Lectures on Cosmological Correlations

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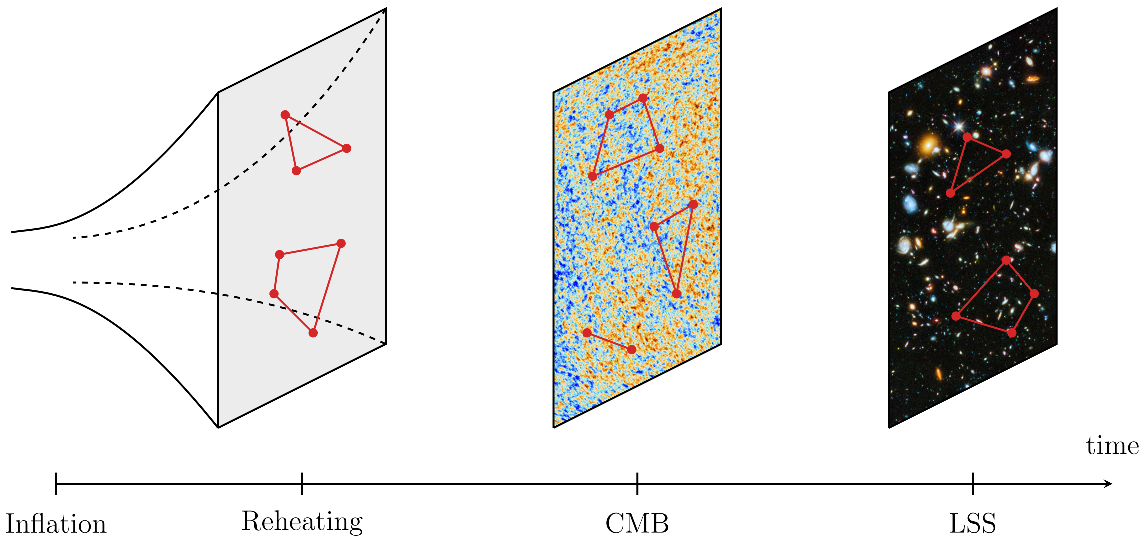


OUTLINE

I. Motivation

II. In-In Formalism

III. Wavefunction Approach

IV. Cosmological Bootstrap

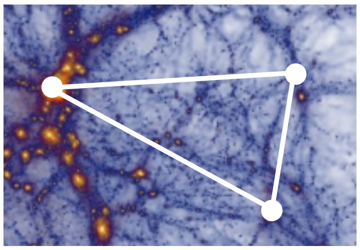
V. Outlook

Lecture notes and lecture scripts can be found at:
https://github.com/ddbaumann/cosmo-correlators

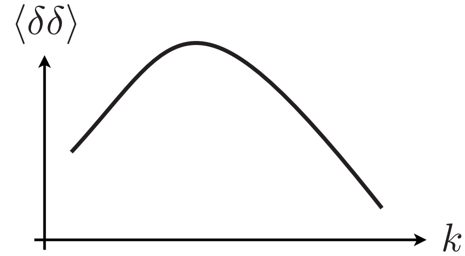
I. MOTIVATION

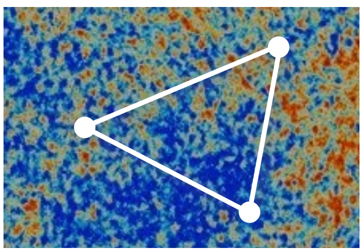
1.1. Cosmological Correlations

In cosmology, we measure **spatial correlations**:

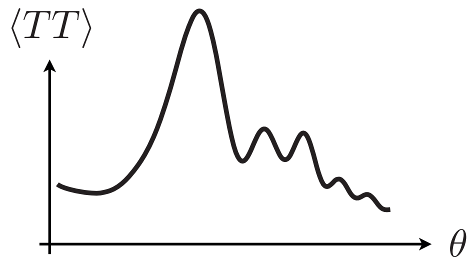


$$\langle \delta\rho(\mathbf{x}_1) \cdots \delta\rho(\mathbf{x}_N) \rangle$$

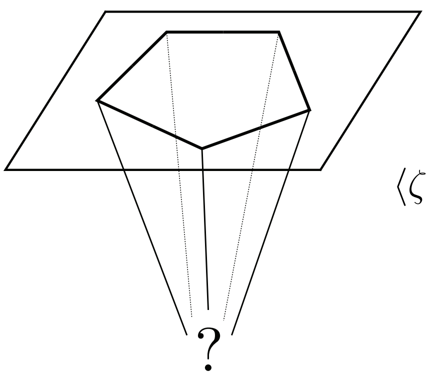




$$\langle \delta T(\theta_1) \cdots \delta T(\theta_N) \rangle$$



These correlations can be traced back to the origin of the hot Big Bang:



$$\langle \zeta(\mathbf{x}_1) \cdots \zeta(\mathbf{x}_N) \rangle$$

Where did the primordial correlations come from?

- Clue 1: The correlations span superhorizon scales.
- Clue 2: They are scale-invariant.

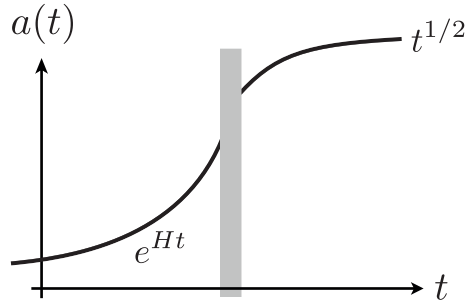
This suggests that the fluctuations were created **before the hot Big Bang**, during a phase of approximate time-translation invariance.

1.2. Inflation and De Sitter Space

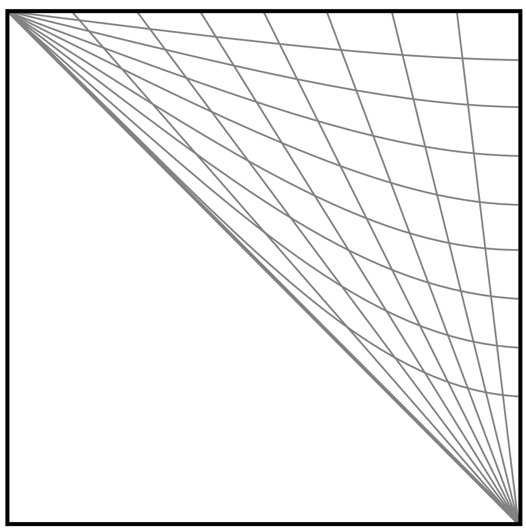
The observed correlations can be explained by an extended period of **accelerated expansion** (= inflation):

$$H(t) \equiv \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} \approx \text{const.}$$

$$\varepsilon(t) \equiv -\frac{\dot{H}}{H^2} \ll 1\,.$$



The spacetime during inflation is approximately **de Sitter space**.



In conformal time, $\mathrm{d}\eta = \mathrm{d}t/a(t)$, the de Sitter metric is

$$\mathrm{d}s^2 = \frac{-\mathrm{d}\eta^2 + \mathrm{d}\mathbf{x}^2}{(H\eta)^2}\,,$$

where $-\infty < \eta < 0$. The primordial correlations live on the **future boundary** of the de Sitter space at $\eta_* \approx 0$.

1.3. Quantum Fluctuations

Consider a massless scalar field during inflation:

$$\begin{aligned} S &= \frac{1}{2}\int \mathrm{d}\eta \mathrm{d}^3x\, a^2 \left(\dot{\phi}^2 - (\partial_i \phi)^2\right) \\ &= \frac{1}{2}\int \mathrm{d}\eta \mathrm{d}^3x \left(\dot{u}^2 - (\partial_i u)^2 + \frac{\ddot{a}}{a}u^2\right), \quad \text{where} \quad u(\eta, \mathbf{x}) \equiv a(\eta)\phi(\eta, \mathbf{x}) . \end{aligned}$$

Classical dynamics

The classical equation of motion (for each Fourier mode) is

$$\boxed{\ddot{u}_{\mathbf{k}} + \left(k^2 - \frac{2}{\eta^2}\right) u_{\mathbf{k}} = 0} \, .$$

- At *early times* $(-k\eta \ll 1)$, we have

$$\ddot{u}_{\mathbf{k}} + k^2 u_{\mathbf{k}} = 0 \qquad \Longrightarrow \qquad u_k \sim \frac{1}{\sqrt{2k}} e^{\pm ik\eta} \, .$$

- At *late times* $(-k\eta \to 0)$, we have

$$\ddot{u}_{\mathbf{k}} - \frac{2}{\eta^2} u_{\mathbf{k}} = 0 \qquad \Longrightarrow \qquad u_k \sim c_1 \eta^{-1} + c_2 \eta^2 \to c_1 \eta^{-1} \, .$$

The exact solution is

$$\boxed{u_{\mathbf{k}}(\eta) = a_{\mathbf{k}}^{\pm} \left(1 \pm \frac{i}{k\eta}\right) \frac{e^{\pm ik\eta}}{\sqrt{2k}}} \, .$$

Canonical quantization

- Promote classical fields $u, \pi = \dot{u}$ to **quantum operators** $\hat{u}, \hat{\pi}$, with

$$[\hat{u}(\eta, \mathbf{x}), \hat{\pi}(\eta, \mathbf{x}')] = i\delta_D(\mathbf{x} - \mathbf{x}') \quad \Longrightarrow \quad [\hat{u}_{\mathbf{k}}(\eta), \hat{\pi}_{\mathbf{k}'}(\eta)] = i(2\pi)^3 \delta_D(\mathbf{k} + \mathbf{k}') \, .$$

- Define the **mode expansion**

$$\hat{u}_{\mathbf{k}}(\eta) = u_k^*(\eta)\, \hat{a}_{\mathbf{k}} + u_k(\eta)\, \hat{a}_{-\mathbf{k}}^\dagger \, ,$$

  where $[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta_D(\mathbf{k} - \mathbf{k}')$ and $\dot{u}_k\, u_k^* - u_k\, \dot{u}_k^* = i$ (Wronksian).

- Define the **vacuum** by (see **Exercise 2.1**)

$$\hat{a}_{\mathbf{k}}|0\rangle = 0 \quad \text{and} \quad u_k(\eta) = \left(1 + \frac{i}{k\eta}\right)\frac{e^{ik\eta}}{\sqrt{2k}} \quad \text{(BD)}\,.$$

- Compute the two-point function of **zero-point fluctuations**:

$$\begin{aligned}\langle 0|\hat{u}_{\mathbf{k}}(\eta)\hat{u}_{\mathbf{k}'}(\eta)|0\rangle &= \langle 0|\left(u_k^*(\eta)\,\hat{a}_{\mathbf{k}} + u_k(\eta)\,\hat{a}^\dagger_{-\mathbf{k}}\right)\left(u_{k'}^*(\eta)\,\hat{a}_{\mathbf{k}'} + u_{k'}(\eta)\,\hat{a}^\dagger_{-\mathbf{k}'}\right)|0\rangle \\ &= (2\pi)^3\delta_D(\mathbf{k}+\mathbf{k}')\,|u_k(\eta)|^2\,.\end{aligned}$$

- The **power spectrum** of ϕ is

$$P_\phi(k) = \frac{|u_k(\eta)|^2}{a^2(\eta)} = \frac{H^2}{2k^3}\left(1+k^2\eta^2\right) \xrightarrow{k\eta\to 0} \boxed{\frac{H^2}{2k^3}}\,.$$

  This is the famous **scale-invariant** spectrum of a massless field in de Sitter.

Curvature perturbations

The initial conditions of the hot Big Bang are typically defined in terms of the comoving curvature perturbation:

$$\delta g_{ij} = a^2(\eta)e^{2\zeta(\eta,\mathbf{x})}\,,$$

which is related to the inflaton fluctuations (in spatially flat gauge) by

$$\zeta = -\frac{H}{\dot{\phi}}\delta\phi\,.$$

The power spectrum of ζ then is

$$\boxed{P_\zeta(k) = \left(\frac{H}{\dot{\phi}}\right)^2\frac{H^2}{2k^3}\bigg|_{k=aH} \equiv A_{\rm s}k^{n_{\rm s}-1}}\,.$$

Massive fields

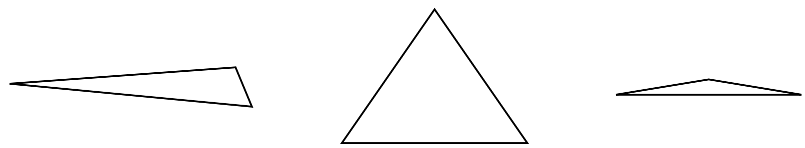
Massive fields are also produced during inflation, but don't survive until late times (see **Exercise 2.2**). Their imprints can, however, be found in the correlations of the light fields (= "cosmological collider physics")

1.4. Primordial Non-Gaussianity

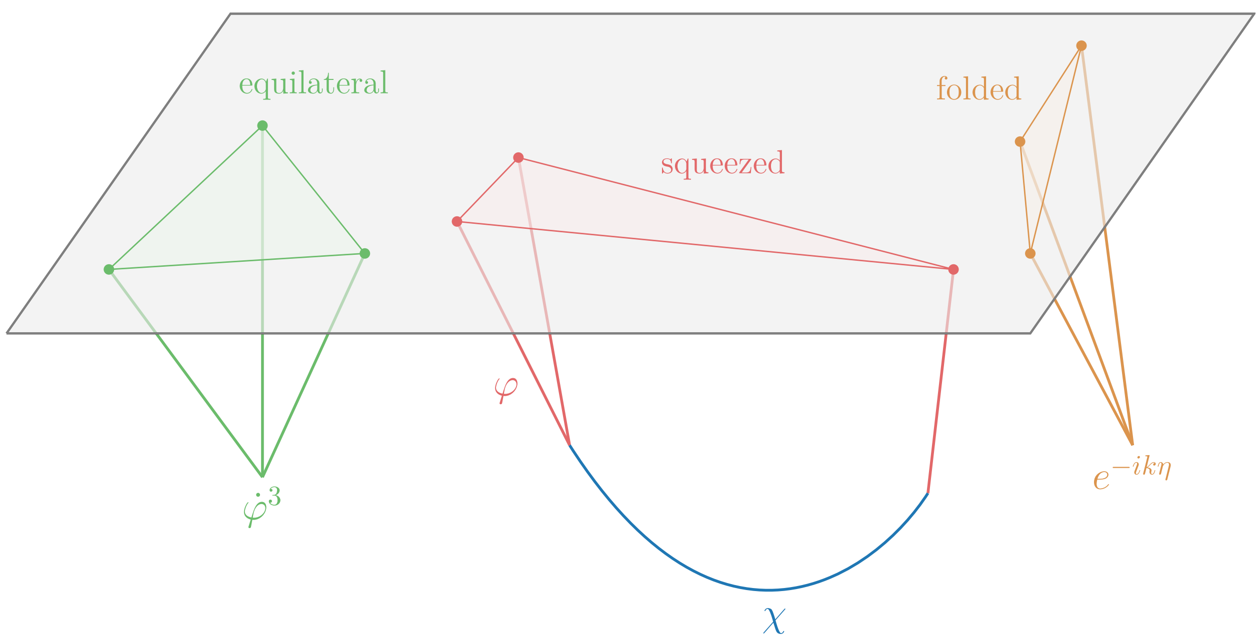
The main diagnostic of primordial non-Gaussianity is the **bispectrum**:

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle = B_\zeta(k_1, k_2, k_3) \times (2\pi)^3 \delta_D(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \, .$$

- amplitude: $f_{\rm NL} \equiv \dfrac{5}{18} \dfrac{B_\zeta(k,k,k)}{P_\zeta^2(k)}$
- shape:



- effect: new particles | new interactions | excited states
- Planck constraints: $|f_{\rm NL}^{\rm loc}| < 5$ | $|f_{\rm NL}^{\rm equil}| < 40$ | $|f_{\rm NL}^{\rm flat}| < 20$



Massive particles can be created by the inflationary expansion.
The **decay** of particles produces distinct correlations.

These correlations are **tracers** of the inflationary dynamics.

In the rest of the lectures, I will describe three methods for computing these higher-order correlations:

- In-In Formalism
- Wavefunction Approach
- Cosmological Bootstrap

I will focus mostly on the latter two.

II. IN-IN FORMALISM

2.1. In-In Correlators

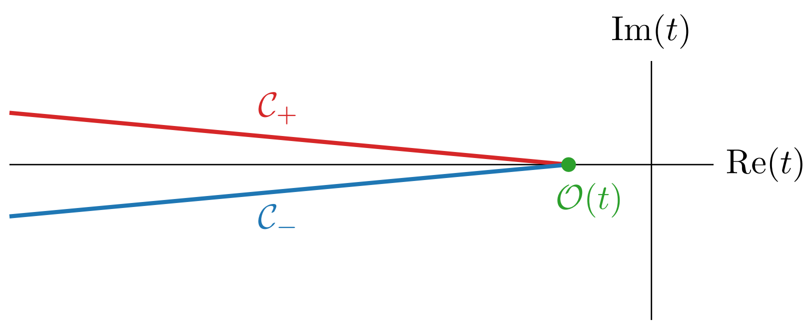
In cosmology, we compute **in-in correlators** (rather than in-out transition amplitudes):

$$\langle \mathcal{O}(t_*) \rangle \equiv \langle \Omega | \mathcal{O}(t_*) | \Omega \rangle \,.$$

In the lecture notes, we derive the analog of **Dyson's formula**:

$$\boxed{\langle \mathcal{O}(t_*) \rangle = \langle 0 | \, \bar{T} e^{i \int_{-\infty}^{t_*} \mathrm{d}t' H_{\mathrm{int}}(t')} \, \mathcal{O}(t_*) \, T e^{-i \int_{-\infty}^{t_*} \mathrm{d}t' H_{\mathrm{int}}(t')} \, |0\rangle} \,,$$

where the integration contour is



In perturbation theory, correlators are evaluated by expanding Dyson's formula in powers of H_{int}:

$$\begin{aligned}
\langle \mathcal{O}(t_*) \rangle &= i \int_{-\infty}^{t_*} \mathrm{d}t' \, \langle 0 | [H_{\mathrm{int}}(t'), \mathcal{O}(t_*)] | 0 \rangle \,, \\
\langle \mathcal{O}(t_*) \rangle &= - \int_{-\infty}^{t_*} \mathrm{d}t' \int_{-\infty}^{t'} \mathrm{d}t'' \, \langle 0 | \big[H_{\mathrm{int}}(t''), [H_{\mathrm{int}}(t'), \mathcal{O}(t_*)] \big] | 0 \rangle \,, \quad \text{etc.}
\end{aligned}$$

To stay sane, it is helpful to organize the computation in **Feynman diagrams**:

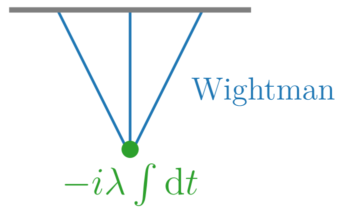
- Operators can be inserted on the two branches of the integration contour.
- We must account for the different time orderings.

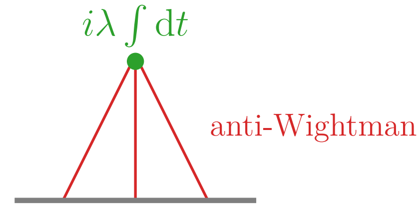
There are two types of **propagators**:

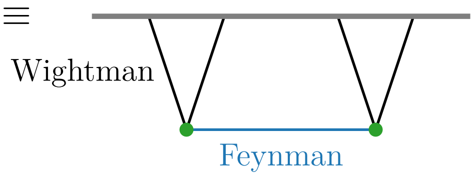
- Wightman: $W_k(t,t') = f_k^*(t) f_k(t')$
- Feynman: $G_F(k; t, t') = W_k(t,t')\, \theta(t-t') + W_k(t',t)\, \theta(t'-t)$

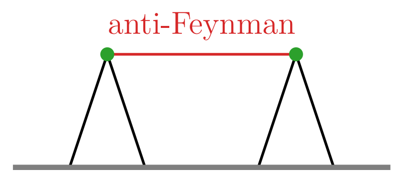
Consider Φ^3 theory.

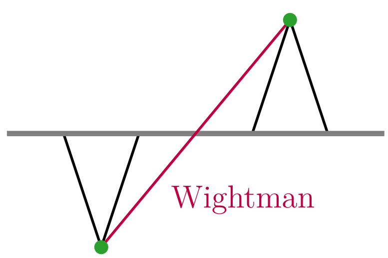
A diagrammatic representation of the Feynman rules is

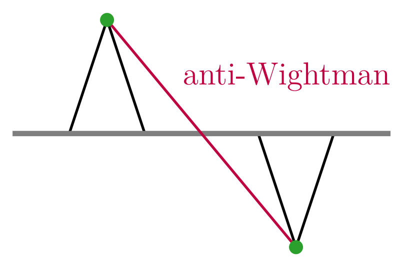
$I_+ \equiv$ 

$I_- \equiv$ 

$I_{++} \equiv$ 

$I_{--} \equiv$ 

$I_{+-} \equiv$ 

$I_{-+} \equiv$ 

2.2. Correlators in Flat Space

Evaluate correlators at $t_* \equiv 0$.
A massless scalar in flat space has the mode function

$$f_k(t) = \frac{e^{ikt}}{\sqrt{2k}}\,.$$

The propagators are

$$\begin{aligned} W_k(t,t') &= \frac{1}{2k}\, e^{-ik(t-t')}\,, \\ G_F(k;t,t') &= \frac{1}{2k}\left(e^{-ik(t-t')}\,\theta(t-t') + e^{ik(t-t')}\,\theta(t'-t)\right). \end{aligned}$$

Let us compute the correlators of Φ^3 theory.

- **Three-point function**

The two in-in integrals are

$$\begin{aligned} I_+ &= -\frac{ig}{8k_1k_2k_3}\int_{-\infty}^0 \mathrm{d}t\, e^{i(k_1+k_2+k_3)t} = -\frac{g}{8k_1k_2k_3}\,\frac{1}{k_1+k_2+k_3}\,, \\ I_- &= \frac{ig}{8k_1k_2k_3}\int_{-\infty}^0 \mathrm{d}t\, e^{-i(k_1+k_2+k_3)t} = -\frac{g}{8k_1k_2k_3}\,\frac{1}{(k_1+k_2+k_3)}\,. \end{aligned}$$

and we get

$$\boxed{\langle \phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2}\phi_{\mathbf{k}_3}\rangle' = -\frac{\lambda}{4k_1k_2k_3}\,\frac{1}{K}}\,, \quad \text{where} \quad K \equiv k_1+k_2+k_3\,.$$

- **Four-point function**

In the lecture notes, we show in detail that

$$\boxed{\langle \phi_{\mathbf{k}_1}\phi_{\mathbf{k}_2}\phi_{\mathbf{k}_3}\phi_{\mathbf{k}_4}\rangle' = \frac{g^2}{8k_1k_2k_3k_4}\left[\frac{1}{EE_LE_R} + \frac{1}{k_IE_LE_R}\right]}\,,$$

where $E \equiv k_1+k_2+k_3+k_4$, $E_L \equiv k_{12}+k_I$, and $E_R \equiv k_{34}+k_I$.

2.3. Correlators in de Sitter

Generically, correlators in de Sitter cannot be computed analytically. Consider the special case of a conformally coupled scalar ($m^2 = 2H^2$). Its mode function is

$$f_k(\eta) = -H\eta \, \frac{e^{ik\eta}}{\sqrt{2k}} \, .$$

We again compute the correlators of Φ^3 theory.

• Three-point function

The two in-in integrals are

$$\begin{aligned} I_+ &= -\frac{i\lambda}{8} \frac{H^2 \eta_*^3}{k_1 k_2 k_3} \int_{-\infty}^{\eta_*} \frac{\mathrm{d}\eta}{\eta} e^{iK\eta} = -\frac{i\lambda}{8} \frac{H^2 \eta_*^3}{k_1 k_2 k_3} \log(iK\eta_*) \, , \\ I_- &= I_+^* \, . \end{aligned}$$

and we get

$$\langle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \rangle' = -\frac{\lambda}{8} \frac{H^2 \eta_*^3}{k_1 k_2 k_3} \Big(i \log(iK\eta_*) - i \log(-iK\eta_*) \Big) = \boxed{\frac{\pi}{8} \, \lambda \, \frac{H^2 \eta_*^3}{k_1 k_2 k_3}} \, .$$

• Four-point function

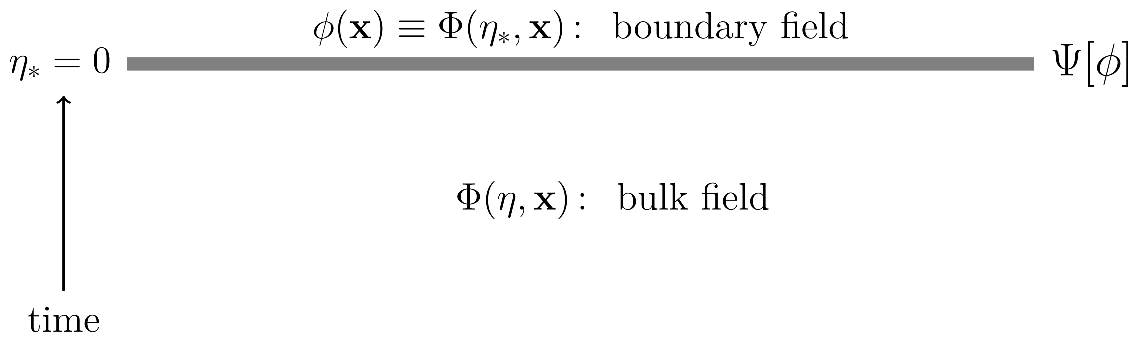
In the lecture notes, we show in detail that

$$\boxed{\begin{aligned} \langle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \phi_{\mathbf{k}_4} \rangle' = \frac{\lambda^2 H^2 \eta_*^4}{16 k_1 k_2 k_3 k_4 k_I} \bigg[\mathrm{Li}_2 \left(\frac{E - E_L}{E} \right) + \mathrm{Li}_2 \left(\frac{E - E_R}{E} \right) \\ + \log \left(\frac{E_L}{E} \right) \log \left(\frac{E_R}{E} \right) + \frac{\pi^2}{3} \bigg] \, , \end{aligned}}$$

where Li_2 is the dilogarithm.

III. WAVEFUNCTION APPROACH

3.1. Wavefunction of the Universe



The "wavefunction of the universe" is

$$\Psi[\phi] \equiv \langle \phi(\mathbf{x}) | 0 \rangle = \int_{\substack{\Phi(\eta_*)=\phi \\ \Phi(-\infty)=0}} \mathcal{D}\Phi \, e^{iS[\Phi]} \approx e^{iS_{\rm cl}[\Phi_{\rm cl}]} \, .$$

It defines boundary correlators

$$\langle \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_N) \rangle = \int \mathcal{D}\phi \, \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_N) \, |\Psi[\phi]|^2 \, .$$

The perturbative expansion of the wavefunction (in momentum space) is

$$\Psi[\phi] \;=\; \exp\left(- \sum_{N=2}^{\infty} \frac{1}{N!} \int \mathrm{d}^3 k_1 \cdots \mathrm{d}^3 k_N \, \Psi_N(\underline{\mathbf{k}}) \, \phi_{\mathbf{k}_1} \cdots \phi_{\mathbf{k}_N} \right) ,$$

where the "wavefunction coefficients" are

$$\Psi_N(\underline{\mathbf{k}}) = (2\pi)^3 \delta^3(\mathbf{k}_1 + \cdots + \mathbf{k}_N) \, \langle \underset{\substack{\uparrow \\ \text{dual operators: } \phi \to O, \; \gamma_{ij} \to T_{ij}}}{O_{\mathbf{k}_1}} \cdots O_{\mathbf{k}_N} \rangle' \, .$$

The relation between correlators and wavefunction coefficients is

$$\langle \phi\phi \rangle = \frac{1}{2\mathrm{Re}\langle OO \rangle} \, ,$$

$$\langle \phi\phi\phi \rangle = \frac{2\mathrm{Re}\langle OOO \rangle}{\prod_{n=1}^{3} 2\mathrm{Re}\langle O_n O_n \rangle} \, ,$$

$$\langle \phi\phi\phi\phi \rangle = \frac{\langle OOOO \rangle}{\langle OO \rangle^4} + \frac{\langle OOX \rangle^3}{\langle XX \rangle \langle OO \rangle^4} \, .$$

3.2. A Warmup

As a warmup, we apply the wavefunction approach to a **harmonic oscillator**:

$$S[\Phi] = \int \mathrm{d}t \left(\frac{1}{2}\dot{\Phi}^2 - \frac{1}{2}\omega^2\Phi^2\right).$$

The classical solution is

$$\Phi_{\rm cl} = \phi e^{i\omega t}\,,$$

and the on-shell action becomes

$$\begin{aligned} S[\Phi_{\rm cl}] &= \int_{t_i}^{t_*} \mathrm{d}t \left[\frac{1}{2}\partial_t(\dot{\Phi}_{\rm cl}\Phi_{\rm cl}) - \frac{1}{2}\Phi_{\rm cl}\underbrace{(\ddot{\Phi}_{\rm cl} + \omega^2\Phi_{\rm cl})}_{=0}\right] \\ &= \frac{1}{2}\dot{\Phi}_{\rm cl}\Phi_{\rm cl}\Big|_{t=t_*} \\ &= \frac{i\omega}{2}\phi^2\,. \end{aligned}$$

The wavefunction then is

$$\Psi[\phi] \approx \exp(iS[\Phi_{\rm cl}]) = \exp\left(-\frac{\omega}{2}\phi^2\right),$$

which implies

$$\boxed{\langle\phi^2\rangle = \frac{1}{2\omega}}\,.$$

In QFT, the same result applies for each Fourier mode:

$$\boxed{\langle\phi_{\mathbf{k}}\phi_{-\mathbf{k}}\rangle' = \frac{1}{2\omega_k}}\,,$$

where $\omega_k = \sqrt{k^2+m^2}$.

To make this more interesting, consider a **time-dependent oscillator**:

$$S[\Phi] = \int \mathrm{d}t \left(\frac{1}{2} A(t) \dot{\Phi}^2 - \frac{1}{2} B(t) \Phi^2 \right) .$$

The classical solution is

$$\Phi_{\mathrm{cl}} = \phi K(t) \, , \quad \text{with} \quad \begin{array}{l} K(0) = 1 \\ K(-\infty) \sim e^{i\omega t} \end{array}$$

and the on-shell action becomes

$$\begin{aligned} S[\Phi_{\mathrm{cl}}] &= \int_{t_i}^{t_*} \mathrm{d}t \left[\frac{1}{2} \partial_t (A \dot{\Phi}_{\mathrm{cl}} \Phi_{\mathrm{cl}}) - \frac{1}{2} \Phi_{\mathrm{cl}} \underbrace{(\partial_t (A \dot{\Phi}_{\mathrm{cl}}) + B \Phi_{\mathrm{cl}})}_{=0} \right] \\ &= \frac{1}{2} A \dot{\Phi}_{\mathrm{cl}} \Phi_{\mathrm{cl}} \Big|_{t=t_*} \\ &= \frac{1}{2} A \phi^2 \partial_t \log K \Big|_{t=t_*} . \end{aligned}$$

The wavefunction then is

$$\Psi[\phi] \approx \exp(iS[\Phi_{\mathrm{cl}}]) = \exp\left(\frac{i}{2} (A \partial_t \log K)\Big|_* \phi^2 \right) ,$$

which implies

$$|\Psi[\phi]|^2 = \exp\left(-\mathrm{Im}(A \partial_t \log K)\big|_* \phi^2 \right) \quad \Longrightarrow \quad \boxed{\langle \phi^2 \rangle = \frac{1}{2\, \mathrm{Im}(A \partial_t \log K)\big|_*}} .$$

3.3. Free Fields in de Sitter

Consider a **free massive field in de Sitter**:

$$
\begin{aligned}
S &= \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}m^2\Phi^2\right] \\
&= \int \mathrm{d}\eta\mathrm{d}^3x\, a^4(\eta)\left[\frac{1}{2a^2(\eta)}\left((\Phi')^2-(\nabla\Phi)^2\right) - \frac{1}{2}m^2\Phi^2\right] \\
&= \frac{1}{2}\int \mathrm{d}\eta\mathrm{d}^3k\left[\frac{1}{(H\eta)^2}\Phi'_{\mathbf{k}}\Phi'_{-\mathbf{k}} - \frac{1}{(H\eta)^2}\left(k^2+\frac{m^2}{(H\eta)^2}\right)\Phi_{\mathbf{k}}\Phi_{-\mathbf{k}}\right],
\end{aligned}
$$

which is the same as the time-dependent oscillator.

The classical solution is

$$
\Phi_{\mathrm{cl}} = \phi K(\eta)\,, \quad \text{with} \quad \begin{array}{l} K(0)=1 \\ K(-\infty)\sim e^{ik\eta} \end{array}
$$

The function $K(\eta)$ is the *bulk-to-boundary propagator*.
For a massless field, we have

$$
\begin{aligned}
K(\eta) &= (1-ik\eta)e^{ik\eta}\,, \\
\log K(\eta) &= \log(1-ik\eta)+ik\eta\,,
\end{aligned}
$$

and hence

$$
\begin{aligned}
\mathrm{Im}(A\partial_\eta \log K)\big|_{\eta=\eta_*} &= \frac{1}{(H\eta_*)^2}\,\mathrm{Im}\left(\frac{-ik}{1-ik\eta_*}+ik\right) \\
&= \frac{1}{(H\eta_*)^2}\,\mathrm{Im}\left(\frac{k^2\eta+ik^3\eta_*^2}{1+k^2\eta_*^2}\right) \xrightarrow{\eta_*\to 0} \boxed{\frac{k^3}{H^2}}\,,
\end{aligned}
$$

The two-point function then is

$$
\boxed{\langle\phi_{\mathbf{k}}\phi_{-\mathbf{k}}\rangle' = \frac{H^2}{2k^3}}\,.
$$

The result for a massive field is derived in the lecture notes.

3.4. Anharmonic Oscillator

Consider the following **anharmonic oscillator**:

$$S[\Phi] = \int \mathrm{d}t \left(\frac{1}{2}\dot{\Phi}^2 - \frac{1}{2}\omega^2\Phi^2 - \frac{1}{3}\lambda\Phi^3 \right).$$

The classical equation of motion is

$$\ddot{\Phi}_{\mathrm{cl}} + \omega^2\Phi_{\mathrm{cl}} = -\lambda\Phi_{\mathrm{cl}}^2 .$$

A formal solution is

$$\Phi_{\mathrm{cl}}(t) = \phi e^{i\omega t} + i\int \mathrm{d}t'\, G(t,t')\big(-\lambda\Phi_{\mathrm{cl}}^2(t')\big),$$

where $G(t,t')$ is the Green's function:

$$G(t,t') = \frac{1}{2\omega}\left(e^{-i\omega(t-t')}\theta(t-t') + e^{i\omega(t-t')}\theta(t'-t) - e^{i\omega(t+t')} \right).$$

Computing the on-shell action is now a bit more subtle.

As before, we first write

$$\begin{aligned} S[\Phi] &= \int_{t_i}^{t_*} \mathrm{d}t \left[\frac{1}{2}\partial_t(\Phi\dot{\Phi}) - \frac{1}{2}\Phi(\ddot{\Phi} + \omega^2\Phi) - \frac{\lambda}{3}\,\Phi^3 \right] \\ &= \frac{1}{2}\Phi\dot{\Phi}\bigg|_{t=t_*} + \int \mathrm{d}t \left[-\frac{1}{2}\Phi(\ddot{\Phi} + \omega^2\Phi) - \frac{\lambda}{3}\,\Phi^3 \right]. \end{aligned}$$

Since

$$\begin{aligned} &\lim_{t\to 0} G(t,t') = 0\,, \\ &\lim_{t\to 0} \partial_t G(t,t') = -ie^{i\omega t'} \neq 0\,, \end{aligned}$$

the boundary term is

$$\begin{aligned} \frac{1}{2}\Phi_{\mathrm{cl}}\dot{\Phi}_{\mathrm{cl}}\bigg|_{t=t_*} &= \frac{1}{2}\phi\left(i\omega\phi - i\lambda\int \mathrm{d}t'\,\big(-ie^{i\omega t'}\big)\,\Phi_{\mathrm{cl}}^2(t') \right) \\ &= \frac{i\omega}{2}\phi^2 - \frac{\lambda}{2}\phi\int \mathrm{d}t'\, e^{i\omega t'}\,\Phi_{\mathrm{cl}}^2(t')\,. \end{aligned}$$

The action then becomes

$$\begin{aligned} S[\Phi_{\rm cl}] &= \frac{i\omega}{2}\phi^2 - \frac{\lambda}{2}\phi \int \mathrm{d}t\, e^{i\omega t}\,\Phi_{\rm cl}^2 \\ &\quad + \int \mathrm{d}t \left[-\frac{1}{2}\left(\phi e^{i\omega t} - i\lambda \int \mathrm{d}t'\, G(t,t')\,\Phi_{\rm cl}^2(t')\right)\left(-\lambda \Phi_{\rm cl}^2(t)\right) - \frac{\lambda}{3}\,\Phi_{\rm cl}^3\right] . \end{aligned}$$

The terms linear in ϕ cancel and we get

$$\boxed{S[\Phi_{\rm cl}] = \frac{i\omega}{2}\phi^2 - \frac{\lambda}{3}\int \mathrm{d}t\,\Phi_{\rm cl}^3(t) - \frac{i\lambda^2}{2}\int \mathrm{d}t\,\mathrm{d}t'\,G(t,t')\,\Phi_{\rm cl}^2(t')\,\Phi_{\rm cl}^2(t)} \, .$$

To evaluate this, we write the classical solution as a perturbative expansion:

$$\Phi_{\rm cl}(t) = \Phi^{(0)}(t) + \lambda \Phi^{(1)}(t) + \lambda^2 \Phi^{(2)}(t) + \cdots \, .$$

where

$$\begin{aligned} \Phi^{(0)}(t) &= \phi e^{i\omega t} \, , \\ \Phi^{(1)}(t) &= i \int \mathrm{d}t'\, G(t,t') \left(-(\Phi^{(0)}(t'))^2 \right) \\ &= i \int \mathrm{d}t'\, G(t,t') \left(-\phi^2 e^{2i\omega t'} \right) = \frac{\phi^2}{3\omega^2} \left(e^{2i\omega t} - e^{i\omega t} \right) \, . \end{aligned}$$

With this, the wavefunction becomes

$$\boxed{\Psi[\phi] \approx e^{iS[\Phi_{\rm cl}]} = \exp\left(-\frac{\omega}{2}\phi^2 - \frac{\lambda}{9\omega}\phi^3 + \frac{\lambda^2}{72\omega^3}\phi^4 + \cdots \right)} \, .$$

From this, we can compute $\langle \phi^3 \rangle$, $\langle \phi^4 \rangle$, etc.

3.5. Interactions in Field Theory

Back to field theory. Consider

$$S[\Phi]=\int \mathrm{d}^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi-\frac{1}{2}m^2\Phi^2-\frac{1}{3}\lambda\Phi^3\right).$$

The classical equation of motion is

$$(\Box-m^2)\Phi=-\lambda\Phi^3\,.$$

A formal solution is

$$\Phi_{\rm cl}(t,\mathbf{x})=\int \mathrm{d}^3x'\,K(t,\mathbf{x},\mathbf{x}')\phi(\mathbf{x}')+i\int \mathrm{d}^4x'\,G(x,x')\big(-\lambda\Phi^2_{\rm cl}(x')\big)\,,$$

where

$$\begin{aligned}K(t_*,\mathbf{x},\mathbf{x}')&=\delta^{(3)}(\mathbf{x}-\mathbf{x}')\,,\\(\Box-m^2)G(x,x')&=i\,\delta(t-t')\delta^{(3)}(\mathbf{x}-\mathbf{x}')\,.\end{aligned}$$

Being careful about the boundary term, the on-shell action is

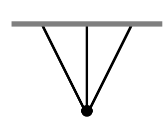
$$\begin{aligned}S[\Phi_{\rm cl}]=&-\frac{1}{2}\int \mathrm{d}^3x\,\mathrm{d}^3x'\,\phi(\mathbf{x})\phi(\mathbf{x}')\,\partial_t\log K(t,\mathbf{x},\mathbf{x}')\Big|_{t=t_*}\\&-\frac{\lambda}{3}\int \mathrm{d}^4x\,\Phi^3_{\rm cl}(x)-\frac{i\lambda^2}{2}\int \mathrm{d}^4x\,\mathrm{d}^4x'\;G(x,x')\,\Phi^2_{\rm cl}(x)\Phi^2_{\rm cl}(x')\,.\end{aligned}$$

In perturbation theory, we write

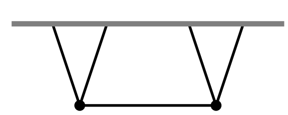
$$\Phi_{\rm cl}(x)=\Phi^{(0)}(x)+\lambda\,\Phi^{(1)}(x)+\cdots\,.$$

The terms in the action then have a diagrammatic interpretation:

$$-\frac{i\lambda}{3}\int \mathrm{d}^4x\,(\Phi^{(0)}(x))^3=$$



$$-\frac{\lambda^2}{8}\int \mathrm{d}^4x\int \mathrm{d}^4x'\,(\Phi^{(0)}(x))^2\,G(x,x')(\Phi^{(0)}(x'))^2=$$

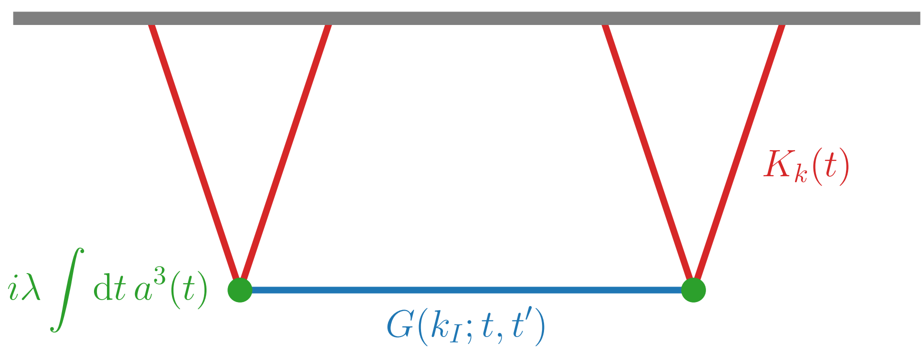


Given a mode function $f_k(t)$, the bulk-to-boundary and bulk-to-bulk propagators are

$$K_k(t) = \frac{f_k(t)}{f_k(t_*)}\,,$$
$$G(k;t,t') = \underbrace{f_k^*(t)f_k(t')\,\theta(t-t') + f_k^*(t')f_k(t)\,\theta(t'-t)}_{=\,G_F(k;t,t')} - \frac{f_k^*(t_*)}{f_k(t_*)} f_k(t)f_k(t')\,.$$

The **Feynman rules** for wavefunction coefficients are:

- bulk-to-boundary propagator K for every external line
- bulk-to-bulk propagator G for every internal line
- integrate each vertex over time.



3.7. Correlators in Flat Space

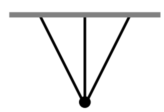
Evaluate correlators at $t_* \equiv 0$. The flat-space mode function is

$$f_k(t) = \frac{1}{\sqrt{2k}} e^{ikt} \,.$$

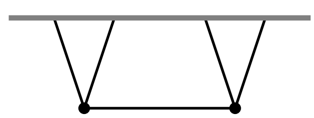
Using this, the relevant propagators are

$$\begin{aligned} K_k(t) &= e^{ikt} \,, \\ G(k;t,t') &= \frac{1}{2k}\left(e^{-ik(t-t')}\,\theta(t-t') + e^{ik(t-t')}\,\theta(t'-t) - e^{ik(t+t')}\right) . \end{aligned}$$

The three- and four-point wavefunction coefficients in Φ^3 theory are

$\langle O_1 O_2 O_3 \rangle \equiv$ 

$$\begin{aligned} &= i\lambda \int_{-\infty}^{0} \mathrm{d}t\, e^{i(k_1+k_2+k_3)t} \\ &= \frac{\lambda}{(k_1+k_2+k_3)} \,. \end{aligned}$$

$\langle O_1 O_2 O_3 O_4 \rangle \equiv$ 

$$\begin{aligned} &= -\lambda^2 \int_{-\infty}^{0} \mathrm{d}t'\,\mathrm{d}t''\, e^{ik_{12}t'} G(k_I;t',t'')\, e^{ik_{34}t''} \,, \\ &= \frac{\lambda^2}{(k_{12}+k_{34})(k_{12}+k_I)(k_{34}+k_I)} \,. \end{aligned}$$

In the lecture notes, we show in detail how this reproduces the correct in-in correlators.

3.8. Correlators in de Sitter

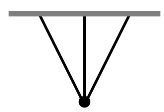
Generically, correlators in de Sitter cannot be computed analytically.

Consider the special case of a **conformally coupled scalar** ($m^2 = 2H^2$). Its mode function is

$$f_k(\eta) = -H\eta\,\frac{e^{ik\eta}}{\sqrt{2k}}\,.$$

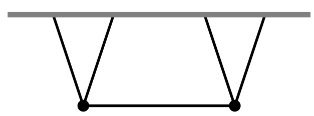
The three- and four-point wavefunction coefficients then are

$$\langle O_1O_2O_3\rangle \equiv$$



$$= \frac{i\lambda}{H^4\eta_*^3}\log(iK\eta_*)\,,$$

$$\langle O_1O_2O_3O_4\rangle \equiv$$



$$= \frac{\lambda^2}{2H^6\eta_*^4}\frac{1}{k_I}\left[\mathrm{Li}_2\left(\frac{k_{12}-k_I}{E}\right) + \mathrm{Li}_2\left(\frac{k_{34}-k_I}{E}\right) + \log\left(\frac{k_{12}+k_I}{E}\right)\log\left(\frac{k_{34}+k_I}{E}\right) - \frac{\pi^2}{6}\right],$$

where $K \equiv k_1 + k_2 + k_3$ and $E \equiv k_1 + k_2 + k_3 + k_4$.

In the lecture notes, we show that this reproduces the correct in-in correlators.

3.9. A Challenge

Consider the exchange of a generic massive scalar χ between two pairs of conformally coupled scalars Φ:

$$F \equiv \langle O_1O_2O_3O_4\rangle = -\lambda^2 \int \frac{\mathrm{d}\eta'}{\eta'^2}\int \frac{\mathrm{d}\eta''}{\eta''^2}\, e^{ik_{12}\eta'}e^{ik_{34}\eta''}\, G(k_I;\eta',\eta'')\,.$$

The time integrals cannot be performed analytically. Instead, we will derive a differential equation for F.

The Green's function satisfies:

$$\left(\eta^2\frac{\partial^2}{\partial\eta^2} - 2\eta\frac{\partial}{\partial\eta} + k_I^2\eta^2 + \frac{m^2}{H^2}\right)G(k_I;\eta,\eta') = -iH^2\eta^2\eta'^2\delta(\eta-\eta')\,.$$

Consider

$$\widetilde{F} \equiv -\lambda^2 \int \frac{\mathrm{d}\eta'}{\eta'^2} \frac{\mathrm{d}\eta''}{\eta''^2} e^{ik_{12}\eta'} e^{ik_{34}\eta''} \left(\eta'^2 \frac{\partial^2}{\partial \eta'^2} - 2\eta' \frac{\partial}{\partial \eta'} + k_I^2 \eta'^2 + \frac{m^2}{H^2} \right) G \,,$$

and evaluate it in two ways:

- First, directly:

$$\widetilde{F} = i\lambda^2 H^2 \int_{-\infty}^{0} \mathrm{d}\eta' \, e^{iE\eta'} = \frac{g^2 H^2}{E} \,, \qquad (\star)$$

with $E \equiv k_{12} + k_{34}$.

- Second, integrate by parts and trade factors of η' for $-i\partial_{k_{12}}$:

$$\begin{aligned} \widetilde{F} &= -\lambda^2 \int_{-\infty}^{0} \frac{\mathrm{d}\eta'}{\eta'^2} \frac{\mathrm{d}\eta''}{\eta''^2} \left((k_I^2 - k_{12}^2)\eta'^2 + 2ik_{12}\eta' + \frac{m^2}{H^2} - 2 \right) e^{ik_{12}\eta'} e^{ik_{34}\eta''} G \\ &= -\lambda^2 \int_{-\infty}^{0} \frac{\mathrm{d}\eta'}{\eta'^2} \frac{\mathrm{d}\eta''}{\eta''^2} \left((k_{12}^2 - k_I^2)\partial_{k_{12}}^2 + 2k_{12}\partial_{k_{12}} + \frac{m^2}{H^2} - 2 \right) e^{ik_{12}\eta'} e^{ik_{34}\eta''} G \\ &= \left((k_{12}^2 - k_I^2)\partial_{k_{12}}^2 + 2k_{12}\partial_{k_{12}} + \frac{m^2}{H^2} - 2 \right) F \,, \end{aligned} \qquad (\star\star)$$

Comparing $(\star\star)$ to $(\star)$, we get

$$\boxed{\left((k_{12}^2 - k_I^2)\partial_{k_{12}}^2 + 2k_{12}\partial_{k_{12}} + \frac{m^2}{H^2} - 2 \right) F = \frac{\lambda^2 H^2}{E}} \,.$$

$\Rightarrow$ Time has disappeared from the problem!

There should be a more efficient way of determine these boundary correlators without ever explicitly referring to the (unobservable) bulk time evolution.