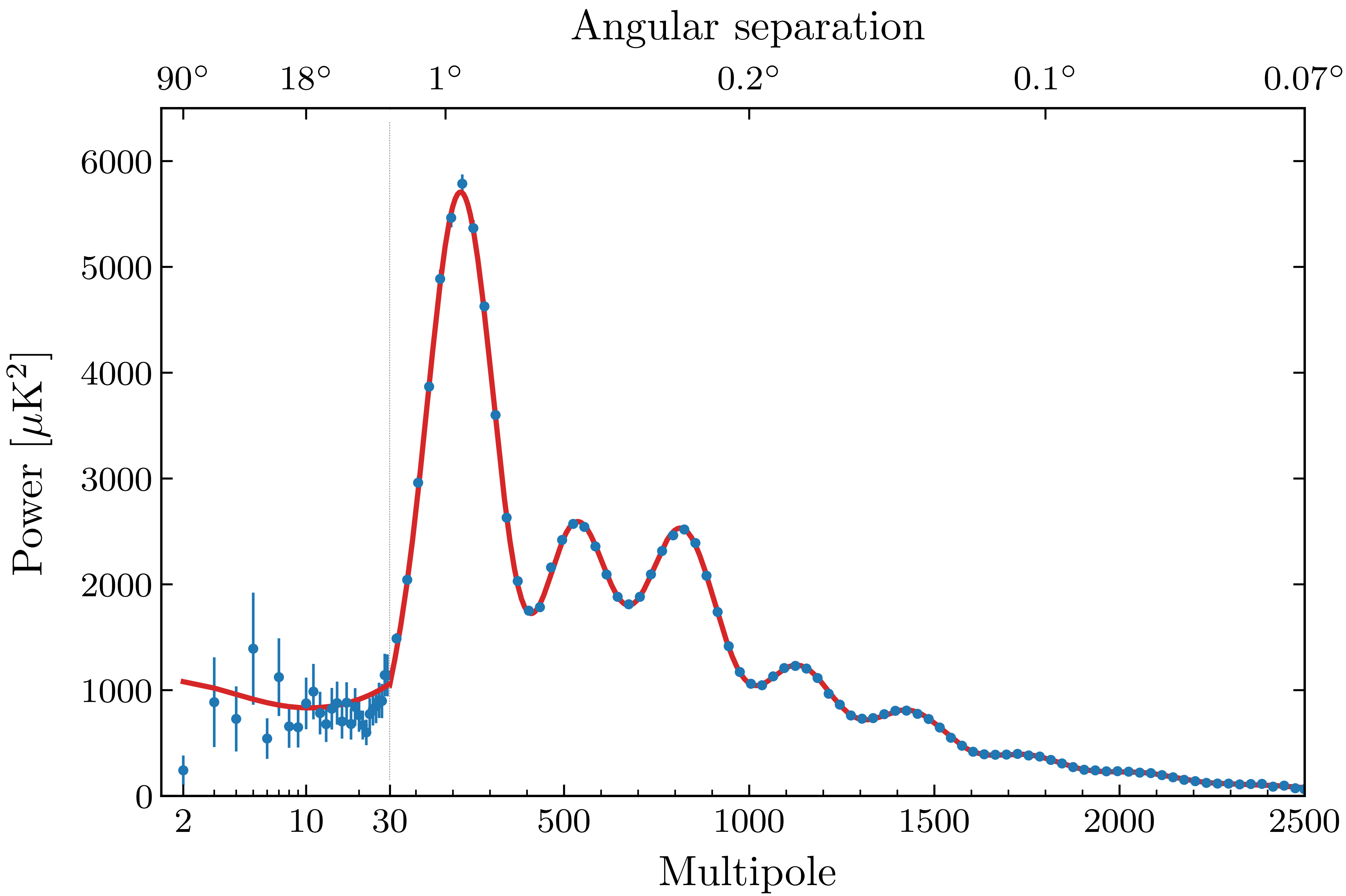
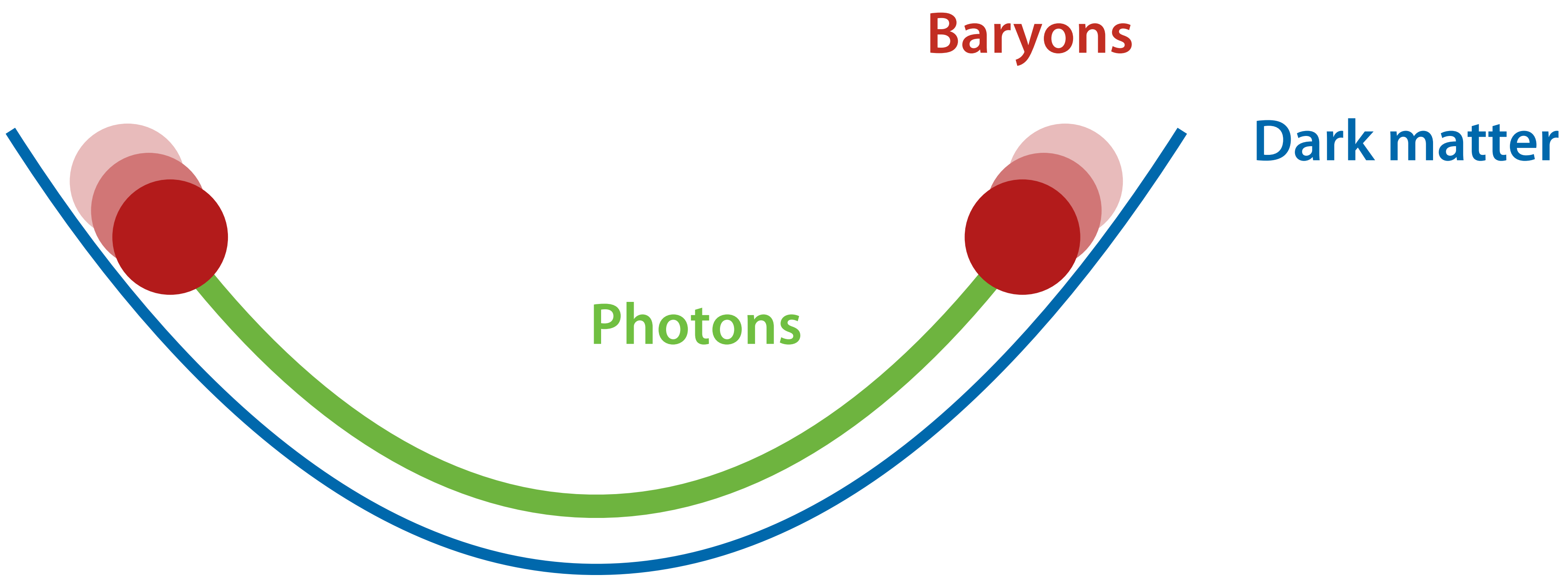
CMB Power Spectrum



What created the features in the power spectrum?

Photon-Baryon Fluid

At early times, photons and baryons (mostly protons and electrons) are strongly coupled and act as a single fluid:



The **photon pressure** prevents the collapse of density fluctuations.
This allows for **sound waves** (like for density fluctuations in air).

Cosmic Sound Waves

Decomposing the sound waves into Fourier modes, we have

$$\boxed{\frac{\delta\rho_\gamma}{\rho_\gamma} = A_{\mathbf{k}}\cos[c_s k\tau_*] + B_{\mathbf{k}}\sin[c_s k\tau_*]} \;\leftarrow\; 4\,\frac{\delta T}{T}$$

Time at recombination

Sound speed $c_s \approx 1/\sqrt{3}$

The wave amplitudes are random variables, with

$$\langle A_{\mathbf{k}}\rangle = \langle B_{\mathbf{k}}\rangle = 0$$

This is why we don't see a wavelike pattern in the CMB map.

To see the sound waves, we need to look at the statistics of the map.

CMB Power Spectrum

The two-point function in Fourier space is

$$\langle \delta T(\mathbf{k}) \delta T(\mathbf{k}') \rangle = \Big[P_A(k) \cos^2(c_s k \tau_*) + 2 C_{AB}(k) \sin(c_s k \tau_*) \cos(c_s k \tau_*) + P_B(k) \sin^2(c_s k \tau_*) \Big] (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}')$$

If the sound waves were created by a random source, we would have

$$C_{AB}(k) = 0 \qquad P_A(k) = P_B(k) \equiv P(k)$$

and hence the two-point function becomes

$$\langle \delta T(\mathbf{k}) \delta T(\mathbf{k}') \rangle = P(k)\, (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}')$$

Where are the oscillations?

Coherent Phases

To explain the oscillations in the CMB power spectrum, we have to assume that the sound waves do **not** have random phases:

$$\delta T(\mathbf{k}) = A_{\mathbf{k}} \cos[c_s k \tau_*]$$

$$\downarrow$$

$$\langle \delta T(\mathbf{k}) \delta T(\mathbf{k}') \rangle' = P_A(k) \cos^2[c_s k \tau_*]$$

Oscillations in the CMB power spectrum

What created the coherent phases in the primordial sound waves?

Before the Big Bang

Phase coherence is created if the fluctuations were produced **before the hot Big Bang**. Let's see why.

Going back in time, all fluctuations have wavelengths > Hubble scale.

The solution on these large scales is

$$\frac{\delta\rho_\gamma}{\rho_\gamma} = C_{\mathbf{k}} + D_{\mathbf{k}}\, a^{-3}$$

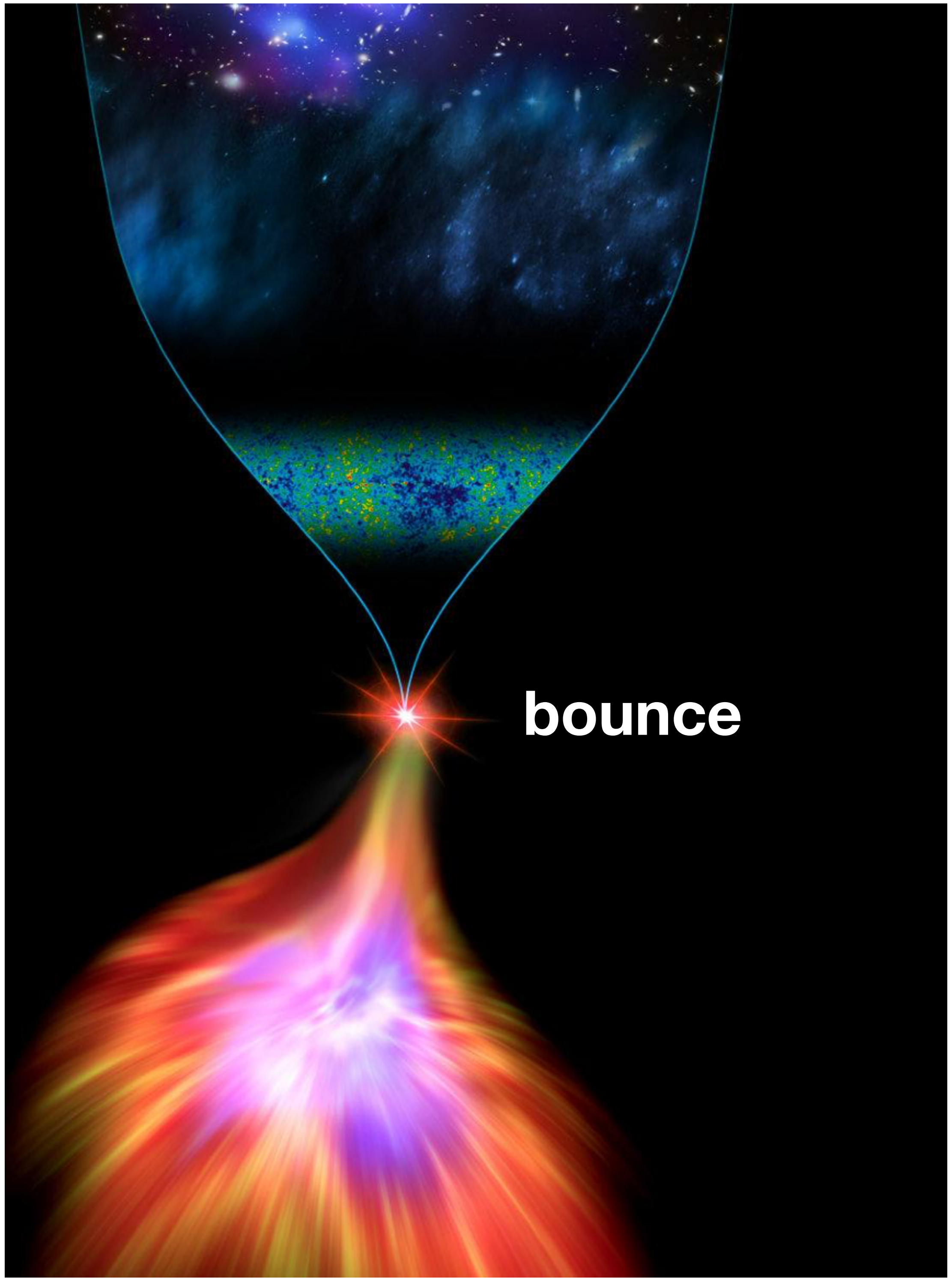
If the fluctuations were created long before the hot Big Bang, then only the constant growing mode survived.

This explains the phase coherence of the primordial sound waves

$$\frac{\delta\rho_\gamma}{\rho_\gamma} = C_{\mathbf{k}} \;\; \to \;\; A_{\mathbf{k}} \cos\left[c_s k\tau\right]$$

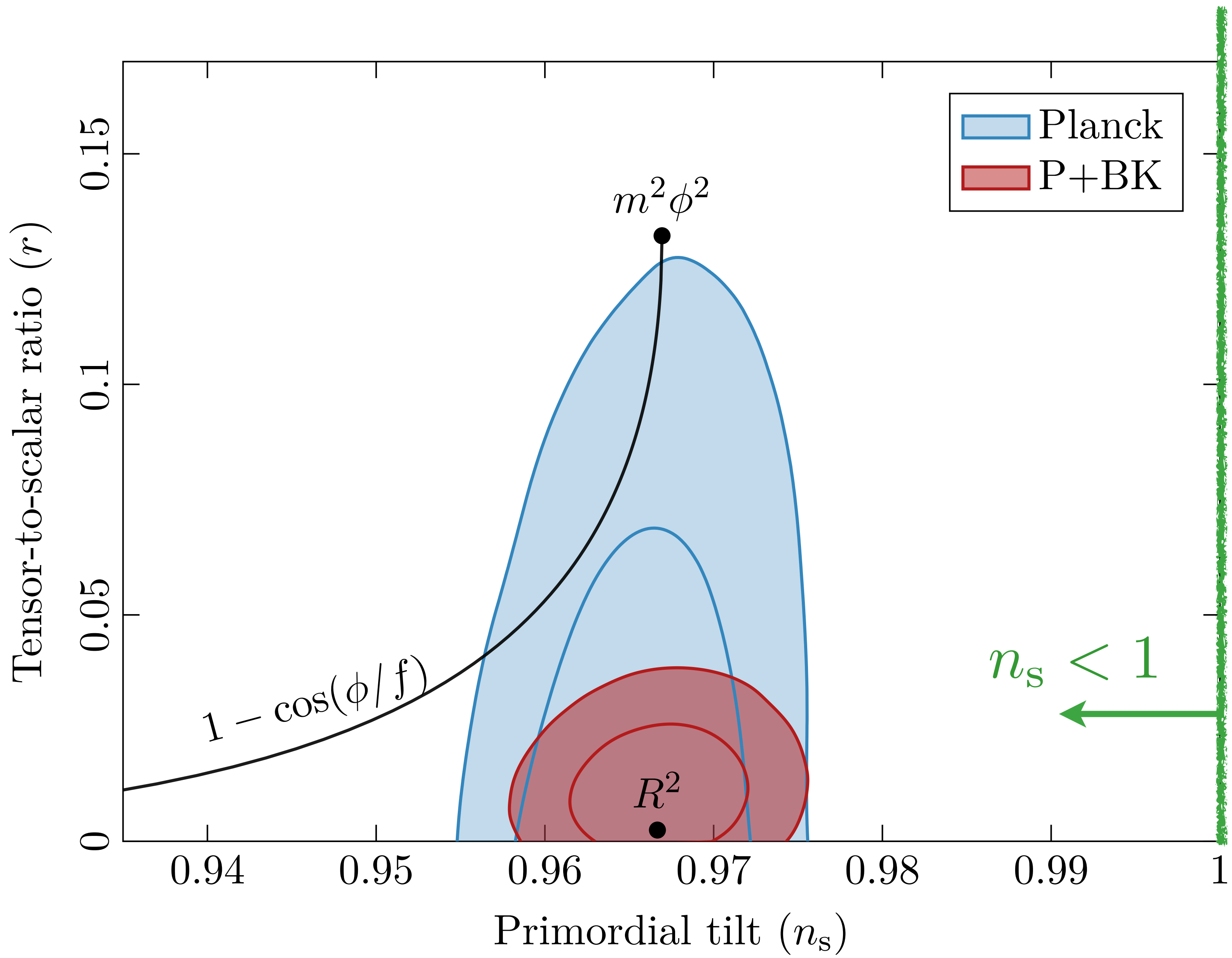
Before the Big Bang

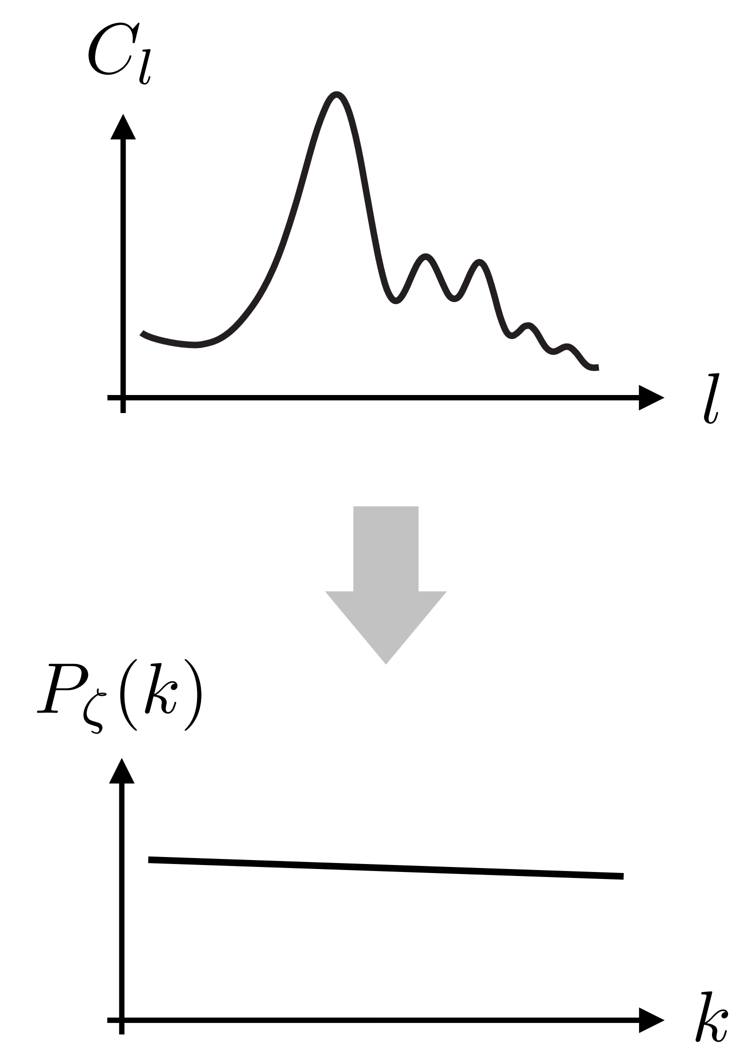
This suggests two options: **rapid expansion** or **slow contraction**?



Scale-Invariance

The primordial fluctuations are approximately **scale-invariant**:





This suggests that the fluctuations were created in a phase of approximate **time-translation invariance** (= inflation).

Inflation

During inflation, the expansion rate is nearly constant:

$$\boxed{H(t) \equiv \frac{1}{a}\frac{da}{dt} \approx \text{const}}$$

- → Exponential expansion: $a(t) \approx \exp(Ht)$
- → Quasi-de Sitter geometry

Inflation needs to end, so there has to be some time-dependence:

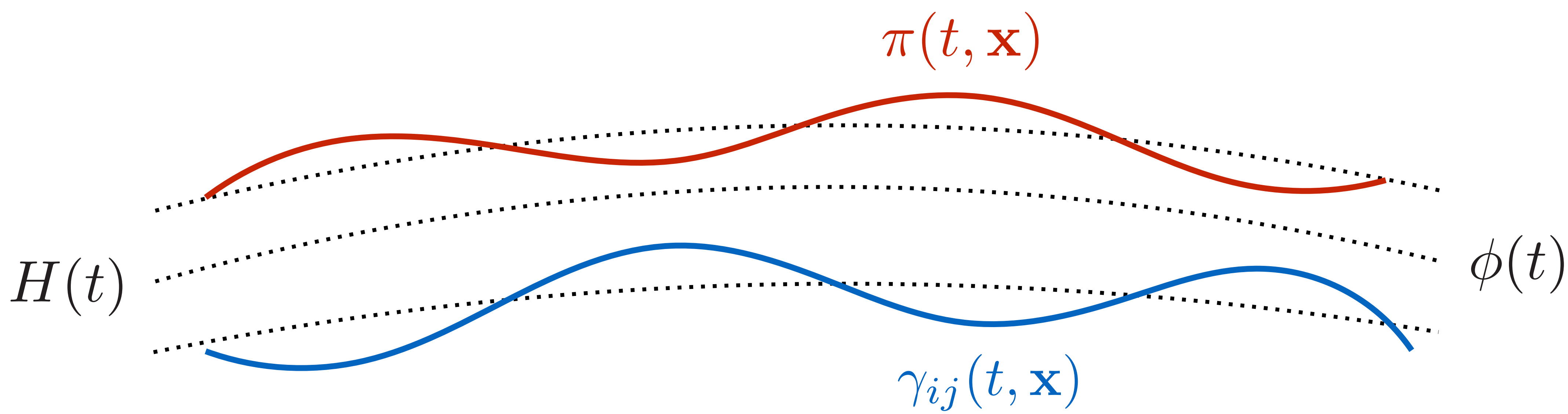
$$\boxed{\varepsilon \equiv -\frac{\dot{H}}{H^2} \ll 1}$$

In slow-roll inflation, this is generated by a field rolling down a flat potential. However, there are other ways to achieve inflation.

EFT of Inflation

Inflation is a **symmetry breaking phenomenon**:

Creminelli et al. [2006]
Cheung et al. [2008]



The low-energy EFT is parameterized by two massless fields:

- **Goldstone boson**
  of broken time translations

$$\delta\phi = \phi(t+\pi) - \bar{\phi}(t)$$

- **Graviton**

EFT of Inflation

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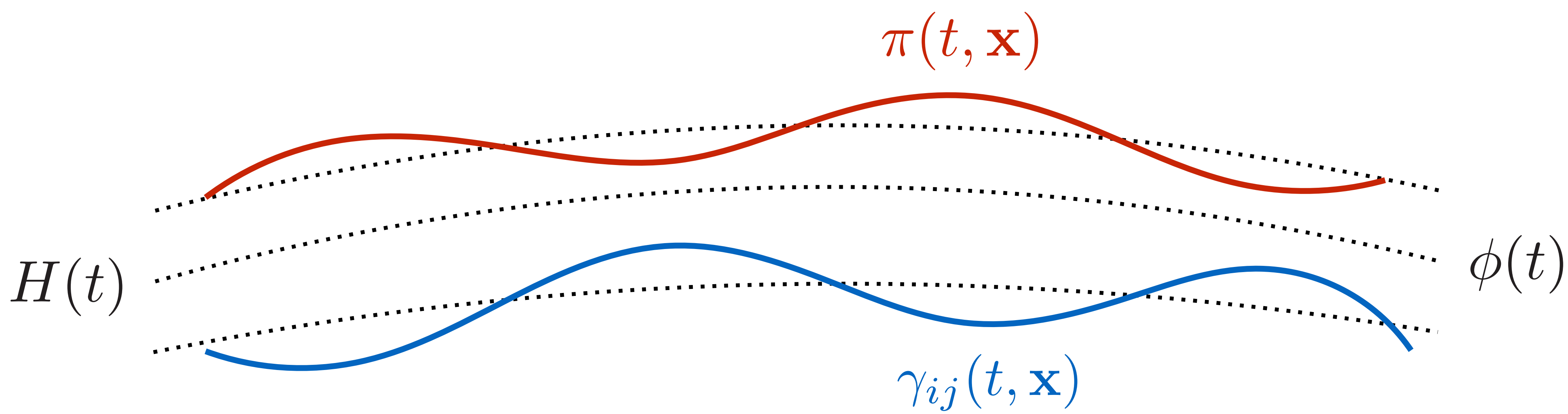


In comoving gauge, the Goldstone boson gets eaten by the metric:

$$g_{ij} = a^2 e^{2\zeta} [e^{\gamma}]_{ij} \qquad \zeta = -H\pi$$

Curvature perturbation

EFT of Inflation

The Goldstone Lagrangian is

Creminelli et al. [2006]
Cheung et al. [2008]

$$\mathcal{L}_\pi = M_{\mathrm{pl}}^2 \dot{H} (\partial\pi)^2 + \sum_{n=2}^{\infty} \frac{M_n^4}{n!}\left[-2\dot{\pi} + (\partial\pi)^2\right]^n + \cdots$$

Slow-roll inflation

$\frac{1}{2}(\partial\phi)^2 - V(\phi)$

Higher-derivative corrections

$(\partial\phi)^{2n}$

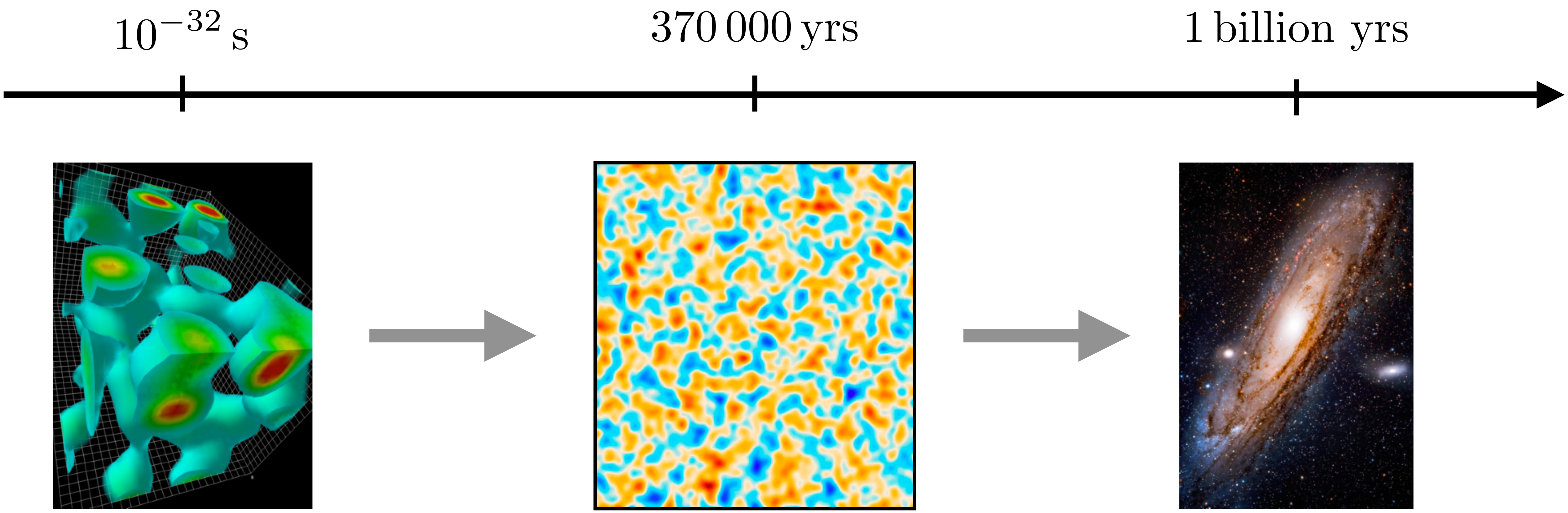
The fluctuations can have a small **sound speed** and large **interactions**:

$$\mathcal{L}_\pi = \frac{M_{\mathrm{pl}}^2|\dot{H}|}{c_s^2}\left[\left(\dot{\pi}^2 - c_s^2(\partial_i\pi)^2\right) - (1-c_s^2)\dot{\pi}(\partial_\mu\pi)^2 + \cdots\right]$$

nonlinearly realized symmetry

Quantum Fluctuations

Quantum fluctuations of the Goldstone produced the density fluctuations in the early universe:



Let us derive this.

Quantum Fluctuations

We start from the quadratic Goldstone action

$$S = \int \mathrm{d}^4x \sqrt{-g}\, M_{\rm Pl}^2 \dot{H}\, g^{\mu\nu} \partial_\mu \pi \partial_\nu \pi$$
$$= \int \mathrm{d}\tau \mathrm{d}^3x\, a^2(\tau)\, M_{\rm Pl}^2 |\dot{H}| \Big((\pi')^2 - (\nabla \pi)^2 \Big)$$

where $a(\tau) \approx -(H\tau)^{-1}$ (de Sitter).

It is convenient to introduce the canonically normalized field

$$u \equiv a(\tau) \sqrt{2 M_{\rm Pl} |\dot{H}|}\, \pi$$

so that $$S = \frac{1}{2} \int \mathrm{d}\tau \mathrm{d}^3x \Big(\underbrace{(u')^2 - (\nabla u)^2}_{\text{Minkowski}} + \frac{a''}{a} u^2 \Big)$$

Time-dependent mass

Classical Dynamics

The equation of motion then is

$$\boxed{u''_{\mathbf{k}} + \left(k^2 - \frac{2}{\tau^2}\right) u_{\mathbf{k}} = 0}$$

- At early times, $-k\tau \gg 1$, this becomes

$$u''_{\mathbf{k}} + k^2 u_{\mathbf{k}} \approx 0 \quad \longrightarrow \quad u_{\mathbf{k}}(\tau) = \frac{c_\pm}{\sqrt{2k}} e^{\pm ik\tau}$$

Harmonic oscillator

- At late times, $-k\tau \to 0$, this becomes

$$u''_{\mathbf{k}} - \frac{2}{\tau^2} u_{\mathbf{k}} \approx 0 \quad \longrightarrow \quad u_{\mathbf{k}}(\tau) = c_1 \tau^{-1} + c_2 \tau^2$$

Growing mode

Fixed by harmonic oscillator

Classical Dynamics

The equation of motion then is

$$\boxed{u_{\mathbf{k}}'' + \left(k^2 - \frac{2}{\tau^2}\right) u_{\mathbf{k}} = 0}$$

- The curvature perturbation becomes a constant at late times:

$$\begin{aligned}\zeta = -H\pi &\propto a^{-1} u \\ &\propto \tau \times \tau^{-1} = \text{const}\end{aligned}$$

- The complete solution is

$$u_{\mathbf{k}}(\tau) = c_1 \frac{1}{\sqrt{2k}} \left(1 + \frac{i}{k\tau}\right) e^{ik\tau} + c_2 \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) e^{-ik\tau}$$

where the constants will be fixed by quantum initial conditions.

Canonical Quantization

Quantization starts from the action of the canonically normalized field:

$$S = \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3 x \left((u')^2 - (\nabla u)^2 + \frac{a''}{a} u^2 \right)$$

- Define the conjugate momentum: $p = \delta \mathcal{L} / \delta u' = u'$
- Promote fields to operators: $u, p \to \hat{u}, \hat{p}$
- Impose canonical commutation relations: $[\hat{u}(\tau, \mathbf{x}), \hat{p}(\tau, \mathbf{x}')] = i\hbar\, \delta(\mathbf{x} - \mathbf{x}')$
- Define the mode expansion of the field operator:

$$\hat{u}(\tau, \mathbf{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left(u_k^*(\tau)\, \hat{a}_{\mathbf{k}} + u_k(\tau)\, \hat{a}^\dagger_{-\mathbf{k}} \right) e^{i\mathbf{k}\cdot\mathbf{x}}$$

where $u_k(\tau)$ is a solution to the classical equation of motion and

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}')$$

Annihilation Creation

Canonical Quantization

- Define the vacuum state: $\hat{a}_{\mathbf{k}}|0\rangle = 0$

- The expectation value of the Hamiltonian in this state is minimized for the Bunch-Davies mode function:

positive frequency

$$u_k(\tau) = \frac{1}{\sqrt{2k}}\left(1 + \frac{i}{k\tau}\right) \boxed{e^{ik\tau}}$$

- The two-point function of the field operator (in that state) is

$$\begin{aligned}\langle 0|\hat{u}_{\mathbf{k}}(\tau)\hat{u}_{\mathbf{k}'}(\tau)|0\rangle &= \langle 0| \left(u_k^*(\tau)\,\hat{a}_{\mathbf{k}} + u_k(\tau)\,\hat{a}^\dagger_{-\mathbf{k}}\right)\left(u_{k'}^*(\tau)\,\hat{a}_{\mathbf{k}'} + u_{k'}(\tau)\,\hat{a}^\dagger_{-\mathbf{k}'}\right)|0\rangle \\ &= (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\,\boxed{|u_k(\tau)|^2}\end{aligned}$$

Power spectrum

Power Spectrum

Using the Bunch-Davies mode function, we obtain the following power spectrum for the primordial curvature perturbations:

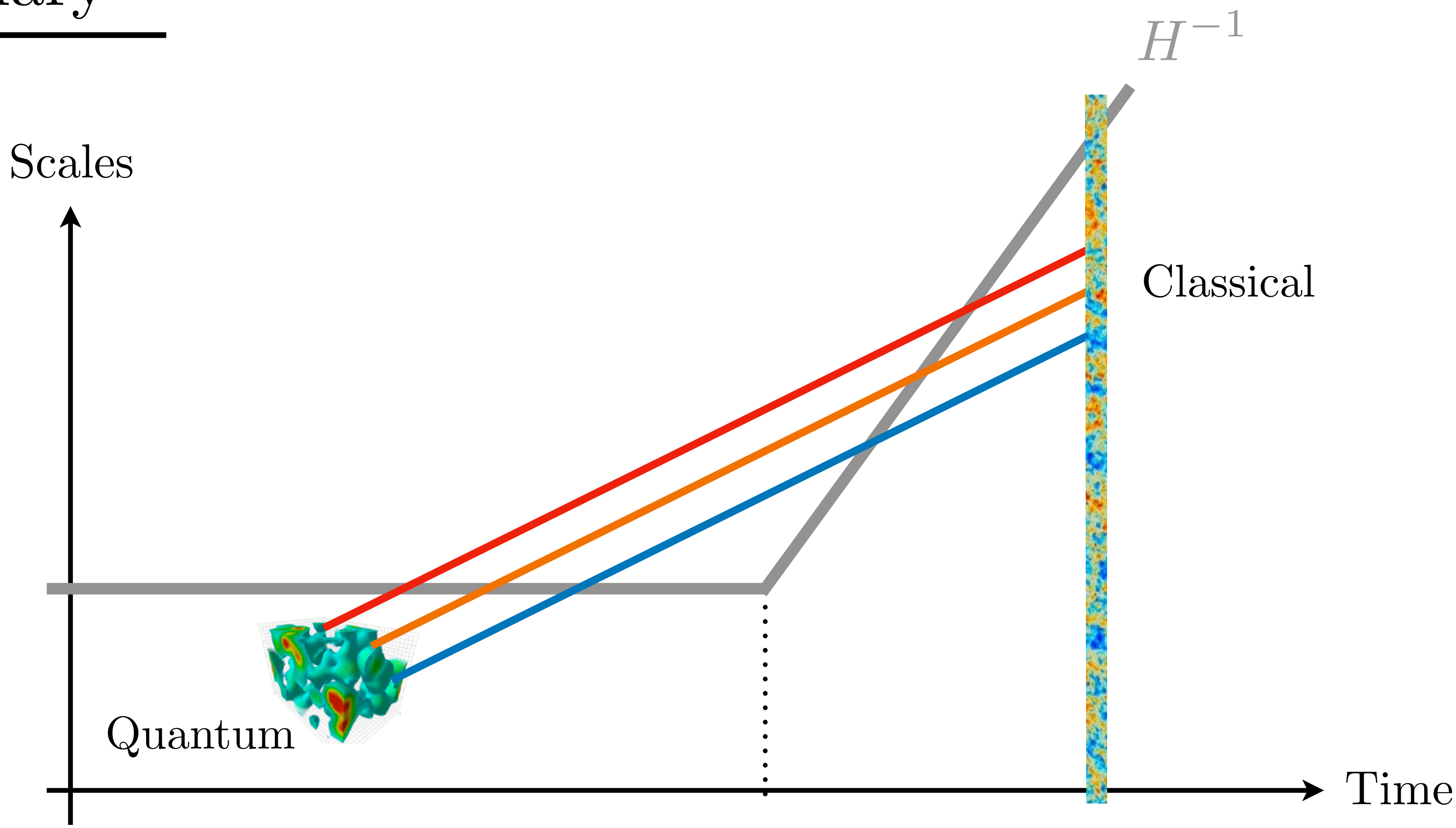
$$P_\zeta(k,\tau) \equiv \frac{H^2}{2M_{\rm Pl}^2|\dot{H}|}\frac{|u_k(\tau)|^2}{a^2(\tau)} = \frac{H^2}{2M_{\rm Pl}^2\varepsilon}\frac{1}{k^3}(1+k^2\tau^2) \xrightarrow{k\tau\to 0} \boxed{\frac{H^2}{2M_{\rm Pl}^2\varepsilon}\frac{1}{k^3}}$$

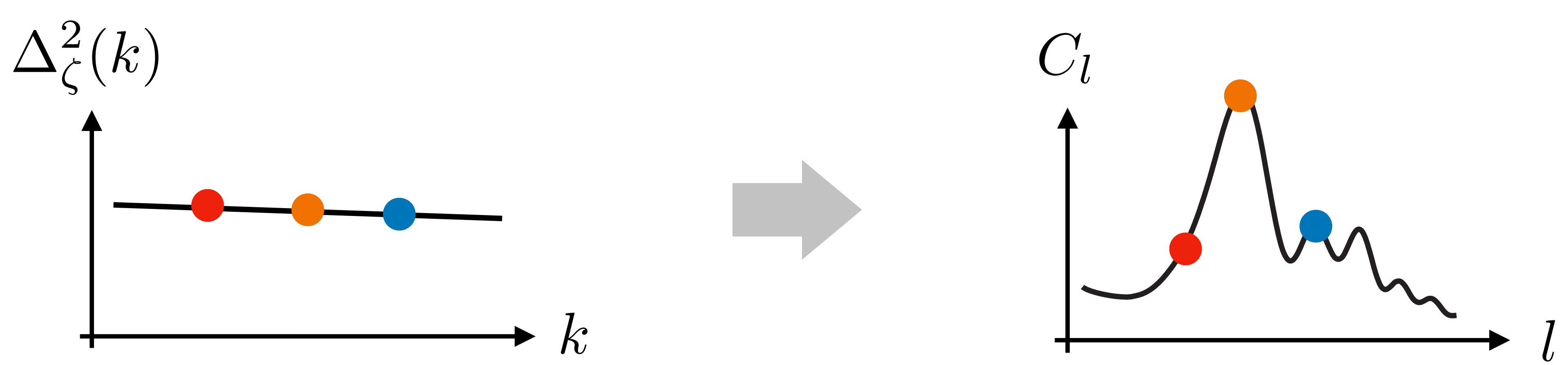
- The k^{-3} scaling is the characteristic of a scale-invariant spectrum.
- During inflation, both $H(t)$ and $\varepsilon(t)$ depend on time, which leads to a small scale-dependence

$$\boxed{\frac{k^3}{2\pi^2}P_\zeta(k) = A_{\rm s}\left(\frac{k}{k_0}\right)^{n_{\rm s}-1}} \equiv \Delta_\zeta^2(k)$$

- Observations find $n_{\rm s} = 0.9667 \pm 0.0040$.

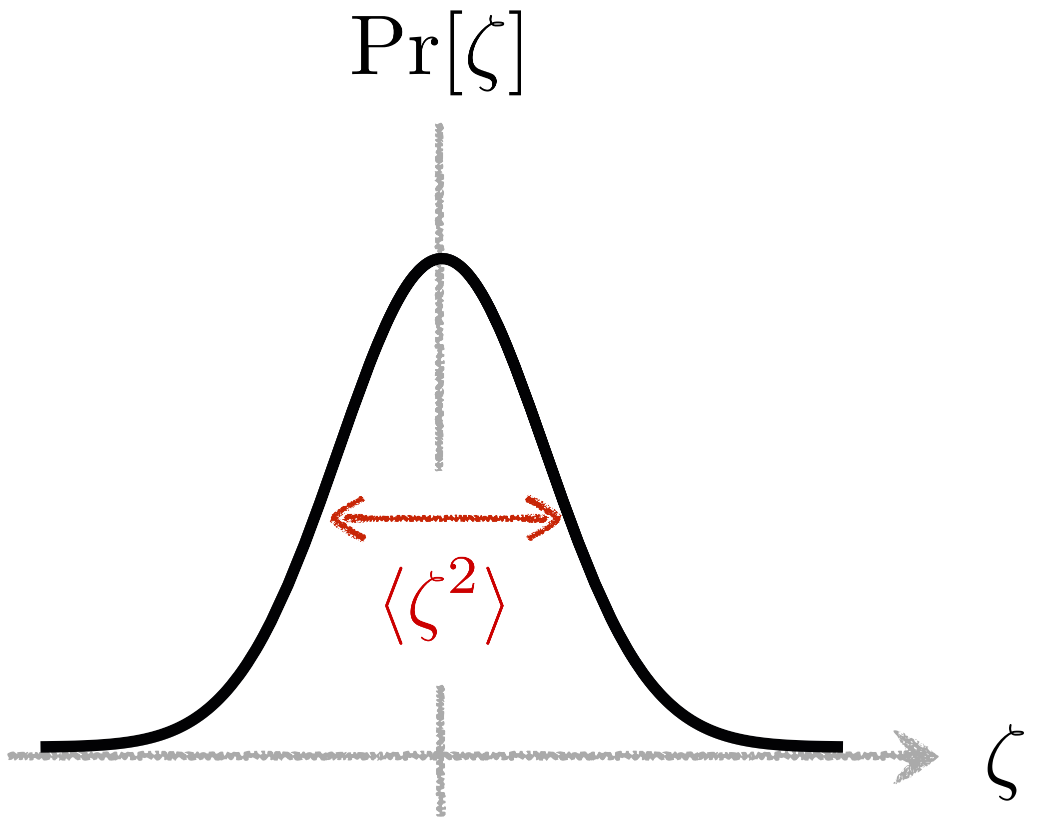
Summary





Gaussianity

The primordial fluctuations were highly **Gaussian** (as expected for the ground state of a harmonic oscillator):



$$F_{\mathrm{NL}} \equiv \frac{\langle\zeta\zeta\zeta\rangle}{\langle\zeta\zeta\rangle^{3/2}} \lesssim 10^{-3}$$

The universe is more Gaussian than flat.

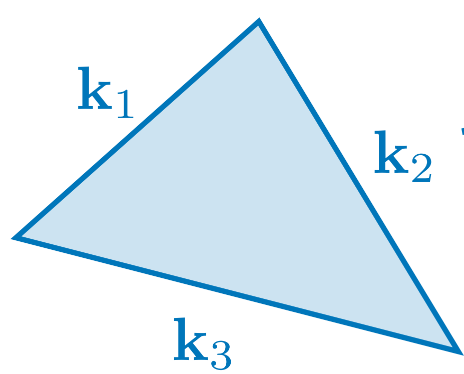
So far, we have only studied the free theory.

Interactions during inflation can lead to **non-Gaussianity**.

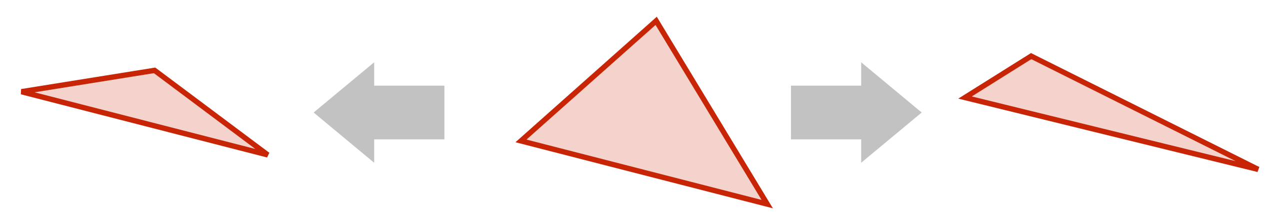
Primordial Non-Gaussianity

The main diagnostic of primordial non-Gaussianity is the **bispectrum**:

$$\boxed{\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle = (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \, \frac{(2\pi^2)^2}{(k_1 k_2 k_3)^2} \, B_\zeta(k_1, k_2, k_3)}$$



Shape

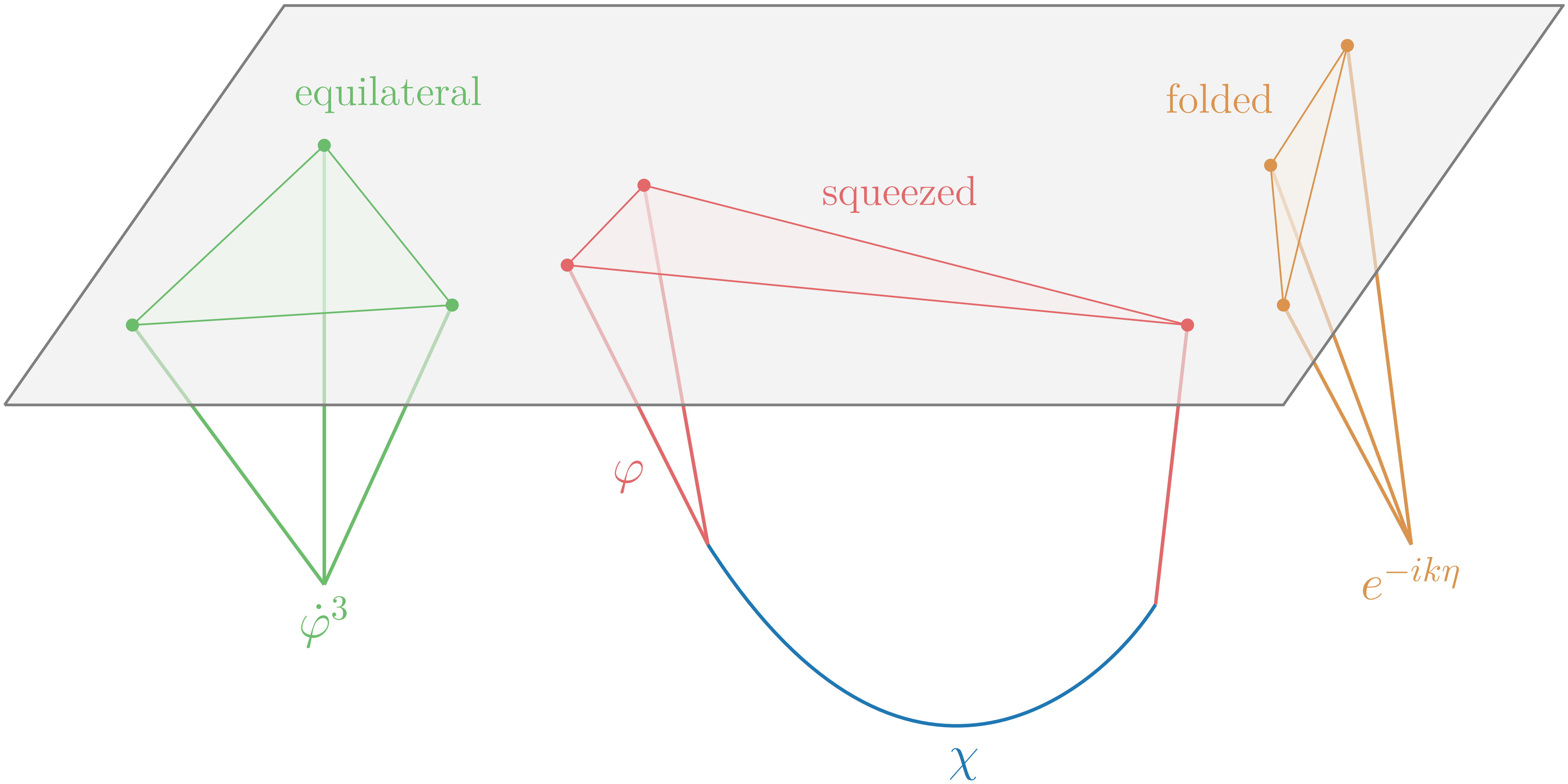


- The **amplitude** of the non-Gaussianity is defined as the size of the bispectrum in the equilateral configuration:

$$\boxed{F_{\rm NL}(k) \equiv \frac{5}{18} \frac{B_\zeta(k,k,k)}{\Delta_\zeta^3(k)}}$$

Shapes of Non-Gaussianity

- The **shape** of the non-Gaussianity contains a lot of information about the microphysics of inflation:



Equilateral Non-Gaussianity

The Goldstone mode during inflation can have large **self-interactions**:

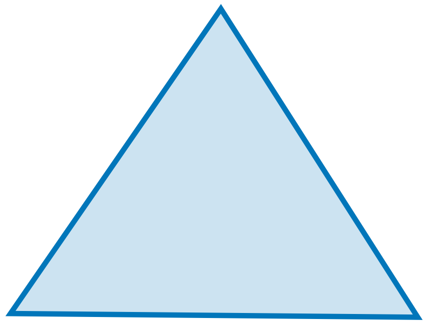
$$\mathcal{L}_\pi = \frac{M_{\rm pl}^2|\dot{H}|}{c_s^2}\Big[\left(\dot{\pi}^2 - c_s^2(\partial_i\pi)^2\right) - (1-c_s^2)\,\dot{\pi}(\partial_\mu\pi)^2 + \cdots\Big]$$

nonlinearly realized symmetry

$$F_{\rm NL} \propto c_s^{-2}$$

Correlations are largest when the fluctuations have comparable wavelengths:

equilateral NG:



$$B_{\dot{\pi}^3} \propto \frac{k_1k_2k_3}{(k_1+k_2+k_3)^3}$$

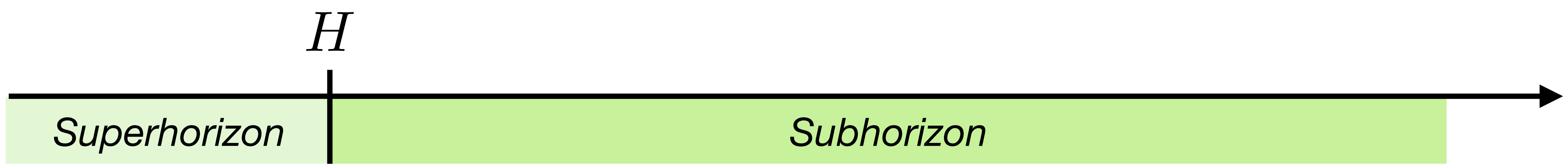
What would measuring (or constraining) equilateral NG teach us about the physics of inflation?

Equilateral Non-Gaussianity

Minimal models of inflation are characterized by three **energy scales**:

Equilateral Non-Gaussianity

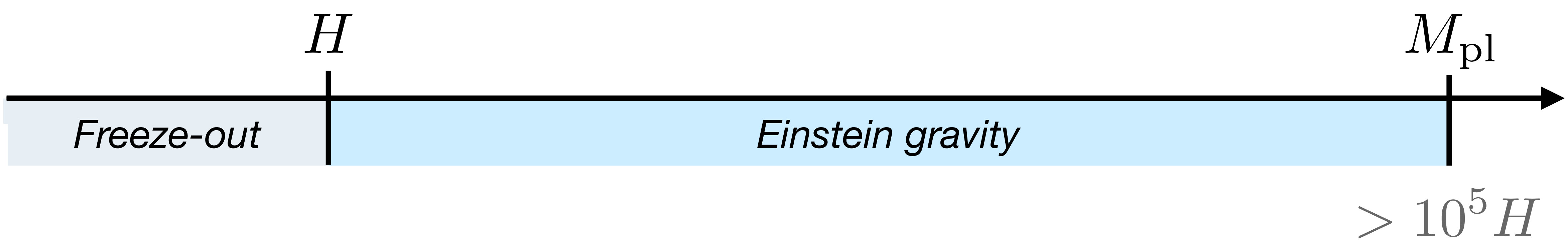
Minimal models of inflation are characterized by three **energy scales**:



- At the **Hubble scale**, modes exit the horizon and freeze out.
- This sets the **energy scale of the experiment**.

Equilateral Non-Gaussianity

Minimal models of inflation are characterized by three **energy scales**:



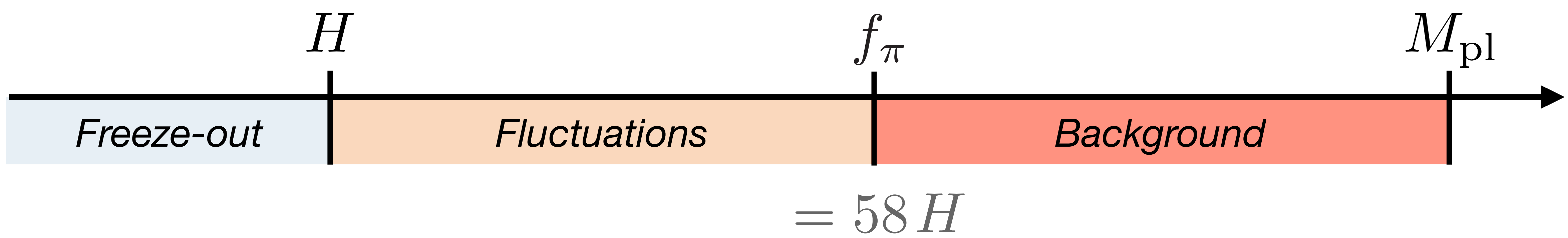
- At the **Planck scale**, gravitational interactions become strong.
- This sets the **amplitude of tensor fluctuations**:

$$\Delta_\gamma^2 = \frac{2}{\pi^2}\left(\frac{H}{M_{\rm pl}}\right)^2 < 0.1\,\Delta_\zeta^2 \approx 10^{-10}$$

No B-modes

Equilateral Non-Gaussianity

Minimal models of inflation are characterized by three **energy scales**:



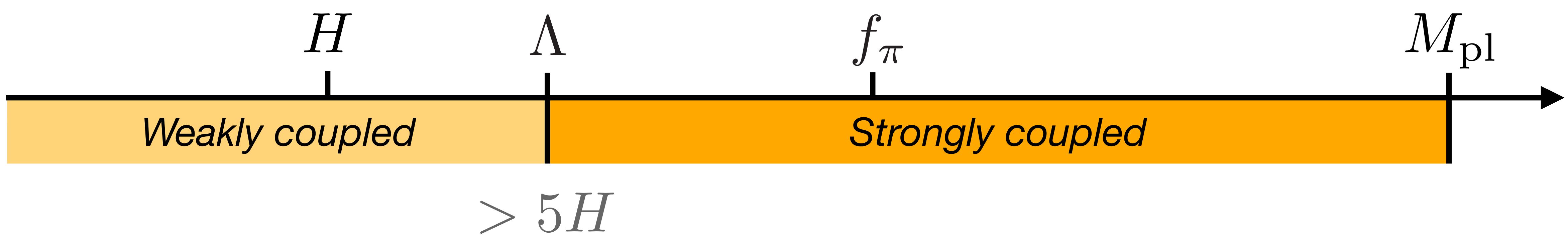
- At the **symmetry breaking scale**, time translations are broken.
- This sets the **amplitude of scalar fluctuations**:

$$\Delta_\zeta^2 = \frac{1}{4\pi^2}\left(\frac{H}{f_\pi}\right)^4 \approx 2.2 \times 10^{-9}$$

$$f_\pi^4 = M_{\rm pl}^2 |\dot{H}| c_s \to \dot{\phi}^2 \ \text{(for SR)}$$

Equilateral Non-Gaussianity

Closer inspection may reveal additional scales:



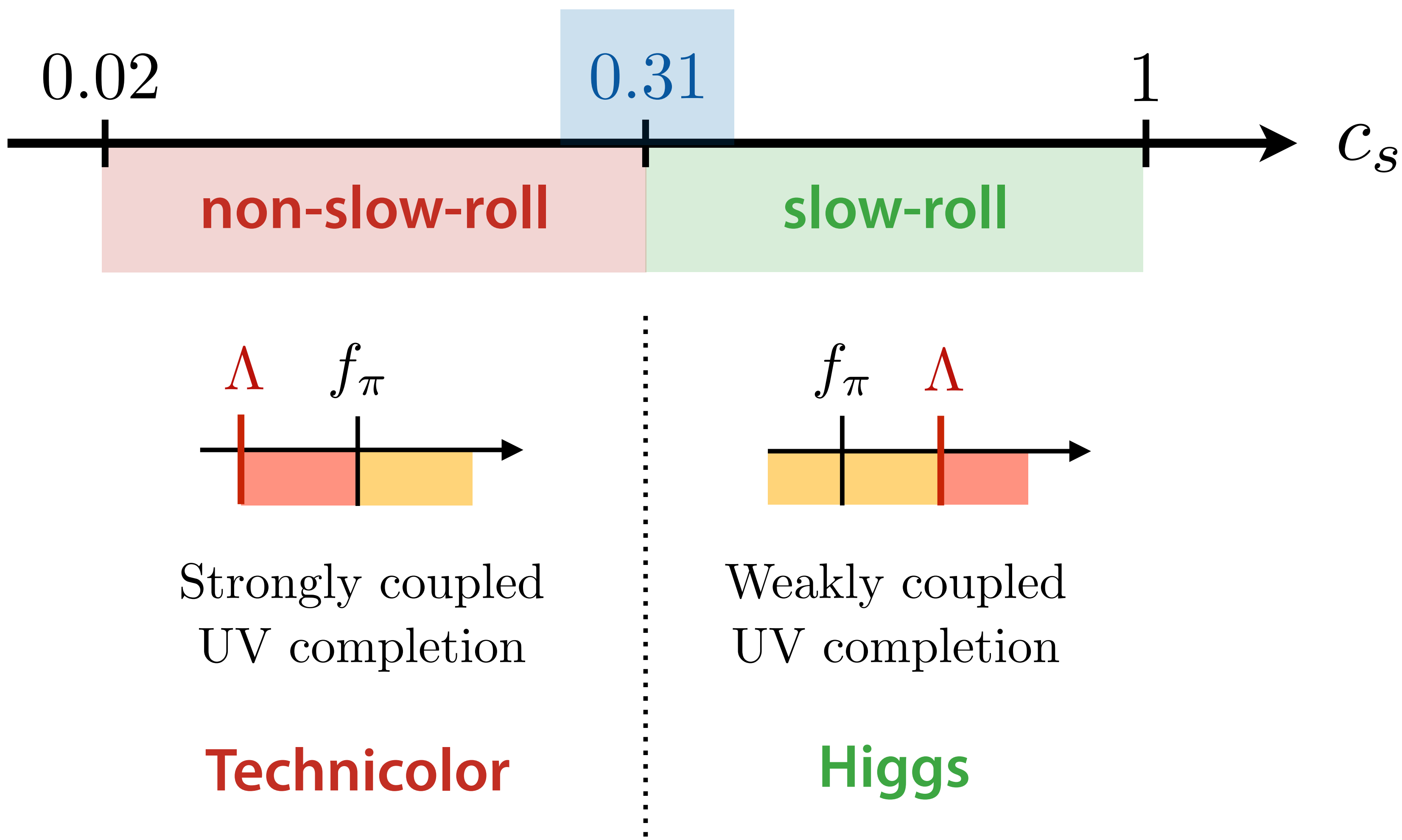
- At the **strong coupling scale**, interactions become strongly coupled.
- This sets the **amplitude of interactions**:

$$F_{\rm NL}^{\rm equil} \sim \frac{H^2}{\Lambda^2} < 0.01$$

No PNG

Equilateral Non-Gaussianity

Asking for the theory to be weakly coupled up to the symmetry breaking scale implies a critical value for the sound speed:



$$F_{\rm NL}^{\rm equil} \sim \Delta_\zeta$$

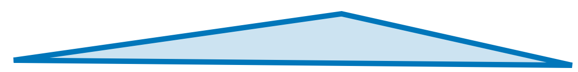
Hard to measure!

DB, Green, Lee and Porto [2015]
Grall and Mellville [2020]

Folded Non-Gaussianity

Excited initial states (negative frequency modes) lead to enhanced correlations when two wave vectors become collinear:

folded NG:



$$B \propto \frac{1}{k_1 + k_2 - k_3}$$

$e^{-ik_3\tau}$

- The inflationary expansion dilutes any pre-inflationary particles. In minimal scenarios, we therefore don't expect to see folded NG.

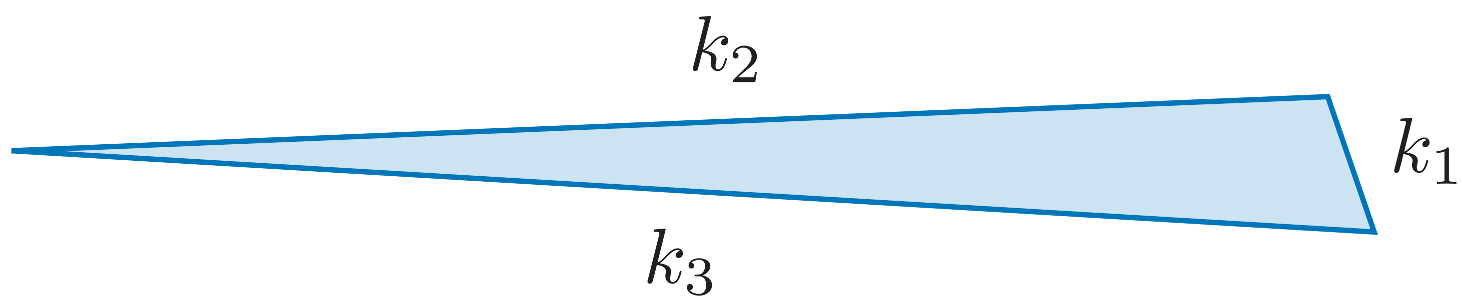
- Models with continuous particle production during inflation can still give a signal in the folded limit.

Flauger, Green and Porto [2013]
Flauger, Mirbabayi, Senatore and Silverstein [2017]
Green and Porto [2020]

Squeezed Non-Gaussianity

In single-field inflation, correlations must vanish in the **squeezed limit**:

$\lim_{k_1 \to 1}$



~ 0

Maldacena [2003]
Creminelli and Zaldarriaga [2004]

The signal in the squeezed limit therefore acts as a **particle detector**.

Chen and Wang [2009]
DB and Green [2011]
Noumi, Yamaguchi andYokoyama [2013]
Arkani-Hamed and Maldacena [2015]
Lee, DB and Pimentel [2016]
DB, Goon, Lee and Pimentel [2017]
Kumar and Sundrum [2018]
Jazayeri and Renaux-Petel [2022]
Pimentel and Wang [2022]

Cosmological Collider Physics

Massive particles are spontaneously created in an expanding spacetime.

Arkani-Hamed and Maldacena [2015]

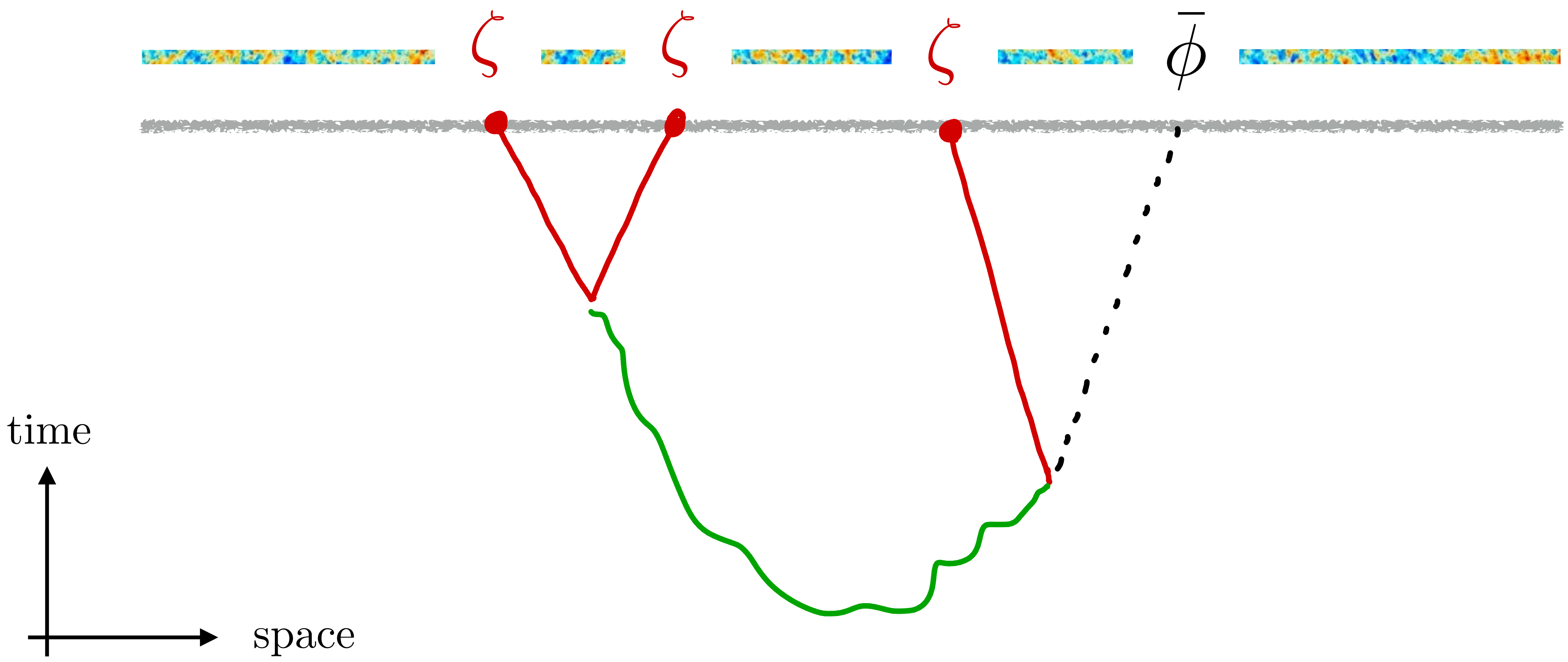
Cosmological Collider Physics

However, they cannot be directly observed at late times.

Arkani-Hamed and Maldacena [2015]

Cosmological Collider Physics

Instead, they decay into light fields.

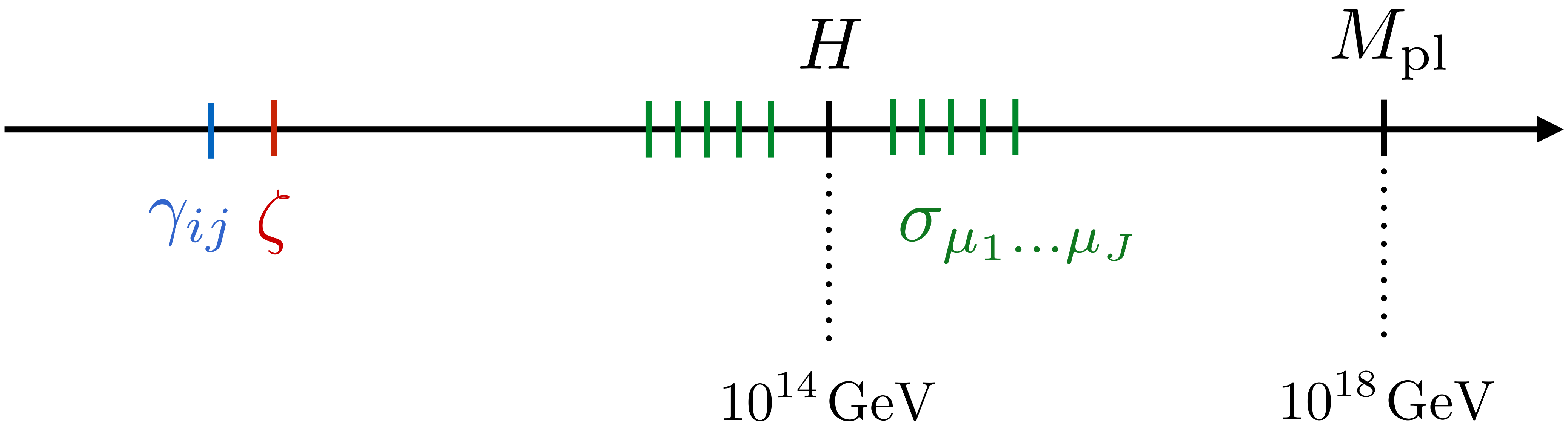


These correlated decays create distinct higher-order correlations in the inflationary perturbations.

Arkani-Hamed and Maldacena [2015]

Cosmological Collider Physics

This allows us to probe the particle spectrum at inflationary energies:

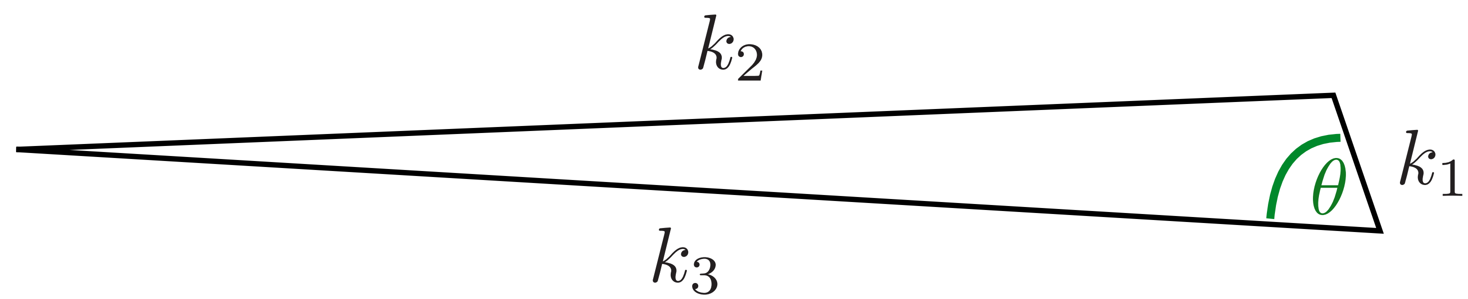


This can be many orders of magnitudes above the scales probed by terrestrial colliders.

Arkani-Hamed and Maldacena [2015]

Cosmological Collider Physics

The signal depends on the masses and spins of the new particles:



$$\lim_{k_1 \to 0} \langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle' \propto \begin{cases} \left(\dfrac{k_1}{k_3}\right)^{\Delta} & m < \frac{3}{2}H \\[2ex] \left(\dfrac{k_1}{k_3}\right)^{3/2} \cos\left[\mu \ln \dfrac{k_1}{k_3}\right] & m > \frac{3}{2}H \end{cases}$$

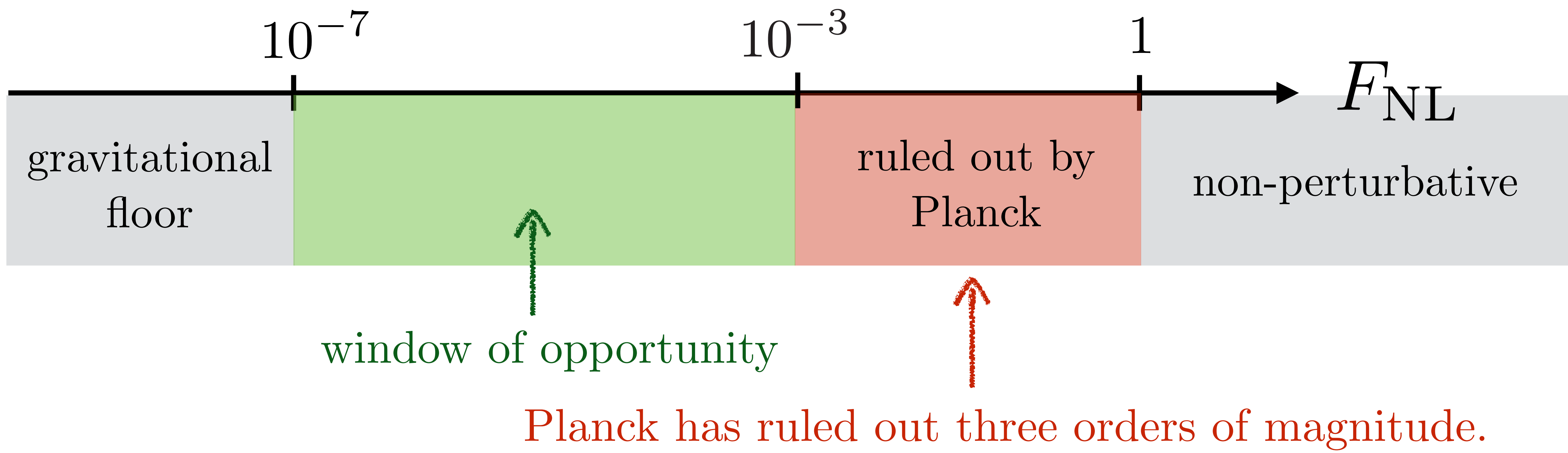
$$\propto \quad P_J(\cos\theta)$$

spin

Arkani-Hamed and Maldacena [2015]

Observational Constraints

The theoretically interesting regime of non-Gaussianity spans about seven orders of magnitude:



There is still room for new particles to leave their mark.

Any Questions?

So far, I have focussed mostly on the **practical motivations** for studying cosmological correlators.

In the rest of these lectures, I will introduce three different ways to compute primordial correlation functions:

- In-In Formalism — Notes
- Wavefunction Approach
- Cosmological Bootstrap — Lectures

I will leads us to more **conceptual questions** about the theory of quantum fields in cosmological spacetimes.

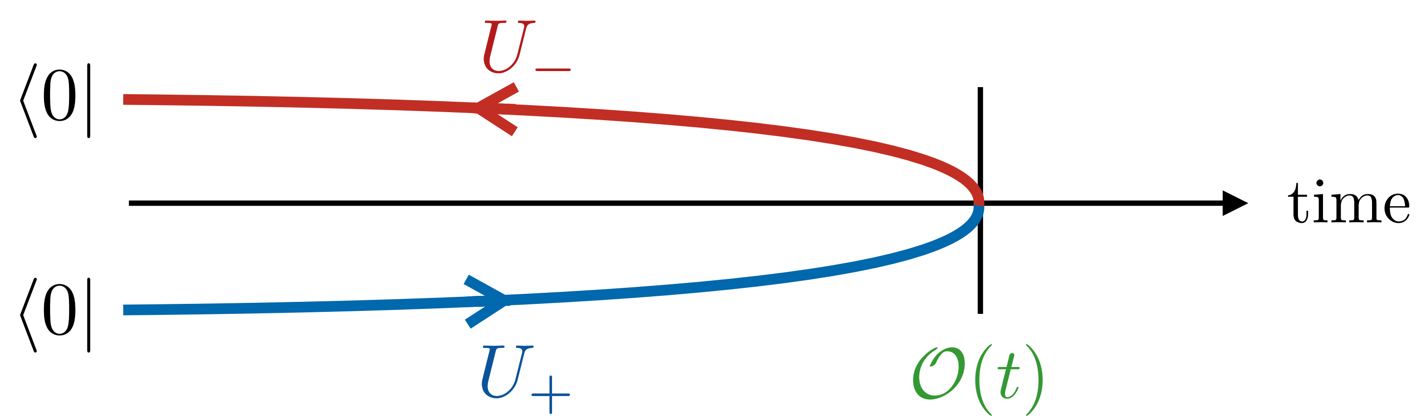
Wavefunction Approach

In-In Formalism

The main observables in cosmology are spatial correlation functions:

$$\langle\Omega|\, \phi(t,\mathbf{x}_1)\phi(t,\mathbf{x}_2)\cdots\phi(t,\mathbf{x}_N)\, |\Omega\rangle$$

These correlators are usually computed in the **in-in formalism**:



which leads to the following master equation

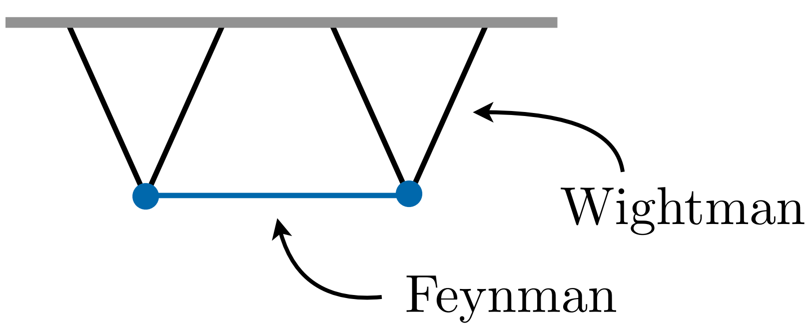
$$\boxed{\langle\mathcal{O}(t)\rangle = \langle 0|\, \bar{T}e^{i\int_{-\infty}^{t}\mathrm{d}t''\, H_{\rm int}(t'')}\, \mathcal{O}(t)\, Te^{-i\int_{-\infty}^{t}\mathrm{d}t'\, H_{\rm int}(t')}|0\rangle}$$

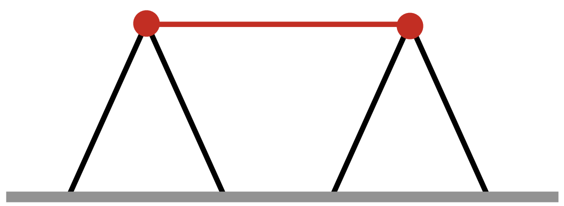
Weinberg [2005]

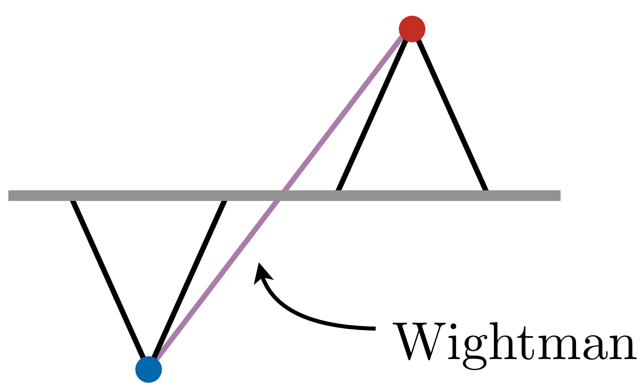
In-In Formalism

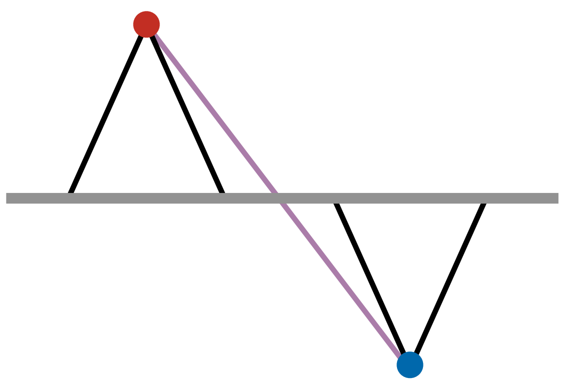
$$\langle \mathcal{O}(t) \rangle = \langle 0 | \, \bar{T} e^{i \int_{-\infty}^{t} \mathrm{d}t'' H_{\mathrm{int}}(t'')} \, \mathcal{O}(t) \, T e^{-i \int_{-\infty}^{t} \mathrm{d}t' H_{\mathrm{int}}(t')} | 0 \rangle$$

Diagrammatically, this can be written as

$I_{++} =$ 

$I_{--} =$ 

$I_{+-} =$ 

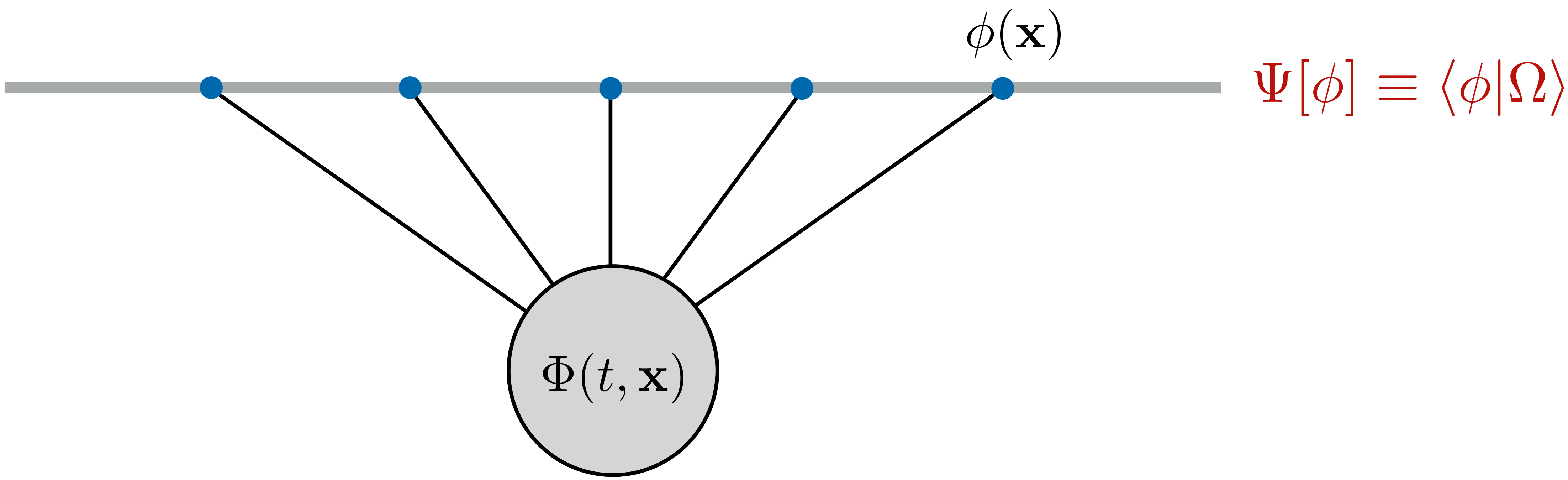
$I_{-+} =$ 

Giddings and Sloth [2010]

In the lecture notes, we apply this to many examples (see Section 3).
Here, we will instead follow the alternative wavefunction approach.

Wavefunction Approach

We first define a **wavefunction** for the late-time fluctuations:



and then use it to compute the **boundary correlations**

$$\boxed{\langle \phi(\mathbf{x}_1) \dots \phi(\mathbf{x}_N) \rangle = \int \mathcal{D}\phi \; \phi(\mathbf{x}_1) \dots \phi(\mathbf{x}_N) \, |\Psi[\phi]|^2}$$

The wavefunction has nicer analytic properties than the correlation functions and it also plays a central role in the cosmological bootstrap.

OUTLINE:

Wavefunction of the Universe	Warmup in QM	Flat-Space Wavefunction	De Sitter Wavefunction

REFERENCE:

DB and Joyce, *Lectures on Cosmological Correlations*

Wavefunction of the Universe

Wavefunction Coefficients

For small fluctuations, we expand the wavefunction as

$$\Psi[\phi] \;=\; \exp\left(-\sum_N \int \frac{\mathrm{d}^3k_1 \ldots \mathrm{d}^3k_N}{(2\pi)^{3N}}\, \Psi_N(\underline{\mathbf{k}})\, \phi_{\mathbf{k}_1} \ldots \phi_{\mathbf{k}_N}\right)$$

Wavefunction coefficient

- By translation invariance, we have

$$\Psi_N(\underline{\mathbf{k}}) = (2\pi)^3 \delta(\mathbf{k}_1 + \cdots + \mathbf{k}_N)\, \psi_N(\underline{\mathbf{k}})\,.$$

- We will also write

$$\psi_N(\underline{\mathbf{k}}) = \langle O_1 \ldots O_N \rangle'\,,$$

where $O_a \equiv O_{\mathbf{k_a}}$ are **dual operators**.

This will just be notation and (so far) does not have a deeper holographic meaning (as in dS/CFT).

Relation to In-In Correlators

In perturbation theory, the correlation functions

$$\langle \phi_1 \dots \phi_N \rangle = \int \mathcal{D}\phi \; \phi_1 \dots \phi_N \, |\Psi[\phi]|^2$$

are related to the wavefunction coefficients:

$$\boxed{\begin{aligned}
\langle \phi_1 \phi_2 \rangle \; &= \; \frac{1}{2\,\mathrm{Re}\langle O_1 O_2 \rangle} \\
\langle \phi_1 \phi_2 \phi_3 \rangle \; &= \; \frac{\mathrm{Re}\langle O_1 O_2 O_3 \rangle}{4 \prod_{a=1}^{3} \mathrm{Re}\langle O_a O_a \rangle} \\
\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle \; &= \; \frac{\mathrm{Re}\langle O_1 O_2 O_3 O_4 \rangle}{8 \prod_{a=1}^{4} \mathrm{Re}\langle O_a O_a \rangle^4} \; + \; \frac{\langle O_1 O_2 X \rangle \langle X O_3 O_4 \rangle + \text{c.c.}}{8\,\mathrm{Re}\langle X X \rangle \prod_{a=1}^{4} \mathrm{Re}\langle O_a O_a \rangle^4}
\end{aligned}}$$

Wavefunction coefficients therefore contain the same information, but are easier to compute and have nicer analytic properties.

Computing the Wavefunction

The wavefunction has the following path integral representation:

$$\Psi[\phi] = \int_{\substack{\Phi(0)=\phi \\ \Phi(-\infty)=0}} \mathcal{D}\Phi \, e^{iS[\Phi]}$$

Action

Boundary conditions:

$\Phi(t = -\infty(1 - i\varepsilon)) = 0$
selects the Bunch-Davies vacuum.

For tree-level processes, we can evaluate this in a saddle-point approximation:

$$\Psi[\phi] \approx e^{iS[\Phi_{\rm cl}]}$$

Classical solution
with $\Phi_{\rm cl}(0) = \phi$ and $\Phi_{\rm cl}(-\infty) = 0$.

To find the wavefunction, we therefore need to find the classical solution for the bulk field with the correct boundary conditions.

Warmup in Quantum Mechanics

Harmonic Oscillator

Consider our old friend the **simple harmonic oscillator**

$$\boxed{S[\Phi] = \int \mathrm{d}t \left(\frac{1}{2}\dot{\Phi}^2 - \frac{1}{2}\omega^2\Phi^2 \right)}$$

- The classical solution (with the correct boundary conditions) is

$$\Phi_{\rm cl}(t) = \phi\, e^{i\omega t}$$

- The on-shell action becomes

$$\begin{aligned} S[\Phi_{\rm cl}] &= \int_{t_i}^{t_*} \mathrm{d}t \left[\frac{1}{2}\partial_t(\dot{\Phi}_{\rm cl}\Phi_{\rm cl}) - \frac{1}{2}\Phi_{\rm cl} \underbrace{(\ddot{\Phi}_{\rm cl} + \omega^2\Phi_{\rm cl})}_{=0} \right] \\ &= \frac{1}{2}\dot{\Phi}_{\rm cl}\Phi_{\rm cl}\Big|_{t=t_*} \\ &= \frac{i\omega}{2}\phi^2 \end{aligned}$$

Harmonic Oscillator

- The wavefunction then is

Gaussian

$$\Psi[\phi] \approx \exp(iS[\Phi_{\rm cl}]) = \exp\left(-\frac{\omega}{2}\phi^2\right)$$

- The quantum variance of the oscillator therefore is

$$\boxed{\langle \phi^2 \rangle = \frac{1}{2\omega}}$$

- In QFT, the same result applies to each Fourier mode

$$\langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle' = \frac{1}{2\omega_k} \, ,$$

where $\omega_k = \sqrt{k^2 + m^2}$.

Time-Dependent Oscillator

To make this more interesting, consider a **time-dependent oscillator**

$$\boxed{S[\Phi] = \int \mathrm{d}t \left(\frac{1}{2} A(t) \dot{\Phi}^2 - \frac{1}{2} B(t) \Phi^2 \right)}$$

- The classical solution is

$$\Phi_{\rm cl}(t) = \phi K(t)\,, \quad \text{with} \quad \begin{array}{l} K(0) = 1 \\ K(-\infty) \sim e^{i\omega t} \end{array}$$

- The on-shell action becomes

$$\begin{aligned} S[\Phi_{\rm cl}] &= \int_{t_i}^{t_*} \mathrm{d}t \left[\frac{1}{2} \partial_t (A \dot{\Phi}_{\rm cl} \Phi_{\rm cl}) - \frac{1}{2} \Phi_{\rm cl} \underbrace{\left(\partial_t (A \dot{\Phi}_{\rm cl}) + B \Phi_{\rm cl} \right)}_{=0} \right] \\ &= \frac{1}{2} A \dot{\Phi}_{\rm cl} \Phi_{\rm cl} \Big|_{t=t_*} \\ &= \frac{1}{2} A \phi^2 \partial_t \log K \Big|_{t=t_*} \end{aligned}$$

Time-Dependent Oscillator

- The wavefunction then is

$$\Psi[\phi] \approx \exp(iS[\Phi_{\rm cl}]) = \exp\left(\frac{i}{2}(A\partial_t \log K)\Big|_* \phi^2\right),$$

which implies

$$|\Psi[\phi]|^2 = \exp\left(-\mathrm{Im}(A\partial_t \log K)\Big|_* \phi^2\right)$$

$$\downarrow$$

$$\boxed{\langle\phi^2\rangle = \frac{1}{2\,\mathrm{Im}(A\partial_t \log K)\Big|_*}}$$

Free Field in de Sitter

The action of a **massless field in de Sitter** is that of a time-dependent oscillator:

$$
\begin{aligned}
S &= \int \mathrm{d}\eta\, \mathrm{d}^3x\, a^2(\eta)\Big[(\Phi')^2 - (\nabla\Phi)^2\Big] \\
&= \frac{1}{2}\int \mathrm{d}\eta\, \frac{\mathrm{d}^3k}{(2\pi)^3}\left[\frac{1}{(H\eta)^2}\Phi'_{\mathbf{k}}\Phi'_{-\mathbf{k}} - \frac{k^2}{(H\eta)^2}\Phi_{\mathbf{k}}\Phi_{-\mathbf{k}}\right]
\end{aligned}
$$

- The classical solution is

$$
\Phi_{\rm cl}(\eta) = \phi K(\eta)\,, \quad \text{with} \quad \begin{aligned} K(\eta) &= (1-ik\eta)e^{ik\eta} \\ \log K(\eta) &= \log(1-ik\eta) + ik\eta \end{aligned}
$$

and hence

$$
\begin{aligned}
\mathrm{Im}(A\partial_\eta \log K)\Big|_{\eta=\eta_*} &= \frac{1}{(H\eta_*)^2}\,\mathrm{Im}\left(\frac{-ik}{1-ik\eta_*} + ik\right) \\
&= \frac{1}{(H\eta_*)^2}\,\mathrm{Im}\left(\frac{k^2\eta_* + ik^3\eta_*^2}{1+k^2\eta_*^2}\right) \xrightarrow{\eta_*\to 0} \boxed{\frac{k^3}{H^2}}
\end{aligned}
$$

Free Field in de Sitter

- Using our result for the variance of a time-dependent oscillator, we then get

$$\boxed{\langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle' = \frac{H^2}{2k^3}}$$

  which we derived in the last lecture using canonical quantization.

- The result for a massive field is derived in the lecture notes.

Anharmonic Oscillator

Consider the following **anharmonic oscillator**

$$\boxed{S[\Phi] = \int \mathrm{d}t \left(\frac{1}{2}\dot{\Phi}^2 - \frac{1}{2}\omega^2\Phi^2 - \frac{1}{3}g\Phi^3 \right)}$$

- The classical equation of motion is

$$\ddot{\Phi}_{\mathrm{cl}} + \omega^2 \Phi_{\mathrm{cl}} = -g\Phi_{\mathrm{cl}}^2$$

- A formal solution is

$$\Phi_{\mathrm{cl}}(t) = \phi\, K(t) + i \int \mathrm{d}t'\, G(t,t') \left(-g\Phi_{\mathrm{cl}}^2(t')\right)$$

where

$$K(t) = e^{i\omega t}\,,$$

$$G(t,t') = \frac{1}{2\omega}\Big(\underbrace{e^{-i\omega(t-t')}\theta(t-t') + e^{i\omega(t-t')}\theta(t'-t)}_{\text{Feynman propagator}} \; - \; e^{i\omega(t+t')} \Big)\,.$$

Anharmonic Oscillator

- Deriving the on-shell action is now a bit more subtle:

$$S[\Phi_{\rm cl}] = \int_{t_i}^{t_*} {\rm d}t \left[\frac{1}{2}\partial_t(\Phi_{\rm cl}\dot{\Phi}_{\rm cl}) - \frac{1}{2}\Phi_{\rm cl}(\ddot{\Phi}_{\rm cl} + \omega^2\Phi_{\rm cl}) - \frac{g}{3}\,\Phi_{\rm cl}^3\right]$$
$$= \frac{1}{2}\Phi_{\rm cl}\dot{\Phi}_{\rm cl}\Big|_{t=t_*} + \int {\rm d}t\left[-\frac{1}{2}\Phi_{\rm cl}(\ddot{\Phi}_{\rm cl} + \omega^2\Phi_{\rm cl}) - \frac{g}{3}\,\Phi_{\rm cl}^3\right].$$

Since $\lim_{t\to 0}\partial_t G(t,t') = -ie^{i\omega t'} \neq 0$, the boundary term is

$$\frac{1}{2}\Phi_{\rm cl}\dot{\Phi}_{\rm cl}\Big|_{t=t_*} = \frac{1}{2}\phi\left(i\omega\phi - ig\int {\rm d}t'\,\big(-ie^{i\omega t'}\big)\,\Phi_{\rm cl}^2(t')\right)$$
$$= \frac{i\omega}{2}\phi^2 - \frac{g}{2}\phi\int {\rm d}t'\,e^{i\omega t'}\,\Phi_{\rm cl}^2(t')\,.$$

The action then becomes

$$S[\Phi_{\rm cl}] = \frac{i\omega}{2}\phi^2 - \frac{g}{2}\phi\int {\rm d}t\,e^{i\omega t}\,\Phi_{\rm cl}^2$$
$$+ \int {\rm d}t\left[-\frac{1}{2}\left(\phi\, e^{i\omega t} - ig\int {\rm d}t'\,G(t,t')\,\Phi_{\rm cl}^2(t')\right)\Big(-g\Phi_{\rm cl}^2(t)\Big) - \frac{g}{3}\,\Phi_{\rm cl}^3\right].$$

The terms linear in ϕ cancel.

Anharmonic Oscillator

- The final on-shell action is

$$\boxed{S[\Phi_{\rm cl}] = \frac{i\omega}{2}\phi^2 - \frac{g}{3}\int \mathrm{d}t\,\Phi_{\rm cl}^3(t) - \frac{ig^2}{2}\int \mathrm{d}t\,\mathrm{d}t'\,G(t,t')\,\Phi_{\rm cl}^2(t')\,\Phi_{\rm cl}^2(t)}$$

- To evaluate this, we write the classical solution as $\Phi_{\rm cl}(t) = \sum_n g^n \Phi^{(n)}(t)$, where

$$\Phi^{(0)}(t) = \phi\, e^{i\omega t}\,,$$

$$\Phi^{(1)}(t) = i\int \mathrm{d}t'\,G(t,t')\left(-\phi^2 e^{2i\omega t'}\right) = \frac{\phi^2}{3\omega^2}\,\left(e^{2i\omega t} - e^{i\omega t}\right).$$

- With this, the wavefunction becomes

$$\boxed{\Psi[\phi] \approx e^{iS[\Phi_{\rm cl}]} = \exp\left(-\frac{\omega}{2}\phi^2 - \frac{g}{9\omega}\phi^3 + \frac{g^2}{72\omega^3}\phi^4 + \cdots\right)}$$

- From this, we can compute $\langle\phi^3\rangle$, $\langle\phi^4\rangle$, etc.

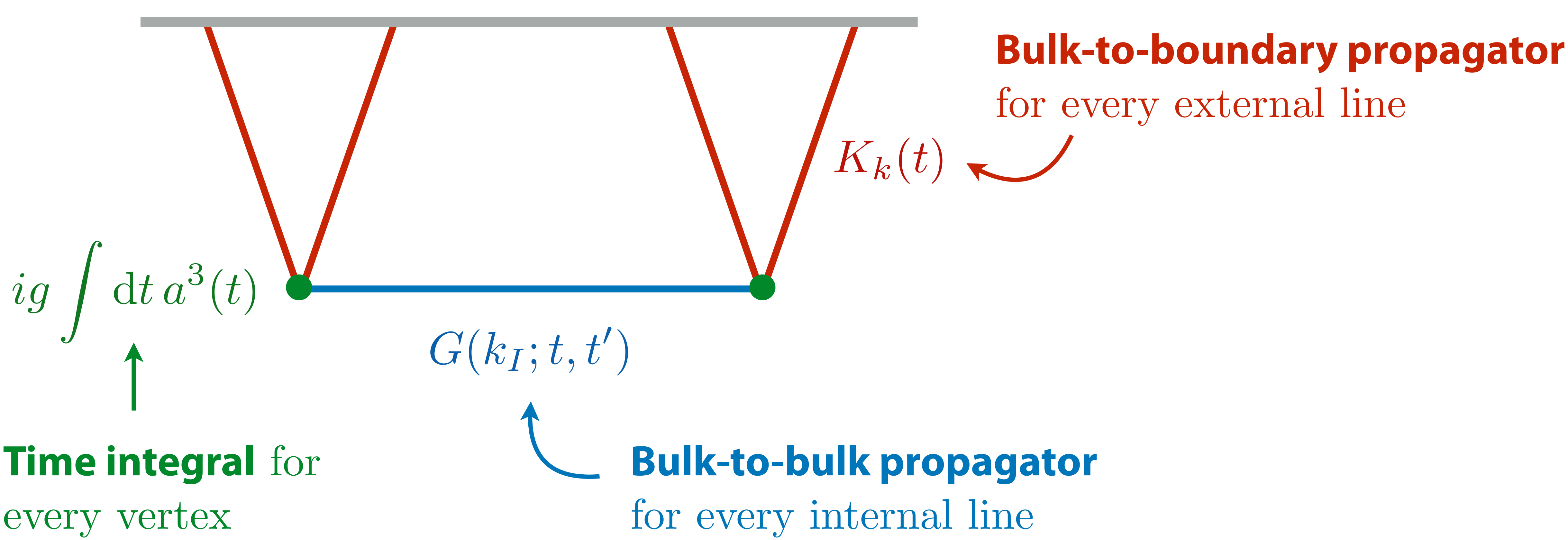
Interacting Fields

Back to field theory:

$$S[\Phi] = \int \mathrm{d}^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}m^2\Phi^2 - \frac{1}{3}g\Phi^3\right)$$

The analysis is similar to that of the anharmonic oscillator (see lecture notes).

In the interest of time, we jump directly to **Feynman rules** for WF coefficients:



Flat-Space Wavefunction

Flat-Space Wavefunction

The wavefunction is an interesting object even in flat space.

- We will study the following simple **toy model**

$$S = \int \mathrm{d}^4x \left(-\frac{1}{2}(\partial\Phi)^2 - \frac{g}{3!}\Phi^3 \right).$$

- All correlators can be evaluated at $t_* \equiv 0$ without loss of generality.

- The **propagators** are the same as for the harmonic oscillator:

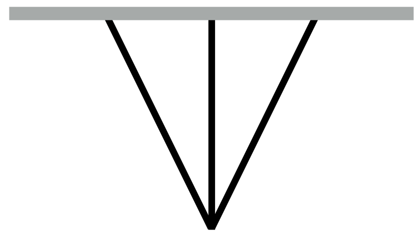
$$\boxed{\begin{aligned} K_k(t) &= e^{ikt} \\ G(k;t,t') &= \frac{1}{2k}\left(e^{-ik(t-t')}\theta(t-t') + e^{ik(t-t')}\theta(t'-t) \; - \; e^{ik(t+t')}\right) \end{aligned}}$$

We will compute the simplest tree-level correlators in this theory.

Contact Diagrams

The three-point wavefunction coefficient is

$$\langle O_1 O_2 O_3 \rangle =$$



$$\begin{aligned} &= ig \int_{-\infty}^{0} \mathrm{d}t\, K_{k_1}(t) K_{k_2}(t) K_{k_3}(t) \\ &= ig \int_{-\infty}^{0} \mathrm{d}t\, e^{i(k_1+k_2+k_3)t} \\ &= \frac{g}{(k_1+k_2+k_3)} \,. \end{aligned}$$

We will have more to say about this singularity.

This is easily generalized to N-point wavefunction coefficients:

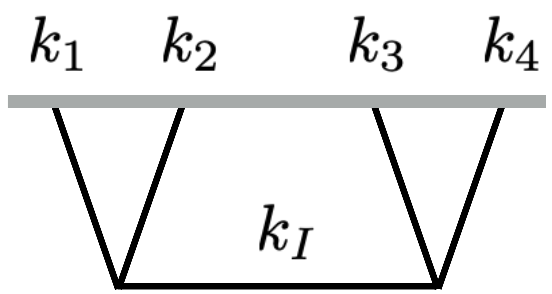
$$\langle O_1 O_2 \dots O_N \rangle = ig \int_{-\infty}^{0} \mathrm{d}t\, e^{i(k_1+k_2+\cdots+k_N)t} = \frac{g}{(k_1+k_2+\cdots+k_N)} \,.$$

Exchange Diagrams

The four-point wavefunction coefficient in Φ^3 theory is

$k_{12} \equiv k_1 + k_2$

$$\langle O_1 O_2 O_3 O_4 \rangle =$$



$$= -g^2 \int_{-\infty}^{0} \mathrm{d}t \, \mathrm{d}t' \; e^{ik_{12}t} \, G(k_I; t, t') \, e^{ik_{34}t'}$$

$$\begin{aligned}
= & -\frac{g^2}{2k_I} \int_{-\infty}^{0} \mathrm{d}t \int_{-\infty}^{t} \mathrm{d}t' \, e^{i(k_{12}-k_I)t} e^{i(k_{34}+k_I)t'} \\
& -\frac{g^2}{2k_I} \int_{-\infty}^{0} \mathrm{d}t \int_{-\infty}^{t} \mathrm{d}t' \, e^{i(k_{12}+k_I)t'} e^{i(k_{34}-k_I)t} \\
& +\frac{g^2}{2k_I} \int_{-\infty}^{0} \mathrm{d}t \int_{-\infty}^{0} \mathrm{d}t' \, e^{i(k_{12}+k_I)t} e^{i(k_{34}+k_I)t'}
\end{aligned}$$

$$= \frac{g^2}{2k_I} \left[\frac{1}{(k_{12}+k_{34})(k_{34}+k_I)} + \frac{1}{(k_{12}+k_{34})(k_{12}+k_I)} - \frac{1}{(k_{12}+k_I)(k_{34}+k_I)} \right]$$

$$= \frac{g^2}{(k_{12}+k_{34})(k_{12}+k_I)(k_{34}+k_I)}$$

Exchange Diagrams

The final answer is remarkably simple and has an interesting singularity structure:

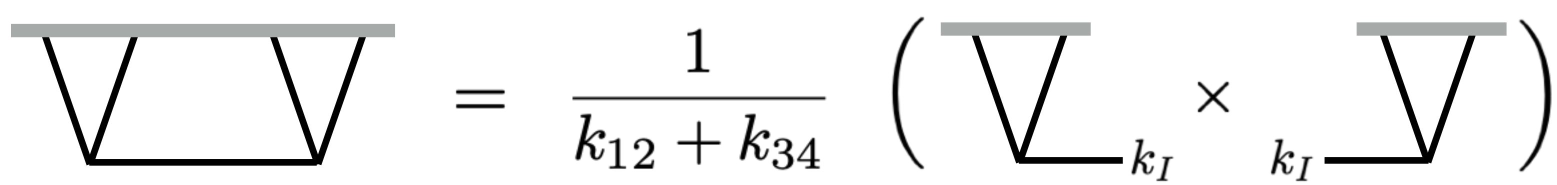
$$\langle O_1 O_2 O_3 O_4 \rangle = \frac{g^2}{\underbrace{(k_{12} + k_{34})}_{\text{Total energy}} \underbrace{(k_{12} + k_I)}_{\substack{\text{Energy entering} \\ \text{the left vertex}}} \underbrace{(k_{34} + k_I)}_{\substack{\text{Energy entering} \\ \text{the right vertex}}}}$$

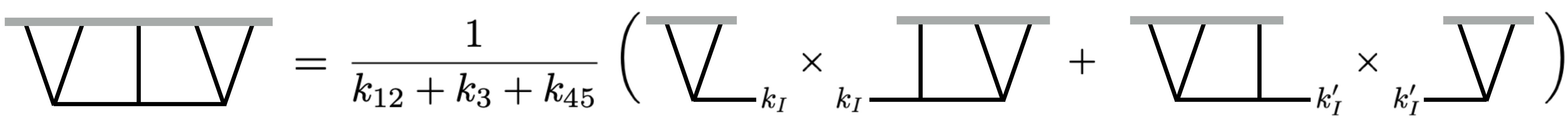
We will have more to say about this later.

Recursion Relation

More complicated graphs can be computed in the same way, but the complexity of the time integrals increases quickly with the number of internal lines.

Fortunately, there is a powerful **recursion relation** that allows complex graphs to be constructed from simpler building blocks:





This provides a simple algorithm to produce the result for any tree graph.

Arkani-Hamed, Benincasa and Postnikov [2017]

De Sitter Wavefunction

Free Field in de Sitter

Consider a **massive scalar** Φ in de Sitter space.

- The action of $u \equiv a(\eta)\Phi$ is

$$S = \frac{1}{2}\int \mathrm{d}\eta\,\mathrm{d}^3x\left[(u')^2 - (\nabla u)^2 - \frac{m^2/H^2 - 2}{\eta^2}u^2\right]$$

- The equation of motion is

$$\boxed{u'' + \left(k^2 + \frac{m^2/H^2 - 2}{\eta^2}\right)u = 0}$$

- The general solution of this equation is

$$u_k(\eta) = \sqrt{\frac{\pi}{4}}(-\eta)^{1/2}H_\nu^{(2)}(-k\eta)\,, \quad \text{where} \quad \nu \equiv \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}\,.$$

Hankel function

Free Field in de Sitter

Consider a **massive scalar** Φ in de Sitter space.

- The action of $u \equiv a(\eta)\Phi$ is

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- The equation of motion is

$$\boxed{u'' + \left(k^2 + \frac{m^2/H^2 - 2}{\eta^2}\right)u = 0}$$

- For $m^2 = 2H^2$, the field is **conformally coupled** and

$$\boxed{u_k(\eta) = \frac{e^{ik\eta}}{\sqrt{2k}}} \longleftarrow \text{Harmonic oscillator}$$

We will often consider this special case, because the wavefunction can then be computed analytically.

Propagators

For a generic field, with mode function $f_k(\eta) \equiv u_k(\eta)/a(\eta)$, the propagators are

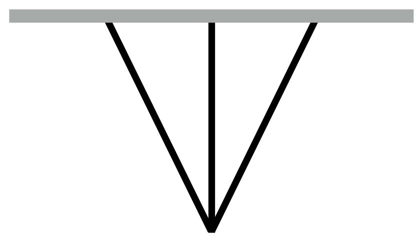
$$\boxed{\begin{aligned} K_k(\eta) &= \frac{f_k(\eta)}{f_k(\eta_*)} \\ G(k;\eta,\eta') &= f_k^*(\eta) f_k(\eta')\,\theta(\eta-\eta') + f_k^*(\eta') f_k(\eta)\,\theta(\eta'-\eta) - \frac{f_k^*(\eta_*)}{f_k(\eta_*)} f_k(\eta) f_k(\eta') \end{aligned}}$$

In general, these are given by Hankel functions which leads to complicated integrals in the wavefunction calculations.

Our explicit calculations will therefore be for conformally coupled scalars. We return to the general case in the cosmological bootstrap.

Contact Diagrams

The **three-point WF coefficient** is

$$\langle O_1 O_2 O_3 \rangle =$$ 

$$= ig \int_{-\infty}^{0} \mathrm{d}\eta \, a^4(\eta) \, K_{k_1}(\eta) K_{k_2}(\eta) K_{k_3}(\eta)$$

$$= \frac{ig}{f_{k_1}(\eta_*) f_{k_2}(\eta_*) f_{k_3}(\eta_*)} \int_{-\infty}^{\eta_*} \frac{\mathrm{d}\eta}{(H\eta)^4} \, f_{k_1}(\eta) f_{k_2}(\eta) f_{k_3}(\eta)$$

Product of
Hankel functions

$$= \text{Appell } F_4$$

Contact Diagrams

For **conformally coupled scalars**, we get

$$\langle O_1 O_2 O_3 \rangle = \frac{ig}{H^4 \eta_*^3} \int_{-\infty}^{\eta_*} \frac{\mathrm{d}\eta}{\eta} \, e^{i(k_1+k_2+k_3)\eta} = \boxed{\frac{ig}{H^4 \eta_*^3} \log(iK\eta_*)}$$

where $K \equiv k_1 + k_2 + k_3$.

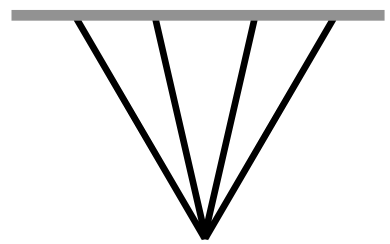
The corresponding **correlator** is

$$\begin{aligned} \langle \phi_1 \phi_2 \phi_3 \rangle &= -\frac{H^6 \eta_*^6}{8 k_1 k_2 k_3} \Big(\langle O_1 O_2 O_3 \rangle + \langle O_1 O_2 O_3 \rangle^* \Big) \\ &= -\frac{g}{8} \frac{H^2 \eta_*^3}{k_1 k_2 k_3} \Big(i \log(iK\eta_*) - i \log(-iK\eta_*) \Big) = \boxed{\frac{\pi}{8} \, g \, \frac{H^2 \eta_*^3}{k_1 k_2 k_3}} \end{aligned}$$

Contact Diagrams

The **four-point WF coefficient** for a contact interaction of conformally coupled scalars is

$$\langle O_1 O_2 O_3 O_4 \rangle =$$

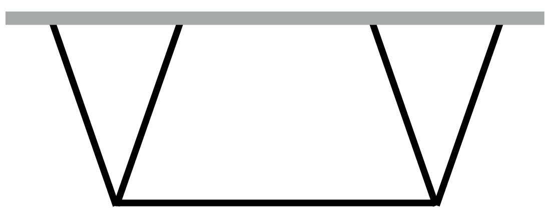


$$= \frac{ig}{H^4 \eta_*^4} \int_{-\infty}^{\eta_*} \mathrm{d}\eta \, e^{i(k_1+k_2+k_3+k_4)\eta} = \boxed{\frac{1}{H^4 \eta_*^4} \frac{g}{(k_1+k_2+k_3+k_4)}}$$

Since the Φ^4 interaction preserves the conformal symmetry of the free theory, this is the same result as in flat space (modulo the prefactor).

Exchange Diagrams

The four-point WF coefficient corresponding to an **exchange diagram** (of conformally coupled scalars) is

$$\langle O_1 O_2 O_3 O_4 \rangle =$$ 

$$= -\frac{g^2}{H^6\eta_*^4}\int_{-\infty}^{0}\frac{\mathrm{d}\eta}{\eta}\int_{-\infty}^{0}\frac{\mathrm{d}\eta'}{\eta'}\, e^{ik_{12}\eta}e^{ik_{34}\eta'}\frac{1}{2k_I}\left[e^{-ik_I|\eta-\eta'|}-e^{ik_I(\eta+\eta')}\right]$$

$$= -\frac{g^2}{H^6\eta_*^4}\int_{k_{12}}^{\infty}\mathrm{d}x\int_{k_{34}}^{\infty}\mathrm{d}y\int_{-\infty}^{0}\mathrm{d}\eta\,\mathrm{d}\eta'\, e^{ix\eta}e^{iy\eta'}\frac{1}{2k_I}\left[e^{-ik_I|\eta-\eta'|}-e^{ik_I(\eta+\eta')}\right]$$

$$= \frac{1}{H^6\eta_*^4}\int_{k_{12}}^{\infty}\mathrm{d}x\int_{k_{34}}^{\infty}\mathrm{d}y\,\langle O_1 O_2 O_3 O_4\rangle_{(\text{flat})}(x,y,k_I)\,,$$

$$= \boxed{-\frac{g^2}{H^6\eta_*^4}\int_{k_{12}}^{\infty}\mathrm{d}x\int_{k_{34}}^{\infty}\mathrm{d}y\,\frac{1}{(x+y)(x+k_I)(y+k_I)}}$$

Exchange Diagrams

We see that this four-point function can be written as an integral of the flat-space result:

$$\langle O_1 O_2 O_3 O_4 \rangle = -\frac{g^2}{H^6 \eta_*^4} \int_{k_{12}}^{\infty} \mathrm{d}x \int_{k_{34}}^{\infty} \mathrm{d}y \, \frac{1}{(x+y)(x+k_I)(y+k_I)}$$

This integral can be performed analytically to give

$$\boxed{\begin{aligned}\langle O_1 O_2 O_3 O_4 \rangle = \frac{g^2}{2H^6 \eta_*^4} \frac{1}{k_I} \Bigg[\mathrm{Li}_2 \left(\frac{k_{12} - k_I}{E} \right) + \mathrm{Li}_2 \left(\frac{k_{34} - k_I}{E} \right) & \\ + \log \left(\frac{k_{12} + k_I}{E} \right) \log \left(\frac{k_{34} + k_I}{E} \right) - \frac{\pi^2}{6} \Bigg] &\end{aligned}}$$

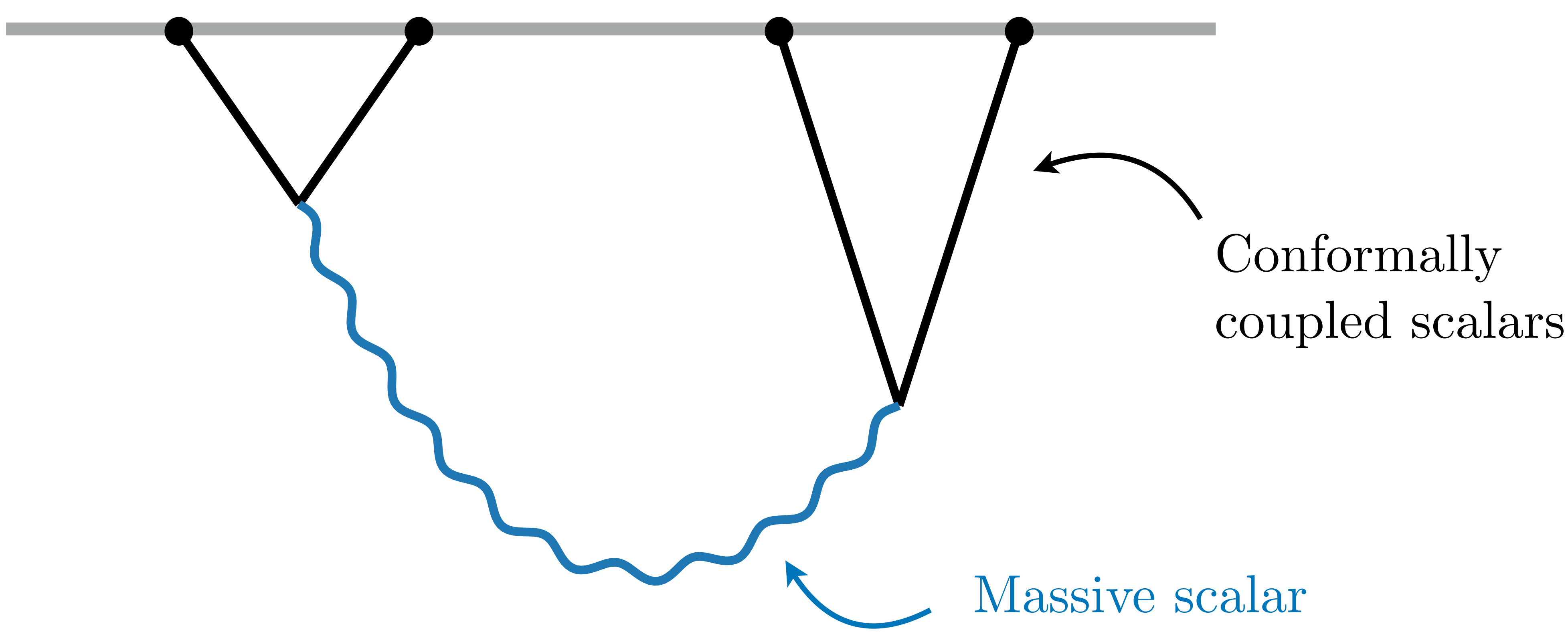
where $E \equiv k_1 + k_2 + k_3 + k_4$ and Li_2 is the dilogarithm.

Arkani-Hamed and Maldacena [2015]

A Challenge

So far, we have only computed the correlators of conformally coupled scalars.
Consider now the **exchange of a generic massive scalar**:

$$\langle O_1 O_2 O_3 O_4 \rangle =$$

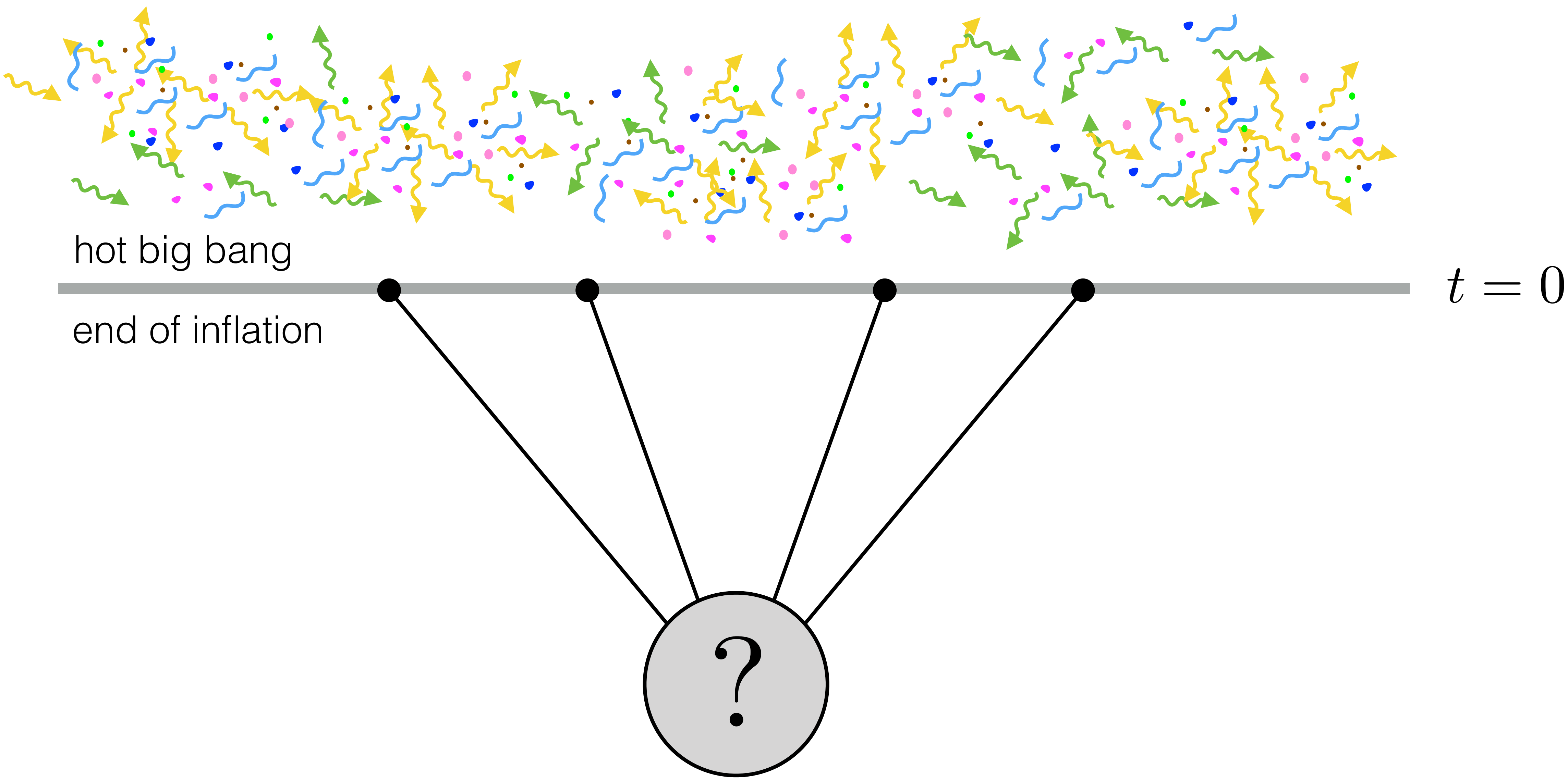


$$= \; -g^2 \int \frac{\mathrm{d}\eta}{\eta^2} \int \frac{\mathrm{d}\eta'}{\eta'^2}\, e^{ik_{12}\eta} e^{ik_{34}\eta'}\, G(k_I;\eta,\eta')$$

In general, the time integrals cannot be performed analytically.
We need a different approach.

Cosmological Bootstrap

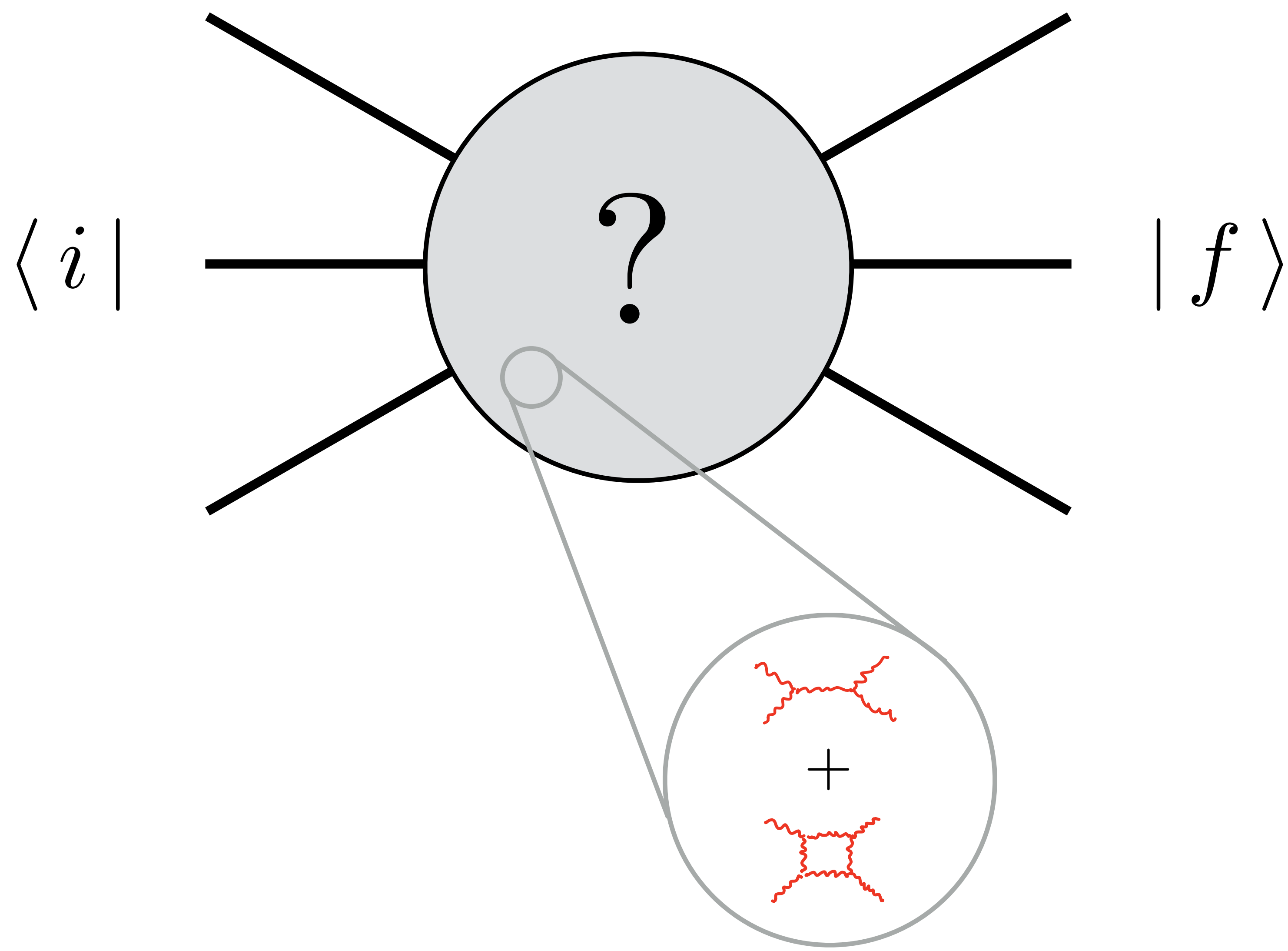
If inflation is correct, then the reheating surface is the future boundary of an approximate **de Sitter spacetime**:



Can we **bootstrap** these boundary correlators directly?

- Conceptual advantage: **focuses directly on observables**.
- Practical advantage: **simplifies calculations**.

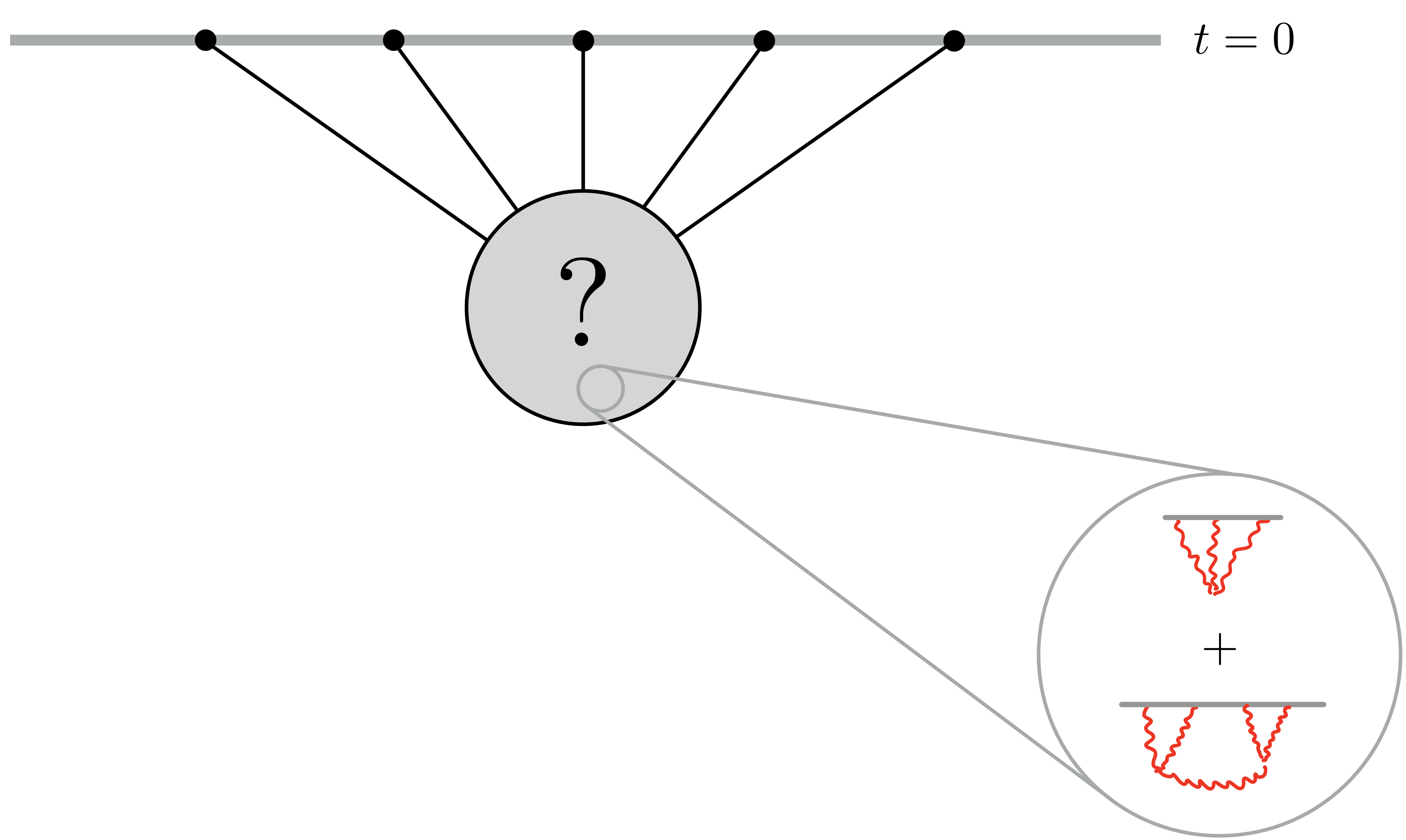
The bootstrap perspective has been very influential for **scattering amplitudes**:



There has be enormous progress in bypassing Feynman diagram expansions to write down on-shell amplitudes directly:

- Practical advantage: **simplifies calculations**.
- Conceptual advantage: **reveals hidden structures**.

I will describe recent progress in developing a similar bootstrap approach for cosmological correlators:



Goal: Develop an understanding of cosmological correlators that parallels our understanding of flat-space scattering amplitudes.

OUTLINE:

The Bootstrap Philosophy	Bootstrapping Tools	Examples

REFERENCES:

DB, Green, Joyce, Pajer, Pimentel, Sleight and Taronna, *Snowmass White Paper: The Cosmological Bootstrap* [2203.08121]

DB and Joyce, *Lectures on Cosmological Correlations*

The Bootstrap Philosophy

The Conventional Approach

How we usually make predictions:

Physical Principles $\longrightarrow$ **Models** $\longrightarrow$ **Observables**

locality, causality,
unitarity, symmetries

Lagrangians
equations of motion
spacetime evolution
Feynman diagrams

Works well if we have a well-established theory (Standard Model, GR, ...) and many observables.

The Conventional Approach

Although conceptually straightforward, this has some drawbacks:

- It involves unphysical gauge degrees of freedom.
- Relation between Lagrangians and observables is many-to-one.
- Even at tree level, the computation of cosmological correlators involves complicated time integrals:

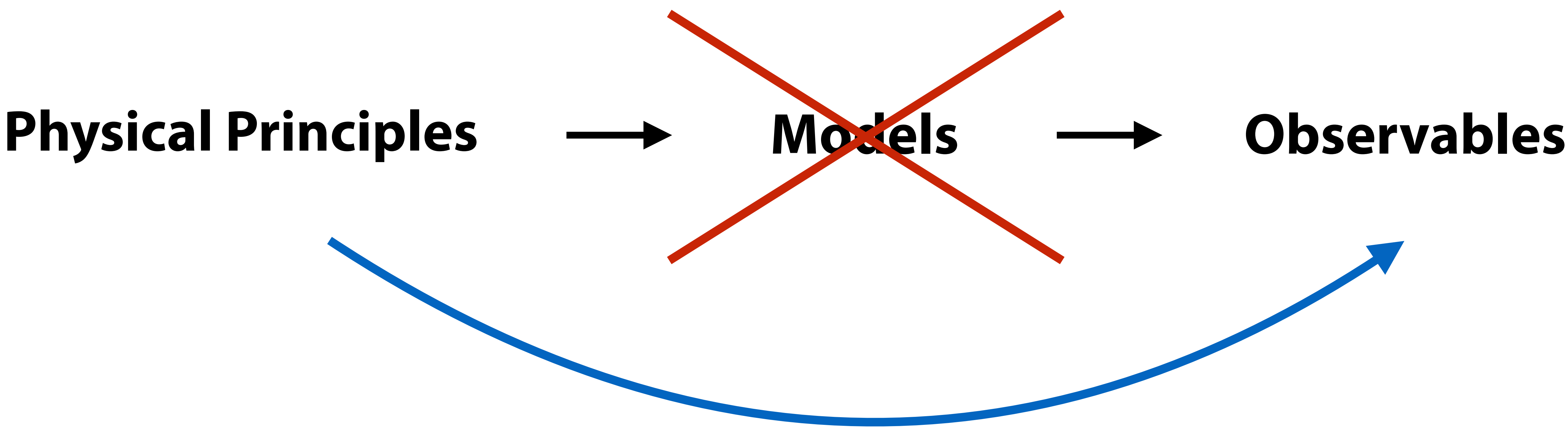
$$\boxed{\psi_N = \sum_{\text{Diagrams}} \left(\prod_{\text{Vertices}} \int \mathrm{d}t \right) \begin{pmatrix} \text{External} \\ \text{propagators} \end{pmatrix} \begin{pmatrix} \text{Internal} \\ \text{propagators} \end{pmatrix} \begin{pmatrix} \text{Vertex} \\ \text{factors} \end{pmatrix}}$$

  Hard to compute!

- Fundamental principles (e.g. locality and unitarity) are obscured.
- To derive nonperturbative correlators and constraints from the UV completion, we need a deeper understanding.

The Bootstrap Method

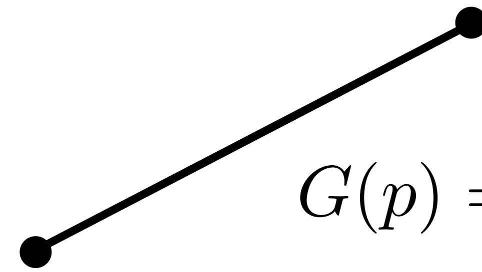
In the bootstrap approach, we cut out the middle man and go directly from fundamental principles to physical observables:



This is particularly relevant when we have many theories (inflation, BSM, ...) and few observables.

S-matrix Bootstrap

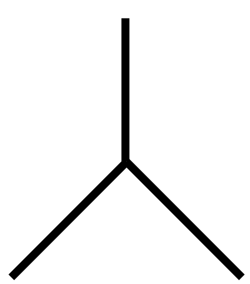
- What stable particles exist?



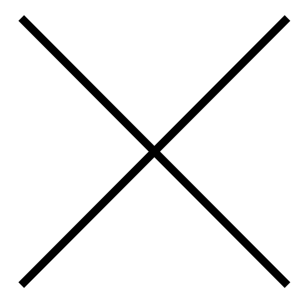
$$G(p) = \frac{1}{p^2 + M^2}$$

fixed by locality and
Lorentz symmetry

- What interactions are allowed?

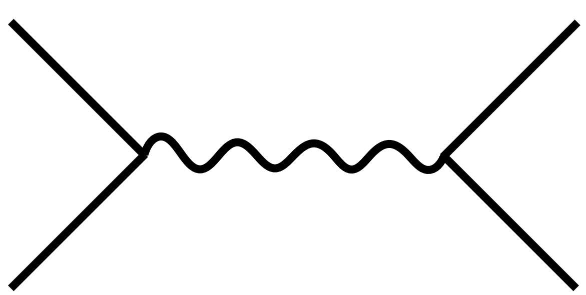


$$= \left(\frac{\langle 12 \rangle^3}{\langle 13 \rangle \langle 23 \rangle} \right)^J$$



$$= \sum_{m,n} a_{nm} \, s^m t^n$$

Lorentz



$$= \frac{g^2 P_J(\cos\theta)}{s - M^2}$$

unitarity

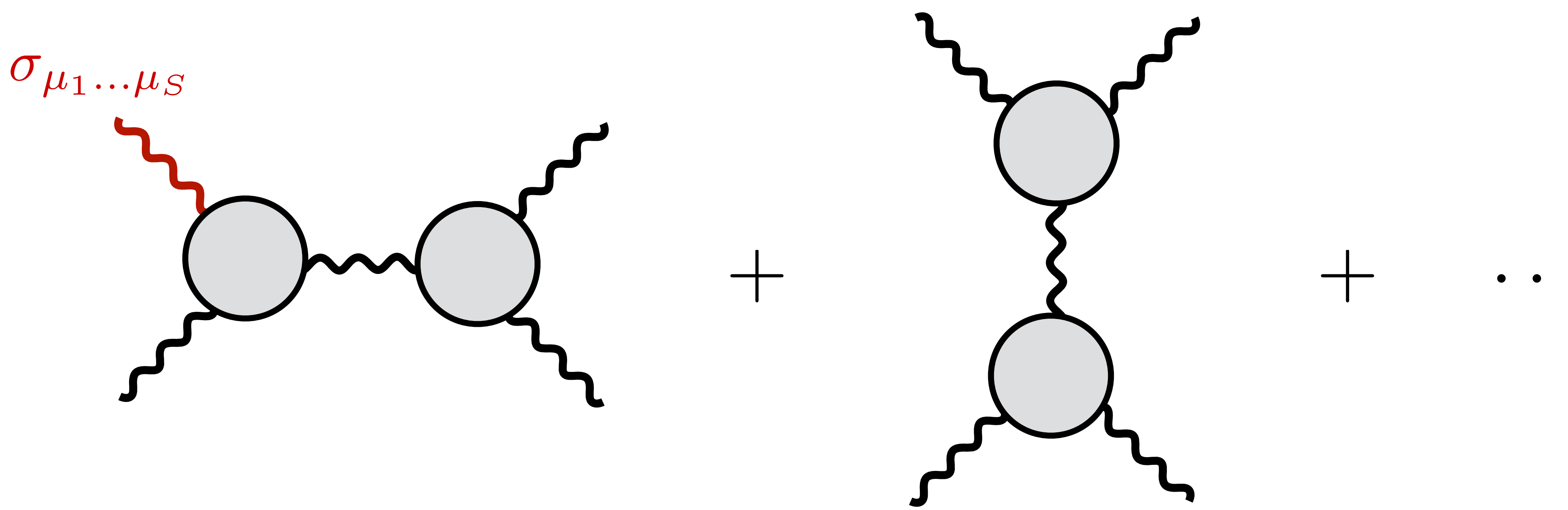
locality

→ No Lagrangian is required to derive this.

→ Basic principles allow only a small menu of possibilities.

S-matrix Bootstrap

Consistent factorization is very constraining for massless particles:



→ Only consistent for spins $S = \{\, 0\,,\ \frac{1}{2}\,,\ 1\,,\ \frac{3}{2}\,,\ 2\,\}$

YM SUSY GR

Benincasa and Cachazo [2007]
McGady and Rodina [2013]

A Success Story

The modern amplitudes program has been very successful:

1. New Computational Techniques
   - Recursion relations
   - Generalized unitarity
   - Soft theorems

2. New Mathematical Structures
   - Positive geometries
   - Amplituhedrons
   - Associahedrons

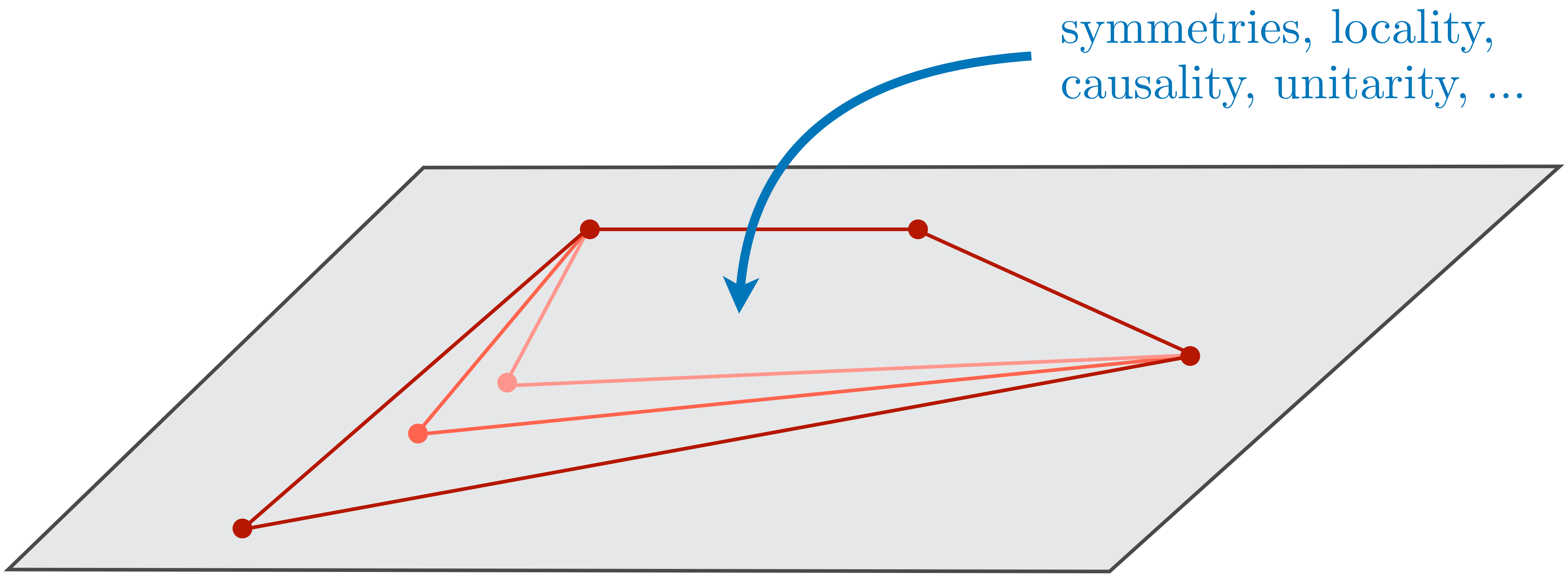
3. New Relations Between Theories
   - Color-kinematics duality
   - BCJ double copy

4. New Constraints on QFTs
   - Positivity bounds
   - EFThedron

5. New Applications
   - Gravitational wave physics
   - Cosmology

Cosmological Bootstrap

Recently, there has been significant progress in **bootstrapping correlators** using physical consistency conditions on the boundary:



In this part of the lectures, I will review these developments.

Bootstrapping Tools

Symmetries	Singularities	Unitarity

Bootstrapping Tools

Symmetries

Singularities

Unitarity

De Sitter Symmetries

D-dimensional de Sitter space can be represented as an **hyperboloid** in $(D+1)$-dimensional Minkowski space:

$$\eta_{AB} X^A X^B = -X_0^2 + X_1^2 + \cdots + X_D^2 = H^{-2}$$

de Sitter radius

The symmetries of de Sitter then correspond to higher-dimensional **Lorentz transformations**:

$$SO(D,1): \quad [J_{AB}, J_{CD}] = \eta_{BC} J_{AD} - \eta_{AC} J_{BD} + \eta_{AD} J_{BC} - \eta_{BD} J_{AC}$$

In a specific time slicing, we define

$$\hat{D} \equiv J_{D0} \longleftarrow \text{Dilatations}$$

$$P_i \equiv J_{Di} - J_{0i} \longleftarrow \text{Translations}$$

$$K_i \equiv J_{Di} + J_{0i} \longleftarrow \text{Boosts}$$

$$J_{ij} \longleftarrow \text{Rotations}$$

De Sitter Symmetries

D-dimensional de Sitter space can be represented as an **hyperboloid** in $(D+1)$-dimensional Minkowski space:

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de Sitter radius

The symmetries of de Sitter then correspond to higher-dimensional **Lorentz transformations**:

$$SO(D,1): \quad [J_{AB}, J_{CD}] = \eta_{BC}J_{AD} - \eta_{AC}J_{BD} + \eta_{AD}J_{BC} - \eta_{BD}J_{AC}$$

The de Sitter algebra then becomes

$$\begin{aligned}
[J_{ij}, J_{kl}] &= \delta_{jk}J_{il} - \delta_{ik}J_{jl} + \delta_{il}J_{jk} - \delta_{jl}J_{ik} \,, & [\hat{D}, P_i] &= P_i \,, \\
[J_{ij}, K_l] &= \delta_{jl}K_i - \delta_{il}K_j \,, & [\hat{D}, K_i] &= -K_i \,, \\
[J_{ij}, P_l] &= \delta_{jl}P_i - \delta_{il}P_j \,, & [K_i, P_j] &= 2\delta_{ij}\hat{D} - 2J_{ij} \,.
\end{aligned}$$

De Sitter Symmetries

We will work in the flat-slicing, where the de Sitter metric is

$$\mathrm{d}s^2 = \frac{1}{H^2\eta^2}\left(-\mathrm{d}\eta^2 + \mathrm{d}\mathbf{x}^2\right)$$

and the symmetry generators (Killing vectors) become

$$P_i = \partial_i \qquad \hat{D} = -\eta\partial_\eta - x^i\partial_i$$

$$J_{ij} = x_i\partial_j - x_j\partial_i \qquad K_i = 2x_i\eta\partial_\eta + \left(2x^j x_i + (\eta^2 - x^2)\delta_i^j\right)\partial_j$$

- **Dilatations** and **boosts** correspond to the following finite transformations:

$$\eta \mapsto \lambda\eta \qquad \mathbf{x} \mapsto \lambda\mathbf{x}$$

$$\eta \mapsto \eta/\Omega \qquad \mathbf{x} \mapsto \left[\mathbf{x} + (x^2 - \eta^2)\,\mathbf{b}\right]/\Omega$$

where $\Omega \equiv 1 + 2(\mathbf{b}\cdot\mathbf{x}) + b^2(x^2 - \eta^2)$

- At late times, the boosts become special conformal transformations (**SCTs**).
- Late-time correlators are therefore constrained by **conformal symmetry**.

Unitarity Representations

Physical states are described by unitary representations of the de Sitter group. These representations are labelled by the eigenvalues of the Casimir operators:

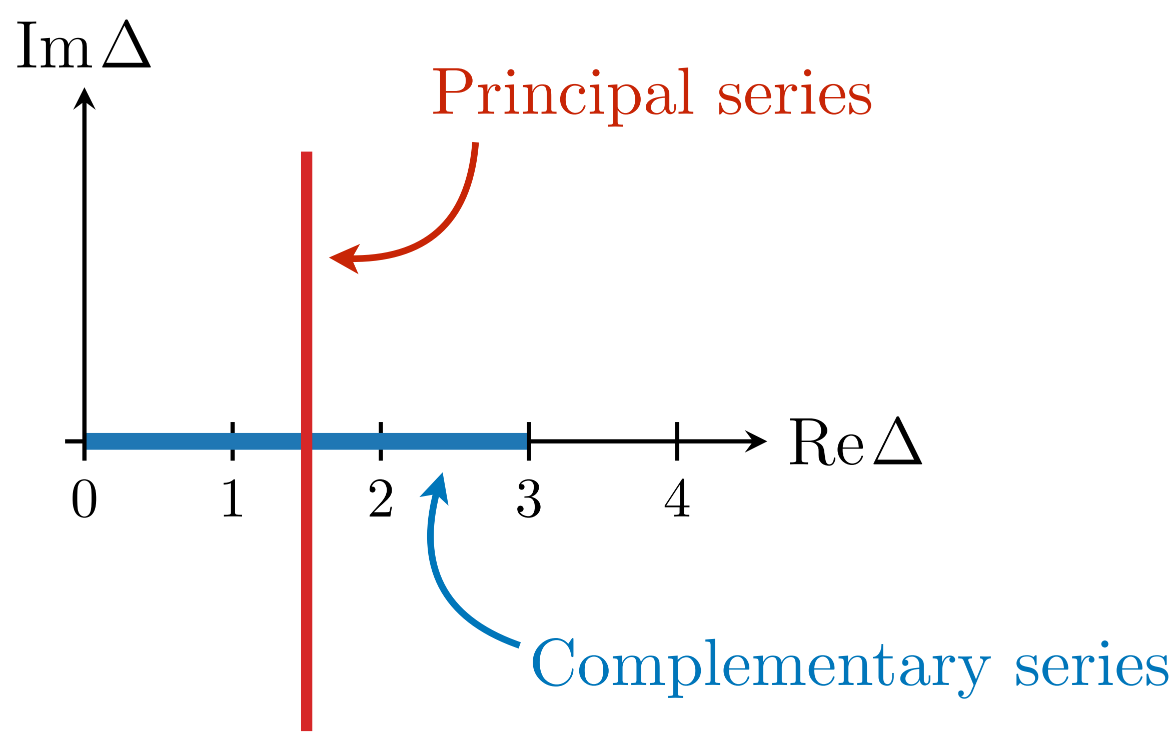
$$\mathcal{C}_2 \equiv \frac{1}{2} J^{AB} J_{AB} = \hat{D}(d - \hat{D}) + P^i K_i + \frac{1}{2} J^{ij} J_{ij}$$
$$\mathcal{C}_4 \equiv J_{AB} J^{BC} J_{CD} J^{DA}$$

$$\longrightarrow \quad |\Delta, J\rangle$$

Scaling dimension

Spin

Unitarity puts constraints on the allowed scaling dimensions.
For scalars, we have



Bulk Fields

These representations can be realized by bulk fields.

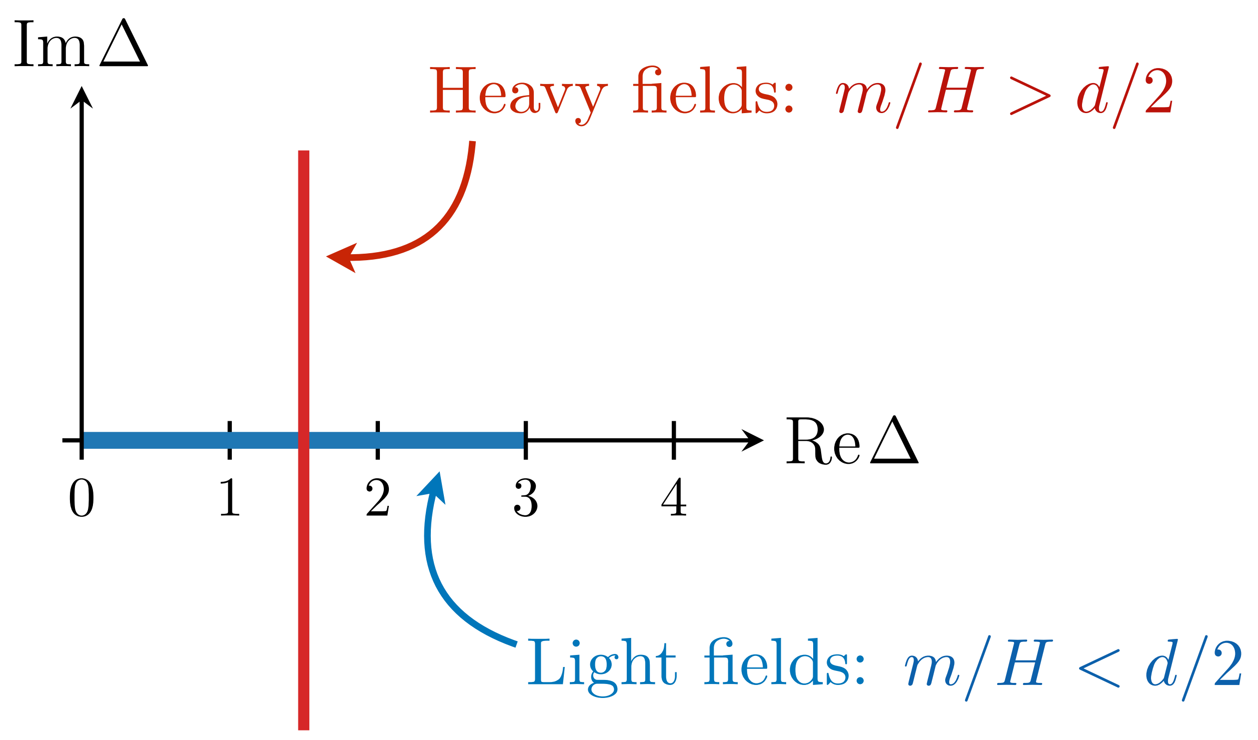
Consider a massive scalar field. Its equation of motion can be written as an eigenvalue equation for the quadratic Casimir:

$$\mathcal{C}_2\Phi = -\eta^2\left(\frac{\partial^2}{\partial\eta^2} - \frac{d-1}{\eta}\frac{\partial}{\partial\eta} - \nabla^2\right)\Phi = \frac{m^2}{H^2}\Phi$$

Using $\mathcal{C}_2 = \Delta(d-\Delta)$, we get

$$\Delta \equiv \frac{d}{2} - \sqrt{\frac{d^2}{4} - \frac{m^2}{H^2}}$$

and $\bar{\Delta} = d - \Delta$.



Boundary Fluctuations

Our main interest are the fluctuations on the late-time boundary.

At late times, the solution for the bulk field is

$$\Phi(\mathbf{x}, \eta \to 0) = \phi(\mathbf{x})\, \eta^{\Delta} + \bar{\phi}(\mathbf{x})\, \eta^{d-\Delta}$$

Boundary field profile

For light fields, the first term of the asymptotic solution dominates.

The de Sitter isometries then act as

$$\boxed{\begin{aligned} P_i \phi &= \partial_i \phi \\ J_{ij} \phi &= (x_i \partial_j - x_j \partial_i)\phi \\ D\phi &= -(\Delta + x^i \partial_i)\phi \\ K_i \phi &= \left[2x_i \Delta + \left(2x^j x_i - x^2 \delta_i^j\right) \partial_j\right] \phi \end{aligned}} \longleftarrow \quad \eta\partial_\eta \mapsto \Delta$$

$\longrightarrow$ Transformations of a primary operator of weight Δ in a **CFT**.

Conformal Ward Identities

The **boundary correlators** are constrained by the conformal symmetry:

$$\sum_{a=1}^{N} \langle \phi_1 \cdots \delta\phi_a \cdots \phi_N \rangle = 0$$

where $\delta\phi_a$ stand for any of the field transformations.

- Translations and rotations require that the correlator is only a function of the separations $|\mathbf{x}_a - \mathbf{x}_b|$.

- Dilatation and SCT invariance imply

$$\boxed{\begin{aligned} 0 &= \sum_{a=1}^{N} \left(\Delta_a + x_a^j \frac{\partial}{\partial x_a^j} \right) \langle \phi_1 \cdots \phi_a \cdots \phi_N \rangle \\ 0 &= \sum_{a=1}^{N} \left(2\Delta_a x_a^i + 2 x_a^i x_a^j \frac{\partial}{\partial x_a^j} - x_a^2 \frac{\partial}{\partial x_{a,i}} \right) \langle \phi_1 \cdots \phi_a \cdots \phi_N \rangle \end{aligned}}$$

Conformal Ward Identities

$$\boxed{\begin{aligned} 0 &= \sum_{a=1}^{N} \left(\Delta_a + x_a^j \frac{\partial}{\partial x_a^j} \right) \langle \phi_1 \cdots \phi_a \cdots \phi_N \rangle \\ 0 &= \sum_{a=1}^{N} \left(2\Delta_a x_a^i + 2 x_a^i x_a^j \frac{\partial}{\partial x_a^j} - x_a^2 \frac{\partial}{\partial x_{a,i}} \right) \langle \phi_1 \cdots \phi_a \cdots \phi_N \rangle \end{aligned}}$$

Exercise: Show that for two- and three-point functions the conformal Ward identities are solved by

$$\langle \phi_1 \phi_2 \rangle = \frac{1}{x_{12}^{2\Delta_1}} \, \delta_{\Delta_1, \Delta_2}$$

$$\langle \phi_1 \phi_2 \phi_3 \rangle = \frac{c_{123}}{x_{12}^{\Delta_t - 2\Delta_3} x_{23}^{\Delta_t - 2\Delta_1} x_{31}^{\Delta_t - 2\Delta_2}}$$

where $\phi_a \equiv \phi_a(\mathbf{x}_a)$, $x_{ab} \equiv |\mathbf{x}_a - \mathbf{x}_b|$ and $\Delta_t \equiv \sum \Delta_a$.

Conformal Ward Identities

Similar Ward identities can be derived for the **wavefunction coefficients**.

Consider

$$\Psi[\phi] = \exp\left(\cdots + \int \mathrm{d}^d x_1 \cdots \mathrm{d}^d x_N \, \Psi_N(\underline{\mathbf{x}})\, \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_N) + \cdots \right)$$

$$\downarrow$$

$$\delta\Psi \supset \sum_{a=1}^{N} \int \mathrm{d}^d x_1 \cdots \mathrm{d}^d x_N \, \Psi_N(\underline{\mathbf{x}})\, \phi(\mathbf{x}_1) \cdots \delta\phi(\mathbf{x}_a) \cdots \phi(\mathbf{x}_N)$$

Integrate by parts, so that the generator of the transformation acts on the wavefunction coefficient. For example, for the dilatation, we have

$$\begin{aligned} \delta\Psi &\supset \sum_{a=1}^{N} \int \mathrm{d}^d x_1 \cdots \mathrm{d}^d x_N \, \Psi_N(\underline{\mathbf{x}})\; \phi(\mathbf{x}_1) \cdots \left(-\Delta_a - x_a^j \partial_{x_a^j}\right) \phi(\mathbf{x}_a) \cdots \phi(\mathbf{x}_N) \\ &= \sum_{a=1}^{N} \int \mathrm{d}^d x_1 \cdots \mathrm{d}^d x_N \left[\left((d - \Delta_a) + x_a^j \partial_{x_a^j} \right) \Psi_N(\underline{\mathbf{x}}) \right] \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_N) \end{aligned}$$

Conformal Ward Identities

Under dilatations, the wavefunction coefficients therefore transform as

$$\delta\Psi_N(\underline{\mathbf{x}}) = \sum_{a=1}^{N} \left((d-\Delta_a) + x_a^j \partial_{x_a^j} \right) \Psi_N(\underline{\mathbf{x}})$$

A similar derivation applies to SCTs.

Imposing $\delta\Psi_N = 0$ leads to the conformal Ward identity for WF coefficients:

$$\boxed{\begin{aligned} 0 &= \sum_{a=1}^{N} \left((d-\Delta_a) + x_a^j \frac{\partial}{\partial x_a^j} \right) \langle O_1 \cdots O_a \cdots O_N \rangle \\ 0 &= \sum_{a=1}^{N} \left(2(d-\Delta_a) x_a^i + 2 x_a^i x_a^j \frac{\partial}{\partial x_a^j} - x_a^2 \frac{\partial}{\partial x_{a,i}} \right) \langle O_1 \cdots O_a \cdots O_N \rangle \end{aligned}}$$

where $\Psi_N \equiv \langle O_1 \cdots O_N \rangle$.

$\longrightarrow$ Same as before, except the dual operators have weight $\bar{\Delta}_a = d - \Delta_a$.

Conformal Ward Identities

In cosmology, we need these Ward identities in **Fourier space**.

Although the correlators themselves are hard to Fourier transform, the symmetry generators are not. This gives

$$\hat{D} O_{\mathbf{k}} = \left[\bar{\Delta} + k^i \partial_{k^i}\right] O_{\mathbf{k}}$$

$$K_i O_{\mathbf{k}} = \left[2\bar{\Delta}\,\partial_{k^i} - k_i \partial_{k^j}\partial_{k^j} + 2k^j \partial_{k^j}\partial_{k^i}\right] O_{\mathbf{k}}$$

Since $\langle O_1 \cdots O_N \rangle = (2\pi)^d \delta^{(d)}(\mathbf{k}_1 + \cdots + \mathbf{k}_N) \langle O_1 \cdots O_N \rangle'$, we need to pay attention to the momentum derivatives hitting the delta function.

This gives an extra term in the dilatation Ward identity, so that

$$\boxed{\begin{aligned} \left[-d + \sum_{a=1}^{N} \hat{D}_a\right] \langle O_1 \cdots O_N \rangle'(\underline{\mathbf{k}}) &= 0 \\ \sum_{a=1}^{N} K_a^i \langle O_1 \cdots O_N \rangle'(\underline{\mathbf{k}}) &= 0 \end{aligned}}$$

Conformal Ward Identities

$$\boxed{\begin{aligned}\left[-d+\sum_{a=1}^{N}\hat{D}_a\right]\langle O_1\cdots O_N\rangle'(\underline{\mathbf{k}})&=0\\ \sum_{a=1}^{N}K_a^i\langle O_1\cdots O_N\rangle'(\underline{\mathbf{k}})&=0\end{aligned}}$$

- An N-point function has scaling dimension $\bar{\Delta}_t \equiv \sum_a \bar{\Delta}_a$ in position space.
- The momentum-space correlator then has dimension $\bar{\Delta}_t - d \times N$.
- After stripping off the delta function, the dimension is $\bar{\Delta}_t - d(N-1)$.

This fixes the overall momentum scaling of the correlator, allowing us to make an ansatz that automatically solves the dilatation Ward identity.

All the juice is then in the SCT Ward identity.

Two-Point Functions

Consider the two-point function of **two arbitrary scalar operators**.

- An ansatz that solves the dilatation Ward identity is

$$\langle O_1 O_2 \rangle' \propto k_1^{\bar{\Delta}_1 + \bar{\Delta}_2 - d}$$

- The SCT Ward identity then forces $\bar{\Delta}_1 = \bar{\Delta}_2$, and hence

$$\boxed{\langle O_1 O_2 \rangle' = A\, k_1^{2\bar{\Delta}_1 - d}\, \delta_{\bar{\Delta}_1, \bar{\Delta}_2}}$$

- A massless field $(\bar{\Delta} = 3)$ in $d = 3$ dimensions has

$$\langle \phi_1 \phi_2 \rangle' = \frac{1}{2\,\mathrm{Re}\langle O_1 O_2 \rangle'} \propto \frac{1}{k^3}$$

as expected for a scale-invariant power spectrum.

Three-Point Functions

Consider the three-point function of **three arbitrary scalar operators**.

- An ansatz that solves the dilatation Ward identity is

$$\langle O_1 O_2 O_3 \rangle' = k_3^{\bar{\Delta}_1+\bar{\Delta}_2+\bar{\Delta}_3-2d}\, \hat{G}(p,q)$$

  where $p \equiv k_1/k_3$ and $q \equiv k_2/k_3$.

- The SCT Ward identity then implies two differential equations for $\hat{G}(p,q)$, which are solved by

$$\boxed{\langle O_1 O_2 O_3 \rangle' = \text{Appell } F_4}$$

$$\propto \int_0^\infty \mathrm{d}x\, x^{\frac{d}{2}-1} K_{\nu_1}(k_1 x) K_{\nu_2}(k_2 x) K_{\nu_3}(k_3 x)$$

Three-Point Functions

It will be useful to work out more explicitly the case of **two conformally coupled scalars** and **a generic scalar** in 3 dimensions:

$$\langle O_1 O_2 X_3 \rangle' = k_3^{\bar{\Delta}-2}\, \hat{G}(u,w)$$

where $u \equiv k_3/(k_1+k_2)$ and $w \equiv k_3/(k_1-k_2)$.

- The SCT Ward identity then implies $\hat{G}(u,w) = \hat{G}(u)$, with

$$\boxed{\left[\Delta_u + \left(\mu^2 + \frac{1}{4}\right)\right]\hat{G}(u) = 0}$$

where $\Delta_u \equiv u^2(1-u^2)\partial_u^2 - 2u^3\partial_u$ and $\mu^2 + \frac{1}{4} = -(\bar{\Delta}-2)(\bar{\Delta}-1)$.

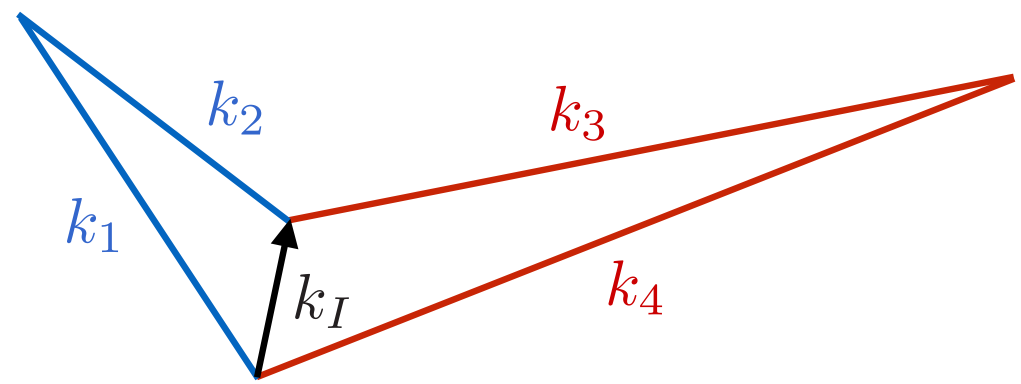
- This is solved by $\hat{G}_+(u) \propto P^0_{i\mu-\frac{1}{2}}(u^{-1})$ and $\hat{G}_-(u) \propto P^0_{i\mu-\frac{1}{2}}(-u^{-1})$.

associated Legendre

Four-Point Functions

Finally, we consider the four-point function of **conformally coupled scalars**:

$$\langle O_1 O_2 O_3 O_4 \rangle' =$$



$$= \frac{1}{k_I} \hat{F}(u, v)$$

where $u \equiv k_I/(k_1 + k_2)$ and $v \equiv k_I/(k_3 + k_4)$.

- The SCT Ward identity then implies

$$\boxed{(\Delta_u - \Delta_v)\hat{F} = 0}$$

Arkani-Hamed, DB, Lee and Pimentel [2018]

where $\Delta_u \equiv u^2(1 - u^2)\partial_u^2 - 2u^3\partial_u$.

- To solve this equation requires additional physical input.

Bootstrapping Tools

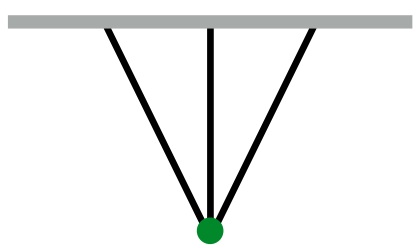
Symmetries	**Singularities**	Unitarity

Much of the physics of scattering amplitudes is fixed by their **singularities**.

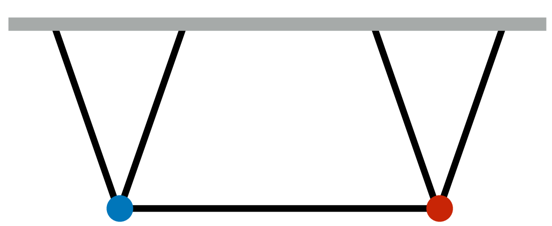
Singularities also play an important role in constraining the structure of cosmological correlators.

Flat-Space Wavefunction

In the last lecture, we computed wavefunction coefficients in flat space. Let us look back at some of these results.



$$= ig \int_{-\infty}^{0} \mathrm{d}t \; e^{iEt} = \frac{g}{E}$$



$$= -g^2 \int_{-\infty}^{0} \mathrm{d}t_1 \mathrm{d}t_2 \, e^{ik_{12}t_1} \, G_{k_I}(t_1, t_2) \, e^{ik_{34}t_2}$$

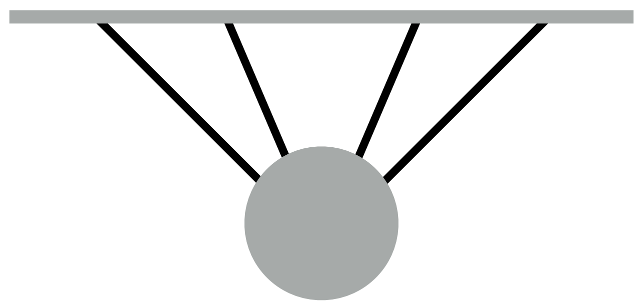
$$= \frac{g^2}{(k_{12} + k_{34})(k_{12} + k_I)(k_{34} + k_I)} \equiv \frac{g^2}{E E_L E_R}$$

$\longrightarrow$ Correlators are singular when energy is conserved.

Total Energy Singularity

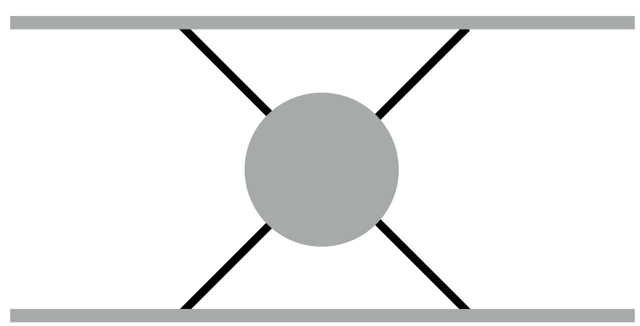
Every correlator has a singularity at vanishing total energy:

$$\psi_N =$$



$$\sim \int_{-\infty}^{0} \mathrm{d}t\, e^{iEt} f(t) = \frac{A_N}{E} + \cdots$$

$$A_N =$$



$$\sim \int_{-\infty}^{\infty} \mathrm{d}t\, e^{iEt} f(t) = M_N \delta(E)$$

- The residue of the total energy singularity is the corresponding amplitude.

Raju [2012]
Maldacena and Pimentel [2011]

Total Energy Singularity

- This is easy to confirm for the tree-exchange diagram:

$$\lim_{E\to 0} \frac{g^2}{E E_L E_R} = \frac{1}{E} \frac{g^2}{(k_{12}+k_I)(-k_{12}+k_I)} = -\frac{1}{E}\frac{g^2}{s} = \frac{A_4}{E}$$

where $s \equiv k_{12}^2 - k_I^2$ is the Mandelstam invariant.

- The total energy singularity arises from the early-time limit of the integration (where the boundary is infinitely far away):

$$\psi_N \sim \int_{-\infty}^{0} \mathrm{d}t\, e^{iEt} f(t) \;=\; \frac{A_N}{E} + \cdots$$

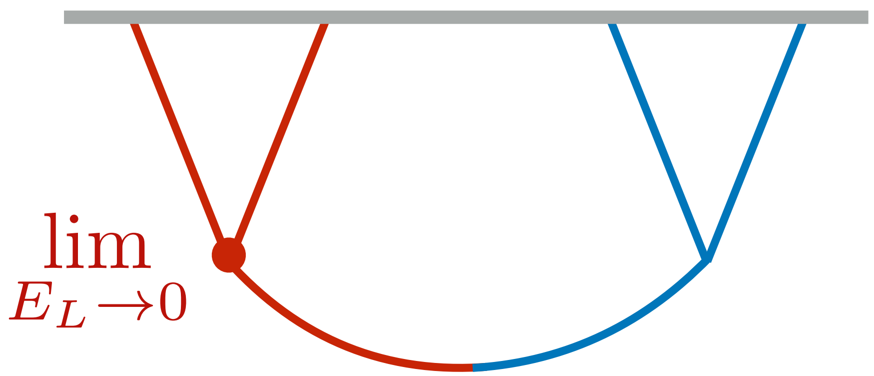
This explains intuitively why this singularity reproduces flat-space physics.

- The singularity can also be seen in the flat-space recursion relation for the wavefunction.

Arkani-Hamed, Benincasa and Postnikov [2017]

Partial Energy Singularities

Exchange diagrams lead to additional singularities:



$$= \quad \frac{A_L \times \tilde{\psi}_R}{E_L} + \cdots$$

- The singularity arises from the early-time limit of the integration, where the **bulk-to-bulk propagator factorizes**:

$$G_{k_I}(t_L, t_R) \xrightarrow{t_L \to -\infty} \frac{e^{ik_I t_L}}{2k_I}\left(e^{-ik_I t_R} - e^{+ik_I t_R}\right)$$

- On the pole, the correlator factorizes into a product of an **amplitude** and a **shifted correlator**:

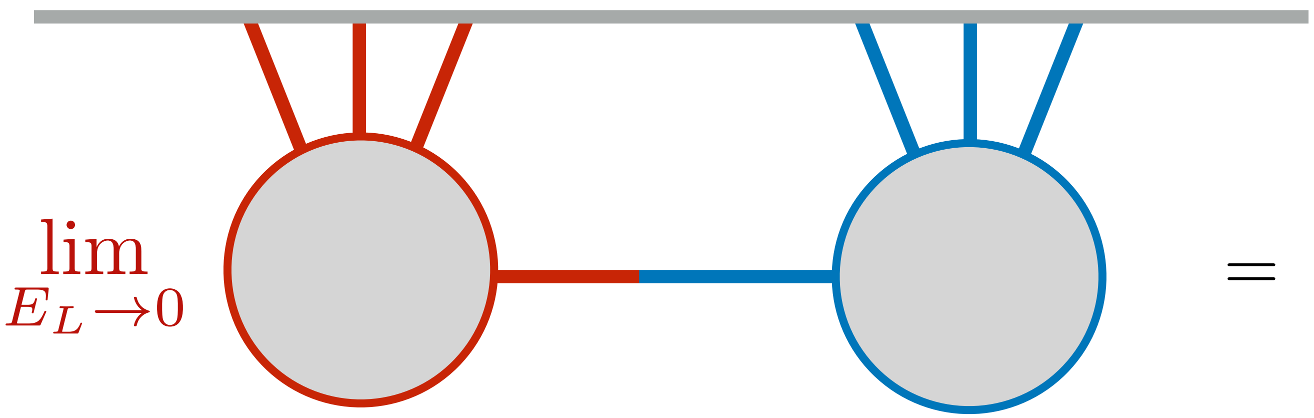
$$\boxed{\tilde{\psi}_R \equiv \frac{1}{2k_I}\Big(\psi_R(k_{34} - k_I) - \psi_R(k_{34} + k_I)\Big)}$$

Partial Energy Singularities

This pattern generalizes to **arbitrary graphs**.

- The wavefunction is singular when the energy flowing into any connected subgraph vanishes:

$$\lim_{E_L \to 0}$$



$$= \frac{A_L \times \tilde{\psi}_R}{E_L} + \cdots$$

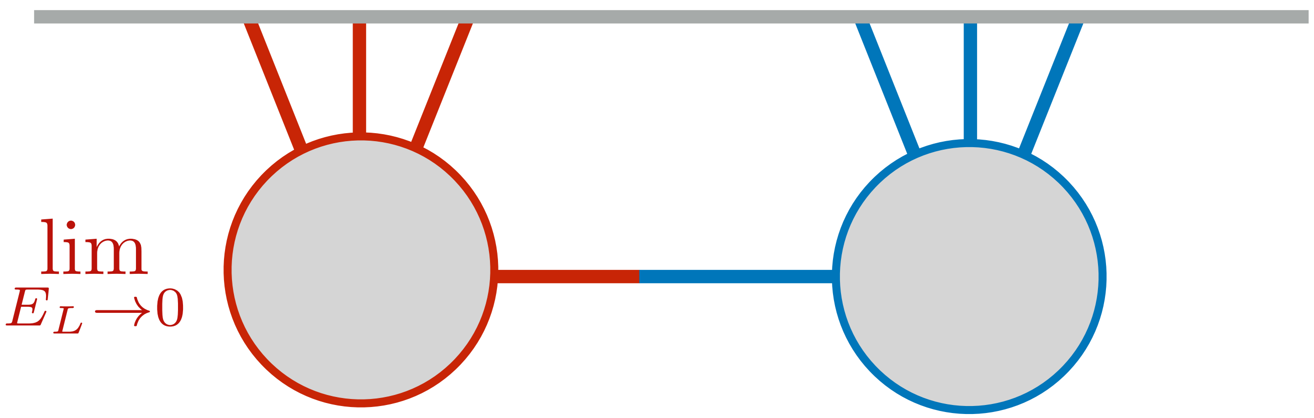
- For loops, the poles can become branch points.

Arkani-Hamed, Benincasa and Postnikov [2017]
DB, Chen, Duaso Pueyo, Joyce, Lee and Pimentel [2021]
Salcedo, Lee, Melville and Pajer [2022]

Generalization to De Sitter

The wavefunction in **de Sitter** also has singularities when the energy flowing into any connected subgraphs vanishes:

$\lim_{E_L \to 0}$



$$= \frac{A_L \times \tilde{\psi}_R}{E_L^P} + \cdots$$

- These singularities again arise from the early-time limit of the integration, where the bulk-to-bulk propagator factorizes.

- In these singular limits, the correlator factorizes into a product of an **amplitude** and a **shifted correlator**:

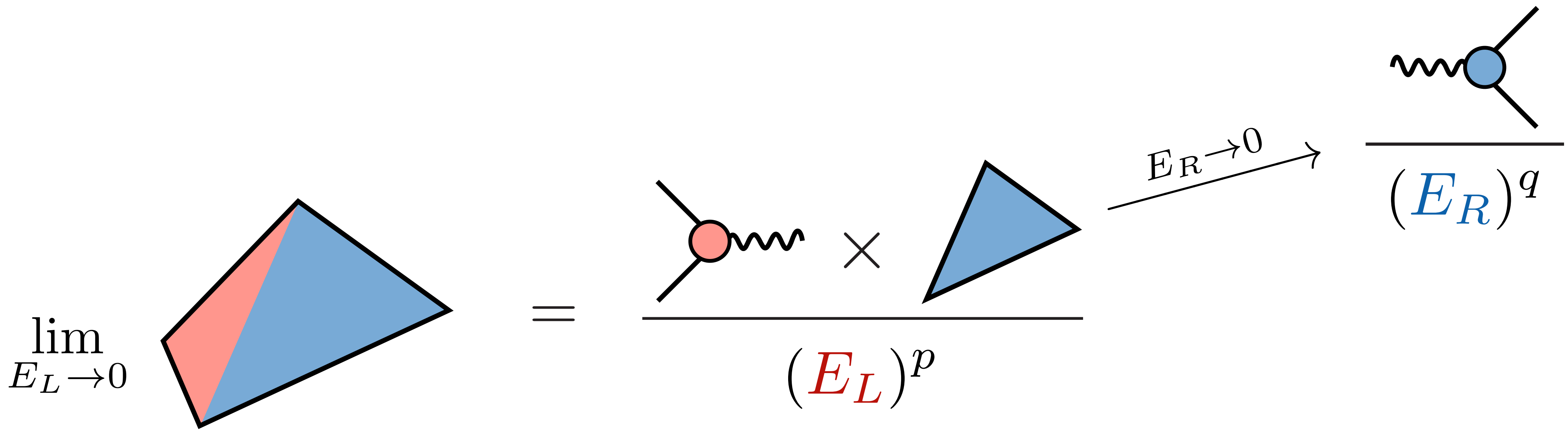
$$\boxed{\tilde{\psi}_R \equiv P(k_I)\Big(\psi_R(k_a, -k_I) - \psi_R(k_a, k_I)\Big)}$$

Power spectrum of exchanged field

Singularities as Input

The singularities of the wavefunction only arise at unphysical kinematics. Nevertheless, they control the structure of cosmological correlators.

- Amplitudes are the **building blocks** of cosmological correlators:



- To construct the full correlator, we need to connect its singularities. We will present two ways of doing this:

1) **Ward identities**
2) **Unitarity**

Bootstrapping Tools

Symmetries	Singularities	**Unitarity**

Unitarity is a fundamental feature of quantum mechanics.

However, the constraints of **bulk unitary** on cosmological correlators were understood only very recently.

Goodhew, Jazayeri and Pajer [2020]
Meltzer and Sivaramakrishnan [2020]

S-matrix Optical Theorem

A consistent theory must have a unitary S-matrix: $S^\dagger S = 1$

Writing $S = 1 + iT$, this implies

$$T - T^\dagger = i T^\dagger T$$

Sandwiching this between the states $\langle f|$ and $|i\rangle$, and inserting a complete set of states, we get

$$\langle f|T|i\rangle - \langle i|T|f\rangle^* = i \sum_X \int \mathrm{d}\Pi_X \, \langle f|T^\dagger|X\rangle\langle X|T|i\rangle$$

Using $\langle f|T|i\rangle \equiv (2\pi)^4\delta(p_i - p_f)A(i \to f)$, we obtain

$$\boxed{2\,\mathrm{Im}\,A(i \to f) = \sum_X \int \mathrm{d}\Pi_X \, (2\pi)^4\delta(p_i - p_X)\, A(i \to X)A^*(f \to X)}$$

The right-hand side can be written in terms of the total cross section, giving the familiar form of the **optical theorem**.

Cosmological Optical Theorem

Recently, an analogous result was derived for cosmological correlators.

Goodhew, Jazayeri and Pajer [2020]

Recall that $\Psi[\Phi] = \langle \Phi | \hat{U} | 0 \rangle$, where

$$\hat{U} = T \exp\left(-i \int_{-\infty}^{\eta_*} \mathrm{d}\eta \, \hat{H}_{\mathrm{int}} \right)$$

Probabilities, $|\Psi(\eta)|^2$, are conserved if $U^\dagger U = 1$.

Writing $\hat{U} = 1 + \hat{V}$, this implies

$$\hat{V} + \hat{V}^\dagger = -\hat{V}^\dagger \hat{V}$$

Sandwiching this between $\langle \Phi |$ and $| 0 \rangle$, and inserting a complete set of states, we get

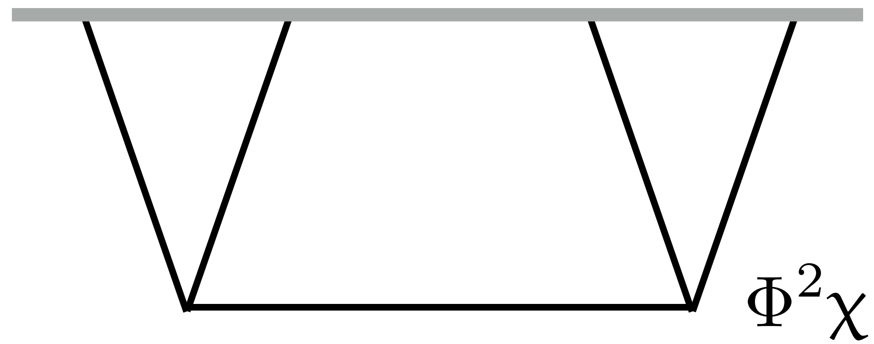
$$\boxed{\langle \Phi | \hat{V} | 0 \rangle + \langle \Phi | \hat{V}^\dagger | 0 \rangle = - \int \mathcal{D}X \, \langle \Phi | \hat{V}^\dagger | X \rangle \langle X | \hat{V} | 0 \rangle}$$

Cosmological Optical Theorem

To use this, we need to relate the matrix elements to wavefunction coefficients.

Example: Four-Point Exchange

As a concrete example, we consider the **four-point exchange diagram**:



To extract a constraint on the WF coefficient from the COT, we let $|\Phi\rangle$ be the momentum eigenstate $|\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4\rangle \equiv |\{\mathbf{k}_a\}\rangle$:

$$\boxed{\langle\{\mathbf{k}_a\}|\hat{V}|0\rangle + \langle\{\mathbf{k}_a\}|\hat{V}^\dagger|0\rangle = -\sum_X \int \prod_{b\in X} \frac{\mathrm{d}^3\mathbf{q}_b}{(2\pi)^3} \langle\{\mathbf{k}_a\}|\hat{V}^\dagger|\{\mathbf{q}_b\}\rangle\langle\{\mathbf{q}_b\}|\hat{V}|0\rangle}$$

where $\quad \hat{V} = -i\int_{-\infty}^{\eta_*} \mathrm{d}\eta\, \hat{H}_{\mathrm{int}}(\eta) - \int_{-\infty}^{\eta_*} \mathrm{d}\eta \int_{-\infty}^{\eta} \mathrm{d}\eta'\, \hat{H}_{\mathrm{int}}(\eta)\hat{H}_{\mathrm{int}}(\eta') + \cdots\,.$

Our task is to relate the matrix elements in this expression to WF coefficients.

Example: Four-Point Exchange

$$\boxed{\langle\{\mathbf{k}_a\}|\hat{V}|0\rangle + \langle\{\mathbf{k}_a\}|\hat{V}^\dagger|0\rangle = -\sum_X \int \prod_{b\in X} \frac{\mathrm{d}^3\mathbf{q}_b}{(2\pi)^3} \langle\{\mathbf{k}_a\}|\hat{V}^\dagger|\{\mathbf{q}_b\}\rangle\langle\{\mathbf{q}_b\}|\hat{V}|0\rangle}$$

- Substituting the mode expansions

$$\hat{\Phi}_{\mathbf{k}}(\eta) = f_k^*(\eta)\,\hat{a}_{\mathbf{k}} + f_k(\eta)\,\hat{a}_{-\mathbf{k}}$$
$$\hat{\chi}_{\mathbf{k}}(\eta) = \chi_k^*(\eta)\,\hat{b}_{\mathbf{k}} + \chi_k(\eta)\,\hat{b}_{-\mathbf{k}}$$

the **first term** on the left-hand side becomes

$$\langle\{\mathbf{k}_a\}|\hat{V}|0\rangle = -g^2 \int \mathrm{d}\eta\mathrm{d}\eta'\, a^4(\eta)a^4(\eta')\, G_F(k_I;\eta,\eta') \prod_{a=1}^{4} f_{k_a}(\eta)$$

where G_F is the Feynman propagator.

Example: Four-Point Exchange

$$\boxed{\langle \{\mathbf{k}_a\} | \hat{V} | 0 \rangle + \langle \{\mathbf{k}_a\} | \hat{V}^\dagger | 0 \rangle = - \sum_X \int \prod_{b \in X} \frac{\mathrm{d}^3 \mathbf{q}_b}{(2\pi)^3} \langle \{\mathbf{k}_a\} | \hat{V}^\dagger | \{\mathbf{q}_b\} \rangle \langle \{\mathbf{q}_b\} | \hat{V} | 0 \rangle}$$

Substituting

$$G_F(k;\eta,\eta') = G(k;\eta,\eta') + \frac{\chi_k^*(\eta_*)}{\chi_k(\eta_*)} \chi_k(\eta)\chi_k(\eta')$$

we get

$$\langle \{\mathbf{k}_a\} | \hat{V} | 0 \rangle = \Big(\psi_4(k_a, k_I) - \psi_3(k_{12}, k_I)\psi_3(k_{34}, k_I)\, P_\chi(k_I) \Big) \left(\prod_{a=1}^4 f_{k_a}(\eta_*) \right).$$

- A similar analysis for the **second term** leads to

$$\langle \{\mathbf{k}_a\} | \hat{V}^\dagger | 0 \rangle = \Big(\big[\psi_4(-k_a, k_I)\big]^* - \psi_3(k_{12}, -k_I)\psi_3(k_{34}, -k_I)\, P_\chi(k_I) \Big) \left(\prod_{a=1}^4 f_{k_a}(\eta_*) \right)$$

Analytic continuation

Example: Four-Point Exchange

$$\boxed{\langle\{\mathbf{k}_a\}|\hat{V}|0\rangle + \langle\{\mathbf{k}_a\}|\hat{V}^\dagger|0\rangle = -\sum_X \int \prod_{b\in X} \frac{\mathrm{d}^3\mathbf{q}_b}{(2\pi)^3} \langle\{\mathbf{k}_a\}|\hat{V}^\dagger|\{\mathbf{q}_b\}\rangle\langle\{\mathbf{q}_b\}|\hat{V}|0\rangle}$$

- Finally, the **third term** leads to

$$\text{RHS} = \Big[\psi_3(k_{12},-k_I)\psi_3(k_{34},k_I) + \psi_3(k_{12},k_I)\psi_3(k_{34},-k_I)\Big] P_\chi(k_I)\left(\prod_{a=1}^{4} f_{k_a}(\eta_*)\right)$$

- Putting it all together, we get

$$\boxed{\psi_4(k_a,k_I) + \big[\psi_4(-k_a,k_I)\big]^* = -P_\chi(k_I)\,\tilde{\psi}_3(k_{12},k_I)\tilde{\psi}_3(k_{34},k_I)}$$

where $\tilde{\psi}_3(k_{12},k_I) \equiv \psi_3(k_{12},k_I) - \psi_3(k_{12},-k_I)$.

Cutting Rules

For more complicated graphs, the explicit evaluation of the COT quickly becomes cumbersome. A simpler alternative are **cutting rules**.

We start with some analytic properties of the propagators defining the perturbative wavefunction:

- Both propagators decay at past infinity, which we can enforce by giving the norm k a negative imaginary part: $k^2 = re^{i\theta}$, with $\theta \in (-2\pi, 0)$.

- A useful concept is the **discontinuity** of a function

$$\mathrm{Disc}[f(k)] \equiv f(k) - f(e^{-i\pi}k)$$

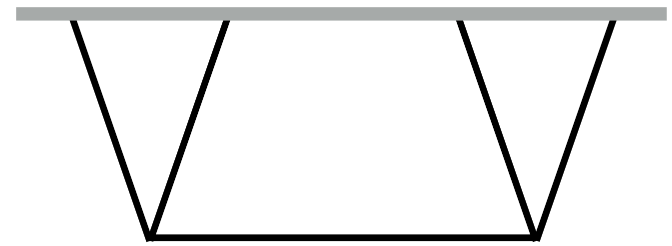
- The discontinuity of the **bulk-to-bulk propagator** is

$$\boxed{\mathrm{Disc}[G(k;\eta,\eta')] = -P(k)\,\mathrm{Disc}[K(k,\eta)] \times \mathrm{Disc}[K(k,\eta')]}$$

Cutting Rules

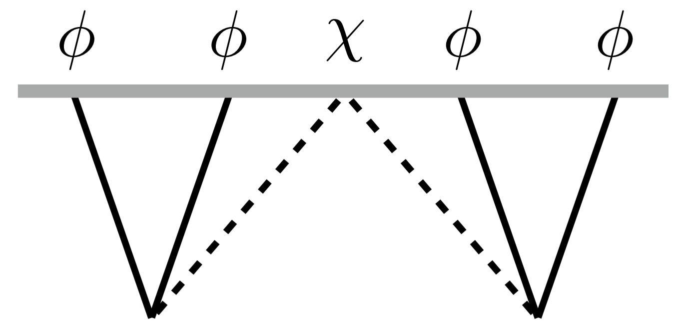
Consider, the four-point exchange diagram

$$\psi_4(k_a, k_I) \;=$$



Taking the discontinuity (with respect to k_I) corresponds to cutting the internal line:

$$\mathrm{Disc}_{k_I}\left[\psi_4(k_a, k_I)\right] \;=\; P_\chi(k)$$



Hence, we find

$$\boxed{\mathrm{Disc}_{k_I}\left[\psi_4(k_a, k_I)\right] = P_\chi(k_I)\,\mathrm{Disc}_{k_I}\left[\psi_3(k_{12}, k_I)\right] \times \mathrm{Disc}_{k_I}\left[\psi_3(k_{34}, k_I)\right]}$$

which is equivalent to the result derived from the COT.

Cutting Rules

This procedure generalizes to systematic **cutting rules**:

$$\mathrm{Disc}\big[\psi_N\big] = \sum \text{all possible cuts}$$

These are the analog of the Cutkosky cutting rules for the S-matrix.

More details can be found in the lecture notes and in the relevant literature.

Goodhew, Jazayeri and Pajer [2020]

Meltzer and Sivaramakrishnan [2020]

Caspedes, Davis and Melville [2020]

Goodhew, Jazayeri, Lee and Pajer [2021]

DB, Chen, Duaso Pueyo, Joyce, Lee and Pimentel [2021]

Melville and Pajer [2021]

Sleight and Taronna [2021]

Jazayeri, Pajer and Stefanyszyn [2021]

Meltzer [2021]

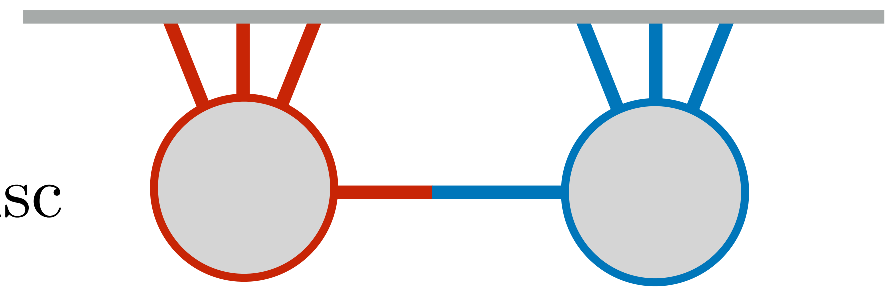
Benincasa [2022]

Unitarity as Input

- The discontinuity of a WF coefficient factorizes into lower-point objects.

- This contains slightly more information that the energy singularities:

$$\lim_{E_L \to 0} \;\text{Disc}$$



$$= \lim_{E_L \to 0} \Big[\psi_N(k_a, k_I) - \psi_N(k_a, -k_I) \Big] = \lim_{E_L \to 0} \psi_N(k_a, k_I)$$

has no singularities in E_L

The cutting rule fixes the entire Laurent series in E_L.

- These unitarity cuts have become an important bootstrapping tool.

Examples

Massive Exchange

Dispersive Integrals

FRW Correlators

Examples

Massive Exchange

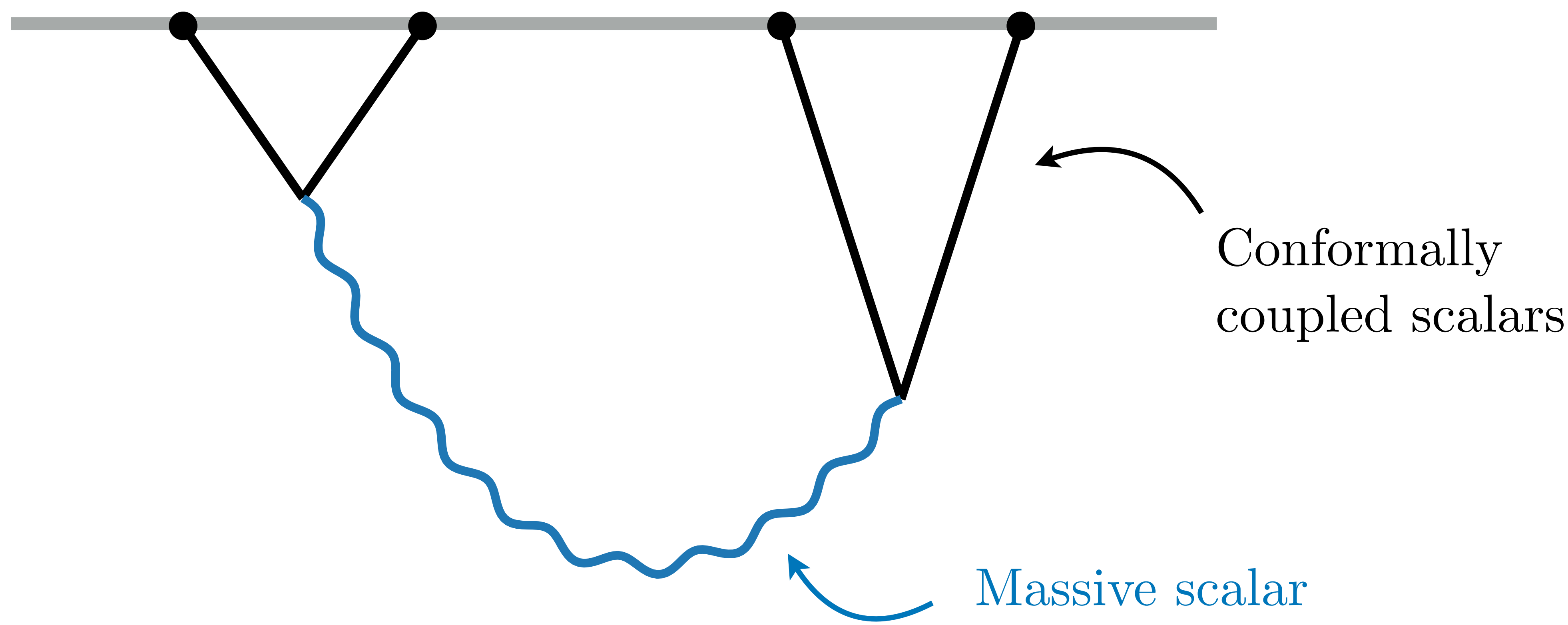
Dispersive Integrals

FRW Correlators

Exchange Four-Point Function

We are now ready to return to the challenge of massive particle exchange.

$$\langle O_1 O_2 O_3 O_4 \rangle' =$$



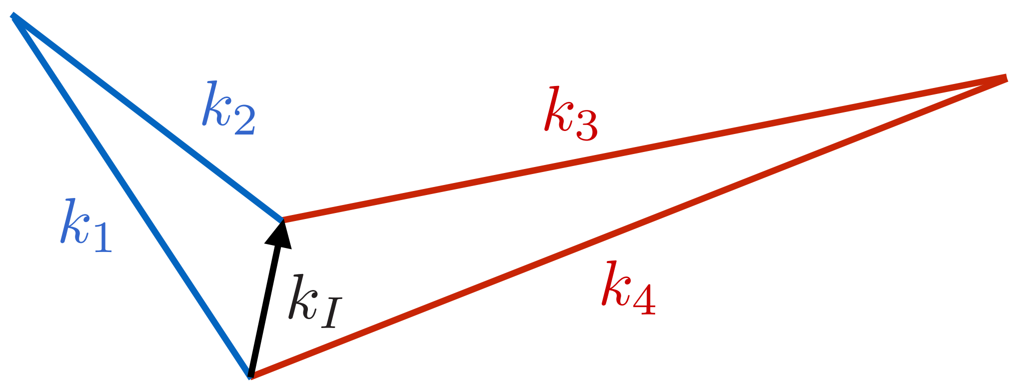
$$= \; -g^2 \int \frac{\mathrm{d}\eta}{\eta^2} \int \frac{\mathrm{d}\eta'}{\eta'^2}\, e^{ik_{12}\eta} e^{ik_{34}\eta'}\, G(k_I; \eta, \eta'')$$

In general, the time integrals cannot be performed analytically.

Conformal Ward Identity

Recall that the s-channel correlator can be written as

$$\langle O_1 O_2 O_3 O_4 \rangle' = $$



$$= \frac{1}{k_I} \hat{F}(u, v)$$

where $u \equiv k_I/(k_1 + k_2)$ and $v \equiv k_I/(k_3 + k_4)$.

We have seen that the **conformal symmetry** implies the following differential equation

$$\boxed{(\Delta_u - \Delta_v)\hat{F} = 0}$$

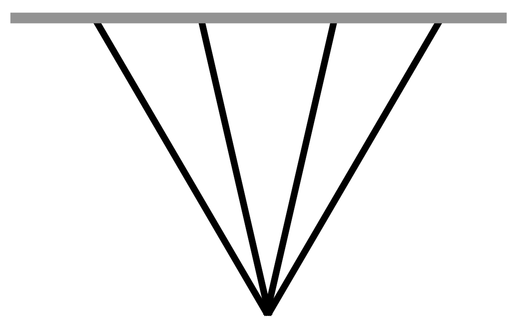
where $\Delta_u \equiv u^2(1 - u^2)\partial_u^2 - 2u^3\partial_u$.

$\longrightarrow$ We will classify solutions to this equation by their **singularities**.

Contact Solutions

The simplest solutions correspond to **contact interactions**:

$\hat{F}_c =$



$$= \sum_n \frac{c_n(u,v)}{E^{2n+1}}$$

Fixed by symmetry

Only total energy singularities

- For Φ^4, we have $\hat{F}_{c,0} = \dfrac{k_I}{E} = \dfrac{uv}{u+v}$.
- Higher-derivative bulk interactions, lead to $\hat{F}_{c,n} = \Delta_u^n \hat{F}_{c,0}$.

$$\hat{F}_{c,1} = -2\left(\frac{k_I}{E}\right)^3 \frac{1+uv}{uv}\,,$$
$$\hat{F}_{c,2} = -4\left(\frac{k_I}{E}\right)^5 \frac{u^2+v^2+uv(3u^2+3v^2-4)-6(uv)^2-6(uv)^3}{(uv)^3}\,.$$

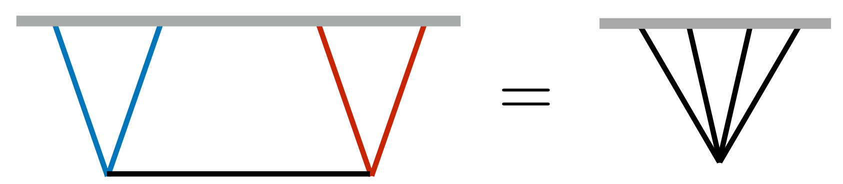
This can get relatively complex, but everything is fixed by symmetry.

Exchange Solutions

For tree exchange, we try

$$\boxed{\begin{aligned}(\Delta_u + M^2)\hat{F}_e &= \hat{C} \\ (\Delta_v + M^2)\hat{F}_e &= \hat{C}\end{aligned}}$$

$(\Delta_u + M^2)$
$(\Delta_v + M^2)$

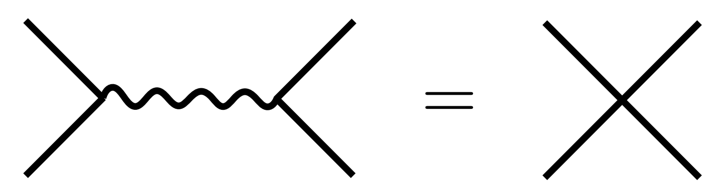


where $\hat{C} = M^2 \hat{F}_c$ is a contact solution and $M^2 \equiv m^2/H^2 - 2$.

For amplitudes, the analog of this is

$$\boxed{(s + m^2)A_e = A_c(s,t)}$$

$(s + m^2)$



This is an algebraic relation, while for correlators we have a differential equation.

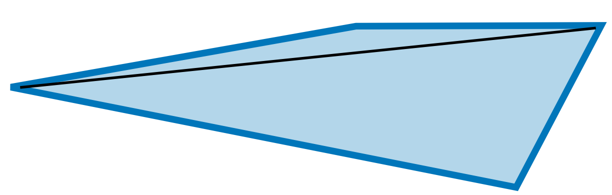
Exchange Solutions

Using the simplest contact interaction as a source, we have

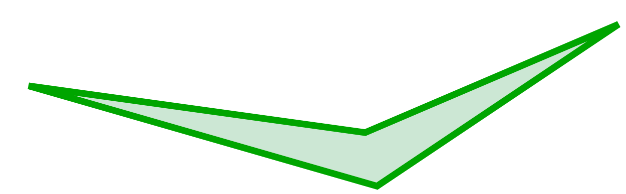
$$\boxed{\left[u^2(1-u^2)\partial_u^2 - 2u^3\partial_u + M^2\right]\hat{F} = \frac{uv}{u+v}}$$

Before solving this equation, let us look at its **singularities**.

- **Flat-space limit**: $\lim_{u\to -v} \hat{F} \propto A_4(u+v)\log(u+v)$
- **Factorization limit**: $\lim_{u,v\to -1} \hat{F} \propto A_3 \log(1+u) \times A_3 \log(1+v)$
- **Folded limit**: $\lim_{u\to +1} \hat{F} \propto \log(1-u)$
- **Collapsed limit**: $\lim_{u\to 0} \hat{F} \propto u^{iM}$



This singularity should be absent in the Bunch-Davies vacuum.



This non-analyticity is a key signature of particle production.

Exchange Solutions

Using the simplest contact interaction as a source, we have

$$\boxed{\left[u^2(1-u^2)\partial_u^2 - 2u^3\partial_u + M^2\right]\hat{F} = \frac{uv}{u+v}}$$

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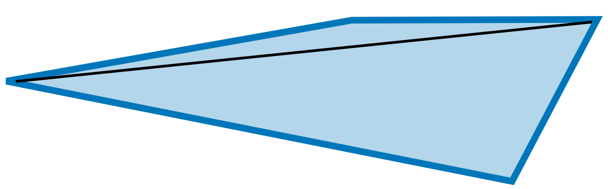
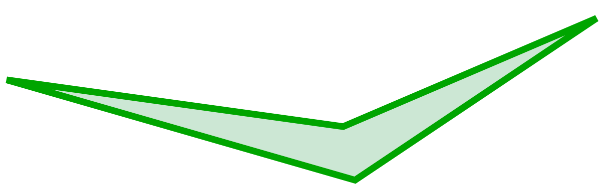




→ Factorization and folded limits uniquely fix the solution.
The remaining singularities become consistency checks.

Forced Harmonic Oscillator

In the limit of small internal momentum ($u < v \ll 1$), the equation becomes

$$\boxed{\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} + M^2\right]\hat{F} = \frac{1}{\cosh t}} \qquad t \equiv \log(u/v)$$

whose solution is

$$\boxed{\hat{F} = \underbrace{\sum_n \frac{(-1)^n}{(n+\frac{1}{2})^2 + M^2}\left(\frac{u}{v}\right)^{n+1}}_{\text{EFT expansion}} + \underbrace{\frac{\pi}{\cosh(\pi M)}\frac{\sin(M\log(u/v))}{M}}_{\text{particle production}}}$$

analytic

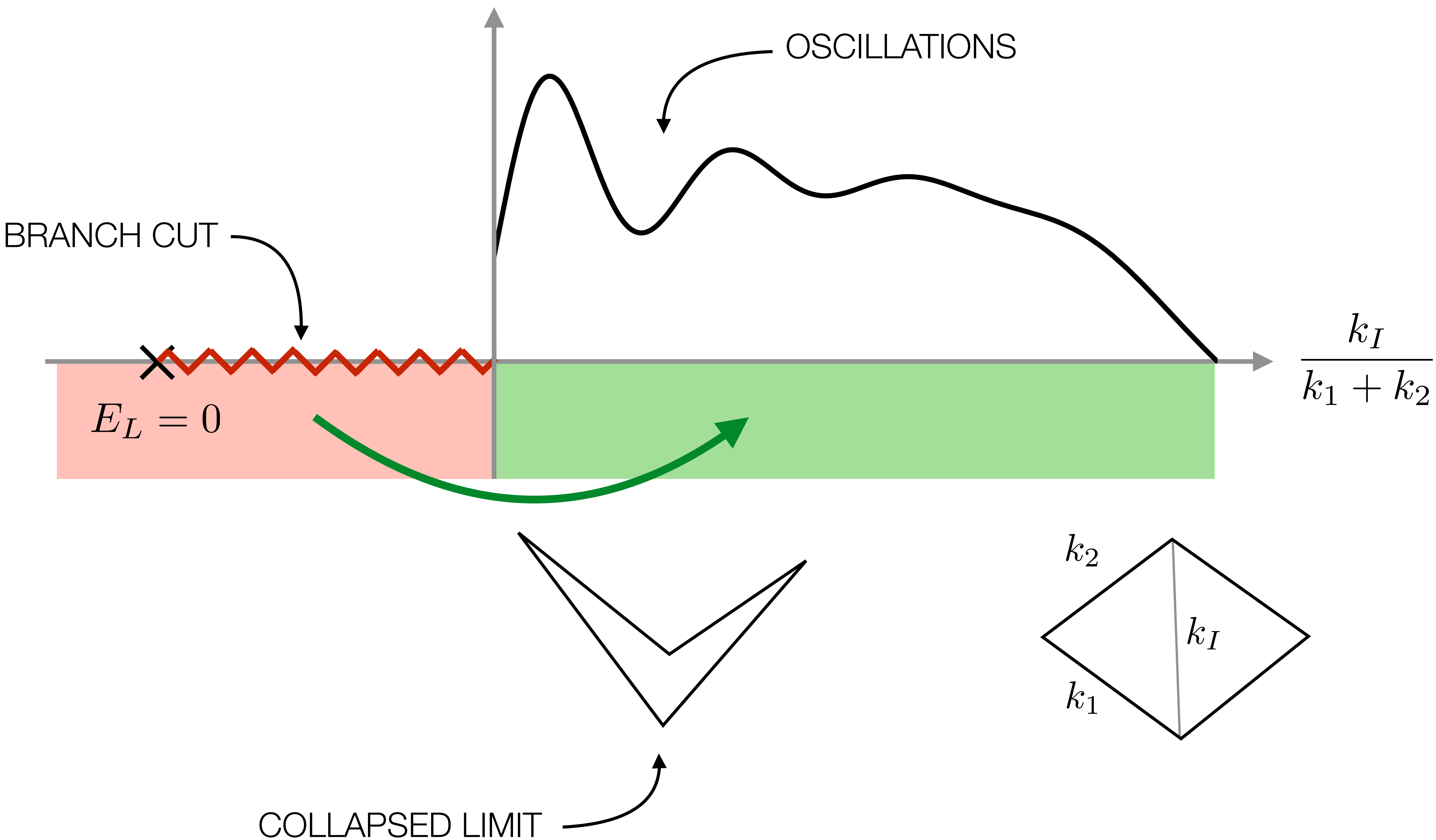
non-analytic

The general solution takes a similar form.

Arkani-Hamed, DB, Lee and Pimentel [2018]

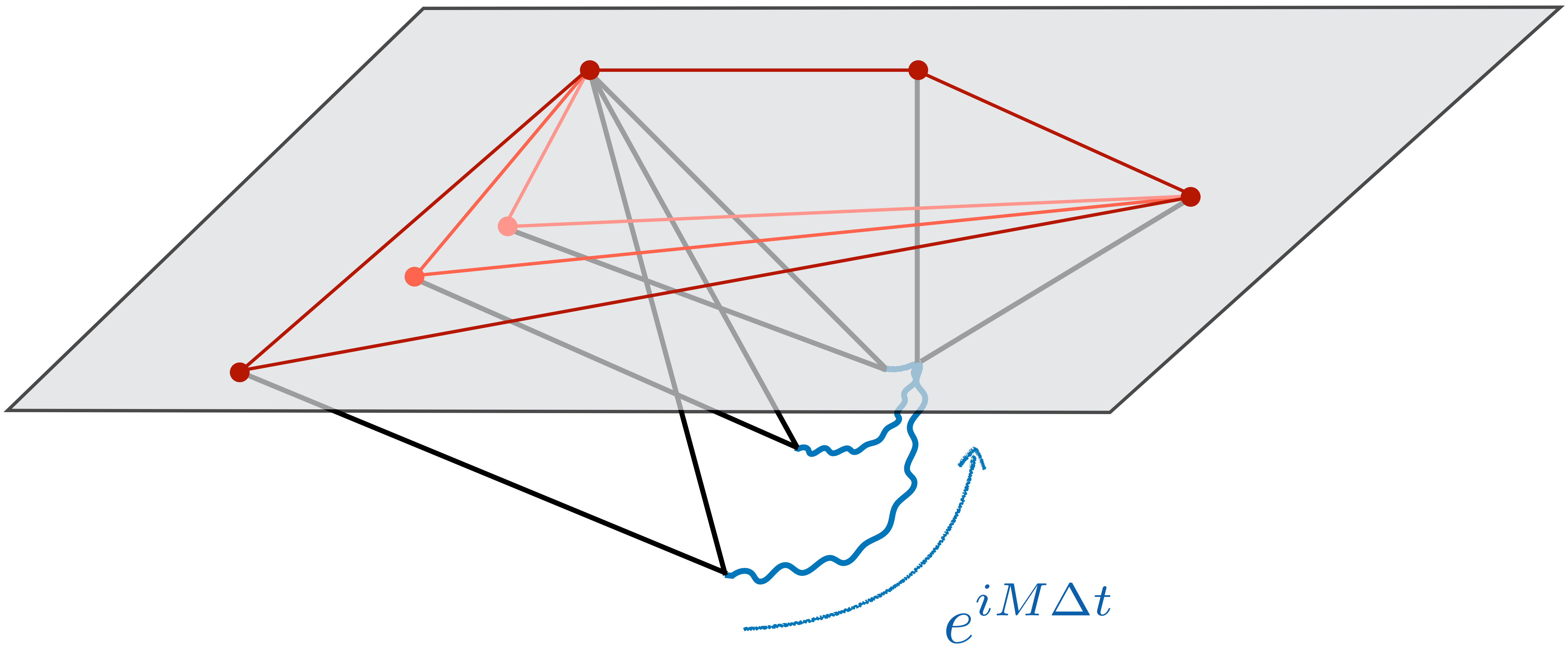
Oscillations

The correlator is forced to have an oscillatory feature in the physical regime:



Particle Production

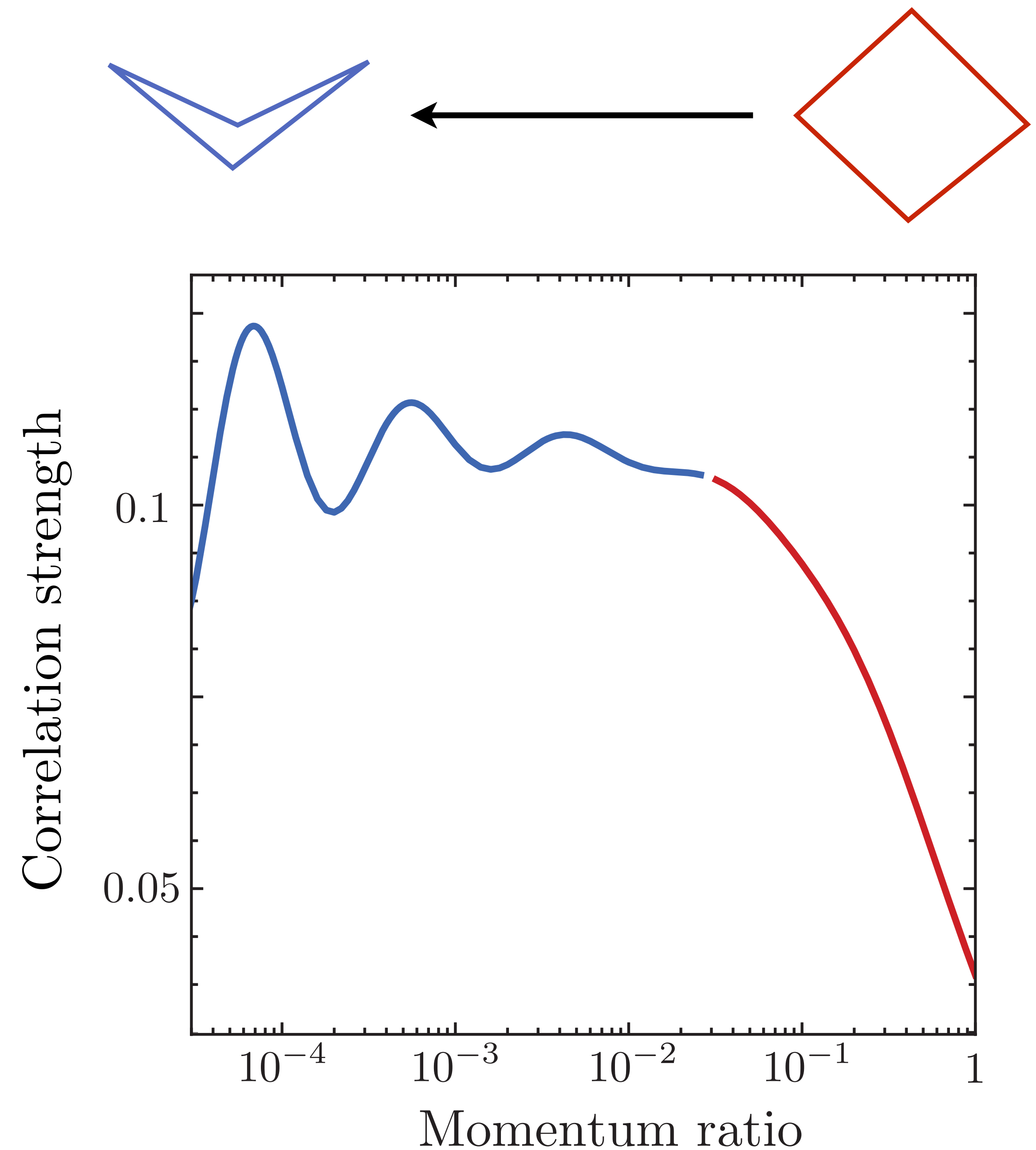
These oscillations reflect the evolution of the massive particles during inflation:

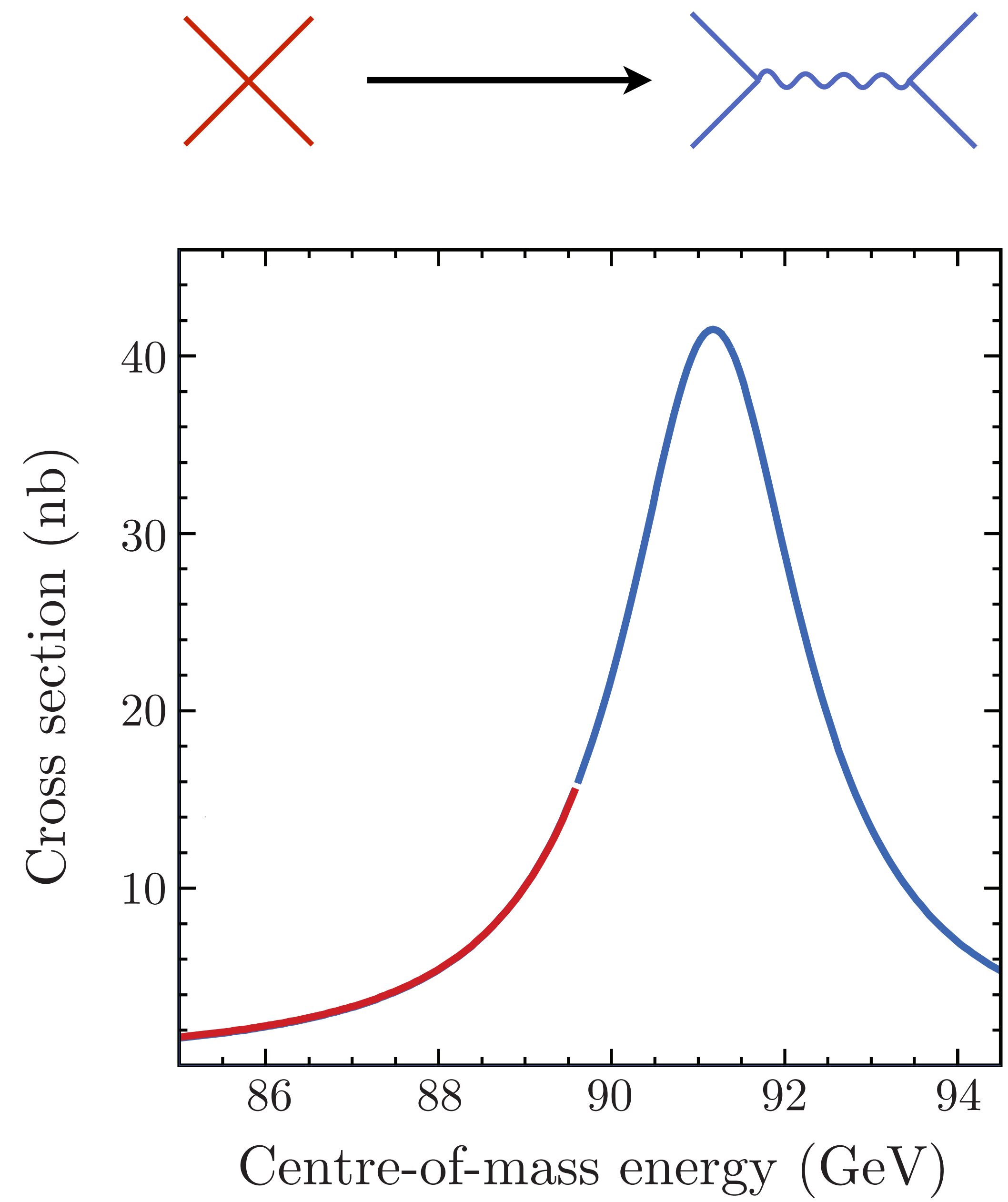


Time-dependent effects have emerged in the solution of the time-independent bootstrap constraints.

Cosmological Collider Physics

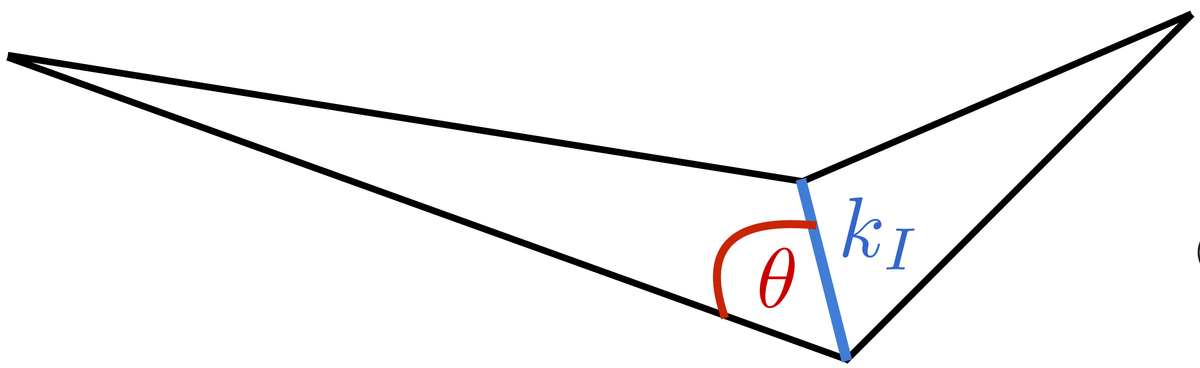
The oscillatory feature is the analog of a **resonance** in collider physics:





Particle Spectroscopy

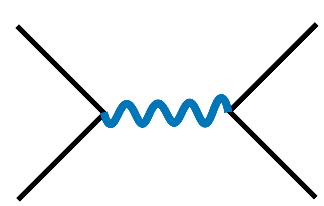
The frequency of the oscillations depends on the **mass** of the particles:



$$\propto \sin[M \log k_I]\, P_J(\cos\theta)$$

The angular dependence of the signal depends on the **spin** of the particles.

This is very similar to what we do in collider physics:



$$= \frac{g^2}{s - M^2}\, P_J(\cos\theta)$$

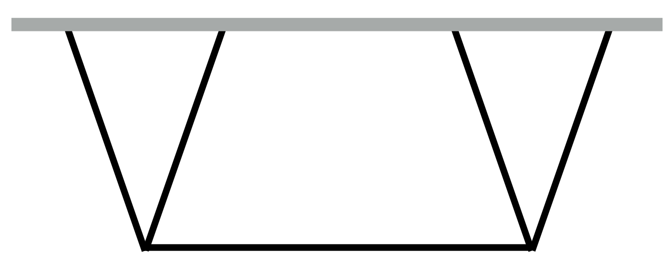
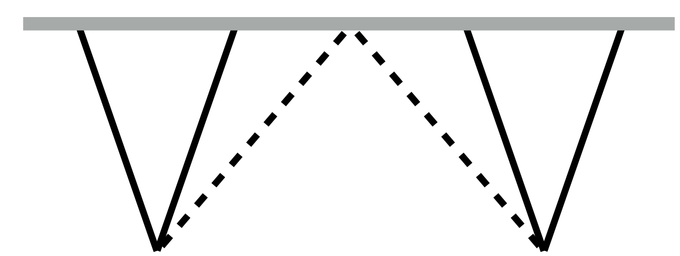
Examples

Massive Exchange

Dispersive Integrals

FRW Correlators

We have seen that the discontinuity of a wavefunction coefficient is related to lower-point objects:

Disc_{k_I}  $=$ $P(k)$ 

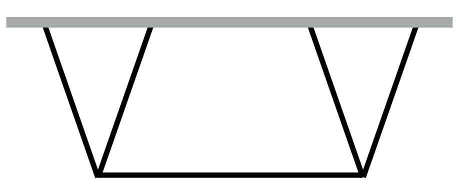
For rational correlators, we can “integrate” the discontinuity to get the full correlator.

Dispersive Integral

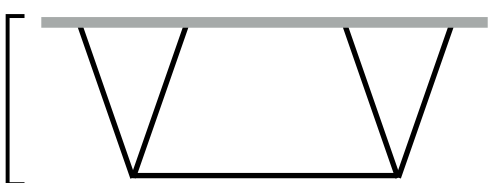
By causality, the **bulk-to-bulk propagator** is an analytic function of k^2 that decays at large k. It can therefore be written as

$$G(k;\eta,\eta') = \frac{1}{2\pi i}\int_0^\infty \frac{\mathrm{d}q^2}{q^2-k^2+i\varepsilon}\,\mathrm{Disc}[G(k;\eta,\eta')]$$

This can be used to integrate the discontinuities of tree-level WF coefficients. As an example, we consider the four-point exchange diagram:



$$= \frac{1}{2\pi i}\int_0^\infty \frac{\mathrm{d}q^2}{q^2-k_I^2+i\varepsilon}\,\mathrm{Disc}_q\Bigg[$$



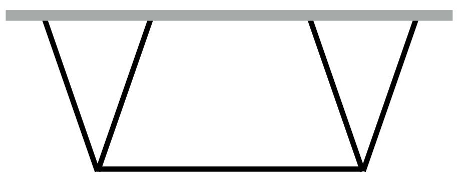
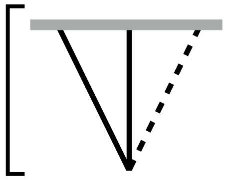
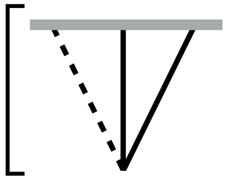
$$\Bigg]$$

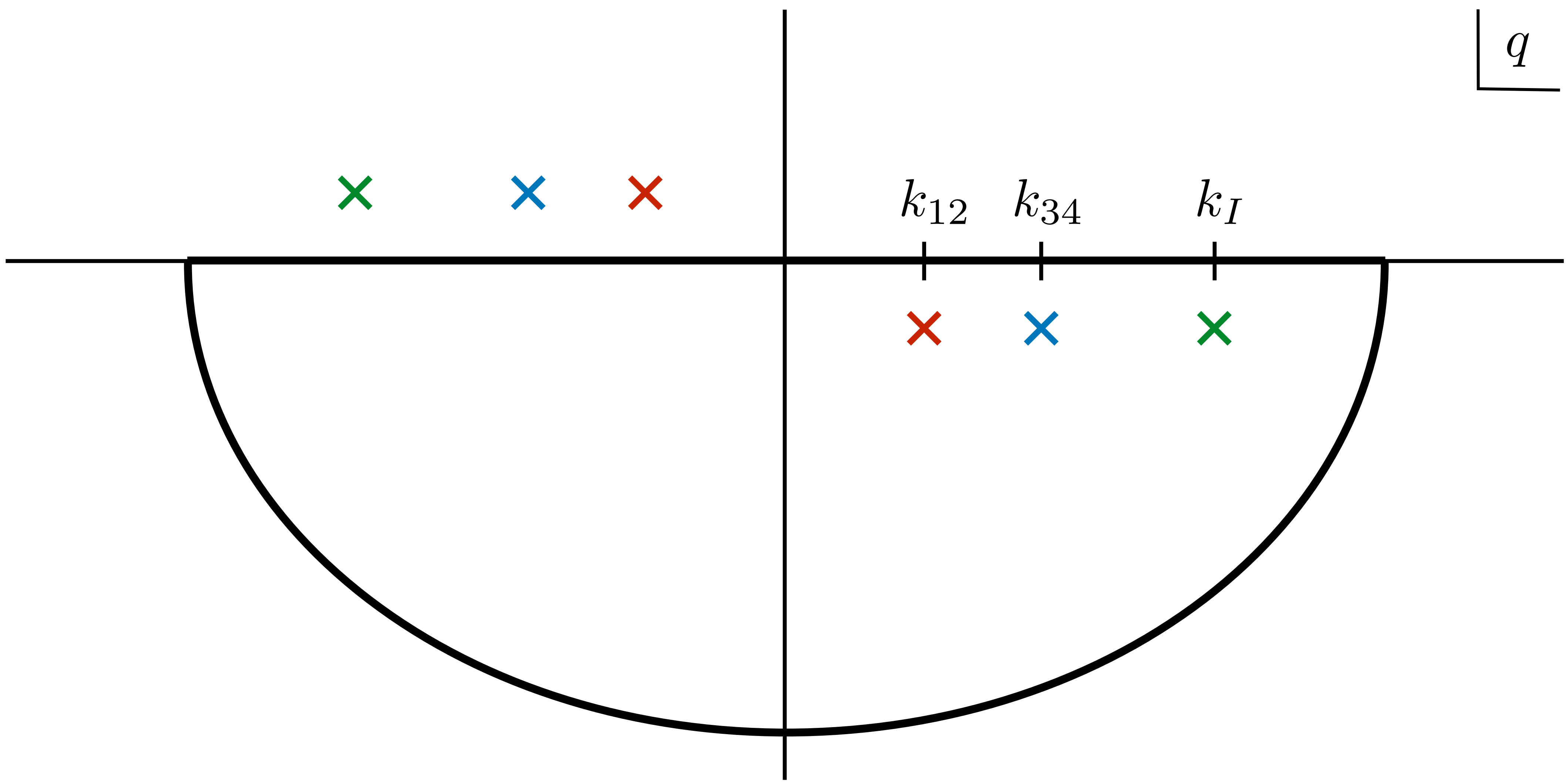
Meltzer [2021]

Often this **dispersive integral** is easier to compute than the bulk time integrals.

Dispersive Integral

Using the unitarity cut, this becomes

 $= \dfrac{1}{2\pi i}\displaystyle\int_0^\infty \dfrac{\mathrm{d}q^2}{q^2 - k_I^2 + i\varepsilon}\, P(q)\ \mathrm{Disc}_q\Big[$  $\Big]\ \mathrm{Disc}_q\Big[$  $\Big]$

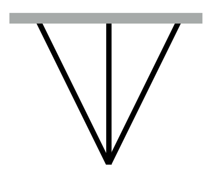


The full correlator can then be written as a sum over residues, using only three-point data.

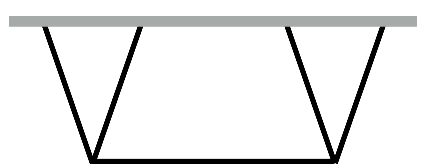
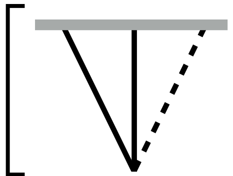
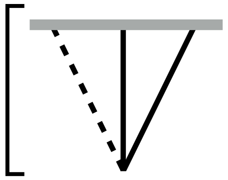
Conformally Coupled Scalar

As an example, let us reproduce the result for conformally coupled scalars.

- Input: three-point function

 $= \; i \log(k_1 + k_2 + k_3)$

- Discontinuity:

$\mathrm{Disc}_q\Big[$  $\Big] \; = \; P(q)\ \mathrm{Disc}_q\Big[$  $\Big]\,\mathrm{Disc}_q\Big[$  $\Big]$

$$= \frac{1}{2q} \log\left(\frac{k_{12} - q}{k_{12} + q}\right) \log\left(\frac{k_{34} - q}{k_{34} + q}\right)$$

$$= 4q \int_{k_{12}}^{\infty} \frac{\mathrm{d}x}{x^2 - q^2} \int_{k_{34}}^{\infty} \frac{\mathrm{d}y}{y^2 - q^2}$$

Conformally Coupled Scalar

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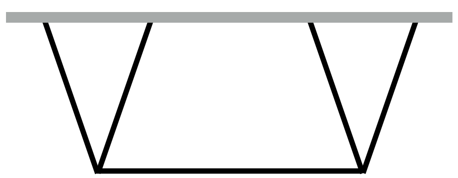
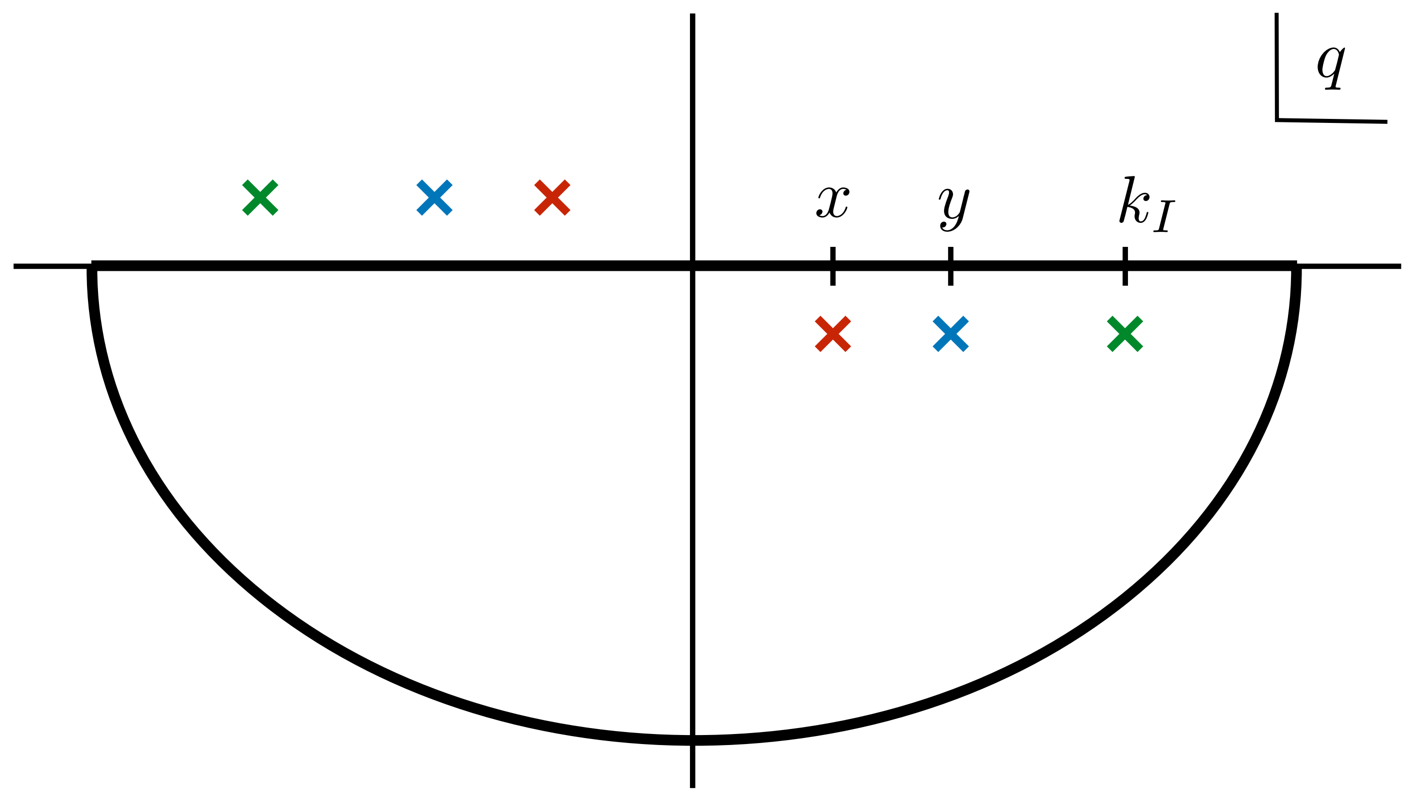
- Dispersive integral



$$= \int_{k_{12}}^{\infty} \mathrm{d}x \int_{k_{34}}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \frac{\mathrm{d}q}{\pi i} \, \frac{q^2}{q^2 - k_I^2} \, \frac{1}{x^2 - q^2} \, \frac{1}{y^2 - q^2}$$



$$= \int_{k_{12}}^{\infty} \mathrm{d}x \int_{k_{34}}^{\infty} \mathrm{d}y \, \frac{1}{(x+y)(x+k_I)(y+k_I)}$$

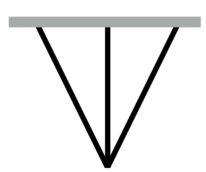
Correct!

EFT of Inflation

One of the important features of using unitarity cuts to bootstrap correlators is that it doesn't rely the interactions being approximately boost-invariant.

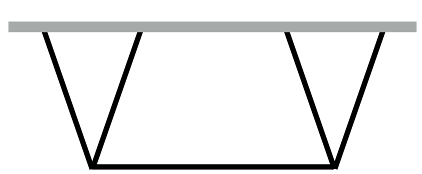
Exercise: Consider $\dot{\pi}^3$ in the EFT of inflation.

- The three-point wavefunction coefficient is



$$= \frac{(k_1 k_2 k_3)^2}{(k_1 + k_2 + k_3)^3}$$

- Using the dispersive integral, show that the exchange 4pt function is



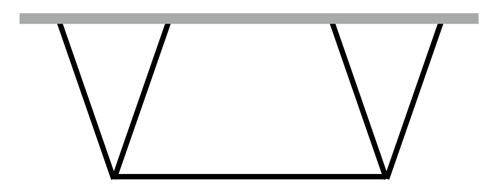
$$= \frac{(k_1 k_2 k_3 k_4)^2}{E^5 E_L^3 E_R^3} \Big[k_I^2 \Big(6 E_L^2 E_R^2 + 3 E E_L E_R (E_L + E_R) + E^2 (E_L + E_R)^2 + E^3 (E_L + E_R) + E^4 \Big) - 6 E_L^3 E_R^3 \Big]$$

Examples

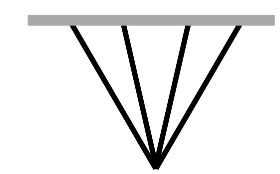
Massive Exchange	Dispersive Integrals	**FRW Correlators**

We have seen that cosmological correlators satisfy
interesting differential equations in kinematic space:

$$\left[u^2(1-u^2)\partial_u^2 - 2u^3\partial_u + \left(\mu^2 + \frac{1}{4}\right)\right]$$



$$=$$



We derived these equations from conformal symmetry,
but is there a deeper reason for their existence?

What classes of correlation functions can arise from consistent cosmological evolution? Surprisingly little is know about this in a systematic fashion.

In an upcoming paper, we will answer this question for conformal scalars in a power-law FRW background.

Arkani-Hamed, DB, Hillman, Joyce, Lee and Pimentel [to appear]

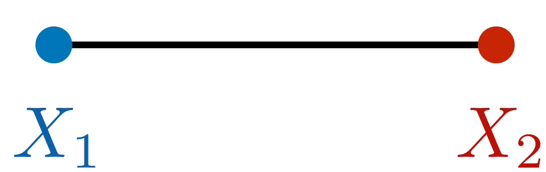
Conformal Scalars in dS

We have seen that the four-point function of conformal scalars in de Sitter space can be written as an integral of the flat-space result:

$$\boxed{\psi \equiv \langle O_1 O_2 O_3 O_4 \rangle \propto \int_{k_{12}}^{\infty} \mathrm{d}x_1 \int_{k_{34}}^{\infty} \mathrm{d}x_2 \, \frac{1}{(x_1+x_2)(x_1+k_I)(x_2+k_I)}}$$

$$\boxed{\psi = \int_0^{\infty} \mathrm{d}x_1 \mathrm{d}x_2 \, \frac{1}{(X_1+X_2+x_1+x_2)(X_1+x_1+1)(X_2+x_2+1)}}$$

where $X_1 \equiv k_{12}/k_I$ and $X_2 \equiv k_{34}/k_I$.



Conformal Scalars in FRW

An interesting generalization of this integral is

Twist

$$\psi = \int_0^\infty \mathrm{d}x_1 \mathrm{d}x_2 \, \frac{(x_1 x_2)^\varepsilon}{(X_1 + X_2 + x_1 + x_2)(X_1 + x_1 + 1)(X_2 + x_2 + 1)}$$

This describes the four-point function of conformal scalars in FRW backgrounds with scalar factor

$$a(\eta) = \left(\frac{\eta}{\eta_0}\right)^{-(1+\varepsilon)}$$

Similar integrals arise for **loop amplitudes** in dimensional regularization.

In that case, powerful mathematics can be used to evaluate these integrals:

- Twisted cohomology
- Canonical differential equations

We will show that a similar approach applies to our FRW correlators.

Family of Integrals

To attack the problem, we introduce a **family of integrals** with the same singularities (divisors):

$$I_{n_1 n_2 n_3 n_4 n_5} \equiv \int (x_1 x_2)^{\varepsilon}\, \Omega\,, \qquad \Omega \equiv \frac{\mathrm{d}x_1 \wedge \mathrm{d}x_2}{L_1^{n_1} L_2^{n_2} L_3^{n_3} L_4^{n_4} L_5^{n_5}}\,,$$

where

$$\begin{aligned} &L_1 \equiv x_1\,, \quad L_3 \equiv X_1 + x_1 + 1\,, \\ &L_2 \equiv x_2\,, \quad L_4 \equiv X_2 + x_2 + 1\,, \quad L_5 \equiv X_1 + X_2 + x_1 + x_2\,. \end{aligned}$$

- Not all of these integrals are independent because of integration-by-parts and partial fraction identities.

- How many independent **master integrals** are there?

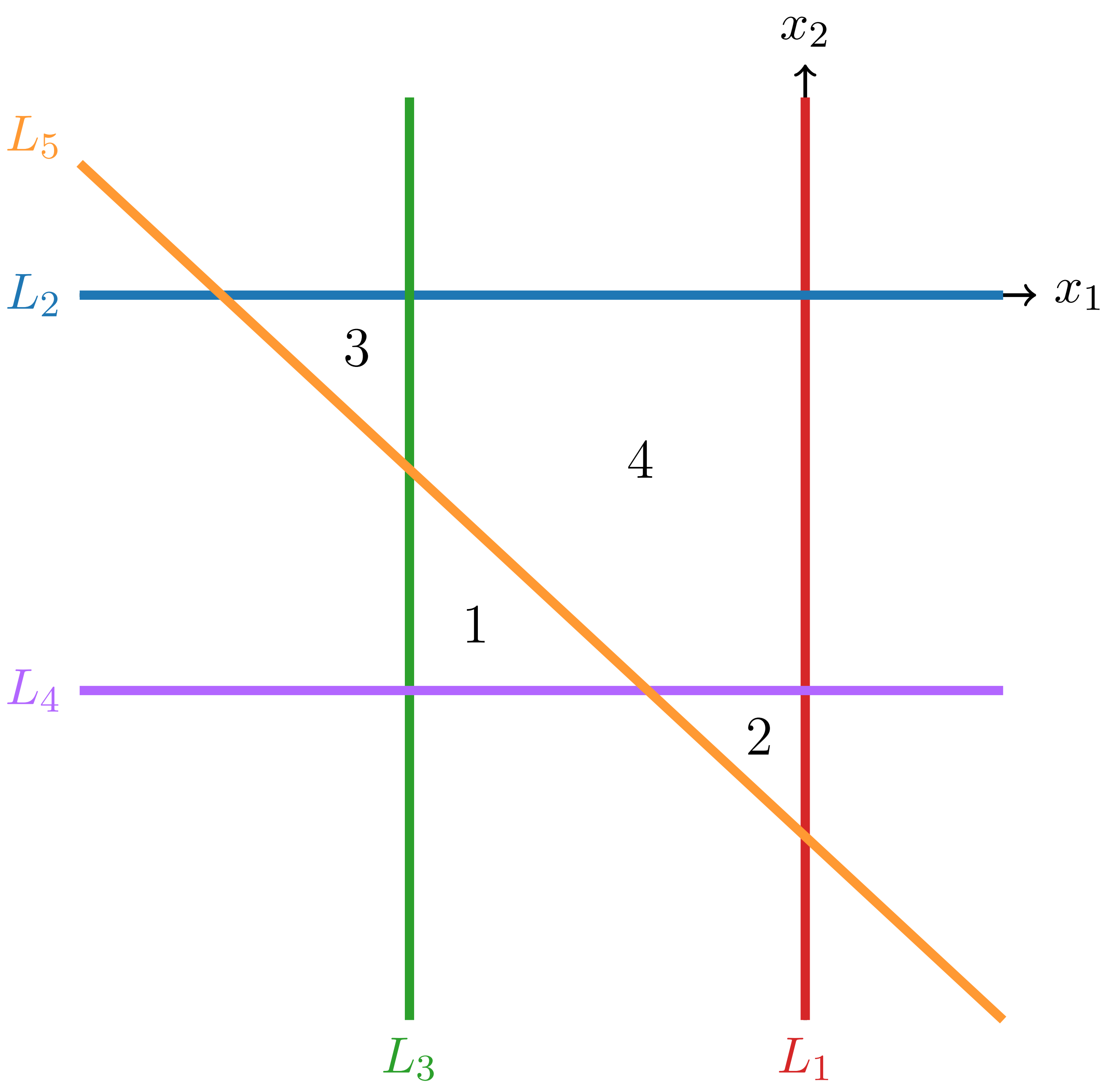
$\longrightarrow$ The answer is provided by a beautiful fact of **twisted cohomology**.

Aomoto [1975]

Mastrolia and Mizera [2018]

A Beautiful Fact

The number of independent master integrals is equal to the number of **bounded regions** defined by the divisors of the integrand.



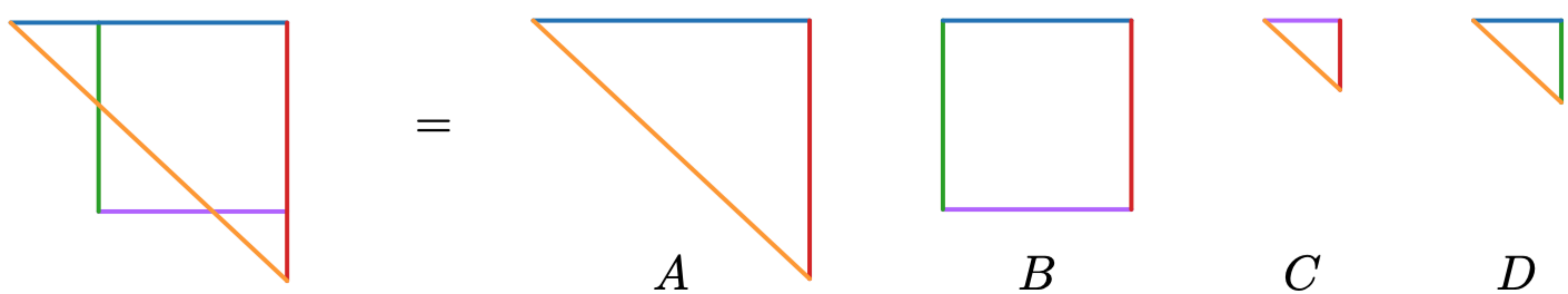
Master Integrals

The **vector space** of independent master integrals is four-dimensional:

$$\mathcal{I} \equiv \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \int (x_1 x_2)^{\varepsilon} \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{bmatrix}$$

Canonical forms

The basis integrals are not unique, but a convenience choice is



Master Integrals

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Canonical forms

The basis integrals are not unique, but a convenience choice is

$$\Omega_A = (X_1 + X_2)\frac{\mathrm{d}x_1 \wedge \mathrm{d}x_2}{L_1 L_2 L_5}\,, \qquad \Omega_C = -(X_1 - 1)\frac{\mathrm{d}x_1 \wedge \mathrm{d}x_2}{L_1 L_4 L_5}\,,$$

$$\Omega_B = -(X_1 + 1)(X_2 + 1)\frac{\mathrm{d}x_1 \wedge \mathrm{d}x_2}{L_1 L_2 L_3 L_4}\,, \qquad \Omega_D = -(X_2 - 1)\frac{\mathrm{d}x_1 \wedge \mathrm{d}x_2}{L_2 L_3 L_5}\,.$$

Divide and Conquer

Our original integral of interest is

$$\psi(X_1, X_2) = \int (x_1 x_2)^{\varepsilon} (\Omega_A + \Omega_B + \Omega_C + \Omega_D)$$

i.e. it is a sum of all basis integrals. Each basis integral is easier to evaluate!

Key idea:

The basis integral forms a finite dimensional vector space, so taking derivatives with respect to X_1 and X_2 must lead to coupled differential equations:

$$\mathrm{d}\mathcal{I} = \varepsilon A\, \mathcal{I}$$

Exterior derivative

4×4 matrix

$$\mathrm{d} \equiv \sum_i \partial_{X_i} \mathrm{d}X_i$$

Solving this differential equation turns out to be easy.

Differential Equation

$$\boxed{\mathrm{d}\mathcal{I} = \varepsilon A\, \mathcal{I}}$$

Explicitly, we find

$$\begin{aligned} A = \alpha_1 \,\mathrm{d}\log(X_1 + X_2) + \alpha_2 \,\mathrm{d}\log(X_1 + 1) + \alpha_3 \,\mathrm{d}\log(X_2 + 1) \\ + \alpha_4 \,\mathrm{d}\log(X_1 - 1) + \alpha_5 \,\mathrm{d}\log(X_2 - 1) \end{aligned}$$

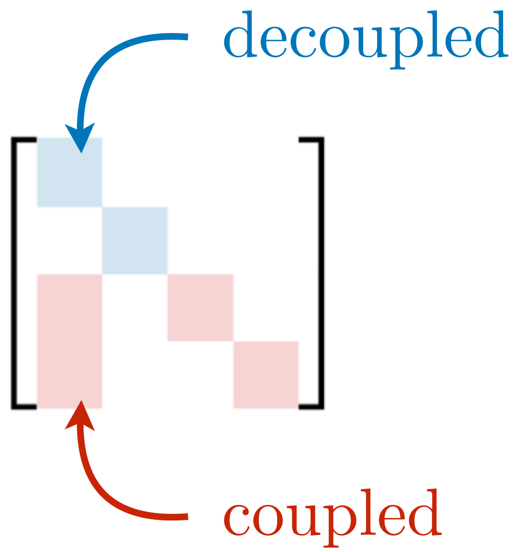
with

$$\alpha_1 \equiv \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \alpha_2 \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad \alpha_3 \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\alpha_4 \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \alpha_5 \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The Solution

More abstractly, we have

$A =$ 

This leads to **two types of functions**:

$$\left.\begin{aligned} I_A &= (X_1 + X_2)^{2\varepsilon} \\ I_B &= (1 + X_1)^{\varepsilon}(1 + X_2)^{\varepsilon} \end{aligned}\right] \quad \text{Power laws}$$

$$\left.\begin{aligned} I_C &= (X_1 + X_2)^{2\varepsilon}\, {}_2F_1\left[\begin{matrix} 1\,,\ \varepsilon \\ 1-\varepsilon \end{matrix} \,\middle|\, \frac{1 - X_1}{1 + X_2} \right] \\ I_D &= I_C(X_1 \leftrightarrow X_2) \end{aligned}\right] \quad \text{Hypergeometric}$$

In the de Sitter limit, these become the logs and dialogs we found earlier.

The Solution

Imposing appropriate boundary conditions (i.e. singularities), the solution for our original integral is

$$\boxed{\begin{aligned} \psi \;=\; & c_1(\varepsilon)\,(1+X_1)^{\varepsilon}(1+X_2)^{\varepsilon} \\ & c_2(\varepsilon)\,(X_1+X_2)^{2\varepsilon}\left(1 - {}_2F_1\left[\begin{matrix} 1\,,\,\varepsilon \\ 1-\varepsilon \end{matrix}\;\middle|\;\frac{1-X_2}{1+X_1}\right] - (X_1 \leftrightarrow X_2)\right) \end{aligned}}$$

Arkani-Hamed, DB, Hillman, Joyce, Lee and Pimentel [to appear]

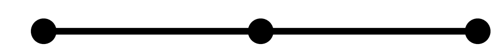
Space of Functions

The number of special functions depends on the number of bulk vertices, and the function have an interesting iterative structure:

Power law

Hypergeometric $= \int$ Power law

Kampé de Fériet $= \int$ Hypergeometric



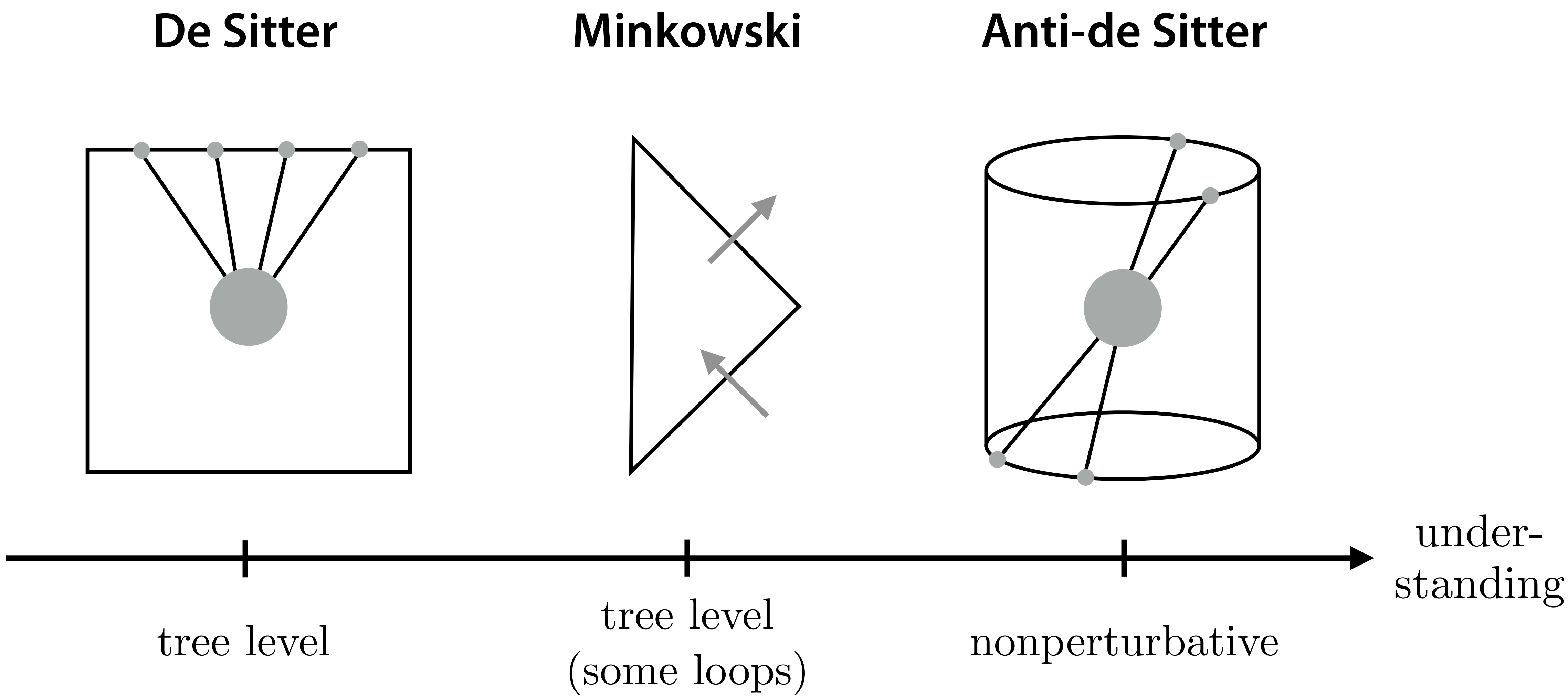
This structure is hidden in bulk perturbation theory.

→ Just the beginning of a systematic exploration of these mathematical structures living inside cosmological correlators.

Arkani-Hamed, DB, Hillman, Joyce, Lee and Pimentel [to appear]

Outlook and Speculations

Despite enormous progress in recent years, we are only at the beginning of a systematic exploration of cosmological correlators:

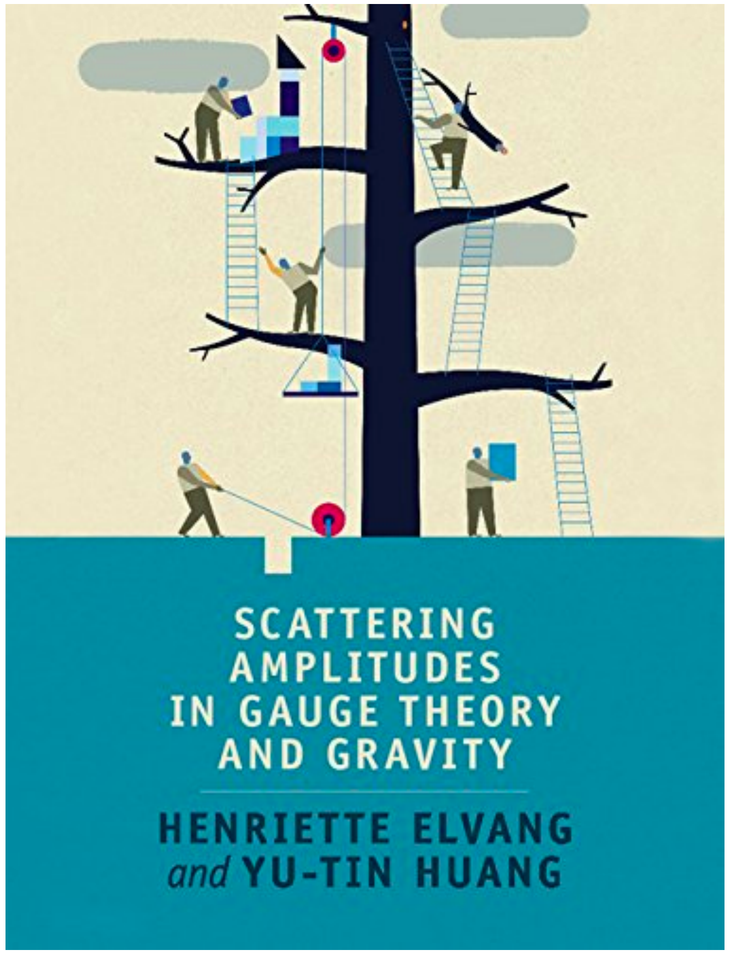


Many fundamental questions still remain unanswered.

Beyond Feynman Diagrams

So far, the cosmological bootstrap has only been applied to individual Feynman diagrams (which by themselves are unphysical).

However, in the **S-matrix bootstrap**, the real magic is found for **on-shell diagrams**:



- Recursion relations
- Generalized unitarity
- Color-kinematics duality
- Hidden positivity
- ...

What is the on-shell formulation of the cosmological bootstrap?
What magic will it reveal?

Beyond Perturbation Theory

So far, the bootstrap constraints (singularities, unitarity cuts, etc.) are only formulated in perturbation theory and implemented only at tree level.

In contrast, we understand the **conformal bootstrap** nonperturbatively:

Recently, some preliminary progress was made to apply unitarity constraints in de Sitter nonperturbatively, but much work remains.

Hogervorst, Penedones and Vaziri [2021]
Di Pietro, Gorbenko and Komatsu [2021]

A nonperturbative bootstrap will be required if we want to understand the UV completion of cosmological correlators.

Beyond Spacetime

In quantum gravity, spacetime is an emergent concept. In cosmology, the notation of "time" breaks down at the Big Bang. What replaces it?

The cosmological bootstrap provides a modest form of emergent time, but there should be a much more radical approach where space and time are outputs, not inputs.

For scattering amplitudes, such a reformulation was achieved in the form of the **amplituhedron**: Arkani-Hamed and Trnka [2013]

What is the analogous object in cosmology?
How does the Big Bang arise from it?

Thank you for your attention!