

Lectures on Cosmological Correlations

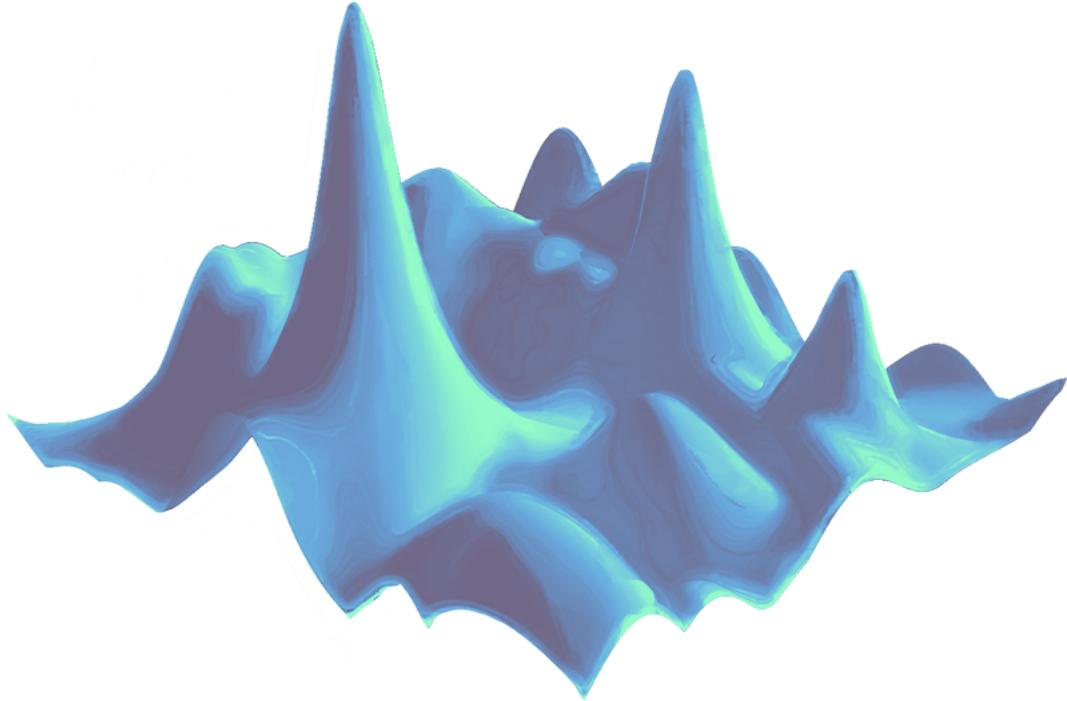
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Abstract

In these lecture notes, we present three approaches to compute cosmological correlation functions: 1) the in-in formalism, 2) the wavefunction approach, and 3) the cosmological bootstrap. We include over 50 exercises, with detailed solutions, to help students get proficient with the first-principles computations of cosmological correlators.

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1 Introduction

One of the biggest questions in all of science is the origin of structure in the Universe. What created everything that we see around us? An important clue lies in the fact that the structures in the Universe aren't distributed randomly, but displays large-scale correlations. These correlations are a fossil record of the early universe, and by measuring them we hope to uncover how the cosmological perturbations formed and evolved.

Our earliest snapshot of the large-scale structure of the Universe comes from measurements of the cosmic microwave background (CMB), whose intensity varies as a function of position on the sky. These CMB anisotropies reflect variations in the matter density at the time when the first atoms formed, about 380 000 years after the Big Bang. Over time, the density fluctuations grew, until clouds of gas started to collapse into stars and galaxies. In principle, the distribution of the collapsed objects still contains correlations that carry information about the primordial universe. Extracting this information is a fundamental challenge of modern cosmology.

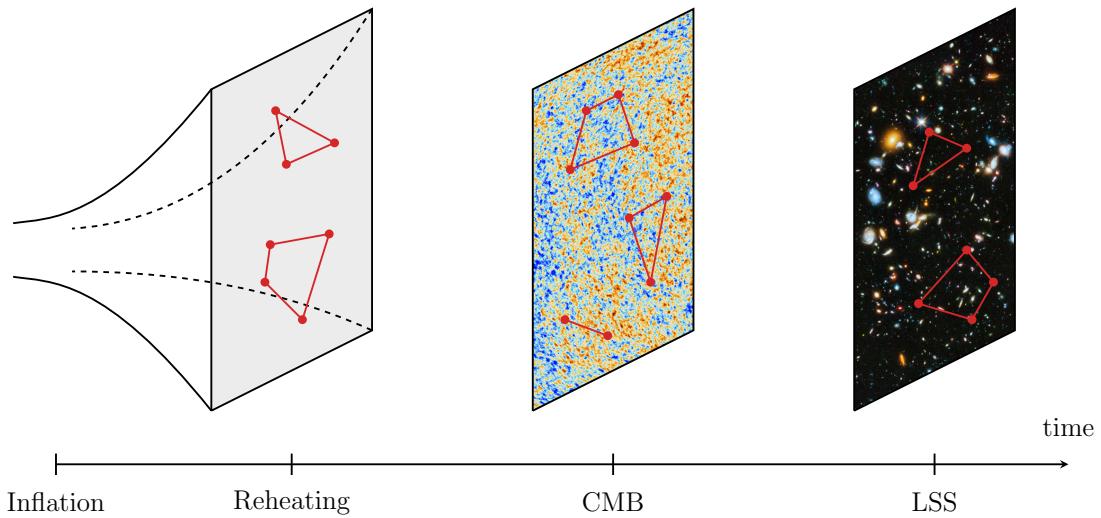


Figure 1: The correlations in late-time cosmological observables—like the distribution of the large-scale structure (LSS) or the anisotropies of the cosmic microwave background (CMB)—can be traced back in time to primordial correlations on the “reheating surface” (marking the initial surface of the hot Big Bang universe or the final surface at the end of inflation). Imprinted in these correlations is the physics of the inflationary era. (Figure adapted from [1].)

We understand the late-time evolution of the Universe well enough to trace it back until the beginning of the hot Big Bang. While doing this, cosmologists recently discovered a remarkable fact: the initial fluctuations which ultimately led to the large-scale structure of the Universe must have been created *before* the hot Big Bang. The conventional Big Bang was therefore *not* the beginning of time, but the end of an earlier high-energy period, and the primordial fluctuations were created in this pre-Big Bang phase. But, how exactly did this happen? The challenge is to extract this information from the pattern of correlations in the late universe.

We have some evidence for a period of inflationary expansion before the hot Big Bang. The details of inflation are still uncertain, but it provides an elegant mechanism to convert quantum

vacuum fluctuations into large-scale cosmological perturbations. The main observable predictions of inflation are spatial correlation functions and our goal in these lectures is to explain how these are computed.

We will describe three different approaches to compute cosmological correlation functions:

- *In-in formalism*—In cosmology, we compute “in-in correlators” (rather than in-out transition amplitudes). This means that we are interested in the expectation values of quantum operators, $\mathcal{O}(t)$ at some fixed time, $\langle \mathcal{O}(t) \rangle \equiv \langle \Omega | \mathcal{O}(t) | \Omega \rangle$, where $|\Omega\rangle$ is a specific state that we evolve from some initial time t_i to the later time t .
- *Wavefunction approach*—Alternatively, we first define a wavefunctional for the late-time fluctuations, $\Psi[\phi]$, whose square determines the probability for a given field configuration $\phi(\mathbf{x})$ at the end of inflation. From this wavefunctional we can then compute all correlations of the late-time field ϕ .
- *Cosmological bootstrap*—A drawback to the in-in and wavefunction methods is that they involve complex time integrals that often cannot be performed analytically. Since the time evolution during inflation is not actually observable, it stands to reason that the late-time correlators can be determined more directly without having to follow this evolution. The main idea of the cosmological bootstrap is to infer the static correlations on the late-time boundary from consistency conditions alone.

Our goal is to explain in detail how to perform computations in all three of these approaches.

Outline The outline of these notes is as follows: In Section 2, we provide a lightning review of elementary concepts of modern cosmology. For most readers, this will hopefully just be a reminder of material they have seen before, and experts can skip this section. The following three sections then present three different ways of computing cosmological correlation functions from first principles: In Section 3, we review the in-in formalism and use it to compute correlation functions both in flat space and in de Sitter space. In Section 4, we introduce the wavefunction approach and use it to compute a wide variety of correlators. We show in detail how the wavefunction computations reproduce the results from the in-in computations. Finally, in Section 5, we present a new “bootstrap” approach to the problem of cosmological correlations. Instead of struggling with complex time integrals, the bootstrap method tries to infer the spatial correlators directly by imposing consistency requirements. The cosmological bootstrap is still an active area of research and we therefore only provide a limited pedagogical introduction to the bootstrapping of simple scalar correlators. We end, in Section 6, with an outlook on the future.

Exercises The notes contain many exercise to help the reader to gain experience in the explicit computations of cosmological correlators. Explicit solutions to all exercises are given in Appendix E. In the pdf, these solutions are hyperlinked via the icon → at the end of each problem.

Further reading These lecture notes are *not* meant to be a review, but rather are intended as a tutorial that teaches students how to compute cosmological correlators. For further reading, we recommend [2, 3] (on the in-in formalism) and [1, 4, 5] (on the cosmological bootstrap).

2 Basics of Cosmology

We begin with a brief review of cosmology, focusing on the elements most relevant for the study of cosmological correlations. Readers who are familiar with the basic concepts of cosmology may skip this section.

2.1 Homogeneous Cosmology

Averaged over long enough distances, the Universe is homogeneous (translation invariant) and isotropic (rotation invariant). The Robertson–Walker metric of such a homogenous and isotropic spacetime is

$$ds^2 = -dt^2 + a^2(t) g_{ij} dx^i dx^j , \quad (2.1)$$

where g_{ij} denotes the metric of a maximally symmetric 3-space and $a(t)$ is the scale factor. Homogeneity and isotropy allow three possibilities for the curvature of the spatial slices (positive, negative or flat), but observations favor a flat universe, $g_{ij} \rightarrow \delta_{ij}$, which we will assume throughout these lectures.

The evolution of the metric is determined by the Einstein equations, which in this context take the form of the Friedmann equation for the scale factor

$$H^2 = \frac{8\pi G}{3}\rho , \quad (2.2)$$

where $H \equiv \dot{a}/a$ is the expansion rate (also called the Hubble parameter) and $\rho = \sum_i \rho_i$ is the total energy density of all constituents of the Universe. The present value of the Hubble parameter is $H_0 \approx 70 \text{ km s}^{-1} \text{Mpc}^{-1}$ and the average density today is

$$\begin{aligned} \rho_0 &= \frac{3H_0^2}{8\pi G} = 0.8 \times 10^{-29} \text{ g cm}^{-3} \\ &\approx 5 \text{ protons m}^{-3} . \end{aligned}$$

In comparison, the best vacuum on Earth has a density of about $1000 \text{ atoms cm}^{-3}$, so the Universe is pretty empty.

The Universe is filled with several different species of particles:

$$\underbrace{\text{photons } (\gamma) \quad \text{neutrinos } (\nu)}_{\text{radiation } (r)} \quad \underbrace{\overbrace{\text{electrons } (e) \quad \text{protons } (p)}^{\text{baryons } (b)}}_{\text{matter } (m)} \quad \text{cold dark matter } (c) .$$

For most of the cosmological history, these particles can be treated in a fluid approximation and each homogeneous matter species is characterized by an energy density, ρ , and a pressure, P . In an expanding universe, these fluid components satisfy the continuity equation

$$\frac{d\rho}{dt} = -3H(\rho + P) . \quad (2.3)$$

The two equations (2.2) and (2.3) together fully specify the homogeneous evolution of the combined metric-matter system once we posit an equation of state relating the density and pressure

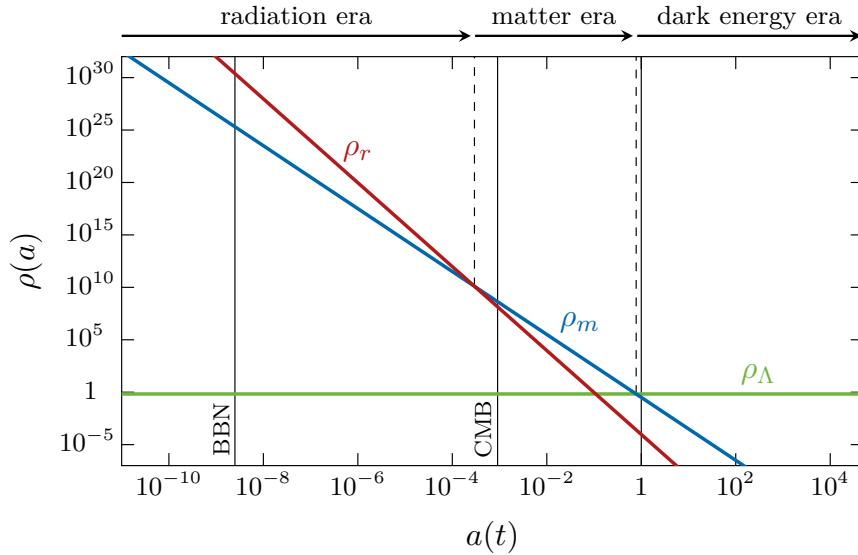


Figure 2: Evolution of the energy densities in the Universe. We see that the Universe was first dominated by radiation, then matter and today dark energy.

of the fluid components. (Typically, this specifies the pressure in terms of the energy as $P = w\rho$, for some constant w , which is called the equation of state parameter.)

Non-relativistic matter is pressureless ($w = 0$) and from (2.3) we infer that its energy density evolves as $\rho_m \propto a^{-3}$. This scaling is fairly intuitive since the energy of non-relativistic particles in a fixed volume is constant, while the latter increases as $V \propto a^3$. In contrast, the energy density of relativistic particles (or “radiation”, with $w = 1/3$) decreases as $\rho_r \propto a^{-4}$. The extra dilution comes from the fact that the energy of relativistic particles isn’t constant, but redshifts as $E \propto a^{-1}$. Feeding these scalings into the Friedmann equation (2.2) gives $a(t) \propto t^{2/3}$ and $a(t) \propto t^{1/2}$ during the matter and radiation eras, respectively.

In 1998, it was discovered that the Universe is accelerating. The source of the acceleration is unknown, and we call it “dark energy.” The simplest form of dark energy is a cosmological constant (or “vacuum energy”), with $\rho_\Lambda = \text{const.}$ Substituting this into the Friedmann equation leads to exponential expansion of the scale factor, $a(t) \propto e^{H_\Lambda t}$.

It is conventional to measure the contribution of each species i as a fraction of the total energy density today, $\Omega_i \equiv \rho_i/\rho_0$. The measured ΛCDM parameters then are

$$\Omega_r = 9.4 \times 10^{-5}, \quad \Omega_b = 0.04, \quad \Omega_m = 0.32, \quad \Omega_\Lambda = 0.68.$$

Why the present Universe has comparable amounts of (dark) matter and dark energy is a mystery called the ”coincidence problem.” Figure 2 shows the evolution of the three main energy components in the Universe. We see that although the Universe today is dominated by dark energy, in past matter and radiation were more important. Many interesting things happened during the early radiation phase of the universe, most notably the formation of the light elements during Big Bang nucleosynthesis (BBN). The fascinating history of the primordial universe is told in any cosmology textbook, but is beyond the scope of these lectures.

2.2 Cosmological Perturbations

Of course, the Universe is not perfectly homogeneous and cosmologists have learned a great deal about the structure and evolution of the Universe by studying its large-scale structure. Mathematically, we describe these inhomogeneities by perturbations of the energy-momentum tensor, $\delta T_{\mu\nu}(t, \mathbf{x})$, and the spacetime metric, $\delta g_{\mu\nu}(t, \mathbf{x})$. The Einstein equations—together with fluid or Boltzmann equations for the matter—then relate these fluctuations and determine their evolution. Not all fluctuations, however, are physical since they can transform under a change of coordinates. This gauge problem is addressed either by working with a fixed set of coordinates—and tracking all fluctuations—or by defining gauge-invariant combinations of the perturbations and following their evolution. Irrespective of which approach is being used, in the end, the result must be related to unambiguous physical observables.

Late-time correlations

We measure cosmological perturbations by observing different *tracers* of the distribution of structure in the Universe. For example, we observe the positions and redshifts of galaxies to determine the late-time distribution of matter, or the angular variations in the CMB temperature to infer the primordial density perturbations at the time of recombination. We will collectively denote the different cosmological observables by $\mathcal{O}(t, \mathbf{x})$. It is a crucial fact that the large-scale structure in the Universe isn't distributed randomly, but has interesting correlations between spatially

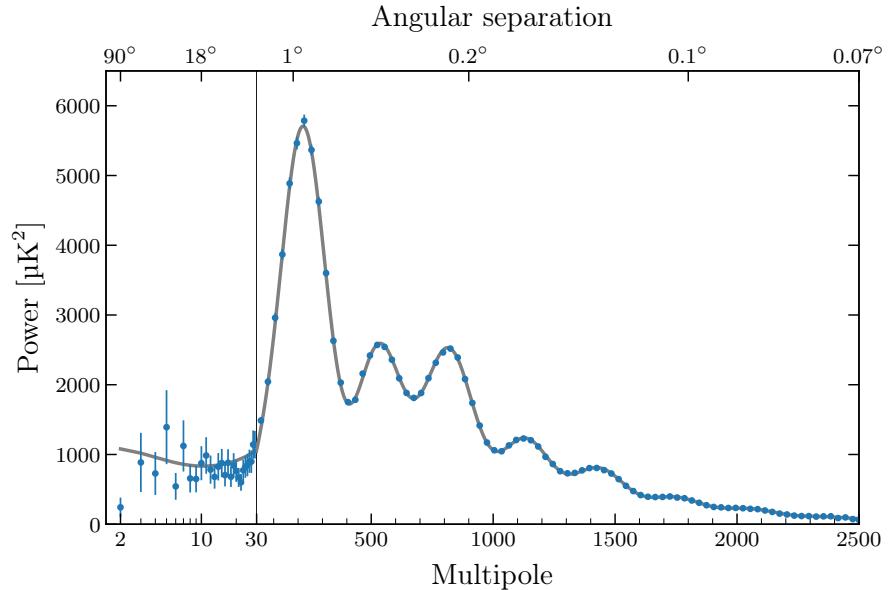


Figure 3: Power spectrum of CMB temperature anisotropies as measured by the Planck satellite [6].

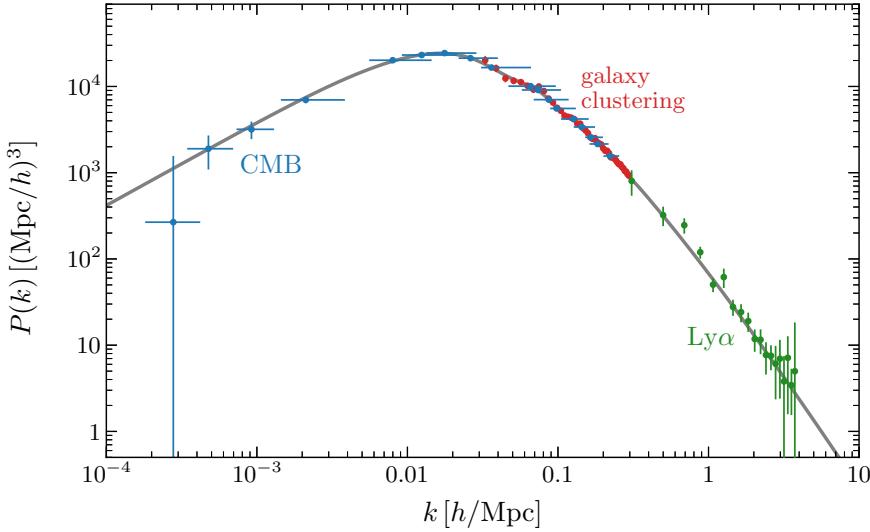


Figure 4: Measurements of the linear matter power spectrum (figure adapted from [7]). The “galaxy clustering” constraints are from the luminous red galaxy sample of the Sloan Digital Sky Survey (SDSS), the “CMB” constraints are derived from the Planck measurements of the CMB anisotropies, and “Ly α ” refers to the Lyman-alpha forest.

separated points:¹

$$\langle \mathcal{O}(t, \mathbf{x}_1) \mathcal{O}(t, \mathbf{x}_2) \dots \mathcal{O}(t, \mathbf{x}_N) \rangle. \quad (2.4)$$

These N -point correlators are to the cosmologist what N -point scattering amplitudes are to the particle physicist.

Figure 3 shows the two-point correlation function of the CMB temperature anisotropies, $\langle \delta T(\boldsymbol{\theta}_1) \delta T(\boldsymbol{\theta}_2) \rangle$. The importance of this graph for cosmology cannot be overstated. The shape of the CMB power spectrum depends sensitively on the cosmological parameters and CMB measurements have therefore been used to determine the parameters of the Λ CDM cosmology to great precision.

Figure 4 displays measurements of the two-point function of matter density fluctuations, $\langle \delta \rho_m(\mathbf{x}_1) \delta \rho_m(\mathbf{x}_2) \rangle$, using different tracers of the matter perturbations. The shape of this two-point function can be understood from the difference in the clustering of subhorizon mode during the radiation and matter eras. (Matter only starts to cluster significantly when the universe becomes matter dominated.)

¹The expectation value $\langle \dots \rangle$ denotes an *ensemble average* of the stochastic process that created the random field $\mathcal{O}(\mathbf{x})$ (e.g. quantum fluctuations during inflation). While the ensemble average is the natural object to compute from a theory of the initial conditions, it is not what we actually measure in observations. The observed large-scale structure of the Universe is a single realization of a random process. To relate these observations to our theoretical predictions, we assume *ergodicity*, which states that ensemble averages are equal to spatial averages as the volume becomes infinitely large. In that case, different parts of the Universe can be viewed as different realizations of the underlying random process. The volume average therefore approximates the average over different members of the ensemble. Of course, in reality, we don’t measure an infinite volume which introduces an irreducible *cosmic variance* in the measurements.

Initial conditions

Under relatively mild assumptions, the cosmological perturbations can be traced back in time to the origin of the hot Big Bang (or the reheating surface after inflation). In principle, the initial fluctuations can be very complicated, with independent fluctuations in each component of the primordial plasma. In practice, however, all fluctuations are synchronized and can be described by a single fluctuating degree of freedom (we call this *adiabatic initial conditions*). We will describe the initial fluctuations at the beginning of the hot Big Bang by the so-called *curvature perturbation*, $\zeta(t_i, \mathbf{x})$, which is defined as a local, isotropic rescaling of the spatial metric

$$g_{ij}(t, \mathbf{x}) = a^2(t) e^{2\zeta(t, \mathbf{x})} \delta_{ij}. \quad (2.5)$$

The name curvature perturbation follows from the fact that the three-dimensional Ricci curvature on the spatial slices is $a^2 R_{(3)} = 4\nabla^2 \zeta$. We assume that the initial fluctuations were drawn from a probability distribution $\mathbb{P}[\zeta]$ and that their correlation functions can therefore be written as

$$\langle \zeta(\mathbf{x}_1) \cdots \zeta(\mathbf{x}_N) \rangle = \int \mathcal{D}\zeta \, \zeta(\mathbf{x}_1) \cdots \zeta(\mathbf{x}_N) \mathbb{P}[\zeta], \quad (2.6)$$

where the integral is a functional integral (or path integral) over field configurations. By tracing the observed correlations $\langle \mathcal{O}_1 \mathcal{O}_2 \dots \mathcal{O}_N \rangle$ back in time, cosmologists try to measure the initial correlations $\langle \zeta_1 \zeta_2 \dots \zeta_N \rangle$ and use them to extract information about the initial probability distribution $\mathbb{P}(\zeta)$ and about the physics that gave rise to it.

If the initial perturbations were drawn from a Gaussian probability distribution then all statistical information is captured by the two-point function, $\langle \zeta(\mathbf{x}) \zeta(\mathbf{x}') \rangle$. Statistical homogeneity and isotropy demand that the two-point function is invariant under translations and rotations, and can therefore only depend on the distance between two points, $r \equiv |\mathbf{x} - \mathbf{x}'|$.

It is also convenient to describe the field $\zeta(\mathbf{x})$ in terms of its Fourier modes:

$$\zeta(t_i, \mathbf{x}) = \int \frac{d^3 k}{(2\pi)^3} \zeta_{\mathbf{k}}(t_i) e^{i\mathbf{k} \cdot \mathbf{x}}. \quad (2.7)$$

Since $\zeta(\mathbf{x})$ is a real field, its Fourier components satisfy $\zeta_{\mathbf{k}} = \zeta_{-\mathbf{k}}^*$. The two-point correlator of the Fourier modes is the *power spectrum*:

$$\langle \zeta_{\mathbf{k}} \zeta_{\mathbf{k}'} \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}') P_{\zeta}(k), \quad (2.8)$$

where the delta function is a consequence of translation invariance. Rotational invariance demands that the power spectrum is a function only of $k \equiv |\mathbf{k}|$ and doesn't depend on the direction of the wavevector. Substituting (2.8) into the definition of the two-point function (in position space), we find

$$\begin{aligned} \langle \zeta(\mathbf{x}) \zeta(\mathbf{x}') \rangle &= \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 k'}{(2\pi)^3} \langle \zeta_{\mathbf{k}} \zeta_{\mathbf{k}'} \rangle e^{i\mathbf{k} \cdot \mathbf{x}} e^{i\mathbf{k}' \cdot \mathbf{x}'} \\ &= \int \frac{d^3 k}{(2\pi)^3} P_{\zeta}(k) e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')}. \end{aligned} \quad (2.9)$$

We see that the power spectrum is the Fourier transform of the two-point function. Performing the angular integral, we find

$$\langle \zeta(\mathbf{x})\zeta(\mathbf{x}') \rangle = \int d\log k \frac{k^3}{2\pi^2} P_\zeta(k) j_0(k|\mathbf{x} - \mathbf{x}'|), \quad (2.10)$$

where $j_0(x) = \sin(x)/x$ is a spherical Bessel function of order zero. Equation (2.10) motivates us to define the dimensionless power spectrum:

$$\Delta_\zeta^2(k) = \frac{k^3}{2\pi^2} P_\zeta(k). \quad (2.11)$$

The variance of the field is $\langle \zeta^2(\mathbf{x}) \rangle = \int d\log k \Delta_\zeta^2(k)$. A *scale-invariant* spectrum has $\Delta_\zeta^2(k) = \text{const.}$ and the variance receives equal contributions from every logarithmic integral of k . (In position space, this translates into $\langle \zeta(\lambda\mathbf{x})\zeta(\lambda\mathbf{x}') \rangle = \langle \zeta(\mathbf{x})\zeta(\mathbf{x}') \rangle$, for a constant λ .) Deviations from scale invariance are often parameterized by a power-law ansatz

$$\Delta_\zeta^2(k) = A_s \left(\frac{k}{k_0} \right)^{n_s - 1}, \quad (2.12)$$

where n_s is the *scalar spectral index* defined relative to a reference scale k_0 . By a historical accident, the scale-invariant spectrum corresponds to $n_s = 1$. The measured spectral index in our Universe is [6]

$$n_s = 0.9667 \pm 0.0040. \quad (2.13)$$

We see that there is a statistically significant deviation from the scale-invariant spectrum.

Transfer functions

As long as the fluctuations are small (i.e. at early times or on large scales), their evolution equations can be linearized. It is convenient to describe this linear evolution in Fourier space, since the individual Fourier modes evolve independently. Each observable \mathcal{O} is then related to the primordial perturbations $\zeta_{\mathbf{k}}(t_i)$ by a *transfer function*:

$$\mathcal{O}(\mathbf{x}, t) = \int \frac{d^3 k}{(2\pi)^3} T_{\mathcal{O}}(k; t, t_i) \zeta_{\mathbf{k}}(t_i) e^{i\mathbf{k}\cdot\mathbf{x}}. \quad (2.14)$$

As indicated by the notation, there are different transfer functions $T_{\mathcal{O}}$ for each component of the Universe (photons, baryons, dark matter, etc.) At early times, photons and baryons were strongly coupled and acted as a single fluid. The large pressure of the radiation supported sound waves in the primordial plasma that left an imprint in the pattern of CMB anisotropies that we can still observe today. All of this interesting physics is captured by the CMB transfer function. Dark matter, on the other hand, is pressureless and therefore started to cluster earlier than the baryons. Only after recombination, did the baryons lose their pressure support and started collapsing into denser structure. It is important that the clustering of dark matter had a head start and then assisted the formation of baryonic structures (like stars and galaxies). Without the scaffolding provided by the dark matter structures wouldn't have formed fast enough to produce the Universe we see around us. All of this is encoded in the (dark) matter transfer function. More on the

linear transfer functions for cosmological fluctuations can be found in any cosmology textbook, e.g. [8, 9]. For the purpose of these lectures, it is only important to know that the physics of the linear evolution is extremely well-understood, so that the fluctuations can be traced back to their primordial origins, $\zeta_{\mathbf{k}}(t_i)$.²

2.3 Inflation and de Sitter Space

A promising (but still speculative) framework for explaining the cosmological initial conditions is the theory of *inflation*, which posits that the early universe experience a brief period of accelerated expansion, $\ddot{a} > 0$. During this phase, small quantum fluctuations were stretched to cosmic scales and became the seeds for the formation of structure in the Universe. Accelerated expansion can occur in GR if the Universe is filled with a substance with nearly constant energy density (like a temporary cosmological constant). The expansion rate is then nearly constant, $H \approx \text{const.}$, and the scale factor grows exponentially $a(t) \approx e^{Ht}$. The metric during inflation is approximately that of *de Sitter space*:

$$\begin{aligned} ds^2 &= -dt^2 + e^{2Ht} d\mathbf{x}^2 \\ &= \frac{1}{H^2 \eta^2} (-d\eta^2 + d\mathbf{x}^2), \end{aligned} \tag{2.15}$$

where we have introduced conformal time, $d\eta = dt/a(t)$. The deviation from the perfect de Sitter limit is characterized by the parameter

$$\varepsilon \equiv -\frac{\dot{H}}{H^2} \ll 1. \tag{2.16}$$

Inflationary models are characterized by the history of the nearly constant Hubble parameter, $H(t)$. The spontaneous breaking of time translation symmetry leads to a scalar Goldstone mode, $\pi(t, \mathbf{x})$, whose quantum fluctuations may have been the seeds for density fluctuations in the early universe. Differential inflationary models can be classify by their symmetry breaking patterns and the associated Goldstone interactions [10, 11].

An influential toy model for the inflationary dynamics is *slow-roll inflation*. In these models, a scalar field ϕ (the *inflaton*) dominates the energy density of the universe and dictates the evolution of the background spacetime. The action of the field, minimally coupled to gravity, is

$$S = \int d^4x \sqrt{-g} \left(\frac{M_{\text{Pl}}^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right). \tag{2.17}$$

The expansion rate of the universe is then determined by the Friedmann equation

$$H^2 = \frac{1}{3M_{\text{Pl}}^2} \left(\frac{1}{2} \dot{\phi}^2 + V(\phi) \right), \tag{2.18}$$

which during inflation has to remain approximately constant. This inflationary condition is satisfied if a nearly constant potential energy density, $V(\phi)$, dominates over the kinetic energy

²At some point—at late times and on small scales—the fluctuations become large and the linearize approximation breaks down. The relation between the observables \mathcal{O} and the initial conditions ζ then isn't described by linear transfer functions, but instead must be described by a phenomenological model.

density of the field. For successful inflation, this slow-roll evolution has to be maintained for a sufficiently long amount of time, which requires the acceleration of the field to be small. We assess this by considering the Klein–Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{dV}{d\phi}. \quad (2.19)$$

Sustained slow-roll inflation occurs when the Hubble friction term, $3H\dot{\phi}$, is large compared to the acceleration of the field, $\ddot{\phi}$. Many models of inflation have been constructed that satisfy these slow-roll conditions (see e.g. [12, 13] for more details).

2.4 Quantum Initial Conditions

So far, we have only discussed the homogeneous dynamics of inflation. One of the most fascinating features of the theory, however, is that it automatically contains a mechanism to explain the origin of fluctuations in the Universe. In this section, we will describe a simplified toy model for this process. More details can be found in [8, 14].

For concreteness, we will describe the quantization of fluctuations in slow-roll inflation, but the same procedure applies to the fluctuations in more general models of inflation. Consider the perturbed inflaton field:

$$\phi(\eta, \mathbf{x}) = \bar{\phi}(\eta) + \varphi(\eta, \mathbf{x}). \quad (2.20)$$

In GR, the field fluctuations will not be independent from metric fluctuations, but will be coupled to them through the Einstein equations. These gravitational couplings are gauge dependent, i.e. depend on the choice of coordinates used to describe the fluctuations. A convenient gauge to describe the quantization of the inflaton fluctuations is the spatially-flat gauge, with $g_{ij} = a^2(\eta)\delta_{ij}$. In this gauge, the mixing between φ and the metric perturbations $\delta g_{0\mu}$ vanishes in the limit $\varepsilon \rightarrow 0$. The inflaton self-interactions are also suppressed due to the flatness of the slow-roll potential. The leading behaviour in the limit $\varepsilon \rightarrow 0$ therefore is that of a free massless scalar in exact de Sitter space, described by the action

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \right) \quad (2.21)$$

$$= \frac{1}{2} \int d\eta d^3x a^2(\eta) \left((\varphi')^2 - (\nabla \varphi)^2 \right), \quad (2.22)$$

where $a(\eta) = -(H\eta)^{-1}$. Our goal is to compute the variance of the field fluctuations induced by quantum mechanics.

Classical dynamics

We begin by studying the classical dynamics of the field φ . From the action (4.58), we obtain the following equation of motion:

$$\varphi'' + 2\frac{a'}{a}\varphi' - \nabla^2\varphi = 0, \quad (2.23)$$

so that each Fourier mode satisfies

$$\varphi''_{\mathbf{k}} - \frac{2}{\eta}\varphi'_{\mathbf{k}} + k^2\varphi_{\mathbf{k}} = 0. \quad (2.24)$$

In order to cast this equation in a more familiar form, we define $\varphi_{\mathbf{k}} \equiv a(\eta)^{-1}u_{\mathbf{k}}$, where the function $u_{\mathbf{k}}(\eta)$ obeys

$$u''_{\mathbf{k}} + \left(k^2 - \frac{2}{\eta^2} \right) u_{\mathbf{k}} = 0 . \quad (2.25)$$

This is the equation of a harmonic oscillator, with a frequency given by the wavenumber k and a time-dependent (tachyonic) mass term.

Before solving (2.25), it is useful to determine the limiting behavior of the solutions:

- At *early times*, most modes of interest had wavelengths much smaller than the comoving Hubble scale, $k^{-1} \ll (aH)^{-1} = -\eta$. In this regime, the mass term can be dropped and equation (2.25) simplifies to

$$u''_{\mathbf{k}} + k^2 u_{\mathbf{k}} = 0 , \quad (2.26)$$

which is exactly the same as the equation satisfied by a massless scalar field in Minkowski space.³ The solutions to this equation are the standard solutions of the simple harmonic oscillator

$$u_{\mathbf{k}} \xrightarrow{k\eta \rightarrow -\infty} \frac{1}{\sqrt{2k}} e^{\pm ik\eta} , \quad (2.27)$$

where we have chosen the overall normalization, so that the solutions have Wronskian $\pm i$.

- At *late times*, the tachyonic mass term becomes relevant, and the fluctuations are amplified. In this limit, equation (2.25) becomes

$$u''_{\mathbf{k}} - \frac{2}{\eta^2} u_{\mathbf{k}} = 0 , \quad (2.28)$$

which is solved by

$$u_{\mathbf{k}} = c_1 \eta^{-1} + c_2 \eta^2 \xrightarrow{k\eta \rightarrow 0} c_1 \eta^{-1} . \quad (2.29)$$

The growing mode solution implies $\varphi_{\mathbf{k}} = a^{-1}u_{\mathbf{k}} \propto \eta \times \eta^{-1} = \text{const}$, so that the original field fluctuations are *frozen*.

Of course, we can also solve (2.25) directly to find

$$u_{\mathbf{k}}(\eta) = c_1 \frac{1}{\sqrt{2k}} \left(1 + \frac{i}{k\eta} \right) e^{ik\eta} + c_2 \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{k\eta} \right) e^{-ik\eta} , \quad (2.30)$$

where the free coefficients c_1, c_2 will be fixed by boundary conditions. From these solutions, we can explicitly see that there is a crossover in behavior when the wavelength of a mode becomes larger than the comoving Hubble radius, where the mode goes from being oscillatory to growing like a power law, reproducing the behaviors we saw above in the relevant regimes. Translating back into the original field, $\varphi_{\mathbf{k}} = a^{-1}u_{\mathbf{k}}$, the solutions are

$$\varphi_{\mathbf{k}} = \frac{H}{\sqrt{2k^3}} (1 \pm ik\eta) e^{\mp ik\eta} . \quad (2.31)$$

Now that we have a handle on the classical dynamics, we turn to the quantization of this field.

³Physically, this is a manifestation of the equivalence principle, which states that at scales much smaller than the radius of curvature of de Sitter space, the effects of the geometry are invisible.

Canonical quantization

We have seen that, at early times, the field fluctuations $u_{\mathbf{k}}$ satisfy the equation of a simple harmonic oscillator. Quantum zero-point fluctuations of this collection of oscillators produce a nonzero variance of the field. The expansion of the Universe amplifies these fluctuations and stretches them to large scales. In the following, we will compute the spectrum of these quantum fluctuations.

We start with the action for the canonically-normalized field $u \equiv a(\eta)\varphi$:

$$S = \frac{1}{2} \int d\eta d^3x \left((u')^2 - (\nabla u)^2 + \frac{a''}{a} u^2 \right), \quad (2.32)$$

whose equation of motion reproduces (2.25). The canonical momentum conjugate to u is

$$\pi \equiv \frac{\delta \mathcal{L}}{\delta u'} = u'. \quad (2.33)$$

We quantize the theory by promoting the classical fields u and π to quantum operators \hat{u} and $\hat{\pi}$, and impose the canonical commutation relation

$$[\hat{u}(\eta, \mathbf{x}), \hat{\pi}(\eta, \mathbf{x}')] = i\delta^{(3)}(\mathbf{x} - \mathbf{x}'), \quad (2.34)$$

in units where $\hbar \equiv 1$. The delta function is required by locality: modes at different points in space are independent and the corresponding operators therefore must commute. In Fourier space, the commutator becomes

$$\begin{aligned} [\hat{u}_{\mathbf{k}}(\eta), \hat{\pi}_{\mathbf{k}'}(\eta)] &= \int d^3x \int d^3x' \underbrace{[\hat{u}(\eta, \mathbf{x}), \hat{\pi}(\eta, \mathbf{x}')]}_{i\delta^{(3)}(\mathbf{x} - \mathbf{x}')} e^{-i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}'\cdot\mathbf{x}'} \\ &= i \int d^3x e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} \\ &= i(2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}'), \end{aligned} \quad (2.35)$$

and we see that modes with different wavelengths commute. We then decompose the field operator in terms of creation and annihilation operators

$$\hat{u}(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \left(u_k^*(\eta) \hat{a}_{\mathbf{k}} + u_k(\eta) \hat{a}_{-\mathbf{k}}^\dagger \right) e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (2.36)$$

where the mode function $u_k(\eta)$ is a solution to the classical equation of motion (2.25), and $u_k^*(\eta)$ is its complex conjugate.⁴ The commutator of the ladder operators is

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \quad (2.37)$$

which is the direct analogue of the commutator for the ladder operators in the ordinary harmonic oscillator. The field operator and its conjugate momentum operator then satisfy the commutation relation (2.35), as long as the mode functions obey the *Wronskian condition*:

$$\partial_\eta u_k(\eta) u_k^*(\eta) - u_k(\eta) \partial_\eta u_k^*(\eta) = i. \quad (2.38)$$

Up to an irrelevant phase, this fixes the normalization of the solution $u_k(\eta)$.

⁴Because the evolution is isotropic, the mode functions depend only on the magnitude of the wavevector, $k \equiv |\mathbf{k}|$, and not its direction. The creation and annihilation operators, on the other hand, do depend on direction and only their statistical properties have to be isotropic.

Vacuum choice

At this point, we have not yet picked a particular solution for the mode function $u_k(\eta)$. In fact, there is a one-parameter family of solutions that satisfy the Wronskian condition (2.38), and our selection of $u_k(\eta)$ is related to choosing the vacuum state of the theory. As in the ordinary harmonic oscillator, the vacuum state is defined to be the state that satisfies

$$\hat{a}_{\mathbf{k}}|0\rangle = 0. \quad (2.39)$$

Effectively, different choices of u_k in (2.36) correspond to different choices of vacuum state. In flat space, there is a natural choice, which is to choose the vacuum to be the (unique) state of lowest energy. However, in de Sitter space there is no time translation invariance, and so no bounded operator that we can use as the Hamiltonian.

The most natural vacuum choice is to require that modes at large k (or, equivalently, in the infinite past) reduce to their Minkowski space counterparts (see Exercise 2.1):

$$\lim_{k\eta \rightarrow -\infty} u_k(\eta) = \frac{1}{\sqrt{2k}} e^{ik\eta}, \quad (2.40)$$

which is known as the *Bunch–Davies* vacuum.⁵ This additional boundary condition fixes the solution to be

$$u_k(\eta) = \frac{1}{\sqrt{2k}} \left(1 + \frac{i}{k\eta} \right) e^{ik\eta}. \quad (2.41)$$

This specifies the ground state of the system and fixes the operator (2.36), so that we can compute its variance.

Exercise 2.1 Consider a simple harmonic oscillator whose action is

$$S = \int dt \left(\frac{1}{2} \dot{q}^2 - \frac{1}{2} \omega^2 q^2 \right), \quad (2.42)$$

where q is the deviation from equilibrium and ω is a constant frequency. The mass of the oscillator has been set to unity, $m \equiv 1$. The position operator is written as

$$\hat{q}(t) = q^*(t) \hat{a} + q(t) \hat{a}^\dagger. \quad (2.43)$$

where $q(t)$ is a solution to the classical equations of motion:

$$q(t) = c_1 \frac{1}{\sqrt{2\omega}} e^{i\omega t} + c_2 \frac{1}{\sqrt{2\omega}} e^{-i\omega t}. \quad (2.44)$$

⁵This vacuum state has numerous names. Other common ones are the Euclidean vacuum, adiabatic vacuum, and the Hartle–Hawking state. In principle we did not have to choose this as the ground state. In contrast to what happens in flat space, there is a one-parameter family of de Sitter-symmetric vacua, where we take $u_k(\eta)$ to be any linear combination of (2.41) and its complex conjugate satisfying the Wronskian condition. These different linear combinations are conventionally labeled by a parameter α , and so are typically called α -vacua [15, 16]. It should be noted that these states are somewhat peculiar and it is unclear whether they make sense beyond free theory.

The Wronskian of the solution is $q^* \dot{q} - \dot{q}^* q = i$, so that $[\hat{a}, \hat{a}^\dagger] = 1$. By considering the vacuum expectation value of the Hamiltonian, $\langle 0 | \hat{H} | 0 \rangle$, show that the vacuum $|0\rangle$ is the minimum energy state iff $c_1 = 1$ and $c_2 = 0$. →

Power spectrum

We now want to compute the two-point function of the field φ :

$$\langle 0 | \hat{\varphi}(\eta, \mathbf{x}) \hat{\varphi}(\eta, \mathbf{0}) | 0 \rangle = \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 k'}{(2\pi)^3} \langle 0 | \hat{\varphi}_{\mathbf{k}}(\eta) \hat{\varphi}_{\mathbf{k}'}(\eta) | 0 \rangle e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (2.45)$$

where we have taken advantage of the translation invariance of spatial slices to place one of the points at the origin. Using

$$\hat{\varphi}_{\mathbf{k}}(\eta) = f_k^*(\eta) \hat{a}_{\mathbf{k}} + f_k(\eta) \hat{a}_{-\mathbf{k}}^\dagger, \quad (2.46)$$

with $f_k(\eta) = u_k(\eta)/a(\eta)$, we can then compute

$$\begin{aligned} \langle 0 | \hat{\varphi}_{\mathbf{k}}(\eta) \hat{\varphi}_{\mathbf{k}'}(\eta) | 0 \rangle &= \langle 0 | \left(f_k^*(\eta) \hat{a}_{\mathbf{k}} + f_k(\eta) \hat{a}_{-\mathbf{k}}^\dagger \right) \left(f_{k'}^*(\eta) \hat{a}_{\mathbf{k}'} + f_{k'}(\eta) \hat{a}_{-\mathbf{k}'}^\dagger \right) | 0 \rangle \\ &= (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}') f_k^*(\eta) f_{k'}(\eta), \end{aligned} \quad (2.47)$$

where we have used the commutator (2.37). The *power spectrum* is therefore simply the square of the mode function:

$$P_\varphi(k) \equiv |f_k(\eta)|^2. \quad (2.48)$$

Substituting (2.47) back into (2.45), we get

$$\langle 0 | \hat{\varphi}(\eta, \mathbf{x}) \hat{\varphi}(\eta, \mathbf{0}) | 0 \rangle = \int \frac{d^3 k}{(2\pi)^3} P_\varphi(k) e^{i\mathbf{k}\cdot\mathbf{x}} \quad (2.49)$$

$$= \int d \log k \frac{k^3}{2\pi^2} P_\varphi(k) \frac{\sin(kr)}{kr}, \quad (2.50)$$

where $r \equiv |\mathbf{x}|$. The first line shows that the two-point function (in position space) is the Fourier transform of the power spectrum, and in the second line we have done the integral over the angular directions of \mathbf{k} . It is also often useful to define the *dimensionless power spectrum*

$$\Delta_\varphi^2(k) \equiv \frac{k^3}{2\pi^2} P_\varphi(k). \quad (2.51)$$

We now want to compute these power spectra explicitly for the massless scalar (2.21). Substituting the solution (2.41) into (2.48), we find

$$P_\varphi(k) = \frac{|u_k(\eta)|^2}{a^2(\eta)} = \frac{H^2}{2k^3} (1 + k^2 \eta^2) \xrightarrow{k\eta \rightarrow 0} \frac{H^2}{2k^3}. \quad (2.52)$$

It is these late-time statistics that are relevant for our Universe. The fluctuations generated during inflation that survive until late times are the things that serve as the initial conditions for the evolution of the universe at the onset of the hot Big Bang. In terms of the dimensionless power spectrum, we find at late times

$$\boxed{\Delta_\varphi^2 = \left(\frac{H}{2\pi} \right)^2}. \quad (2.53)$$

The fact that this is independent of k is typically called *scale invariance*. Looking back at (2.50) we see that this scale invariance means that there is equal power per logarithmic interval in fluctuations of φ . The (near) scale-invariance of fluctuations is one of the critical predictions of the inflationary universe.⁶

Exercise 2.2 Repeat the canonical quantization procedure for a massive scalar field in de Sitter space, starting from the action

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{2}(\partial\chi)^2 - \frac{m^2}{2}\chi^2 \right). \quad (2.54)$$

Determine the Bunch–Davies mode function for the field, $\chi_k(\eta)$, and then derive the late-time power spectrum

$$\lim_{k\eta \rightarrow 0} P_\chi(k, \eta) = \lim_{k\eta \rightarrow 0} |\chi_k(\eta)|^2. \quad (2.55)$$

What is the qualitative difference of the results for small masses and large masses? 

Curvature perturbations

As we described above, the primordial fluctuations are usually characterized in terms of the comoving curvature perturbation ζ . Performing a gauge transformation from the spatially flat gauge to comoving gauge gives the following relation between the inflaton fluctuations and the curvature perturbations:

$$\zeta = -\frac{\dot{H}}{\dot{\phi}}\varphi. \quad (2.56)$$

From the result (2.53), we then obtain

$$\boxed{\Delta_\zeta^2(k) = \left(\frac{\dot{H}}{\dot{\phi}}\right)^2 \left(\frac{H}{2\pi}\right)^2 \Big|_{k=aH}}, \quad (2.57)$$

where the right-hand side is evaluated at the horizon crossing of each mode. Since $H(t)$ and $\dot{\phi}(t)$ are weakly time-dependent during inflation, the dimensionless power spectrum $\Delta_\zeta^2(k)$ inherits a weak scale-dependence.

2.5 Primordial Non-Gaussianity

We have seen that inflation predicts that the initial correlations are highly Gaussian and therefore described well by the power spectrum. Nevertheless, in principle, a significant amount of information about the physics of inflation can be encoded in small levels of non-Gaussianity. In particular, the higher-order correlations associated with non-Gaussian initial conditions are

⁶This scale invariance follows directly from the symmetries of de Sitter space (as we will see in Section 5). This is one of the amazing aspects of inflation, features of the bulk time evolution get encoded in objects that live on the future boundary and which make no reference to the inflationary spacetime. In this case, the symmetries of the power spectrum are encoding the dilatation (essentially time translation) invariance of de Sitter dynamics.

sensitive to nonlinear interactions, while the power spectrum only probes the free theory. Measurements of primordial non-Gaussianity would therefore probe the particle content and the interactions during inflation.⁷

The leading signature of non-Gaussianity is a nonzero three-point correlation function, or its Fourier equivalent, the *bispectrum*:

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \frac{(2\pi^2)^2}{(k_1 k_2 k_3)^2} B_\zeta(k_1, k_2, k_3), \quad (2.58)$$

where the delta function is again a consequence of the homogeneity of the background. The sum of the three wavevectors must therefore form a closed triangle and the strength of the signal will depend on the shape of the triangle. As indicated by the notation, the bispectrum is a function of the magnitudes of the wavevectors, $k_{n=1,2,3}$, as required by the isotropy of the background. The factor of $(k_1 k_2 k_3)^{-2}$ has been extracted in to make the function $B_\zeta(k_1, k_2, k_3)$ dimensionless. The *amplitude* of the non-Gaussianity is then defined as the size of the bispectrum in the equilateral configuration:

$$f_{\text{NL}}(k) \equiv \frac{5}{18} \frac{B_\zeta(k, k, k)}{\Delta_\zeta^4(k)}, \quad (2.59)$$

where the numerical factor is a historical accident. In general, the parameter f_{NL} can depend on the overall wavenumber (or the size of the triangle), but for scale-invariant initial conditions it would be a constant. The bispectrum can then be written as

$$B_\zeta(k_1, k_2, k_3) = \frac{18}{5} f_{\text{NL}} \times S(x_2, x_3) \times \Delta_\zeta^4, \quad (2.60)$$

where $x_2 \equiv k_2/k_1$ and $x_3 \equiv k_3/k_1$. The shape function $S(x_2, x_3)$ is normalized so that $S(1, 1) \equiv 1$. As we will see below, the *shape* of the non-Gaussianity contains a lot of information about the microphysics of inflation (see Fig. 5). This is to be contrasted with the power spectrum, which is described by just two numbers, A_s and n_s , and not a whole function.

Equilateral In slow-roll inflation, the flatness of the inflationary potential constrains the size of the inflaton self-interactions. However, interesting models of inflation have been suggested in which higher-derivative corrections—such as $(\partial\phi)^4$ —play an important role during inflation. These interactions lead to cubic interactions of the inflaton perturbations—like $\dot{\varphi}^3$ and $\dot{\varphi}(\partial_i \varphi)^2$ —and hence a nonzero bispectrum.⁸ Since the inflaton fluctuations interact locally at points in the bulk spacetime, this produces a bispectrum with an enhanced signal for “equilateral” triangles, with $k_1 \approx k_2 \approx k_3$.

⁷An analogy with particle physics may be informative: the measurements of the power spectrum are the analog of measuring the propagators of particles. While these propagators are important features of long-lived particles, their form is completely fixed by Lorentz symmetry and therefore doesn't contain dynamical information. To learn about the dynamics of the theory, we collide particles and study the resulting interactions. Measurements of higher-order correlations (or non-Gaussianity) are the analog of measuring collisions in particle physics. The study of non-Gaussianity therefore also goes by the name of “cosmological collider physics” [17].

⁸A systematic way to classify these derivative interactions is in terms of an EFT for the inflationary fluctuations [10, 11].

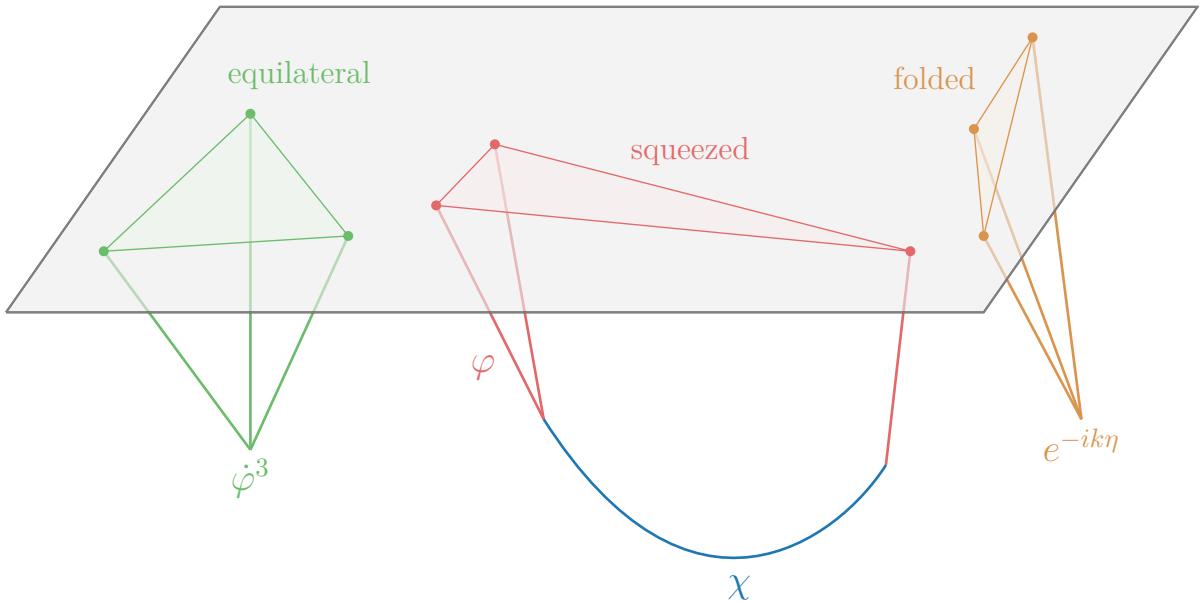


Figure 5: Illustration of the different types of non-Gaussianity described in the text: 1) Local interactions in the bulk produce the “equilateral” shape; 2) Excited initial states create an enhanced signal in the “folded” configuration; and 3) The production and decay of massive particles leave an imprint in the “squeezed” limit.

Folded The Gaussianity of slow-roll inflation also relies on the fact that we evaluated the quantum fluctuations in the Bunch–Davies vacuum (corresponding to the ground state of the harmonic oscillator). In contrast, starting from an excited initial state would lead to non-Gaussianity. The detailed shape of this non-Gaussianity depends on the model for the excited initial state. A universal feature is that the correlations are enhanced for “folded” triangles where two of the wavevectors become colinear, so that $k_1 + k_2 \approx k_3$. The signal in the folded configuration also provides an interesting test of the quantum origin of the fluctuations [18]. While classical fluctuations would generically have non-vanishing correlations in the folded limit, quantum fluctuations in the Bunch–Davies vacuum are characterized by the absence of such a signal.

Squeezed Finally, non-Gaussianity is a sensitive detector of new particles during inflation. These particles are created by the rapid expansion of the spacetime and, through their decays, produce distinct correlations in the primordial fluctuations. The massive particles can decay at widely separated points in the bulk spacetime, leading to an imprint in the so-called “squeezed” limit, $k_3 \ll k_1 \approx k_2$.

In the rest of these lectures, we will describe three methods for computing cosmological correlators: 1) the in-in formalism, 2) the wavefunction approach, and 3) the cosmological bootstrap.

3 In-In Formalism

The standard way to compute cosmological correlation functions uses the so-called “in-in” (or Schwinger–Keldysh) formalism [19–23]. In this section, we will introduce the in-in formalism and apply it to a few illustrative examples. The use of the in-in formalism in cosmology was reinvigorated by Maldacena, who applied it to compute the three-point function in slow-roll inflation [24]. The approach was further systematized by Weinberg [25] in order to study loop corrections. Nice reviews of the in-in formalism with cosmological applications in mind can be found in [2, 3].⁹

3.1 In-In Correlators

In flat-space quantum field theory, one is often interested in the transition amplitudes between some specified initial (“in”) and final (“out”) states, $S = \langle \text{out} | \text{in} \rangle$.¹⁰ More precisely, we can write this “ S -matrix” as

$$S_{\beta\alpha} = \langle \beta | \alpha \rangle, \quad (3.1)$$

which captures the probability for the state prepared in the far past with particles α to evolve into a state with particles β in the far future. In order to actually compute the S -matrix, we specify the initial state in the asymptotic past, $|\alpha_0\rangle \equiv |\alpha_0(t_i)\rangle$, where the theory is effectively free and evolve this state using the interaction Hamiltonian H_{int} :

$$|\alpha_0(t_f)\rangle = T \exp \left(-i \int_{t_i}^{t_f} dt H_{\text{int}} \right) |\alpha_0(t_i)\rangle. \quad (3.2)$$

where T denotes time-ordering. We then measure the overlap of $|\alpha_0(t_f)\rangle$ with some other multi-particle final state, $|\beta_0\rangle \equiv |\beta_0(t_f)\rangle$, specified again in the free theory. The S -matrix for this process is

$$S_{\beta\alpha} = \langle \beta_0 | T e^{-i \int_{-\infty}^{\infty} dt H_{\text{int}}} | \alpha_0 \rangle. \quad (3.3)$$

Importantly, to specify the S -matrix we choose boundary conditions for our measurement *both* in the past and in the future.

In cosmology, the situation is different. Roughly speaking, none of the conditions we require to define the S -matrix hold in cosmological backgrounds. It is not possible to localize wave packets and separate them asymptotically in order to map interacting states to free states (at least not for light fields). Relatedly, the gravitational production of particles cannot be shut off in asymptotic regions, so in and out states with definite particle numbers are not particularly well-defined. Moreover, in de Sitter space a single observer cannot access an entire Cauchy slice of the spacetime, so they cannot even in principle set up a scattering experiment. Finally, as far as we know, the Universe did not start in some state specified by an experimentalist, so even if they could be defined, transition amplitudes are of questionable practical utility.¹¹

⁹ Aside from cosmology, the in-in formalism is useful any time one is interested in a question where only the initial state can be prepared, and is frequently used in the study of thermal [?] or out-of-equilibrium systems [?].

¹⁰ See Chapter 3 of [26] for an authoritative discussion.

¹¹ Such an object would still be of formal interest. See [?], for some attempts to define an S -matrix-like quantity in de Sitter space.

This motivates us to ask a different question. Given some initial “in-state” of the Universe, $|0\rangle$, which we evolve from an early time t_i to a late time t , what is the quantum-mechanical expectation value of an operator $\mathcal{O}(t)$ in this (time-evolved) state? This expectation value is called an *in-in correlator* and can be written as

$$\langle \mathcal{O}(t) \rangle = \langle 0 | \bar{T} e^{i \int_{-\infty}^t dt' H_{\text{int}}} \mathcal{O}(t) T e^{-i \int_{-\infty}^t dt' H_{\text{int}}} | 0 \rangle, \quad (3.4)$$

where again T denotes time-ordering and \bar{T} denotes anti time-ordering. While this formula bears some formal similarity to (3.3), the important difference is that we can only specify the initial state of the system, and compute expectation values sandwiched between this in-state on both sides (hence the name “in-in”).

We now elaborate on the the meaning and derivation of the formula (3.4), and then utilize it to compute expectation values in a number of situations.

3.1.1 A Master Formula

The fundamental quantities that we are interested in computing are expectation values of quantum operators, $\mathcal{O}(t)$ at some time t in a state $|\Omega\rangle$:

$$\langle \mathcal{O}(t) \rangle \equiv \langle \Omega | \mathcal{O}(t) | \Omega \rangle. \quad (3.5)$$

The operator \mathcal{O} can be either a single operator or a string of operators evaluated at the same time.¹² Typically, we take the state $|\Omega\rangle$ to be the in vacuum state of the full interacting theory. Much as in the construction of the S -matrix, we compute these expectation values by envisioning $|\Omega\rangle$ as the time-evolution of some state specified in the infinite past as a state of the corresponding free theory (typically the free vacuum).

In the Heisenberg picture, states are time-independent and operators evolve according to the full Hamiltonian H of the theory:

$$\mathcal{O}(t) = U^\dagger(t, t_i) \mathcal{O}(t_i) U(t, t_i), \quad (3.6)$$

where t_i is some initial time, and the unitary time-evolution operator $U(t, t_i)$ satisfies

$$\frac{dU}{dt} = -iHU, \quad \text{with} \quad U(t_i, t_i) = 1. \quad (3.7)$$

The exact evolution of $\mathcal{O}(t)$ is complicated by the fact that H contains interaction terms (when written in terms of the field perturbations), which leads to nonlinear equations of motion. We deal with this by working in the *interaction picture*, where operators evolve according to the free Hamiltonian H_0 and states according to the interaction Hamiltonian $H_{\text{int}} = H - H_0$. The interaction picture operator at time t is then given by

$$\mathcal{O}_I(t) = U_0^\dagger(t, t_i) \mathcal{O}(t_i) U_0(t, t_i), \quad (3.8)$$

¹²It is not difficult to generalize the formalism to compute expectation values involving operator insertions at different times. One just inserts time evolution operators between the various operator insertions. We will specialize to equal-time correlators though because they are typically the observables of interest in cosmology.

where $U_0(t, t_i)$ satisfies

$$\frac{dU_0}{dt} = -iH_0 U_0, \quad \text{with} \quad U_0(t_i, t_i) = 1. \quad (3.9)$$

The main idea is that (3.8) encodes the leading evolution in terms of interaction picture fields and the effects of interactions will be treated perturbatively as a power series in H_{int} .

We now write (3.5) in terms of interaction-picture operators as

$$\begin{aligned} \langle \Omega | \mathcal{O}(t) | \Omega \rangle &= \langle \Omega | U^\dagger(t, t_i) \mathcal{O}(t_i) U(t, t_i) | \Omega \rangle \\ &= \langle \Omega | F^\dagger(t, t_i) U_0^\dagger(t, t_i) \mathcal{O}(t_i) U_0(t, t_i) F(t, t_i) | \Omega \rangle \\ &= \langle \Omega | F^\dagger(t, t_i) \mathcal{O}_I(t) F(t, t_i) | \Omega \rangle, \end{aligned} \quad (3.10)$$

where we have defined

$$F(t, t_i) \equiv U_0^\dagger(t, t_i) U(t, t_i). \quad (3.11)$$

Using (3.7) and (3.9), we find that

$$\begin{aligned} \frac{dF}{dt} &= \frac{d}{dt}(U_0^\dagger U) = \frac{dU_0^\dagger}{dt} U + U_0^\dagger \frac{dU}{dt} = (iH_0 U_0^\dagger) U + U_0^\dagger (-iH U) \\ &= -iU_0^\dagger (H - H_0) U_0 U_0^\dagger U \\ &= -i(U_0^\dagger H_{\text{int}} U_0) U_0^\dagger U \\ &= -iH_{\text{int}}^I F, \end{aligned} \quad (3.12)$$

where the superscript on H_{int}^I indicates that the fields inside H_{int} are evolved in the interaction picture (using the free Hamiltonian H_0). We can think of F as evolving the state using the interaction Hamiltonian: $|\Omega_I(t)\rangle = F(t, t_i)|\Omega_I(t_i)\rangle$, where $|\Omega_I(t_i)\rangle \equiv |\Omega\rangle$. Equation (3.10) then implies consistency of the expectation value in the Heisenberg and interaction pictures: $\langle \Omega | \mathcal{O}(t) | \Omega \rangle = \langle \Omega_I(t) | \mathcal{O}_I(t) | \Omega_I(t) \rangle$.

A formal solution of (3.12) is

$$F(t, t_i) = T \exp \left(-i \int_{t_i}^t dt' H_{\text{int}}^I(t') \right), \quad (3.13)$$

where T denotes the time-ordering operation. The expectation value (3.5) can then be written as

$$\langle \mathcal{O}(t) \rangle = \langle \Omega | \overline{T} e^{i \int_{t_i}^t dt' H_{\text{int}}^I(t')} \mathcal{O}_I(t) T e^{-i \int_{t_i}^t dt' H_{\text{int}}^I(t')} | \Omega \rangle, \quad (3.14)$$

where \overline{T} denotes anti time-ordering.

We would like the state $|\Omega\rangle$ to be the “in” vacuum of the *interacting theory*. In the interaction picture, all states can be expressed in terms of the Fock states of the free Hamiltonian (which can be constructed by applying creation operators on the free vacuum $|0\rangle$). We would therefore like to express the interacting vacuum at early times in terms of this Fock (or *Bunch–Davies*) vacuum $|0\rangle$. We first write $|0\rangle$ in terms of a complete set of energy eigenstates of the *full* theory:

$$|0\rangle = \sum_N |N\rangle \langle N|0\rangle. \quad (3.15)$$

Near $t_i \rightarrow -\infty$, the evolution of the Fock vacuum with the full Hamiltonian H gives

$$\begin{aligned} e^{-iH(t-t_i)}|0\rangle &= \sum_N e^{-iE_N(t-t_i)}|N\rangle\langle N|0\rangle \\ &= e^{-iE_\Omega(t-t_i)}|\Omega\rangle\langle\Omega|0\rangle + \sum_{N' \neq \Omega} e^{-iE_{N'}(t-t_i)}|N'\rangle\langle N'|0\rangle, \end{aligned} \quad (3.16)$$

where $|N'\rangle$ denotes all states excluding the vacuum. We can now employ the same trick as in standard in-out field theory: we rotate time slightly into the imaginary direction, so that

$$t \mapsto t(1 - i\epsilon) \equiv t^-, \quad (3.17)$$

where $\epsilon > 0$ is an infinitesimal constant. Near $t_i \rightarrow -\infty$, the $i\epsilon$ factor leads to factors of the form $e^{-\infty \times \epsilon E_N}$. Since E_Ω is the smallest eigenvalue of the Hamiltonian H , only the state $|\Omega\rangle$ survives in this limit, and will project the interacting vacuum onto the free vacuum:

$$e^{-iH(t^- - t_i^-)}|\Omega\rangle = \frac{e^{-iH(t^- - t_i^-)}|0\rangle}{\langle\Omega|0\rangle}. \quad (3.18)$$

Equation (3.14) can then be written as

$$\langle\Omega|\mathcal{O}(t)|\Omega\rangle = \frac{1}{|\langle\Omega|0\rangle|^2} \langle 0 | \overline{T} e^{i \int_{-\infty+}^t dt' H_{\text{int}}^I(t')} \mathcal{O}_I(t) T e^{-i \int_{-\infty-}^t dt' H_{\text{int}}^I(t')} | 0 \rangle, \quad (3.19)$$

where we have introduced the shorthand $-\infty^\pm \equiv -\infty(1 \pm i\epsilon)$. Note that the anti-time-ordering changes the sign of the $i\epsilon$ prescription. Taking $\mathcal{O}(t)$ to be the identity operator, equation (A.11) implies

$$|\langle\Omega|0\rangle|^2 = \frac{\langle 0 | F^\dagger F | 0 \rangle}{\langle\Omega|\Omega\rangle} = \frac{\langle 0 | 0 \rangle}{\langle\Omega|\Omega\rangle} = 1. \quad (3.20)$$

We can therefore drop the prefactor in (A.11) and obtain our final *master in-in formula*

$$\langle\mathcal{O}(t)\rangle = \langle 0 | \overline{T} e^{i \int_{-\infty+}^t dt' H_{\text{int}}(t')} \mathcal{O}(t) T e^{-i \int_{-\infty-}^t dt' H_{\text{int}}(t')} | 0 \rangle. \quad (3.21)$$

To avoid clutter, we have dropped the labels I on the right-hand side, but evolution in the interaction picture will always be implied. A graphical illustration of the in-in contour is given in Figure 6. Operators will be time-ordered on \mathcal{C}_+ and anti-time-ordered on \mathcal{C}_- .

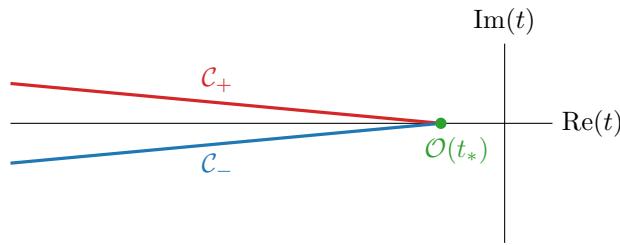


Figure 6: Graphical illustration of the in-in contour. The contour has small imaginary parts at early times to project the interacting vacuum onto the free vacuum.

3.1.2 Perturbative Expansion

Equation (3.21) is the in-in analogue of Dyson's formula. In practice, we use it to compute correlation functions in perturbation theory by expanding in powers of H_{int} and evaluating the resulting expressions by writing a mode expansion of the field operators in canonical quantization.

At lowest order, we find that the expectation value of \mathcal{O} is given by

$$\langle \mathcal{O}(t) \rangle = 2 \text{Im} \left(\int_{-\infty}^t dt' \langle 0 | \mathcal{O}(t) H_{\text{int}}(t') | 0 \rangle \right), \quad (3.22)$$

where we have assumed that $\langle 0 | \mathcal{O}(t) | 0 \rangle = 0$. This expression captures the leading-order perturbative effect of interactions, which from a diagrammatic viewpoint correspond to contact interactions. To make this less abstract, we work out an explicit example in the box below.

Example To show how to use (3.22), let us consider the three-point function coming from a ϕ^3 interaction in an arbitrary FRW spacetime. We take the interaction Lagrangian to be

$$L_{\text{int}} = -\frac{1}{3!} g \phi^3. \quad (3.23)$$

In this case, the interaction Hamiltonian is then just minus the interaction Lagrangian

$$\begin{aligned} H_{\text{int}}(t) &= \frac{g}{3!} \int d^3x a^3(t) \phi^3(\mathbf{x}, t) \\ &= \frac{g}{3!} a^3(t) \int d^3x \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \frac{d^3q_3}{(2\pi)^3} \phi_{\mathbf{q}_1}(t) \phi_{\mathbf{q}_2}(t) \phi_{\mathbf{q}_3}(t) e^{-i(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3)\mathbf{x}} \\ &= \frac{g}{3!} a^3(t) \int \frac{d^3q_1}{(2\pi)^3} \frac{d^3q_2}{(2\pi)^3} \frac{d^3q_3}{(2\pi)^3} \phi_{\mathbf{q}_1}(t) \phi_{\mathbf{q}_2}(t) \phi_{\mathbf{q}_3}(t) (2\pi)^3 \delta^{(3)}(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3), \end{aligned} \quad (3.24)$$

where g is the coupling constant. Equation (3.22) then implies

$$\langle \phi_{\mathbf{k}_1}(t) \phi_{\mathbf{k}_2}(t) \phi_{\mathbf{k}_3}(t) \rangle = 2 \text{Im} \left(\int_{-\infty}^t dt' \langle 0 | \phi_{\mathbf{k}_1}(t) \phi_{\mathbf{k}_2}(t) \phi_{\mathbf{k}_3}(t) H_{\text{int}}(t') | 0 \rangle \right), \quad (3.25)$$

and we therefore need to compute the following expectation value

$$\langle 0 | \phi_{\mathbf{k}_1}(t) \phi_{\mathbf{k}_2}(t) \phi_{\mathbf{k}_3}(t) \phi_{\mathbf{q}_1}(t') \phi_{\mathbf{q}_2}(t') \phi_{\mathbf{q}_3}(t') | 0 \rangle. \quad (3.26)$$

The computation should be familiar from your courses on quantum field theory. Each field operator is written in terms of annihilation and creation operators,

$$\phi_{\mathbf{k}}(t) = f_k^*(t) \hat{a}_{\mathbf{k}} + f_k(t) \hat{a}_{-\mathbf{k}}^\dagger, \quad (3.27)$$

and we evaluate (3.26) like we did the two-point function in Section 2.4 by shuffling the $\hat{a}_{\mathbf{k}}$ s all the way to the right using the commutator (2.37), so that they can annihilate the vacuum. This somewhat tedious procedure can be streamlined using Wick's theorem.¹³ We define the *Wick contraction* between two operators as

$$\overline{\phi_{\mathbf{k}}(t) \phi_{\mathbf{q}}(t')} = f_k^*(t) f_k(t') \delta^{(3)}(\mathbf{k} + \mathbf{q}) \equiv W_k(t, t') \delta^{(3)}(\mathbf{k} + \mathbf{q}), \quad (3.28)$$

where W_k is the *Wightman function*. Wick's theorem then says that (3.26) is equal to the sum of all possible such contractions. We are interested in the *connected* part of the correlation function, which consists of contractions between the external legs (i.e., one of the $\phi_{\mathbf{k}}$'s) and the vertex (i.e., one of the $\phi_{\mathbf{q}}$'s). Here is an example:

$$\begin{aligned} & \langle 0 | \overbrace{\phi_{\mathbf{k}_1}(t) \phi_{\mathbf{k}_2}(t) \phi_{\mathbf{k}_3}(t) \phi_{\mathbf{q}_1}(t') \phi_{\mathbf{q}_2}(t') \phi_{\mathbf{q}_3}(t')} | 0 \rangle \\ &= f_{k_1}^*(t) f_{k_1}(t') \delta^{(3)}(\mathbf{k}_1 + \mathbf{q}_3) f_{k_2}^*(t) f_{k_2}(t') \delta^{(3)}(\mathbf{k}_2 + \mathbf{q}_1) f_{k_3}^*(t) f_{k_3}(t') \delta^{(3)}(\mathbf{k}_3 + \mathbf{q}_2). \end{aligned} \quad (3.29)$$

Integrating over the \mathbf{q} 's, the delta function $\delta^{(3)}(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3)$ in (3.24) becomes $\delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)$. It doesn't matter how the \mathbf{q} 's are paired with the \mathbf{k} 's, they will all lead to the same result. As there are $3!$ different ways of pairing the \mathbf{q} 's and the \mathbf{k} 's, we just need to multiply the result of one way of contracting fields by $3!$. After this, (3.25) becomes

$$\langle \phi_{\mathbf{k}_1}(t) \phi_{\mathbf{k}_2}(t) \phi_{\mathbf{k}_3}(t) \rangle' = 2g \operatorname{Im} \left[f_{k_1}^*(t) f_{k_2}^*(t) f_{k_3}^*(t) \int_{-\infty}^t dt' a^3(t') f_{k_1}(t') f_{k_2}(t') f_{k_3}(t') \right]. \quad (3.30)$$

In a specific theory, with some given scale factor $a(t)$ and mode functions $f_k(t)$, we could compute the time integral and get an explicit answer for the correlator.

In order to compute to higher orders in perturbation theory—which is necessary to capture the effects of exchanges or loops—we need to expand (3.21) to higher order. For example, to compute the tree-level four-point function in ϕ^3 theory, we have to expand (3.21) to second order in H_{int} , which leads to

$$\begin{aligned} \langle \mathcal{O}(t) \rangle &= \int_{-\infty}^t dt' \int_{-\infty}^t dt'' \langle 0 | H_{\text{int}}(t') \mathcal{O}(t) H_{\text{int}}(t'') | 0 \rangle \\ &\quad - 2 \operatorname{Re} \left(\int_{-\infty}^t dt' \int_{-\infty}^{t'} dt'' \langle 0 | H_{\text{int}}(t') H_{\text{int}}(t'') \mathcal{O}(t) | 0 \rangle \right). \end{aligned} \quad (3.31)$$

In principle, this expression can be evaluated using Wick contractions as in the example above. However, like in ordinary QFT, this operator-based approach is somewhat cumbersome and it pays to work with Feynman diagrams and associated Feynman rules to simplify calculations.

Exercise 3.1 Derive (3.22) and (3.31). Verify that they are equivalent to

$$\langle \mathcal{O}(t) \rangle = i \int_{-\infty}^t dt' \langle 0 | [H_{\text{int}}(t'), \mathcal{O}(t)] | 0 \rangle, \quad (3.32)$$

$$\langle \mathcal{O}(t) \rangle = - \int_{-\infty}^t dt' \int_{-\infty}^{t'} dt'' \langle 0 | [H_{\text{int}}(t''), [H_{\text{int}}(t'), \mathcal{O}(t)]] | 0 \rangle, \quad (3.33)$$

respectively. Try to find an all-orders expression for the expansion of (3.21) of this form. →

3.1.3 Feynman Rules

Diagrammatic rules for the computation of in-in correlation functions were developed in [25, 28]. We will state these rules without explicit derivation and then illustrate them in a few examples.

1. Draw the final time surface ($t = t_*$).
2. Draw all diagrams contributing to the relevant process, allowing vertices both above and below this surface.
3. Lines that cross or end on the time surface represent a *Wightman propagator*,

$$W_k(t, t') = f_k^*(t) f_k(t'), \quad (3.34)$$

where the left argument is given by the time of the uppermost vertex and the right argument is given by the lowermost vertex. Lines which lie entirely below the fixed time surface represent *Feynman propagators*,

$$G_F(k; t, t') = W_k(t, t') \theta(t - t') + W_k(t', t) \theta(t' - t), \quad (3.35)$$

while lines which lie entirely above the surface are replaced with time-reversed Feynman propagators, $G_F(k; t', t) = G_F^*(k; t, t')$.

4. Vertices below the time surface get vertex factors $-iV$, where V is obtained by functional differentiation of the interaction Hamiltonian.¹⁴ Vertices above the time surface get factors of iV^\dagger . In Fourier space, momentum is conserved at each vertex, and there is an overall momentum-conserving delta function.
5. All vertex insertions must be integrated over time, using the measure $Dt \equiv dt a^3(t)$. If the diagram involves loops, integrate over the loop momenta.
6. If there are symmetry factors in diagrams, divide by them. In the second step above, we can mod out by reflections across the final time surface, but then we have to take 2Re of the answer.

Example Let us apply these rules to the three-point example worked out in Section 3.1.2. We are instructed to consider two diagrams—one with the interaction vertex above and one with it below the final time surface. Using the Feynman rules presented above, the first diagram gives

$$I_+ \equiv \begin{array}{c} \text{Diagram of } I_+ \\ \text{with a vertex above the time surface} \end{array} = -iV \int_{-\infty}^{t_*} Dt W_{k_1}(t_*, t) W_{k_2}(t_*, t) W_{k_3}(t_*, t) \quad (3.36)$$

$$= -iV f_{k_1}^*(t_*) f_{k_2}^*(t_*) f_{k_3}^*(t_*) \int_{-\infty}^{t_*} Dt f_{k_1}(t) f_{k_2}(t) f_{k_3}(t), \quad (3.37)$$

¹⁴For example, a theory with $L_{\text{int}} = -g\phi^n/n!$ would get a factor of $-iV = -ig$ at an n -point vertex.

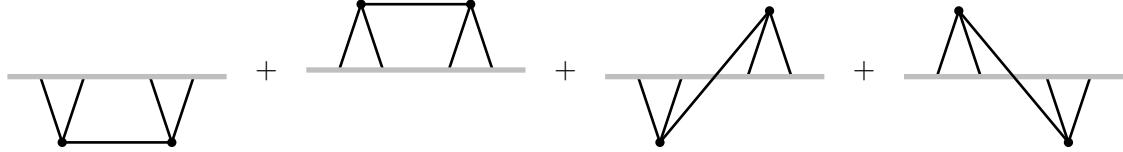
where we are being a bit schematic about the meaning of the vertex factor V . In cases that involve derivatives, we have to include the relevant factors of spatial momenta and appropriately act with time derivatives on the mode functions inside the integral. Similarly, the second diagram leads to

$$I_- \equiv \begin{array}{c} \text{Diagram of a triangle with vertices on a horizontal line} \\ \text{with two internal vertical lines} \end{array} = +iV \int_{-\infty}^{t_*} Dt W_{k_1}(t, t_*) W_{k_2}(t, t_*) W_{k_3}(t, t_*) \quad (3.38)$$

$$= +iV f_{k_1}(t_*) f_{k_2}(t_*) f_{k_3}(t_*) \int_{-\infty}^{t_*} Dt f_{k_1}^*(t) f_{k_2}^*(t) f_{k_3}^*(t). \quad (3.39)$$

The sum of the two diagrams, $\langle\phi\phi\phi\rangle = I_+ + I_-$, therefore indeed reproduces the result in (3.30).

Things become more interesting for exchange diagrams. For the tree-level four-point function in ϕ^3 theory, there are four different time orderings for the two vertices of the exchange diagram, corresponding to four distinct diagrams (in the s -channel):



Rather than writing complicated abstract formulas for each of the contributions, we will evaluate them in specific examples.

3.2 Correlators in Flat Space

Though our primary interest is in cosmology, an instructive warmup is to consider in-in correlators in flat space. In that case, the mode functions are simple enough that all time integrals can be evaluated explicitly. The computation in de Sitter space will then be conceptually exactly the same, but just algebraically more complicated.

We assume that the correlations are computed at a fixed time $t_* = 0$. This time slice is the analog of the future boundary in the de Sitter case.¹⁵ The interactions which create the late-time correlations live at $-\infty < t < 0$. For simplicity, we first consider a massless ϕ^3 theory, with the action

$$S = \int d^4x \left(-\frac{1}{2}(\partial\phi)^2 - \frac{g}{3!}\phi^3 \right). \quad (3.40)$$

The flat-space mode function for this massless scalar is

$$f_k(t) = \frac{e^{ikt}}{\sqrt{2k}}, \quad (3.41)$$

¹⁵Unlike the de Sitter case, there is no privileged time slice in Minkowski space. The choice of such a slice does break the time translation symmetry of the spacetime, but all such constant time slices are equivalent because they can be transformed into each other by time translations. It is therefore convenient to choose the slice $t_* = 0$.

which implies that the Wightman and Feynman propagators are given by

$$W_k(t, t') = \frac{1}{2k} e^{-ik(t-t')}, \quad (3.42)$$

$$G_F(k; t, t') = \frac{1}{2k} \left(e^{-ik(t-t')} \theta(t - t') + e^{ik(t-t')} \theta(t' - t) \right). \quad (3.43)$$

The vertex factors are simple because $H_{\text{int}} = -L_{\text{int}}$, so that $V = g$. We can now combine together these building blocks to compute some example correlators.

Three-point function We first compute the three-point function $\langle \phi \phi \phi \rangle$. Substituting the flat-space mode function into the expressions (3.37) and (3.39), and taking the limit $t_* \rightarrow 0$, gives

$$I_+ = -\frac{ig}{8k_1 k_2 k_3} \int_{-\infty}^0 dt e^{iKt} = -\frac{g}{8k_1 k_2 k_3} \frac{1}{K}, \quad (3.44)$$

$$I_- = \frac{ig}{8k_1 k_2 k_3} \int_{-\infty}^0 dt e^{-iKt} = -\frac{g}{8k_1 k_2 k_3} \frac{1}{K}, \quad (3.45)$$

where we have defined the sum of energies $K \equiv k_1 + k_2 + k_3$. Notice that in order to get these oscillatory integrals to converge as $t \rightarrow -\infty$, we have implicitly implemented the $i\epsilon$ prescription of (3.21) and rotated the contour slightly. Summing these answers, we get the final correlator

$$\boxed{\langle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \rangle' = -\frac{g}{4k_1 k_2 k_3} \frac{1}{(k_1 + k_2 + k_3)}}.$$

(3.46)

Notice from (3.37) and (3.39) that I_+ and I_- in general are complex conjugates of each other. In this specific case, however, both contributions are purely real, so we just get two times the contribution of one of the graphs.

Four-point function As a more nontrivial example, we next compute the four-point function $\langle \phi \phi \phi \phi \rangle$, which involves the exchange of a particle.¹⁶ In the s -channel, there are four relevant diagrams (corresponding to the four different time orderings of the interaction vertices), which we will now evaluate in turn.

We first consider the diagram where both vertices are below the final time surface. Using the Feynman rules, we find

$$\begin{aligned} I_{++} &\equiv \begin{array}{c} \text{Diagram: Two vertical lines meeting at a point below the time surface, connected by a horizontal line above the surface.} \end{array} \\ &= (-ig)^2 \int_{-\infty}^0 dt' dt'' W_{k_1}(0, t') W_{k_2}(0, t') G_F(k_I; t', t'') W_{k_3}(0, t'') W_{k_4}(0, t'') \\ &= -\frac{g^2}{16k_1 k_2 k_3 k_4} \int_{-\infty}^0 dt' dt'' e^{ik_{12}t'} G_F(k_I; t', t'') e^{ik_{34}t''}, \end{aligned} \quad (3.47)$$

¹⁶There can, of course, also be contributions to the four-point function coming from contact interactions, which will involve a single insertion of the quartic interaction Hamiltonian. See Section 3.3, for an example.

where we have defined $k_{12} \equiv k_1 + k_2$, $k_{34} \equiv k_3 + k_4$ and $k_I \equiv |\mathbf{k}_1 + \mathbf{k}_2|$. (Due to momentum conservation at the bulk vertices, the momentum of the internal line is $\mathbf{k}_I = \mathbf{k}_1 + \mathbf{k}_2 = -\mathbf{k}_3 - \mathbf{k}_4$.) Substituting the Feynman propagator (3.43), this becomes

$$\begin{aligned} I_{++} &= -\frac{g^2}{16k_1 k_2 k_3 k_4} \frac{1}{2k_I} \int_{-\infty}^0 dt' dt'' e^{ik_{12}t'} e^{-ik_I(t'-t'')} \theta(t' - t'') e^{ik_{34}t''} + (k_{12} \leftrightarrow k_{34}) \\ &= -\frac{g^2}{16k_1 k_2 k_3 k_4} \frac{1}{2k_I} \int_{-\infty}^0 dt' e^{i(k_{12}-k_I)t'} \int_{-\infty}^{t'} dt'' e^{i(k_{34}+k_I)t''} + (k_{12} \leftrightarrow k_{34}) \\ &= -\frac{g^2}{16k_1 k_2 k_3 k_4} \frac{1}{2k_I} \int_{-\infty}^0 dt' \frac{e^{i(k_{12}+k_{34})t'}}{i(k_{34}+k_I)} + (k_{12} \leftrightarrow k_{34}) \\ &= \frac{g^2}{16k_1 k_2 k_3 k_4} \frac{1}{2k_I} \left(\frac{1}{k_{12}+k_{34}} + \frac{1}{k_{34}+k_I} \right) \end{aligned} \quad (3.48)$$

$$= \frac{g^2}{16k_1 k_2 k_3 k_4} \left(\frac{1}{EE_L E_R} + \frac{1}{2k_I E_L E_R} \right), \quad (3.49)$$

where we have defined $E \equiv k_{12} + k_{34}$, $E_L \equiv k_{12} + k_I$ and $E_R \equiv k_{34} + k_I$. While in more general situations in cosmology these nested time integrals are challenging to compute, in this flat-space example the final result is a relatively simple rational function of the energies involved in the process.

Similarly, when both vertices are above the final time surface, the relevant integral is

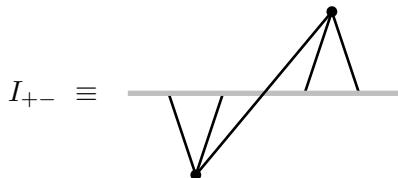
$$\begin{aligned} I_{--} &\equiv \text{Diagram showing two vertices above the final time surface connected by a horizontal line.} \\ &= (ig)^2 \int_{-\infty}^0 dt' dt'' W_{k_1}(t', 0) W_{k_2}(t', 0) G_F(k_I; t'', t') W_{k_3}(t'', 0) W_{k_4}(t'', 0). \end{aligned} \quad (3.50)$$

Note the difference in the orderings of the arguments of the propagators in (3.47) and (3.50). Using that $W_k(t', t) = W_k^*(t, t')$ and $G_F(k; t'', t') = G_F^*(k; t', t'')$, we see that $I_{--} = I_{++}^*$ and hence $I_{++} + I_{--} = 2 \operatorname{Re}(I_{++})$. In this specific case, both contributions are purely real (so $I_{--} = I_{++}$) and the sum of the two diagrams is just twice the answer in (3.49):

$$I_{++} + I_{--} = \frac{g^2}{8k_1 k_2 k_3 k_4} \left(\frac{1}{EE_L E_R} + \frac{1}{2k_I E_L E_R} \right). \quad (3.51)$$

We have separated the answer into a “connected part” (with an $E = 0$ pole) and a “disconnected part” (without an $E = 0$ pole).

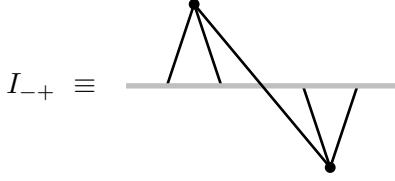
Next, we consider the case where one vertex is below the final time surface and one is above:



$$\begin{aligned}
&= (-ig)(ig) \int_{-\infty}^0 dt' dt'' W_{k_1}(0, t') W_{k_2}(0, t') W_{k_I}(t'', t') W_{k_3}(t'', 0) W_{k_4}(t'', 0) \\
&= \frac{g^2}{16k_1 k_2 k_3 k_4} \frac{1}{2k_I} \int_{-\infty}^0 dt' dt'' e^{ik_{12}t'} e^{-ik_I(t''-t')} e^{-ik_{34}t''} \\
&= \frac{g^2}{16k_1 k_2 k_3 k_4} \frac{1}{2k_I} \int_{-\infty}^0 dt' e^{i(k_{12}+k_I)t'} \int_{-\infty}^0 dt'' e^{-i(k_{34}+k_I)t''} \\
&= \frac{g^2}{16k_1 k_2 k_3 k_4} \frac{1}{2k_I} \frac{1}{(k_{12}+k_I)(k_{34}+k_I)} \\
&= \frac{g^2}{16k_1 k_2 k_3 k_4} \frac{1}{2k_I E_L E_R}. \tag{3.52}
\end{aligned}$$

Since all propagators are Wightman propagators, the two time integrations decoupled and could be performed separately. The answer therefore only includes a “disconnected part.” We will see later that it is useful to combine this with the disconnect part inside I_{++} .

Finally, we consider the contribution I_{-+} , with opposite time ordering of the interaction vertices:



$$\begin{aligned}
I_{-+} &\equiv \text{Diagram above} \\
&= ig(-ig) \int_{-\infty}^0 dt' dt'' W_{k_1}(t', 0) W_{k_2}(t', 0) W_{k_I}(t', t'') W_{k_3}(0, t'') W_{k_4}(0, t''). \tag{3.54}
\end{aligned}$$

It is straightforward to check that $I_{-+} = I_{+-}^*$, and since I_{+-} is purely real, it will just give us a factor of 2:

$$I_{+-} + I_{-+} = \frac{g^2}{8k_1 k_2 k_3 k_4} \frac{1}{2k_I E_L E_R}. \tag{3.55}$$

Notice that this contribution to the correlator only has the singularities when the energy flowing into a vertex is conserved, but does not have a singularity when the total energy is conserved.

Putting everything together, the final result for the s -channel exchange is then the sum of (3.51) and (3.55):

$$\langle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \phi_{\mathbf{k}_4} \rangle' = \frac{g^2}{8k_1 k_2 k_3 k_4} \left[\frac{1}{E E_L E_R} + \frac{1}{k_I E_L E_R} \right]. \tag{3.56}$$

One of the benefits of this simplified situation is that we can obtain completely explicit expressions for all the constituent pieces of the correlator, and get a sense of how they fit together in the final answer. When it comes time to examine the singularity structure of correlation functions, we will see that this is enormously useful.

One can proceed to compute higher-point correlation functions in this model, or consider more nontrivial types of interactions. One such example is given in the following exercise.

Exercise 3.2 Compute the four-point function arising from a cubic interaction involving derivatives

$$S = \int d^4x \left(-\frac{1}{2}(\partial\Phi)^2 - \frac{\dot{\Phi}^3}{6\Lambda^2} \right). \quad (3.57)$$

Note that the interaction involves $\dot{\Phi}$, so be careful about the definition of H_{int} . 

3.3 Correlators in de Sitter

Having gained some experience with the computation of in-in correlators in flat space, we are now ready to attack the corresponding computations in de Sitter space. Conceptually, these computations are exactly the same, only algebraically do they become a bit more involved. In fact, in many cases, the resulting time integrals cannot be performed explicitly in de Sitter space, which will motivate us to develop a bootstrap method that avoids these integrals altogether (see Section 5).

For concreteness, we again consider Φ^3 theory, where the field Φ is a generic scalar field of mass m . As we showed in Exercise 2.2, the Bunch–Davies mode function of a massive field is

$$f_k(\eta) = H \sqrt{\frac{\pi}{4}} e^{-\frac{i\pi}{4}(1+2\nu)} (-\eta)^{3/2} H_\nu^{(2)}(-k\eta), \quad \text{where } \nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}. \quad (3.58)$$

We will often specialize to *conformally coupled scalars*, with $m^2 = 2H^2$. In that case, the order of the Hankel function becomes $\nu = 1/2$ and the mode function is almost as simple as in flat space:

$$f_k(\eta) = H(-\eta) \frac{e^{ik\eta}}{\sqrt{2k}}. \quad (3.59)$$

The in-in computations simplify drastically for conformally coupled scalars and can often be performed explicitly.¹⁷

Three-point function We first compute the three-point function $\langle\phi\phi\phi\rangle$. Equation (3.37) still applies, so we have

$$I_+ = I_-^* = -ig f_{k_1}^*(\eta_*) f_{k_2}^*(\eta_*) f_{k_3}^*(\eta_*) \int_{-\infty}^{\eta_*} D\eta f_{k_1}(\eta) f_{k_2}(\eta) f_{k_3}(\eta), \quad (3.60)$$

where $D\eta \equiv d\eta a^4(\eta)$. One conceptual difference with the flat-space case is that there is a distinguished time slice in de Sitter space ($\eta_* = 0$), where it is most natural to compute correlators. This future boundary is particularly nice because it is mapped to itself under de Sitter transformations. In the following we will always compute correlators in the $\eta_* \rightarrow 0$ limit for this reason.

¹⁷ Aside from being simple to compute, it is often possible to transform the correlation functions of conformally coupled scalar fields into those of fields with other mass values, corresponding to half-integer values of ν , by a “weight-shifting” procedure [29]. The correlators of conformal scalars therefore become seeds for more complicated correlators.

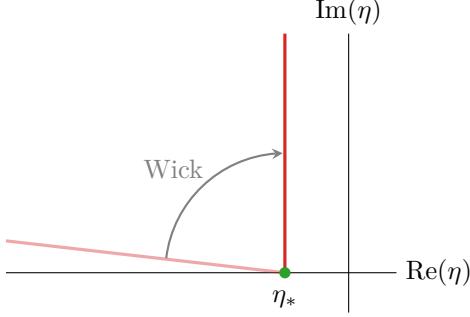


Figure 7: Graphical illustration of Wick rotation of the contour used in the integral in (3.61).

For the totally generic case involving massive fields, this involves an integral over a product of three Hankel functions. This integral can be evaluated in terms of Appell's F_4 function [30], but unfortunately the properties of this special function are still somewhat mysterious, making this representation somewhat less useful than it could be. However, for special masses the integral simplifies and can be evaluated explicitly.

One such example is the three-point function of conformally coupled scalars. In this case, the relevant integral (in the limit $\eta_* \rightarrow 0$) is

$$I_+ = -\frac{ig}{8} \frac{H^2 \eta_*^3}{k_1 k_2 k_3} \int_{-\infty^-}^{\eta_*} \frac{d\eta}{\eta} e^{iK\eta} = -\frac{ig}{8} \frac{H^2 \eta_*^3}{k_1 k_2 k_3} \log(-iK\eta_*), \quad (3.61)$$

where again $K \equiv k_1 + k_2 + k_3$. Two aspects of this expression merit discussion. First, this is our first encounter with a correlator that diverges as $\eta_* \rightarrow 0$. We regulate this IR divergence by cutting the integration off at finite $\eta = \eta_*$. Second, the answer has a factor of i in the argument of the log. This arises from the $i\epsilon$ prescription used to define the integral (see Fig. 7 for a graphical illustration of the Wick rotation of the integration contour).¹⁸

Combining I_+ with its complex conjugate I_- , we obtain the full correlator

$$\langle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \rangle' = -\frac{g}{8} \frac{H^2 \eta_*^3}{k_1 k_2 k_3} \left(i \log(-iK\eta_*) - i \log(iK\eta_*) \right) = \boxed{\frac{\pi}{8} g \frac{H^2 \eta_*^3}{k_1 k_2 k_3}}. \quad (3.62)$$

Notice that the two branches of the in-in contour conspire to cancel the $\log \eta_*$ divergence, and to remove the singularity as $K \rightarrow 0$ that I_+ and I_- have individually.

Another particularly relevant mass value is $m^2 = 0$, where the field Φ is related to the fluctuations of the inflaton field in the conformal limit, $\varepsilon \rightarrow 0$. This means that correlation functions in slow-roll inflation can often be obtained from those involving massless scalars.

¹⁸The $i\epsilon$ prescription for I_+ selects a contour that extends to infinity above the x -axis. Since the integrand does not have any singularities in the upper left quadrant of the complex plane, we can rotate it to run along the imaginary axis, $\eta \rightarrow -i\eta$.

Exercise 3.3 Consider a massless scalar field with a Φ^3 interaction in de Sitter space. Show that the bispectrum, in the limit $\eta_* \rightarrow 0$, is [31] **[DB: CHECK]**

$$\langle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \rangle' = -\frac{gH^2}{12} \frac{(\gamma + \log(|K\eta_*|)) \sum_i k_i^3 - 3K \sum_{i < j} k_i k_j + K^3}{(k_1 k_2 k_3)^3}, \quad (3.63)$$

where γ is the Euler–Mascheroni constant. What is the origin of the log divergence?

Hint: You may use that

$$\int_{-\infty}^{\eta_*} d\eta \frac{e^{iK\eta}}{\eta} \rightarrow \gamma + \log(|K\eta_*|), \quad (3.64)$$

in the limit $\eta_* \rightarrow 0$. →

It is also possible to directly integrate (3.60) when two of the fields involved are conformally coupled and the third has an arbitrary mass, as you will see in the following exercise. This correlator is useful as a building block to construct more complicated processes, and gives us some intuition for how correlators involving massive fields behave.

Exercise 3.4 Derive an expression for the three-point function of two conformally coupled scalars and one scalar of generic mass. You should find

$$\begin{aligned} \langle \phi \phi \chi \rangle' = & -\frac{g(H\eta_*)^2}{16\sqrt{\pi}} \frac{1}{k_1 k_2 k_3^2} \Gamma[\frac{1}{2} - \nu] \Gamma[\frac{1}{2} + \nu] P_{\nu - \frac{1}{2}}^0(u^{-1}) \\ & \times \left[\cos\left(\frac{\pi}{4}(1+2\nu)\right) \Gamma[\nu] \left(\frac{-k_3\eta_*}{2}\right)^{\frac{3}{2}-\nu} + \cos\left(\frac{\pi}{4}(1-2\nu)\right) \Gamma[-\nu] \left(\frac{-k_3\eta_*}{2}\right)^{\frac{3}{2}+\nu} \right], \end{aligned} \quad (3.65)$$

where $u \equiv k_3/(k_1 + k_2)$ and $P_\nu^\mu(x)$ is an associated Legendre function. Show that you recover (3.62) in the appropriate limit.

Hints: You may find the following integral representations of the Hankel function and the hypergeometric function useful

$$H_\nu^{(2)}(x) = \frac{i}{\sqrt{\pi}} \frac{2^{\nu+1} e^{i\nu\pi}}{\Gamma[\nu + \frac{1}{2}]} x^\nu e^{-ix} \int_0^\infty ds e^{-2ixs} s^{\nu - \frac{1}{2}} (1+s)^{\nu - \frac{1}{2}}, \quad (3.66)$$

$${}_2F_1 \left[\begin{matrix} a, & b \\ c & \end{matrix} \middle| z \right] = \frac{\Gamma[c]}{\Gamma[b]\Gamma[c-b]} \int_0^\infty dt t^{-b+c-1} (t+1)^{a-c} (t-z+1)^{-a}. \quad (3.67)$$

Moreover, you may use that

$$P_\nu^\mu(x) = \left(\frac{x+1}{x-1}\right)^{\mu/2} \frac{1}{\Gamma[1-\mu]} {}_2F_1 \left[\begin{matrix} \nu+1, & -\nu \\ 1-\mu & \end{matrix} \middle| \frac{1}{2} - \frac{x}{2} \right], \quad (3.68)$$

and assume that the Legendre function has no branch cut in the physical region. →

So far, we have only considered the simplest ϕ^3 interaction. If the cubic coupling between the scalar fields respects de Sitter symmetry, this choice is actually fully general—any de Sitter

invariant scalar cubic interaction can be reduced to this form via integration by parts and field redefinitions [32]. However, during inflation, the self-interactions of the inflaton generically break de Sitter symmetry, essentially because it can be thought of as a Nambu–Goldstone mode for the breaking of these symmetries. It is therefore interesting to consider more general cubic interactions that violate de Sitter symmetry, while keeping the background spacetime de Sitter. In the following exercise, you will investigate the simplest such interactions that arise in the effective field theory description of inflation [10, 11, 33].

Exercise 3.5 Consider the following action for a massless scalar field, π , in de Sitter space:

$$S = \int d^4x a^3(t) \left(\frac{1}{2} \dot{\pi}^2 - \frac{1}{2a^2(t)} (\nabla\pi)^2 + \frac{c_{\dot{\pi}^3}}{f_\pi^2} \dot{\pi}^3 + \frac{c_{\dot{\pi}(\nabla\pi)^2}}{f_\pi^2} \frac{1}{a^2(t)} \dot{\pi} (\nabla\pi)^2 \right), \quad (3.69)$$

where $c_{\dot{\pi}^3}$ and $c_{\dot{\pi}(\nabla\pi)^2}$ are dimensionless constants and f_π has mass dimension one. Compute the in-in three-point function arising from each of these interactions. →

Four-point function We next consider the four-point function $\langle \phi\phi\phi\phi \rangle$. The simplest such four-point functions arise from contact interactions, much like the three-point functions that have studied so far. In terms of (3.21), they arise from a single insertion of the quartic interactions inside H_{int} , such as the bulk interaction Φ^4 . As a concrete example, we can consider the four-point function of a conformally coupled scalar field. From the Feynman rules, we find that the I_+ contribution is given by

$$\begin{aligned} I_+ &= \overline{\text{V}} \text{V} = -iV \int_{-\infty}^{\eta_*} D\eta W_{k_1}(\eta_*, \eta) W_{k_2}(\eta_*, \eta) W_{k_3}(\eta_*, \eta) W_{k_4}(\eta_*, \eta) \\ &= -i \frac{\lambda(H\eta_*)^4}{16 k_1 k_2 k_3 k_4} \int_{-\infty}^{\eta_*} d\eta e^{i(k_1+k_2+k_3+k_4)\eta} \\ &= -\frac{\lambda(H\eta_*)^4}{16 k_1 k_2 k_3 k_4 (k_1 + k_2 + k_3 + k_4)}. \end{aligned} \quad (3.70)$$

This contribution is again purely real, so the full correlator is just twice the above answer:

$$\langle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \phi_{\mathbf{k}_4} \rangle' = -\frac{\lambda(H\eta_*)^4}{8 k_1 k_2 k_3 k_4} \frac{1}{E}, \quad (3.71)$$

where $E \equiv k_1 + k_2 + k_3 + k_4$. In this particular case, the flat-space version of the correlator is very simply related to the one that we have just calculated—one just removes the $(H\eta_*)^4 = a^{-4}(\eta_*)$ factor. This is special to four dimensions where the theory is conformal.¹⁹ Notice that (3.70) decays as $\eta_* \rightarrow 0$, but the coefficient of this power-law decay is still physically meaningful, as we will see more explicitly in Section 5.

¹⁹Since a conformally coupled scalar with a $\lambda\Phi^4$ interaction is conformally invariant, it can be related to the corresponding flat-space theory by a Weyl rescaling with $g_{\mu\nu}^{\text{dS}} = a^2(\eta)\eta_{\mu\nu}$ and $\Phi_{\text{flat}} = a(\eta)\Phi_{\text{dS}}$.

Exercise 3.6 Consider a conformally coupled scalar field in D -dimensional de Sitter space with a quartic interaction

$$S = \int d^D x \sqrt{-g} \left(-\frac{1}{2} (\partial \Phi)^2 - \frac{D(D-2)H^2}{8} \Phi^2 - \frac{\lambda}{4!} \Phi^4 \right). \quad (3.72)$$

Compute the in-in four-point function at tree level. Do you notice a difference between even and odd D ? →

We can also compute tree-level exchange contributions to four-point functions in the same way that we did in flat space. It is again useful to begin by considering a conformally coupled scalar. We can write the Wightman and Feynman propagators explicitly using (3.59):

$$W_k(\eta, \eta') = H^2 \eta \eta' \frac{1}{2k} e^{-ik(\eta-\eta')}, \quad (3.73)$$

$$G_F(k; \eta, \eta') = H^2 \eta \eta' \frac{1}{2k} \left(e^{-ik(\eta-\eta')} \theta(\eta - \eta') + e^{ik(\eta-\eta')} \theta(\eta' - \eta) \right). \quad (3.74)$$

Notice that these are exactly $H^2 \eta \eta'$ times their flat-space counterparts (3.42) and (3.43). (This is another manifestation of the conformal invariance of the conformally coupled scalar.) The I_{++} contribution is

$$\begin{aligned} I_{++} &\equiv \text{Diagram: Two vertical lines meeting at a central horizontal line, forming a V-shape.} \\ &= (-ig)^2 \int_{-\infty}^{\eta_*} D\eta' D\eta'' W_{k_1}(\eta_*, \eta') W_{k_2}(\eta_*, \eta') G_F(k_I; \eta', \eta'') W_{k_3}(\eta_*, \eta'') W_{k_4}(\eta_*, \eta'') \\ &= -\frac{g^2 H^2 \eta_*^4}{16 k_1 k_2 k_3 k_4} \int_{-\infty}^{\eta_*} \frac{d\eta'}{\eta'} \int_{-\infty}^{\eta_*} \frac{d\eta''}{\eta''} e^{ik_{12}\eta'} G_F^{(\text{flat})}(k_I; \eta', \eta'') e^{ik_{34}\eta''}, \end{aligned} \quad (3.75)$$

where, in the second line, we have suggestively written the expression in terms of the flat-space Feynman propagator (3.43). This integral is a bit tricky to evaluate because it is divergent. As we will see, the divergences in I_{--} and I_{++} cancel against similar divergences in I_{-+} and I_{+-} . In order to isolate the divergent piece, it is therefore useful to add and subtract a contribution to the integral (3.75) as²⁰

$$I_{++} = -\frac{g^2 H^2 \eta_*^4}{16 k_1 k_2 k_3 k_4} \int_{-\infty}^{\eta_*} \frac{d\eta'}{\eta'} \int_{-\infty}^{\eta_*} \frac{d\eta''}{\eta''} e^{ik_{12}\eta'} \left(G_B^{(\text{flat})}(k_I; \eta', \eta'') + \frac{1}{2k_I} e^{ik_I(\eta'+\eta'')} \right) e^{ik_{34}\eta''}, \quad (3.76)$$

where we have defined²¹

$$G_B^{(\text{flat})}(k_I; \eta', \eta'') \equiv G_F^{(\text{flat})}(k_I; \eta', \eta'') - \frac{1}{2k_I} e^{ik_I(\eta'+\eta'')}. \quad (3.77)$$

We can now perform the integral involving G_B via a trick. We note that

$$-i \int_a^\infty dx e^{ix\eta} = \frac{1}{\eta} e^{ian}, \quad (3.78)$$

²⁰We have dropped a factor of $e^{-iE\eta_*}$ which becomes unity in the limit $\eta_* \rightarrow 0$.

²¹In Section 4, we will encounter this object as the *bulk-to-bulk propagator*.

so that (3.76) can be written as

$$I_{++} = \frac{g^2 H^2 \eta_*^4}{16k_1 k_2 k_3 k_4} \left[\int_{k_{12}}^\infty dx \int_{k_{34}}^\infty dy \int_{-\infty}^{\eta_*} d\eta' d\eta'' e^{ix\eta'} e^{iy\eta''} G_B^{(\text{flat})}(k_I; \eta', \eta'') - \frac{1}{2k_I} \int_{-\infty}^{\eta_*} \frac{d\eta'}{\eta'} \int_{-\infty}^{\eta_*} \frac{d\eta''}{\eta''} e^{i(k_{12}+s)\eta'} e^{i(k_{34}+s)\eta''} \right], \quad (3.79)$$

The time integrals in the first line now only involve flat-space quantities and can therefore easily be computed

$$I(x, y) \equiv \int_{-\infty}^0 d\eta' d\eta'' e^{ix\eta'} e^{iy\eta''} G_B^{(\text{flat})}(k_I; \eta', \eta'') = -\frac{1}{(x+y)(x+k_I)(y+k_I)}. \quad (3.80)$$

It is then useful to split I_{++} into two pieces: a “connected” component, $I_{++}^{(c)}$, which is the energy integral of $I(x, y)$, and a “disconnected” component, $I_{++}^{(d)}$, which involves un-ordered time integrals.

Performing the energy integrals, the connected piece becomes [17]

$$\begin{aligned} I_{++}^{(c)} &\equiv \frac{g^2 H^2 \eta_*^4}{16k_1 k_2 k_3 k_4} \int_{k_{12}}^\infty dx \int_{k_{34}}^\infty dy I(x, y) \\ &= \boxed{\frac{g^2 H^2 \eta_*^4}{32k_1 k_2 k_3 k_4 k_I} \left[\text{Li}_2\left(\frac{E - E_L}{E}\right) + \text{Li}_2\left(\frac{E - E_R}{E}\right) + \log\left(\frac{E_L}{E}\right) \log\left(\frac{E_R}{E}\right) - \frac{\pi^2}{6} \right]}, \end{aligned} \quad (3.81)$$

where Li_2 is the dilogarithm and we have defined the variables $E_L \equiv k_{12} + k_I$ (the energy flowing into the left vertex), $E_R \equiv k_{34} + k_I$ (same for the right vertex) and the total energy $E \equiv k_{12} + k_{34}$.

Exercise 3.7 Derive the result (3.81). →

In order to compute the disconnected piece, it is simplest to do the time integrals directly, rather than trying to write them as energy integrals of the flat-space result. This is because we have to be careful about the $i\epsilon$ prescriptions for the contours, much like we did for (3.61). In this case, we can rotate the contours in the same direction as (3.61), so we get

$$\begin{aligned} I_{++}^{(d)} &\equiv -\frac{g^2 H^2 \eta_*^4}{32k_1 k_2 k_3 k_4 k_I} \int_{-\infty}^{\eta_*} \frac{d\eta'}{\eta'} e^{i(k_{12}+s)\eta'} \int_{-\infty}^{\eta_*} \frac{d\eta''}{\eta''} e^{i(k_{34}+s)\eta''} \\ &= -\frac{g^2 H^2 \eta_*^4}{32k_1 k_2 k_3 k_4 k_I} \log(-iE_L \eta_*) \log(-iE_R \eta_*). \end{aligned} \quad (3.82)$$

As advertised, this contribution is divergent as $\eta_* \rightarrow 0$, which is why we split it off from $I_{++}^{(c)}$ to begin with. These divergences cancel with the $I_{\mp\pm}$ pieces, which we now want to calculate.

Using the Feynman rules, we find

$$\begin{aligned}
I_{+-} &\equiv \text{Feynman diagram for } I_{+-} \\
&= g^2 \int_{-\infty}^{\eta_*} D\eta' D\eta'' W_{k_1}(\eta_*, \eta') W_{k_2}(\eta_*, \eta') W_{k_I}(\eta'', \eta') W_{k_3}(\eta'', \eta_*) W_{k_4}(\eta'', \eta_*) \\
&= \frac{g^2 H^2 \eta_*^4}{32 k_1 k_2 k_3 k_4 k_I} \int_{-\infty}^{\eta_*} \frac{d\eta'}{\eta'} e^{i(k_{12}+k_I)\eta'} \int_{-\infty}^{\eta_*} \frac{d\eta''}{\eta''} e^{-i(k_{34}+k_I)\eta''} \\
&= \frac{g^2 H^2 \eta_*^4}{32 k_1 k_2 k_3 k_4 k_I} \log(-iE_L \eta_*) \log(iE_R \eta_*) . \tag{3.83}
\end{aligned}$$

In the last line, we evaluated the integrals, noting that the different $i\epsilon$ prescriptions required for convergence of the integrals lead to different signs of the arguments of the log for the η' and η'' integrals. Adding (3.83) to the result (3.82) for $I_{++}^{(d)}$, we get

$$I_{++}^{(d)} + I_{+-} = \frac{g^2 H^2 \eta_*^4}{32 k_1 k_2 k_3 k_4 k_I} (-i\pi \log(-iE_L \eta_*)) . \tag{3.84}$$

Notice that this is similar in structure to what we found for the three-point function in (3.61). The sum of the $I_{--}^{(d)}$ and I_{-+} contributions gives the complex conjugate of (3.84). Putting everything together, we therefore find

$$I_{++}^{(d)} + I_{+-} + I_{--}^{(d)} + I_{-+} = \boxed{\frac{g^2 H^2 \eta_*^4}{32 k_1 k_2 k_3 k_4 k_I} \pi^2} . \tag{3.85}$$

Note that there is a similar effect as in (3.62) where the kinematic part of the log drops out and we are left with just factors of π .

Assembling all the pieces, the full correlator is

$$\begin{aligned}
\langle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \phi_{\mathbf{k}_4} \rangle' &= I_{++}^{(c)} + (I_{++}^{(d)} + I_{+-}) + \text{c.c.} \\
&= \boxed{\frac{g^2 H^2 \eta_*^4}{16 k_1 k_2 k_3 k_4 k_I} \left[\text{Li}_2\left(\frac{E - E_L}{E}\right) + \text{Li}_2\left(\frac{E - E_R}{E}\right) + \log\left(\frac{E_L}{E}\right) \log\left(\frac{E_R}{E}\right) + \frac{\pi^2}{3} \right]} . \tag{3.86}
\end{aligned}$$

Notice that this correlator has transcendentality two, compared to its flat-space counterpart (3.56), which is a rational function. This is rather interesting, and is in some sense a signature of the time evolution of the spacetime that gave rise to it. Aside from its intrinsic interest, and serving as a useful testing ground, this correlator is closely related to many correlation functions of practical interest, including those involving exchanges of massless particles with spin, which can be generated by differentiating this correlator in a suitable way [29, 34].

As a last example, we consider the four-point function that arises from the exchange of a massive particle $m > \frac{3}{2}H$ between two pairs of conformally coupled scalar fields. In this case,

the order ν of the Hankel function in (3.58) is purely imaginary, so it is useful to write the mode functions as

$$f_k(\eta) = e^{-\frac{i\pi}{4}} H \sqrt{\frac{\pi}{4}} (-\eta)^{3/2} e^{\frac{\pi}{2}\mu} H_{i\mu}^{(2)}(-k\eta), \quad \text{where } i\mu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}. \quad (3.87)$$

The definition of μ is such that it is a real number for $m > \frac{3}{2}H$. From this mode function, we can build the Wightman and Feynman propagators:²²

$$W_k(\eta, \eta') = \frac{H^2\pi}{4} (\eta\eta')^{3/2} H_{i\mu}^{(1)}(-k\eta) H_{i\mu}^{(2)}(-k\eta'), \quad (3.89)$$

$$G_F(k; \eta, \eta') = W_k(\eta, \eta') \theta(\eta - \eta') + W_k(\eta', \eta) \theta(\eta' - \eta). \quad (3.90)$$

With these building blocks, we can write down the various exchange contributions. For example, I_{++} is

$$I_{++} = -\frac{g^2\eta_*^4}{16k_1k_2k_3k_4} \int_{-\infty}^{\eta_*} \frac{d\eta'}{(\eta')^2} \int_{-\infty}^{\eta_*} \frac{d\eta''}{(\eta'')^2} e^{ik_{12}\eta'} G_F(k_I; \eta', \eta'') e^{ik_{34}\eta''}, \quad (3.91)$$

and I_{--} is just the complex conjugate of this expression. Similarly, I_{+-} is

$$I_{+-} = \frac{g^2\eta_*^4}{16k_1k_2k_3k_4} \int_{-\infty}^{\eta_*} \frac{d\eta'}{(\eta')^2} \int_{-\infty}^{\eta_*} \frac{d\eta''}{(\eta'')^2} e^{ik_{12}\eta'} W_{k_I}(\eta'', \eta') e^{-ik_{34}\eta''}, \quad (3.92)$$

whose complex conjugate is I_{-+} .

The integral in (3.92) can be done by the same method employed in Exercise 3.4, but doing the nested time integrals involved in (3.91) is a challenge that we will return to later. However, in the limit that the exchanged momentum is small $k_I \rightarrow 0$, the computation is quite tractable.

Exercise 3.8 In the long-wavelength limit $k_I \rightarrow 0$, there is no difference between the Wightman and Feynman propagators:

$$\begin{aligned} \lim_{k_I \rightarrow 0} G(k_I; \eta', \eta'') &= \lim_{k_I \rightarrow 0} W_{k_I}(\eta', \eta'') \\ &= \frac{H^2}{4\pi} (\eta'\eta'')^{3/2} \left[\Gamma[-i\mu]^2 \left(\frac{k_I^2 \eta' \eta''}{4} \right)^{i\mu} + \Gamma[i\mu]^2 \left(\frac{k_I^2 \eta' \eta''}{4} \right)^{-i\mu} \right]. \end{aligned} \quad (3.93)$$

This is basically because we are probing separations between the two vertices that are outside each others' lightcones, so that the time-ordering does not matter. (This still captures the interesting non-analytic signature of particle exchange.) Use this fact to perform the time integrals (3.91) and (3.92), and combine them together with I_{--} and I_{-+} to obtain the in-in correlator in the $k_I \rightarrow 0$ limit. →

²²Here, we have used the fact that

$$\left(H_{i\mu}^{(2)}(x) \right)^* = H_{-i\mu}^{(1)}(x) = e^{-\pi\mu} H_{i\mu}^{(1)}(x), \quad (3.88)$$

which causes the explicit exponential factors of μ to cancel in the Wightman function.

4 Wavefunction Approach

Cosmological correlations can also be computed by means of the so-called “wavefunction of the universe.” The wavefunction is a slightly more primitive object than correlators themselves—the relation between them is roughly the same as that between the S -matrix and scattering cross-sections. As a result, the wavefunction is somewhat simpler than correlators in certain ways that we will see. In this section, we will describe the wavefunction approach and show, in specific examples, how it reproduces the in-in results of the previous section.

4.1 Wavefunction of the Universe

Consider a set of bulk fields, $\Phi(t, \mathbf{x})$, which can include both matter fields (such as the inflaton) and metric fluctuations (such as the graviton). We are interested in their spatial correlations. The information about these correlations is contained in the quantum-mechanical state of the system at a given time. It is convenient to describe states in the basis of field eigenstates, which (in the Heisenberg picture) satisfy

$$\hat{\Phi}(t, \mathbf{x})|\phi(\mathbf{x})\rangle = \phi(\mathbf{x})|\phi(\mathbf{x})\rangle, \quad (4.1)$$

where $\hat{\Phi}$ is a field operator and $\phi(\mathbf{x}) \equiv \phi(t, \mathbf{x})$ is the spatial profile of the field at a given time t . Any state of the system can be projected onto this basis, but we will primarily be interested in the (interacting) vacuum state of the system. Writing this state in the basis of field eigenstates, the overlap coefficients are

$$\Psi[\phi(\mathbf{x})] \equiv \langle \phi(\mathbf{x}) | 0 \rangle. \quad (4.2)$$

This *wavefunctional* expresses the overlap between the vacuum state and a given state of the late-time fluctuations, and therefore provides a probability density for spatial field configurations. Equal-time correlation functions of the field are then defined as

$$\langle \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_n) \rangle = \int \mathcal{D}\phi \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_n) |\Psi[\phi]|^2, \quad (4.3)$$

which is a familiar formula from quantum mechanics applied to this field theory setup. This formula makes it clear that the wavefunctional and (equal-time) correlation functions contain the same information, though the translation between the two can be somewhat complicated in general. The fact that the wavefunction is in a sense a “square root” of correlation functions will lead to some simplifications in its structure, as we will see.

Since the wavefunctional is just a way of representing the state of a quantum field theory, it can be employed in a variety of situations, including in flat space. What is special about cosmology is that there is a distinguished time slice on which to consider the wavefunction—the future boundary of de Sitter space. In this $\eta_* \rightarrow 0$ limit, fields take on time-independent spatial profiles and the fact that the symmetries of de Sitter space preserve this spatial slice means that the invariance of the vacuum wavefunctional under de Sitter symmetry is easy to implement. In many cases this makes it possible to reconstruct the wavefunctional directly on the boundary from these symmetry principles (see Section 5).

The wavefunction at a given time (which we can set to $t_* = 0$ without loss of generality) is formally computed by the following path integral

$$\Psi[\phi] = \int \mathcal{D}\Phi e^{iS[\Phi]}, \quad \begin{array}{l} \Phi(0) = \phi \\ \Phi(-\infty^+) = 0 \end{array} \quad (4.4)$$

which defines a sum over field configurations that interpolate between the free theory vacuum in the far past (selected via the usual $i\epsilon$ prescription) and a Dirichlet boundary condition that the field have some specified spatial profile at $t_* = 0$, $\phi(\mathbf{x}) \equiv \phi(t_*, \mathbf{x})$. Clearly, Ψ depends on this choice of late-time boundary condition for the field, making it a functional.

In practice, we compute the path integral (4.4) in perturbation theory. At tree level, the path integral can then be approximated by its saddle point

$$\Psi[\phi] \approx \exp(iS[\Phi_{\text{cl}}]), \quad (4.5)$$

where Φ_{cl} is a solution to the classical equations of motion with future boundary condition that $\Phi_{\text{cl}}(t_*) = \phi$. We will almost exclusively be interested in tree-level processes, so that this lowest-order approximation of the wavefunction will be sufficient for our purposes, but it can be systematically corrected by including quantum fluctuations around the classical saddle, as is typical in path-integral computations.

We begin by considering the evaluation of the wavefunction (4.4) in free theory, where the saddle-point approximation (4.5) is exact. We then turn to the perturbative evaluation of the wavefunction in interacting theories, and show how computations utilizing the wavefunction are equivalent to the in-in formalism.

4.2 Warmup in Quantum Mechanics

To get an intuitive understanding of the wavefunction approach, it is instructive to first apply it to a few simple examples in quantum mechanics.

4.2.1 Harmonic Oscillator

Consider a *simple harmonic oscillator*, whose action is

$$S[\Phi] = \int dt \left(\frac{1}{2} \dot{\Phi}^2 - \frac{1}{2} \omega^2 \Phi^2 \right), \quad (4.6)$$

where Φ is the deviation from equilibrium (which we called q in Exercise 2.1) and ω is the constant frequency of the oscillator. On-shell, the action can equivalently be written as a pure boundary term

$$\begin{aligned} S[\Phi_{\text{cl}}] &= \int_{t_i}^{t_*} dt \left[\frac{1}{2} \partial_t (\dot{\Phi}_{\text{cl}} \Phi_{\text{cl}}) - \frac{1}{2} \Phi_{\text{cl}} \underbrace{(\ddot{\Phi}_{\text{cl}} + \omega^2 \Phi_{\text{cl}})}_{=0} \right] \\ &= \frac{1}{2} \dot{\Phi}_{\text{cl}} \Phi_{\text{cl}} \Big|_{t=t_*}, \end{aligned} \quad (4.7)$$

where we have integrated by parts and used the fact that the classical solution satisfies the equation of motion $\ddot{\Phi}_{\text{cl}} + \omega^2 \Phi_{\text{cl}} = 0$.

To determine the classical solution $\Phi_{\text{cl}}(t)$ we have to specify two boundary conditions: We require: 1) the late-time value of the oscillator position is $\Phi_{\text{cl}}(t_* \equiv 0) = \phi$, and 2) the early-time limit is the minimum-energy solution $\Phi_{\text{cl}}(t) \sim e^{i\omega t}$. The unique solution satisfying these boundary conditions is $\Phi_{\text{cl}}(t) = \phi e^{i\omega t}$. Substituting this into (4.7), we get

$$S = \frac{i\omega}{2} \phi^2 \implies \exp(iS) = \exp\left(-\frac{\omega}{2}\phi^2\right) \implies |\Psi[\phi]|^2 = e^{-\omega\phi^2}. \quad (4.8)$$

We see the familiar fact that the ground state wavefunction of the simple harmonic oscillator is a Gaussian.²³ The width of this Gaussian determines the size of the zero-point fluctuations of the oscillator:

$$\langle \phi^2 \rangle = \frac{1}{2\omega}. \quad (4.9)$$

In Exercise 2.1, we derived the same result using canonical quantization.

Free quantum field theory in Fourier space is essentially equivalent to a set of harmonic oscillators indexed by the continuous wavenumber \mathbf{k} . To see this, we write the action of a free scalar in Fourier space:

$$\begin{aligned} S &= \int dt d^3x \left(\frac{1}{2} \dot{\Phi}^2 - \frac{1}{2} (\nabla \Phi)^2 - \frac{1}{2} m^2 \Phi^2 \right) \\ &= \int dt \frac{d^3k}{(2\pi)^3} \left(\frac{1}{2} \dot{\Phi}_{\mathbf{k}} \dot{\Phi}_{-\mathbf{k}} - \frac{1}{2} (k^2 + m^2) \Phi_{\mathbf{k}} \Phi_{-\mathbf{k}} \right), \end{aligned} \quad (4.10)$$

where $\Phi_{\mathbf{k}}(t)$ are the Fourier modes of the field. We see that these Fourier modes each satisfy the equation of a simple harmonic oscillator with frequencies $\omega_k \equiv \sqrt{k^2 + m^2}$. The wavefunction, and the associated zero-point fluctuations, can then be written as a sum over the corresponding results for each harmonic oscillator:

$$\Psi[\phi] = \exp\left(-\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \omega_k \phi_{\mathbf{k}} \phi_{-\mathbf{k}}\right) \implies \langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle' = \frac{1}{2\omega_k}. \quad (4.11)$$

The treatment of free fields in quantum field theory is therefore as simple as that of an infinite collection of uncoupled harmonic oscillators in quantum mechanics.

4.2.2 Time-Dependent Oscillator

In Section 2.4, we showed that a free field in de Sitter space (and also the fluctuations during inflation) satisfy the equation of motion of a harmonic oscillator with a time-dependent frequency (for each Fourier mode); cf. equation (2.25). It is therefore useful to consider the the following *time-dependent oscillator*:

$$S[\Phi] = \int dt \left(\frac{1}{2} \textcolor{red}{A(t)} \dot{\Phi}^2 - \frac{1}{2} \textcolor{blue}{B(t)} \Phi^2 \right), \quad (4.12)$$

²³If we wanted to reproduce the correct phase of the wavefunction $\sim e^{i\omega t/2}$, we would have to compute the Gaussian integral for fluctuations around this classical trajectory. See, e.g. Appendix A of [35] for a discussion. Since this phase is independent of ϕ , it is not important for our purposes.

where $A(t)$ and $B(t)$ are arbitrary functions of time. On-shell, this action can again be written as a boundary term

$$\begin{aligned} S[\Phi_{\text{cl}}] &= \int_{t_i}^{t_*} dt \left[\frac{1}{2} \partial_t (A \dot{\Phi}_{\text{cl}} \Phi_{\text{cl}}) - \frac{1}{2} \Phi_{\text{cl}} \underbrace{(\partial_t (A \dot{\Phi}_{\text{cl}}) + B \Phi_{\text{cl}})}_{=0} \right] \\ &= \frac{1}{2} A \dot{\Phi}_{\text{cl}} \Phi_{\text{cl}} \Big|_{t=t_*}, \end{aligned} \quad (4.13)$$

where we have used that the classical solution satisfies $\partial_t (A \dot{\Phi}_{\text{cl}}) + B \Phi_{\text{cl}} = 0$.

We again need to find the classical solution with the correct boundary conditions to describe the ground state. We can write the desired solution abstractly as $\Phi_{\text{cl}}(t) = \phi K(t)$. Substituting this into the equation of motion, we obtain a differential equation for K :

$$\partial_t (A(t) \dot{K}(t)) + B(t) K(t) = 0. \quad (4.14)$$

The boundary conditions that we impose on the solution are that $K(t_*) = 1$ and $K(-\infty) \sim e^{i\omega t}$. Of course, the precise form of $K(t)$ depends on $A(t)$ and $B(t)$, but given those functions we can then solve the equation of motion subject to the prescribed boundary conditions to obtain K .

Substituting this solution into (4.13), we have the following expression for the ground state wavefunction

$$\Psi[\phi] = \exp(iS) = \exp \left(\frac{i}{2} (A \partial_t \log K) \Big|_{t=t_*} \phi^2 \right), \quad (4.15)$$

and hence we can compute the two-point correlator using

$$|\Psi[\phi]|^2 = \exp(-\text{Im}(A \partial_t \log K) \Big|_{t=t_*} \phi^2) \implies \boxed{\langle \phi^2 \rangle = \frac{1}{2 \text{Im}(A \partial_t \log K) \Big|_{t=t_*}}}. \quad (4.16)$$

As we will see in the next section, this result will be directly applicable to fields in de Sitter space.

4.3 Free Fields in de Sitter

Having gained some intuition for the wavefunction in the quantum-mechanical setting, we are now ready to use it to compute de Sitter correlators.

We start with the action of a free massive scalar field in a curved background:

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} m^2 \Phi^2 \right). \quad (4.17)$$

Substituting the de Sitter metric (2.15) (in conformal time), this becomes

$$\begin{aligned} S &= \int d\eta d^3x a^4(\eta) \left[\frac{1}{2a^2(\eta)} ((\Phi')^2 - (\nabla \Phi)^2) - \frac{1}{2} m^2 \Phi^2 \right] \\ &= \frac{1}{2} \int d\eta \frac{d^3k}{(2\pi)^3} \left[\frac{1}{(H\eta)^2} \Phi'_k \Phi'_{-k} - \frac{1}{(H\eta)^2} \left(k^2 + \frac{m^2}{(H\eta)^2} \right) \Phi_k \Phi_{-k} \right], \end{aligned} \quad (4.18)$$

where, in the second line, we have written the field in terms of its Fourier components and used $a(\eta) = -1/(H\eta)$. We see that the action for each Fourier mode is the same as that of the time-dependent oscillator in (4.12), with the identifications

$$A(\eta) \equiv \frac{1}{(H\eta)^2}, \quad (4.19)$$

$$B(\eta) \equiv \frac{1}{(H\eta)^2} \left(k^2 + \frac{m^2}{(H\eta)^2} \right). \quad (4.20)$$

We can therefore apply the result (4.16) to compute the wavefunction and the variance of the field.

The classical equation of motion is

$$(A\Phi'_{\text{cl}})' + B\Phi_{\text{cl}} = 0 \implies \Phi''_{\text{cl}} + \frac{2}{\eta}\Phi'_{\text{cl}} + \left(k^2 + \frac{(m/H)^2}{\eta^2} \right)\Phi_{\text{cl}} = 0. \quad (4.21)$$

The qualitative features of the solutions to this equation were discussed in Section 2.4. We saw that on small scales, $|k\eta| \gg 1$, the equation reduces to that of simple harmonic oscillator with frequency k . On large scales, $|k\eta| \rightarrow 0$, the harmonic oscillator becomes over-damped and the fluctuations freeze-out (for $m = 0$) or decay slowly (for $m > 0$).

We write the solution to (4.21) as

$$\Phi_{\text{cl}}(\eta) \equiv \phi K(\eta), \quad (4.22)$$

where the function $K(\eta)$ approaches unity at late times and oscillates like $e^{ik\eta}$ at early times. For massless fields, we have

$$\begin{aligned} K(\eta) &= (1 - ik\eta)e^{ik\eta}, \\ \log K(\eta) &= \log(1 - ik\eta) + ik\eta. \end{aligned} \quad (4.23)$$

Substituting this into (4.16) gives

$$\begin{aligned} \text{Im}(A\partial_\eta \log K)|_{\eta=\eta_*} &= \frac{1}{(H\eta_*)^2} \text{Im} \left(\frac{-ik}{1 - ik\eta_*} + ik \right) \\ &= \frac{1}{(H\eta_*)^2} \text{Im} \left(\frac{k^2\eta_* + ik^3\eta_*^2}{1 + k^2\eta_*^2} \right) \xrightarrow{\eta_* \rightarrow 0} \boxed{\frac{k^3}{H^2}}, \end{aligned} \quad (4.24)$$

where we have taken $\eta_* \rightarrow 0$ at the end. The late-time two-point function of the field therefore is

$$\boxed{\langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle' = \frac{H^2}{2k^3}}, \quad (4.25)$$

which is the same as the result (2.52) derived by canonical quantization in Section 2.4.

For massive fields, the relevant solution to (4.21) has

$$\begin{aligned} K(\eta) &= \left(\frac{\eta}{\eta_*} \right)^{3/2} \frac{H_\nu^{(2)}(-k\eta)}{H_\nu^{(2)}(-k\eta_*)}, \\ \log K(\eta) &= \frac{3}{2} \log \eta + \log [H_\nu^{(2)}(-k\eta)], \end{aligned} \quad (4.26)$$

and hence the width of the wavefunction is given by

$$\begin{aligned}\text{Im}(A\partial_\eta \log K)|_{\eta=\eta_*} &= \frac{1}{(H\eta_*)^2} \text{Im} \left(\frac{\partial_\eta H_\nu^{(2)}(-k\eta)}{H_\nu^{(2)}(-k\eta)} \right) \Big|_{\eta=\eta_*} \\ &= \frac{1}{(H\eta_*)^2} \frac{1}{|H_\nu^{(2)}(-k\eta_*)|^2} \text{Im} \left(H_\nu^{(2)*}(-k\eta) \partial_\eta H_\nu^{(2)}(-k\eta) \right) \Big|_{\eta=\eta_*}. \quad (4.27)\end{aligned}$$

Consider first the case $m < \frac{3}{2}H$, where the order ν of the Hankel function is real. We then have $H_\nu^{(2)*} = H_\nu^{(1)}$ and hence

$$\text{Im} \left(H_\nu^{(2)*} \partial_\eta H_\nu^{(2)} \right) \Big|_{\eta=\eta_*} = \frac{1}{2i} \left(H_\nu^{(1)} \partial_\eta H_\nu^{(2)} - H_\nu^{(2)} \partial_\eta H_\nu^{(1)} \right). \quad (4.28)$$

We recognize the term in the brackets as the Wronskian of the Hankel function:

$$H_\nu^{(1)} \partial_\eta H_\nu^{(2)} - H_\nu^{(2)} \partial_\eta H_\nu^{(1)} = -\frac{4i}{\pi} \frac{1}{\eta}. \quad (4.29)$$

Substituting this into (4.27), we get

$$\text{Im}(A\partial_\eta \log K)|_{\eta=\eta_*} = \frac{1}{(H\eta_*)^2} \frac{1}{|H_\nu^{(2)}(-k\eta_*)|^2} \left(-\frac{2}{\pi} \frac{1}{\eta_*} \right). \quad (4.30)$$

The two-point function of the field is therefore

$$\langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle' = \frac{\pi}{4} H^2 (-\eta_*)^3 H_\nu^{(1)}(-k\eta_*) H_\nu^{(2)}(-k\eta_*) \Bigg], \quad (4.31)$$

which is consistent with the result obtained in Exercise 2.2.

Exercise 4.1 Prove equation (4.29).

Hint: Take a derivative of the Wronskian W and use the Bessel equation to obtain a first-order differential equation for W . Find the solution of this equation, fixing the integration constant by considering the large argument limit of the Hankel functions. →

The two-point function (4.31) can actually be analytically continued to $m > \frac{3}{2}H$ by sending $\nu \rightarrow i\mu$, though showing this requires a little bit of work.

Exercise 4.2 Show that for $m > \frac{3}{2}H$, where $\nu \equiv i\mu$ is pure imaginary, the power spectrum becomes

$$\begin{aligned}\langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle' &= \frac{\pi}{4} H^2 (-\eta_*)^3 e^{\pi\mu} |H_{i\mu}^{(2)}(-k\eta_*)|^2 \\ &= \frac{\pi}{4} H^2 (-\eta_*)^3 H_{i\mu}^{(2)}(-k\eta_*) H_{i\mu}^{(1)}(-k\eta_*) . \quad (4.32)\end{aligned}$$

The last expression holds both for real and imaginary orders of the Hankel function. →

4.4 Interactions

Our main interest in these lectures is not in free fields, but in their interactions and the associated higher-point correlations. In this section, we will use the wavefunction approach to study higher-point correlations and show how this reproduces the in-in results of Section 3.

4.4.1 Anharmonic Oscillator

Before we study interacting field theory, it is again useful to first study a simpler example within quantum mechanics—the *anharmonic oscillator*. Consider the following action

$$S[\Phi] = \int dt \left(\frac{1}{2} \dot{\Phi}^2 - \frac{1}{2} \omega^2 \Phi^2 - \frac{1}{3} g \Phi^3 \right), \quad (4.33)$$

which is the ordinary harmonic oscillator perturbed by the addition of a cubic interaction.²⁴ As before, we want to determine its ground state wavefunction by computing the on-shell action, $S[\Phi_{\text{cl}}]$.²⁵ The classical solution must obey the same boundary conditions as for the unperturbed oscillator: $\Phi_{\text{cl}}(t=0) = \phi$ and $\Phi_{\text{cl}}(t \rightarrow -\infty) = e^{i\omega t}$. The complication in this case is that the equation of motion is nonlinear

$$\ddot{\Phi} + \omega^2 \Phi = -g \Phi^2, \quad (4.34)$$

and the solutions cannot be written in closed form. We can, however, write a formal solution to this equation by treating the $g\Phi^2$ term as a source:

$$\Phi_{\text{cl}}(t) = \phi e^{i\omega t} + i \int dt' G(t, t') (-g \Phi_{\text{cl}}^2(t')), \quad (4.35)$$

where $\phi e^{i\omega t}$ is the solution to the free equation of motion and $G(t, t')$ is a Green's function for the harmonic oscillator, which satisfies $(\partial_t^2 + \omega^2)G(t, t') = -i\delta(t - t')$. The boundary conditions we specify for Φ_{cl} determine what Green's function we should use. In this case, the nontrivial boundary condition that we have to impose is $G(0, t') = G(t, 0) = 0$. This guarantees that only the $\phi e^{i\omega t}$ part of the solution contributes at $t = 0$, as this piece already satisfies the boundary condition $\Phi_{\text{cl}}(0) = \phi$ by itself. The relevant Green's function is then

$$G(t, t') = \frac{1}{2\omega} \left(e^{-i\omega(t-t')} \theta(t-t') + e^{i\omega(t-t')} \theta(t'-t) - e^{i\omega(t+t')} \right), \quad (4.36)$$

which can be derived in the usual way.

Exercise 4.3 Derive the Green's function (4.36) for the the harmonic oscillator by imposing the usual continuity and jump conditions along with requiring that the field oscillate as $e^{i\omega t}$ in the far past and that the Green's function vanishes at $t = 0$.



²⁴This model is, of course, unstable. However, since we are only interested in perturbatively correcting the ground state of the ordinary harmonic oscillator, this instability will not be important. One could alternatively consider the ϕ^4 anharmonic oscillator if this instability is troubling.

²⁵[DB: footnote on classical approximation, reintroducing \hbar]

To derive the wavefunction of the oscillator in the saddle point approximation, we have to compute the on-shell action, $S[\Phi_{\text{cl}}]$. It is convenient to first write (4.33) in the following form

$$\begin{aligned} S[\Phi] &= \int dt \left[\frac{1}{2} \partial_t(\Phi \dot{\Phi}) - \frac{1}{2} \Phi (\ddot{\Phi} + \omega^2 \Phi) - \frac{g}{3} \Phi^3 \right] \\ &= \frac{1}{2} \Phi \dot{\Phi} \Big|_{t=0} + \int dt \left[-\frac{1}{2} \Phi (\ddot{\Phi} + \omega^2 \Phi) - \frac{g}{3} \Phi^3 \right]. \end{aligned} \quad (4.37)$$

We have to be somewhat careful when we evaluate the boundary term for the classical solution (4.35). Although the Green's function vanishes for $t = 0$ (so that $\Phi_{\text{cl}}(0) = \phi$), its derivative does not, but instead satisfies the identity

$$\lim_{t \rightarrow 0} \partial_t G(t, t') = -ie^{i\omega t'}. \quad (4.38)$$

Using this, the boundary term becomes

$$\begin{aligned} \frac{1}{2} \Phi_{\text{cl}} \dot{\Phi}_{\text{cl}} \Big|_{t=0} &= \frac{1}{2} \phi \left(i\omega \phi - ig \int dt' (-ie^{i\omega t'}) \Phi_{\text{cl}}^2(t') \right) \\ &= \frac{i\omega}{2} \phi^2 - \frac{g}{2} \phi \int dt' e^{i\omega t'} \Phi_{\text{cl}}^2(t'), \end{aligned} \quad (4.39)$$

and substituting (4.35) into (4.37) gives

$$\begin{aligned} S[\Phi_{\text{cl}}] &= \frac{i\omega}{2} \phi^2 - \frac{g}{2} \phi \int dt e^{i\omega t} \Phi_{\text{cl}}^2 \\ &\quad + \int dt \left[-\frac{1}{2} \left(\phi e^{i\omega t} - ig \int dt' G(t, t') \Phi_{\text{cl}}^2(t') \right) \left(-g \Phi_{\text{cl}}^2(t) \right) - \frac{g}{3} \Phi_{\text{cl}}^3 \right]. \end{aligned} \quad (4.40)$$

The terms linear in ϕ cancel and we get the following *on-shell action*:

$$S[\Phi_{\text{cl}}] = \frac{i\omega}{2} \phi^2 - \frac{g}{3} \int dt \Phi_{\text{cl}}^3(t) - \frac{ig^2}{2} \int dt dt' G(t, t') \Phi_{\text{cl}}^2(t') \Phi_{\text{cl}}^2(t). \quad (4.41)$$

This way of organizing the terms makes it clear that the first term contributes only to the two-point function. At higher orders both of the last two terms in (4.41) can be important.

Exercise 4.4 Consider an anharmonic oscillator with a general interaction Lagrangian L_{int} . Show that the on-shell action can be written as

$$S[\Phi_{\text{cl}}] = \frac{i\omega}{2} \phi^2 + \int dt L_{\text{int}} - \frac{i}{2} \int dt dt' G(t, t') \frac{\delta S_{\text{int}}}{\delta \Phi_{\text{cl}}(t')} \frac{\delta S_{\text{int}}}{\delta \Phi_{\text{cl}}(t)}, \quad (4.42)$$

which for $L_{\text{int}} = -\frac{1}{3}g \Phi^3$ reduces to (4.41). →

Given the classical solution to the equations of motion, we can use (4.41) to compute the wavefunction. We obtain such a solution in perturbation theory by solving (4.35) order-by-order in the parameter g . To do so, we expand

$$\Phi_{\text{cl}}(t) = \Phi^{(0)}(t) + g \Phi^{(1)}(t) + g^2 \Phi^{(2)}(t) + \dots \quad (4.43)$$

The lowest order ($g = 0$) solution is just the solution to the unperturbed oscillator

$$\Phi^{(0)}(t) = \phi e^{i\omega t}. \quad (4.44)$$

At first order in g , we find

$$\begin{aligned} \Phi^{(1)}(t) &= i \int dt' G(t, t') \left(-(\Phi^{(0)}(t'))^2 \right) \\ &= i \int dt' G(t, t') \left(-\phi^2 e^{2i\omega t'} \right) = \frac{\phi^2}{3\omega^2} (e^{2i\omega t} - e^{i\omega t}), \end{aligned} \quad (4.45)$$

where we have used the lowest-order solution as a source and performed the integral using the Green's function (4.36). By plugging this solution back into (4.35) and iterating, we can perturbatively solve to any order in g desired.

Substituting (4.43) into (4.41), we obtain a perturbative expansion of the on-shell action:

$$\begin{aligned} S[\Phi_{\text{cl}}] &= \frac{i\omega}{2}\phi^2 - \frac{g}{3} \int dt \left[(\Phi^{(0)})^3 + 3g(\Phi^{(0)})^2\Phi^{(1)} + \dots \right] \\ &\quad + \frac{g^2}{2} \int dt (\Phi^{(0)}(t))^2 \int dt' iG(t, t') \left(-(\Phi^{(0)}(t'))^2 \right) + \dots, \end{aligned} \quad (4.46)$$

where we have displayed all terms up to $\mathcal{O}(g^2)$. Identifying the last integral in the second line with the solution $\Phi^{(1)}$, we can also write

$$S[\Phi_{\text{cl}}] = \frac{i\omega}{2}\phi^2 - \frac{g}{3} \int dt (\Phi^{(0)})^3 - \frac{g^2}{2} \int dt (\Phi^{(0)})^2 \Phi^{(1)} + \dots. \quad (4.47)$$

Using the explicit solutions for the zeroth and first-order solutions—equations (4.44) and (4.45)—we find that the wavefunction of the anharmonic oscillator up to this order is

$$\Psi[\phi] \approx e^{iS[\Phi_{\text{cl}}]} = \exp \left(-\frac{\omega}{2}\phi^2 - \frac{g}{9\omega}\phi^3 + \frac{g^2}{72\omega^3}\phi^4 + \dots \right). \quad (4.48)$$

We see that the anharmonic interaction induces a deviation from the Gaussian shape of the wavefunction. These terms will contribute to the moments of the distribution of ϕ defined by $|\Psi[\phi]|^2$. For example, we could compute $\langle \phi^3 \rangle$ in this theory and would get a nonzero answer, in contrast to its Gaussian counterpart where $g = 0$. Similarly, higher moments $\langle \phi^n \rangle$ receive contributions order-by-order in g . It is the analogue of these moments that we want to compute in the field theory context.

Exercise 4.5 Derive the first-order solution (4.45) by explicitly integrating the Green's function for the harmonic oscillator. Note that $\Phi^{(1)}(0) = 0$ as expected. Using (4.44) and (4.45), confirm the result (4.48) for the wavefunction. →

4.4.2 Wavefunction in Field Theory

After this brief detour into quantum mechanics, we return to our main problem of interest, computing the wavefunctional in interacting field theories. In perturbation theory, we expect fluctuations to be small, and so it is convenient to expand the wavefunction (4.2) in powers of the field fluctuations as:

$$\begin{aligned} \Psi[\phi] = \exp & \left(-\frac{1}{2} \int d^3x_1 d^3x_2 \Psi_2(\mathbf{x}_1, \mathbf{x}_2) \phi(\mathbf{x}_1) \phi(\mathbf{x}_2) \right. \\ & \left. + \sum_{n=3}^{\infty} \int d^3x_1 \cdots d^3x_n \Psi_n(\underline{\mathbf{x}}) \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_n) \right), \end{aligned} \quad (4.49)$$

where $\underline{\mathbf{x}} \equiv \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ denotes the set of all spatial coordinates and the kernels $\Psi_n(\underline{\mathbf{x}})$ are called *wavefunction coefficients*. These functions capture the statistics of the fluctuations ϕ . It is useful to write the wavefunction coefficients in momentum space,

$$\int d^3x_1 \cdots d^3x_n \Psi_n(\underline{\mathbf{x}}) \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_n) = \int \frac{d^3k_1 \cdots d^3k_n}{(2\pi)^{3n}} \Psi_n(\underline{\mathbf{k}}) \phi_{\mathbf{k}_1} \cdots \phi_{\mathbf{k}_n}, \quad (4.50)$$

which is the natural habitat of cosmological correlations. As a consequence of translation invariance, the momentum-space wavefunction coefficients contain a momentum-conserving delta function. It is often useful to extract this delta function and write

$$\Psi_n(\underline{\mathbf{k}}) = (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \cdots + \mathbf{k}_n) \psi_n(\underline{\mathbf{k}}). \quad (4.51)$$

These wavefunction coefficients can also be interpreted as correlation functions of operators, O , which are dual to the bulk fields, $\Psi_n = \langle O_{\mathbf{k}_1} \cdots O_{\mathbf{k}_n} \rangle$, because they have the same kinematic properties [24]. [AJ: maybe be careful about previous statement because its really only true in dS, more complicated in flat space] As we will see in Section 5.4, in the context of cosmology in de Sitter space, these dual operators transform like primaries in a 3d conformal field theory.²⁶

Correlators

The wavefunction coefficients $\psi_n(\underline{\mathbf{k}})$ then contain all the information about correlations in the quantum state. To see this, we parameterize the wavefunction as

$$\begin{aligned} \Psi[\phi] = \exp & \left(-\frac{1}{2} \int \frac{d^3k_1 d^3k_2}{(2\pi)^6} \langle O_{\mathbf{k}_1} O_{\mathbf{k}_2} \rangle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} + \int \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^9} \langle O_{\mathbf{k}_1} O_{\mathbf{k}_2} O_{\mathbf{k}_3} \rangle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \right. \\ & \left. + \int \frac{d^3k_1 d^3k_2 d^3k_3 d^3k_4}{(2\pi)^{12}} \langle O_{\mathbf{k}_1} O_{\mathbf{k}_2} O_{\mathbf{k}_3} O_{\mathbf{k}_4} \rangle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \phi_{\mathbf{k}_4} + \cdots \right). \end{aligned} \quad (4.52)$$

In order to connect to our previous discussion of correlation functions, we now want to see in perturbation theory how these wavefunction coefficients are related to correlators. To do this, we

²⁶Note that there are three sets of variables: the bulk field Φ , its boundary value ϕ and the dual boundary operator O . It is worth emphasizing that we are not requiring these O 's to literally be operators in some exact dual QFT, but rather are just exploiting the fact that they kinematically behave as such.

expand the wavefunction around a Gaussian:

$$\begin{aligned}\Psi[\phi] = \exp\left(-\frac{1}{2}\int\frac{d^3k_1 d^3k_2}{(2\pi)^6} \langle O_{\mathbf{k}_1} O_{\mathbf{k}_2} \rangle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2}\right) \\ \times \left[1 + \int\frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^9} \langle O_{\mathbf{k}_1} O_{\mathbf{k}_2} O_{\mathbf{k}_3} \rangle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \right. \\ + \frac{1}{2} \int\frac{d^3k_1}{(2\pi)^3} \cdots \frac{d^3k_6}{(2\pi)^3} \langle O_{\mathbf{k}_1} O_{\mathbf{k}_2} O_{\mathbf{k}_3} \rangle \langle O_{\mathbf{k}_4} O_{\mathbf{k}_5} O_{\mathbf{k}_6} \rangle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \phi_{\mathbf{k}_4} \phi_{\mathbf{k}_5} \phi_{\mathbf{k}_6} \\ \left. + \int\frac{d^3k_1 d^3k_2 d^3k_3 d^3k_4}{(2\pi)^{12}} \langle O_{\mathbf{k}_1} O_{\mathbf{k}_2} O_{\mathbf{k}_3} O_{\mathbf{k}_4} \rangle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \phi_{\mathbf{k}_4} + \dots \right], \quad (4.53)\end{aligned}$$

where we are displaying the terms that will contribute up to the four-point function at tree level. We can then use the quantum-mechanics formula (4.3) to compute correlation functions by performing Gaussian integrals over the fields ϕ .

For example, the simplest correlation function we could compute is the two-point function, which is given by

$$\langle \phi_1 \phi_2 \rangle' = \frac{1}{2 \text{Re} \langle O_1 O_2 \rangle'}, \quad (4.54)$$

where we have introduced some condensed notation $\phi_a \equiv \phi_{\mathbf{k}_a}$ (and similar for O), and where the prime on the wavefunction coefficient $\langle \dots \rangle'$ indicates that we have removed the momentum-conserving delta function. Similarly, the three-point function is

$$\langle \phi_1 \phi_2 \phi_3 \rangle' = \frac{\text{Re} \langle O_1 O_2 O_3 \rangle'}{2 \text{Re} \langle O_1 O_1 \rangle' \text{Re} \langle O_2 O_2 \rangle' \text{Re} \langle O_3 O_3 \rangle'}, \quad (4.55)$$

and the four-point function becomes

$$\begin{aligned}\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle' = & \frac{\text{Re} \langle O_1 O_2 O_3 O_4 \rangle'}{8 \text{Re} \langle O_1 O_1 \rangle' \text{Re} \langle O_2 O_2 \rangle' \text{Re} \langle O_3 O_3 \rangle' \text{Re} \langle O_4 O_4 \rangle'} \\ & + \frac{\langle O_1 O_2 O_I \rangle' \langle O_I O_3 O_4 \rangle'^* + \text{c.c.}}{32 \text{Re} \langle O_I O_I \rangle' \text{Re} \langle O_1 O_1 \rangle' \text{Re} \langle O_2 O_2 \rangle' \text{Re} \langle O_3 O_3 \rangle' \text{Re} \langle O_4 O_4 \rangle'} + \text{perms.} \quad (4.56) \\ & + \frac{\langle O_1 O_2 O_I \rangle' \langle O_I O_3 O_4 \rangle' + \text{c.c.}}{32 \text{Re} \langle O_I O_I \rangle' \text{Re} \langle O_1 O_1 \rangle' \text{Re} \langle O_2 O_2 \rangle' \text{Re} \langle O_3 O_3 \rangle' \text{Re} \langle O_4 O_4 \rangle'} + \text{perms.},\end{aligned}$$

where “perms.” denotes the sum over other permutations of the external operators (i.e. other exchange channels) because we have taken all the operators to be identical for simplicity, and we have dropped the primes on the various wavefunction coefficients to simplify notation.

Exercise 4.6 Derive the formulas (4.54)–(4.56). →

On-shell action

Now that we know how to use the wavefunction, we turn to calculating it. Like we did for the harmonic oscillator, we are going to use the fact that, at tree level, the wavefunction is simply

related to the on-shell action

$$\Psi[\phi] \approx \exp(iS[\Phi_{\text{cl}}]). \quad (4.57)$$

For concreteness, we consider the action of a massive scalar field in an arbitrary spacetime background:

$$S[\Phi] = \int d^4x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} m^2 \Phi^2 + \mathcal{L}_{\text{int}} \right), \quad (4.58)$$

where we have isolated the free part of the action and grouped all of the interactions schematically into \mathcal{L}_{int} . The equation of motion following from this action is

$$(\square - m^2)\Phi = -\frac{1}{\sqrt{-g}} \frac{\delta S_{\text{int}}}{\delta \Phi}, \quad (4.59)$$

where $\square \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the curved-space d'Alembert operator. As before, we can write a formal solution to this equation as [DB: Do all of this in Fourier space?]

$$\Phi_{\text{cl}}(t, \mathbf{x}) = \int d^3x' \sqrt{\gamma} K(t, \mathbf{x}, \mathbf{x}') \phi(\mathbf{x}') + i \int d^4x' G(x, x') \frac{\delta S_{\text{int}}}{\delta \Phi(x')} \Big|_{\Phi=\Phi_{\text{cl}}}, \quad (4.60)$$

where γ_{ij} is the spatial metric and $K(t, \mathbf{x}, \mathbf{x}')$ is a solution to the free equation of motion that satisfies the boundary condition

$$K(t_*, \mathbf{x}, \mathbf{x}') = \frac{1}{\sqrt{\gamma}} \delta^{(3)}(\mathbf{x} - \mathbf{x}'), \quad (4.61)$$

and reduces to the vacuum solution as $t \rightarrow -\infty$ (we will see that this second boundary condition is most simply implemented in Fourier space). The solution $K(t, \mathbf{x}, \mathbf{x}')$ is often called the *bulk-to-boundary propagator*.²⁷ The function $G(x, x')$ is a Green's function satisfying the equation

$$(\square - m^2)G(x, x') = \frac{i}{\sqrt{-g}} \delta(t - t') \delta^{(3)}(\mathbf{x} - \mathbf{x}'). \quad (4.62)$$

The precise Green's function of interest also reduces to the vacuum in the far past and vanishes if we take one of the points to the boundary $G(x, x')|_{t=t_*} = G(x, x')|_{t'=t_*} = 0$. The latter constraint ensures that the formal solution (4.60) will satisfy the boundary condition $\Phi_{\text{cl}}(t_*, \mathbf{x}) = \phi(\mathbf{x})$. The function $G(x, x')$ is often called the *bulk-to-bulk propagator*.²⁸ The two propagators K and G are related by the identity

$$\lim_{t \rightarrow t_*} g^{tt} \partial_t G(x, x') = iK(t', \mathbf{x}, \mathbf{x}'). \quad (4.63)$$

As in the case of the anharmonic oscillator, we can substitute the formal solution (4.60) into the action (4.58). This is again most simply done by first integrating the kinetic term by parts:

$$S[\Phi] = - \int d^3x \sqrt{\gamma} \frac{1}{2} g^{tt} \Phi \dot{\Phi} \Big|_{t=t_*} + \int d^4x \sqrt{-g} \left(\frac{1}{2} \Phi (\square - m^2) \Phi + \mathcal{L}_{\text{int}}[\Phi] \right), \quad (4.64)$$

²⁷As we will see, in the diagrammatic interpretation of the computation of the wavefunction, this object is associated to lines connecting bulk vertices to the $t = t_*$ boundary slice. The nomenclature comes from AdS/CFT, where one computes the bulk on-shell action using essentially the same procedure [?].

²⁸We will see that it is associated to internal lines in diagrams that connect two spacetime interaction vertices.

which generates a boundary term on the $t = t_*$ slice (with induced metric γ). As in the case of the anharmonic, we have to be careful when evaluating the boundary term. In particular, although $\Phi_{\text{cl}}(t_*, \mathbf{x}) = \phi(\mathbf{x})$, the time derivative of the solution (4.60) is

$$\begin{aligned}\partial_t \Phi_{\text{cl}}(t_*, \mathbf{x}) &= \int d^3x' \sqrt{\gamma} \partial_t K(t, \mathbf{x}, \mathbf{x}') \phi(\mathbf{x}') \Big|_{t=t_*} + i \int d^4x' \partial_t G(x, x') \frac{\delta S_{\text{int}}}{\delta \Phi_{\text{cl}}(x')} \\ &= \int d^3x' \sqrt{\gamma} \partial_t K(t, \mathbf{x}, \mathbf{x}') \phi(\mathbf{x}') \Big|_{t=t_*} - \int d^4x' \frac{K(t', \mathbf{x}, \mathbf{x}')}{g^{tt}} \frac{\delta S_{\text{int}}}{\delta \Phi_{\text{cl}}(x')},\end{aligned}\quad (4.65)$$

where we have used the identity (4.63). Evaluating the boundary term in (4.64) on the classical solution therefore gives

$$\begin{aligned}- \int d^3x \sqrt{\gamma} \frac{1}{2} g^{tt} \Phi_{\text{cl}} \dot{\Phi}_{\text{cl}} \Big|_{t=t_*} &= - \frac{1}{2} \int d^3x \sqrt{\gamma} d^3x' \sqrt{\gamma} \phi(\mathbf{x}) \phi(\mathbf{x}') \left(g^{tt} \partial_t K(t, \mathbf{x}, \mathbf{x}') \right) \Big|_{t=t_*} \\ &\quad + \frac{1}{2} \int d^4x' d^3x \sqrt{\gamma} K(t', \mathbf{x}, \mathbf{x}') \phi(\mathbf{x}) \frac{\delta S_{\text{int}}}{\delta \Phi_{\text{cl}}(x')},\end{aligned}\quad (4.66)$$

which should be compared to equation (4.39) in the anharmonic oscillator example. Note that the second line in (4.66) involves a bulk integral over x' . Equation (4.64) can then be written as

$$\begin{aligned}S[\Phi_{\text{cl}}] &= - \int d^3x \sqrt{\gamma} d^3x' \sqrt{\gamma} \frac{1}{2} \phi(\mathbf{x}) \phi(\mathbf{x}') \left(g^{tt} \partial_t K(t, \mathbf{x}, \mathbf{x}') \right) \Big|_{t=t_*} \\ &\quad - \frac{1}{2} \int d^4x d^3x' \sqrt{\gamma} K(t, \mathbf{x}', \mathbf{x}) \phi(\mathbf{x}') \frac{\delta S_{\text{int}}}{\delta \Phi_{\text{cl}}(x)} \\ &\quad + \int d^4x \left(-\frac{1}{2} \Phi_{\text{cl}} \frac{\delta S_{\text{int}}}{\delta \Phi_{\text{cl}}} + \sqrt{-g} \mathcal{L}_{\text{int}}[\Phi_{\text{cl}}] \right).\end{aligned}\quad (4.67)$$

We now see that if we substitute (4.60) into the first term in the last line, the term involving ϕK will cancel with the second line and we will be left with the *on-shell action* [?]

$$\begin{aligned}S[\Phi_{\text{cl}}] &= - \int d^3x \sqrt{\gamma} d^3x' \sqrt{\gamma} \frac{1}{2} \phi(\mathbf{x}) \phi(\mathbf{x}') \left(g^{tt} \partial_t K(t, \mathbf{x}, \mathbf{x}') \right) \Big|_{t=t_*} + \int d^4x \sqrt{-g} \mathcal{L}_{\text{int}}[\Phi_{\text{cl}}] \\ &\quad - \frac{i}{2} \int d^4x d^4x' \frac{\delta S_{\text{int}}}{\delta \Phi_{\text{cl}}(x)} G(x, x') \frac{\delta S_{\text{int}}}{\delta \Phi_{\text{cl}}(x')},\end{aligned}\quad (4.68)$$

which is the analogue of (4.41) for the anharmonic oscillator. Given a formula for the classical solution Φ_{cl} , we can then plug it into (4.68) to obtain the tree-level wavefunction.

4.4.3 Perturbation Theory

We can solve (4.60) iteratively in perturbation theory. Like for the anharmonic oscillator, we imagine that there is some perturbative parameter g that we can expand the solution in

$$\Phi_{\text{cl}}(x) = \Phi^{(0)}(x) + g \Phi^{(1)}(x) + g^2 \Phi^{(2)}(x) + \dots,\quad (4.69)$$

where $\Phi^{(0)}(x)$ is the solution to the free equation of motion, $\Phi^{(1)}(x)$ is the first order correction,

$$\Phi^{(1)}(t, \mathbf{x}) = i \int d^4x' G(x, x') \frac{\delta S_{\text{int}}}{\delta \Phi(x')} \Big|_{\Phi=\Phi^{(0)}},\quad (4.70)$$

and so on.

In order to systematize the calculation, it is possible to assign a diagrammatic set of rules to the entire procedure of computing the classical solution order-by-order in perturbation theory and substituting this solution back into the action (4.68). To make this slightly more intuitive, consider the special case where $\mathcal{L}_{\text{int}} = -\frac{1}{3!}g\Phi^3$. Up to $\mathcal{O}(g^2)$, equation (4.68) can then be written as

$$\begin{aligned} iS[\Phi_{\text{cl}}] = & -i \int d^3x \sqrt{\gamma} d^3x' \sqrt{\gamma} \frac{1}{2} \phi(\mathbf{x}) \phi(\mathbf{x}') \left(g^{tt} \partial_t K(t, \mathbf{x}, \mathbf{x}') \right) \Big|_{t=t_*} - \frac{ig}{3!} \int d^4x \sqrt{-g} (\Phi^0(x))^3 \\ & - \frac{g^2}{8} \int d^4x \sqrt{-g} \int d^4x' \sqrt{-g} (\Phi^0(x))^2 G(x, x') (\Phi^0(x'))^2 + \dots . \end{aligned} \quad (4.71)$$

The first term is just the ordinary Gaussian part of the action. The other terms have the following diagrammatic interpretation:

$$-ig \int d^4x \sqrt{-g} (\Phi^{(0)}(x))^3 = \begin{array}{c} \text{---} \\ \backslash \quad / \\ \bullet \end{array} , \quad (4.72)$$

$$-g^2 \int d^4x \sqrt{-g} \int d^4x' \sqrt{-g} (\Phi^{(0)}(x))^2 G(x, x') (\Phi^{(0)}(x'))^2 = \begin{array}{c} \text{---} \\ \backslash \quad / \quad \backslash \quad / \\ \bullet \quad \bullet \end{array} , \quad (4.73)$$

and this can be continued to higher order in the boundary conditions ϕ .

In order to state the Feynman rules most simply, it is convenient to express the computation in Fourier space to trivialize the integrals over space. At this point we are specializing the background to assume that spatial translations are a symmetry. For concreteness, we will further assume isotropy so that the spacetime is of FRW form. We can then express G and K in terms of the solutions to the free equation of motion for the field Φ in Fourier space. Given the solution to the (Fourier space) free equation of motion, $f_k(t)$, which satisfies the conditions

$$f_k^*(t) \partial_t f_k(t) - \partial_t f_k^*(t) f_k(t) = i, \quad (4.74)$$

$$\lim_{t \rightarrow -\infty} f_k(t) \sim e^{ikt}, \quad (4.75)$$

we can write the bulk-to-boundary and bulk-to-bulk propagators as

$$K_k(t) = \frac{f_k(t)}{f_k(t_*)}, \quad (4.76)$$

$$G(k; t, t') = f_k^*(t) f_k(t') \theta(t - t') + f_k^*(t') f_k(t) \theta(t' - t) - \frac{f_k^*(t_*)}{f_k(t_*)} f_k(t) f_k(t'). \quad (4.77)$$

One can check that the function $G(k; t, t')$ satisfies the equation

$$(\square_k - m^2) G(k; t, t') = \frac{i}{\sqrt{-g}} \delta(t - t'), \quad (4.78)$$

where \square_k is the spatial Fourier transform of the wave operator, while $K_k(t)$ solves the homogeneous version of the same equation. The classical solution (4.60) is then expressed in Fourier space as

$$\Phi_{\text{cl}}(t, \mathbf{k}) = K_k(t) \phi_{\mathbf{k}} + i \int dt' G(k; t, t') \frac{\delta S_{\text{int}}}{\delta \Phi_{\mathbf{k}}(t')} \Big|_{\Phi=\Phi_{\text{cl}}} . \quad (4.79)$$

We can now state the rules for the computation of wavefunction coefficients.

Feynman rules

Similar to the computation of in-in correlators, wavefunctional calculations can be simplified by organizing things into a systematic expansion in Feynman diagrams. Like in-in, wavefunctional diagrams have two different types of propagators, which makes them slightly more complicated than their flat-space counterparts. However, compared to the in-in diagrams we encountered above, the wavefunction is a bit simpler, which is a benefit. Concretely, the Feynman rules are:

1. Draw the fixed-time boundary surface ($t = t_*$) on which you want to compute the wavefunction.
2. To compute the wavefunction coefficient multiplying ϕ^n , draw all diagrams with a fixed number of lines, n , ending on the boundary.
3. Assign a vertex factor, iV , to each bulk interaction, where V is obtained by functionally differentiating the action as usual. (Note that this can lead to factors of momenta or time derivatives acting on propagators if there are derivative interactions.)
4. Assign a bulk-to-bulk propagator, G , to each internal line.
5. Assign a bulk-to-boundary propagator, K , to each external line (lines that end on the boundary).
6. Integrate over the time insertions of all bulk vertices using the measure $Dt \equiv dt a^3(t)$. For diagrams involving loops, integrate over the loop momenta.

We see that all told, the Feynman rules are fairly similar to the in-in case, except that the propagators that we use are slightly different, and that there are about half as many diagrams.

A small warning: As we saw above, there are some normalization factors that appear in the on-shell action multiplying the outputs of these Feynman diagrams to give the true wavefunction coefficients. For most of our purposes, the normalizations will not end up being particularly important, but if one wants to match exactly the result of an in-in calculation, the normalizations do matter. Working out these relative normalizations in complete generality is somewhat tedious, but for the case where one is computing the wavefunction coefficient coming from a $\mathcal{L}_{\text{int}} = -g\Phi^n$ interaction, with I internal lines (at tree level), one should multiply the result of the Feynman diagram calculation by

$$c_{n,I} = \frac{\Gamma[I(n-1) + n]}{\Gamma[I+2]\Gamma[I(n-2) + n+1]\Gamma[n]^{I+1}}, \quad (4.80)$$

in order to convert it into a true wavefunction coefficient. In the following, we will often ignore these normalization factors and refer to the output of the Feynman diagram calculation as wavefunction coefficients.

Exercise 4.7 Using these Feynman rules, show explicitly that the expressions (4.55) and (4.56) are equivalent to the correlators derived in Section 3 using the in-in formalism.

Hint: You should find that the second line in (4.56) is equal to $I_{+-} + I_{-+}$, and that the first line is equal to $I_{++} + I_{--} + I_{\text{extra}}$, where I_{extra} precisely cancels against the third line. →

4.4.4 Examples in Flat Space

We are now ready to compute correlators using the wavefunction formalism. We begin with examples of flat-space correlators where all time integrals can be performed explicitly. All correlators can be evaluated at $t_* \equiv 0$ without loss of generality.

We will consider the same toy model as in Section 3.2

$$S = \int d^4x \left(-\frac{1}{2}(\partial\Phi)^2 - \frac{g}{3!}\Phi^3 \right). \quad (4.81)$$

The free equation of motion in Fourier space is $(\partial_t^2 + k^2)\phi_{\mathbf{k}} = 0$, so that the properly-normalized mode functions are

$$f_k(t) = \frac{1}{\sqrt{2k}} e^{ikt}. \quad (4.82)$$

Using this, the relevant flat-space propagators are

$$K_k(t) = e^{ikt}, \quad (4.83)$$

$$G(k; t, t') = \frac{1}{2k} \left(e^{-ik(t-t')} \theta(t-t') + e^{ik(t-t')} \theta(t'-t) - e^{ik(t+t')} \right). \quad (4.84)$$

The two-point wavefunction coefficient was computed in (4.11) and is $\langle O_1 O_2 \rangle = k_1$, which by (4.54) implies the correct two-point function $\langle \phi_1 \phi_2 \rangle = 1/(2k_1)$. Let's now compute the simplest tree-level correlators in this theory.

Three-point function Applying the Feynman rules to the three-point wavefunction coefficient, we get

$$\begin{aligned} \langle O_1 O_2 O_3 \rangle' &\equiv \begin{array}{c} \diagup \quad \diagdown \\ \text{---} \quad \text{---} \\ \diagdown \quad \diagup \end{array} \\ &= ig \int_{-\infty}^0 dt K_{k_1}(t) K_{k_2}(t) K_{k_3}(t) \\ &= ig \int_{-\infty}^0 dt e^{i(k_1+k_2+k_3)t} = \boxed{\frac{g}{(k_1+k_2+k_3)}}. \end{aligned} \quad (4.85)$$

Substituting this into (4.55), we obtain the following three-point function

$$\langle \phi_1 \phi_2 \phi_3 \rangle' = -\frac{\text{Re} \langle O_1 O_2 O_3 \rangle'}{4k_1 k_2 k_3} = -\frac{g}{4k_1 k_2 k_3} \frac{1}{k_1 + k_2 + k_3}, \quad (4.86)$$

which agrees with (3.46), the result of the in-in computation.

Four-point function Applying the Feynman rules to the four-point wavefunction coefficient, we get

$$\begin{aligned} \langle O_1 O_2 O_3 O_4 \rangle' &\equiv \text{Diagram: A horizontal line with two vertices connected to it by diagonal lines, forming a V-shape.} \\ &= -g^2 \int_{-\infty}^0 dt' dt'' K_{k_1}(t') K_{k_2}(t') G(k_I; t', t'') K_{k_3}(t'') K_{k_4}(t'') \\ &= -g^2 \int_{-\infty}^0 dt' dt'' e^{ik_{12}t'} \left[G_F(k_I; t', t'') - \frac{1}{2k_I} e^{ik_I(t'+t'')} \right] e^{ik_{34}t''}, \end{aligned} \quad (4.87)$$

where we have split the bulk-to-bulk propagator into the Feynman propagator and an extra term. Up to a rescaling, the integral involving the Feynman propagator is the same as the one we computed in (3.47) to get the I_{++} contribution to the in-in correlator. We therefore have

$$\begin{aligned} \langle O_1 O_2 O_3 O_4 \rangle' &= 16 k_1 k_2 k_3 k_4 \times I_{++} + \frac{g^2}{2k_I} \int_{-\infty}^0 dt' e^{i(k_{12}+k_I)t'} \int_{-\infty}^0 dt'' e^{i(k_{34}+k_I)t''} \\ &= 16 k_1 k_2 k_3 k_4 \times I_{++} - \frac{g^2}{2k_I} \frac{1}{(k_{12}+k_I)(k_{34}+k_I)}. \end{aligned} \quad (4.88)$$

Substituting (3.49) for I_{++} , this gives

$$\begin{aligned} \langle O_1 O_2 O_3 O_4 \rangle' &= \frac{g^2}{2k_I} \left(\frac{1}{k_{12}+k_{34}} \left(\frac{1}{k_{12}+k_I} + \frac{1}{k_{34}+k_I} \right) - \frac{1}{(k_{12}+k_I)(k_{34}+k_I)} \right) \\ &= \boxed{\frac{g^2}{(k_{12}+k_{34})(k_{12}+k_I)(k_{34}+k_I)}}. \end{aligned} \quad (4.89)$$

In summary, we have obtained the following wavefunction coefficients [AJ: Fix normalizations]

$$\langle O_1 O_2 \rangle' = k_1, \quad (4.90)$$

$$\langle O_1 O_2 O_3 \rangle' = \frac{g}{k_1 + k_2 + k_3} \quad (4.91)$$

$$\langle O_1 O_2 O_3 O_4 \rangle' = \frac{g^2}{(k_{12}+k_{34})(k_{12}+k_I)(k_{34}+k_I)}. \quad (4.92)$$

Note the simplicity of these expressions relative to the corresponding in-in correlation functions.

Finally, let us confirm that these results reproduce the four-point function that we derived in Section 3.2 using the in-in formalism. Since the wavefunction coefficients are real, equation (4.56) reduces to

$$\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle' = \frac{1}{8 k_1 k_2 k_3 k_4} \left(\langle O_1 O_2 O_3 O_4 \rangle' + \frac{\langle O_1 O_2 O_1 \rangle' \langle O_1 O_3 O_4 \rangle'}{k_I} \right) \quad (4.93)$$

$$= \frac{g^2}{8 k_1 k_2 k_3 k_4} \left(\frac{1}{(k_{12}+k_{34})(k_{12}+k_I)(k_{34}+k_I)} + \frac{1}{k_I} \frac{1}{(k_{12}+k_I)(k_{34}+k_I)} \right), \quad (4.94)$$

where we substituted (4.91) and (4.92) in the second line. It is not hard to confirm that (4.94) is the same as (3.56), so that we precisely recover the result of the in-in computation.

Higher-point functions* To simplify the calculation of higher-order wavefunction coefficients, we can take advantage of a hidden structure underlying the flat-space wavefunction. In particular, as a consequence of the time-translation invariance of the vacuum, the wavefunction coefficients can be recursively generated from lower-point objects [36].

Consider an arbitrary diagram consisting of vertices v and bulk-to-bulk edges e . The wavefunction coefficient is calculated by the time integral

$$\psi = \int \prod_{v \in \mathcal{V}} dt_v \left[\prod_v e^{iE_v t_v} \right] \prod_{e \in \mathcal{E}} G(k_e; t_{v_e}, t_{v'_e}), \quad (4.95)$$

where t_{v_e} and $t_{v'_e}$ are associated with the pair of vertices (v, v') connected by the edge e . Here, E_v is the sum of the energies in the bulk-to-boundary propagators connected to the vertex v . We then consider an auxiliary object, F , created by inserting the time-translation operator Δ in such a way that it acts only on the bulk-to-boundary propagators:

$$F \equiv \int \prod_v dt_v \left[\prod_v \Delta e^{iE_v t_v} \right] \prod_e G(k_e; t_{v_e}, t_{v'_e}), \quad \Delta \equiv -i \sum_v \partial_{t_v}. \quad (4.96)$$

This expression can be computed in two different ways: First, by acting directly on the bulk-to-boundary propagators $e^{iE_v t_v}$, which simply gives $E \equiv \sum_v E_v$ times the original wavefunction ψ . Second, by first integrating by parts, so that Δ acts on the products of bulk-to-bulk propagators. We therefore get [36]

$$\begin{aligned} F &= \left(\sum_v E_v \right) \psi = - \int \left[\prod_v dt_v e^{iE_v t_v} \right] \prod_e \Delta G(k_e; t_{v_e}, t_{v'_e}) \\ &= - \int \left[\prod_{v \in \mathcal{V}} dt_v e^{iE_v t_v} \right] \sum_{e \in \mathcal{E}} \left[\Delta G_e \prod_{\bar{e} \in \mathcal{E}/\{e\}} G_{\bar{e}} \right], \end{aligned} \quad (4.97)$$

where G_e is shorthand for $G(k_e; t_{v_e}, t_{v'_e})$. Since the time-ordered part of the propagator (??) is time-translation invariant, the operator Δ only acts nontrivially on the non-time-ordered part,

$$\Delta G_e = \Delta \left[-\frac{e^{ik_e(t_{v_e}+t_{v'_e})}}{2k_e} \right] = -e^{ik_e(t_{v_e}+t_{v'_e})}, \quad (4.98)$$

so that (4.97) becomes

$$\left(\sum_v E_v \right) \psi = \sum_{e \in \mathcal{E}} \int \left[\prod_{v \in \mathcal{V}} dt_v e^{iE_v t_v} \right] e^{ik_e(t_{v_e}+t_{v'_e})} \prod_{\bar{e} \in \mathcal{E}/\{e\}} G_{\bar{e}}. \quad (4.99)$$

The right-hand side of (4.99) represents the sum over functions with one edge removed from the definition of ψ in (4.95), and the energies of the vertices connected by this edge shifted by $+k_e$. Diagrammatically, this can be written as

$$\left(\sum_v E_v \right) \psi = \sum_e \left(\psi_L^{+k_e} \right) \cdots \left(\psi_R^{+k_e} \right) + \sum_e \left(\psi^{+k_e} \right)_e \quad (4.100)$$

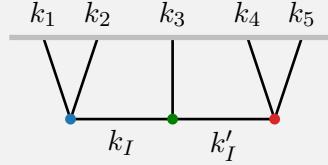
where the sum runs over all edges and the dashed lines indicate the edges which got erased. Equation (4.100) expresses a recursion relation between a wavefunction coefficient and all possible ways of partitioning it into two sub-diagrams at tree level, along with all possible ways to cut open a loop. From now on, we will focus on the tree-level recursion, for which only the first term on the right-hand side of (4.100) contributes. By continuing the recursive procedure (each time removing another edge), we can then reduce any graph to single-site graphs with shifted energies.

The recursion relation is initialized using the result for single-site graph: $\bullet = E^{-1}$. The recursion for the two-site chain is then performed in a single step:

$$\begin{aligned} (k_{12} + k_{34}) \bullet &= \bullet \underset{k_I}{\text{---}} \bullet = \bullet \underset{k_{12} + k_I}{\textcircled{---}} \mid \bullet \underset{k_{34} + k_I}{\textcircled{---}} \\ &= \frac{1}{(k_{12} + k_I)(k_{34} + k_I)}, \end{aligned} \quad (4.101)$$

which yields precisely equation (4.92).

Exercise 4.8 Use the recursion relation to derive the wavefunction coefficient for the following five-point diagram:



You should find:

$$\langle O_1 O_2 O_3 O_4 O_5 \rangle' = \frac{1}{E} \frac{1}{E_L E_M E_R} \left[\frac{1}{k_{123} + k'_I} + \frac{1}{k_{345} + k_I} \right], \quad (4.102)$$

where $E_L \equiv k_{12} + k_I$, $E_M \equiv k_3 + k_I + k'_I$ and $E_R \equiv k_{45} + k'_I$ are the energies running into the three different vertices. →

Combinatorial definition: The above results can also be summarized in a simple combinatorial prescription for generating the wavefunction coefficient associated to a graph. To explain this, we must first introduce the concept of *graph tubings*. [DB: re-write the rest]

Given a graph, we define a *tube* to be a subset of vertices of the graph which form a connected subgraph of the original graph. It is convenient to denote the tube by circling the relevant subgraph. For example, the following is a tube of the three-site graph:

$$\bullet \textcircled{---} \bullet \quad (4.103)$$

For our purposes, the only compatibility condition needed for tubes is that they do *not* intersect. (Two tubes intersect if their intersection is non-empty and neither is a subset of the other.)

Finally, a *complete tubing* is a maximal set of compatible (non-overlapping) tubes of a graph.²⁹

To compute the flat-space wavefunction coefficients, we first generate the set of compatible complete tubings of a given graph. For each tube, we sum up all the energies of vertices enclosed by the tube, along with the energy of any lines that intersect the tube. (This is the energy associated to the subgraph enclosed by the tube.) We then assign a rational function to the complete tubing which is the inverse of the product of all of the energy sums associated to the constituent tubings, and sum over all possible complete tubings of the graph. The resulting rational function is precisely the wavefunction coefficient associated to the graph [36].

It is simplest to understand this algorithm through examples. Above we derived the wavefunction of the two-site graph, which has a single complete tubing:

$$\psi_{\text{flat}}^{(2)} = \text{ (Diagram of a single tube)} = \frac{1}{E_{\text{L}} E_{\text{R}}}, \quad (4.104)$$

where on the right-hand side we have written the product of the sums of energies associated to each tube of the same color. We see that this reproduces exactly the wavefunction coefficient (4.92). A more nontrivial example is provided by the three-site graph. In this case, there are two inequivalent complete tubings. Summing them, we obtain the wavefunction coefficient

$$\begin{aligned} \psi_{\text{flat}}^{(3)} &= \text{ (Diagram of two tubes: one purple, one green)} + \text{ (Diagram of two tubes: one blue, one green)} \\ &= \frac{1}{E_{\text{L}} E_{\text{M}} E_{\text{R}}} \left(\frac{1}{k_{123} + k'_I} + \frac{1}{k_{345} + k_I} \right), \end{aligned} \quad (4.105)$$

which matches the result of a bulk computation. It is straightforward to consider more complicated graphs and derive their wavefunction coefficients in the same way.

4.4.5 Examples in de Sitter Space

It is now straightforward to extend the computations to de Sitter space. The relevant propagators are given by (4.76) and (4.77), with

$$f_k(\eta) = H \sqrt{\frac{\pi}{4}} e^{-\frac{i\pi}{4}(1+2\nu)} (-\eta)^{3/2} H_\nu^{(2)}(-k\eta). \quad (4.106)$$

While the computations are conceptually exactly the same as in flat space, for generic fields the time integrals involving Hankel functions typically cannot be performed explicitly. As in Section 3.3, we will therefore often specialize to situations involving scalars with conformal mass $m^2 = 2H^2$, whose mode functions are

$$f_k(\eta) = H(-\eta) \frac{e^{ik\eta}}{\sqrt{2k}}, \quad (4.107)$$

corresponding to $\nu = 1/2$ in (4.106). The simplicity of these functions will enable analytic results in many cases.

²⁹We should warn the reader that a different notion of compatibility is used in the mathematical literature [?]. There, tubes are considered to be compatible only if they neither intersect nor are adjacent. For us, the restriction that tubes not be adjacent would be rather artificial. An interesting avenue for investigation is to explore the mathematical properties of the compatibility conditions that are natural from the cosmological perspective.

Contact diagrams We first consider contact diagrams coming from polynomial bulk interactions of the form Φ^n . For example, the three-point wavefunction coefficient is [AJ: check sign]

$$\begin{aligned} \langle O_1 O_2 O_3 \rangle' &\equiv \text{Diagram: a horizontal line at the top with two vertical lines pointing down to a single point} \\ &= ig \int_{-\infty}^{\eta_*} D\eta K_{k_1}(\eta) K_{k_2}(\eta) K_{k_3}(\eta) \\ &= \frac{ig}{f_{k_1}(\eta_*) f_{k_2}(\eta_*) f_{k_3}(\eta_*)} \int_{-\infty}^{\eta_*} \frac{d\eta}{(H\eta)^4} f_{k_1}(\eta) f_{k_2}(\eta) f_{k_3}(\eta). \end{aligned} \quad (4.108)$$

In the generic case, this involves an integral over a product of three Hankel functions whose result in terms of an Appell F_4 function is not very illuminating.

Specializing to conformal scalars, and taking the limit $\eta_* \rightarrow 0$, we get

$$\langle O_1 O_2 O_3 \rangle' = -\frac{ig}{H^4(-\eta_*)^3} \int_{-\infty}^{\eta_*} \frac{d\eta}{\eta} e^{iK\eta} = \boxed{\frac{ig}{H^4\eta_*^3} \log(-iK\eta_*)}, \quad (4.109)$$

where $K = k_1 + k_2 + k_3$. As in (3.61) the factors of i come from the $i\epsilon$ prescription used to regulate the integral. Using $2\text{Re}\langle O_n O_n \rangle' = |f_{k_n}(\eta_*)|^{-2}$, the three-point correlator (4.55) becomes [AJ: fix normalization]

$$\begin{aligned} \langle \phi_1 \phi_2 \phi_3 \rangle' &= -\frac{H^6 \eta_*^6}{8k_1 k_2 k_3} (\langle O_1 O_2 O_3 \rangle' + \langle O_1 O_2 O_3 \rangle'^*) \\ &= -\frac{g}{8} \frac{H^2 \eta_*^3}{k_1 k_2 k_3} (i \log(-iK\eta_*) - i \log(iK\eta_*)) = \boxed{\frac{\pi}{8} g \frac{H^2 \eta_*^3}{k_1 k_2 k_3}}. \end{aligned} \quad (4.110)$$

which is the same as the in-in result (3.62). Notice that the two contributions from the different branches of the in-in contour are reproduced from the wavefunction perspective by the contributions from Ψ and Ψ^* .

Another case where the integrals can be explicitly computed involves two conformally coupled scalar fields and one scalar with general mass.

Exercise 4.9 Derive the three-point wavefunction coefficient of two conformally coupled scalars and one massive scalar with $m < \frac{3}{2}H$. You should find

$$\langle O_1 O_2 X_3 \rangle' = \frac{\sqrt{\pi}}{2^{\nu-\frac{1}{2}}} \frac{ig}{H^4} \frac{\Gamma[1-\nu]}{\cot(\pi\nu)} \frac{(-ik_3\eta_*)^{\nu-\frac{1}{2}}}{(-\eta_*)^3} {}_2F_1 \left[\begin{matrix} \frac{1}{2} + \nu, & \frac{1}{2} - \nu \\ 1 & \end{matrix} \middle| \frac{u-1}{2u} \right], \quad (4.111)$$

where $u = k_3/(k_1 + k_2)$ and ${}_2F_1$ is the Gauss hypergeometric function.

Hint: You may use the same integral representations of the Hankel and hypergeometric functions as in Exercise 3.4. →

Exercise 4.10 Show that the contact n -point wavefunction coefficient and correlator for conformal scalars arising from a $-g\Phi^n/n!$ interaction are ($n \geq 4$):

$$\langle O_1 \cdots O_n \rangle'_c = -i^n \frac{g}{H^4 \eta_*^n} \frac{\Gamma[n-3]}{n!} \frac{1}{K^{n-3}}, \quad (4.112)$$

$$\langle \phi_1 \cdots \phi_n \rangle'_c = \begin{cases} (-1)^{\frac{n}{2}+1} \Gamma[n-3] \frac{g H^{2n-4} \eta_*^n}{2^{n-1} k_1 \cdots k_4} \frac{1}{K^{n-3}}, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \quad (4.113)$$

where $K \equiv k_1 + k_2 + \cdots + k_n$. Since the Φ^4 interaction preserves the conformal symmetry of the free theory, the result for $n=4$ is essentially the same as in flat space (except for a factor of $H\eta_*$ for each external field). →

Exchange diagrams Next, we look at exchange diagrams. For example, the four-point wavefunction coefficient arising from the exchange of a massive scalar:

$$\begin{aligned} \langle O_1 O_2 O_3 O_4 \rangle' &\equiv \begin{array}{c} \text{---} \\ \backslash \quad / \\ \bullet \quad \bullet \\ \text{---} \end{array} \\ &= -g^2 \int_{-\infty}^{\eta_*} D\eta' D\eta'' K_{k_1}(\eta') K_{k_2}(\eta') G(k_I; \eta', \eta'') K_{k_3}(\eta'') K_{k_4}(\eta''). \end{aligned} \quad (4.114)$$

We first consider the case of a conformal scalar, for which the bulk-to-boundary and bulk-to-bulk propagators (for $\eta_* \rightarrow 0$) are

$$K_k(\eta) = \frac{\eta}{\eta_*} e^{ik\eta}, \quad (4.115)$$

$$G(k_I; \eta', \eta'') = \frac{H^2 \eta' \eta''}{2k_I} \left[e^{-ik|\eta' - \eta''|} - e^{ik_I(\eta' + \eta'')} \right]. \quad (4.116)$$

Equation (4.114) then becomes

$$\begin{aligned} \langle O_1 O_2 O_3 O_4 \rangle' &= -\frac{g^2}{H^6 \eta_*^4} \int_{-\infty}^0 \frac{d\eta'}{\eta'} \int_{-\infty}^0 \frac{d\eta''}{\eta''} e^{ik_{12}\eta'} e^{ik_{34}\eta''} \frac{1}{2k_I} \left[e^{-ik_I|\eta' - \eta''|} - e^{ik_I(\eta' + \eta'')} \right] \\ &= -\frac{g^2}{H^6 \eta_*^4} \int_{k_{12}}^\infty dx \int_{k_{34}}^\infty dy \int_{-\infty}^0 d\eta' d\eta'' e^{ix\eta'} e^{iy\eta''} \frac{1}{2k_I} \left[e^{-ik_I|\eta' - \eta''|} - e^{ik_I(\eta' + \eta'')} \right] \\ &= \frac{1}{H^6 \eta_*^4} \int_{k_{12}}^\infty dx \int_{k_{34}}^\infty dy \langle O_1 O_2 O_3 O_4 \rangle'_{(\text{flat})}(x, y, k_I), \end{aligned} \quad (4.117)$$

where we have used the same trick as in Section 3.3 to write the expression as an integral over the corresponding flat-space result. Substituting (4.89) for $\langle O_1 O_2 O_3 O_4 \rangle'_{(\text{flat})}$, gives the same integral

as for $I_{++}^{(c)}$ in (3.81). We therefore get

$$\begin{aligned}\langle O_1 O_2 O_3 O_4 \rangle' &= -\frac{g^2}{H^6 \eta_*^4} \int_{k_{12}}^\infty dx \int_{k_{34}}^\infty dy \frac{1}{(x+y)(x+k_I)(y+k_I)} \\ &= \boxed{\frac{g^2}{2H^6 \eta_*^4 k_I} \left[\text{Li}_2\left(\frac{k_{12}-k_I}{E}\right) + \text{Li}_2\left(\frac{k_{34}-k_I}{E}\right) + \log\left(\frac{k_{12}+k_I}{E}\right) \log\left(\frac{k_{34}+k_I}{E}\right) - \frac{\pi^2}{6} \right]}. \end{aligned}\quad (4.118)$$

Notice that the wavefunction coefficient (4.118) is very similar to the full correlator (3.86). The primary difference is in the π^2 term (along with some overall normalization factors), which comes from the disconnected contributions to the correlator from $\langle O_1 O_2 O_3 \rangle$.

Exercise 4.11 Using the wavefunction coefficient that you just computed, along with the three-point result (4.109), check that the correlator derived from the wavefunction reproduces the in-in result (3.86). →

It is also straightforward to include the effects of derivative interactions in the vertices of diagrams. In many of the most phenomenologically interesting cases, the interactions of fields break the de Sitter symmetries of the background. Processes involving time derivatives of massless scalar fields are particularly simple [?].

Exercise 4.12 Consider the (de Sitter breaking) interaction

$$L_{\text{int}} = -\frac{g}{2} a^3(t) \dot{\Phi}^2 \chi, \quad (4.119)$$

where Φ is a massless scalar and χ is a conformally coupled scalar. Compute the four-point wavefunction coefficient $\langle OOOO \rangle$ for massless scalars coming from the exchange of χ . You should find

$$\langle O_1 O_2 O_3 O_4 \rangle' = -\frac{g^2}{H^2} k_1^2 k_2^2 k_3^2 k_4^2 \frac{2E_L E_R + E(E_L + E_R) + E^2}{E^3 E_L^2 E_R^2}, \quad (4.120)$$

where $E \equiv k_{12} + k_{34}$, $E_L \equiv k_{12} + k_I$, $E_R \equiv k_{34} + k_I$. Can you relate this to the four-point function of a massless scalar in flat space? →

4.5 A Challenge

So far, most of our explicit computations in de Sitter space were for conformally coupled scalars. However, as soon as we allow for generic massive fields, the resulting integrals typically cannot be evaluated analytically. A simple example is the exchange of a generic massive scalar χ between two pairs of conformally coupled scalars Φ . The corresponding wavefunction coefficient is of the form (4.114) and can explicitly be written as

$$\psi \equiv \langle O_1 O_2 O_3 O_4 \rangle' = -g^2 \int \frac{d\eta'}{\eta'^2} \int \frac{d\eta''}{\eta''^2} e^{ik_{12}\eta'} e^{ik_{34}\eta''} G_\nu(k_I; \eta', \eta''), \quad (4.121)$$

where the propagator of the massive field is

$$G_\nu(k_I; \eta', \eta'') = \frac{\pi}{4} (\eta' \eta'')^{3/2} \left[H_\nu^{(1)}(-k\eta') H_\nu^{(2)}(-k\eta'') \right. \\ \left. - \frac{H_\nu^{(1)}(-k\eta_*)}{H_\nu^{(2)}(-k\eta_*)} H_\nu^{(2)}(-k\eta') H_\nu^{(2)}(-k\eta'') \right], \quad \text{for } \eta'' < \eta'. \quad (4.122)$$

The expression for $\eta' < \eta''$ is the same but with η' and η'' interchanged. Here, ν can either be real or complex ($\nu = i\mu$), depending on the mass of the exchanged field.

The wavefunction coefficient (4.121) captures some of the physics of the on-shell production and decay of massive particles in de Sitter space, and can be used as a seed to compute the phenomenological signatures of these heavy particles [17, 29, 34]. Unfortunately, however, the time integrals involved cannot be evaluated by elementary means. Instead of trying to compute the integrals directly, we note that G_ν satisfies the differential equation

$$\left(\eta^2 \frac{\partial^2}{\partial \eta^2} - 2\eta \frac{\partial}{\partial \eta} + k_I^2 \eta^2 + \frac{m^2}{H^2} \right) G_\nu(k_I; \eta, \eta') = -iH^2 \eta^2 \eta'^2 \delta(\eta - \eta'), \quad (4.123)$$

which is just the explicit form of the Green's function equation (4.78). The fact that G_ν satisfies this differential equation can be used to derive a differential equation for the function F . To do this, we consider the auxiliary object

$$\tilde{F} \equiv -g^2 \int \frac{d\eta'}{\eta'^2} \frac{d\eta''}{\eta''^2} e^{ik_{12}\eta'} e^{ik_{34}\eta''} \left(\eta'^2 \frac{\partial^2}{\partial \eta'^2} - 2\eta' \frac{\partial}{\partial \eta'} + k_I^2 \eta'^2 + \frac{m^2}{H^2} \right) G_\nu(k_I, \eta', \eta''), \quad (4.124)$$

and evaluate it in two ways. First, we use (4.123) to write

$$\tilde{F} = ig^2 H^2 \int_{-\infty}^0 d\eta' e^{iE\eta'} = \frac{g^2 H^2}{E}, \quad (4.125)$$

with $E \equiv k_{12} + k_{34}$. Next, we integrate the time derivatives by parts in (4.124) and trade factors of η' for $-i\partial_{k_{12}}$ to obtain³⁰

$$\begin{aligned} \tilde{F} &= -g^2 \int_{-\infty}^0 \frac{d\eta'}{\eta'^2} \frac{d\eta''}{\eta''^2} \left((k_I^2 - k_{12}^2) \eta'^2 + 2ik_{12}\eta' + \frac{m^2}{H^2} - 2 \right) e^{ik_{12}\eta'} e^{ik_{34}\eta''} G_\nu(k_I, \eta', \eta'') \\ &= -g^2 \int_{-\infty}^0 \frac{d\eta'}{\eta'^2} \frac{d\eta''}{\eta''^2} \left((k_{12}^2 - k_I^2) \partial_{k_{12}}^2 + 2k_{12} \partial_{k_{12}} + \frac{m^2}{H^2} - 2 \right) e^{ik_{12}\eta'} e^{ik_{34}\eta''} G_\nu(k_I, \eta', \eta'') \\ &= \left((k_{12}^2 - k_I^2) \partial_{k_{12}}^2 + 2k_{12} \partial_{k_{12}} + \frac{m^2}{H^2} - 2 \right) \psi, \end{aligned} \quad (4.126)$$

where, in the second line, we have pulled the differential operator outside the integral, which we then recognize as being precisely (4.121). Comparing (4.125) and (4.126), we find that ψ satisfies the following differential equation

$$\left((k_{12}^2 - k_I^2) \partial_{k_{12}}^2 + 2k_{12} \partial_{k_{12}} + \frac{m^2}{H^2} - 2 \right) \psi = \frac{g^2 H^2}{E}. \quad (4.127)$$

³⁰We can drop the boundary term, because the bulk-to-bulk propagator vanishes on the boundary.

This equation is rather interesting. Time evolution in the bulk spacetime has been traded for a differential operator in terms of the external energy k_{12} . When this operator acts on the exchange solution it produces a conformally coupled scalar contact solution; cf. (4.112). We can therefore think of the operator as collapsing the internal propagator, creating a contact interaction.

Exercise 4.13 Show that $\hat{F} = k_I \psi$ satisfies

$$\left(u^2(1-u^2)\partial_u^2 - 2u^3\partial_u + \frac{m^2}{H^2} - 2 \right) \hat{F} = g^2 H^2 \frac{uv}{u+v}, \quad (4.128)$$

where $u \equiv k_I/k_{12}$ and $v = k_I/k_{34}$.



This strategy of turning the integrals for the late-time wavefunction coefficients into differential equations is applicable in a wide range of situations. In the following exercise, you will derive an analogous differential equation for a three-point function.

Exercise 4.14 Derive a differential equation for the three-point wavefunction coefficient involving two conformally coupled scalars and one massive scalar, $G \equiv \langle OOX \rangle'$. Write the equation in terms of $u \equiv k_3/k_{12}$.



Notice that “time” has completely disappeared from the problem. The differential equation (4.127) is formulated purely in terms of the boundary momenta (or energies k_{12} , k_{34} and k_I). This suggests that there should be a more efficient way of understanding the correlations on the late-time boundary of de Sitter without ever explicitly referring to the (unobservable) bulk time evolution. In the next section, we will first show how the differential equation (4.127) can be derived directly on the boundary and then discuss its solutions.

5 Cosmological Bootstrap

In the previous two sections, we have presented two methods for computing inflationary correlations by evolving the fluctuations through the bulk spacetime and evaluating the resulting correlators on the future boundary. This has resulted in complex time integrals (one for each vertex in a diagram) that in general cannot be computed analytically. Numerical integration is possible, but comes at the cost of masking the physical structure of the answers.

In this section, we describe a new approach to deriving the correlations directly on the boundary by imposing consistency with unitarity, locality and symmetry assumptions. Although “time” will not make any explicit appearance, the effects of the time-dependent bulk physics (such as the production and decay of massive particles) will be reflected in the momentum scalings of the resulting correlators (see Fig. 8). The cosmological bootstrap is a very active area of current research which we will *not* be able to fully review in these lectures. Instead, we will try to give a pedagogical introduction to a few selected topics that hopefully will provide students with a useful entry point into the subject. More details and further references can be found in [1, 4, 5].

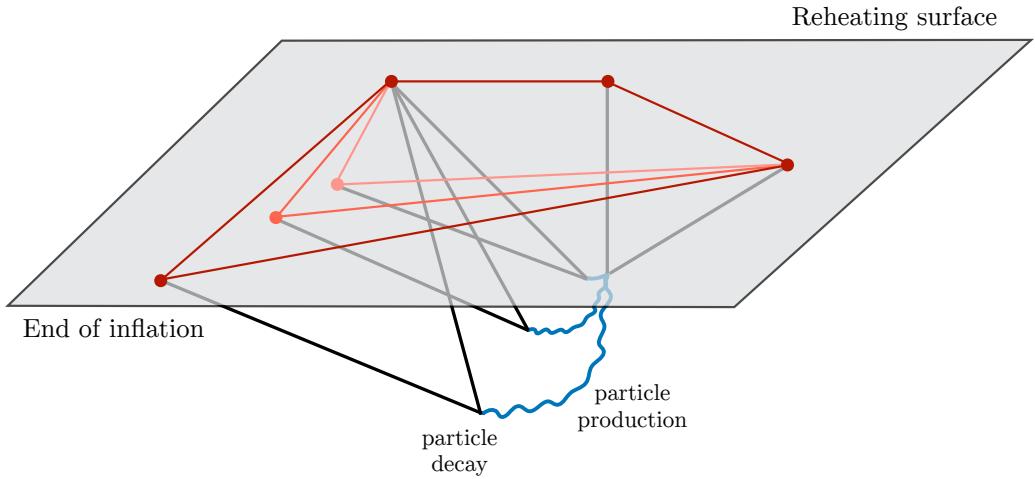


Figure 8: All cosmological correlations can be traced back to the spacelike boundary of the inflationary quasi-de Sitter spacetime. The time dependence of bulk interactions is encoded in the momentum dependence of these boundary correlators. The cosmological bootstrap is a way to derive these correlators purely from a boundary perspective.

5.1 Bootstrap Philosophy

Let us start with a few general remarks on the bootstrap philosophy and then illustrate them with the example of the S-matrix bootstrap.

You have probably learned that theories are defined by their Lagrangians. In particular, Lagrangians are a transparent way of encoding local and causal interactions between particles, which in relativistic quantum mechanics requires the introduction of fields. However, as illustrated in Fig. 9, Lagrangians are not really the fundamental starting point, but instead they are intermediate objects that take us from a set of fundamental principles to physical observables. To make locality and causality manifest, the Lagrangian formalism often requires the introduction

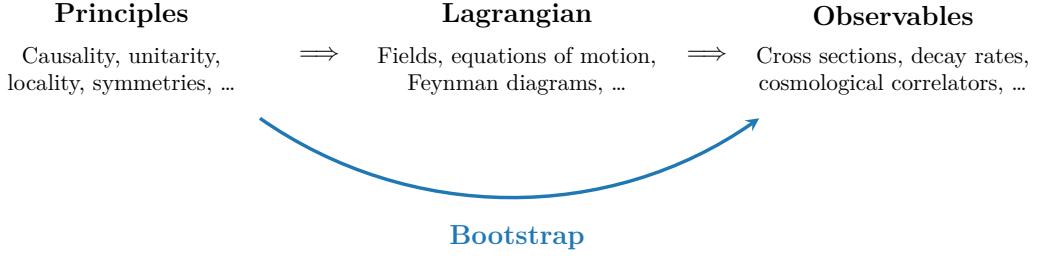


Figure 9: Graphical illustration of the bootstrap philosophy. Instead of encoding physical principles in a Lagrangian, the basic idea of the bootstrap is to take us directly from these principles to the final observables.

of additional, unphysical, degrees of freedom that are not observable, but nevertheless must be tracked in detail throughout the calculation.³¹ This introduces a substantial complication, and so the basic bootstrap philosophy is to go directly from the physical principles to final observables, without the intermediate step of introducing a Lagrangian and computing Feynman diagrams.

5.1.1 S-Matrix Bootstrap

The textbook approach to computing scattering amplitudes involves a perturbative expansion in terms of Feynman diagrams. Unfortunately, these calculations can quickly become very complicated, especially in gauge theories. For example, the five-point gluon amplitude involves 25 diagrams and the associated mathematical expressions span many pages. Similar, the four-point gravity amplitude involves thousands of diagrams, each involving hundreds of terms. In these cases, it required truly heroic efforts (by Parke and Taylor [37] and DeWitt [38], respectively) to compute these amplitudes explicitly. Yet, when restricted to physical (“on-shell”) configurations, the final answers are remarkably simple one-line expressions. This suggested that there should be a better way to obtain these results. Part of the problem is that individual Feynman diagrams are not completely physical, in the sense that they carry along unphysical gauge degrees of freedom that must cancel in the final answers.³² In the modern amplitudes bootstrap, the focus is instead on the external on-shell data (like the momenta and helicities of the external particles) and the space of consistent amplitudes is reconstructed from consistency requirements alone, dispensing with the need for Lagrangians, Feynman diagrams and unphysical gauge degrees of freedom.

We describe the bootstrap of massless particles with spin in Appendix C (following closely the excellent TASI lectures by Cliff Cheung [39]). In the following, we summarize the main results of this appendix in mostly qualitative terms.

Three-particle amplitudes The simplest interactions involve three particles. For massless particles with spin, the form of these interactions is uniquely fixed by *Lorentz symmetry* and *little group scaling*. The latter means that three-point amplitude has to transform in the expected

³¹For example, the two degrees of freedom of a photon are encoded in the four components of the electromagnetic vector potential A_μ . The additional (gauge) degrees of freedom are unphysical, but can appear at intermediate stages of the computations.

³²Even in scalar theories, Feynman diagrams are not invariant under field redefinitions. An infinite number of different Lagrangians and their corresponding Feynman diagrams can map to the same S-matrix. In that case, a poor choice of field basis can complicate the calculation and obscure important structures of the theory.

way when we rotate the polarizations of the three particles. In Section C.3, we present compact expressions for the allowed three-particles amplitudes A_3 in terms of the so-called spinor helicity variables:

$$\begin{aligned} A(1^-2^-3^-) &= \left(\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle \right)^S, & A(1^-2^-3^+) &= \left(\frac{\langle 12 \rangle^3}{\langle 13 \rangle \langle 32 \rangle} \right)^S, \\ A(1^+2^+3^+) &= \left([12] [23] [31] \right)^S, & A(1^+2^+3^-) &= \left(\frac{[12]^3}{[13][32]} \right)^S, \end{aligned} \quad (5.1)$$

where $\langle ij \rangle = \sqrt{p_i \cdot p_j} e^{i\delta}$ and $[ij] = \sqrt{p_i \cdot p_j} e^{-i\delta}$.

Four-particle amplitudes The challenge then is to construct consistent four-particle interactions out of these three-particle interactions. There is no guarantee that the resulting particle exchange will be local [40, 41].

Locality is reflected in the analytic structure of scattering amplitudes. At tree level (i.e. leading order in the perturbative expansion), amplitudes can have *poles* when intermediate particles go on-shell and satisfy $p_I^2 = 0$. These poles must be simple poles, since higher-order poles in momentum space translate to non-locality in position space. For example, in the s -channel, the limit $s \rightarrow 0$ can diverge as $1/s$, but not $1/s^2$. On these poles, the amplitudes must *factorize* into a product of lower-point amplitudes. This is easy to understand physically: when an intermediate particle goes on-shell it becomes long-lived and can propagate a large distance in spacetime before decaying. This propagation connects particles at separated points in spacetime, which explains the factorization of the corresponding scattering amplitude.

The first non-trivial example is the scattering of four particles. In that case, poles can appear in three different kinematic limits: the s , t , and u -channels. Locality fixes the residues of these poles in terms of products of the known three-particle amplitudes:

$$\begin{aligned} \lim_{s \rightarrow 0} A_4 &= \frac{A_3 \times A_3}{s} \equiv \frac{R_s}{s}, \\ \lim_{t \rightarrow 0} A_4 &= \frac{A_3 \times A_3}{t} \equiv \frac{R_t}{t}, \\ \lim_{u \rightarrow 0} A_4 &= \frac{A_3 \times A_3}{u} \equiv \frac{R_u}{u}. \end{aligned} \quad (5.2)$$

The challenge is to find a function that satisfies all these limits simultaneously. This *four-particle test* turns out to be a surprisingly stringent constraint on consistent theories of Nature [40].

In Section C.4, we consider the amplitude $A_4(1^-2^-3^+4^+)$. Using the results in (5.1), it can be shown that the residue of the s -channel pole is

$$R_s = (\langle 12 \rangle^2 [34]^2)^S \times \frac{1}{u^S}. \quad (5.3)$$

We see that the s -channel residue has a secret singularity for $u \rightarrow 0$. This is only allowed if this singularity can be interpreted as the propagation of the intermediate particle in the u -channel.

A similar analysis of the t and u -channel residues gives

$$\begin{aligned} R_t &= (\langle 12 \rangle^2 [34]^2)^S \times \frac{1}{s^S}, \\ R_u &= (\langle 12 \rangle^2 [34]^2)^S \times \frac{1}{t^S}. \end{aligned} \quad (5.4)$$

We see that each residue has a singularity in another channel. The challenge is to find a function that combines these different limits without having any higher-order poles. For the example above, we can write the following ansatz

$$A_4 = (\langle 12 \rangle^2 [34]^2)^S F(s, t, u). \quad (5.5)$$

Our task then is to find a function $F(s, t, u)$ that has the correct factorization limits, but doesn't have any unphysical higher-order poles.

Discovering YM and GR The four-particle test is very restrictive and highly constrains the allowed interactions of massless particles with spin. The details are given in Appendix C. Here, we just provide a brief snapshot of the argument.

- $S = 1$: Let us begin with a single massless spin-1 particle (*photon*). The ansatz (5.5) becomes

$$A_4 = \langle 12 \rangle^2 [34]^2 \left[\frac{c_s}{su} + \frac{c_t}{ts} + \frac{c_u}{ut} \right]. \quad (5.6)$$

By dimensional analysis, the coefficients c_s , c_t , c_u are dimensionless. Computing the s , t , u -channel residues of this ansatz and comparing the result to (5.3) and (5.4), one finds

$$\begin{aligned} c_s - c_t &= 1, \\ c_t - c_u &= 1, \\ c_u - c_s &= 1. \end{aligned} \quad (5.7)$$

However, adding these three expressions, we find a contradiction

$$0 = 3. \quad (5.8)$$

We conclude that there is no function that consistently factorizes in all channels. We have therefore discovered that photons are not self-interacting. Maxwell's equation are linear. This is why light waves don't scatter off each other and we can see over macroscopic distances.

To obtain a non-trivial four-particle amplitude, we consider multiple species of spin-1 particles (*gluons*). The three-particle interaction now has a coupling constant f_{abc} . We consider the amplitude $A(1_a^- 2_b^- 3_c^+ 4_d^+)$. The residues in (5.3) and (5.4) now get modified by products of the three-point couplings like $R_s \propto \sum_e f_{abe} f_{ecd}$. Comparing this to the residues of the ansatz (5.6) now gives

$$\begin{aligned} c_s - c_t &= \sum_e f_{abe} f_{ecd}, \\ c_t - c_u &= \sum_e f_{bce} f_{ead}, \\ c_u - c_s &= \sum_e f_{cae} f_{ebd}. \end{aligned} \quad (5.9)$$

Adding these three expressions, we then get

$$0 = \sum_e (f_{abe}f_{ecd} + f_{bce}f_{ead} + f_{cae}f_{ebd}), \quad (5.10)$$

which is the *Jacobi identity*. We see that gluons can have a consistent four-particle amplitude if the structure constant f_{abc} satisfies the Jacobi identity.

But there is more. We can couple the gluons A_a^μ to additional matter particles ϕ_i with a coupling $(T^a)_{ij}$ and consider the case of Compton scattering. This only passes the four-particle test if these couplings satisfy a *Lie algebra*, $[T^a, T^b]_{ij} = f^{abc}T^c_{ij}$. We have discovered the structure of *Yang-Mills theory*.

- **$S = 2$:** Next, we consider a single massless spin-2 particle (*graviton*). The s -channel residue (5.3) in this case is scale as $R_s \propto 1/u^2$. At first sight, the second-order pole at $u = 0$ looks like trouble. However, since $s + t + u = 0$, we have $u = -t$ when $s = 0$, so that (5.3) can also be written as $R_s \propto 1/(tu)$. A similar analysis for the t and u -channel residues gives $R_t \propto 1/(su)$ and $R_u \propto 1/(st)$. The unique function with the correct residues in all channels and only simple poles is³³

$$A_4 = -\frac{\langle 12 \rangle^4 [34]^4}{stu}. \quad (5.11)$$

We see that gravity has just barely escaped the perils of non-locality. It is also worth highlighting how simple it was to bootstrap the result (5.11). This is to be contrasted with the explicit calculation using Feynman diagram which is incredible complicated [38]. Moreover, the answer (5.11) is unique and corresponds to the result for *General Relativity*.

Finally, we can also consider Coulomb scattering of two gravitons and two matter particles. The four-particle test is only successful if the coupling to the matter particles is universal and equal to the graviton self-couplings. We have discovered the *equivalence principle* (without ever thinking about falling elevators).

- **$S \geq 3$:** Finally, we consider massless higher-spin particles. Now, we are doomed. All massless particles with spin greater than 2 fail the four-particle test. Such theories are not consistent with both Lorentz symmetry and locality.

It is remarkable how much can be derived from so little input. By imposing just Lorentz symmetry and consistent factorization, we have recovered the structure of all known long-range interactions. Derivations of these results, as well as further interesting consequences of the four-particle test, are presented in Appendix C.

5.1.2 Cosmological Bootstrap

In the following, we will present attempts to develop a similar bootstrap method for cosmological correlators. This is a vibrant and evolving subject, which we will not be able to review fully. Instead, we will take a specific path through the topic. We apologize for having to leave out many interesting recent developments.

³³In fact, this is unique function that satisfies the ansatz (5.6), has the correct dimensions and only simple poles in the s , t and u -channels—i.e. we can arrived at this result without even imposing factorization.

- **Singularities**—In the previous section, we have seen that singularities play an important role in determining the structure of scattering amplitudes. The same holds for cosmological correlators, although they have a richer structure of singularities. We will discuss this in Section 5.2. As we will show, every correlator has a singularity when the sum of all external energies vanishes, $E \equiv |\vec{k}_1| + |\vec{k}_2| + \dots + |\vec{k}_N| \rightarrow 0$. Moreover, the residue of this “total energy singularity” is the corresponding flat-space scattering amplitude. Rather remarkably, amplitudes therefore “live inside” cosmological correlators.

More generally, cosmological correlators have singularities whenever the energies entering any subgraph add up to zero. On these “partial energy singularities” the correlator factorizes into an amplitude and a lower-point correlators. This is similar to the factorization of amplitudes that we discussed in the previous section. Like for amplitudes, consistent factorization on all partial energy singularities is an important constraint on cosmological correlators. As in the S-matrix bootstrap, the challenge is to find a function with the correct singular limits, without introducing additional spurious singularities or other physical inconsistencies.

- **Unitarity cuts**—Unitarity plays a fundamental role in defining consistent observables in any quantum-mechanical theory. Until recently, it wasn’t known how the constraints of bulk unitarity are encoded in cosmological boundary correlators. In [42, 43], it was pointed out that unitarity implies specific relations that cosmological correlators have to satisfy in perturbation theory, quite similar to the optical theorem for scattering amplitudes in flat space. Moreover, these relations can be systematized into a set of cutting rules for cosmological correlators [44–47] that are similar to the Cutkosky rules for the S-matrix [48]. As we will explain in Section 5.3, cutting rules have since become an important tool to bootstrap cosmological correlators.

The main difficulty faced by the standard way of computing cosmological correlators is the presence of complicated nested time integrals. We will introduce the concept of “cut wavefunctions” for which these nested integrals factorize. We will also show that, in the limit of vanishing partial energies, the cut wavefunction reduces to the original wavefunction. In this way, unitarity fixes the residues of all partial energy singularities (not just the leading singularity) and therefore put an important constraint on the wavefunction coefficients. Especially for rational correlators, the constraints imposed by unitarity cuts go a long way towards determining the full correlators.

- **Symmetries**—As we describe in Section 5.4, one way to link the singularities of cosmological correlators is to use symmetry. During inflation, the spacetime is approximately de Sitter space, and the isometries of this bulk spacetime become conformal symmetries on the late-time boundary. Assuming that the bulk interactions don’t break this symmetry, the boundary correlators must then satisfy the Ward identities associated to this conformal symmetry. These Ward identities take the form of differential equations in terms of the external energies, which characterize how strength of the correlations depend on the energies. The singular limits of the correlator become the boundary conditions of the dif-

ferential equations. By solving the differential equations under these boundary conditions we therefore connect the singularities of the cosmological correlators.

A drawback of assuming conformal symmetry on the boundary is that it suppresses the strength on the non-Gaussian signals. Large non-Gaussianity requires the de Sitter boost symmetry to be strongly violated by the bulk interactions (at least in single-field inflation). The “bootless bootstrap” therefore studies correlators without assuming de Sitter boosts. The above constraints on the singularity structure of correlators, and the consequences of unitarity, still hold in these less symmetric settings and can be used to construct correlators in these models [49–53].

In the following, we will first describe these bootstrap constraints in more details and then apply them to a few concrete examples.

5.2 Singularities

An important feature of cosmology is that energy is not conserved—indeed this is a direct consequence of the time dependence of the dynamics. As we will see, this leads to cosmological correlators possessing characteristic singularities at special points in the space of kinematic variables. We will see how these singularities arise and discuss how they are an important input for the bootstrap construction of wavefunctions and correlators.

Roughly speaking, wavefunction coefficients are singular when the sum of the energies flowing into a subgraph vanishes, and the coefficients of these singularities are fixed in terms of simpler building blocks (lower-point amplitudes and correlators). It is important to point out that these singularities cannot be accessed for physical kinematics, since for correlators all energies have to be positive. (For amplitudes, in contrast, energies can be positive or negative corresponding to incoming or outgoing particles, respectively.) Nevertheless, the fact that the analytic continuation of correlation functions must have these singularities will be an important constraint. Using this as an input for the bootstrap procedure will allow us to reconstruct the full correlator and extract answers in the physical domain.

5.2.1 Flat-Space Examples

Even the flat-space wavefunction has these energy singularities. In that case, the presence of a spatial boundary leads to a breaking of time translations which implies that energy is not conserved. It is useful to start with a few concrete examples and then try to draw some general lessons from them. In Section 4.4.4, we computed wavefunction coefficients in flat space. Let us look back at some of the results.

One-site graph

The three-point wavefunction coefficient is

$$\langle O_1 O_2 O_3 \rangle' \equiv \begin{array}{c} \diagup \\ \diagdown \\ \text{---} \end{array} = ig \int_{-\infty}^0 dt e^{i(k_1+k_2+k_3)t} = \frac{g}{k_1 + k_2 + k_3}. \quad (5.12)$$

We see that this has a singularity when the total energy of the process vanishes, $K \equiv k_1 + k_2 + k_3 \rightarrow 0$. For a general N -point contact interaction, we would get a similar result with a singularity at $E \equiv k_1 + k_2 + \dots + k_N \rightarrow 0$. We call these *total energy singularities* [54, 55].

We can understand this singularity as an avatar of the energy-conserving delta functions that appear in the computations of scattering amplitudes:

$$\int_{-\infty}^{\infty} dt e^{i(E_1 + \dots + E_N)t} = 2\pi\delta(E_1 + \dots + E_N). \quad (5.13)$$

(Recall that the time integral factorizes from the rest of the computation because of time-translation invariance.) In contrast, the flat-space wavefunction involves integration from $t = -\infty$ to $t = 0$, so instead of a delta function, we schematically find

$$i \int_{-\infty}^0 dt e^{i(E_1 + \dots + E_N)t} = \frac{1}{E_1 + \dots + E_N}, \quad (5.14)$$

which is indeed singular as $E_1 + \dots + E_N \rightarrow 0$. Physically, this singularity comes from the lower limit of the integration. This domain is normally regulated by the oscillatory factors in the integrand, but these vanish exactly when the sum of the energies vanishes, so the wavefunction diverges.

The integrals that we have to evaluate in realistic situations are typically more complicated than the one appearing in (5.14), but they will still (almost always) have singularities when the total energy vanishes. More remarkably, the residues of these singularities are the corresponding scattering amplitudes. For the above contact examples, this is rather trivial since the residue of the total energy singularity is the coupling constant g , which is the amplitude for a polynomial contact interaction. However, the fact that amplitudes live at the loci of energy singularities will be true more generally.³⁴

Two-site graph

A more nontrivial example is the four-point wavefunction coefficient arising from an exchange diagram (4.87):

$$\langle O_1 O_2 O_3 O_4 \rangle' \equiv \begin{array}{c} \text{Feynman diagram: two external lines } O_1 O_2 \text{ and } O_3 O_4 \text{ meeting at a central vertex.} \\ \text{The central vertex is connected to two internal lines forming a loop.} \end{array} = -\frac{g^2}{2k_I} \int_{-\infty}^0 dt' \int_{-\infty}^{t'} dt'' e^{i(k_{12}-k_I)t'} e^{i(k_{34}+k_I)t''} - \frac{g^2}{2k_I} \int_{-\infty}^0 dt' \int_{-\infty}^{t'} dt'' e^{i(k_{12}+k_I)t''} e^{i(k_{34}-k_I)t'} + \frac{g^2}{2k_I} \int_{-\infty}^0 dt' \int_{-\infty}^0 dt'' e^{i(k_{12}+k_I)t'} e^{i(k_{34}+k_I)t''}. \quad (5.15)$$

³⁴In flat space, this is not so mysterious. The Feynman rules that we use to compute both the wavefunction coefficients and scattering amplitudes are nearly identical, the only material difference being the domain of integration over time; we integrate over all time to compute the S-matrix, while for the wavefunction we only integrate from $t = -\infty$ to $t = 0$. (The Green's functions that we use for internal lines also differ, but this difference is regular as $E \rightarrow 0$.)

This combination of time integrals also has a singularity as $E \equiv k_{12} + k_{34} \rightarrow 0$. To make this manifest, it is useful to define $t' \equiv T$ and $t'' - t' \equiv \delta T$ in the first two integrals. We can then combine these integrals as

$$\begin{aligned}\langle O_1 O_2 O_3 O_4 \rangle' &= -\frac{g^2}{2k_I} \int_{-\infty}^0 dt e^{iET} \int_{-\infty}^0 d\delta T \left(e^{iE_L \delta T} + e^{iE_R \delta T} \right) \\ &\quad + \frac{g^2}{2k_I} \int_{-\infty}^0 dt' e^{iE_L t'} \int_{-\infty}^0 dt'' e^{iE_R t''},\end{aligned}\tag{5.16}$$

where $E_L \equiv k_{12} + k_I$ and $E_R \equiv k_{34} + k_I$. Since we have decoupled the integration over T , it is clear that there will be a singularity when $E \rightarrow 0$. (It is also easy to see that only the time-ordered parts of the Feynman diagram contribute to the total energy singularity, which is both necessary and expected in order for it to be related to a scattering amplitude.) Indeed, performing the integrals in (5.16), we find

$$\langle O_1 O_2 O_3 O_4 \rangle' = \frac{g^2}{EE_L E_R}.\tag{5.17}$$

Taking the limit $E \rightarrow 0$, it is easy to confirm that the residue of the total energy singularity is indeed the corresponding exchange amplitude in the s -channel: [AJ: be careful about factors of i in the amplitude]

$$\lim_{E \rightarrow 0} \frac{g^2}{EE_L E_R} = \frac{1}{E} \frac{g^2}{(k_{12} + k_I)(-k_{12} + k_I)} = -\frac{1}{E} \frac{g^2}{s} \equiv \frac{A_4}{E},$$

where $s \equiv k_{12}^2 - k_I^2$ is the Mandelstam invariant.

In addition to the total energy singularity at $E = 0$, the result in (5.17) also has *partial energy singularities* at $E_L = 0$ and $E_R = 0$. In the limit $E_L \rightarrow 0$, the wavefunction becomes

$$\lim_{E_L \rightarrow 0} \frac{g^2}{EE_L E_R} = \frac{g}{E_L} \frac{g}{(k_{34} + k_I)(k_{34} - k_I)} = \frac{A_3 \times \tilde{\psi}_3}{E_L}.\tag{5.18}$$

We see that the residue of this partial energy singularity is the product of a three-point scattering amplitude $A_3 = g$ and a new object which we call a *shifted wavefunction coefficient*

$$\tilde{\psi}_3 \equiv \frac{1}{2k_I} [\psi_3(k_{34}, -k_I) - \psi_3(k_{34}, k_I)],\tag{5.19}$$

where ψ_3 is the three-point wavefunction coefficient given in (5.12). (The E_R singularity behaves similarly.) In fact, these simple flat-space correlators are completely fixed by their singularities. In particular, the result (5.17) is the unique function that has the correct factorization singularities. This is rather interesting, because it implies that the answer can be assembled from simpler building blocks (amplitudes and lower-point shifted wavefunctions).

By inspection of (5.16), we see that these energy singularities again come from the early-time regions of the integration. It is also easy to see why the residues of the partial energy singularities take the form shown in (5.18). In the early-time limit, the bulk-to-bulk propagator factorizes and the time-ordering disappears [36, 56]:

$$G(k_I; t', t'') \xrightarrow{t' \rightarrow -\infty} \frac{e^{ik_I t'}}{2k_I} (e^{-ik_I t''} - e^{+ik_I t''}).\tag{5.20}$$

On the partial energy poles, the correlator therefore factorizes into a product of an amplitude and a shifted correlator.

Exercise 5.1 In Exercise 4.8, you were asked to derive the five-point wavefunction coefficient in flat space. You should have found the following result:

$$\begin{aligned} \langle O_1 O_2 O_3 O_4 O_5 \rangle' &\equiv \begin{array}{c} k_1 \quad k_2 \quad k_3 \quad k_4 \quad k_5 \\ \hline \diagdown \quad \diagup \quad | \quad \diagdown \quad \diagup \\ \text{---} \quad k_I \quad k'_I \quad \text{---} \end{array} \\ &= \frac{1}{E_L E_M E_R} \left[\frac{1}{k_{123} + k'_I} + \frac{1}{k_{345} + k_I} \right], \end{aligned} \quad (5.21)$$

where $E_L \equiv k_{12} + k_I$, $E_M \equiv k_3 + k_I + k'_I$ and $E_R \equiv k_{45} + k'_I$ are the energies running into the three different vertices. Discuss the different singularities of the solution. →

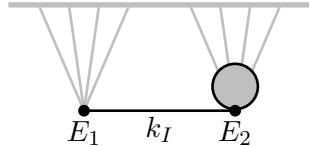
Arbitrary graphs

This pattern generalizes to arbitrary graphs. Flat-space wavefunction coefficients are singular whenever the energy flowing into a connected subgraph vanishes [36]. At tree level, the singularities are just (simple) poles, but for loops the poles can become branch points. We can see this structure of singularities directly from the time integral representation of the wavefunction.

For simplicity, we will focus on tree-level wavefunction coefficients, but loops can be understood similarly. We will restrict to a massless scalar with a polynomial self-interaction Φ^n , but introducing more complicated interaction vertices does not materially change the discussion. We proceed inductively. Assume that the wavefunction coefficient associated to some general graph



has a total energy singularity at $E \rightarrow 0$, whose coefficient is the corresponding flat-space scattering amplitude. (We have already shown that this is true for both the one-site and two-site graphs above as base cases.) Note that the number of external lines does not matter, just the total energy flowing into and out of each site of the graph. We now attach an internal line to this graph:



where E_1 is the energy flowing out to the boundary from the new vertex that we have added, E_2 is the energy flowing to the boundary from the vertex that the internal line attaches to, and k_I is the energy carried by the internal line. The time integral that computes the wavefunction

coefficient for this graph can then schematically be written as

$$\psi = i \int_{-\infty}^0 dt_2 e^{iE_2 t_2} \text{ } \bullet \text{ } \int_{-\infty}^0 dt_1 e^{iE_1 t_1} G(k_I; t_1, t_2), \quad (5.23)$$

where \bullet stands for the time integral that computes the wavefunction coefficient (5.22), minus the plane wave factor that connects the vertex 2 to the boundary, which we have pulled out explicitly, for clarity. The time t_1 only appears in the last factor of (5.23), so we can use the expression (4.84) to do the time integral explicitly:

$$\int_{-\infty}^0 dt_1 e^{iE_1 t_1} G(k_I; t_1, t_2) = \frac{-i}{k_I^2 - E_1^2} \left(e^{iE_1 t_2} - e^{ik_I t_2} \right). \quad (5.24)$$

We see that the addition of an internal line shifts the energy flowing out of the vertex 2 vertex in two different ways. Plugging this back into (5.23), we find the following recursive formula:

$$\begin{array}{c} \text{---} \\ | \\ E_1 \end{array} \xrightarrow{k_I} \begin{array}{c} \text{---} \\ | \\ E_2 \end{array} = \frac{1}{k_I^2 - E_1^2} \left[\begin{array}{c} \text{---} \\ | \\ E_1 + E_2 \end{array} - \begin{array}{c} \text{---} \\ | \\ E_2 + k_I \end{array} \right]. \quad (5.25)$$

If \bullet had a total energy singularity at $E \rightarrow 0$ (where E is the total energy of the blob), the first term on the right-hand side of (5.25) will have a corresponding singularity at $E_1 + E_2$. This is precisely the total energy of the graph on the left-hand side of (5.25). Further, we see that the prefactor is precisely the flat-space propagator for the internal line we have added, so the coefficient of this singularity is the flat-space scattering amplitude.

We can also understand the structure of partial energy singularities from (5.25). Consider building up any tree-level graph by iteratively adding edges and using the formula (5.25). The total energy singularities of the constituent subgraphs will then appear as partial energy singularities of the full graph. It is then clear that the coefficient of this singularity will be the amplitude corresponding to the graph whose energy is conserved times a shifted wavefunction, which arises in the same way as in (5.18) from the limit of the bulk-to-bulk propagator (5.20). Since we can imagine starting with the one-site graph—which is only singular when the total energy vanishes—we see that these are the only singularities.³⁵ (See also [57] for an explanation of this point.)

The above characterization of singularities is manifestly rooted in the time integral representation of the wavefunction, since this will be most easily adapted to the cosmological setting. Of course, part of the philosophy of the cosmological bootstrap is to view the wavefunction as an intrinsically boundary object, that does not intrinsically require the presence of the bulk. One might therefore expect that it is possible to understand the structure of singularities of the wavefunction directly only the boundary. Indeed this is possible—the singularities of the flat-space wavefunction can be simply characterized using the recursion relation (4.100).³⁶ This recursive

³⁵One might be concerned that the prefactor of (5.25) is singular at $E_1 = k_I$, which does not seem to be a legal singularity. However, at this kinematic point, the two terms in brackets exactly cancel, so this limit is regular.

³⁶Note that this relation can be derived without any reference to time, and further has a beautiful geometric interpretation in terms of cosmological polytopes [36].

formula makes manifest the total energy singularity of wavefunction coefficients, and we can further understand the partial energy singularities as a consequence of the recursive construction from simpler subgraphs.

5.2.2 Generalization to de Sitter

The singularity structure of the wavefunction in de Sitter space is very similar to that of its flat-space analog. It is still true that the wavefunction is singular at loci where the energy flowing into a subgraph vanishes, though now the singularities can be more complicated: they can be poles of higher order or branch points (even at tree level).

The reason that the singularities of the de Sitter wavefunction have the same structure is that the divergences in the time integrals are localized at early times, where the fluctuations evolve in the same way as in Minkowski space. Concretely, the bulk-to-bulk and bulk-to-boundary propagators are built from the mode functions (4.106). In the early-time limit, these can be expanded as³⁷

$$f_k(\eta) \xrightarrow{k\eta \rightarrow -\infty} \frac{(-H\eta)}{\sqrt{2k}} e^{ik\eta} \left((4\nu^2 - 1) + \frac{i}{8}(4\nu^2 - 1)(4\nu^2 - 9) \frac{1}{k\eta} + \dots \right), \quad (5.27)$$

where as usual $\nu \equiv \sqrt{9/4 - m^2/H^2}$. We see that in the region where the divergences are localized, the integrand takes the form of powers of η times the corresponding flat-space integrand. These additional factors of η can change the nature of the divergences, but not their locations. Perhaps the simplest example is the three-point wavefunction of conformally coupled scalars that we computed in (4.109):

$$\psi_{\varphi^3} = \frac{ig}{H^4 \eta_*^3} \log(iK\eta_*), \quad (5.28)$$

which has a branch point singularity when its total energy (K) vanishes. Similarly, the four-point wavefunction (4.118) has branch points both when its total energy and partial energies vanish.

The general characterization of singularities is similar to that in flat space: wavefunctions can be singular whenever the energy flowing into a subgraph, E_Γ , vanishes. At these singularities, the wavefunction factorizes [AJ: fix phase] [DB: explain factors of k_i in $\tilde{A}_\Gamma^{(m)}$]

$$\boxed{\psi \xrightarrow{E_\Gamma \rightarrow 0} \frac{1}{E_\Gamma^p} \tilde{A}_\Gamma^{(m)} \times \tilde{\psi}_{\Gamma'}^{(n-m)}}, \quad (5.29)$$

where Γ denotes a subgraph with m external lines and Γ' is the complementary subgraph with $n - m$ external lines. We have define a rescaled amplitude and a shifted wavefunction coefficient:

$$\tilde{A}_\Gamma \equiv k_1^{\nu_1 - \frac{1}{2}} \dots k_m^{\nu_m - \frac{1}{2}} A_\Gamma^{(m)}, \quad (5.30)$$

$$\tilde{\psi}_{\Gamma'} \equiv P(k_I) \left(\psi_{\Gamma'}(k_a, -k_I) - \psi_{\Gamma'}(k_a, k_I) \right), \quad (5.31)$$

³⁷Here, we are using the following asymptotic expansion for the Hankel function:

$$H_\nu^{(2)}(x) \xrightarrow{x \rightarrow \infty} \left(\frac{2}{\pi x} \right)^{\frac{1}{2}} e^{-ix + \frac{i\pi}{4}(1+2\nu)} \left((4\nu^2 - 1) - \frac{i}{8}(4\nu^2 - 1)(4\nu^2 - 9)x^{-1} + \dots \right). \quad (5.26)$$

See, for example, the DLMF.

where $P(k_I)$ is the power spectrum of the field connecting the two graphs Γ and Γ' . The order of the singularity, p , in (5.29) is set by the number of derivatives involved in the interactions that gave rise to it. For de Sitter-invariant interactions, the power of the divergence, is $p = \dots$ [AJ: add]. (Note that when $p = 0$, the singularity is a branch point.)

It is not difficult to check that all the examples of wavefunction coefficients that we have computed in de Sitter space only have singularities at these locations.

[AJ: mention examples like all plus YM that don't have singularities.]

[AJ: Explain singularities of correlators—additional singularities at k_I etc.]

5.2.3 Folded Singularities

[DB: Add.]

5.2.4 Linking Singularities

So far, we have discovered the energy singularities of the wavefunction and the associated residues directly from the time-integral computation of the wavefunction. However, as is familiar from complex analysis, functions are often controlled by their structure of singularities.³⁸ We are therefore motivated to try inverting the logic—we promote this singularity structure to a *principle*, and see to what extent we can use this information to bootstrap the full cosmological correlation function from its singularities.

In this respect a simple success story is illuminating: Consider the exchange four-point function of massless scalars in flat space (4.89). We expect that this wavefunction can be assembled from the three-point wavefunction coefficient and the corresponding three-point amplitude:

$$\psi_3 = \frac{1}{k_1 + k_2 + k_3}, \quad A_3 = 1, \quad (5.32)$$

where we have normalized the coupling constant to 1. The four-point wavefunction coefficient should be singular when either its left partial energy $E_L \equiv k_{12} + k_I$ or right partial energy $E_R \equiv k_{34} + k_I$ vanishes:

$$\psi_4 \xrightarrow{E_L \rightarrow 0} \frac{1}{E_L} A_L \times \tilde{\psi}_{3,R} = \frac{1}{k_{12} + k_I} \frac{1}{(k_{34} + k_I)(k_{34} - k_I)}, \quad (5.33)$$

$$\psi_4 \xrightarrow{E_R \rightarrow 0} \frac{1}{E_R} \tilde{\psi}_{3,L} \times A_R = \frac{1}{k_{34} + k_I} \frac{1}{(k_{12} + k_I)(k_{12} - k_I)}, \quad (5.34)$$

where we have written the residues in terms of the appropriate lower-point amplitudes and (shifted) wavefunction coefficients. Interestingly, this is enough information to reconstruct the complete exchange four-point function. The nontrivial constraint is that the residue of each partial energy singularity has a pole at the location of the other singularity, and so we have to find a function that consistently factorizes in both locations. In order to do this, we must interpret the factors of $k_{12} - k_I$ and $k_{34} - k_I$ as the total energy, evaluated in these particular

³⁸The analogy is more exact than it might appear because the singularities we are discussing really are singularities of the wavefunction thought of as a complex function of its arguments.

kinematic configurations. (This also has the benefit of removing the apparently illegal folded singularities.) We therefore have

$$\psi_4 = \frac{1}{EE_L E_R}, \quad (5.35)$$

which is precisely (5.17). There is, of course, an ambiguity in this answer, which is that we could add terms that are regular in the partial energy limits (corresponding to contact interactions). However, it is interesting that a total energy singularity arises as an inevitable output in this context.

The previous example is so simple that we can completely reconstruct it only from knowledge of its factorization singularities. In this respect, it is somewhat similar to four-particle amplitudes in theories of massless spinning particles in flat space (see Section 5.1.1). However, in more complicated examples, or at higher points, it becomes quite cumbersome (or even sometimes impossible), to extend the wavefunction away from its singularities from pure thought. We therefore need some additional organizing principles to help connect the singularities. In some sense, we have traded the challenge of time evolution for this problem. We will present two strategies for linking the singularities of cosmological correlations: symmetry and unitarity.

5.3 Unitarity Cuts

[DB: Most of this section is copied from our paper. Needs to be re-written.]

In quantum mechanics, any consistent theory must have a unitary S-matrix, $S^\dagger S = 1$. This simple constraint has remarkable consequences [58, 59]. ... optical theorem ... cutting rules ...

	nonperturbative	perturbative
S-matrix	optical theorem	cutting rules
correlator	??	cutting rules

5.3.1 Cutting Rules in Flat Space

[DB: FIX:] Conjugate wavefunction coefficients, $\bar{\psi}_n$, are computed using the same Feynman rules as for the wavefunction coefficients, except that we assign the *anti*-bulk-to-bulk propagator to internal lines, $\bar{G} = G^*$, and replace each vertex factor with $-iV$, but use the same bulk-to-boundary propagators as for the ordinary wavefunction. An important feature of the conjugate wavefunction coefficients is that in a *unitary* theory, they are related to the complex conjugate of

the ordinary wavefunction coefficients, but with analytically continued external energies [42, 43]³⁹

$$\bar{\psi}_n(k_1, \dots, k_n) = \psi_n^*(-k_1, \dots, -k_n) \quad (\text{unitarity}). \quad (5.36)$$

This relation will play an important role in the cutting rules for the wavefunction coefficients.

[DB: combine] We can do this by considering a suitable combination of the wavefunction coefficients ψ and conjugate wavefunction coefficients $\bar{\psi}$, engineered so that the bulk-to-bulk propagator and the anti-bulk-to-bulk propagator, $\bar{G}(k; t, t') = -G(-k; t, t')$, appear as the sum

$$\tilde{G}(k; t, t') \equiv G(k; t, t') + \bar{G}(k; t, t') = -\frac{1}{2k} (e^{-ikt} - e^{ikt}) (e^{-ikt'} - e^{ikt'}). \quad (5.37)$$

We will refer to \tilde{G} as the *cut propagator*. This combination trivializes the time-ordering involved in the bulk integrations and therefore simplifies the computation of particular sums of ψ and $\bar{\psi}$. We will see that introducing this combination of propagators has the interpretation of cutting the internal line of a graph, as in the S-matrix cutting rules. The cutting rules essentially systematize finding combinations that simplify time integrals, and naturally produce identities involving both ψ and $\bar{\psi}$. In a unitary theory, $\bar{\psi}$ can be related back to ψ which produces identities satisfied by the wavefunction ψ alone.

A simple example

To illustrate the cutting rule in a simple example, we again consider the four-point function arising from tree-level exchange in ϕ^3 theory. The wavefunction coefficient corresponding to the *s*-channel contribution was given in (??):

$$\psi_4(k_{12}, k_{34}) = - \int_{-\infty}^0 dt dt' e^{ik_{12}t} G(k_I; t, t') e^{ik_{34}t'}. \quad (5.38)$$

Note that the bulk-to-bulk propagator in (??) leads to nested time integrals, which can be done explicitly in this simple example, but are harder to perform in more general situations. However, we can simplify the integrals involved by considering the sum of ψ_4 and its conjugate $\bar{\psi}_4$:

$$\begin{aligned} \psi_4(k_{12}, k_{34}) + \bar{\psi}_4(k_{12}, k_{34}) &= - \int_{-\infty}^0 dt dt' e^{ik_{12}t} \tilde{G}(k_I; t, t') e^{ik_{34}t'} \\ &= -\frac{1}{2k_I} \int_{-\infty}^0 dt e^{ik_{12}t} (e^{-ik_I t} - e^{ik_I t}) \int_{-\infty}^0 dt' e^{ik_{34}t'} (e^{-ik_I t'} - e^{ik_I t'}) \\ &\equiv 2k_I \tilde{\psi}_3(k_{12}, k_I) \tilde{\bar{\psi}}_3(k_{34}, k_I), \end{aligned} \quad (5.39)$$

³⁹As we will discuss more in detail in Section ??, this analytic continuation to negative energies is performed by rotating clockwise in the complex plane. Moreover, when dealing with external spinning particles, the wavefunction coefficients can have an explicit dependence on spatial momenta \mathbf{k}_a , contracted with polarization vectors. In such cases, we also flip the signs of external momenta $\mathbf{k}_a \mapsto -\mathbf{k}_a$, which can be understood as analytically continuing the energies $k_a \mapsto -k_a$, while keeping the directions of \hat{k}_a fixed in the real domain. The polarization vectors can be either stripped off in the process or kept unchanged under the replacement rule $[\xi_a^\pm(-\hat{k}_a)]^* \mapsto \xi_a^\pm(\hat{k}_a)$; see also [45]. With this being understood, in what follows, we will only show energies in the argument of ψ for brevity.

where we have introduced the *shifted three-point function*

$$\tilde{\psi}_3(k_{12}, k_I) \equiv \frac{i}{2k_I} \int_{-\infty}^0 dt e^{ik_{12}t} \left(e^{-ik_I t} - e^{ik_I t} \right) = \frac{1}{2k_I} \left(\psi_3(k_{12} - k_I) - \psi_3(k_{12} + k_I) \right). \quad (5.40)$$

We can interpret the right-hand side as cutting the internal line connecting the two vertices, and then shifting the energies of these nodes by the energy of the cut line. At this point, the relation (??) is just an algebraic identity between ψ_4 and the auxiliary object $\bar{\psi}_4$. However, using (5.36)—which holds in a unitary theory—we can write the left-hand side in terms of ψ_4 alone⁴⁰

$$\boxed{\psi_4(k_{12}, k_{34}) + \psi_4^*(-k_{12}, -k_{34}) = -2k_I \tilde{\psi}_3(k_{12}, k_I) \tilde{\psi}_3(-k_{34}, k_I)}. \quad (5.41)$$

For later convenience, we will use the following shorthand for the left-hand side of the cutting rule

$$\text{Disc}[\psi_4] \equiv \psi_4(k_{12}, k_{34}) + \psi_4^*(-k_{12}, -k_{34}), \quad (5.42)$$

although we must emphasize that this operation is not precisely the same as extracting the discontinuity of a complexified function.⁴¹ [DB: fix:] Note that $\tilde{\psi}_3(k_{34} \mp s) = \tilde{\psi}_3(-k_{34} \mp s)$, but it is convenient to write things as in (5.41) in order to give the final formula a diagrammatic interpretation. The formula (5.41) also provides an additional perspective on the partial energy singularities discussed in Section ???. By taking partial energy limits of this formula we can reproduce the partial energy singularity formulas (5.18).

5.3.2 Generalization to De Sitter

[DB: include]

5.3.3 Cosmological Cutting Rules

[DB: include]

5.3.4 Dispersive Integrals

In [53], Meltzer derived an interesting tree-level dispersion relation that, in some cases, allows us to integrate the unitary cut to obtain the full exchange four-point function from its discontinuity. We will first derive this relation and then apply it to a few examples.

By causality, the bulk-to-bulk propagator is an analytic function of k^2 that decays at large k . This implies that we can write the following dispersion formula for the bulk-to-bulk propagator:

$$G(k; \eta, \eta') = \frac{1}{2\pi i} \int_0^\infty \frac{dq^2}{q^2 - k^2 + i\epsilon} \text{Disc}_{q^2}[G(q; \eta, \eta')], \quad (5.43)$$

⁴⁰In the simple flat-space examples that we are considering, we will never encounter branch cuts, so we do not have to specify the precise analytic continuation to negative energies, but this will be important in the generalization to de Sitter space, as we discuss in Section 5.3.2.

⁴¹The notation follows that of [43, 45, 46, 50], and is meant to recall the discontinuity operation involved in the cuts of scattering amplitudes. While the way that we have defined this operation it is not a true discontinuity, at least at tree level it can be related to a true discontinuity with respect to internal energy variables [53]. It is also important to note that our definition of this operation differs slightly from that of [43, 45, 46, 50], because of different conventions for factors of i .

where $\text{Disc}_{q^2}[G(q)] \equiv G(e^{-2\pi i k^2}) - G(k^2)$ is the discontinuity of the propagator. From this result, we can derive dispersion relations for tree-level exchange correlators. As a concrete example, consider the exchange diagram arising from a non-derivative $\Phi^2 \chi$ interaction, where χ is a generic scalar. Using the Feynman rules, the tree-level four-point function is

$$\psi_4 = -g^2 \int \frac{d\eta}{\eta^4} \frac{d\eta'}{\eta'^4} K_{k_1}(\eta) K_{k_2}(\eta) G(k_I; \eta, \eta') K_{k_3}(\eta') K_{k_4}(\eta'), \quad (5.44)$$

where $k_I \equiv |\mathbf{k}_1 + \mathbf{k}_2|$. Writing the bulk-to-bulk propagator as the dispersive integral (5.43), and changing the order of integration, then leads to the following dispersion relation for the four-point function [53]:

$$\psi_4(k_a, k_I) = \frac{1}{2\pi i} \int \frac{dq^2}{q^2 - k_I^2 + i\epsilon} \text{Disc}_{q^2}[\psi_4(k_a, q)]. \quad (5.45)$$

In Section 5.3, we used unitarity to derive a cutting rule for the discontinuity of the exchange diagram [DB: Needs to be coordinated with the previous section]

$$\text{Disc}_{q^2}[\psi_4(k_a, q)] = -P(q) \text{Disc}_{q^2}[\psi_3(k_1, k_2, q)] \times \text{Disc}_{s^2}[\psi_3(k_3, k_4, q)]. \quad (5.46)$$

Feeding this into (5.45) then expresses the four-point function as an integral over a product of three-point data. As we will see, in many case, the dispersion integral (5.45) is simpler to compute than the bulk time integrals in (5.44). [DB: mention generalization to derivative interactions.]

Conformally coupled scalar As a first example, we would like to reproduce the exchange four-point function of a conformally coupled scalar; see Section 4.4.5.

We start from the three-point wavefunction coefficient (4.109):

$$\psi_3(k_1, k_2, k_3) = i \log(k_1 + k_2 + k_3) + \text{const}. \quad (5.47)$$

Feeding this into the cutting rule (5.46), together with $P(k) = (2k)^{-1}$, we get the discontinuity of the exchange four-point function:

$$\text{Disc}_{k_I^2}[\psi_4(k_a, k_I)] = \frac{1}{2s} \log\left(\frac{k_{12} - k_I}{k_{12} + k_I}\right) \log\left(\frac{k_{34} - k_I}{k_{34} + k_I}\right), \quad (5.48)$$

and dispersion relation (5.45) reads

$$\psi_4 = \frac{1}{2\pi i} \int_0^\infty \frac{dq^2}{q^2 - k_I^2 + i\epsilon} \frac{1}{2q} \log\left(\frac{k_{12} - q}{k_{12} + q}\right) \log\left(\frac{k_{34} - q}{k_{34} + q}\right). \quad (5.49)$$

We would like to show that this is equivalent to the result derived previously in Section 4.4.5. To deal with the dispersive integral, we write the logarithmic factors using

$$\log\left(\frac{k_{12} - q}{k_{12} + q}\right) = 2q \int_{k_{12}}^\infty \frac{dx}{x^2 - q^2}. \quad (5.50)$$

Equation (5.49) then becomes

$$\begin{aligned}\psi_4 &= \frac{2}{\pi i} \int_0^\infty \frac{q^2 dq}{q^2 - k_I^2} \int_{k_{12}}^\infty \frac{dx}{x^2 - q^2} \int_{k_{34}}^\infty \frac{dy}{y^2 - q^2} \\ &= \int_{k_{12}}^\infty dx \int_{k_{34}}^\infty dy \underbrace{\int_{-\infty}^\infty \frac{dq}{\pi i} \frac{q^2}{q^2 - k_I^2 + i\epsilon} \frac{1}{x^2 - q^2} \frac{1}{y^2 - q^2}}_{\equiv \mathcal{I}},\end{aligned}\quad (5.51)$$

where, in the last line, we have used that the integrand is an even function of q to extend the range of integration from $(0, \infty)$ to $(-\infty, \infty)$. To perform the q -integral, we note that the integrand has poles at

$$\begin{aligned}q &= \pm \sqrt{k_I^2 - i\epsilon}, \\ q &= \pm x, \\ q &= \pm y,\end{aligned}\quad (5.52)$$

where the Bunch-Davies boundary condition implies that x and y have a small, negative imaginary parts. We close the contour in the lower half-plane to find,

$$\begin{aligned}\mathcal{I} &= - \left(\text{Res}_{q=\sqrt{k_I^2-i\epsilon}} + \text{Res}_{q=x} + \text{Res}_{q=y} \right) \frac{2q^2}{q^2 - k_I^2} \frac{1}{x^2 - q^2} \frac{1}{y^2 - q^2} \\ &= - \frac{2q^2}{q + k_I} \frac{1}{x^2 - q^2} \frac{1}{y^2 - q^2} \Big|_{q=k_I} - \frac{2q^2}{q^2 - k_I^2} \frac{1}{x + q} \frac{1}{y^2 - q^2} \Big|_{q=x} - \frac{2q^2}{q^2 - k_I^2} \frac{1}{x^2 - q^2} \frac{1}{y + q} \Big|_{q=y} \\ &= - \frac{k_I}{(x^2 - k_I^2)(y^2 - k_I^2)} - \frac{x}{(x^2 - k_I^2)(y^2 - x^2)} - \frac{y}{(y^2 - k_I^2)(x^2 - y^2)} \\ &= \frac{1}{(x + y)(x + k_I)(y + k_I)}.\end{aligned}\quad (5.53)$$

We therefore get that

$$\boxed{\psi_4 = \int_{k_{12}}^\infty dx \int_{k_{34}}^\infty dy \frac{1}{(x + y)(x + k_I)(y + k_I)}},\quad (5.54)$$

which is the integral representation of the expected result; cf. (4.118).

EFT of inflation One of the important features of using unitarity cuts to bootstrap correlators is that it doesn't rely the interactions being approximately boost-invariant. In the following exercise, you will derive the exchange four-point function arising from a $\dot{\Phi}^3$ interaction in de Sitter [53]. This is one of the relevant interactions in the EFT of inflation [11].

Exercise 5.2 Consider a theory with boost-breaking $\dot{\Phi}^3$ interaction in de Sitter space. The three-point wavefunction coefficient is

$$\psi_3 = 2g \frac{(k_1 k_2 k_3)^2}{(k_1 + k_2 + k_3)^3}.\quad (5.55)$$

Using (5.45), derive the exchange four-point function. You should find:

$$\begin{aligned}\psi_4 = 4g^2 \frac{(k_1 k_2 k_3 k_4)^2}{E^5 E_L^3 E_R^3} & \left(k_I^2 \left(6E_L^2 E_R^2 + 3EE_L E_R(E_L + E_R) + E^2 (E_L + E_R)^2 \right. \right. \\ & \left. \left. + E^3 (E_L + E_R) + E^4 \right) - 6E_L^3 E_R^3 \right),\end{aligned}\quad (5.56)$$

where E, E_L, E_R are defined as before. →

5.4 Symmetries

In this section, we review the role of symmetries to constrain the functional form of the boundary correlators in the inflationary quasi-de Sitter spacetime. We will first present the set of symmetries of a perfect de Sitter space and then describe some of their important consequences: We first derive the unitary representations associate to the de Sitter group and show how these representations are realized by bulk fields and boundary operators. We then demonstrate that the de Sitter isometries become a conformal symmetry on the late-time boundary and derive the associated Ward identities. This leads to differential equations describing the constrained deformations of the correlators in momentum space.

5.4.1 De Sitter Isometries

We can view D -dimensional de Sitter space as a hypersurface in $(D+1)$ -dimensional Minkowski space defined by the embedding constraint

$$\eta_{AB} X^A X^B = -X_0^2 + X_1^2 + \cdots + X_D^2 = H^{-2}. \quad (5.57)$$

The symmetries of de Sitter space then correspond to the higher-dimensional Lorentz transformations that leave the embedded hypersurface invariant. The generators of these Lorentz symmetries, J_{AB} , satisfy the $\text{so}(D, 1)$ algebra:

$$[J_{AB}, J_{CD}] = \eta_{BC} J_{AD} - \eta_{AC} J_{BD} + \eta_{AD} J_{BC} - \eta_{BD} J_{AC}. \quad (5.58)$$

It is natural, particularly in the inflationary slicing, to define⁴²

$$\begin{aligned}\hat{D} &\equiv J_{D0}, \\ P_i &\equiv J_{Di} - J_{0i}, \\ K_i &\equiv J_{Di} + J_{0i},\end{aligned}\quad (5.59)$$

so that the de Sitter algebra is given by

$$\begin{aligned}[J_{ij}, J_{kl}] &= \delta_{jk} J_{il} - \delta_{ik} J_{jl} + \delta_{il} J_{jk} - \delta_{jl} J_{ik}, & [\hat{D}, P_i] &= P_i, \\ [J_{ij}, K_l] &= \delta_{jl} K_i - \delta_{il} K_j, & [\hat{D}, K_i] &= -K_i, \\ [J_{ij}, P_l] &= \delta_{jl} P_i - \delta_{il} P_j, & [K_i, P_j] &= 2\delta_{ij} \hat{D} - 2J_{ij}.\end{aligned}\quad (5.60)$$

⁴²The hat on the dilatation operator \hat{D} was introduced to avoid confusion with the dimension D of the spacetime.

The generators P_i and J_{ij} create spatial translations and rotations, while \hat{D} and K_i are associated with dilatations and special conformal transformations.

We will usually work with conformal coordinates, in which the de Sitter line element (in the flat slicing) takes the form

$$ds^2 = \frac{1}{H^2\eta^2} (-d\eta^2 + \delta_{ij}dx^i dx^j). \quad (5.61)$$

This is obviously invariant under spatial translations and rotations on the spatial slices \mathbb{R}^d . The addition, the line element has a *dilatation symmetry*:

$$\begin{aligned} \eta &\mapsto \lambda\eta, \\ x^i &\mapsto \lambda x^i, \end{aligned} \quad (5.62)$$

where λ is a constant parameter. Slightly harder to see (but easy to verify) is the symmetry under *special conformal transformations* (SCTs):

$$\begin{aligned} \eta &\mapsto \frac{\eta}{1 + 2b_i x^i + b^2(x^2 - \eta^2)}, \\ x^i &\mapsto \frac{x^i + (x^2 - \eta^2)b^i}{1 + 2b_i x^i + b^2(x^2 - \eta^2)}. \end{aligned} \quad (5.63)$$

where b^i is a constant vector. As we will see below, the transformations (5.63) act precisely as special conformal transformations on the spacelike boundary of de Sitter, motivating the naming convention. This is no accident, and is a manifestation of the fact that the de Sitter group $\text{SO}(D, 1)$ is exactly the Euclidean conformal group in $D - 1$ dimensions.

The generators of infinitesimal transformations can be found by pulling back the ambient space Lorentz generators to find explicit expressions for the Killing vectors (5.59). This leads to

$$\begin{aligned} P_i &= \partial_i, \\ J_{ij} &= x_i \partial_j - x_j \partial_i, \\ \hat{D} &= -\eta \partial_\eta - x^i \partial_i, \\ K_i &= 2x_i \eta \partial_\eta + \left(2x^j x_i + (\eta^2 - x^2)\delta_i^j\right) \partial_j. \end{aligned} \quad (5.64)$$

It is easy to check that these satisfy the commutation relations in (5.60).

Exercise 5.3 The flat slicing (5.61) corresponds to the embedding

$$\begin{aligned} X^0 &= \frac{L}{2(-\eta)} (1 - \eta^2 + x^2), & X^i &= \frac{L}{(-\eta)} x^i, \\ X^D &= \frac{L}{2(-\eta)} (1 + \eta^2 - x^2), \end{aligned} \quad (5.65)$$

which foliates $\mathbb{R}^{D,1}$ by de Sitter hyperboloids of radius $L = H^{-1}$. Using these embedding functions, pull back the ambient space Lorentz transformations to derive the Killing vectors

We want to understand how the symmetries of de Sitter space manifest themselves in the physics of cosmological correlations. In Section 5.4.2, we will present a brief summary of the unitary representations of the de Sitter group, with further details given in Appendix B. In Section 5.4.3, we then show how the symmetries of de Sitter space are realized on bulk fields. We use these results, in Section 5.4.4, to determine how the symmetries act on fluctuations on the late-time boundary of the spacetime. We discover the famous fact that the isometries of the bulk spacetime become conformal symmetries on the spatial boundary. In Section 5.4.5, we present the associated conformal Ward identities, both for correlators and wavefunction coefficients. Finally, we solve these Ward identities for a few selected examples in Section 5.4.6.

5.4.2 Unitary Representations

[DB: Flat space: Wigner classification of particles]

The Hilbert space of a quantum field theory in de Sitter space arranges itself into representations of the de Sitter group. In this section, we will present the essential properties of these unitary irreducible representations (in the bosonic case), leaving their derivations to Appendix B. Further details can be found in [60–62].

The goal is to realize the generators J_{AB} as anti-Hermitian operators on a vector space with a positive-definite norm. It is natural to label representations by their eigenvalues of the quadratic and quartic Casimir operators [60, 62, 63]

$$\begin{aligned}\mathcal{C}_2 &\equiv \frac{1}{2} J^{AB} J_{AB} = \hat{D}(d - \hat{D}) + P^i K_i + \frac{1}{2} J^{ij} J_{ij}, \\ \mathcal{C}_4 &\equiv J_{AB} J^{BC} J_{CD} J^{DA},\end{aligned}\tag{5.66}$$

since these commute with all the generators of the de Sitter symmetries. (The expression for \mathcal{C}_4 in terms of the generators in (5.59) is fairly long and unilluminating.)

Unitary representations of the de Sitter group can then be constructed by starting from a *primary state* $|\bar{\Delta}\rangle_s$, which is annihilated by the SCT operator K_i , has dilatation eigenvalue $\bar{\Delta} = d - \Delta$, and so(D) spin weight s . Since Δ and s completely determine the Casimirs in (5.66), we can label representations by these quantum numbers. In Appendix B.2, we describe the explicit construction of unitary scalar representations (the techniques are similar for spinning representations). Here, we just record the list of unitary representations (specializing to $D \geq 3$ spacetime dimensions).

Scalar representations: There are three families of unitary irreducible scalar representations on de Sitter space:⁴³

⁴³Note that this is already richer than in flat space, where there are only two families: the ordinary scalar representations with $m^2 \geq 0$ and tachyons with $m^2 < 0$.

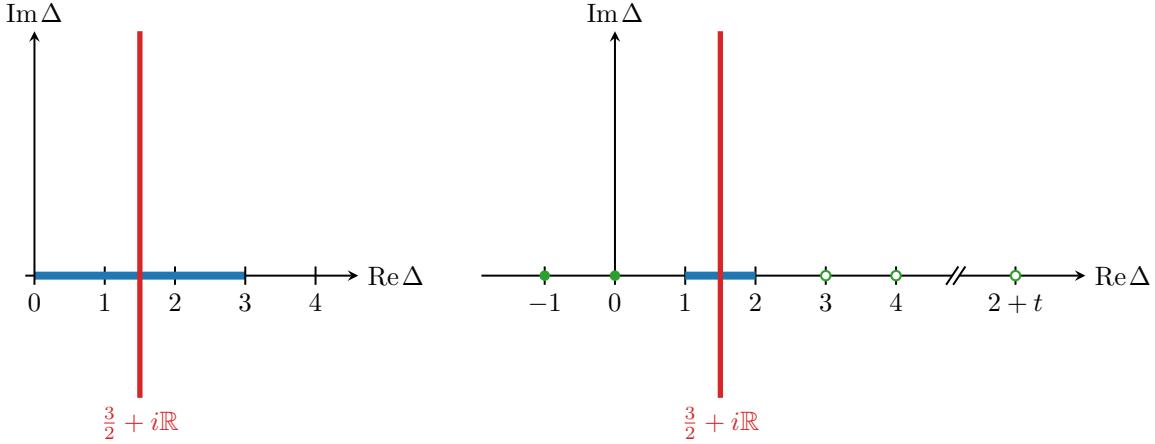


Figure 10: Illustration of scalar (*left*) and spin- ℓ (*right*) representations of $\text{SO}(4, 1)$ in the complex Δ -plane. The red lines corresponds to the principal series, while the blue lines mark the complementary series. The green circles for the spin- ℓ case denote the representations of the exceptional series (which coincide with the discrete series in this case). Open circles are the shadows of the filled ones.

- **Principal series:** This family is generated by a primary with conformal dimension $\Delta = \frac{d}{2} + i\mu$, where $\mu \in \mathbb{R}$. We will see that these representations correspond to *heavy fields* in de Sitter with $m^2 > \frac{d^2}{4} H^2$.
- **Complementary series:** This family is generated by a primary with conformal dimension in the range $0 < \Delta < d$. We will see that these representations are carried by *light fields* with $0 < m^2 < \frac{d}{4} H^2$.
- **Exceptional series:** These representations have $\Delta = -k$, with $k \in \mathbb{Z}_{\geq 0}$. The representation obtained from this primary state is reducible and so must be projected. It is unclear what field theory carries these representations.⁴⁴

It is interesting to compare these representations to the unitary scalar representations of the group $\text{SO}(D - 1, 2)$, which is the group of isometries of anti-de Sitter space. The representations are built in a similar manner, starting from a primary state, and in the AdS context, unitarity places a lower bound of the value of Δ for scalar representations: $\Delta \geq \frac{d}{2} - 1$.

Spinning representations: Fields with spin have a slightly different classification, which is somewhat richer than that of scalars. Along with the principal and complementary series, there are additional families of unitary representations, whose weights take on discrete values. The classification of general mixed-symmetry fields is rather intricate [60, 61], so we will focus on spin- ℓ fields whose spin weight is $\mathbf{s} = (\ell, 0, \dots, 0)$:⁴⁵

⁴⁴A reasonable guess is that the shift-symmetric scalar fields considered in [64] carry these representations if they are quantized treating the shift symmetries like gauge redundancies. The mass values of these fields are $m^2 = -k(d + k)H^2$.

⁴⁵In odd d , general mixed-symmetry fields have another family of representations, called the *discrete series*. However, these representations require all spin weights to be activated [61], so they do not appear for spin- ℓ representations, except in $d = 3$, where they happen to coincide with the exceptional series.

- **Principal series:** Spinning representations in the principal series have conformal dimensions satisfying $\Delta = \frac{d}{2} + i\mu$, with $\mu \in \mathbb{R}$. The corresponding bulk fields are heavy fields with

$$m^2 \geq (\frac{d}{2} + \ell - 2)^2 H^2. \quad (5.67)$$

- **Complementary series:** Fields in this family have conformal dimensions in the range $1 < \Delta < d - 1$. Note that this is bounded away from $\Delta = 0$. These representations correspond to bulk fields with masses in the range

$$(\ell - 1)(\ell + d - 3)H^2 < m^2 < (\frac{d}{2} + \ell - 2)^2 H^2. \quad (5.68)$$

A qualitative difference from the scalar case is that spinning representations with $\ell \geq 2$ have a lower bound on their mass in order to remain unitary. This lower bound is typically called the *Higuchi bound* [65].

- **Exceptional series:** These representations have a primary state with $\Delta = 1 - t$, where $t \in \{0, 1, \dots, \ell - 1\}$ is the *depth*. The corresponding bulk gauge fields have spin ℓ and mass

$$m^2 = (\ell - 1 - t)(\ell + t + d - 3)H^2. \quad (5.69)$$

The representation with $t = \ell - 1$ is a *massless* field, the other depths are known as *partially massless* fields, have gauge invariances with t -index gauge parameters, and have no flat-space analogues.

The unitary scalar and spin- ℓ representations of $\text{SO}(4, 1)$ —which is the physically relevant case—are depicted in Fig. 10.

5.4.3 Realization on Bulk Fields

Next, we want to understand how the symmetries of de Sitter space are realized on bulk fields, and in particular what representations of $\text{SO}(D, 1)$ they carry. As before, we will focus mostly on the scalar case and comment only briefly on the generalization to fields with spin.

Consider a real scalar field Φ , with mass m , in de Sitter. It is straightforward to derive the transformation of Φ under the isometries generated by the Killing vectors (5.64); it just transforms by the corresponding *Lie derivative*⁴⁶

$$\begin{aligned} P_i \Phi &= \partial_i \Phi, \\ J_{ij} \Phi &= (x_i \partial_j - x_j \partial_i) \Phi, \\ \hat{D} \Phi &= -(\eta \partial_\eta + x^i \partial_i) \Phi, \\ K_i \Phi &= 2x_i \eta \partial_\eta \Phi + \left(2x^j x_i + (\eta^2 - x^2) \delta_i^j\right) \partial_j \Phi. \end{aligned} \quad (5.70)$$

From this, we can work out the action of the quadratic Casimir (5.66):

$$\mathcal{C}_2 \Phi = -\eta^2 \left(\frac{\partial^2}{\partial \eta^2} - \frac{d-1}{\eta} \frac{\partial}{\partial \eta} - \nabla^2 \right) \Phi. \quad (5.71)$$

⁴⁶Recall that the Lie derivative of a function is simply its directional derivative, $\mathcal{L}_\xi f = \xi(f) = \xi^a \partial_a f$.

This is exactly the wave operator acting on the scalar: $H^2 \mathcal{C}_2 \Phi = \square \Phi$, so that the equation of motion can be read as an eigenvalue equation

$$\mathcal{C}_2 \Phi = \frac{m^2}{H^2} \Phi. \quad (5.72)$$

Equating this to the Casimir of a scalar representation $\mathcal{C}_2 = \Delta(d - \Delta)$, we can solve for Δ :

$$\Delta_{\pm} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} - \frac{m^2}{H^2}}. \quad (5.73)$$

Notice first that the two branches Δ_+ and Δ_- are related by $\Delta_+ = d - \Delta_-$. Scaling weights with this relation are so-called *shadows* of each other (see Appendix B.2), and correspond to equivalent representations, which makes sense because we anticipate that a scalar should carry a single representation that could be labeled by either of these two choices.

Notice that there are two qualitative regimes:

- **Complementary series:** For $0 \leq m^2/H^2 < d^2/4$, the values of Δ_{\pm} lie in the interval $[0, d]$, which is the range corresponding to the complementary series. We therefore see that “light” fields in de Sitter carry complementary series representations.
- **Principal series:** For $m^2/H^2 > d^2/4$, the values of Δ_{\pm} are of the form $\frac{d}{2} + i\mu$, with μ a real number. These are precisely the principal series values. Hence, “heavy” fields in de Sitter carry principal series representations.

We note in passing that fields with the (tachyonic) masses $m^2/H^2 = -k(D - 1 + k)$ have the correct weights to be the exceptional series scalars, but they have shift symmetries that are naively de Sitter symmetry-violating zero modes. It would therefore seem that they have to be quantized in some nonstandard way. Other tachyonic mass values do not correspond to any representations on the list in Section 5.4.2 and so are not unitary.

Cosmological correlations naturally live on the boundary of de Sitter space, so we now want to understand the action of these symmetries on the boundary.

5.4.4 Boundary Conformal Symmetry

There is a distinguished spatial slice in de Sitter space—the future boundary—which is invariant under the symmetries of de Sitter space. The points on this surface get mapped into each other, but the surface itself does not move. In the flat slicing, this is the $\eta = 0$ surface. In the idealized inflationary picture, this is where the universe reheats and the standard hot Big Bang cosmology starts. The fluctuations on this surface serve as the initial conditions for the universe, and therefore the correlations at $\eta = 0$ have a distinguished status. We would like to understand how the de Sitter symmetries act on them.

Consider a scalar field, Φ , with mass m . Its equation of motion is

$$\Phi'' - \frac{(d-1)}{\eta} \Phi' - \nabla^2 \Phi + \frac{m^2}{H^2} \frac{1}{\eta^2} \Phi = 0. \quad (5.74)$$

At late times ($\eta \rightarrow 0$), we can neglect the spatial gradient term and the solutions become power laws in time:

$$\Phi(\mathbf{x}, \eta \rightarrow 0) = \phi(\mathbf{x}) \eta^\Delta + \bar{\phi}(\mathbf{x}) \eta^{\bar{\Delta}}, \quad (5.75)$$

where $\bar{\Delta} = d - \Delta$ and the *scaling dimension* (sometimes also called the scaling weight) is

$$\Delta \equiv \frac{d}{2} - \sqrt{\frac{d^2}{4} - \frac{m^2}{H^2}}. \quad (5.76)$$

Note that Δ is the same as Δ_- in (5.73). For “light fields”, with $m < \frac{d}{2}H$, the scaling dimension is purely real and the first term in (5.75) dominates at late times. Near the boundary, we can then replace time derivatives by $\eta\partial_\eta \mapsto \Delta, \bar{\Delta}$ and the de Sitter isometries (in the limit $\eta \rightarrow 0$) act as

$$\begin{aligned} P_i \phi &= \partial_i \phi, \\ J_{ij} \phi &= (x_i \partial_j - x_j \partial_i) \phi, \\ D \phi &= -(\Delta + x^i \partial_i) \phi, \\ K_i \phi &= \left[2x_i \Delta + \left(2x^j x_i - x^2 \delta_i^j \right) \partial_j \right] \phi, \end{aligned} \quad (5.77)$$

and similarly for $\bar{\phi}$. As you will show in the following exercise, these are precisely the transformation rules for a primary operator of conformal weight Δ in a conformal field theory (CFT). The late-time correlations of the field Φ can be thought of as being captured by the correlations of the spatial coefficients $\phi, \bar{\phi}$. Since these coefficients transform like operators in a CFT, late-time correlators in dS are constrained by conformal symmetry.⁴⁷ The constraints of conformal symmetry are well understood in position space (see Appendix D), but, in cosmology, we need results in Fourier space. Unfortunately, the Fourier transforms of CFT correlations are highly nontrivial [30] and in practice is easier to redo the analysis directly in Fourier space.

5.4.5 Conformal Ward Identities

Our primary interest is in late-time cosmological correlators, and so we want to understand the constraints of de Sitter symmetry for these objects. In general, symmetries imply that correlation functions satisfy differential equations called *Ward identities*. We will now derive these Ward identities for both cosmological correlation functions and wavefunction coefficients.

Correlators: We can think of in-in correlation functions as directly correlating the late-time field profiles of the bulk fields Φ . As such, they are essentially correlation functions of the coefficients of the field fall-offs in (5.75), which transform as in (5.77) under a de Sitter transformation. For light fields (in the complementary series), only the leading fall-off ϕ survives at late times, because the other branch decays more quickly. In this case, the requirements of de Sitter invariance can be phrased as

$$\sum_{a=1}^N \langle \phi_1 \cdots \delta\phi_a \cdots \phi_N \rangle = 0, \quad (5.78)$$

⁴⁷This assumes that the bulk interactions don't strongly break the symmetry. Especially in a time-dependent inflationary background this may not be the case. We will return to this issue below.

where $\delta\phi_a$ can stand for any of the field transformations in (5.77). The constraints of translation and rotation invariance are very easy to implement and simply require that the correlator is a function only of the separations $|\mathbf{x}_a - \mathbf{x}_b|$. The constraints of dilatation and SCT invariance are more nontrivial, and lead to the equations

$$D : 0 = \sum_{a=1}^N \left(\Delta_a + x_a^j \frac{\partial}{\partial x_a^j} \right) \langle \phi_1 \cdots \phi_a \cdots \phi_N \rangle, \quad (5.79)$$

$$SCT : 0 = \sum_{a=1}^N \left(2\Delta_a x_a^i + 2x_a^i x_a^j \frac{\partial}{\partial x_a^j} - x_a^2 \frac{\partial}{\partial x_{a,i}} \right) \langle \phi_1 \cdots \phi_a \cdots \phi_N \rangle. \quad (5.80)$$

Correlation functions computed directly from the bulk will automatically satisfy these equations. The idea of the bootstrap approach is to shortcut this procedure and solve these equations directly. However, solving these equations in general is rather difficult, and there are many solutions, so we require additional input in order to know which solutions are physically relevant.

Wavefunction coefficients: Next, we will determine how these conformal transformations act on the wavefunction coefficients. Consider the N -th order term in the perturbative expansion of the late-time wavefunction (4.49):

$$\Psi[\phi] = \exp \left(\cdots + \int d^d x_1 \cdots d^d x_N \Psi_N(\underline{\mathbf{x}}) \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_N) + \cdots \right). \quad (5.81)$$

Under a de Sitter transformation, we have

$$\delta\Psi \supset \sum_{a=1}^N \int d^d x_1 \cdots d^d x_N \Psi_N(\underline{\mathbf{x}}) \phi(\mathbf{x}_1) \cdots \delta\phi(\mathbf{x}_a) \cdots \phi(\mathbf{x}_N), \quad (5.82)$$

where the transformations of the late-time field profiles $\delta\phi_a$ are given by (5.77). (We are again assuming a field in the complementary series, so that it only has a single fall-off, otherwise we should also keep track of the second mode $\bar{\phi}$.) We now integrate this expression by parts, so that the generators of the transformation acts on the wavefunction coefficient $\Psi_N(\underline{\mathbf{x}})$. For example, for the dilatation symmetry, we get

$$\begin{aligned} \delta\Psi &\supset \sum_{a=1}^N \int d^d x_1 \cdots d^d x_N \Psi_N(\underline{\mathbf{x}}) \phi(\mathbf{x}_1) \cdots \left(-\Delta_a - x_a^j \partial_{x_a^j} \right) \phi(\mathbf{x}_a) \cdots \phi(\mathbf{x}_N) \\ &= \sum_{a=1}^N \int d^d x_1 \cdots d^d x_N \left[\left((d - \Delta_a) + x_a^j \partial_{x_a^j} \right) \Psi_N(\underline{\mathbf{x}}) \right] \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_N), \end{aligned} \quad (5.83)$$

and hence the wavefunction coefficients transform as

$$\delta\Psi_N(\underline{\mathbf{x}}) = \sum_{a=1}^N \left((d - \Delta_a) + x_a^j \partial_{x_a^j} \right) \Psi_N(\underline{\mathbf{x}}). \quad (5.84)$$

If we think of the wavefunction coefficients as being correlation functions in some putative dual theory, $\Psi_N(\underline{\mathbf{x}}) = \langle O(\mathbf{x}_1) \cdots O(\mathbf{x}_N) \rangle$, then the operators, O_a , transform as fields of weight $\bar{\Delta}_a =$

$d - \Delta_a$. (There is also a flip of sign of the parameter of the transformation, but this is not important.) The same is true for the special conformal transformations, so that O_a behaves like a primary operator of weight $d - \Delta_a$. Explicitly, the action of de Sitter symmetry on the late-time wavefunction coefficients is captured by the following transformation rules for the dual operators

$$\hat{D}O = ((d - \Delta) + x^j \partial_{x^j}) O, \quad (5.85)$$

$$K_i O = \left(2(d - \Delta)x^i + 2x^i x^j \frac{\partial}{\partial x^j} - x^2 \frac{\partial}{\partial x_i} \right) O. \quad (5.86)$$

If we assume that the wavefunction Ψ is *invariant* under the conformal symmetry, then the wavefunction coefficients must be annihilated by the action of the symmetry generators:

$$\sum_{a=1}^N \hat{D}_a \langle O_1 \cdots O_N \rangle = 0, \quad (5.87)$$

$$\sum_{a=1}^N K_a^i \langle O_1 \cdots O_N \rangle = 0, \quad (5.88)$$

where the operators \hat{D}_a and K_a^i are given by (5.85) and (5.86), respectively. These equations are the analogue for the wavefunction coefficients of equations (5.79) and (5.80).

For the most part, we will be interested in solving these equations for the wavefunction because, as we have seen, wavefunction coefficients are somewhat simpler than in-in correlators and contain the same information.

Fourier space: In cosmology, we typically work in Fourier space, so it will be useful to have expressions for the action of the dilatation (5.85) and special conformal (5.86) symmetries in Fourier space.⁴⁸ On a scalar operator, they act as [17, 30, 54]

$$\hat{D}O_{\mathbf{k}} = \left[-(\Delta - d) + k^i \partial_{k^i} \right] O_{\mathbf{k}}, \quad (5.89)$$

$$K_i O_{\mathbf{k}} = - \left[2(\Delta - d) \partial_{k^i} + k_i \partial_{k^j} \partial_{k^j} - 2k^j \partial_{k^j} \partial_{k^i} \right] O_{\mathbf{k}}. \quad (5.90)$$

One further subtlety arises when writing the Ward identities (5.87) and (5.88) in Fourier space. The derivatives with respect to the momenta can hit the momentum-conserving delta function which appears in front of a correlator:

$$\langle O_1 \cdots O_N \rangle = (2\pi)^d \delta^{(d)}(\mathbf{k}_1 + \cdots + \mathbf{k}_N) \langle O_1 \cdots O_N \rangle'. \quad (5.91)$$

We can deal with this subtlety by treating the delta function as a distribution and integrating by parts the terms where the derivatives hit the delta function. The SCT operator (5.90) ends up passing through the delta function unchanged, but the dilatation operator becomes

$$\sum_{a=1}^N \hat{D}_a \langle O_1 \cdots O_N \rangle' = \left[d(N-1) - \sum_{a=1}^N \Delta_a + \sum_{a=1}^{N-1} k_a^i \partial_{k_a^i} \right] \langle O_1 \cdots O_N \rangle', \quad (5.92)$$

⁴⁸Reference [66] has an explanation of how to derive these transformations in Fourier space, though beware of some sign errors. See also [67] for details.

where we have imposed the constraint that the delta function implements by setting $\mathbf{k}_N = -\sum_{a=1}^{N-1} \mathbf{k}_a$. Concretely, this means that wavefunction coefficients (in Fourier space) satisfy

$$\left[-d + \sum_{a=1}^N \hat{D}_a \right] \langle O_1 \cdots O_N \rangle'(\underline{\mathbf{k}}) = 0, \quad (5.93)$$

$$\sum_{a=1}^N K_a^i \langle O_1 \cdots O_N \rangle'(\underline{\mathbf{k}}) = 0, \quad (5.94)$$

where \hat{D}_a and K_a^i are given by (5.89) and (5.90), respectively.

Exercise 5.4 Derive the Fourier-space action of the dilatation (5.89) and SCT (5.90) operators. Then, verify that their action on the wavefunction coefficients (with the momentum-conserving delta function removed) is given by (5.93) and (5.94).

Warning: This is relatively tedious. →

Dilatation symmetry is reflected simply in the overall scaling dimension of the correlator: a general N -point function has scaling dimension $\bar{\Delta}_t \equiv \sum_a \bar{\Delta}_a$ in position space. This implies that the momentum-space correlator has dimension $\bar{\Delta}_t - d \times N$. After stripping off the momentum-conserving delta function, the dimension of the remaining function is $\bar{\Delta}_t - d(N-1)$. This fixes the overall momentum scaling of the correlator, allowing us to make an ansatz for $\langle O_1 \cdots O_N \rangle'$ that automatically solves the dilatation Ward identity (5.93). All the juice is therefore in the SCT Ward identity (5.94).

5.4.6 Selected Examples

We now want to see how solving this Ward identity directly in a couple simple situations can reproduce the de Sitter physics that we previously derived directly from the bulk.

Two-point functions

We first derive the constraints of conformal symmetry on the two-point functions of scalar operators. Following the above argument, it is straightforward to see that an ansatz that solves the dilatation Ward identity (5.93) is that

$$\langle O_1 O_2 \rangle' \propto k_1^\alpha, \quad (5.95)$$

with $\alpha \equiv \bar{\Delta}_1 + \bar{\Delta}_2 - d$. Since the two-point function depends only on the magnitude of momenta, the SCT operator takes the simplified form:

$$b_i K^i O_{\mathbf{k}} = \mathbf{b} \cdot \mathbf{k} \left[\frac{\partial^2}{\partial k^2} + \frac{(d+1-2\bar{\Delta})}{k} \frac{\partial}{\partial k} \right] O_{\mathbf{k}}, \quad (5.96)$$

where b^i is an arbitrary vector. Demanding SCT invariance of the two-point function then implies

$$\alpha + d - 2\bar{\Delta}_1 = \bar{\Delta}_2 - \bar{\Delta}_1 = 0, \quad (5.97)$$

which forces the scaling dimensions of the two operators to be equal, $\bar{\Delta}_1 = \bar{\Delta}_2$. We therefore get

$$\langle O_1 O_2 \rangle' = A k_1^{2\bar{\Delta}_1 - d} \delta_{\bar{\Delta}_1, \bar{\Delta}_2}. \quad (5.98)$$

We have learned that the two-point function of two scalar vanishes unless their scaling dimensions are equal and that their momentum scaling is fixed in terms of that scaling dimension.

Exercise 5.5 Show that (5.98) is consistent with the explicit result for the power spectrum of a massive field derived in Exercise 2.2, which consists of one term with scaling weight Δ_+ and one with weight Δ_- .



Three-point functions

As a more nontrivial example—which is still completely fixed by symmetry—we consider the three-point function of scalar fields interacting in exact de Sitter space. The constraints of conformal invariance on the corresponding momentum space three-point function were treated comprehensively in [30]. An ansatz that solves the dilatation Ward identity (5.93) is

$$\langle O_1 O_2 O_3 \rangle' = k_3^{\bar{\Delta}_1 + \bar{\Delta}_2 + \bar{\Delta}_3 - 2d} \hat{G}(p, q), \quad (5.99)$$

where $p \equiv k_1/k_3$ and $q \equiv k_2/k_3$. Although it looks like k_3 is special here, it is not because of the presence of an arbitrary function, \hat{G} . The SCT Ward identity (5.94) then implies two differential equations for the dimensionless function $\hat{G}(p, q)$, whose solution can be expressed in terms of the Appell F_4 function.⁴⁹

For our purpose, it will be useful to work out more explicitly the special case of two conformally coupled fields Φ (with $\bar{\Delta} = 2$ for the dual operator O) and a generic field χ (with arbitrary $\bar{\Delta}$ for the dual operator X), in $d = 3$ dimensions. A dilatation-invariant ansatz for the wavefunction coefficient is

$$\langle O_1 O_2 X_3 \rangle' = k_3^{\bar{\Delta}-2} \hat{G}(u, w), \quad (5.101)$$

where $u \equiv k_3/(k_1 + k_2)$ and $w \equiv k_3/(k_1 - k_2)$. This is a slightly different ansatz from (5.99), but is convenient for this situation because the variables u, w have definite parity under $1 \leftrightarrow 2$. We can now act with the SCT operator in the form (5.96) and choose $\mathbf{b} \cdot \mathbf{k}_3 = 0$. After using momentum conservation, we then find

$$\left(\frac{\partial^2}{\partial k_1^2} - \frac{\partial^2}{\partial k_2^2} \right) \hat{G}(u, w) = 0 \implies \frac{4u^2 w^2}{k_3^2} \partial_u \partial_w \hat{G}(u, w) = 0. \quad (5.102)$$

⁴⁹This special function is not so familiar, but there is a relatively nice representation in terms of an integral over products of modified Bessel functions. Using this integral representation, one can derive the Fourier space expression for the correlation function of three scalar operators [30]:

$$\langle O_1 O_2 O_3 \rangle' \propto k_1^{\bar{\Delta}_1 - \frac{d}{2}} k_2^{\bar{\Delta}_2 - \frac{d}{2}} k_3^{\bar{\Delta}_3 - \frac{d}{2}} \int_0^\infty dx x^{\frac{d}{2}-1} K_{\nu_1}(k_1 x) K_{\nu_2}(k_2 x) K_{\nu_3}(k_3 x), \quad (5.100)$$

where K_ν , with $\nu = \bar{\Delta} - \frac{d}{2}$, is the modified Bessel function of the second kind. See [68, 69], for more details of the evaluation of integrals of this type for special mass values.

We can therefore take $\hat{G}(u, w) = \hat{G}(u)$.⁵⁰ Next, we consider the SCT Ward identity with $\mathbf{b} \cdot \mathbf{k}_2 = 0$, which leads to

$$\left(\frac{\partial^2}{\partial k_1^2} - \frac{\partial^2}{\partial k_3^2} - \frac{4 - 2\bar{\Delta}}{k_3} \frac{\partial}{\partial k_3} \right) \left(k_3^{\bar{\Delta}-2} \hat{G}(u) \right) = 0. \quad (5.103)$$

Changing the derivatives to ones with respect to u , this implies the equation

$$\left[\Delta_u + \left(\mu^2 + \frac{1}{4} \right) \right] \hat{G}(u) = 0, \quad (5.104)$$

where $\Delta_u \equiv u^2(1-u^2)\partial_u^2 - 2u^3\partial_u$ is the same differential operator as in (4.128) and $\mu^2 + \frac{1}{4} \equiv -(\bar{\Delta}-2)(\bar{\Delta}-1)$. In terms of the mass parameter of the bulk field χ , we have $\mu^2 + \frac{1}{4} = M^2/H^2 - 2$. Notice that this differential equation is precisely the one obtained for this wavefunction coefficient in Exercise 4.14 from the bulk perspective.

Equation (5.104) is of the hypergeometric type and is solved by

$$\hat{G}_+(u) \propto {}_2F_1 \left[\begin{matrix} \frac{1}{2} + i\mu, \frac{1}{2} - i\mu \\ 1 \end{matrix} \middle| \frac{u-1}{2u} \right] = P_{i\mu-\frac{1}{2}}^0(u^{-1}), \quad (5.105)$$

where we have written the solution both in terms of the Gauss hypergeometric function and as an associated Legendre function. A second linearly-independent solution is $\hat{G}_-(u) \propto P_{i\mu-\frac{1}{2}}^0(-u^{-1})$. The solution G_+ can then be written as [DB: FIX]

$$\begin{aligned} G_+(u) &= \left(\frac{\pi}{2 \cosh(\pi\mu)} \right)^{\frac{1}{2}} k_3^{i\mu-\frac{1}{2}} {}_2F_1 \left[\begin{matrix} \frac{1}{2} + i\mu, \frac{1}{2} - i\mu \\ 1 \end{matrix} \middle| \frac{u-1}{2u} \right] \\ &= \left(\frac{\pi}{2 \cosh(\pi\mu)} \right)^{\frac{1}{2}} k_3^{i\mu-\frac{1}{2}} P_{i\mu-\frac{1}{2}}^0(u^{-1}), \end{aligned} \quad (5.106)$$

where G has been normalized so that its Wronskian is $1/(1-u^2)$.⁵¹ This solution is exactly the same as the wavefunction coefficient obtained in Exercise 4.9, and is simply related to the in-in correlator from Exercise 3.4.

The solution (5.106) has a more complicated analytic structure than many of the examples we have encountered so far. In particular, it has branch points at $u = 0, 1$ (it is natural to choose the cut to not lie in the physical region, which is the interval between $u = 0$ and $u = 1$). This branch cut can be thought of as a signature of the production of massive particles with a continuum of possible relative momenta in the bulk spacetime. Note also that for special values of μ , the equation (5.120) is not satisfied exactly in momentum space, due to the presence of anomalies.⁵²

⁵⁰There is another solution that depends only on w , but we can obtain it from this one by sending $k_2 \rightarrow -k_2$.

⁵¹The Wronskian of the associated Legendre function satisfies

$$(P_\nu^{-\mu}(x), P_\nu^{-\mu}(-x)) = P_\nu^{-\mu}(x)\partial_x P_\nu^{-\mu}(-x) - \partial_x P_\nu^{-\mu}(x)P_\nu^{-\mu}(-x) = \frac{2}{\Gamma[\mu-\nu]\Gamma[\nu+\mu+1]} \frac{1}{1-x^2}, \quad (5.107)$$

which we can use to normalize the solution of interest.

⁵²For example, this happens when the third field also has $\bar{\Delta} = 2$. In Section 4.4.5, we derived the three-point wavefunction coefficient of three conformally coupled fields:

$$\langle O_1 O_2 O_3 \rangle' \propto \log(K/\bar{\mu}). \quad (5.108)$$

Four-point functions

In Section 5.5, we will bootstrap the four-point function of conformally coupled scalars, which have $\bar{\Delta} = 2$. It will be useful to have at hand the conformal Ward identity for this case, so we will derive it now.

A priori, the four-point function $\psi \equiv \langle O_1 O_2 O_3 O_4 \rangle'$ depends on four three-momenta, $\mathbf{k}_{n=1,2,3,4}$, and hence 12 scalar variables. Translational and rotational invariance reduce this to six independent variables before imposing conformal symmetry. It is convenient to take these variables to be $k_{n=1,2,3,4} \equiv |\mathbf{k}_n|$, $k_I \equiv |\mathbf{k}_1 + \mathbf{k}_2|$ and $k'_I \equiv |\mathbf{k}_2 + \mathbf{k}_3|$. Constraints from dilatation symmetry and SCTs reduce these six independent variables to just two, which in position space are the conformally-invariant cross ratios. In momentum space, we will use the dimensionless variables

$$u \equiv \frac{k_I}{k_1 + k_2}, \quad v \equiv \frac{k_I}{k_3 + k_4}. \quad (5.109)$$

We explained above that dilatation invariance forces an N -point function in d dimensional momentum space to have scaling dimension $\bar{\Delta}_t - d(N - 1)$. For the four-point function of conformally coupled scalars ($\bar{\Delta} = 2$) in $d = 3$ dimensions, this implies that ψ has dimension $4 \times 2 - 3 \times (4 - 1) = -1$. This motivates us to define

$$\langle O_1 O_2 O_3 O_4 \rangle' = \frac{1}{k_I} \hat{F}(u, v), \quad (5.110)$$

which automatically solves the dilatation Ward identity.

At this point, the dimensionless function $\hat{F}(u, v)$ is still completely undetermined. Its functional form will be constrained by the special conformal Ward identity and the allowed singularities of a given perturbative process. After quite some work (see Appendix A of [34]), the conformal Ward identity (5.94) can be written as

$$(\Delta_u - \Delta_v) \hat{F} = 0, \quad (5.111)$$

where $\Delta_u \equiv u^2(1-u^2)\partial_u^2 - 2u^3\partial_u$ is our old friend the hypergeometric differential operator.

Exercise 5.6 Derive the result (5.111).

Warning: This is algebraically very involved and tedious. →

This expression solves the Ward identities “anomalously.” The scale variation of the logarithm does not vanish, but is instead a function that is analytic in the momenta (in this case it is just a constant), indicating that it is a contact term in position space. This correlation function therefore satisfies the conformal Ward identities at separated points, which is all that is required. (The failure of the conformal Ward identities at coincident points is a reflection of an anomaly [?].) Note that we can freely add an arbitrary constant to this correlation function by shifting the (arbitrary) scale $\bar{\mu}$.

5.5 Cosmological Collider Physics

As a concrete example of the bootstrap approach, we now revisit the challenge posed in Section 4.5: We would like to derive the four-point function of conformally coupled scalars, mediated by the tree-level exchange of a generic massive scalar. We will focus on the s -channel contribution, from which the t and u -channels can be obtained as simple permutations. The description below follows closely the treatment in [34], which should be consulted for further details.

In Section 5.4.5, we derived the Ward identity satisfied by the relevant wavefunction coefficient. Defining $\psi \equiv k_I^{-1} \hat{F}(u, v)$, with $u \equiv k_I/(k_1 + k_2)$ and $v \equiv k_I/(k_3 + k_4)$, we found

$$(\nabla_u - \nabla_v) \hat{F} = 0, \quad (5.112)$$

where $\Delta_u \equiv u^2(1-u^2)\partial_u^2 - 2u^3\partial_u$. Equation (5.112) is a purely kinematic constraint that must be satisfied by *all* de Sitter-invariant four-point functions. We will classify the physical distinct solutions to (5.112) by the expected singularities of different processes.

Contact solutions

The simplest solutions to equation (5.112) correspond to the four-point functions arising from contact interactions:

$$\hat{F}_c = \begin{array}{c} \text{---} \\ \backslash \diagup \diagdown / \\ \backslash \diagup \diagdown / \end{array}$$

These solutions are characterized by the simplest possible singularity structure with only total energy singularities. Consider first the case of a bulk Φ^4 interaction. As we have seen in Exercise 4.10, the four-point wavefunction coefficient coming from a Φ^4 interaction in both flat space and de Sitter space is proportional to $1/E$, where $E \equiv k_1 + k_2 + k_3 + k_4$ is the total energy. Up to an overall constant factor, we therefore have

$$\hat{F}_{c,0} = \frac{k_I}{E} = \frac{uv}{u+v}. \quad (5.113)$$

It is easy to confirm that this is indeed a solution of (5.112).

Solutions corresponding to higher-derivative contact interactions in the bulk are created by the repeated application of the operator Δ_u :⁵³

$$\hat{F}_{c,n} \equiv \Delta_u^n \hat{F}_{c,0}, \quad (5.114)$$

It is easy to see that these are indeed solutions to (5.111). For example,

$$\begin{aligned} (\Delta_u - \Delta_v) \hat{F}_{c,1} &= (\Delta_u - \Delta_v) \Delta_u \hat{F}_{c,0} \\ &= (\Delta_u^2 - \Delta_v \Delta_u) \hat{F}_{c,0} \\ &= (\Delta_u^2 - \Delta_u \Delta_v) \hat{F}_{c,0} \\ &= \Delta_u (\Delta_u - \Delta_v) \hat{F}_{c,0} = 0, \end{aligned} \quad (5.115)$$

where we used that $\Delta_v \Delta_u \hat{F}_{c,0} = \Delta_u \Delta_v \hat{F}_{c,0}$ because the seed solution $\hat{F}_{c,0}$ is symmetric in u and v .

⁵³This only captures the contact solutions arising from integrating out massive scalars. Integrating out particles with spin can produce contact solutions with dependence on additional kinematic invariants.

Exercise 5.7 By acting with Δ_u on the seed solution (5.113), show that

$$\hat{F}_{c,1} = -2 \left(\frac{uv}{u+v} \right)^3 \frac{1+uv}{uv}, \quad (5.116)$$

$$\hat{F}_{c,2} = -4 \left(\frac{uv}{u+v} \right)^5 \frac{u^2 + v^2 + uv(3u^2 + 3v^2 - 4) - 6(uv)^2 - 6(uv)^3}{(uv)^3}. \quad (5.117)$$

We see that even these contact solutions are relatively intricate functions of the momenta, but are fully controlled by symmetries and singularities without any reference to a Lagrangian. →

The sum of contact solutions can be written as

$$\hat{F}_c = \sum_{n=0}^{\infty} \left(\frac{k_I}{E} \right)^{2n+1} \hat{f}_n(u, v) + \text{perms}, \quad (5.118)$$

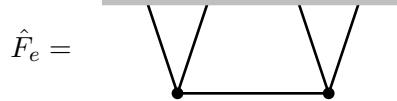
where the functional form of the dimensionless functions $\hat{f}_n(u, v)$ is fixed by conformal invariance. The expansion in powers of E^{-1} corresponds to the derivative expansion of the bulk EFT, with higher-order poles corresponding to higher-derivative interactions. The limit $E \rightarrow 0$ of the individual contact solutions is

$$\lim_{E \rightarrow 0} \hat{F}_{c,n} = (2n)! \frac{s^{2n}}{E^{2n+1}}, \quad (5.119)$$

where s is the Mandelstam invariant. This confirms the expected relation between the singularity of the correlator at $E = 0$ and the scattering amplitude for contact interactions.

Exchange solutions

Next, we look for solutions of the Ward identity corresponding to the tree-level exchange of a scalar particle:



We will proceed by inspired guesswork. In Section 5.4.5, we have seen that the three-point function of two conformally coupled scalars and a generic scalar, $\langle O_1 O_2 X_3 \rangle' = k_3^{\bar{\Delta}-2} \hat{G}(u)$, with $u \equiv k_3/k_{12}$, satisfies

$$\left[\Delta_u + \left(\mu^2 + \frac{1}{4} \right) \right] \hat{G} = 0, \quad (5.120)$$

where Δ_u is the same operator as that appearing in the four-point Ward identity (5.112). It is therefore easy to see that a product of two three-point functions, $\hat{F}(u, v) = \hat{G}(u)\hat{H}(v)$, is also a solution of (5.112):

$$\begin{aligned} (\Delta_u - \Delta_v) \hat{G}(u) \hat{H}(v) &= (\Delta_u \hat{G}) \hat{H} - \hat{G}(\Delta_v \hat{H}) \\ &= -\left(\mu^2 + \frac{1}{4} \right) \hat{G} \hat{H} - \hat{G} \left[-\left(\mu^2 + \frac{1}{4} \right) \hat{H} \right] \\ &= -\left(\mu^2 + \frac{1}{4} \right) (\hat{G} \hat{H} - \hat{G} \hat{H}) \\ &= 0. \end{aligned} \quad (5.121)$$

This isn't surprising since a four-point function factorizes into a product of three-point functions in the OPE limit.

This motivates us to find a more general solution of (5.112), not of the factorized form, but obeying

$$\begin{aligned} \left[\Delta_u + \left(\mu^2 + \frac{1}{4} \right) \right] \hat{F}_e &= \hat{C}, \\ \left[\Delta_v + \left(\mu^2 + \frac{1}{4} \right) \right] \hat{F}_e &= \hat{C}. \end{aligned} \quad (5.122)$$

Note that these are two *ordinary* differential equations in u and v separately. The solution corresponding to a product of three-point functions are the homogeneous solutions to these equations, which explains the introduction of the μ -dependent terms, to dictate the mass of the exchanged particle.

Exercise 5.8 Show that for consistency the function $\hat{C}(u, v)$ appearing on the right-hand side of (5.122) must satisfy $(\Delta_u - \Delta_v)\hat{C} = 0$ and is thus itself conformally invariant. →

What is the physical interpretation of the inhomogeneous source function $\hat{C}(u, v)$? Consider the limit of large μ , for which we expect the interaction to effectively reduce to a contact interaction, $\hat{F}_e \rightarrow \hat{F}_c = \mu^{-2}\hat{C}$. The source function on the right-hand side of (5.122) is therefore just one of our previously found contact solutions. Given our classification of these contact solutions, we can therefore also classify the corresponding exchange solutions. In Lagrangian language, this classification corresponds to choosing the bulk interactions of the theory.

Another way of understanding (5.122) is by considering the example of tree-exchange amplitudes. Multiplying the s -channel amplitude by the inverse propagator, we get

$$(s + m^2)A_e = A_c(s, t), \quad (5.123)$$

where the right-hand side is a rational function of the Mandelstam variables, i.e. a sum of contact amplitudes. The structure of (5.123) is similar to (5.122) except that it is an algebraic equation rather than a differential equation.

Exercise 5.9 In Section 4.4.5, we derived the four-point wavefunction coefficient of conformally coupled scalars:

$$\langle O_1 O_2 O_3 O_4 \rangle'_c = \frac{\lambda}{H^4 \eta_*^4} \frac{1}{E}, \quad (5.124)$$

$$\begin{aligned} \langle O_1 O_2 O_3 O_4 \rangle'_e &= \frac{g^2}{2H^6 \eta_*^4 k_I} \frac{1}{E} \left[\text{Li}_2 \left(\frac{k_{12} - k_I}{E} \right) + \text{Li}_2 \left(\frac{k_{34} - k_I}{E} \right) \right. \\ &\quad \left. + \log \left(\frac{k_{12} + k_I}{E} \right) \log \left(\frac{k_{34} + k_I}{E} \right) - \frac{\pi^2}{6} \right], \end{aligned} \quad (5.125)$$

where the subscripts denote the contact (c) and exchange (e) contributions. Show that these solutions solve the differential equations in (5.122) for $\lambda = g^2/H^2$. →

A formal solution of (5.111) can be obtained by inverting the differential operator:

$$\begin{aligned}\hat{F} = \frac{\hat{C}_0}{\Delta_u + M^2} &= \sum_n \frac{1}{n!} \left(\frac{\Delta_u}{M^2} \right)^n \frac{\hat{C}_0}{M^2} \\ &= \hat{F}_{c,0} + \frac{\hat{F}_{c,1}}{M^2} + \frac{1}{2} \frac{\hat{F}_{c,2}}{M^4} + \dots,\end{aligned}\tag{5.126}$$

where $M^2 \equiv \mu^2 + \frac{1}{4}$. We see that the solution has been written as an infinite expansion of contact solutions, which corresponds to the *EFT expansion* of the correlator as expected from integrating out a massive particle. However, as we will see, this EFT expansion misses nonperturbative terms associated with particle production in the expanding spacetime.

In the rest of this section, we will find an explicit solution to (5.122) corresponding to the exchange of a generic massive particle, thereby addressing the challenge posed in Section 4.5. For concreteness, we consider the simplest contact interaction, $\hat{F}_{c,0}$, as a source. The solutions for other sources, $\hat{F}_{c,n}$, can then simply be found by acting with Δ_u . The equation we would like to solve then is

$$\boxed{\left[u^2(1-u^2)\partial_u^2 - 2u^3\partial_u + \left(\mu^2 + \frac{1}{4}\right) \right] \hat{F} = \frac{uv}{u+v}},\tag{5.127}$$

where we have rescaled \hat{F} to set the normalization of the source function to unity. Note that this is the same as (4.128) if we recall that $\mu^2 + 1/4 = m^2/H^2 - 2$.

The homogeneous solutions of (5.127), which we will denote by \hat{F}_\pm , can be expressed as hypergeometric functions (or associated Legendre functions). The Wronksian of these solutions, $W \equiv \hat{F}_+ \partial_u \hat{F}_- - \hat{F}_- \partial_u \hat{F}_+$, satisfies $\partial_u W = 2u/(1-u^2)W$, so it is natural to normalize the solutions, so that $W = 1/(1-u^2)$. The correctly normalized homogeneous solutions are then

$$\hat{F}_\pm(u) = \left(\frac{iu}{2\mu} \right)^{\frac{1}{2} \pm i\mu} {}_2F_1 \left[\begin{matrix} \frac{1}{4} \pm \frac{i\mu}{2}, \frac{3}{4} \pm \frac{i\mu}{2} \\ 1 \pm i\mu \end{matrix} \middle| u^2 \right],\tag{5.128}$$

and equivalent solutions for the differential equation in terms of v . [DB: Relate back to $\langle O_1 O_2 X_3 \rangle'$]

Singularities To find the full solution for the inhomogeneous equation, we must impose boundary conditions for specific values of u and v . We will do this by looking at the singular points of the differential equation.⁵⁴ First, we note that the differential operator on the left-hand side of (5.127) becomes singular at $u \rightarrow 0$ and $u \rightarrow \pm 1$. Let us discuss the physical meaning of these singularities:

- *Factorization limit:* In the limit $u \rightarrow -1$, the equation has a logarithmic singularity

$$\lim_{u \rightarrow -1} \hat{F} \propto A_3 \log(1+u).\tag{5.129}$$

⁵⁴Alternatively, one can also use the correct normalization of the unitarity cut as a boundary condition. This was described in [?].

This is precisely the partial energy singularity at $E_L = 0$ that we discussed in Section 5.2. Correctly normalizing this singularity (together with the corresponding singularity at $E_R = 0$ or $v \rightarrow -1$) will be one of the boundary conditions for our differential equation.

- *Folded limit:* Similarly, in the limit $u \rightarrow +1$, we have

$$\lim_{u \rightarrow +1} \hat{F} \propto \log(1-u). \quad (5.130)$$

This is a singularity in the folded limit, $k_I \rightarrow k_1 + k_2$, where two of the momenta become collinear. As we discussed in Section 5.2, this singularity should be absent for Bunch–Davies initial conditions. We will take this as a second boundary condition for our differential equation.

- *Collapsed limit:* Imposing the boundary conditions at $u = \pm 1$ will completely fix the solution and the behavior at the remaining singularities becomes an output. For example, in the limit $u \rightarrow 0$, the solution becomes

$$\lim_{u \rightarrow 0} \hat{F} \propto u^{\frac{1}{2}+i\mu}. \quad (5.131)$$

This is the so-called collapsed limit of the correlator where the internal momentum becomes soft, $k_I \rightarrow 0$. As we will see, the boundary conditions of the problem will force us add a fixed amount of the homogeneous solutions (5.128) which has this oscillatory behavior in the collapsed limit. Physically, the non-analytic scaling for $u \rightarrow 0$ is the signature of particle production in the bulk spacetime. We will have more to say about this below.

- *Flat-space limit:* Finally, the source term on the right-hand side of (5.127) has a singularity for $u = -v$. Of course, this is simply the total energy singularity of the contact solution. We therefore expect the full exchange solution also to have this singularity:

$$\lim_{u \rightarrow -v} \hat{F} = A_4 (u+v) \log(u+v). \quad (5.132)$$

The coefficient of this singularity must be the correct four-point exchange amplitude (in the high-energy limit), $A_4 = 1/s$. Reproducing this singularity will be an important consistency check for our solution.

Harmonic oscillator limit Before discussing the general solution of (5.127), it is instructive to consider the limit $v \rightarrow 0$, where the source term is small. Writing $u \equiv \xi v$ and $\tilde{F} \equiv v \tilde{F}$, the differential equation (5.127) then takes the form

$$\left[\xi^2 \partial_\xi^2 + \left(\mu^2 + \frac{1}{4} \right) \right] \tilde{F} = \frac{\xi}{1+\xi}, \quad (5.133)$$

and defining $t \equiv \ln \xi$ and $f \equiv \tilde{F}/\sqrt{\xi}$, it becomes

$$\left[\frac{d^2}{dt^2} + \mu^2 \right] f = \frac{1}{2 \cosh(\frac{1}{2}t)}, \quad (5.134)$$

which is the equation of a *forced harmonic oscillator*.

- The *homogeneous solutions* of this equation are those of the simple harmonic oscillator:

$$f_{\pm} = e^{\pm i\mu t} = \xi^{\pm i\mu}, \quad (5.135)$$

where we have used $\xi = e^t$ in the final equality.

- We would now like to construct a *particular solution* of the inhomogeneous equation. Our strategy will be to find two series solutions around $\xi = 0$ and $\xi = \infty$, and then match the two solutions at $\xi = 1$ (where both solutions are convergent). The matching will force us to add a fixed amount of the homogeneous solution.

Exercise 5.10 Show that, around $\xi = 0$, the inhomogeneous equation has the following series solution

$$f_<(\xi) = \sqrt{\xi} \sum_{n=0}^{\infty} (-1)^n \frac{\xi^n}{(n + \frac{1}{2})^2 + \mu^2}, \quad (5.136)$$

which is convergent for $\xi \leq 1$ and divergent for $\xi > 1$. →

Modulo the overall factor of $\sqrt{\xi}$, the solution (5.136) is analytic around $\xi = 0$. We would like to extend this solution into the regime $\xi > 1$. In this simple case, the solution can be written as a hypergeometric function and we could rely on the known analytic structure of the hypergeometric functions to find the solution for general ξ . However, as a warmup for the general case it will be useful to proceed by finding a series solution to the differential equation ourselves.

Exercise 5.11 Show that, around $\xi = \infty$, the solution can be written as a series in ξ^{-1} :

$$f_>(\xi) = \frac{1}{\sqrt{\xi}} \sum_{n=0}^{\infty} (-1)^n \frac{\xi^{-n}}{(n + \frac{1}{2})^2 + \mu^2}, \quad (5.137)$$

which is convergent for $\xi \geq 1$ and divergent for $\xi < 1$. →

Modulo the overall factor of $1/\sqrt{\xi}$, this solution is analytic around $\xi = \infty$. Importantly, the two solutions $f_<(\xi)$ and $f_>(\xi)$ are both convergent at $\xi = 1$. We therefore find the full solution by matching the solutions at $\xi = 1$. Consider first the difference $f_<(\xi) - f_>(\xi)$, which must be a solution of the homogeneous equation:

$$f_<(\xi) - f_>(\xi) = \sum_{\pm} A_{\pm} \xi^{\pm i\mu}, \quad (5.138)$$

where continuity at $\xi = 1$ requires that $A_+ = -A_- \equiv A$ and

$$2i\mu A = \sum_n^{\infty} (-1)^n \frac{(2n+1)}{(n + \frac{1}{2})^2 + \mu^2} = \frac{\pi}{\cosh \pi\mu}. \quad (5.139)$$

We have therefore obtained an explicit form for the solution of the differential equation which is analytic around the origin:

$$\tilde{F}_<(\xi) = \begin{cases} \sum_{n=0}^{\infty} (-1)^n \frac{\xi^{n+1}}{(n + \frac{1}{2})^2 + \mu^2} & \xi \leq 1, \\ \sum_{n=0}^{\infty} (-1)^n \frac{\xi^{-n}}{(n + \frac{1}{2})^2 + \mu^2} + \frac{\pi}{\cosh \pi \mu} \frac{\xi^{\frac{1}{2}-i\mu} - \xi^{\frac{1}{2}+i\mu}}{2i\mu} & \xi \geq 1. \end{cases} \quad (5.140)$$

We see that it is impossible for the solution which is analytic around $\xi = 0$ to be analytic around $\xi = \infty$. This important feature will persist in the general solution.

General solution The general solution can be derived in a similar way. To avoid losing the forest for the trees, we will delegate some of the algebraic details to the exercises. The solutions to these exercises can be found in Appendix E.

Exercise 5.12 Show that, around $u = 0$, the inhomogeneous solution is [34]

$$\hat{F}_<(u, v) \sum_{m,n=0}^{\infty} c_{mn}(\mu) u^{2m+1} (u/v)^n, \quad (5.141)$$

where the series coefficients are

$$c_{mn}(\mu) = \frac{(-1)^n (n+1)(n+2)\cdots(n+2m)}{[(n + \frac{1}{2})^2 + \mu^2][(n + \frac{5}{2})^2 + \mu^2]\cdots[(n + \frac{1}{2} + 2m)^2 + \mu^2]}. \quad (5.142)$$

This solution is convergent for $u < v$. →

Note that (5.141) is the unique particular solution that is regular at the origin, since both homogeneous solutions are non-analytic at $u = 0$. In contrast, regularity around $u = \infty$ does not uniquely fix $\hat{F}_>(u, v)$. Instead, we will demand that the full particular solution is symmetric under the exchange $u \leftrightarrow v$. For $u > v$, the solution therefore is $\hat{F}_>(u, v) = \hat{F}_<(v, u)$, or

$$\hat{F}_>(u, v) = \sum_{m,n=0}^{\infty} c_{mn}(\mu) v^{2m+1} (v/u)^n, \quad (5.143)$$

where $c_{mn}(\mu)$ are same coefficients as in (5.142). It is straightforward to check that (5.143) solves (5.127).

Exercise 5.13 By matching the two solutions (5.141) and (5.143) at $u = v$, show the particular solution that is regular around $u = 0$ is

$$\hat{F}_<(u, v) = \begin{cases} \sum_{m,n=0}^{\infty} c_{mn}(\mu) u^{2m+1} (u/v)^n & u \leq v, \\ \sum_{m,n=0}^{\infty} c_{mn}(\mu) v^{2m+1} (v/u)^n + \frac{\pi}{\cosh \pi \mu} \hat{F}_h(u, v) & u \geq v, \end{cases} \quad (5.144)$$

where $\hat{F}_h(u, v) \equiv \hat{F}_+(v)\hat{F}_-(u) - \hat{F}_-(v)\hat{F}_+(u)$.

Hint: You may use the identity

$$\sum_{n=0}^{\infty} (2m+2n+1)c_{mn}(\mu) = \frac{\pi}{\cosh \pi\mu}, \quad (5.145)$$

which can be proven by matching the residues on both sides. →

We still have the freedom to add homogeneous solutions to (5.144). In fact, we *must* add them to satisfy the boundary conditions [DB: fix]

$$\lim_{u \rightarrow +1} \hat{F} = \text{regular}, \quad (5.146)$$

$$\lim_{u,v \rightarrow -1} \hat{F} = \frac{1}{2} \log(1+u) \log(1+v). \quad (5.147)$$

It is easy to check that the homogeneous solution (5.128) has the following singular behavior in the limit $u \rightarrow 1$:

$$\lim_{u \rightarrow 1} \hat{F}_{\pm}(u) = \alpha_{\pm} \log(1-u), \quad \text{where} \quad \alpha_{\pm} \equiv -\left(\frac{i}{2\mu}\right)^{\frac{1}{2} \pm i\mu} \frac{\Gamma(1 \pm i\mu)}{\Gamma(\frac{1}{4} \pm \frac{i\mu}{2})\Gamma(\frac{3}{4} \pm \frac{i\mu}{2})}. \quad (5.148)$$

By adding an appropriate amount of the homogeneous solution, we can cancel off any potential singularity in this limit. Finally, the solution in (5.144) isn't symmetric in $u \leftrightarrow v$, as required by consistency of the bulk evolution (and the symmetry of the conformally-invariant contact terms). In particular, the nonperturbative correction is absent as $u \rightarrow 0$, and as a result the solution is analytic in this limit. In the following exercise you will fix these deficiencies.

Exercise 5.14 Show that the final result is [34, 45]

$$\hat{F}(u, v) = \begin{cases} \sum_{m,n=0}^{\infty} c_{mn} u^{2m+1} (u/v)^n + \frac{\pi}{2 \cosh \pi\mu} \hat{g}(u, v) & u \leq v, \\ \sum_{m,n=0}^{\infty} c_{mn} v^{2m+1} (v/u)^n + \frac{\pi}{2 \cosh \pi\mu} \hat{g}(v, u) & u \geq v, \end{cases} \quad (5.149)$$

where we have defined [DB: fix]

$$\begin{aligned} \hat{g}(u, v) \equiv & \hat{F}_h(u, v) - \frac{\alpha_-}{\alpha_+} (\beta_0 + 1) \hat{F}_+(u) \hat{F}_+(v) - \frac{\alpha_+}{\alpha_-} (\beta_0 - 1) \hat{F}_-(u) \hat{F}_-(v) \\ & + \beta_0 [\hat{F}_-(u) \hat{F}_+(v) + \hat{F}_-(v) \hat{F}_+(u)], \end{aligned} \quad (5.150)$$

with $\beta_0 = (i \sinh \pi\mu)^{-1}$ and the parameters α_{\pm} were defined in (5.148). →

It is interesting that the only solutions that are symmetric and free of spurious singularities are necessarily non-analytic around $u \rightarrow 0$ and $v \rightarrow 0$. This is how particle production in the time-dependent bulk spacetime is encoded in the boundary correlators.

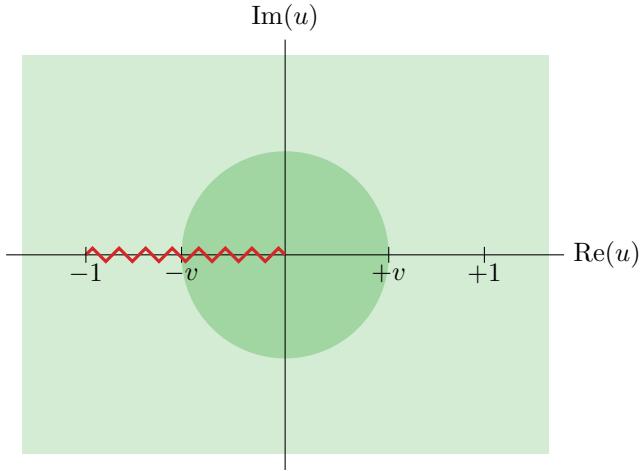


Figure 11: Analytic structure of the result (5.149). Note that the physical regime corresponds to $0 < u < 1$, and the branch cut for negative u is a signal of particle production.

Analytic structure The solution (5.149) has an interesting analytic structure, illustrated in Fig. 11. The function is analytic in the physical domain, $0 < u < 1$, but has a branch cut for $-1 < u \leq 0$. The existence of this branch cut is to be contrasted with the case of scattering amplitude which at tree level can only have poles. As we alluded to above, the discontinuity at $E = 0$ (or $u \rightarrow -v$) contains the flat-space scattering amplitude. In the following, we will show this explicitly.

To study the $E \rightarrow 0$ limit, we go back to the differential equation (5.127), which, in the limit $u \rightarrow -v$, becomes

$$\lim_{u \rightarrow -v} \frac{\partial^2 \hat{F}}{\partial u^2} = -\frac{1}{1-v^2} \frac{1}{u+v}. \quad (5.151)$$

This integrates to

$$\lim_{u \rightarrow -v} \hat{F} = -\frac{1}{(1-v^2)} (u+v) \log(u+v) = -\frac{k_I}{s} E \log E, \quad (5.152)$$

where $s = k_{12}^2 - k_I^2$ is the Mandelstam variable. We see that the coefficient of the $E = 0$ singularity is indeed given by the high-energy limit of the flat-space amplitude, $A_4 = 1/s$.

Particle production In the collapsed limit $u \rightarrow 0$, the particle production piece dominates and the function (5.149) becomes

$$\lim_{u,v \rightarrow 0} \hat{F}(u,v) = g^2 \left(\frac{uv}{4} \right)^{\frac{1}{2}+i\mu} (1+i \sinh \pi \mu) \frac{\Gamma(\frac{1}{2}+i\mu)^2 \Gamma(-i\mu)^2}{2\pi} + \text{c.c.} \quad (5.153)$$

We see that the signal oscillates with a frequency given by the mass of the new particles. This is the analog of resonances in collider physics (see Fig. 12). [DB: more]

[DB: bispectrum; weight-shifting; spectroscopy]

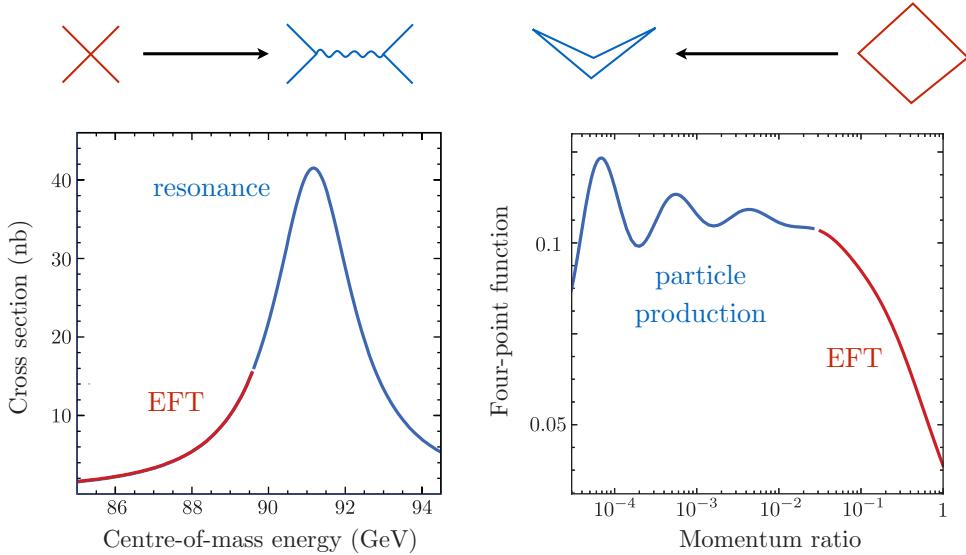


Figure 12: Oscillations in the soft limit of cosmological correlators (*right*) are the analog of resonances in scattering experiments (*left*).

6 Conclusions and Outlook

The theory of cosmological correlations is an important part of modern cosmology. In these lectures, we presented three different approaches to compute cosmological correlation functions: 1) the in-in formalism, 2) the wavefunction approach, and 3) the cosmological bootstrap. We included many examples and exercises to show how these computations are performed and to demonstrate explicitly that they give the same results in cases where comparison is possible.

There are a number of relevant topics that we didn't cover in much detail. In particular, our focus in these lectures has mostly been on conceptual aspects in the computations of cosmological correlations, while important observational and phenomenological aspects were not described in as much depth as they deserve. In recent years, there have been relevant parallel developments in the phenomenology of “cosmological collider physics” [70–80], including interesting observational studies [81–85]. We encourage the reader to learn about this important work in the references cited above.

Quantum field theory in cosmological spacetimes is still an active area of research. While we have a mature understanding of QFT in flat space ($\Lambda = 0$) and in anti-de Sitter spacetime ($\Lambda < 0$), in spacetimes with a positive cosmological constant ($\Lambda > 0$), we don't even know how to rigorously define a theory. It is ironic that the spacetimes most similar to our own—asymptotically de Sitter space—are still the most mysterious (see Fig. 13). Despite significant progress in recent years, we are only at the beginning of a systematic exploration of cosmological correlators in this context. There are still many puzzles to decipher and challenges to overcome. These challenges provide opportunities for further progress, and we close by listing some important open directions to pursue in the coming years.

From trees to loops In the last few years, we have significantly improved our understanding of tree-level correlators in cosmological spacetimes. We have learned how to efficiently bootstrap these correlators and gained insight into the space of functions that can arise from tree-level processes. In contrast, our understanding of loop-level correlators is much less developed. Very few explicit results exist even just at one-loop, and very little is known more generally about the analytic structure of correlators arising from loops in the bulk spacetime. Besides being of conceptual value, understanding loops in cosmology can sometimes also be practically important. For example, the leading couplings of the inflaton to fermions (including Standard Model fields) only arise at one-loop [86–89]. Moreover, loops are important in assessing the quantum stability of de Sitter space [90] and the dynamics of eternal inflation [91–100]. Finally, in the effective theory of large-scale structure (EFT-of-LSS) [101, 102], gravitational nonlinearities are captured by loops, and computing these loops is relevant for the analysis of data from future galaxy surveys.

Non-perturbative bootstrap So far, the bootstrap constraints (singularities, unitarity cuts, etc.) have only been formulated in perturbation theory. In particular, the existence of the energy singularities of cosmological correlators (described in Section 5.2) relies on perturbative theory and the associated effective field theory expansion. In a UV-complete theory of quantum gravity, we expect these singularities to be absent, but the precise consequences of this fact remain to be explored. Similarly, the constraints of unitarity on cosmological correlators (described in Section 5.3) are only understood perturbatively. This is to be contrasted with the case for scattering amplitudes where the unitarity constraints can be formulated nonperturbatively.

Interesting progress has recently been made in deriving the nonperturbative consequences of unitarity on the Hilbert space of de Sitter representations [103, 104]. Assuming invariance under the full dS isometry group, a Källén–Lehmann representation of the two-point function and a conformal partial wave decomposition of the four-point function were derived. In both cases,

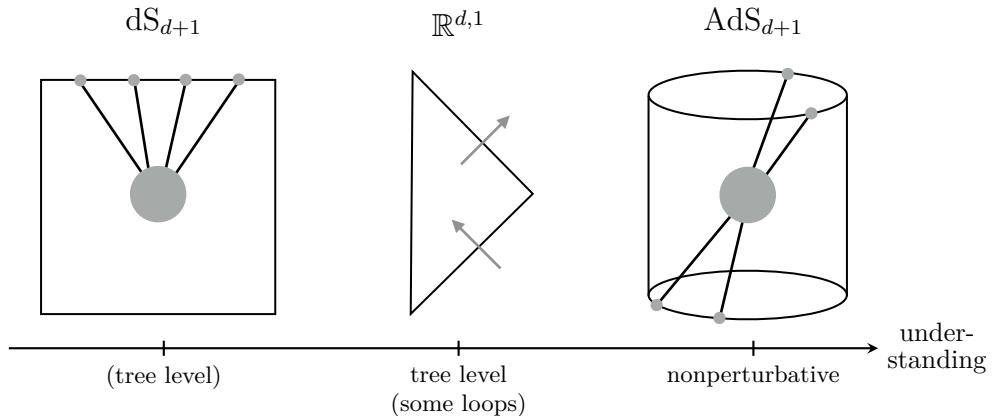


Figure 13: Remarkably, the closer we get to spacetimes that describe the Universe we live in (de Sitter space), the less we understand about how to rigorously define a theory in that space. In contrast, we have a perfect nonperturbative definition of quantum gravity in AdS spacetimes in terms of a boundary CFT. In flat space, we have some understanding of the perturbative structure of QFT and partial understanding of some nonperturbative effects. On the other hand, we are just beginning to scratch the surface of the rules underlying consistent theories in dS, where our understanding is not even complete at tree level.

unitarity implies positivity of the corresponding spectral densities.

Beyond Feynman diagrams Another important shortcoming of the cosmological bootstrap is that the correlators are still described in terms of their separation into Feynman diagrams, even for gauge theory and gravity where these diagrams themselves aren't even physical. This is in stark contrast to the case of the S-matrix bootstrap, where the real magic is found in the sum of diagrams. In 1986, Parke and Taylor famously found a remarkably simple formula describing the scattering of n gluons [37]:

$$A(1^- 2^- 3^+ \cdots n^+) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle (n-1)n \rangle \langle n1 \rangle}, \quad (6.1)$$

where the brackets $\langle ij \rangle$ represent the momenta of the external particles in terms of spinor helicity variables (see Appendix C). Subsequently, it was shown that the Parke–Taylor formula can be bootstrapped from basic physical consistency requirements, avoiding the crippling complexity of Feynman diagrams. This started a revolution in our understanding of scattering amplitudes.

Cosmology has not yet had its Parke–Taylor moment, meaning that the analog of a formula like (6.1) has not been discovered. Part of the problem is that, in the cosmological context, we don't have kinematic variables that describe the sum of Feynman diagrams as a unified physical object.

Hidden structures Going beyond Feynman diagrams and focussing directly on observable (on-shell) amplitudes has greatly simplified the computation of scattering amplitudes. More importantly, it has also led to an ocean of theoretical data (i.e. explicit results for amplitudes in different theories up to high multiplicity and loop orders) that could then be explored for new structures. [DB: add] [105, 106]

Beyond spacetime For scattering amplitudes, the development of new bootstrap methods was only the beginning of a much larger story. Using the results of the S-matrix bootstrap, it was realized that there are radically new ways of conceptualizing what scattering amplitudes are. It was discovered that scattering amplitudes in certain special theories can be thought of as the ‘volume’ of a new geometrical object called the ‘amplituhedron’ [107]. This geometric shape is an example of a ‘positive geometry’, a concept that generalizes the familiar convex polyhedra that have been studied since antiquity. The amplituhedron is described by the energies, momenta and spins of the particles involved in the scattering process, but makes no reference to their trajectories in spacetime and to the evolution of their quantum state. It therefore gives an existence proof of a new framework for particle physics in which spacetime and quantum mechanics emerge from more basic mathematical concepts.

The amplituhedron is described by the energies, momenta and spins of the particles involved in the scattering process, but makes no reference to their trajectories in spacetime and to the evolution of their quantum state. It therefore gives an existence proof of a new framework for particle physics in which spacetime and quantum mechanics emerge from more basic mathematical concepts. This resonates with the fact that, in quantum gravity, spacetime is an emergent concept.

Nowhere is the need for an understanding of emergent space-*time* more pressing than in cosmology, where the birth of spacetime and the Universe itself are intimately connected at the Big Bang singularity. If the notion of “time” breaks down at the Big Bang, what replaces it? [DB: more]

As we see from this list of open problems, the field of cosmological correlators is still a fruitful area of current research. [DB: final paragraph]

Acknowledgments We are grateful to our friends and collaborators for teaching us most of what we know about the theory of cosmological correlators. Thanks to Nima Arkani-Hamed, Hayden Lee, Guilherme Pimentel, Carlos Duaso Pueyo, Garrett Goon, Manuel Loparco, Wei-Ming Chen, Dionysios Anninos, Tarek Anous, Frederik Denef, Enrico Pajer, Paolo Benincasa, Zimo Sun, Daniel Green, Yu-tin Huang, Chia-Kai Kuo, Zi-Xun Huang, Yohan Liu and Kamran Salehi Vaziri. Laura Engelbrecht provided many corrections to a draft version of these notes.

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A Special Functions

In this appendix, we collect properties of a few selected special functions that will be useful for the computations and the exercises presented in the main text. Further details and derivations can be found in [? ?].

A.1 Legendre Functions

Legendre functions are solutions of

$$\boxed{\left[(1-x^2) \frac{d^2}{dx^2} - 2x \frac{d}{dx} \right] P_l = -l(l+1)P_l}, \quad (\text{A.1})$$

where l is a real number. In physics, these equations arise when we solve the Laplace equation in spherical coordinates. The variable x is then the cosine of the polar angle and its value is constrained to the range $-1 \leq x \leq 1$.

When the parameter l is an integer, the solutions of (A.1) are polynomials in x . The lowest-order *Legendre polynomials* are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad (\text{A.2})$$

while higher-order solutions can be obtained from the recursion relation

$$(l+1)P_{l+1} = (2l+1)xP_l - lP_{l-1}. \quad (\text{A.3})$$

These solutions can also be obtained from Rodrigues' formula

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l. \quad (\text{A.4})$$

Depending on the order, the Legendre polynomials have odd or even parity: $P_l(-x) = (-1)^l P_l(x)$.

The Legendre polynomials are orthogonal on the interval $[-1, 1]$:

$$\int_{-1}^1 dx P_l(x) P_{l'}(x) = \frac{2}{2+l} \delta_{ll'}. \quad (\text{A.5})$$

Any function defined on the interval can therefore be written as

$$f(x) = \sum_l a_l P_l(x), \quad \text{where} \quad a_l = \frac{2l+1}{2} \int_{-1}^1 dx f(x) P_l(x). \quad (\text{A.6})$$

Orthogonality of the Legendre polynomials guarantees that this expansion is unique.

A second class of solutions to (A.1) are the *Legendre functions of the second kind*. Even for integer l , these solutions are not polynomials in x . The first few lowest-order solutions are

$$\begin{aligned} Q_0(x) &= \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right), \\ Q_1(x) &= \frac{x}{2} \ln \left(\frac{1+x}{1-x} \right) - 1, \\ Q_2(x) &= \frac{3x^2 - 1}{4} \ln \left(\frac{1+x}{1-x} \right) - \frac{3x}{2}. \end{aligned} \quad (\text{A.7})$$

The functions $Q_l(x)$ satisfy the same recurrence relation (A.3) as the Legendre polynomials $P_l(x)$. This allows us to find higher-order solutions. Alternatively, we can use the Wronskian method to find

$$Q_l(x) = P_l(x) \int^x \frac{du}{(1-u^2)(P_l(u))^2}, \quad (\text{A.8})$$

where $P_l(x)$ are the Legendre polynomials (of the first kind).

An important generalization are the *associated Legendre functions* which are solutions of

$$\boxed{\left[(1-x^2) \frac{d^2}{dx^2} - 2x \frac{d}{dx} \right] P_l^m(x) + \left[l(l+1) - \frac{m^2}{1-x^2} \right] P_l^m(x) = 0}, \quad (\text{A.9})$$

where l and m are real parameters. For $m = 0$, this equation reduces to the ordinary Legendre equation (A.1).

In physical applications, the parameters l and m are often integers, with $|m| \leq l$. When m is even, the solution is a polynomial. For non-negative integer parameters l and m , we have

$$P_l^m(x) = (1-x^2)^{m/2} \frac{d^m P_l}{dx^m}. \quad (\text{A.10})$$

Sometimes a factor of $(-1)^m$ is included in this expression. Since $P_l(x)$ is a polynomial of order l , we have $P_l^m(x) = 0$ for $|m| > l$. Moreover, $P_l^m(x)$ is then also a polynomial of order l if m is even. Using Rodrigues' formula (A.4), this can be written as

$$P_l^m(x) = \frac{(-1)^m}{2^l l!} (1-x^2)^{m/2} \frac{d^{l+m}}{dx^{l+m}} (x^2 - 1)^l. \quad (\text{A.11})$$

The first few associated Legendre functions (of the first kind) then are

$$\begin{aligned} P_1^1(x) &= (1-x^2)^{1/2}, \\ P_2^1(x) &= 3x(1-x^2)^{1/2}, \quad P_2^2(x) = 3(1-x^2), \\ P_3^1(x) &= \frac{3}{2}(5x^2-1)(1-x^2)^{1/2}, \quad P_3^2(x) = 15x(1-x^2), \quad P_3^3(x) = 15(1-x^2)^3/2. \end{aligned} \quad (\text{A.12})$$

From (A.11), one can also show that

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x), \quad (\text{A.13})$$

which relates functions with positive and negative m .

For complex parameters and complex argument, the associated Legendre functions can be generalized via the hypergeometric function (see below).

A.2 Bessel and Hankel Functions

Ordinary Bessel functions satisfy

$$\boxed{\frac{d^2 J_\nu}{dz^2} + \frac{1}{z} \frac{dJ_\nu}{dz} + \left(1 - \frac{\nu^2}{z^2}\right) J_\nu = 0}, \quad (\text{A.14})$$

where ν is a constant parameter. A series solution of this equation is

$$J_\nu(z) = \left(\frac{z}{2}\right)^{-\nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(\nu + n + 1)} \left(\frac{z}{2}\right)^{2n}. \quad (\text{A.15})$$

As long as ν is not an integer, then a second linearly independent solution is $J_{-\nu}(z)$. The functions $J_\nu(z)$ and $J_{-\nu}(z)$ are called *Bessel functions of the first kind*. If ν is an integer n , then $J_n(z)$ and $J_{-n}(z)$ are not independent, but related by

$$J_{-n}(z) = (-1)^n J_n(z). \quad (\text{A.16})$$

A function that remains linearly independent of $J_\nu(z)$, even for integer ν , is the *Bessel function of the second kind* (sometimes called the *Neumann function*):

$$Y_\nu(z) = \frac{J_\nu(z) \cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}. \quad (\text{A.17})$$

For integer ν , this definition must be understood as the limit $\nu \rightarrow n$. The Bessel functions have the following limiting behavior for $z \rightarrow 0$:

$$\begin{aligned} J_\nu(z) &\xrightarrow{z \rightarrow 0} \frac{1}{\Gamma(\nu + 1)} \left(\frac{z}{2}\right)^\nu, \\ Y_\nu(z) &\xrightarrow{z \rightarrow 0} -\frac{1}{\pi} \Gamma(\nu) \left(\frac{z}{2}\right)^{-\nu}. \end{aligned} \quad (\text{A.18})$$

These limits make manifest that the Bessel equation (A.14) has a scaling symmetry for $z \rightarrow 0$. We see that $J_\nu(z)$ is regular for small argument, while $Y_\nu(z)$ diverges.

In the limit $z \rightarrow \infty$, the Bessel equation exhibits a translation symmetry and should have solutions that scale as $e^{\pm iz}$. This is made more manifest by defining the *Hankel functions* (or *Bessel functions of the third kind*):

$$\begin{aligned} H_\nu^{(1)}(z) &\equiv J_\nu(z) + iY_\nu(z), \\ H_\nu^{(2)}(z) &\equiv J_\nu(z) - iY_\nu(z). \end{aligned} \quad (\text{A.19})$$

We encountered Hankel functions when we derived the mode functions for inflationary fluctuations. The asymptotic limits of the Hankel functions for large arguments are

$$\begin{aligned} H_\nu^{(1)}(z) &\xrightarrow{z \rightarrow \infty} \sqrt{\frac{2}{\pi z}} e^{i(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)}, \\ H_\nu^{(2)}(z) &\xrightarrow{z \rightarrow \infty} \sqrt{\frac{2}{\pi z}} e^{-i(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)}. \end{aligned} \quad (\text{A.20})$$

These limits were used to define the initial conditions of the inflationary fluctuations. The scaling for small arguments follows from (A.18):

$$\begin{aligned} H_\nu^{(1)}(z) &\xrightarrow{z \rightarrow 0} -\frac{i}{\pi} \Gamma(\nu) \left(\frac{z}{2}\right)^{-\nu}, \\ H_\nu^{(2)}(z) &\xrightarrow{z \rightarrow 0} +\frac{i}{\pi} \Gamma(\nu) \left(\frac{z}{2}\right)^{-\nu}. \end{aligned} \quad (\text{A.21})$$

This limit is used to determine the superhorizon behavior of the inflationary fluctuations.

A.3 Polylogarithms

When computing the tree-level correlators of conformally coupled scalars in de Sitter space, we encountered polylogarithms.

A series definition of the polylogarithm is

$$\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} = z + \frac{z^2}{2^s} + \frac{z^3}{3^s} + \dots \quad (\text{A.22})$$

which is valid for arbitrary complex order s and for all complex arguments z with $|z| < 1$. The special case $s = 1$ is the ordinary natural logarithm

$$\text{Li}_1(z) = -\ln(1-z), \quad (\text{A.23})$$

while $s = 2$ and $s = 3$ are called the dilogarithm and the trilogarithm, respectively. For non-positive integers s , polylogarithms are rational functions:

$$\text{Li}_0(z) = \frac{z}{1-z}, \quad \text{Li}_{-1}(z) = \frac{z}{(1-z)^2}, \quad \text{Li}_{-2}(z) = \frac{z(1+z)}{(1-z)^3}, \quad \dots \quad (\text{A.24})$$

Evaluated at $z = 1$, the polylogarithm becomes the Riemann zeta function: $\text{Li}_s(1) = \zeta(s)$.

Polylogarithms satisfy the following iterative integral

$$\text{Li}_{s+1}(z) = \int_0^z dt \frac{\text{Li}_s(t)}{t}. \quad (\text{A.25})$$

The dilogarithm is therefore an integral of the natural logarithm, the trilogarithm is an integral of the dilogarithm, and so on. The derivatives of the polylogarithm satisfy

$$z \frac{\partial \text{Li}_s(z)}{\partial z} = \text{Li}_{s-1}(z), \quad (\text{A.26})$$

which follows directly from (A.25).

The polylogarithm function can also be expressed in terms of integrals of the Bose–Einstein and Fermi–Dirac distributions

$$\text{Li}_s(z) = \frac{1}{\Gamma(s)} \int_0^\infty dt \frac{t^{s-1}}{e^t/z - 1}, \quad (\text{A.27})$$

$$-\text{Li}_s(-z) = \frac{1}{\Gamma(s)} \int_0^\infty dt \frac{t^{s-1}}{e^t/z + 1}. \quad (\text{A.28})$$

It therefore also makes an appearance in statistical mechanics.

A.4 Hypergeometric Functions

The hypergeometric function is a solution of

$$\boxed{z(1-z)\frac{d^2F}{dz^2} + [c - (1+a+b)z]\frac{dF}{dz} - abF = 0}, \quad (\text{A.29})$$

where a, b and c are real numbers. The equation has regular singular points at $z = 0, 1, \infty$. In fact, it can be shown that any second-order differential equation with three regular singularities can be transformed into the hypergeometric equation by a change of variables. This allows other special functions—like the (associated) Legendre functions—to be written as special cases of hypergeometric functions. Often the (Gauss) hypergeometric function is denoted by ${}_2F_1$, while solutions of the ‘generalized hypergeometric equation’ are written as ${}_pF_q$, for integers p and q .

A series solution of (A.29) is

$$F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!} = 1 + \frac{ab}{c} \frac{z}{1!} + \frac{a(a+1)b(b_1)}{c(c+1)} \frac{z^2}{2!} + \dots, \quad (\text{A.30})$$

where the Pochhammer symbol is defined by

$$(a)_{n>0} = a(a+1)\cdots(a+n-1), \quad (a)_0 = 1. \quad (\text{A.31})$$

Equation (A.30) can also be written as

$$F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}, \quad (\text{A.32})$$

where $\Gamma(a)$ is the Euler gamma function.

A very useful integral representation of the hypergeometric function is

$$F(a, b, c; z) = \frac{\Gamma[c]}{\Gamma[b]\Gamma[c-b]} \int_0^\infty dt t^{-b+c-1} (t+1)^{a-c} (t-z+1)^{-a}. \quad (\text{A.33})$$

which converges for $c > b > 0$. This integral representation can be used to prove many general properties of the hypergeometric functions.

There are many relationships between hypergeometric functions with different arguments. Two particularly useful ones are

$$F(a, b, c; z) = (1-z)^{c-a-b} F(c-a, c-b, c; z), \quad (\text{A.34})$$

$$\frac{d}{dz} F(a, b, c; z) = \frac{ab}{c} F(a+1, b+1, c+1; z). \quad (\text{A.35})$$

In addition, there are relations between hypergeometric functions with shifted parameters. The six functions

$$F(a \pm 1, b, c; z), \quad F(a, b \pm 1, c; z), \quad F(a, b, c \pm 1; z) \quad (\text{A.36})$$

are called ‘contiguous’ to $F(a, b, c; z)$. As shown by Gauss, $F(a, b, c; z)$ can be written as a linear combination of any two of its contiguous functions, with rational coefficients depending on a, b, c and z . In total, there are 15 such relations. One example is

$$(c-a)F(a-1, b, c; z) + (2a-c-ax+bx)F(a, b, c; z) + a(x-1)F(a+1, b, c; z). \quad (\text{A.37})$$

These ‘contiguous relations’ provide recurrence relations for the hypergeometric functions.

Many of the above special functions can also be written in terms of hypergeometric functions. Some examples are:

$$\begin{aligned} F(a, b, b; x) &= (1 - x)^{-a}, & F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; x\right) &= x^{-1} \sin^{-1} x, \\ F(1, 1, 2; -x) &= x^{-1} \ln(1 + x), & F\left(\frac{1}{2}, 1, \frac{3}{2}; -x^2\right) &= x^{-1} \tan^{-1} x, \\ F\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}; \sin^2 x\right) &= \cos x, & F(l+1, -l, 1; \frac{1}{2}(1-x)) &= P_l(x), \end{aligned} \quad (\text{A.38})$$

where $P_l(x)$ is the Legendre polynomial. The associated Legendre functions are

$$P_l^m(z) = \frac{1}{\Gamma(1-m)} \left(\frac{1+z}{1-z}\right)^{m/2} {}_2F_1\left(-l, l+1, 1-m; \frac{1-z}{2}\right), \quad (\text{A.39})$$

$$Q_l^m(z) = \frac{\sqrt{\pi}}{2^{l+1}\Gamma(l+3/2)} \frac{1}{z^{l+m+1}} (1-z^2)^{m/2} {}_2F_1\left(\frac{l+m+1}{2}, \frac{l+m+2}{2}, l+\frac{3}{2}; \frac{1}{z^2}\right), \quad (\text{A.40})$$

which holds for complex parameters l, m and complex arguments z .

A.5 Gamma Functions

The Euler gamma function is

$$\boxed{\Gamma(z) \equiv \int_0^\infty dt e^{-t} t^{z-1}}, \quad (\text{A.41})$$

where $\text{Re}(z) > 0$. It is an analytic function with simple poles at $z = -n$, for integer n . A useful relation is

$$\Gamma(z+1) = z\Gamma(z), \quad (\text{A.42})$$

which implies that for integer arguments the gamma function reduces to factorials, $\Gamma(n+1) = n!$.

B De Sitter Space

In this appendix, we provide a few useful technical facts about de Sitter spacetime. [DB: More intro]

B.1 Classical Geometry

In Section 5.4.1, we introduced the idea that D -dimensional de Sitter space can be viewed as a hyperboloid embedded in $(D + 1)$ -dimensional Minkowski space:

$$\eta_{AB} X^A X^B = -X_0^2 + X_1^2 + \cdots + X_D^2 = L^2, \quad (\text{B.1})$$

where L is the de Sitter radius. The de Sitter metric can be derived by projecting the metric of the embedding space, $ds^2 = \eta_{AB} dX^A dX^B$, onto the hyperboloid. In this section, we will present this metric in several different coordinate systems. We will follow the treatment in [108, 109].

Global coordinates: The solution to the embedding constraint (B.1) can be written as

$$X_0 = L \sinh(\tau/L), \quad X_i = L \cosh(\tau/L) \omega_i, \quad (\text{B.2})$$

where $-\infty < \tau < \infty$ and $\omega^i \omega_i = 1$. The vector ω_i defines the points on the unit sphere S_{D-1} . The coordinates (τ, ω_i) cover the entire hyperboloid and are therefore called the *global coordinates* of de Sitter space.⁵⁵ For $D = 4$, we can use the following parameterization in terms of polar angles

$$\begin{aligned} \omega_1 &= \cos \psi, \\ \omega_2 &= \sin \psi \cos \theta, \\ \omega_3 &= \sin \psi \sin \theta \cos \phi, \\ \omega_4 &= \sin \psi \sin \theta \sin \phi, \end{aligned} \quad (\text{B.3})$$

where $0 \leq \psi < \pi$, $0 \leq \theta < \pi$ and $0 \leq \phi < 2\pi$. The metric on the unit three-sphere S_3 is

$$d\Omega_3^2 = d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2). \quad (\text{B.4})$$

In general, it is often useful to foliate the spatial $(D - 1)$ -sphere by $(D - 2)$ -spheres. The metric on S_{D-1} is the written as

$$d\Omega_{D-1}^2 = d\psi^2 + \sin^2 \psi d\Omega_{D-2}^2. \quad (\text{B.5})$$

where $0 \leq \psi < \pi$.

Using (B.2), the induced metric on dS_D is

$$ds^2 = -d\tau^2 + L^2 \cosh^2(\tau/L) d\Omega_{D-1}^2. \quad (\text{B.6})$$

The global coordinates are also called the *closed slicing* of de Sitter, because the metric takes the form of a closed FRW universe. The scale factor $a(\tau) = L \cosh(\tau/L)$ first decreases exponentially, then reaches the minimum $a(0) = L$ at $t = 0$, and finally expands exponentially into the future.

⁵⁵Calling these coordinates “global” is a slight exaggeration since it impossible to cover the spheres S_{D-1} with a single coordinate patch. In particular, the standard polar coordinates have isolated singularities at the poles.

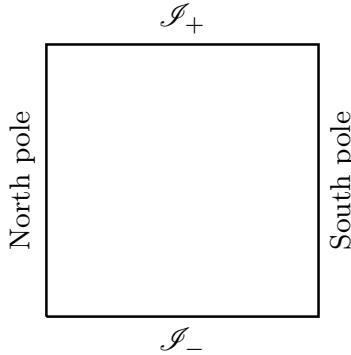


Figure 14: Penrose diagram of D -dimensional de Sitter space (for $D \neq 2$). Each time slice is a $(D - 1)$ -sphere and each point is a $(D - 2)$ -sphere. The left and right “edges” aren’t boundaries, but are the north and south poles of the S_{D-1} . The bottom and top are the past and future boundaries \mathcal{I}_- and \mathcal{I}_+ .

Consider the following analytic continuation of the time coordinate $\tau \rightarrow i\rho$ and define

$$\psi \equiv \frac{\rho}{L} + \frac{\pi}{2}, \quad (\text{B.7})$$

where the shift by $\pi/2$ is just for convenience. The metric (B.6) then becomes

$$ds^2 = L^2(d\psi^2 + \sin^2 \psi d\Omega_{D-1}^2) \equiv L^2 d\Omega_D^2, \quad (\text{B.8})$$

which we recognize as the metric of S_D .

Conformal coordinates: Consider the transformation $\cosh(\tau/L) = (\cos T)^{-1}$, where $-\pi/2 \leq T < \pi/2$. The metric (B.6) take the form

$$ds^2 = \frac{L^2}{\cos^2 T} (-dT^2 + d\Omega_{D-1}^2). \quad (\text{B.9})$$

The conformal coordinates are useful because they make the causal structure of the spacetime manifest. Null geodesics on ds^2 are also null with respect to the conformally related metric $d\tilde{s}^2 = -dT^2 + d\Omega_{D-1}^2$.

Using (B.5), we have

$$ds^2 = \frac{L^2}{\cos^2 T} (-dT^2 + d\psi^2 + \sin^2 \psi d\Omega_{D-2}^2). \quad (\text{B.10})$$

Suppressing the sphere S_{D-2} , this is conformally related to two-dimensional Minkowski space, $d\tilde{s}^2 = -dT^2 + d\psi^2$. Since the coordinates T and ψ both have range π , the corresponding *Penrose diagram* is a square (see Fig. 14).⁵⁶ The fixed T slices of the Penrose diagram are $(D - 1)$ -spheres. Each point on the inside of the diagram is a $(D - 2)$ -sphere. The north and south poles (at $\psi = 0$ and $\psi = \pi$, respectively) are points where the size of the suppressed $(D - 2)$ -spheres shrink to zero. The left and right “edges” of the diagram therefore aren’t real boundaries, but are just the poles of the $(D - 1)$ -spheres at each time slice. Remembering that light rays travel at 45° in a Penrose diagram, it is easy to determine the causal relations of points in the spacetime (see Fig. 15).

⁵⁶The case $D = 2$ is special and the Penrose diagram for dS_2 is twice as wide.

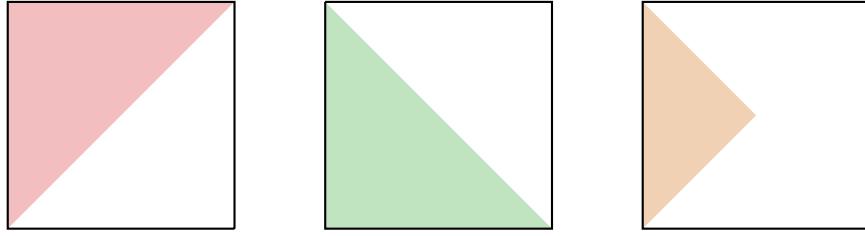


Figure 15: Illustration of the parts of the spacetime that an observer at the North pole can influence (red), from which it can receive signals (green) and with which it can communicate (orange). The latter is called the *causal patch*.

Planar coordinates: Another solution to the embedding constraint (B.1) is

$$X_0 = L \sinh(t/L) + \frac{r^2}{2L} e^{t/L}, \quad X_1 = L \cosh(t/L) - \frac{r^2}{2L} e^{t/L}, \quad X_i = e^{t/L} x_i, \quad (\text{B.11})$$

where $i = 2, \dots, D$ and $r^2 = x^i x_i$. Note that these coordinates only cover $X_0 + X_1 > 0$, i.e. half of the spacetime. Using (B.11), the induced metric is

$$ds^2 = -dt^2 + e^{2t/L} \delta_{ij} dx^i dx^j, \quad (\text{B.12})$$

These coordinates are also called the *flat slicing* of de Sitter, because the metric takes the form of a flat FRW universe. The scale factor $a(t) = e^{2t/L}$ grows exponentially.

Both global and planar coordinates, of course, describe the same de Sitter spacetime. There is no physics in the choice of coordinates. Beyond the idealized de Sitter approximation, however, an FRW universe contains additional matter contributions (like the inflaton field in single-field slow-roll inflation). The average rest frame of the matter selects a preferred frame and breaks the time translation symmetry of de Sitter spacetime. The closed and flat slicings then become inequivalent FRW universes. (The spacetimes are approximately identical, but the initial conditions for the matter are different.) Inflation picks out the flat slicing as the preferred foliation of the spacetime.

Exercise B.1 Using spherical polar coordinates on the spatial slices, we can write (B.12) as

$$ds^2 = -dt^2 + e^{2t/L} (dr^2 + r^2 d\Omega_{D-2}^2). \quad (\text{B.13})$$

Comparing (B.2) and (B.11), show that the global and planar coordinates are related by

$$e^{t/L} = \cos \psi \cosh(\tau/L) + \sinh(\tau/L), \quad \frac{L}{r} e^{t/L} = \sin \psi \cosh(\tau/L). \quad (\text{B.14})$$

Note that the time coordinates becomes the same at late times.

Static coordinates: De Sitter space has a time translation symmetry (i.e. all physical quantities are time-independent constants), but this isn't manifest in the global and planar coordinates. Consider therefore the following alternative solution of the embedding constraint (B.1):

$$X_0 = \sqrt{L^2 - r^2} \sinh(t/L), \quad X_1 = \sqrt{L^2 - r^2} \cosh(t/L), \quad X_i = r \omega_i, \quad (\text{B.15})$$

where $i = 2, \dots, D$ and $\omega^i \omega_i = 1$. In these coordinates, the induced metric on dS_D is

$$ds^2 = -\left(1 - \frac{r^2}{L^2}\right) dt^2 + \left(1 - \frac{r^2}{L^2}\right)^{-1} dr^2 + r^2 d\Omega_{D-1}^2, \quad (\text{B.16})$$

with $0 < r < L$. These ‘‘static coordinates’’ cover the *causal patch* of an observer sitting at the North pole (see Fig. 15). They are the analog of the ‘‘Schwarzschild coordinates’’ for a black hole. The metric is now manifestly static, since its components are independent of t . The horizon is at $r = L$.

Exercise B.2 Let (t_f, r_f) be the coordinates in the flat slicing. Comparing (B.11) and (B.15), show that the coordinates on the static slices are related by

$$r = r_f e^{t_f/L}, \quad e^{-2t_f/L} = e^{-2t_f/L} - \frac{r_f^2}{L^2}. \quad (\text{B.17})$$

We see that the horizon at $r = L$ corresponds to $r_f = Le^{-t_f/L}$ in the flat slicing.

B.2 Unitary Representations

In this section, we describe the construction of unitary representations of the de Sitter group in dimension $D \geq 3$. See the excellent review [62], for more details.

The fact that $J_{AB}^\dagger = -J_{AB}$ implies that the generators satisfy the reality conditions

$$\begin{aligned} P_i^\dagger &= -P_i, & \hat{D}^\dagger &= -\hat{D}, \\ J_{ij}^\dagger &= -J_{ij}, & K_i^\dagger &= -K_i. \end{aligned} \quad (\text{B.18})$$

This is the appropriate reality condition to construct unitary representations of the Euclidean conformal group $\text{SO}(D, 1)$.⁵⁷

The representations of interest are built from a *primary state*, $|\bar{\Delta}, 0\rangle_s$, which is annihilated by the SCT operator K_i , is an eigenstate of the dilatation operator \hat{D} , with eigenvalue $\bar{\Delta} = d - \Delta$, and transforms in an irreducible representation of the $\text{so}(D)$ algebra generated by J_{ij} , which is captured by the spin weights, s . Representations are therefore naturally labeled by these quantum numbers. For simplicity, we will restrict our explicit construction to scalar representations (on which J_{ij} acts trivially) and present the generalization to spinning representations without derivation. The primary state that we start with therefore satisfies⁵⁸

$$\begin{aligned} K_i |\bar{\Delta}, 0\rangle &= 0, \\ \hat{D} |\bar{\Delta}, 0\rangle &= \bar{\Delta} |\bar{\Delta}, 0\rangle, \end{aligned} \quad (\text{B.19})$$

⁵⁷In the study of Lorentzian conformal field theory, representations are commonly constructed in Euclidean signature by Wick rotating and utilizing a reality condition where $P_i^\dagger = K_i$. Representations with a positive norm with respect to this reality condition arise in reflection-positive Euclidean quantum field theories, which analytically continue to unitary Lorentzian CFTs. As such, this alternative reality condition is appropriate to construct unitary representations of the group $\text{SO}(D - 1, 2)$.

⁵⁸Note that this state is in general not normalizable (as we will see) and so is not really part of the Hilbert space of states, but nevertheless can be used to generate the normalizable elements of the Hilbert space.

and the quadratic Casimir of this representation will be $\mathcal{C}_2 = \Delta(d - \Delta)$. Acting with $e^{\mathbf{x} \cdot \mathbf{P}}$ defines the state

$$|\bar{\Delta}, \mathbf{x}\rangle \equiv e^{\mathbf{x} \cdot \mathbf{P}} |\bar{\Delta}, 0\rangle. \quad (\text{B.20})$$

Using the Baker–Campbell–Hausdorff formula and the algebra (5.60), we can determine how the generators act on this state:

$$P_i |\bar{\Delta}, \mathbf{x}\rangle = \partial_i |\bar{\Delta}, \mathbf{x}\rangle, \quad (\text{B.21})$$

$$\hat{D} |\bar{\Delta}, \mathbf{x}\rangle = (\bar{\Delta} + x^i \partial_i) |\bar{\Delta}, \mathbf{x}\rangle, \quad (\text{B.22})$$

$$J_{ij} |\bar{\Delta}, \mathbf{x}\rangle = (x_j \partial_i - x_i \partial_j) |\bar{\Delta}, \mathbf{x}\rangle, \quad (\text{B.23})$$

$$K_i |\bar{\Delta}, \mathbf{x}\rangle = (2\bar{\Delta}x_i + 2x_i x^j \partial_j - x^2 \partial_i) |\bar{\Delta}, \mathbf{x}\rangle. \quad (\text{B.24})$$

A general state can then be defined by integrating over \mathbf{x} :

$$|\psi\rangle = \int d^d x \psi(\mathbf{x}) |\bar{\Delta}, \mathbf{x}\rangle, \quad (\text{B.25})$$

where the (smooth) coefficient wavefunction $\psi(\mathbf{x})$ contains the information about the state of interest.⁵⁹ Integrating by parts, we infer the action of the generators on $\psi(\mathbf{x})$:

$$\begin{aligned} P_i \psi(\mathbf{x}) &= -\partial_i \psi(\mathbf{x}), \\ J_{ij} \psi(\mathbf{x}) &= (x_i \partial_j - x_j \partial_i) \psi(\mathbf{x}), \\ \hat{D} \psi(\mathbf{x}) &= -(\Delta + x^i \partial_i) \psi(\mathbf{x}), \\ K_i \psi(\mathbf{x}) &= (-2\Delta x_i x^j \partial_j - 2x_i x^j \partial_j + x^2 \partial_i) \psi(\mathbf{x}). \end{aligned} \quad (\text{B.26})$$

So far, we have only constructed a representation of the algebra $\text{so}(D, 1)$, we have to ensure that it can be exponentiated to a representation of the group $\text{SO}(D, 1)$. The fact that $\psi(\mathbf{x})$ is a smooth function ensures that translations, rotations, and dilations can be exponentiated, but the exponentiation of SCTs is nontrivial. It turns out that functions that allow us to exponentiate K_i must admit the following asymptotic expansion [62]

$$\psi(\mathbf{x}) \xrightarrow{\mathbf{x} \rightarrow \infty} \frac{1}{|\mathbf{x}|^{2\Delta}} \sum_{n=0}^{\infty} C^{(n)}(\mathbf{x}/x^2), \quad (\text{B.27})$$

where $C^{(n)}$ is a homogeneous polynomial of degree n in \mathbf{x}/x^2 . We will denote the space of smooth functions that satisfy (B.27) by \mathcal{F}_{Δ} . This space is a representation of $\text{SO}(D, 1)$, where the group action is given by exponentiating the action (B.26) on the functions that are smooth and satisfy (B.27). The representations \mathcal{F}_{Δ} are irreducible, except when $\Delta \in \{0, -1, -2, \dots\} \cup \{d, d+1, d+2, \dots\}$ [62?], and all scalar unitary irreducible representations of the de Sitter group can be realized as an invariant subspace of some \mathcal{F}_{Δ} (usually the full space) [60].

⁵⁹This wavefunction should not be confused with either the cosmological wavefunction or the wavefunction coefficients introduced earlier. They are unrelated.

Shadow transform: Not all of the representations \mathcal{F}_Δ are distinct. In particular, there is an important isomorphism of representations which comes from the *shadow transform*, which is an intertwining map between the representations \mathcal{F}_Δ and $\mathcal{F}_{\bar{\Delta}}$. We look for a map between representations $\mathcal{S}_{\Delta,\Delta'} : \mathcal{F}_\Delta \rightarrow \mathcal{F}_{\Delta'}$ of the form

$$\psi_{\Delta'}(\mathbf{x}) = \int d^d y S_{\Delta,\Delta'}(\mathbf{x}, \mathbf{y}) \psi_\Delta(\mathbf{y}), \quad (\text{B.28})$$

where $\psi_\Delta(\mathbf{y}) \in \mathcal{F}_\Delta$, $\psi_{\Delta'}(\mathbf{x}) \in \mathcal{F}_{\Delta'}$ and $S_{\Delta,\Delta'}(\mathbf{x}, \mathbf{y})$ is a kernel that implements the mapping. Given that $\psi_\Delta(\mathbf{y})$ transforms as (B.26), requiring that $\psi_{\Delta'}(\mathbf{x})$ transforms in the representation $\mathcal{F}_{\Delta'}$ implies some conditions on $S_{\Delta,\Delta'}(\mathbf{x}, \mathbf{y})$. For example, we can act on both sides of (B.28) with a translation

$$P_i \psi_{\Delta'}(\mathbf{x}) = -\frac{\partial \psi_{\Delta'}(\mathbf{x})}{\partial x^i} = \int d^d y S_{\Delta,\Delta'}(\mathbf{x}, \mathbf{y}) P_i \psi_\Delta(\mathbf{y}) = -\int d^d y S_{\Delta,\Delta'}(\mathbf{x}, \mathbf{y}) \frac{\partial \psi_\Delta(\mathbf{y})}{\partial y^i}. \quad (\text{B.29})$$

Integrating the last expression by parts, we find that $S_{\Delta,\Delta'}(\mathbf{x}, \mathbf{y})$ must satisfy the differential equation

$$\left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i} \right) S_{\Delta,\Delta'}(\mathbf{x}, \mathbf{y}) = 0. \quad (\text{B.30})$$

Repeating the same exercise for the other symmetries, we find that $S_{\Delta,\Delta'}(\mathbf{x}, \mathbf{y})$ must also [DB: ADD] solve [?]

$$\begin{aligned} & \left(x_{[j} \frac{\partial}{\partial x^{i]} + y_{[j} \frac{\partial}{\partial y^{i]} \right) S_{\Delta,\Delta'}(\mathbf{x}, \mathbf{y}) = 0, \\ & \left(\Delta' + x^i \frac{\partial}{\partial x^i} + \bar{\Delta} + y^i \frac{\partial}{\partial y^i} \right) S_{\Delta,\Delta'}(\mathbf{x}, \mathbf{y}) = 0, \\ & \left(2\Delta' x_i + 2x_i x^j \frac{\partial}{\partial x^j} - x^2 \frac{\partial}{\partial x^i} + 2\bar{\Delta} y_i + 2y_i y^j \frac{\partial}{\partial y^j} - y^2 \frac{\partial}{\partial y^i} \right) S_{\Delta,\Delta'}(\mathbf{x}, \mathbf{y}) = 0. \end{aligned} \quad (\text{B.31})$$

As we will see in Section 5.4.5, these are the Ward identities for a two-point function involving an operator of dimension Δ' and one of $\bar{\Delta}$. We will see that they are solved by

$$S_{\Delta,\Delta'}(\mathbf{x}, \mathbf{y}) = \delta_{\Delta', \bar{\Delta}} \frac{\Gamma[\bar{\Delta}]}{2^{\Delta-\bar{\Delta}} \pi^{\frac{d}{2}} \Gamma[\frac{d}{2} - \bar{\Delta}]} \frac{1}{|\mathbf{x} - \mathbf{y}|^{2\bar{\Delta}}}, \quad (\text{B.32})$$

where we have chosen the normalization for later convenience.⁶⁰ We see that the shadow transform (B.28) maps the representation \mathcal{F}_Δ to $\mathcal{F}_{\bar{\Delta}}$ by convolving with the two-point function of scalar primary operators with dimension $\bar{\Delta} = d - \Delta$. Since this map is equivariant with respect to the group operation, by Schur's lemma these two representations are equivalent. It is often convenient to implement the shadow transform in momentum space, where it just amounts to multiplication by the kernel

$$S_{\Delta, \bar{\Delta}}(p) = p^{2\bar{\Delta}-d}. \quad (\text{B.33})$$

We will often call states with weights related by $\bar{\Delta} = d - \Delta$ shadows of each other. The shadow transform will play an important role in the definition of unitary representations, and in their realization by fields in de Sitter space.

⁶⁰With this normalization convention, composing the shadow transform twice acts as the identity map. This requirement does not uniquely fix the normalization, so we have made a convenient choice.

Unitarity: We are now prepared to examine the constraints that unitarity places on the representations \mathcal{F}_Δ . The natural inner product on the representation comes from the definition (B.25)

$$\langle \psi_1 | \psi_2 \rangle = \int d^d x d^d y \psi_1^*(\mathbf{x}) \langle \mathbf{x}, \bar{\Delta} | \bar{\Delta}, \mathbf{y} \rangle \psi_2(\mathbf{y}), \quad (\text{B.34})$$

and so to define a norm all we have to do is choose the kernel $\langle \mathbf{x}, \bar{\Delta} | \bar{\Delta}, \mathbf{y} \rangle \equiv K_\Delta(\mathbf{x}, \mathbf{y})$. This kernel has to be compatible with the reality condition (B.18). For example, we require that

$$\langle \mathbf{x}, \bar{\Delta} | P_i^\dagger | \bar{\Delta}, \mathbf{y} \rangle = -\langle \mathbf{x}, \bar{\Delta} | P_i | \bar{\Delta}, \mathbf{y} \rangle \implies \frac{\partial}{\partial x^i} \langle \mathbf{x}, \bar{\Delta} | \bar{\Delta}, \mathbf{y} \rangle = -\frac{\partial}{\partial y^i} \langle \mathbf{x}, \bar{\Delta} | \bar{\Delta}, \mathbf{y} \rangle. \quad (\text{B.35})$$

Working out the similar constraints from J_{ij}, D, K_i , we obtain the set of equations that $K_\Delta(\mathbf{x}, \mathbf{y})$ has to solve:

$$\begin{aligned} \left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i} \right) K_\Delta(\mathbf{x}, \mathbf{y}) &= 0, \\ \left(x_{[j} \frac{\partial}{\partial x^{i]}} + y_{[j} \frac{\partial}{\partial y^{i]} \right) K_\Delta(\mathbf{x}, \mathbf{y}) &= 0, \\ \left(\bar{\Delta}^* + x^i \frac{\partial}{\partial x^i} + \bar{\Delta} + y^i \frac{\partial}{\partial y^i} \right) K_\Delta(\mathbf{x}, \mathbf{y}) &= 0, \\ \left(2\bar{\Delta}^* x_i + 2x_i x^j \frac{\partial}{\partial x^j} - x^2 \frac{\partial}{\partial x^i} + 2\bar{\Delta} y_i + 2y_i y^j \frac{\partial}{\partial y^j} - y^2 \frac{\partial}{\partial y^i} \right) K_\Delta(\mathbf{x}, \mathbf{y}) &= 0. \end{aligned} \quad (\text{B.36})$$

The first two equations imply that $K_\Delta(\mathbf{x}, \mathbf{y})$ is translation and rotation invariant, so we can write it as $K_\Delta(|\mathbf{x} - \mathbf{y}|)$. Defining $\mathbf{z} \equiv \mathbf{x} - \mathbf{y}$, we can write the third equation as

$$|\mathbf{z}| K'_\Delta(|\mathbf{z}|) = -(\bar{\Delta}^* + \bar{\Delta}) K_\Delta(|\mathbf{z}|), \quad (\text{B.37})$$

and substituting this into the last equation, we obtain the constraint

$$(\Delta - \Delta^*) (\mathbf{x} - \mathbf{y}) K_\Delta(|\mathbf{x} - \mathbf{y}|) = 0. \quad (\text{B.38})$$

This can be satisfied either if $K_\Delta(|\mathbf{x} - \mathbf{y}|) \propto \delta^{(d)}(\mathbf{x} - \mathbf{y})$, or if $\Delta = \Delta^*$. In the former case, the constraint from dilatation invariance implies $\Delta + \Delta^* = d$ because the delta function has scaling weight d . In the latter case, the equations (B.36) imply $K_\Delta(|\mathbf{x} - \mathbf{y}|) \propto |\mathbf{x} - \mathbf{y}|^{-2\bar{\Delta}}$. Since this has the same coordinate dependence as the kernel of the shadow transform (B.32), it makes sense to choose the normalization so that $K_\Delta(|\mathbf{x} - \mathbf{y}|) = S_{\Delta, \bar{\Delta}}(|\mathbf{x} - \mathbf{y}|)$. We therefore see that there are two possible inner products and conditions on Δ that are compatible with the reality condition (B.18):

- $\Delta = \frac{d}{2} + i\mu$, for $\mu \in \mathbb{R}$, with the inner product given by the kernel $K_\Delta(|\mathbf{x} - \mathbf{y}|) = \delta^{(d)}(\mathbf{x} - \mathbf{y})$.
- $\Delta \in \mathbb{R}$, with the inner product given by the kernel $K_\Delta(|\mathbf{x} - \mathbf{y}|) = S_{\Delta, \bar{\Delta}}(|\mathbf{x} - \mathbf{y}|)$.

We still have to ensure that these inner products are finite and positive. In the case where $K_\Delta(|\mathbf{x} - \mathbf{y}|) = \delta^{(d)}(\mathbf{x} - \mathbf{y})$, this is clearly the case because (B.34) is just the ordinary L^2 norm:

$$\langle \psi_1 | \psi_2 \rangle = \int d^d x \psi_1^*(\mathbf{x}) \psi_2(\mathbf{x}). \quad (\text{B.39})$$

Since the elements of \mathcal{F}_Δ fall off like (B.27), they are normalizable and so \mathcal{F}_Δ , with $\Delta = \frac{d}{2} + i\mu$ provide unitary representations of $\text{SO}(D, 1)$ for any choice of μ , which are known as the *principal series*.

The situation for real values of Δ is more subtle because the inner product does not always converge. This is simplest to see in Fourier space,⁶¹ where we have

$$\langle \psi_1 | \psi_2 \rangle = \int \frac{d^d p}{(2\pi)^d} \psi_1^*(\mathbf{p}) \psi_2(\mathbf{p}) p^{2\bar{\Delta}-d}. \quad (\text{B.40})$$

This integral is manifestly positive, but it does not always converge around $\mathbf{p} \rightarrow 0$. (Around infinity there is no problem because $\psi(\mathbf{x})$ is smooth, so its Fourier transform goes to zero rapidly at large momentum.) Near zero, $\psi(\mathbf{p})$ can have two characteristic fall-offs $\psi(\mathbf{p}) \simeq c_1 p^{2\Delta-d} + c_2$. The first of these is just the Fourier transform of the asymptotic behavior (B.27), while the other is a consequence of the regularity of $\psi(x)$ in real space. From this scaling, we see that the integral converges in the range $0 \leq \Delta \leq d$. The space \mathcal{F}_Δ with Δ in this range therefore produces a unitary irreducible representation, known as the *complementary series*.

There is actually one more family of unitary scalar representations, but their construction is more obscure. They have $\Delta = -k$ with $k \in \mathbb{Z}_{\geq 0}$ and are obtained from \mathcal{F}_{-k} . These representations are reducible, and they have an invariant subspace that furnishes a unitary irreducible representation obtained by quotienting by the kernel of the shadow transform $\mathcal{S}_{-k,d+k}$ (meaning the elements that get mapped to zero by the shadow transform). The inner product on this representation is the same as for the complementary series $K_\Delta(|\mathbf{x} - \mathbf{y}|) = S_{-k,d+k}(|\mathbf{x} - \mathbf{y}|)$. This family of representations is called the *exceptional series*. See [60, 62] for a more detailed discussion and construction of these representations.

It is interesting to compare this list of scalar unitary representations with their analogues in anti-de Sitter space. In that case the unitarity bounds can be derived in a somewhat similar way starting from a primary state.

Exercise B.3 In order to construct unitary representations of $\text{SO}(D - 1, 2)$, it is common to analytically continue and consider representations of the group $\text{SO}(D, 1)$, but with a non-standard reality condition on the generators, so that positivity with respect to this notion of conjugation guarantees that a representation is unitary with respect to the natural $\text{SO}(D - 1, 2)$ inner product. For the purposes of this problem, all you need to know is that

$$P_i^\dagger = K_i, \quad \hat{D}^\dagger = \hat{D}, \quad J_{ij}^\dagger = -J_{ij}. \quad (\text{B.41})$$

Consider a scalar primary state $|\Delta\rangle$. The states obtained by acting with P_i are called *descendant states*. By looking at first descendant $P_i|\Delta\rangle$ and the particular second descendant $P_i P^i |\Delta\rangle$, derive the unitarity bound $\Delta \geq \frac{d}{2} - 1$ by requiring that they have positive norm with respect to the inner product defined by (B.41).

Hint: Use the algebra of generators and the fact that the primary state satisfies $K_i |\Delta\rangle = 0$.

⁶¹See [62], for an argument in real space.

B.3 Källén–Lehmann Representation

[DB: Add.]

C S-Matrix Bootstrap

In this appendix, we describe the S-matrix bootstrap for massless particles with spin, following closely the brilliant TASI lectures by Cliff Cheung [39], as well as the online lectures by Nima Arkani-Hamed [?].

C.1 Spinor Helicity Variables

The kinematics of a scattering process is described by the four-momenta, $p_\mu = (p_0, \vec{p})$, of the particles. For massless particles, it will be convenient to arrange components of these four-momenta in a two-by-two matrix

$$p_{\alpha\dot{\alpha}} = p_\mu \sigma_{\alpha\dot{\alpha}}^\mu = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix}, \quad (\text{C.1})$$

where $\sigma^\mu = (1, \vec{\sigma})$ are the Pauli matrices. For massless particles, the determinant of the above matrix vanishes on-shell, $\det(p) = p^\mu p_\mu = 0$. The matrix is therefore of rank one and can be written as an outer product of two-component spinors

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}. \quad (\text{C.2})$$

Given two particles i and j , we can define the following Lorentz-invariant “angle” and “square” brackets:

$$\langle ij \rangle \equiv \lambda_{i\alpha} \lambda_{j\beta} \epsilon^{\alpha\beta}, \quad (\text{C.3})$$

$$[ij] \equiv \tilde{\lambda}_{i\dot{\alpha}} \tilde{\lambda}_{j\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}}, \quad (\text{C.4})$$

where $\epsilon^{\alpha\beta}$ and $\epsilon^{\dot{\alpha}\dot{\beta}}$ are the anti-symmetric Levi-Civita symbol. Note that these brackets are anti-symmetric under exchange of the particles, $\langle ij \rangle = -\langle ji \rangle$ and $[ij] = -[ji]$, which implies that $\langle ii \rangle = [ii] = 0$. The familiar Mandelstam variables can be expressed in terms of the spinor brackets as [DB: Fix signs]

$$\begin{aligned} s &\equiv (p_1 + p_2)^2 = 2p_1 p_2 = \langle 12 \rangle [12], \\ t &\equiv (p_2 + p_3)^2 = 2p_2 p_3 = \langle 23 \rangle [23], \\ u &\equiv (p_1 + p_3)^2 = 2p_1 p_3 = \langle 13 \rangle [13]. \end{aligned} \quad (\text{C.5})$$

In fact, the kinematic data of all massless amplitudes can be written in terms of only these brackets. As we will see, scattering amplitudes for massless particles simplify dramatically when written in terms of spinor helicity variables.

C.2 Polarization and Little Group

The use of spinor brackets makes the Lorentz symmetry of the amplitudes manifest. There is a subset of Lorentz transformations, called the *little group*, which leave the momentum p_μ of a particle invariant. Heuristically, we can think of these transformations as rotations in the plane transverse to the particle’s trajectory. The finite-dimensional representations of the little group are eigenvectors of this rotation group and the corresponding eigenvalues are the *helicities*.

Given the definition (C.2), it is easy to see that the four-momentum is invariant under the following little group transformation of the spinors:

$$\lambda_i \rightarrow t_i \lambda_i \quad \text{and} \quad \tilde{\lambda}_i \rightarrow t_i^{-1} \tilde{\lambda}_i. \quad (\text{C.6})$$

Not everything is invariant under this transformation however. Polarization vectors transform nontrivially under the little group rescaling, which leads to nontrivial constraints on fixed-helicity amplitudes.

In terms of spinors, the polarization vectors can be written as

$$e_{\alpha\dot{\alpha}}^+ \equiv \frac{\eta_\alpha \tilde{\lambda}_{\dot{\alpha}}}{\langle \eta \lambda \rangle} \quad \text{and} \quad e_{\alpha\dot{\alpha}}^- \equiv \frac{\lambda_\alpha \tilde{\eta}_{\dot{\alpha}}}{[\tilde{\lambda} \tilde{\eta}]}, \quad (\text{C.7})$$

where the superscripts denote the helicity and $\eta_\alpha, \tilde{\eta}_{\dot{\alpha}}$ are arbitrary reference spinors. It can be shown that a redefinition of the reference spinors corresponds to a gauge transformation of the corresponding polarization vectors. Under the little group, the polarization vectors transform as

$$e_{\alpha\dot{\alpha}}^+ \rightarrow t^{-2} e_{\alpha\dot{\alpha}}^+ \quad \text{and} \quad e_{\alpha\dot{\alpha}}^- \rightarrow t^2 e_{\alpha\dot{\alpha}}^-. \quad (\text{C.8})$$

These transformations imply a specific little group covariance for fixed-helicity amplitudes.

We define an n -particle *helicity-amplitude* as

$$A(1^{h_1} \dots n^{h_n}) = e_{\mu_1}^{h_1} \dots e_{\mu_n}^{h_n} A^{\mu_1 \dots \mu_n}, \quad (\text{C.9})$$

where $h_i = \pm 1$ is the helicity of each particle. The tensorial amplitude $A^{\mu_1 \dots \mu_n}$ is the complicated object that Feynman diagrams try to compute, while in the bootstrap approach we try to constraint the fixed-helicity amplitude $A(1^{h_1} \dots n^{h_n})$ directly. Under the little group, the latter must transform as

$$A(1^{h_1} \dots n^{h_n}) \rightarrow \prod_i t_i^{-2h_i} A(1^{h_1} \dots n^{h_n}). \quad (\text{C.10})$$

Note that for real kinematics the parameters t_i are pure phases and hence they disappear in the square of the amplitude (which is all that can be observed). Nevertheless, little group covariance is an important constraint on the form of the fixed-helicity amplitudes.

C.3 Three-Particle Amplitudes

We will first show that all three-particle amplitudes are uniquely fixed by Lorentz invariance and little group covariance.

First notice that momentum conservation, $p_1 + p_2 + p_3 = 0$, implies

$$\begin{aligned} (p_1 + p_2)^2 &= \langle 12 \rangle [12] = p_3^2 = 0, \\ (p_2 + p_3)^2 &= \langle 23 \rangle [23] = p_1^2 = 0, \\ (p_3 + p_1)^2 &= \langle 31 \rangle [31] = p_2^2 = 0. \end{aligned} \quad (\text{C.11})$$

In the following exercise, you will show that this leads to two possible kinematic configurations⁶²

$$\begin{aligned} [12] &= [23] = [31] = 0 \quad \Rightarrow \quad \tilde{\lambda}_1 \propto \tilde{\lambda}_2 \propto \tilde{\lambda}_3, \\ \text{or} \quad \langle 12 \rangle &= \langle 23 \rangle = \langle 31 \rangle = 0 \quad \Rightarrow \quad \lambda_1 \propto \lambda_2 \propto \lambda_3, \end{aligned} \quad (\text{C.12})$$

⁶²This requires complex momenta, so that λ_i and $\tilde{\lambda}_i$ are independent. For real momenta, three-particle amplitudes actually vanish.

called holomorphic and anti-holomorphic, respectively. The expressions for three-particle amplitudes can therefore only depend on angle or square brackets, but not both simultaneously.

Exercise C.1 Show that (C.11) implies (C.12). You will have to use the Schouten identity

$$\langle ij \rangle \lambda_k + \langle ki \rangle \lambda_j + \langle jk \rangle \lambda_i = 0, \quad (\text{C.13})$$

which reflects the fact that, for two-dimensional spinors, any three spinors must be linearly dependent.

For holomorphic kinematics, we write the helicity amplitudes as

$$A(1^{h_1} 2^{h_2} 3^{h_3}) = \langle 12 \rangle^{n_3} \langle 23 \rangle^{n_1} \langle 31 \rangle^{n_2}. \quad (\text{C.14})$$

This expression is little group covariant iff

$$\begin{aligned} n_1 &= h_1 - h_2 - h_3, \\ n_2 &= h_2 - h_3 - h_1, \\ n_3 &= h_3 - h_1 - h_2. \end{aligned} \quad (\text{C.15})$$

Ignoring the dimensionality of the coupling constants, the mass dimension of the amplitudes are

$$[A(1^{h_1} 2^{h_2} 3^{h_3})] = n_1 + n_2 + n_3 = -(h_1 + h_2 + h_3) \equiv -h. \quad (\text{C.16})$$

By locality, this must be non-negative, so we have $h \leq 0$. For anti-holomorphic, we instead find

$$A(1^{h_1} 2^{h_2} 3^{h_3}) = [12]^{-n_3} [23]^{-n_1} [31]^{-n_2}, \quad (\text{C.17})$$

where the exponents n_i are the those defined in (C.15). This corresponds to a local amplitude iff $h \geq 0$. In summary, we therefore have

$$A(1^{h_1} 2^{h_2} 3^{h_3}) = \begin{cases} \langle 12 \rangle^{h_3 - h_1 - h_2} \langle 23 \rangle^{h_1 - h_2 - h_3} \langle 31 \rangle^{h_2 - h_3 - h_1} & h \leq 0, \\ [12]^{h_1 + h_2 - h_3} [23]^{h_2 + h_3 - h_1} [31]^{h_3 + h_1 - h_2} & h \geq 0. \end{cases} \quad (\text{C.18})$$

Using symmetry alone, we have been able to determine the three-particle amplitudes of all massless particles with spin. The results are valid nonperturbatively.

Vectors Consider three identical spin-1 particles with helicities $h_i = \pm 1$. The exponents in (C.18) at then odd integers. This implies that the amplitude is anti-symmetric under the exchange of any two particles, which is inconsistent with spin-1 particles being bosons. The amplitude must therefore vanish. We have discovered that photons don't have any self-interactions.

To obtain a non-zero result, we add a color index to each vector. The three-particle amplitudes corresponding to the interactions between multiple species of spin-1 particles (gluons) then are

$$\begin{aligned} A(1_a^- 2_b^- 3_c^+) &= f_{abc} \frac{\langle 12 \rangle^3}{\langle 13 \rangle \langle 32 \rangle}, \\ A(1_a^+ 2_b^+ 3_c^-) &= f_{abc} \frac{[12]^3}{[13][32]}, \end{aligned} \quad (\text{C.19})$$

where the structure constants f_{abc} are anti-symmetric in its indices, so that the amplitude is symmetric under the exchange of any two particles. If all helicities are negative or positive, we obtain

$$\begin{aligned} A(1_a^- 2_b^- 3_c^-) &= f_{abc} \langle 12 \rangle \langle 23 \rangle \langle 31 \rangle, \\ A(1_a^+ 2_b^+ 3_c^+) &= f_{abc} [12] [23] [31]. \end{aligned} \quad (\text{C.20})$$

These amplitudes come from the three-derivative interactions among the gluons, $f_{abc} F_\mu^{a\nu} F_\nu^{b\sigma} F_\sigma^{c\mu}$.

Tensors Next, we consider the case of three identical spin-2 particles (gravitons) with helicities $h_i = \pm 2$. The exponents in (C.18) are then even integers and the amplitude is symmetric under the exchange of any two particles. A single species of gravitons (without colors) therefore has nonvanishing three-particle amplitudes

$$\begin{aligned} A(1^- 2^- 3^+) &= \frac{\langle 12 \rangle^6}{\langle 13 \rangle^2 \langle 32 \rangle^2}, \\ A(1^+ 2^+ 3^-) &= \frac{[12]^6}{[13]^2 [32]^2}. \end{aligned} \quad (\text{C.21})$$

Note that the dependence on the spinor brackets is precisely the square of that found in (C.19) for massless spin-1 particles. This is a baby version of the so-called *double copy* relation between gravity and Yang-Mills amplitudes.

Similarly, if all helicities are negative or positive, we obtain the square of (C.20):

$$\begin{aligned} A(1^- 2^- 3^-) &= \langle 12 \rangle^2 \langle 23 \rangle^2 \langle 31 \rangle^2, \\ A(1^+ 2^+ 3^+) &= [12]^2 [23]^2 [31]^2. \end{aligned} \quad (\text{C.22})$$

These amplitudes come from the six-derivative interactions among the gravitons living inside the cubic curvature correction,⁶³ $R_{\mu\nu}^{\sigma\tau} R_{\sigma\tau}^{\alpha\beta} R_{\alpha\beta}^{\mu\nu}$.

C.4 Four-Particle Test

We have seen that all three-particle amplitudes are fixed purely by kinematics. Dynamical constraints only arise when we try to combine these three-particle amplitudes into higher-point amplitudes. It is not guaranteed that this can be done in a way that is consistent with locality. Imposing consistency with locality selects a smaller subset of allowed theories and rules out other hypothetical theories.

Locality is reflected in the analytic structure of scattering amplitudes. At tree level (i.e. leading order in the perturbative expansion), amplitudes can have *poles* when intermediate particles go on-shell and satisfy $p_I^2 = 0$. These poles must be *simple poles*, since higher-order poles in momentum space translate to non-locality in position space. For example, in the s -channel, the limit $s \rightarrow 0$ can diverge as $1/s$, but not $1/s^2$.

On these poles, the amplitudes must *factorize* into a product of lower-point amplitudes. This is easy to understand physically: when an intermediate particle goes on-shell it becomes long-lived and can propagate a large distance in spacetime before decaying. This propagation connects

⁶³Curvature-squared operators do not contribute to the tree-level amplitudes.

particles at separated points in spacetime, which explains the factorization of the corresponding scattering amplitude.

The first non-trivial example is the scattering of four particles. In that case, poles can appear in three different kinematic limits: the s , t , and u -channels. Locality then fixes the residues of these poles in terms of products of the known three-particle amplitudes:

$$\begin{aligned}\lim_{s \rightarrow 0} A_4 &= \frac{A_3 \times A_3}{s} \equiv \frac{R_s}{s}, \\ \lim_{t \rightarrow 0} A_4 &= \frac{A_3 \times A_3}{t} \equiv \frac{R_t}{t}, \\ \lim_{u \rightarrow 0} A_4 &= \frac{A_3 \times A_3}{u} \equiv \frac{R_u}{u}.\end{aligned}\tag{C.23}$$

The challenge is to find a function that satisfies all these limits simultaneously. This *four-particle test* turns out to be a surprisingly stringent constraint on consistent theories of Nature [40].

For scalar particles, it is trivial to find an amplitude that satisfies the four-particle test. In that case, the three-particle amplitude is a constant, $A_3 = g$, and a consistent four-particle amplitude is simply the sum of the different channels

$$A_4 = g^2 \left[\frac{1}{s} + \frac{1}{t} + \frac{1}{u} \right],\tag{C.24}$$

where we could add an arbitrary polynomial function $P(s, t, u)$ —corresponding to higher-derivative contact interactions—without changing the limits in (C.23).

For particles with spin, however, the different channels are *not* independent and satisfying the four-particle test becomes much more non-trivial. We will now illustrate this for the case of identical spin- S particles. Further examples can be explored in the exercises at the end of this appendix.

Consider the four-particle amplitude corresponding to two incoming particles with negative helicities and two outgoing particles with positive helicities, $A_4(1^- 2^- 3^+ 4^+)$. The residue in the s -channel is

$$R_s = \sum_h A_3(1^- 2^- I^h) \times A_3(I^{-h} 3^+ 4^+),\tag{C.25}$$

where we have summed over the helicities of the intermediate particle.⁶⁴ Using (C.18), we get

$$R_s = \left(\frac{\langle 12 \rangle^3}{\langle 1I \rangle \langle I2 \rangle} \right)^S \left(\frac{[34]^3}{[3I][I4]} \right)^S,\tag{C.26}$$

where we have only shown the contribution coming from $A_3(1^- 2^- I^+) A_3(I^- 3^+ 4^+)$.

⁶⁴Note that the intermediate particle I must have opposite helicity on the two factors in (C.25). Mathematically, this is because the final object has vanishing helicity weight in I . Physically, the intermediate particle is outgoing on the first factor and incoming on the second factor. By convention, we take all particles to be incoming, so the helicity on the outgoing leg must be flipped.

Exercise C.2 Using $p_I = p_1 + p_2 = -(p_3 + p_4)$, show that

$$\frac{\langle 12 \rangle^3}{\langle 1I \rangle \langle I2 \rangle} \frac{[34]^3}{[3I][I4]} = \frac{\langle 12 \rangle^2 [34]^2}{u}, \quad (\text{C.27})$$

where $u = \langle 13 \rangle [13]$ is the Mandelstam invariant.

Equation (C.26) then becomes

$$R_s = (\langle 12 \rangle^2 [34]^2)^S \times \frac{1}{u^S}. \quad (\text{C.28})$$

We see that the s-channel residue has a u -channel pole. For $S > 1$, this is a higher-order pole that should make us nervous. A similar analysis of the t and u -channel residues gives

$$\begin{aligned} R_t &= (\langle 12 \rangle^2 [34]^2)^S \times \frac{1}{s^S}, \\ R_u &= (\langle 12 \rangle^2 [34]^2)^S \times \frac{1}{t^S}. \end{aligned} \quad (\text{C.29})$$

The challenge is to find a function that has the correct residues in all channels. Unlike that case of the scalar four-particle amplitude in (C.24), for particles with spin we can't just add the three different limits. Instead, we must find a function that unifies these limits in a more non-trivial way.

Photons Let us begin with the case of massless spin-1 particles. We call them *photons*. A natural ansatz to try for the four-point functions is the sum of the s , t and u -channel limits with arbitrary coefficients:

$$A_4 = \langle 12 \rangle^2 [34]^2 \left[\frac{c_s}{su} + \frac{c_t}{ts} + \frac{c_u}{ut} \right]. \quad (\text{C.30})$$

The s-channel residue of this ansatz is

$$R_s = \langle 12 \rangle^2 [34]^2 \left[\frac{c_s}{u} + \frac{c_t}{t} \right] = \langle 12 \rangle^2 [34]^2 \frac{c_s - c_t}{u}, \quad (\text{C.31})$$

where we used that $s + t + u = 0$ implies $t = -u$ if $s = 0$. Comparing this to (C.28), we find that $c_s - c_t = 1$. Repeating this exercise for the t and u -channels, we are led to the following set of constraints:

$$\begin{aligned} c_s - c_t &= 1, \\ c_t - c_u &= 1, \\ c_u - c_s &= 1. \end{aligned} \quad (\text{C.32})$$

Adding these three expressions, however, we find a contradiction

$$0 = 3. \quad (\text{C.33})$$

We conclude that there is no function that consistently factorizes in all channels. We have therefore discovered that photons are not self-interacting.

Gluons To obtain a non-trivial four-particle amplitude, we consider multiple species of spin-1 particles. We call them *gluons*. We consider the amplitude $A(1_a^- 2_b^- 3_c^+ 4_d^+)$. Using (C.19), the *s*-channel residue is

$$\begin{aligned} R_s &= \sum_h \sum_e A_3(1_a^- 2_b^- I_e^h) \times A_3(I_e^{-h} 3_c^+ 4_d^+) \\ &= \sum_e f_{abe} f_{ecd} \frac{\langle 12 \rangle^3}{\langle 1I \rangle \langle I2 \rangle} \frac{[34]^3}{[3I][I4]} = \sum_e f_{abe} f_{ecd} \frac{\langle 12 \rangle^2 [34]^2}{u}. \end{aligned} \quad (\text{C.34})$$

Comparing this to (C.31), we have $c_s - c_t = \sum_e f_{abe} f_{ecd}$. Repeating this exercise for the *t* and *u*-channels, we now find the following set of constraints:

$$\begin{aligned} c_s - c_t &= \sum_e f_{abe} f_{ecd}, \\ c_t - c_u &= \sum_e f_{bce} f_{ead}, \\ c_u - c_s &= \sum_e f_{cae} f_{ebd}. \end{aligned} \quad (\text{C.35})$$

Adding these three expressions, we find

$$0 = \sum_e (f_{abe} f_{ecd} + f_{bce} f_{ead} + f_{cae} f_{ebd}), \quad (\text{C.36})$$

which is the *Jacobi identity*. We see that gluons can have a consistent four-particle amplitude if the structure constant f_{abc} satisfies the Jacobi identity.

Gravitons Finally, we consider the case of massless spin-2 particles. We call them *gravitons*. The *s*-channel residue in this case is

$$R_s = \langle 12 \rangle^4 [34]^4 \times \frac{1}{u^2}. \quad (\text{C.37})$$

At first sight, the second-order pole at $u = 0$ looks like trouble. However, then we remember that $u = -t$ when $s = 0$, so that (C.37) can also be written as $R_s \propto 1/(tu)$. A similar analysis for the *t* and *u*-channel residues gives $R_t \propto 1/(su)$ and $R_u \propto 1/(st)$. The unique function with the correct residues in all channels and only simple poles is

$$A_4 = -\frac{\langle 12 \rangle^4 [34]^4}{stu}. \quad (\text{C.38})$$

We see that gravity has just barely escaped the perils of non-locality. For higher-spin massless particles with $S > 2$ it will not be possible. Theories of massless higher-spin particles are inconsistent in flat space.

Couplings to matter Applying the four-particle test to the couplings to matter leads to additional interesting constraints on theories of massless particles with spin. We will explore this in the following exercises.

Exercise C.3 Consider the Compton scattering of massless spin- S particles with a massless spin-1 particle. Show that this fails the four-particle test if $S \geq 3/2$. We have therefore discovered that Nature doesn't allow for charged massless particles with spin $S \geq 3/2$.

Exercise C.4 Consider the coupling between a species of massless spin-1 particles, A_a^μ , and two scalars ϕ_i and ϕ_j (where the subscript denotes the species of the scalar). The coupling constant associated to this interaction vertex is T_{ij}^a and the spin-1 particle has the self-coupling f^{abc} . Show that this theory only passes the four-particle test for Compton scattering if the couplings satisfy the *Lie algebra*

$$[T^a, T^b]_{ij} = f^{abc} T_{ij}^c. \quad (\text{C.39})$$

We have discovered the structure of non-Abelian Yang-Mills theory.

Exercise C.5 Repeat Exercise C.3 for the case of massless spin- S particles coupled to a massless spin-2 particle. Show that this fails the four-particle test if $S \geq 5/2$. Nature doesn't have any massless particles with spin $S \geq 5/2$, because these particles cannot couple consistently to gravity.

Exercise C.6 Repeat Exercise C.4 for the coupling of a massless spin-2 particle to a species of scalars. Show that this only passes the four-particle test if the coupling is universal (i.e. the same for all species). We have discovered the *equivalence principle*.

Summary It is beautiful to see how the structure of the physical world is fixed by consistency with basic physical principles. In this appendix, we showed first that the three-point interactions of massless particles are fixed by Lorentz symmetry and little group scaling. We then saw that connecting these three-point interactions into consistent higher-point amplitudes can be non-trivial. Imposing locality—in the form of the four-particle test—led to the following interesting conclusions:

- A single species of massless spin-1 particles (photons) cannot have self-interactions.
- Multiple species of massless spin-1 particles (gluons) are only consistent if the three-point couplings f^{abc} satisfy the *Jacobi identity*.
- Massless spin-1 particles can only couple consistently to massless particles with spin less than $3/2$. There are no charged massless particles with spin $S \geq 3/2$.
- The coupling to multiple spin- S particles if the three-point couplings satisfy a *Lie algebra*.
- A single species of massless spin-2 particles (gravitons) is only consistent if the interactions are those of GR.

- Multiple species of massless spin-2 particles cannot interact with each other, i.e. live in separate worlds.
- The couplings of a massless spin-2 particle to different species of matter particles must be universal, i.e. satisfy the *equivalence principle*.

We find it remarkable how all of this follows from basic consistency of four-particle scattering.

D Conformal Field Theory

In this appendix, we review the basics of conformal field theory in $d > 2$ dimensions. We will follow mainly the excellent lecture notes by Alday [110]. Other standard references are [111–114].

D.1 Conformal Symmetry

A conformal transformation is a coordinate transformation $x^\mu \mapsto x'^\mu$ that leaves the metric invariant up to a local rescaling,

$$g_{\mu\nu}(x) \mapsto g'_{\mu\nu}(x') = \Omega(x)g_{\mu\nu}(x). \quad (\text{D.1})$$

The special case $\Omega(x) = 1$ is an *isometry*, while $\Omega(x) = \text{const}$ corresponds to a *scale transformation* (dilatation). For flat space, $g_{\mu\nu} = \eta_{\mu\nu}$, the group of isometries are translations and (Lorentz) rotations. Locally, conformal transformations therefore look like a scale transformation combines with a translation and rotation. It therefore preserves the angles between intersecting curves.

Exercise D.1 Consider an infinitesimal coordinate transformation, $x^\mu \mapsto x'^\mu = x^\mu + \epsilon^\mu(x)$, so that the metric transforms as

$$g_{\mu\nu} \mapsto g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) + \mathcal{O}(\epsilon^2), \quad (\text{D.2})$$

Show that this is a conformal transformation if

$$\epsilon_\mu(x) = a_\mu + r_{\mu\nu}x^\nu + \lambda x_\mu + x^2 b_\mu - 2(b \cdot x)x_\mu, \quad (\text{D.3})$$

where $r_{\mu\nu} = -r_{\nu\mu}$.

The transformations in (D.3) correspond to infinitesimal translations (a_μ), rotations ($r_{\mu\nu}$), dilatations (λ) and special conformal transformation (b_μ). Exponentiation leads to the following finite transformations:

$$\text{T: } x'_\mu = x_\mu + a_\mu \quad \Omega(x) = 1, \quad (\text{D.4})$$

$$\text{R: } x'_\mu = R_{\mu\nu}x^\nu \quad \Omega(x) = 1, \quad (\text{D.5})$$

$$\text{D: } x'_\mu = \lambda x_\mu \quad \Omega(x) = \lambda^{-2}, \quad (\text{D.6})$$

$$\text{SCT: } x'_\mu = \frac{x_\mu - b_\mu x^2}{1 - 2 b \cdot x + b^2 x^2} \quad \Omega(x) = 1 - 2 b \cdot x + b^2 x^2. \quad (\text{D.7})$$

Note that the special conformal transformation can also be written as

$$\frac{x'_\mu}{x'^2} = \frac{x_\mu}{x^2} - b_\mu, \quad (\text{D.8})$$

i.e. it can be thought of as a translation, preceded and followed by an inversion.

Next, we will show how the conformal transformations act on fields. For concreteness, we consider a scalar field $\phi(x)$. Under a general coordinate transformation, the transformed field

satisfies

$$\phi'(x') = \phi(x). \quad (\text{D.9})$$

An infinitesimal transformation can be written as

$$x'^\mu = x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a}, \quad (\text{D.10})$$

where ω_a is a small parameter. The role of the index a will become clear momentarily. The change in the transformed field (at a fixed position x) is

$$\begin{aligned} \delta\phi(x) &\equiv \phi'(x) - \phi(x) \\ &= \phi\left(x - \omega_a \frac{\delta x}{\delta \omega_a}\right) - \phi(x) = -\omega_a \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu \phi(x) \equiv -i\omega_a G_a \phi(x), \end{aligned} \quad (\text{D.11})$$

where we have defined the *generator* of the transformation:

$$G_a \equiv -i \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu. \quad (\text{D.12})$$

As a concrete example, consider the infinitesimal translation $x'^\mu = x^\mu + \omega^\mu$. The index a on ω_a is then simply the spacetime index and $\delta x^\mu / \delta \omega^\nu = \delta_\nu^\mu$. The generator of translations then takes the familiar form

$$P_\mu = -i \partial_\mu. \quad (\text{D.13})$$

An infinitesimal Lorentz transformation can be written as $x'^\mu = x^\mu + \eta^{\mu\alpha} \omega_{\alpha\nu} x^\nu$, where $\omega_{\alpha\nu} = -\omega_{\nu\alpha}$. The index on ω_a now represents $a = \alpha\nu$ and

$$\frac{\delta x^\mu}{\delta \omega_{\alpha\nu}} = \frac{1}{2} (\eta^{\alpha\mu} x^\nu - \eta^{\nu\mu} x^\alpha). \quad (\text{D.14})$$

Substituting this into (D.12) gives the generator of Lorentz transformations

$$J_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu). \quad (\text{D.15})$$

Exercise D.2 Repeating the same analysis for dilatation and special conformal transformations, show that the generators of the conformal group are

$$\begin{aligned} \text{T: } P_\mu &= -i \partial_\mu, \\ \text{R: } J_{\mu\nu} &= i(x_\mu \partial_\nu - x_\nu \partial_\mu), \\ \text{D: } \hat{D} &= -ix^\mu \partial_\mu, \\ \text{SCT: } K_\mu &= -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu). \end{aligned} \quad (\text{D.16})$$

It is a straightforward (but tedious) exercise to show that these generators obey the following

conformal algebra

$$\begin{aligned} [J_{\mu\nu}, J_{\rho\sigma}] &= \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\rho} J_{\nu\sigma} + \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\nu\sigma} J_{\mu\rho}, & [\hat{D}, P_\mu] &= P_\mu, \\ [J_{\mu\nu}, K_\rho] &= \eta_{\nu\rho} K_\mu - \eta_{\mu\rho} K_\nu, & [\hat{D}, K_\mu] &= -K_\mu, \\ [J_{\mu\nu}, P_\rho] &= \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu, & [K_\mu, P_\nu] &= 2\eta_{\mu\nu} \hat{D} - 2J_{\mu\nu}. \end{aligned} \quad (\text{D.17})$$

What subgroups does the conformal group have?

Next, we would like to see how the conformal generators act on local operators in a quantum field theory. In the Heisenberg picture, the evolution of a scalar operator can be written as⁶⁵

$$O(x) = e^{-iPx} O(0) e^{iPx}. \quad (\text{D.18})$$

Let us work out how the conformal generators act on $O(x)$.

The spacetime derivative of (D.18) is

$$\partial_\mu O(x) = e^{-iPx} (-iP_\mu O(0) + O(0)iP_\mu) e^{iPx} = -i[P_\mu, O(x)], \quad (\text{D.19})$$

so that the action of the generator P_μ is

$$[P_\mu, O(x)] = i\partial_\mu O(x). \quad (\text{D.20})$$

To find the actions of the other generators, we first define their actions at the origin:⁶⁶

$$\begin{aligned} [\hat{D}, O(0)] &= i\Delta O(0), \\ [J_{\mu\nu}, O(0)] &= 0, \\ [K_\mu, O(0)] &= 0, \end{aligned} \quad (\text{D.22})$$

where Δ is the *scaling dimension* of the operator. Equation (D.22) is the definition of a *primary operator* (more on this in Section D.2). Combining this with (D.19) allows us to derive the action of the conformal generators at a general spacetime point.

Exercise D.3 Show that

$$\begin{aligned} [P_\mu, O(x)] &= i\partial_\mu O(x), \\ [\hat{D}, O(x)] &= i(\Delta + x^\mu \partial_\mu) O(x), \\ [J_{\mu\nu}, O(x)] &= -i(x^\mu \partial_\nu - x_\nu \partial_\mu) O(x), \\ [K_\mu, O(x)] &= 2ix_\mu \Delta O(x) + i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu) O(x). \end{aligned} \quad (\text{D.23})$$

[DB: finite transformations here?]

⁶⁵Operators with spin would have more than one component $O_\alpha(x)$.

⁶⁶For operators with spin, the action of the Lorentz generator is

$$[J_{\mu\nu}, O_\alpha(0)] = i(S_{\mu\nu})^\beta_\alpha O_\beta(0), \quad (\text{D.21})$$

where $S_{\mu\nu}$ is a matrix representing the spin of the field.

D.2 Conformal Representations

Now that we have acquainted ourselves with the conformal algebra, we would like to study its representations. We have already built some of the groundwork by analyzing the transformation properties of local operators under conformal transformations. The construction of conformal representations somewhat parallels the construction of irreps of the Poincaré algebra. We first consider a representation of the isotropy subalgebra $\{\hat{D}, K_\mu, J_{\mu\nu}\}$ that preserve the origin, and then find the representation of the full algebra induced by this.

Local operators at a point furnish a representation of the conformal algebra. Looking at (D.17), we see that K_μ acts a lowering operator for conformal dimension

$$\hat{D}K_\mu O(0) = K_\mu \hat{D}O(0) + [\hat{D}, K_\mu]O(0) = (\Delta - 1)K_\mu O(0). \quad (\text{D.24})$$

Since the conformal dimension is something like an energy, we might expect that in physically relevant theories the conformal dimension will be bounded below. This is indeed the case in almost all situations of interest.⁶⁷ If we keep applying K_μ to lower Δ , at some point we must reach an operator which is annihilated by K_μ :

$$[K_\mu, O(0)] = 0. \quad (\text{D.25})$$

We call such operators *primary*. Conformal representations are then built from these lowest-weight states by acting with P_μ , which acts as a raising operator:

$$\hat{D}P_\mu O(0) = P_\mu \hat{D}O(0) + [\hat{D}, P_\mu]O(0) = (\Delta + 1)P_\mu O(0). \quad (\text{D.26})$$

Note that in addition to increasing the conformal dimension by 1, the raising operator P_μ increases the spin of the operator $O(0)$. Operators obtained by acting on a primary operator with P_μ are called *descendants*. A primary plus all its descendants provides an irreducible representation of the conformal algebra. We have constructed this representation at a point, in order to make contact with the normal way we think about field theory, note that the operator at a general spacetime point follows from (D.18), which leads to a complicated combination of many descendant operators of $O(0)$.

[DB: add comments on spin and unitarity]

D.3 Constraints of Conformal Invariance

In the following, we will describe some of the powerful constraints that conformal invariance imposed on the structure of a theory. [DB: more]

Stress tensor We have seen that under an infinitesimal coordinate transformation, $x^\mu \mapsto x'^\mu = x^\mu + \epsilon^\mu(x)$, the metric transforms as

$$g_{\mu\nu} \mapsto g'_{\mu\nu} = g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu). \quad (\text{D.27})$$

⁶⁷There are some notable exceptions: in particular the conformal representations that seem to have something to do with de Sitter physics have complex conformal weights and so are not highest-weight representations.

The action of a theory then changes as

$$\delta S = -\frac{1}{2} \int d^d x T^{\mu\nu} (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu), \quad (\text{D.28})$$

where $T_{\mu\nu}$ is the stress tensor. If the theory is invariant under translations, $\epsilon^\mu = \text{const}$, then the stress tensor is conserved, $\partial_\mu T^{\mu\nu} = 0$. A conformal transformation satisfies $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x) \eta_{\mu\nu}$, so that (D.28) becomes

$$\delta S = -\frac{1}{2} \int d^d x f(x) T^\mu{}_\mu. \quad (\text{D.29})$$

For a scale-invariant theory, with $f(x) = \text{const}$, the condition $\delta S = 0$ implies

$$T^\mu{}_\mu = \partial_\nu J^\nu, \quad (\text{D.30})$$

where J^μ is called the *virial current*. A theory is conformally invariant, with general $f(x)$, if the trace of the stress tensor vanishes $T^\mu{}_\mu = 0$.

Exercise D.4 Consider the action of a free scalar field

$$S = -\frac{1}{2} \int d^d x \partial_\mu \phi \partial^\mu \phi. \quad (\text{D.31})$$

Show that the trace of the corresponding stress tensor is

$$T^\mu{}_\mu = -\frac{d-2}{2} \partial_\mu (\phi \partial^\mu \phi). \quad (\text{D.32})$$

This vanishes—and hence the theory is conformal—for $d = 2$. Even for $d \neq 2$, we can define an “improved” stress $\Theta^{\mu\nu} \equiv T^{\mu\nu} + \partial_\rho \Gamma^{\rho\mu\nu}$, with $\Gamma^{\rho\mu\nu} = -\Gamma^{\mu\rho\nu}$, whose trace vanishes as well.

Exercise D.5 Consider the Maxwell action

$$S = -\frac{1}{4} \int d^d x F_{\mu\nu} F^{\mu\nu}. \quad (\text{D.33})$$

Show that the trace of the corresponding stress tensor vanishes for $d = 4$ (i.e. in our world). This time, we can’t find an improved stress tensor with vanishing trace in general dimensions.

Conformal correlators Next, we will consider the constraints imposed by conformal invariance on the N -point functions of a quantum field theory.

To describe finite conformal transformations, it is useful to introduce the *Jacobian* of the transformation

$$\left| \frac{\partial x'}{\partial x} \right| = \frac{1}{\sqrt{\det g'_{\mu\nu}}} = \Omega(x)^{-d/2}, \quad (\text{D.34})$$

where $\Omega(x)$ is the scaling factor introduced in (D.1). Recall that $\Omega(x) = \lambda^{-1}$ for a dilatation and $\Omega(x) = 1 - 2 b \cdot x + b^2 x^2$ for a special conformal transformation. A *primary field* transforms as

$$\phi(x) \mapsto \phi(x') = \left| \frac{\partial x'}{\partial x} \right|^{-\Delta/d} \phi(x) = \Omega(x)^{\Delta/2} \phi(x), \quad (\text{D.35})$$

where Δ is the scaling dimension of ϕ . Correlators in a CFT must then satisfy

$$\langle \phi_1(x'_1) \dots \phi_n(x'_n) \rangle = \Omega(x_1)^{\Delta_1/2} \dots \Omega(x_N)^{\Delta_N/2} \langle \phi_1(x_1) \dots \phi_n(x_n) \rangle. \quad (\text{D.36})$$

We will now prove the famous facts that the two- and three-point functions of primary operators in a CFT are completely fixed by symmetry [115], while four-point functions depend on a free function of two variables (the conformal cross ratios).

Two-point functions We first consider the form of the two-point function of two scalar fields (or operators) ϕ_1 and ϕ_2 . Invariance under translations and rotations implies that the two-point function can only be a function of $|x_1 - x_2|$, i.e.

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = f(|x_1 - x_2|). \quad (\text{D.37})$$

Specializing to dilatations, the identity (E.244) implies that $f(\lambda x) = \lambda^{-\Delta_1 - \Delta_2} f(x)$ and hence

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \frac{c_{12}}{|x_1 - x_2|^{\Delta_1 + \Delta_2}}, \quad (\text{D.38})$$

where c_{12} is a constant coefficient, which can be set to $c_{12} \equiv 1$ by a constant rescaling of the operators. To impose the constraints of special conformal invariance, it is useful to first note that the distance between two points transforms as

$$|x_n - x_m| \mapsto |x'_n - x'_m| = \frac{|x_n - x_m|}{(\Omega_n \Omega_m)^{1/2}}, \quad (\text{D.39})$$

where $\Omega_n = \Omega(x_n)$. For a SCT, the constraint (E.244) then implies

$$\frac{1}{|x'_1 - x'_2|^{\Delta_1 + \Delta_2}} = \frac{(\Omega_1 \Omega_2)^{(\Delta_1 + \Delta_2)/2}}{|x_1 - x_2|^{\Delta_1 + \Delta_2}} = \frac{\Omega_1^{\Delta_1} \Omega_2^{\Delta_2}}{|x_1 - x_2|^{\Delta_1 + \Delta_2}}, \quad (\text{D.40})$$

where we have used (D.39) in the first equality. We see that this is only satisfied if $\Delta_1 = \Delta_2$, so that

$$\langle \phi_1(x_1) \phi_2(x_2) \rangle = \frac{1}{|x_1 - x_2|^{2\Delta_1}} \delta_{\Delta_1, \Delta_2}. \quad (\text{D.41})$$

The two-point function of scalar operator is fixed completely in terms of the scaling dimensions of the operators (and is only nonzero if the dimensions of the two operators are equal).

Three-point functions Invariance under rotations, translations, and dilatations force three-point functions to have the form

$$\langle \phi_1 \phi_2 \phi_3 \rangle = \frac{c_{123}}{x_{12}^a x_{23}^b x_{31}^c}, \quad (\text{D.42})$$

where $\phi_n \equiv \phi_n(x_n)$, $x_{nm} \equiv |x_n - x_m|$ and

$$a + b + c = \Delta_1 + \Delta_2 + \Delta_3. \quad (\text{D.43})$$

Invariance under SCTs demands that

$$\frac{c_{123}}{|x_{12}|^a|x_{23}|^b|x_{13}|^c} = \frac{(\Omega_1\Omega_2)^{a/2}(\Omega_2\Omega_3)^{b/2}(\Omega_1\Omega_3)^{c/2}}{\Omega_1^{\Delta_1}\Omega_2^{\Delta_2}\Omega_3^{\Delta_3}} \frac{c_{123}}{|x_{12}|^a|x_{23}|^b|x_{13}|^c}, \quad (\text{D.44})$$

which leads to

$$\begin{aligned} a &= \Delta_1 + \Delta_2 - \Delta_3, \\ b &= \Delta_2 + \Delta_3 - \Delta_1, \\ c &= \Delta_3 + \Delta_1 - \Delta_2, \end{aligned} \quad (\text{D.45})$$

and hence

$$\langle\phi_1\phi_2\phi_3\rangle = \frac{c_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{23}^{\Delta_2+\Delta_3-\Delta_1}x_{31}^{\Delta_3+\Delta_1-\Delta_2}}. \quad (\text{D.46})$$

We see that the form of three-point functions is completely fixed by conformal invariance.

Four-point functions [DB: more] The four-point function of scalar operators is constrained to take the form

$$\langle\phi_1\phi_2\phi_3\phi_4\rangle = G(u, v) \prod_{n < m}^4 x_{nm}^{\Delta/3 - \Delta_n - \Delta_m}, \quad (\text{D.47})$$

where $\Delta \equiv \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4$ and we have introduced the conformally-invariant ‘‘cross ratios’’

$$u \equiv \left(\frac{x_{12}x_{34}}{x_{13}x_{24}} \right)^2, \quad v \equiv \left(\frac{x_{12}x_{34}}{x_{23}x_{14}} \right)^2. \quad (\text{D.48})$$

[DB: fix; coordinate with conformal bootstrap discussion] Notice that the left-hand side of (D.47) is invariant under permutations of all of the operators, but the right-hand side is not manifestly invariant. Consider swapping the points 2 and 4, what constraint does this place on the function $G(u, v)$?

D.4 OPE and Conformal Blocks

[DB: intro]

Operator product expansion Consider the state $|\psi\rangle \equiv \phi_1(x)\phi_2(0)|0\rangle$, which can be written in terms of eigenstates of the dilatation operator, $|\psi\rangle = \sum_n c_n(x)|\Delta_n\rangle$. Since each state $|\Delta_n\rangle$ can be expressed as a linear combination of primary operators and a tower of descendants, we get

$$\phi_1(x)\phi_2(0)|0\rangle = \sum_{\Delta, \ell} C_{\Delta, \ell}(x, \partial)\phi_{\Delta, \ell}(0)|0\rangle, \quad (\text{D.49})$$

where $\phi_{\Delta, \ell}$ are primary operators of dimension Δ and spin ℓ . This *operator product expansion* (OPE) is valid inside any correlation function and only if any other operators $\phi_n(x_n)$ are separated by $|x_{2n}| \geq |x_{12}|$. Conformal invariance strongly restricts the form of the differential operator $C_{\Delta, \ell}(x, \partial)$ in (D.49).

Exercise D.6 Consider the following ansatz for the OPE:

$$\phi_1(x)\phi_2(0)|0\rangle = \sum_{\Delta,\ell} \frac{\text{const}}{x^{\Delta_1+\Delta_2-\Delta}} (1 + \alpha x^\mu \partial_\mu + \dots) \phi_{\Delta,\ell}(x) |0\rangle, \quad (\text{D.50})$$

where the overall scaling with x is fixed by dilatation symmetry. Show that special conformal invariance requires that

$$\alpha = \frac{\Delta_1 - \Delta_2 + \Delta}{2\Delta}. \quad (\text{D.51})$$

Higher-order terms are similarly fixed by conformal symmetry.

An efficient way to derive $C_{\Delta,\ell}(x, \partial)$ is to consider the three-point function $\langle \phi_1(x)\phi_2(0)\phi_\Delta(z) \rangle$ and write it in terms of the OPE of the first two operators:

$$\langle \phi_1(x)\phi_2(0)\phi_\Delta(z) \rangle = \sum_{\Delta'} C_{\Delta'}(x, \partial) \langle \phi_{\Delta'}(0)\phi_\Delta(z) \rangle \quad (\text{D.52})$$

$$= C_\Delta(x, \partial) \langle \phi_\Delta(0)\phi_\Delta(z) \rangle \quad (\text{D.53})$$

where we have suppressed the dependence on spin. Using the known form of $\langle \phi_\Delta(0)\phi_\Delta(z) \rangle$ and $\langle \phi_1(x)\phi_2(0)\phi_\Delta(z) \rangle$ —cf. (D.41) and (D.46)—and expanding the left-hand side around $x = 0$ fixes $C_{\Delta,\ell}(x, \partial)$ in terms of the operator dimensions and the coefficients $c_{\phi_1\phi_2\phi_\Delta}$. The latter are called OPE coefficients or structure constants.

Conformal blocks Consider the four-point function of identical scalar operators, which takes the form

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{G(u, v)}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}}, \quad (\text{D.54})$$

where u and v are the cross ratios defined in (D.48). [DB: more words] We then write the OPEs between the operators (12) and (34):

$$\phi(x_1)\phi(x_2) = \sum_{\Delta,\ell} c_{\Delta,\ell} C_{\Delta,\ell}(x_{12}, \partial_y) \phi_{\Delta,\ell}(y) \Big|_{y=\frac{x_1+x_2}{2}}, \quad (\text{D.55})$$

$$\phi(x_3)\phi(x_4) = \sum_{\Delta,\ell} c_{\Delta,\ell} C_{\Delta,\ell}(x_{34}, \partial_z) \phi_{\Delta,\ell}(z) \Big|_{z=\frac{x_3+x_4}{2}}, \quad (\text{D.56})$$

where the sum runs over all primary fields and $c_{\Delta,\ell} \equiv c_{\phi\phi\phi_{\Delta,\ell}}$ are the OPE coefficients. Substituting these OPEs into (D.54), we get

$$\langle \overline{\phi(x_1)\phi(x_2)} \overline{\phi(x_3)\phi(x_4)} \rangle = \sum_{\Delta,\ell} c_{\Delta,\ell}^2 [C_{\Delta,\ell}(x_{12}, \partial_y) C_{\Delta,\ell}(x_{34}, \partial_z) \langle \phi_{\Delta,\ell}(y)\phi_{\Delta,\ell}(z) \rangle], \quad (\text{D.57})$$

where the function in the square brackets is fixed by conformal symmetry. We define

$$C_{\Delta,\ell}(x_{12}, \partial_y) C_{\Delta,\ell}(x_{34}, \partial_z) \langle \phi_{\Delta,\ell}(y)\phi_{\Delta,\ell}(z) \rangle \equiv \frac{G_{\Delta,\ell}(u, v)}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}}, \quad (\text{D.58})$$

where the functions $G_{\Delta,\ell}(u, v)$ are called *conformal blocks*. This allows us to write the four-point function (D.54) as

$$G(u, v) = \sum_{\Delta, \ell} c_{\Delta, \ell}^2 G_{\Delta, \ell}(u, v). \quad (\text{D.59})$$

[DB: explain: conformal blocks are fixed by conformal symmetry and lower-point correlators]

D.5 Conformal Bootstrap

A conformal field theory is defined by a choice of operators (with dimensions Δ_i and spins ℓ_i), as well as a list of OPE coefficients c_{ijk} . The rest is dictated by conformal symmetry. Does every choice of this CFT data lead to a consistent theory? Can some theories be ruled out in the abstract?

Above we have shown that the four-point function of identical scalars can be written as an expansion in conformal partial waves using the OPEs for the OPEs for the operators (12) and (34). Diagrammatically, this expansion can be written as

$$\langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle = \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \sum_{\Delta, \ell} \begin{array}{c} \text{Diagram showing a four-point function with indices } \\ \text{underlined: } \overbrace{\phi(x_1)\phi(x_2)}^{\text{indices}}, \overbrace{\phi(x_3)\phi(x_4)}^{\text{indices}}. \end{array} G_{\Delta, \ell}(u, v) \quad (\text{D.60})$$

Alternatively, we could also have used the OPEs for the operators (14) and (23) to define the expansion

$$\langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle = \frac{1}{x_{23}^{2\Delta_\phi} x_{14}^{2\Delta_\phi}} \sum_{\Delta, \ell} \begin{array}{c} \text{Diagram showing a four-point function with indices } \\ \text{underlined: } \overbrace{\phi(x_1)\phi(x_2)}^{\text{indices}}, \overbrace{\phi(x_3)\phi(x_4)}^{\text{indices}}. \end{array} G_{\Delta, \ell}(v, u) \quad (\text{D.61})$$

Note that the cross ratios have been exchanged in this expression $u \leftrightarrow v$ (since we are effectively exchanging the points x_1 and x_3). The expressions in (D.60) and (D.61), however, should give the same result. This *crossing symmetry* implies the following constraint

$$\frac{G(u, v)}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} = \frac{G(v, u)}{x_{23}^{2\Delta_\phi} x_{14}^{2\Delta_\phi}}. \quad (\text{D.62})$$

It is convenient to extract the identity operator from the conformal block expansion and write

$$G(u, v) = 1 + \sum_{\Delta, \ell} c_{\Delta, \ell}^2 \tilde{G}_{\Delta, \ell}(u, v). \quad (\text{D.63})$$

Equation (D.62) then implies the following constraint

$$\sum_{\Delta, \ell} c_{\Delta, \ell}^2 F_{\Delta, \ell}(u, v) = 1, \quad (\text{D.64})$$

where we have defined the functions

$$F_{\Delta, \ell}(u, v) \equiv \frac{v^{\Delta_\phi} \tilde{G}_{\Delta, \ell}(u, v) - u^{\Delta_\phi} \tilde{G}_{\Delta, \ell}(v, u)}{u^{\Delta_\phi} - v^{\Delta_\phi}}. \quad (\text{D.65})$$

The constraint (D.62) becomes particularly powerful when combined with unitarity. The latter implies: 1) lower bounds on the operator dimensions (see Section D.2) and 2) the reality of the OPE coefficients, so that $c_{\Delta,\ell}^2 \geq 0$. Equation (D.64) can be viewed as the expansion of the identity in terms of a set of basis functions $F_{\Delta,\ell}(u, v)$, given an assumed spectrum of operators. If any of the expansion coefficients is negative, the theory is inconsistent with unitarity.

[DB: one more paragraph on the modern *conformal bootstrap*; [113]]

E Solutions to the Exercises

Solution to Exercise 2.1

The Hamiltonian of the simple harmonic oscillator (with unit mass) is

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2q^2, \quad (\text{E.1})$$

where $p = \dot{q}$ is the momentum. The position operator is written as

$$\hat{q}(t) = q^*(t)\hat{a} + q(t)\hat{a}^\dagger. \quad (\text{E.2})$$

where $q(t)$ is a solution to the classical equation of motion, $\ddot{q} + \omega^2q = 0$. The most general such solution is

$$q(t) = \frac{1}{\sqrt{2\omega}}(c_1 e^{i\omega t} + c_2 e^{-i\omega t}), \quad (\text{E.3})$$

$$p(t) = i\sqrt{\frac{\omega}{2}}(c_1 e^{i\omega t} - c_2 e^{-i\omega t}). \quad (\text{E.4})$$

The correct normalization of the Wronskian implies

$$\begin{aligned} i &\equiv \dot{q}q^* - \dot{q}^*q \\ &= pq^* - p^*q \\ &= \frac{i}{2}(c_1 e^{i\omega t} - c_2 e^{-i\omega t})(c_1^* e^{-i\omega t} + c_2^* e^{i\omega t}) + \frac{i}{2}(c_1^* e^{-i\omega t} - c_2^* e^{i\omega t})(c_1 e^{i\omega t} + c_2 e^{-i\omega t}) \\ &= \frac{i}{2}(|c_1|^2 - |c_2|^2 + c_1 c_2^* e^{2i\omega t} - c_1^* c_2 e^{-2i\omega t}) + \frac{i}{2}(|c_1|^2 - |c_2|^2 + c_1^* c_2 e^{-2i\omega t} - c_1 c_2^* e^{2i\omega t}) \\ &= i(|c_1|^2 - |c_2|^2), \end{aligned} \quad (\text{E.5})$$

and hence

$$|c_1|^2 - |c_2|^2 = 1. \quad (\text{E.6})$$

Next, we consider the Hamiltonian operator

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{q}^2, \quad (\text{E.7})$$

and the vacuum expectation value $\langle 0|\hat{H}|0\rangle$. Substituting (E.2) into $\langle 0|\hat{q}^2|0\rangle$, we get

$$\begin{aligned} \langle 0|\hat{q}^2|0\rangle &= \langle 0|(q^*(t)\hat{a} + q(t)\hat{a}^\dagger)(q^*(t)\hat{a} + q(t)\hat{a}^\dagger)|0\rangle \\ &= |q(t)|^2\langle 0|\hat{a}\hat{a}^\dagger|0\rangle \\ &= |q(t)|^2\langle 0|[\hat{a}, \hat{a}^\dagger]|0\rangle \\ &= |q(t)|^2\langle 0|0\rangle \\ &= |q(t)|^2, \end{aligned} \quad (\text{E.8})$$

where we have used that \hat{a} annihilates $|0\rangle$ to the right and \hat{a}^\dagger annihilates $\langle 0|$ to the left. Similarly, we get $\langle 0|\hat{p}^2|0\rangle = |\dot{q}(t)|^2$ and the expectation value of the Hamiltonian is

$$\begin{aligned}\langle 0|\hat{H}|0\rangle &= \frac{1}{2}(|\dot{q}|^2 + \omega^2|q|^2) \\ &= \frac{\omega}{4}(c_1 e^{i\omega t} - c_2 e^{-i\omega t})(c_1^* e^{-i\omega t} - c_2^* e^{i\omega t}) + \frac{\omega}{4}(c_1 e^{i\omega t} + c_2 e^{-i\omega t})(c_1^* e^{-i\omega t} + c_2^* e^{i\omega t}) \\ &= \frac{\omega}{4}(|c_1|^2 + |c_2|^2 - c_1 c_2^* e^{2i\omega t} - c_1^* c_2 e^{-2i\omega t}) + \frac{\omega}{4}(|c_1|^2 + |c_2|^2 + c_1^* c_2 e^{-2i\omega t} + c_1 c_2^* e^{2i\omega t}) \\ &= \frac{\omega}{2}(|c_1|^2 + |c_2|^2).\end{aligned}\quad (\text{E.9})$$

Using (E.6), we then get

$$\boxed{\langle 0|\hat{H}|0\rangle = \frac{\omega}{2}(1 + 2|c_2|^2)}.\quad (\text{E.10})$$

Since $|c_2|^2$ is positive semi-definite, the expectation value of the Hamiltonian is minimized for $c_2 = 0$ (and hence $|c_1| = 1$). Up to an irrelevant phase, the mode function then is

$$\boxed{q(t) = \frac{1}{\sqrt{2\omega}} e^{i\omega t}},\quad (\text{E.11})$$

which is the unique ground state solution of the simple harmonic oscillator.

Solution to Exercise 2.2

The action for a massive scalar field in an FRW background is

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - \frac{1}{2} m^2 \chi^2 \right] \quad (\text{E.12})$$

$$= \frac{1}{2} \int d\eta d^3x \left[a^2(\eta) ((\chi')^2 - (\nabla \chi)^2) - m^2 a^4(\eta) \chi^2 \right]. \quad (\text{E.13})$$

Introducing the canonically-normalized field $u \equiv a\chi$, and substituting $a(\eta) = -(H\eta)^{-1}$, this becomes

$$S = \frac{1}{2} \int d\eta d^3x \left[(u')^2 - (\nabla u)^2 - \left(m^2 a^2 - \frac{a''}{a} \right) u^2 \right] \quad (\text{E.14})$$

$$= \frac{1}{2} \int d\eta d^3x \left[(u')^2 - (\nabla u)^2 - \frac{m^2/H^2 - 2}{\eta^2} u^2 \right]. \quad (\text{E.15})$$

The equation of motion (in Fourier space) then is

$$u_k'' + \left(k^2 + \frac{m^2/H^2 - 2}{\eta^2} \right) u_k = 0. \quad (\text{E.16})$$

Defining $x \equiv -k\eta$, this becomes

$$x^2 \frac{d^2 u_k}{dx^2} + \left(x^2 - \nu^2 + \frac{1}{4} \right) u_k = 0, \quad \text{where } \nu \equiv \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}. \quad (\text{E.17})$$

Writing $u_k(\eta) \equiv \sqrt{x} g(x)$, this takes the form of a *Bessel equation*

$$x^2 \frac{d^2 g}{dx^2} + x \frac{dg}{dx} + (x^2 - \nu^2)g(x) = 0, \quad (\text{E.18})$$

which has the following solution in terms of Hankel functions:

$$g(x) = c_1 H_\nu^{(1)}(x) + c_2 H_\nu^{(2)}(x). \quad (\text{E.19})$$

Using

$$H_\nu^{(1)}(x) \xrightarrow{x \rightarrow \infty} \sqrt{\frac{2}{\pi x}} e^{i(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)}, \quad (\text{E.20})$$

$$H_\nu^{(2)}(x) \xrightarrow{x \rightarrow \infty} \sqrt{\frac{2}{\pi x}} e^{-i(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)}, \quad (\text{E.21})$$

the early-time limit of the canonically-normalized field becomes

$$\lim_{-k\eta \rightarrow \infty} u_k(\eta) = \sqrt{\frac{2}{\pi}} \left(c_1 e^{-\frac{i}{4}\pi(1+2\nu)} e^{-ik\eta} + c_2 e^{\frac{i}{4}\pi(1+2\nu)} e^{ik\eta} \right). \quad (\text{E.22})$$

This matches the Bunch–Davies initial condition,

$$\lim_{-k\eta \rightarrow \infty} u_k(\eta) = \frac{1}{\sqrt{2k}} e^{ik\eta}, \quad (\text{E.23})$$

if $c_1 = 0$ and $c_2 = e^{-\frac{i}{4}\pi(1+2\nu)} \sqrt{\pi/(4k)}$, which fixes the solution to be

$$u_k(\eta) = e^{-\frac{i}{4}\pi(1+2\nu)} \sqrt{\frac{\pi}{4}} \sqrt{-\eta} H_\nu^{(2)}(-k\eta). \quad (\text{E.24})$$

The de Sitter mode function for the massive field, $\chi_k = u_k/a$, then is

$$\boxed{\chi_k(\eta) = H \sqrt{\frac{\pi}{4}} e^{-\frac{i}{4}\pi(1+2\nu)} (-\eta)^{3/2} H_\nu^{(2)}(-k\eta)}. \quad (\text{E.25})$$

For small argument, the Hankel functions have the following expansions:

$$H_\nu^{(2)}(x) \xrightarrow{x \rightarrow 0} \frac{i}{\pi} \left(\Gamma(\nu) \left(\frac{x}{2}\right)^{-\nu} + e^{i\pi\nu} \Gamma(-\nu) \left(\frac{x}{2}\right)^\nu \right), \quad (\text{E.26})$$

$$H_\nu^{(1)}(x) \xrightarrow{x \rightarrow 0} -\frac{i}{\pi} \left(\Gamma(\nu) \left(\frac{x}{2}\right)^{-\nu} + e^{-i\pi\nu} \Gamma(-\nu) \left(\frac{x}{2}\right)^\nu \right). \quad (\text{E.27})$$

Note that these expressions hold also for complex ν . If ν has a positive real part, we can drop the x^ν contribution.

Consider first the case $m < \frac{3}{2}H$, for which ν is purely real (and positive). The complex conjugate of the Hankel function is $(H_\nu^{(2)}(x))^* = H_\nu^{(1)}(x)$, so that

$$\begin{aligned} \lim_{k\eta \rightarrow 0} |H_\nu^{(2)}(-k\eta)|^2 &= \lim_{k\eta \rightarrow 0} H_\nu^{(2)}(-k\eta) H_\nu^{(1)}(-k\eta) \\ &= \frac{1}{\pi^2} \left(\Gamma^2(\nu) \left(\frac{-k\eta}{2}\right)^{-2\nu} + \Gamma^2(-\nu) \left(\frac{-k\eta}{2}\right)^{2\nu} + 2\Gamma(\nu)\Gamma(-\nu) \cos(\pi\nu) \right) \\ &= \frac{1}{\pi^2} \left(\Gamma^2(\nu) \left(\frac{-k\eta}{2}\right)^{-2\nu} + \Gamma^2(-\nu) \left(\frac{-k\eta}{2}\right)^{2\nu} - \frac{2\pi \cot(\pi\nu)}{\nu} \right), \end{aligned} \quad (\text{E.28})$$

where we used Euler's reflection formula $\Gamma(\nu)\Gamma(-\nu) = -\pi/(\nu \sin(\pi\nu))$. The late-time limit of the power spectrum then becomes

$$\begin{aligned} \lim_{k\eta \rightarrow 0} P_\chi(k, \eta) &= \lim_{k\eta \rightarrow 0} |\chi_k(\eta)|^2 \\ &= \frac{\pi}{4} H^2(-\eta)^3 \lim_{k\eta \rightarrow 0} |H_\nu^{(2)}(-k\eta)|^2 = \boxed{\frac{4\Gamma^2(\nu)}{\pi} \frac{H^2}{2k^3} \left(\frac{-k\eta}{2}\right)^{3-2\nu}}. \end{aligned} \quad (\text{E.29})$$

As a consistency check, we take the limit of a massless field, $m^2 = 0$ or $\nu = 3/2$. Using $\Gamma(3/2) = \sqrt{\pi}/2$, we find

$$\lim_{k\eta \rightarrow 0} P_\chi(k, \eta) = \frac{H^2}{2k^3}, \quad (\text{E.30})$$

which matches the result (2.52).

Next, we take $m > \frac{3}{2}H$, for which ν is purely imaginary. It is convenient to write $\nu = i\mu$, where μ is purely real (and positive). The complex conjugate of the Hankel function then is $(H_{i\mu}^{(2)}(x))^* = H_{-i\mu}^{(1)}(x) = e^{-\pi\mu} H_{i\mu}^{(1)}(x)$, so that

$$\begin{aligned} \lim_{k\eta \rightarrow 0} |H_{i\mu}^{(2)}(-k\eta)|^2 &= e^{-\pi\mu} \lim_{k\eta \rightarrow 0} H_{i\mu}^{(2)}(-k\eta) H_{i\mu}^{(1)}(-k\eta) \\ &= \frac{e^{-\pi\mu}}{\pi^2} \left(\Gamma^2(i\mu) \left(\frac{-k\eta}{2}\right)^{-2i\mu} + \Gamma^2(-i\mu) \left(\frac{-k\eta}{2}\right)^{2i\mu} - \frac{2\pi \cot(i\pi\mu)}{i\mu} \right), \end{aligned} \quad (\text{E.31})$$

and the late-time limit of the power spectrum becomes

$$\begin{aligned} \lim_{k\eta \rightarrow 0} P_\chi(k, \eta) &= \lim_{k\eta \rightarrow 0} |\chi_k(\eta)|^2 \\ &= \frac{\pi}{4} e^{\pi\mu} H^2(-\eta)^3 \lim_{k\eta \rightarrow 0} |H_\nu^{(2)}(-k\eta)|^2 \\ &= \frac{1}{4\pi} H^2(-\eta)^3 \left(\Gamma^2(i\mu) \left(\frac{-k\eta}{2}\right)^{-2i\mu} + \Gamma^2(-i\mu) \left(\frac{-k\eta}{2}\right)^{2i\mu} + \frac{2\pi \coth(\pi\mu)}{\mu} \right). \end{aligned} \quad (\text{E.32})$$

Writing $\Gamma(\pm i\mu) \equiv |\Gamma(\pm i\mu)|e^{\pm i\delta}$, and using

$$|\Gamma(\pm i\mu)|^2 = \frac{\pi}{\mu \sinh(\pi\mu)}, \quad (\text{E.33})$$

we can write this as

$$\begin{aligned} \lim_{k\eta \rightarrow 0} P_\chi(k, \eta) &= \frac{H^2(-\eta)^3}{4\mu \sinh(\pi\mu)} \left[e^{2i\delta(\mu)} \left(\frac{-k\eta}{2}\right)^{-2i\mu} + e^{-2i\delta(\mu)} \left(\frac{-k\eta}{2}\right)^{2i\mu} + 2 \cosh(\pi\mu) \right] \\ &= \boxed{\frac{H^2}{2k^3} \frac{(-k\eta)^3}{\mu \sinh(\pi\mu)} \left[\cos\left(2\mu \log\left(\frac{-k\eta}{2}\right) - 2\delta\right) + \cosh(\pi\mu) \right]}. \end{aligned} \quad (\text{E.34})$$

For large masses, $\mu \gg 1$, the power spectrum scales as $1/\sinh(\pi\mu) \rightarrow e^{-\pi\mu}$, corresponding to the Boltzmann suppression of the spontaneous production of massive particles. We also see that the signal oscillates (as a function of $-k\eta$), with a frequency that depends on the mass of the particle.

Solution to Exercise 3.1

We start from the master formula (3.21):

$$\langle \mathcal{O}(t) \rangle = \langle 0 | \bar{T} e^{i \int_{-\infty+}^t dt' H_{\text{int}}(t')} \mathcal{O}(t) T e^{-i \int_{-\infty-}^t dt' H_{\text{int}}(t')} | 0 \rangle , \quad (\text{E.35})$$

which to linear order in H_{int} becomes

$$\langle \mathcal{O}(t) \rangle = -i \int_{-\infty-}^t dt' \langle 0 | \mathcal{O}(t) H_{\text{int}}(t') | 0 \rangle + i \int_{-\infty+}^t dt' \langle 0 | H_{\text{int}}(t') \mathcal{O}(t) | 0 \rangle . \quad (\text{E.36})$$

We notice that

$$\left(\int_{-\infty-}^t dt' \langle 0 | \mathcal{O}(t) H_{\text{int}}(t') | 0 \rangle \right)^\dagger = \int_{-\infty+}^t dt' \langle 0 | H_{\text{int}}(t') \mathcal{O}(t) | 0 \rangle , \quad (\text{E.37})$$

so that (E.36) is of the form $-iA + iA^\dagger = 2 \text{Im } A$, or

$$\langle \mathcal{O}(t) \rangle = 2 \text{Im} \left(\int_{-\infty-}^t dt' \langle 0 | \mathcal{O}(t) H_{\text{int}}(t') | 0 \rangle \right) , \quad (\text{E.38})$$

which is precisely (3.22). Returning to (E.36), we see that this is exactly of the form of the commutator

$$\langle \mathcal{O}(t) \rangle = i \int_{-\infty}^t dt' \langle 0 | [H_{\text{int}}(t'), \mathcal{O}(t)] | 0 \rangle , \quad (\text{E.39})$$

as desired.⁶⁸

Next, we expand (E.35) to second order in H_{int} :

$$\begin{aligned} \langle \mathcal{O}(t) \rangle &= \int_{-\infty}^t dt' \int_{-\infty}^t dt'' \langle 0 | H_{\text{int}}(t') \mathcal{O}(t) H_{\text{int}}(t'') | 0 \rangle \\ &\quad - \int_{-\infty}^t dt' \int_{-\infty}^{t'} dt'' \left(\langle 0 | \mathcal{O}(t) H_{\text{int}}(t') H_{\text{int}}(t'') | 0 \rangle + \langle 0 | H_{\text{int}}(t'') H_{\text{int}}(t') \mathcal{O}(t) | 0 \rangle \right) , \end{aligned} \quad (\text{E.40})$$

where the time integrals in the second line have the same limits because one term is time-ordered and the other is anti-time-ordered. As before, the integrand second line can be combined to give twice the real part, reproducing (3.31). In order to write this expression in commutator form, we note that

$$\int_{-\infty}^t dt' \int_{-\infty}^{t'} dt'' \langle 0 | H_{\text{int}}(t') \mathcal{O}(t) H_{\text{int}}(t'') | 0 \rangle = 2 \int_{-\infty}^t dt' \int_{-\infty}^{t'} dt'' \langle 0 | H_{\text{int}}(t') \mathcal{O}(t) H_{\text{int}}(t'') | 0 \rangle , \quad (\text{E.41})$$

because it is symmetric under the interchange of $t' \leftrightarrow t''$. Now that all the terms are under the same integrals, it is then straightforward to rearrange (E.40) into the form (3.33).

[**AJ:** finish, show commutator form for general n , proof in Weinberg and appendix of [120] add refs to other versions of expanded formula.]

⁶⁸In writing this commutator form, we have glossed over a small technical subtlety associated to the $i\epsilon$ prescription: looking back at (E.36), we see that the two terms have different $i\epsilon$ prescriptions. In order to combine them into (E.39), we are implicitly integrating both terms along the same contour, which seems a bit suspect [116, 117]. In practice, this is not much of a problem because we typically break up the commutator into its constituent pieces and perform the integrals so that they converge. More formally, we can choose a different regularization scheme that is compatible with the commutator grouping. See [118, 119], for detailed discussions.

Solution to Exercise 3.2

Starting from the action

$$S = \int d^4x \left(-\frac{1}{2}(\partial\Phi)^2 - \frac{1}{6\Lambda^2}\dot{\Phi}^3 \right), \quad (\text{E.42})$$

we can derive the canonical momentum conjugate to Φ :

$$\pi \equiv \frac{\delta\mathcal{L}}{\delta\dot{\Phi}} = \dot{\Phi} - \frac{1}{2\Lambda^2}\dot{\Phi}^2. \quad (\text{E.43})$$

The Hamiltonian is then given by

$$\begin{aligned} H &= \int d^3x \left(\pi\dot{\Phi} - \mathcal{L} \right) \\ &= \int d^3x \left(\frac{1}{2}\dot{\Phi}^2 + (\nabla\Phi)^2 - \frac{1}{3\Lambda^2}\dot{\Phi}^3 \right) \\ &= \int d^3x \left(\frac{1}{2}\pi^2 + (\nabla\Phi)^2 + \frac{1}{6\Lambda^2}\pi^3 + \frac{1}{8\Lambda^4}\pi^4 + \dots \right), \end{aligned} \quad (\text{E.44})$$

where, in the last line, we have used the relation between $\dot{\Phi}$ and the canonical momentum

$$\frac{1}{\Lambda^2}\dot{\Phi} = 1 - \sqrt{1 - \frac{2\pi}{\Lambda^2}}, \quad (\text{E.45})$$

expanded out perturbatively for $\pi \ll \Lambda^2$. We therefore see that the interaction Hamiltonian up to quartic order is

$$H_{\text{int}} = \int d^3x \left(\frac{1}{6\Lambda^2}\pi^3 + \frac{1}{8\Lambda^4}\pi^4 \right), \quad (\text{E.46})$$

which, importantly, is not just $-L_{\text{int}}$ appearing in (E.42). To quantize the theory, we treat these interactions perturbatively in the interaction picture, which amounts to setting $\pi = \dot{\Phi}$ and expanding Φ in modes as usual, so that

$$H_{\text{int}} = \int d^3x \left(\frac{1}{6\Lambda^2}\dot{\Phi}^3 + \frac{1}{8\Lambda^4}\dot{\Phi}^4 \right), \quad (\text{E.47})$$

where we are suppressing interaction-picture labels.

In order to compute the four-point function in this theory, we can employ the Feynman rules from Section 3.1.3. There will now be both exchange and contact contributions. Moreover, we have to keep track of one additional subtlety: since there are derivatives acting on the fields in the interaction vertices, lines connected to interactions should now be associated with factors of

$$\dot{f}_k(t) = ik e^{ikt}, \quad (\text{E.48})$$

the time derivative of the mode function. For lines that are not time-ordered (i.e. external lines or lines that cross the $t = 0$ surface) we can proceed in the naive way by differentiating the Wightman propagator. For bulk internal lines, we now have to keep in mind that they should be associated to

$$\langle T\dot{\phi}_{\mathbf{k}}(t)\dot{\phi}_{\mathbf{k}'}(t') \rangle = \partial_t\partial_{t'}W_k(t, t')\theta(t - t') + \partial_t\partial_{t'}W_k(t', t)\theta(t' - t), \quad (\text{E.49})$$

which is the time-ordered two-point function of $\dot{\phi}$. Importantly, this is *not* the time derivative of the Feynman propagator:

$$\partial_t \partial_{t'} G_F(k; t, t') = \partial_t \partial_{t'} \langle T \phi_{\mathbf{k}}(t) \phi_{\mathbf{k}}(t') \rangle = \langle T \dot{\phi}_{\mathbf{k}}(t) \dot{\phi}_{\mathbf{k}}(t') \rangle + i\delta(t - t'). \quad (\text{E.50})$$

With this in mind, we can compute

$$\begin{aligned} I_{++} &\equiv \begin{array}{c} \text{---} \\ \backslash \quad / \\ \text{---} \end{array} \\ &= \left(\frac{-i}{\Lambda^2}\right)^2 \int_{-\infty}^0 dt' dt'' \partial_{t'} W_{k_1}(0, t') \partial_{t'} W_{k_2}(0, t') \langle T \dot{\phi}_{\mathbf{k}_I}(t') \dot{\phi}_{\mathbf{k}_I}(t'') \rangle \partial_{t''} W_{k_3}(0, t'') \partial_{t''} W_{k_4}(0, t'') \\ &= -\frac{1}{\Lambda^4} k_1 k_2 k_3 k_4 k_I^2 \int_{-\infty}^0 dt' dt'' W_{k_1}(0, t') W_{k_2}(0, t') G_F(k_I; t', t'') W_{k_3}(0, t'') W_{k_4}(0, t'') \\ &= \frac{1}{32\Lambda^4} \frac{k_I}{k_{12} + k_{34}} \left(\frac{1}{k_{12} + k_I} + \frac{1}{k_{34} + k_I} \right), \end{aligned} \quad (\text{E.51})$$

where we evaluated the derivatives to put the time integral in the form (3.49). We can similarly evaluate

$$\begin{aligned} I_{+-} &\equiv \begin{array}{c} \text{---} \\ \backslash \quad / \\ \diagup \quad \diagdown \\ \bullet \end{array} \\ &= \frac{1}{\Lambda^4} \int_{-\infty}^0 dt' dt'' \partial_{t'} W_{k_1}(0, t') \partial_{t'} W_{k_2}(0, t') \partial_{t'} \partial_{t''} W_{k_I}(t'', t') \partial_{t''} W_{k_3}(t'', 0) \partial_{t''} W_{k_4}(t'', 0) \\ &= \frac{1}{\Lambda^4} k_1 k_2 k_3 k_4 k_I^2 \int_{-\infty}^0 dt' dt'' W_{k_1}(0, t') W_{k_2}(0, t') W_{k_I}(t'', t') W_{k_3}(t'', 0) W_{k_4}(t'', 0) \\ &= \frac{1}{32\Lambda^4} \frac{k_I}{(k_{12} + k_I)(k_{34} + k_I)}. \end{aligned} \quad (\text{E.52})$$

Finally, we have to compute the contact contribution

$$\begin{aligned} I_+ &= \begin{array}{c} \text{---} \\ \backslash \quad / \\ \diagup \quad \diagdown \\ \diagup \quad \diagdown \\ \bullet \end{array} \\ &= -\frac{3i}{\Lambda^4} \int_{-\infty}^0 dt \partial_t W_{k_1}(0, t) \partial_t W_{k_2}(0, t) \partial_t W_{k_3}(0, t) \partial_t W_{k_4}(0, t) \\ &= -\frac{3}{16\Lambda^4} \frac{1}{k_1 + k_2 + k_3 + k_4}. \end{aligned} \quad (\text{E.53})$$

All of these contributions are purely real, so the other time-orderings will just lead to a factor of two. Adding everything up, we obtain

$$\boxed{\langle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \phi_{\mathbf{k}_4} \rangle' = -\frac{2(k_{12} + k_I)(k_{34} + k_I) + k_{12}k_{34}}{8(k_{12} + k_{34})(k_{12} + k_I)(k_{34} + k_I)}}. \quad (\text{E.54})$$

There were two important subtleties in this computation involving (time) derivative interactions. First, we saw that starting at quartic order, the interaction Hamiltonian was no longer just

minus the interaction Lagrangian. Additionally, we saw that we have to be careful about how time derivatives act on internal lines; we cannot just replace the time-ordered two-point function of $\dot{\phi}$ with the time derivative of the Feynman propagator. Here, we were careful about both of these subtleties. Interestingly enough, most often one can disregard both of these things and the mistakes will cancel out. Roughly speaking, the way that this happens is that the time derivatives acting on the theta functions in the Feynman propagator will produce a delta function term as in (E.50), which will reproduce the contribution from the quartic interaction Hamiltonian. If you are motivated, you should check that this happens for this simple example. See [121–123], for a more detailed discussion.

Solution to Exercise 3.3

We start from equation (3.30):

$$\langle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \rangle'(\eta_*) = 2g \operatorname{Im} \left[f_{k_1}^*(\eta_*) f_{k_2}^*(\eta_*) f_{k_3}^*(\eta_*) \int_{-\infty}^{\eta_*} d\eta a^4(\eta) f_{k_1}(\eta) f_{k_2}(\eta) f_{k_3}(\eta) \right]. \quad (\text{E.55})$$

Substituting the Bunch-Davies mode function,

$$f_k(\eta) = \frac{u_k(\eta)}{a(\eta)} = \frac{-iH}{\sqrt{2k^3}} (1 - ik\eta) e^{ik\eta}, \quad (\text{E.56})$$

the integral in (E.55) can be expressed as

$$\begin{aligned} I(\eta_*) &\equiv \int_{-\infty}^{\eta_*} a^4(\eta) f_{k_1}(\eta) f_{k_2}(\eta) f_{k_3}(\eta) \\ &= \frac{i}{H} \frac{1}{\sqrt{8k_1^3 k_2^3 k_3^3}} \int_{-\infty}^{\eta_*} \frac{d\eta}{\eta^4} (1 - ik_1\eta)(1 - ik_2\eta)(1 - ik_3\eta) e^{iK\eta} \\ &= \frac{i}{H} \frac{1}{\sqrt{8k_1^3 k_2^3 k_3^3}} [\mathcal{I}_4(\eta_*) - iK \mathcal{I}_3(\eta_*) - (k_1 k_2 + k_2 k_3 + k_3 k_1) \mathcal{I}_2(\eta_*) + ik_1 k_2 k_3 \mathcal{I}_1(\eta_*)], \end{aligned} \quad (\text{E.57})$$

where $K \equiv k_1 + k_2 + k_3$ and

$$\mathcal{I}_N(\eta_*) \equiv \int_{-\infty}^{\eta_*} d\eta \frac{e^{iK\eta}}{\eta^N}. \quad (\text{E.58})$$

Substituting $iKe^{iK\eta} = d(e^{iK\eta})/d\eta$ into this expression, and integrating by parts, we get the following recursion relation

$$\mathcal{I}_{N+1}(\eta_*) = \frac{iK}{N} \mathcal{I}_N(\eta_*) - \frac{1}{N} \frac{e^{iK\eta_*}}{\eta_*^N}. \quad (\text{E.59})$$

The integrals \mathcal{I}_2 , \mathcal{I}_3 , \mathcal{I}_4 can therefore all be expressed in terms of \mathcal{I}_1 :

$$\mathcal{I}_2(\eta_*) = iK \mathcal{I}_1(\eta_*) - \frac{e^{iK\eta_*}}{\eta_*}, \quad (\text{E.60})$$

$$\mathcal{I}_3(\eta_*) = -\frac{K^2}{2} \mathcal{I}_1(\eta_*) - \frac{iK}{2} \frac{e^{iK\eta_*}}{\eta_*} - \frac{1}{2} \frac{e^{iK\eta_*}}{\eta_*^2}, \quad (\text{E.61})$$

$$\mathcal{I}_4(\eta_*) = -\frac{iK^3}{6} \mathcal{I}_1(\eta_*) + \frac{K^2}{6} \frac{e^{iK\eta_*}}{\eta_*} - \frac{iK}{6} \frac{e^{iK\eta_*}}{\eta_*^2} - \frac{1}{3} \frac{e^{iK\eta_*}}{\eta_*^3}. \quad (\text{E.62})$$

Equation (E.57) can then be written as

$$I(\eta_*) = \frac{i}{H} \frac{1}{\sqrt{8k_1^3 k_2^3 k_3^3}} \left[i \left(\frac{K^3}{3} - (k_1 k_2 + k_2 k_3 + k_3 k_1) K + k_1 k_2 k_3 \right) \mathcal{I}_1(\eta_*) - \left(\frac{K^2}{3} - (k_1 k_2 + k_2 k_3 + k_3 k_1) \right) \frac{e^{iK\eta_*}}{\eta_*} + \frac{iK}{3} \frac{e^{iK\eta_*}}{\eta_*^2} - \frac{1}{3} \frac{e^{iK\eta_*}}{\eta_*^3} \right]. \quad (\text{E.63})$$

In addition, we have

$$f_{k_1}^* f_{k_2}^* f_{k_3}^*(\eta_*) = \frac{iH^3}{\sqrt{8k_1^3 k_2^3 k_3^3}} \left(1 - iK\eta_* - (k_1 k_2 + k_2 k_3 + k_3 k_1)\eta_*^2 + ik_1 k_2 k_3 \eta_*^3 \right) e^{-iK\eta_*}. \quad (\text{E.64})$$

We now multiply (E.64) and (E.63), and extract the imaginary part. Dropping terms that vanish in the limit $\eta_* \rightarrow 0$, we find [DB: Fix sign and coefficients]

$$\boxed{\langle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \rangle' = -\frac{gH^2}{12} \frac{(\gamma + \log(|K\eta_*|)) \sum_i k_i^3 - 3K \sum_{i < j} k_i k_j + K^3}{(k_1 k_2 k_3)^3}}, \quad (\text{E.65})$$

where we used that $\mathcal{I}_1(\eta_*) \rightarrow \gamma + \log(|K\eta_*|)$.

Solution to Exercise 3.4

We consider an interaction of the form $\mathcal{L}_{\text{int}} = -\frac{1}{2}g\Phi^2\chi$, where Φ is a conformally coupled scalar and χ is a scalar of general mass. Formula (3.60) still applies, so that

$$I_+ = -ig f_{k_1}^*(\eta_*) f_{k_2}^*(\eta_*) \chi_{k_3}^*(\eta_*) \int_{-\infty}^{\eta_*} \frac{d\eta}{(H\eta)^4} f_{k_1}(\eta) f_{k_2}(\eta) \chi_{k_3}(\eta), \quad (\text{E.66})$$

where $f_k(\eta)$ are the mode functions of a conformally coupled scalar (3.59) and $\chi_k(\eta)$ is the mode function of a general mass scalar (3.58):

$$f_k(\eta) = H(-\eta) \frac{e^{ik\eta}}{\sqrt{2k}}, \quad (\text{E.67})$$

$$\chi_k(\eta) = H \sqrt{\frac{\pi}{4}} e^{-\frac{i\pi}{4}(1+2\nu)} (-\eta)^{3/2} H_\nu^{(2)}(-k\eta). \quad (\text{E.68})$$

Using the integral representation of the Hankel function (3.66), we can write the mode function of the massive field as

$$\chi_k(\eta) = \frac{2^\nu e^{\frac{i\pi}{4}(1+2\nu)}}{\Gamma[\nu + \frac{1}{2}]} \frac{H}{k^{3/2}} (-k\eta)^{3/2+\nu} e^{ik\eta} \int_0^\infty ds e^{2ik\eta s} s^{\nu-\frac{1}{2}} (1+s)^{\nu-\frac{1}{2}}. \quad (\text{E.69})$$

Equation (E.66) then becomes

$$I_+ = \frac{e^{-\frac{i\pi}{4}(1-2\nu)}}{\Gamma[\nu + \frac{1}{2}]} g H \eta_*^2 \chi_{k_3}^*(\eta_*) \frac{(2k_3)^\nu}{4k_1 k_2} \times \int_0^\infty ds s^{\nu-\frac{1}{2}} (1+s)^{\nu-\frac{1}{2}} \int_{-\infty}^{\eta_*} d\eta e^{i[k_{12} + (1+2s)k_3]\eta} (-\eta)^{\nu-\frac{1}{2}}, \quad (\text{E.70})$$

where we have dropped the factor of $e^{-ik_{12}\eta_*}$ (which becomes unity in the limit $\eta_* \rightarrow 0$). To perform the time integral in this expression, we use the following identity

$$\int_{-\infty}^0 d\eta e^{i\alpha\eta}(-\eta)^{\Delta-2} = (-i)^{\Delta-1}\Gamma[\Delta-1]\alpha^{1-\Delta}. \quad (\text{E.71})$$

Setting $\Delta = \frac{3}{2} + \nu$ and $\alpha = k_{12} + (1+2s)k_3$, this gives

$$\begin{aligned} \int_{-\infty}^0 d\eta e^{i[k_{12}+(1+2s)k_3]\eta}(-\eta)^{\nu-\frac{1}{2}} &= (-i)^{\frac{1}{2}+\nu}\Gamma[\nu+\frac{1}{2}][k_{12}+k_3(2s+1)]^{-\frac{1}{2}-\nu} \\ &= (-i)^{\frac{1}{2}+\nu}\Gamma[\nu+\frac{1}{2}](2k_3)^{-\frac{1}{2}-\nu}\left(s+\frac{1+u}{2u}\right)^{-\frac{1}{2}-\nu}, \end{aligned} \quad (\text{E.72})$$

where we have defined $u \equiv k_3/k_{12}$. Equation (E.70) then becomes

$$I_+ = -ig H\eta_*^2 \chi_{k_3}^*(\eta_*) \frac{(2k_3)^{-1/2}}{4k_1 k_2} \int_0^\infty ds s^{\nu-\frac{1}{2}}(1+s)^{\nu-\frac{1}{2}} \left(s+\frac{1+u}{2u}\right)^{-\frac{1}{2}-\nu}. \quad (\text{E.73})$$

Recalling the integral formula for the hypergeometric function (valid for $\text{Re } c > \text{Re } b > 0$ and $|\arg(1-z)| < \pi$),

$${}_2F_1 \left[\begin{matrix} a, & b \\ c & \end{matrix} \middle| z \right] = \frac{\Gamma[c]}{\Gamma[b]\Gamma[c-b]} \int_0^\infty dt t^{-b+c-1}(t+1)^{a-c}(t-z+1)^{-a}, \quad (\text{E.74})$$

we see that (E.73) can be written as

$$I_+ = -ig H\eta_*^2 \chi_{k_3}^*(\eta_*) \Gamma[\frac{1}{2}-\nu]\Gamma[\frac{1}{2}+\nu] \frac{(2k_3)^{-1/2}}{4k_1 k_2} {}_2F_1 \left[\begin{matrix} \frac{1}{2}+\nu, & \frac{1}{2}-\nu \\ 1 & \end{matrix} \middle| \frac{u-1}{2u} \right]. \quad (\text{E.75})$$

We can simplify this expression by noting that this particular hypergeometric function is essentially how the associated Legendre function is defined:

$$P_\nu^\mu(x) = \left(\frac{x+1}{x-1}\right)^{\mu/2} \frac{1}{\Gamma[1-\mu]} {}_2F_1 \left[\begin{matrix} \nu+1, & -\nu \\ 1-\mu & \end{matrix} \middle| \frac{1-x}{2} \right]. \quad (\text{E.76})$$

The Legendre function has a branch cut running from $x = -\infty$ to $x = 1$, and is real along the positive real interval $x \in (1, \infty)$. We can then write (E.75) as

$$I_+ = -ig H\eta_*^2 \chi_{k_3}^*(\eta_*) \Gamma[\frac{1}{2}-\nu]\Gamma[\frac{1}{2}+\nu] \frac{(2k_3)^{-1/2}}{4k_1 k_2} P_{\nu-\frac{1}{2}}^0(u^{-1}), \quad (\text{E.77})$$

where the physical region is $u \in (0, 1)$, so that there is no branch cut of the Legendre function in the physical region. The full correlator is given by $I_+ + I_-$, where I_- is the complex conjugate:

$$\langle \phi\phi\chi \rangle' = -\left(i\chi_{k_3}^*(\eta_*) - i\chi_{k_3}(\eta_*)\right) g H\eta_*^2 \Gamma[\frac{1}{2}-\nu]\Gamma[\frac{1}{2}+\nu] \frac{(2k_3)^{-1/2}}{4k_1 k_2} P_{\nu-\frac{1}{2}}^0(u^{-1}). \quad (\text{E.78})$$

Note that this formula holds for both real and complex ν . For real ν it is obvious, for complex $\nu = i\mu$, one has to use the fact that (see p. 140 of [124])

$$P_\nu^\mu(x) = P_{-\nu-1}^\mu(x). \quad (\text{E.79})$$

To finally extract the late-time correlator, we use (E.26) and (E.27) to write

$$\begin{aligned} i\chi_{k_3}^*(\eta_*) - i\chi_{k_3}(\eta_*) &= \\ &= H(-\eta_*)^{3/2} \sqrt{\frac{\pi}{4}} \left(e^{\frac{i\pi}{4}(1+2\nu)} iH_\nu^{(1)}(-k\eta_*) - e^{-\frac{i\pi}{4}(1+2\nu)} iH_\nu^{(1)}(-k\eta_*) \right) \\ &\rightarrow \frac{H(-\eta_*)^{3/2}}{\sqrt{\pi}} \left[\cos\left(\frac{\pi}{4}(1+2\nu)\right) \Gamma[\nu] \left(\frac{-k_3\eta_*}{2}\right)^{-\nu} + \cos\left(\frac{\pi}{4}(1-2\nu)\right) \Gamma[-\nu] \left(\frac{-k_3\eta_*}{2}\right)^\nu \right]. \end{aligned} \quad (\text{E.80})$$

Substituting this into (E.78), we find

$$\begin{aligned} \langle\phi\phi\chi\rangle' &= -\frac{g(H\eta_*)^2}{2\sqrt{\pi}} \frac{1}{k_1 k_2 k_3^2} \Gamma[\frac{1}{2} - \nu] \Gamma[\frac{1}{2} + \nu] P_{\nu-\frac{1}{2}}^0(u^{-1}) \\ &\quad \left[\cos\left(\frac{\pi}{4}(1+2\nu)\right) \Gamma[\nu] \left(\frac{-k_3\eta_*}{2}\right)^{\frac{3}{2}-\nu} + \cos\left(\frac{\pi}{4}(1-2\nu)\right) \Gamma[-\nu] \left(\frac{-k_3\eta_*}{2}\right)^{\frac{3}{2}+\nu} \right], \end{aligned} \quad (\text{E.81})$$

which is valid for both real and imaginary ν . When ν is real, the branch with $\eta_*^{-\nu}$ is more important as $\eta_* \rightarrow 0$.

Let us check that (E.81) reproduces (3.62). A conformally coupled scalar has $\nu = 1/2$. In this case, the first term in brackets decays the most slowly at late times. Using the fact that $P_0^0(x) = 1$, we see that the divergence in $\Gamma[\frac{1}{2} - \nu]$ cancels against the zero in $\cos\left(\frac{\pi}{4}(1+2\nu)\right)$, leaving

$$\boxed{\langle\phi\phi\chi\rangle' \xrightarrow{\nu \rightarrow \frac{1}{2}} \frac{\pi}{8} g \frac{H^2 \eta_*^3}{k_1 k_2 k_3}}, \quad (\text{E.82})$$

as required.

Solution to Exercise 3.5

In this problem, we are computing the simplest non-Gaussianities in the effective field theory of inflation [11]. The action for the scalar (Goldstone) fluctuations, up to cubic order, is

$$S = \int d^4x a^3(t) \frac{f_\pi^4}{c_s^3} \left(\frac{1}{2} \dot{\pi}^2 - \frac{c_s^2}{2a^2} (\nabla\pi)^2 + \frac{(1-c_s^2)}{2} \left(1 + \frac{2c_3}{3} \right) \dot{\pi}^3 - \frac{(1-c_s^2)}{2} \dot{\pi} (\nabla\pi)^2 \right), \quad (\text{E.83})$$

where c_s is the sound speed of the fluctuations and $f_\pi^4 \equiv 2c_s M_{\text{Pl}}^2 |\dot{H}|$ is the symmetry breaking scale. The fact that the cubic $\dot{\pi}(\nabla\pi)^2$ interaction is fixed in terms of the speed of sound is a consequence of the nonlinearly realized symmetries of the Goldstone mode.

It is convenient to absorb the (constant) sound speed into a rescaling of the spatial coordinates, $\mathbf{x} \rightarrow c_s \mathbf{x}$, which restores a fake Lorentz invariance of the free theory. Writing $c_3 = -3/2 + 3c_{\dot{\pi}^3}/(1-c_s^2)$ and calling $c_{\dot{\pi}(\nabla\pi)^2} = \frac{1}{2} (c_s^2 - 1)$, along with $\pi \rightarrow \pi/f_\pi^2$, the action becomes

$$S = \int d^4x a^3(t) \left(\frac{1}{2} \dot{\pi}^2 - \frac{1}{2a^2(t)} (\nabla\pi)^2 + \frac{c_{\dot{\pi}^3}}{f_\pi^2} \dot{\pi}^3 + \frac{c_{\dot{\pi}(\nabla\pi)^2}}{f_\pi^2} \frac{1}{a^2(t)} \dot{\pi} (\nabla\pi)^2 \right), \quad (\text{E.84})$$

which was given as the starting point of the problem. As usual, it is convenient to write this in conformal time

$$S = \int d\eta d^3x a^2(\eta) \left(\frac{1}{2} \pi'^2 - \frac{1}{2} (\nabla \pi)^2 + \frac{c_{\dot{\pi}^3}}{f_\pi^2} \frac{1}{a(\eta)} \pi'^3 + \frac{c_{\dot{\pi}(\nabla \pi)^2}}{f_\pi^2} \frac{1}{a(\eta)} \pi' (\nabla \pi)^2 \right). \quad (\text{E.85})$$

The free part of the action is just that of a massless scalar in de Sitter space, so the mode functions are the familiar ones

$$f_k(\eta) = \frac{H}{\sqrt{2k^3}} (1 - ik\eta) e^{ik\eta}. \quad (\text{E.86})$$

The interaction Hamiltonian is

$$H_{\text{int}} = - \int d^3x \left(\frac{c_{\dot{\pi}^3}}{f_\pi^2} \frac{1}{H(-\eta)} \pi'^3 + \frac{c_{\dot{\pi}(\nabla \pi)^2}}{f_\pi^2} \frac{1}{H(-\eta)} \pi' (\nabla \pi)^2 \right). \quad (\text{E.87})$$

Let's treat each of the two interactions in turn:

$\dot{\pi}^3$ interaction: Using (3.22), and following the same steps as for a ϕ^3 interaction, we obtain the analogue of (3.30). (This can also be obtained by employing the Feynman rules.) The only real subtlety is that time derivatives now act on the mode functions because the interaction involves derivatives:

$$\langle \pi_{\mathbf{k}_1} \pi_{\mathbf{k}_2} \pi_{\mathbf{k}_3} \rangle' = \frac{12c_{\dot{\pi}^3}}{f_\pi^2} \text{Im} \left(f_{k_1}^*(\eta_*) f_{k_2}^*(\eta_*) f_{k_3}^*(\eta_*) \int_{-\infty_-}^{\eta_*} \frac{d\eta}{H\eta} f'_{k_1}(\eta) f'_{k_2}(\eta) f'_{k_3}(\eta) \right), \quad (\text{E.88})$$

where we have summed over all the permutations of the external legs. Since the interaction is symmetric, this just produces a factor of $3! = 6$. In order to evaluate this integral, we note that the derivative of the mode function (E.86) is very simple

$$f'_k(\eta) = \frac{H}{\sqrt{2k^3}} k^2 \eta e^{ik\eta}. \quad (\text{E.89})$$

Using this fact, and taking the $\eta_* \rightarrow 0$ limit, we get

$$\langle \pi_{\mathbf{k}_1} \pi_{\mathbf{k}_2} \pi_{\mathbf{k}_3} \rangle' = \frac{3c_{\dot{\pi}^3}}{f_\pi^2} \frac{H^5}{2k_1 k_2 k_3} \text{Im} \left(\int_{-\infty_-}^{\eta_*} d\eta \eta^2 e^{i(k_1+k_2+k_3)\eta} \right). \quad (\text{E.90})$$

This time integral is straightforward to do by differentiating under the integral:

$$\int_{-\infty}^0 d\eta \eta^2 e^{i\alpha\eta} = - \frac{d^2}{d\alpha^2} \int_{-\infty}^0 d\eta e^{i\alpha\eta} = i \frac{d^2}{d\alpha^2} \frac{1}{\alpha} = \frac{2i}{\alpha^3}. \quad (\text{E.91})$$

All together, we find

$$\langle \pi_{\mathbf{k}_1} \pi_{\mathbf{k}_2} \pi_{\mathbf{k}_3} \rangle' = \frac{3c_{\dot{\pi}^3}}{f_\pi^2} \frac{H^5}{k_1 k_2 k_3 (k_1 + k_2 + k_3)^3}. \quad (\text{E.92})$$

For later, it is convenient to write this in the basis of elementary symmetric polynomials in three variables because it is symmetric under the interchange of k_1, k_2, k_3 . These polynomials are defined by

$$K_{(1)} \equiv k_1 + k_2 + k_3, \quad K_{(2)} \equiv k_1 k_2 + k_2 k_3 + k_3 k_1, \quad K_{(3)} \equiv k_1 k_2 k_3. \quad (\text{E.93})$$

We can then write the correlator as

$$\langle \pi_{\mathbf{k}_1} \pi_{\mathbf{k}_2} \pi_{\mathbf{k}_3} \rangle' = \frac{3c_{\dot{\pi}^3}}{f_\pi^2} \frac{H^5}{K_{(3)} K_{(1)}^3}. \quad (\text{E.94})$$

$\dot{\pi}(\nabla\pi)^2$ interaction: Next, we consider the $\dot{\pi}(\nabla\pi)^2$ interaction. Following the same steps as for the $\dot{\pi}^3$ interaction, we obtain

$$\begin{aligned}\langle \pi_{\mathbf{k}_1} \pi_{\mathbf{k}_2} \pi_{\mathbf{k}_3} \rangle' = & -\frac{2c_{\dot{\pi}(\nabla\pi)^2}}{f_\pi^2} (\mathbf{k}_2 \cdot \mathbf{k}_3) \operatorname{Im} \left(f_{k_1}^*(\eta_*) f_{k_2}^*(\eta_*) f_{k_3}^*(\eta_*) \int_{-\infty}^{\eta_*} \frac{d\eta}{H\eta} f'_{k_1}(\eta) f_{k_2}(\eta) f_{k_3}(\eta) \right) \\ & + 5 \text{ perms.}\end{aligned}\quad (\text{E.95})$$

The primary difference from the previous case is that the spatial derivatives turn into factors of momenta, $\nabla \mapsto -i\mathbf{k}$, and that we now have to sum explicitly over all permutations of the external fields in order to make the final correlator symmetric. In this case, the integral that we need to do is

$$\begin{aligned}\langle \pi_{\mathbf{k}_1} \pi_{\mathbf{k}_2} \pi_{\mathbf{k}_3} \rangle' = & -\frac{H^5 c_{\dot{\pi}^3}}{f_\pi^2} \frac{(\mathbf{k}_2 \cdot \mathbf{k}_3)}{4k_1 k_2^3 k_3^3} \operatorname{Im} \left(\int_{-\infty}^0 d\eta (1 - ik_2\eta)(1 - ik_3\eta) e^{i(k_1+k_2+k_3)\eta} \right) \\ & + 5 \text{ perms.}\end{aligned}\quad (\text{E.96})$$

This integral is much easier than it looks:

$$\begin{aligned}\int_{-\infty}^0 d\eta (1 - ik_2\eta)(1 - ik_3\eta) e^{i(k_1+k_2+k_3)\eta} &= (1 - k_2 \partial_{k_2})(1 - k_3 \partial_{k_3}) \int_{-\infty}^0 d\eta e^{i(k_1+k_2+k_3)\eta} \\ &= (1 - k_2 \partial_{k_2})(1 - k_3 \partial_{k_3}) \frac{-i}{(k_1 + k_2 + k_3)} \\ &= -i \frac{k_1^2 + 3k_1(k_2 + k_3) + 2(k_2^2 + k_3^2) + 3k_2 k_3}{(k_1 + k_2 + k_3)^3}.\end{aligned}\quad (\text{E.97})$$

We can also use the fact that $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$ to write $\mathbf{k}_2 + \mathbf{k}_3 = -\mathbf{k}_1$ and square both sides:

$$k_2^2 + k_3^2 + 2\mathbf{k}_2 \cdot \mathbf{k}_3 = k_1^2 \implies \mathbf{k}_2 \cdot \mathbf{k}_3 = \frac{1}{2} (k_1^2 - k_2^2 - k_3^2). \quad (\text{E.98})$$

Putting this all together, we find

$$\begin{aligned}\langle \pi_{\mathbf{k}_1} \pi_{\mathbf{k}_2} \pi_{\mathbf{k}_3} \rangle' = & \frac{H^5 c_{\dot{\pi}(\nabla\pi)^2}}{f_\pi^2} \frac{(k_1^2 - k_2^2 - k_3^2)}{8k_1 k_2^3 k_3^3} \left(\frac{k_1^2 + 3k_1(k_2 + k_3) + 2(k_2^2 + k_3^2) + 3k_2 k_3}{(k_1 + k_2 + k_3)^3} \right) \\ & + 5 \text{ perms.}\end{aligned}\quad (\text{E.99})$$

Finally, we have to sum over all the permutations of the external momenta to symmetrize the correlator. After doing this, the correlator can uniquely be written in the basis (E.93):

$$\langle \pi_{\mathbf{k}_1} \pi_{\mathbf{k}_2} \pi_{\mathbf{k}_3} \rangle' = \frac{H^5 c_{\dot{\pi}(\nabla\pi)^2}}{f_\pi^2} \frac{A_{\dot{\pi}(\nabla\pi)^2}}{4K_{(3)} K_{(1)}^3},$$

(E.100)

where

$$A_{\dot{\pi}(\nabla\pi)^2} \equiv K_{(1)}^6 - 3K_{(1)}^4 K_{(2)} - 4K_{(1)}^2 K_{(2)}^2 + 11K_{(1)}^3 K_{(3)} - 4K_{(1)} K_{(2)} K_{(3)} + 12K_{(3)}^2. \quad (\text{E.101})$$

This completes the derivation of the bispectrum in the EFT of inflation (at leading order in derivatives).

Solution to Exercise 3.6

The D -dimensional equation of motion for a generic massive scalar is

$$\Phi_k'' - \frac{(D-2)}{\eta} \Phi_k' + \left(k^2 + \frac{m^2}{H^2 \eta^2} \right) \Phi_k = 0. \quad (\text{E.102})$$

Choosing the conformally coupled mass,

$$\frac{m^2}{H^2} = \frac{D(D-2)}{4}, \quad (\text{E.103})$$

the relevant mode function is

$$f_k(\eta) = i H^{(D-2)/2} (-\eta)^{(D-2)/2} \frac{e^{ik\eta}}{\sqrt{2k}}. \quad (\text{E.104})$$

The Wightman function is then given by

$$W_k(\eta, \eta') = \frac{H^{D-2}}{2k} (\eta \eta')^{\frac{D-2}{2}} e^{-ik(\eta-\eta')}. \quad (\text{E.105})$$

The two diagrams we then want to calculate are

$$\begin{aligned} I_+ &= i\lambda \int_{-\infty}^{\eta_*} d\eta a^D(\eta) W_{k_1}(\eta, \eta_*) W_{k_2}(\eta, \eta_*) W_{k_3}(\eta, \eta_*) W_{k_4}(\eta, \eta_*) \\ &= i \frac{\lambda H^{3D-8} (-\eta_*)^{2(D-2)}}{16k_1 k_2 k_3 k_4} e^{iK\eta_*} \int_{-\infty}^{\eta_*} d\eta (-\eta)^{D-4} e^{-iK\eta} \\ &= i^{D-2} \frac{\lambda H^{3D-8} (-\eta_*)^{2(D-2)}}{16k_1 k_2 k_3 k_4 K^{D-3}} \Gamma[D-3], \end{aligned} \quad (\text{E.106})$$

$$\begin{aligned} I_- &= -i\lambda \int_{-\infty}^{\eta_*} d\eta a^D(\eta) W_{k_1}(\eta_*, \eta) W_{k_2}(\eta_*, \eta) W_{k_3}(\eta_*, \eta) W_{k_4}(\eta_*, \eta) \\ &= -i \frac{\lambda H^{3D-8} (-\eta_*)^{2(D-2)}}{16k_1 k_2 k_3 k_4} e^{-iK\eta_*} \int_{-\infty}^{\eta_*} d\eta (-\eta)^{D-4} e^{iK\eta} \\ &= (-i)^{D-2} \frac{\lambda H^{3D-8} (-\eta_*)^{2(D-2)}}{16k_1 k_2 k_3 k_4 K^{D-3}} \Gamma[D-3]. \end{aligned} \quad (\text{E.107})$$

Adding these two contributions together, we obtain the result for the four-point function:

$$\langle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \phi_{\mathbf{k}_4} \rangle' = \cos \left(\frac{(D-2)\pi}{2} \right) \Gamma[D-3] \frac{\lambda H^{3D-8} \eta_*^{2(D-2)}}{8k_1 k_2 k_3 k_4} \frac{1}{K^{D-3}}.$$

(E.108)

This has the interesting feature that it vanishes for odd D due to interference between the two branches of the in-in contour. This effect was pointed out in [125], which has more examples.

Solution to Exercise 3.7

[DB: Austin, please add.]

Solution to Exercise 3.8

We start from (3.91) and (3.92):

$$I_{++} = -\frac{g^2 \eta_*^4}{16k_1 k_2 k_3 k_4} \int_{-\infty}^{\eta_*} \frac{d\eta'}{(\eta')^2} \int_{-\infty}^{\eta_*} \frac{d\eta''}{(\eta'')^2} e^{ik_{12}\eta'} G_F(k_I; \eta', \eta'') e^{ik_{34}\eta''}, \quad (\text{E.109})$$

$$I_{+-} = \frac{g^2 \eta_*^4}{16k_1 k_2 k_3 k_4} \int_{-\infty}^{\eta_*} \frac{d\eta'}{(\eta')^2} \int_{-\infty}^{\eta_*} \frac{d\eta''}{(\eta'')^2} e^{ik_{12}\eta'} W_{k_I}(\eta'', \eta') e^{-ik_{34}\eta''}. \quad (\text{E.110})$$

In the $k_I \rightarrow 0$ limit, we can use that both the Feynman and Wightman propagators are given by

$$\begin{aligned} \lim_{k_I \rightarrow 0} G(k_I; \eta', \eta'') &= \lim_{k_I \rightarrow 0} W_{k_I}(\eta', \eta'') \\ &= \frac{H^2}{4\pi} (\eta' \eta'')^{3/2} \left[\Gamma[-i\mu]^2 \left(\frac{k_I^2 \eta' \eta''}{4} \right)^{i\mu} + \Gamma[i\mu]^2 \left(\frac{k_I^2 \eta' \eta''}{4} \right)^{-i\mu} \right]. \end{aligned} \quad (\text{E.111})$$

With this simplification, all integrals will be of the form

$$\int_{-\infty}^0 d\eta' e^{\pm i\alpha\eta'} (-\eta')^{\Delta-2} = (\mp i)^{\Delta-1} \alpha^{1-\Delta} \Gamma[\Delta-1]. \quad (\text{E.112})$$

Using this formula, we find

$$I_{++} = -\frac{g^2 H^2 \eta_*^4}{64\pi k_1 k_2 k_3 k_4 (k_{12} k_{34})^{1/2}} \left[(-i)^{1+2i\mu} \Gamma[\frac{1}{2} + i\mu]^2 \Gamma[-i\mu]^2 \left(\frac{k_I^2}{4k_{12} k_{34}} \right)^{i\mu} + \text{c.c.} \right], \quad (\text{E.113})$$

$$I_{+-} = \frac{g^2 H^2 \eta_*^4}{64\pi k_1 k_2 k_3 k_4 (k_{12} k_{34})^{1/2}} \left[\Gamma[\frac{1}{2} + i\mu]^2 \Gamma[-i\mu]^2 \left(\frac{k_I^2}{4k_{12} k_{34}} \right)^{i\mu} + \text{c.c.} \right]. \quad (\text{E.114})$$

Combining these contributions, along with $I_{--} = I_{++}^*$ and $I_{-+} = I_{+-}^*$, we obtain

$$\begin{aligned} \langle \phi_{\mathbf{k}_1} \phi_{\mathbf{k}_2} \phi_{\mathbf{k}_3} \phi_{\mathbf{k}_4} \rangle' &= \frac{g^2 H^2 \eta_*^4}{64\pi k_1 k_2 k_3 k_4 (k_{12} k_{34})^{1/2}} \left[(1 - i \sinh \pi\mu) \Gamma[\frac{1}{2} + i\mu]^2 \Gamma[-i\mu]^2 \left(\frac{k_I^2}{4k_{12} k_{34}} \right)^{i\mu} \right. \\ &\quad \left. + (1 + i \sinh \pi\mu) \Gamma[\frac{1}{2} - i\mu]^2 \Gamma[i\mu]^2 \left(\frac{k_I^2}{4k_{12} k_{34}} \right)^{-i\mu} \right]. \end{aligned} \quad (\text{E.115})$$

One of the interesting features of this expression is that it oscillates as a function of $k_I^2/(k_{12} k_{34})$, where the period of the oscillation is set by μ . Since μ is in turn fixed by the mass of the exchanged particle, this oscillatory behavior can be used to perform a kind of spectroscopy on the exchanged particles to extract their mass. See [17], for more details on this ‘cosmological collider physics.’

Solution to Exercise 4.1

Recall the Bessel equation

$$x^2 \frac{d^2 g(x)}{dx^2} + x \frac{dg(x)}{dx} + (x^2 - \nu^2)g(x) = 0, \quad (\text{E.116})$$

which has the Hankel functions $H_\nu^{(1,2)}(x)$ as its solutions. The Wronskian of the two Hankel functions is defined as

$$W \equiv H_\nu^{(1)} \partial_x H_\nu^{(2)} - H_\nu^{(2)} \partial_x H_\nu^{(1)}. \quad (\text{E.117})$$

Taking a derivative of W gives

$$\begin{aligned} \frac{dW}{dx} &= H_\nu^{(1)} \partial_x^2 H_\nu^{(2)} - H_\nu^{(2)} \partial_x^2 H_\nu^{(1)} \\ &= \frac{H_\nu^{(1)}}{x^2} \left(-x \partial_x H_\nu^{(2)} - (x^2 - \nu^2) H_\nu^{(2)} \right) - \frac{H_\nu^{(2)}}{x^2} \left(-x \partial_x H_\nu^{(1)} - (x^2 - \nu^2) H_\nu^{(1)} \right) \\ &= -\frac{1}{x} \left(H_\nu^{(1)} \partial_x H_\nu^{(2)} - H_\nu^{(2)} \partial_x H_\nu^{(1)} \right) \\ &= -\frac{W}{x}. \end{aligned} \quad (\text{E.118})$$

We have therefore found a first-order differential equation for the Wronskian, which has the following general solution

$$W(x) = \frac{A}{x}. \quad (\text{E.119})$$

To fix the normalization A , we consider the large argument limit of the Hankel functions:

$$H_\nu^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}} e^{i(x-\nu\pi/2-\pi/4)} + O(x^{-3/2}), \quad (\text{E.120})$$

$$H_\nu^{(2)}(x) \sim \sqrt{\frac{2}{\pi x}} e^{-i(x-\nu\pi/2-\pi/4)} + O(x^{-3/2}). \quad (\text{E.121})$$

The derivations of these limiting functions are

$$\partial_x H_\nu^{(1)} \sim i H_\nu^{(1)}, \quad (\text{E.122})$$

$$\partial_x H_\nu^{(2)} \sim -i H_\nu^{(2)}, \quad (\text{E.123})$$

and the Wronksian becomes

$$\begin{aligned} W &= H_\nu^{(1)} \partial_x H_\nu^{(2)} - H_\nu^{(2)} \partial_x H_\nu^{(1)} \\ &\sim \sqrt{\frac{2}{\pi x}} (-i) \sqrt{\frac{2}{\pi x}} - \sqrt{\frac{2}{\pi x}} i \sqrt{\frac{2}{\pi x}} = -\frac{4i}{\pi} \frac{1}{x}. \end{aligned} \quad (\text{E.124})$$

This fixed the constant A in (E.125), so that

$$W(x) = -\frac{4i}{\pi} \frac{1}{x},$$

(E.125)

which is the identity we used in the text.

Solution to Exercise 4.2

For purely imaginary order, $\nu = i\mu$, the complex conjugate of the Hankel function satisfies $H_{i\mu}^{(2)*} = e^{-\pi\mu} H_{i\mu}^{(1)}$. We therefore have

$$\begin{aligned}\text{Im} \left(H_\nu^{(2)*} \partial_\eta H_\nu^{(2)} \right) \Big|_{\eta=\eta_*} &= \frac{1}{2i} e^{-\pi\mu} \left(H_\nu^{(1)} \partial_\eta H_\nu^{(2)} - H_\nu^{(2)} \partial_\eta H_\nu^{(1)} \right) \\ &= -\frac{2}{\pi} \frac{1}{\eta_*} e^{-\pi\mu},\end{aligned}\quad (\text{E.126})$$

where we have used the Wronskian of the Hankel function to obtain the second line. Substituting this into (4.27), we get

$$\text{Im}(A \partial_\eta \log K) \Big|_{\eta=\eta_*} = \frac{1}{(H\eta_*)^2} \frac{1}{|H_\nu^{(2)}(-k\eta_*)|^2} \left(-\frac{2}{\pi} \frac{1}{\eta_*} e^{-\pi\mu} \right). \quad (\text{E.127})$$

The two-point function of the field therefore is

$$\boxed{\langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle' = \frac{\pi}{4} H^2(-\eta_*)^3 e^{\pi\mu} |H_{i\mu}^{(2)}(-k\eta_*)|^2}. \quad (\text{E.128})$$

Using $H_{i\mu}^{(2)*} = e^{-\pi\mu} H_{i\mu}^{(1)}$, we may also write

$$\boxed{\langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle' = \frac{\pi}{4} H^2(-\eta_*)^3 H_\nu^{(1)}(-k\eta_*) H_\nu^{(2)}(-k\eta_*)}. \quad (\text{E.129})$$

The latter form of writing the result holds for both real and imaginary ν .

Solution to Exercise 4.3

The Green's function solves the equation

$$(\partial_t^2 + \omega^2) G(t, t') = -i\delta(t - t'), \quad (\text{E.130})$$

as well as the same equation with $t \leftrightarrow t'$. This implies that the Green's function will be symmetric under interchanging t and t' , as it should be. Away from $t = t'$, the most general solution to (E.130) for t is

$$G(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t}, \quad (\text{E.131})$$

and similarly for t' . The general Green's function is then a product of these two solutions

$$G(t, t') = (c_1 e^{i\omega t} + c_2 e^{-i\omega t})(c_3 e^{i\omega t'} + c_4 e^{-i\omega t'}). \quad (\text{E.132})$$

We now impose boundary conditions to select the Green's function of interest:

- *Asymptotics:* When either $t \rightarrow -\infty$ or $t' \rightarrow -\infty$, we want the field to scale as $e^{i\omega t}$. The most general solution compatible with this boundary condition is

$$G(t, t') = \begin{cases} (c_1 e^{i\omega t} + c_2 e^{-i\omega t}) e^{i\omega t'} & \text{when } t' < t, \\ (c_3 e^{i\omega t'} + c_4 e^{-i\omega t'}) e^{i\omega t} & \text{when } t < t'. \end{cases} \quad (\text{E.133})$$

- *Continuity:* We want the Green's function to be continuous at $t = t'$. This requires that $c_1 = c_3$ and $c_2 = c_4$.
- *Jump condition:* Integrating (E.130) across the jump, we find that the discontinuity in the first derivative of G must be [AJ: I think should be $-i$, check sign]

$$\Delta \partial_t G(k; t, t') \Big|_{t=t'-\epsilon}^{t=t'+\epsilon} = i. \quad (\text{E.134})$$

From (E.133), with $c_1 = c_3$ and $c_2 = c_4$, we find

$$\Delta \partial_t G(t, t') \Big|_{t=t'-\epsilon}^{t=t'+\epsilon} = 2i\omega c_2, \quad (\text{E.135})$$

which implies that

$$c_2 = \frac{1}{2\omega}. \quad (\text{E.136})$$

- *Vanishing on boundary:* Finally, the solution must go to zero as $t \rightarrow t_*$.⁶⁹ This fixes $c_1 = -(2\omega)^{-1}e^{-2\omega t_*}$ for $t > t'$, so that

$$G(t, t') = \begin{cases} \frac{1}{2\omega}e^{-i\omega(t-t')} - \frac{1}{2\omega}e^{i\omega(t-t_*)}e^{i\omega k(t'-t_*)} & \text{when } t > t', \\ \frac{1}{2\omega}e^{i\omega(t-t')} - \frac{1}{2\omega}e^{i\omega(t-t_*)}e^{i\omega(t'-t_*)} & \text{when } t' > t. \end{cases} \quad (\text{E.137})$$

This can be written as a single expression

$$G(t, t') = \frac{1}{2\omega} \left(e^{-i\omega(t-t')} \theta(t - t') + e^{i\omega(t-t')} \theta(t' - t) - e^{i\omega(t-t_*)} e^{i\omega(t'-t_*)} \right). \quad (\text{E.138})$$

For $t_* \rightarrow 0$, this becomes

$$G(t, t') = \frac{1}{2\omega} \left(e^{-i\omega(t-t')} \theta(t - t') + e^{i\omega(t-t')} \theta(t' - t) - e^{i\omega(t+t')} \right), \quad (\text{E.139})$$

which is the answer quoted in the text.

Solution to Exercise 4.4

We consider the following anharmonic oscillator

$$S[\Phi] = \int dt \left(\frac{1}{2} \dot{\Phi}^2 - \frac{1}{2} \omega^2 \Phi^2 + L_{\text{int}} \right), \quad (\text{E.140})$$

for a general interaction Lagrangian $L_{\text{int}}[\Phi]$. The classical equation of motion is

$$\ddot{\Phi} + \omega^2 \Phi = \frac{\delta S_{\text{int}}}{\delta \Phi}. \quad (\text{E.141})$$

⁶⁹Note that this is kind of a nonstandard boundary condition, but it is important. We are forced to impose this condition because we are trying to compute the wavefunctional as a function of some boundary conditions specified on some time slice at t_* . This Dirichlet problem requires this set of boundary condition.

A formal solution of this equation is

$$\Phi_{\text{cl}}(t) = \phi e^{i\omega t} + i \int dt' G(t, t') \frac{\delta S_{\text{int}}}{\delta \Phi(t')} \Big|_{\Phi=\Phi_{\text{cl}}}, \quad (\text{E.142})$$

where $G(t, t')$ is the Green's function of the harmonic oscillator. Since we have extracted the solution of the harmonic oscillator, $\phi e^{i\omega t}$, the boundary condition $\Phi_{\text{cl}}(t \rightarrow t_* = 0) = \phi$ implies $G(0, t') = G(t, 0) = 0$. The solution of $(\partial_t^2 + \omega^2)G(t, t') = -i\delta(t - t')$, with this boundary condition, is

$$G(t, t') = \frac{1}{2\omega} \left(e^{-i\omega(t-t')} \theta(t-t') + e^{i\omega(t-t')} \theta(t'-t) - e^{i\omega(t+t')} \right), \quad (\text{E.143})$$

which you were invited to derive in Exercise 4.3.

Although equation (E.142) is a bit formal, it is useful because we can easily substitute it back into the action. In order to do this, it is convenient to integrate (E.140) by parts:

$$S[\Phi] = \frac{1}{2} \Phi \dot{\Phi} \Big|_{t=t_*} + \int dt \left(-\frac{1}{2} \Phi (\ddot{\Phi} + \omega^2 \Phi) + L_{\text{int}} \right). \quad (\text{E.144})$$

Using (E.141), the on-shell action then is

$$S[\Phi_{\text{cl}}] = \frac{1}{2} \Phi_{\text{cl}} \dot{\Phi}_{\text{cl}} \Big|_{t=t_*} + \int dt \left(-\frac{1}{2} \Phi_{\text{cl}} \frac{\delta S_{\text{int}}}{\delta \Phi_{\text{cl}}} + L_{\text{int}} \right). \quad (\text{E.145})$$

For the moment, let us focus on the boundary term. When we substitute in (E.142), the first term produces the expected Gaussian part of the wavefunction. The second term in (E.142) is, however, more subtle. In order to evaluate it, we have to note that the Green's function (E.143) satisfies the identity

$$\lim_{t \rightarrow t_*} \partial_t G(t, t') = -ie^{i\omega t'}, \quad (\text{E.146})$$

for $t_* = 0$. Using this, we find

$$S[\Phi_{\text{cl}}] = \frac{i}{2} \omega \phi^2 + \int dt \left(\frac{1}{2} \phi e^{i\omega t} \frac{\delta S_{\text{int}}}{\delta \Phi_{\text{cl}}} - \frac{1}{2} \Phi_{\text{cl}} \frac{\delta S_{\text{int}}}{\delta \Phi_{\text{cl}}} + L_{\text{int}} \right), \quad (\text{E.147})$$

where we have relabeled $t' \rightarrow t$ in the first term in the integral. Plugging (E.142) into the second term inside the integral, the terms proportional to ϕ cancel and we are left with

$$S[\Phi_{\text{cl}}] = \frac{i}{2} \omega \phi^2 + \int dt L_{\text{int}} - \frac{i}{2} \int dt dt' G(t, t') \frac{\delta S_{\text{int}}}{\delta \Phi_{\text{cl}}(t')} \frac{\delta S_{\text{int}}}{\delta \Phi_{\text{cl}}(t)}, \quad (\text{E.148})$$

as required.

Solution to Exercise 4.5

We have seen that the solution of the anharmonic oscillator at order was g is

$$\Phi^{(1)}(t) = i \int_{-\infty}^0 dt' G(t, t') \left(-\phi^2 e^{2i\omega t'} \right). \quad (\text{E.149})$$

Substituting the Green's function (E.139), this becomes

$$\begin{aligned}
\Phi^{(1)}(t) &= -\frac{i\phi^2}{2\omega} \int_{-\infty}^0 dt' \left(e^{-i\omega(t-t')} \theta(t-t') + e^{i\omega(t-t')} \theta(t'-t) - e^{i\omega(t+t')} \right) e^{2i\omega t'} \\
&= -\frac{i\phi^2}{2\omega} \int_{-\infty}^0 dt' \left(e^{-i\omega(t-3t')} \theta(t-t') + e^{i\omega(t+t')} \theta(t'-t) - e^{i\omega(t+3t')} \right) \\
&= -\frac{i\phi^2}{2\omega} \left[e^{-i\omega t} \int_{-\infty}^t dt' e^{3i\omega t'} + e^{i\omega t} \int_t^0 dt' e^{i\omega t'} - e^{i\omega t} \int_{-\infty}^0 dt' e^{3i\omega t'} \right] \\
&= -\frac{i\phi^2}{2\omega} \left[\frac{1}{3i\omega} e^{2i\omega t} + \frac{e^{i\omega t}}{i\omega} (1 - e^{i\omega t}) - \frac{e^{i\omega t}}{3i\omega} \right] \\
&= \boxed{\frac{\phi^2}{3\omega^2} (e^{2i\omega t} - e^{i\omega t})}. \tag{E.150}
\end{aligned}$$

In equation (4.47), we wrote the on-shell action as

$$\begin{aligned}
S[\Phi_{\text{cl}}] &= \frac{i\omega}{2} \phi^2 - \frac{g}{3} \int dt (\Phi^{(0)})^3 - \frac{g^2}{2} \int dt (\Phi^{(0)})^2 \Phi^{(1)} + \dots \\
&\equiv S^{(0)} + gS^{(1)} + g^2 S^{(2)} + \dots
\end{aligned} \tag{E.151}$$

Substituting the explicit solutions for $\Phi^{(0)}$ and $\Phi^{(1)}$, we can compute this up to order g^2 :

$$\begin{aligned}
S^{(1)} &\equiv -\frac{1}{3} \int dt (\Phi^{(0)})^3 = -\frac{1}{3} \int_{-\infty}^0 dt \phi^3 e^{3i\omega t} = -\frac{\phi^3}{3} \frac{1}{3i\omega} = \boxed{i \frac{\phi^3}{9\omega}}, \tag{E.152} \\
S^{(2)} &\equiv -\frac{1}{2} \int dt (\Phi^{(0)})^2 \Phi^{(1)} = -\frac{1}{2} \int_{-\infty}^0 dt \phi^2 e^{2i\omega t} \frac{\phi^2}{3\omega^2} (e^{2i\omega t} - e^{i\omega t}) \\
&= -\frac{\phi^4}{6\omega^2} \int_{-\infty}^0 dt (e^{4i\omega t} - e^{3i\omega t}) \\
&= -\frac{\phi^4}{6\omega^2} \left(\frac{1}{4i\omega} - \frac{1}{3i\omega} \right) = \boxed{-i \frac{\phi^4}{72\omega^3}}. \tag{E.153}
\end{aligned}$$

Up to order g^2 , the perturbative wavefunction therefore is

$$\Psi[\phi] \approx e^{iS[\Phi_{\text{cl}}]} = \exp \left(-\frac{\omega}{2} \phi^2 - \frac{g}{9\omega} \phi^3 + \frac{g^2}{72\omega^3} \phi^4 + \dots \right), \tag{E.154}$$

which is the result we quoted in the text.

Solution to Exercise 4.6

[DB: Austin, please add]

Solution to Exercise 4.7

We start by writing the bulk-to-boundary propagator (4.76) and the bulk-to-bulk propagator (4.77) in terms of the Wightman propagator (3.34) and Feynman propagator (3.35):

$$K_k(t) = \frac{W_k(t_*, t)}{|f_k(t_*)|^2}, \quad (\text{E.155})$$

$$G(k; t, t') = G_F(k; t, t') - \frac{W_k(t_*, t)W_k(t_*, t')}{|f_k(t_*)|^2}. \quad (\text{E.156})$$

We will now use these to express (4.55) and (4.56) in terms of the building blocks of the in-in formalism.

Three-point function The three-point wavefunction coefficient is

$$\begin{aligned} \langle O_1 O_2 O_3 \rangle' &= ig \int_{-\infty}^{t_*} Dt' K_{k_1}(t') K_{k_2}(t') K_{k_3}(t') \\ &= \frac{ig}{|f_{k_1}(t_*)|^2 |f_{k_2}(t_*)|^2 |f_{k_3}(t_*)|^2} \int_{-\infty}^{t_*} Dt' W_{k_1}(t_*, t') W_{k_2}(t_*, t') W_{k_3}(t_*, t'), \end{aligned} \quad (\text{E.157})$$

where $Dt' \equiv dt' a^3(t')$. The integral is the same that appeared in the computation of I_+ in (3.37), so we have

$$\langle O_1 O_2 O_3 \rangle' = 8 \langle O_1 O_1 \rangle' \langle O_2 O_2 \rangle' \langle O_3 O_3 \rangle' \times I_+. \quad (\text{E.158})$$

where we have used that $2 \langle O_n O_n \rangle' = |f_{k_n}(t_*)|^{-2}$. Equation (4.55) then reads

$$\langle \phi_1 \phi_2 \phi_3 \rangle' = \frac{\langle O_1 O_2 O_3 \rangle' + \langle O_1 O_2 O_3 \rangle'^*}{8 \langle O_1 O_1 \rangle' \langle O_2 O_2 \rangle' \langle O_3 O_3 \rangle'} = I_+ + I_-, \quad (\text{E.159})$$

which is indeed the in-in expression for the three-point function.

Four-point function We first consider the second line of (4.56). Using (E.157), we have

$$\begin{aligned} \langle O_1 O_2 O_I \rangle' \langle O_I O_3 O_4 \rangle'^* &= (-ig) \int Dt' \frac{W_{k_1}(t_*, t')}{|f_{k_1}(t_*)|^2} \frac{W_{k_2}(t_*, t')}{|f_{k_2}(t_*)|^2} \frac{W_{k_I}(t_*, t')}{|f_{k_I}(t_*)|^2} \\ &\quad \times (ig) \int Dt'' \frac{W_{k_I}^*(t_*, t'')}{|f_{k_I}(t_*)|^2} \frac{W_{k_3}^*(t_*, t'')}{|f_{k_3}(t_*)|^2} \frac{W_{k_4}^*(t_*, t'')}{|f_{k_4}(t_*)|^2}, \end{aligned} \quad (\text{E.160})$$

and hence get

$$\begin{aligned} I_2 &\equiv \frac{\langle O_1 O_2 O_I \rangle' \langle O_I O_3 O_4 \rangle'^* + c.c.}{32 \langle O_I O_I \rangle' \prod_{n=1}^4 \langle O_n O_n \rangle'} = g^2 \int Dt' Dt'' W_{k_1}(t_*, t') W_{k_2}(t_*, t') \\ &\quad \times \frac{W_{k_I}(t_*, t') W_{k_I}^*(t_*, t'')}{|f_{k_I}(t_*)|^2} W_{k_3}^*(t_*, t'') W_{k_4}^*(t_*, t'') + c.c. \end{aligned} \quad (\text{E.161})$$

Since

$$\begin{aligned} \frac{W_{k_I}(t_*, t') W_{k_I}^*(t_*, t'')}{|f_{k_I}(t_*)|^2} &= \frac{f_{k_I}^*(t_*) f_{k_I}(t') (f_{k_I}^*(t_*) f_{k_I}(t''))^*}{f_{k_I}(t_*) f_{k_I}^*(t_*)} \\ &= f_{k_I}^*(t'') f_{k_I}(t') \\ &= W_{k_I}(t'', t'), \end{aligned} \quad (\text{E.162})$$

we see that the integral in (E.161) is equal to $I_{-+}^* = I_{+-}$; cf. (3.54). Adding the complex conjugate, we find that the second line of (4.56) is

$$I_2 = I_{+-} + I_{-+}. \quad (\text{E.163})$$

Next, we consider the first line in (4.56). The four-point wavefunction coefficient can be written as

$$\begin{aligned} \langle O_1 O_2 O_3 O_4 \rangle' &= -g^2 \int Dt' Dt'' K_{k_1}(t') K_{k_2}(t') G(k_I; t', t'') K_{k_3}(t'') K_{k_4}(t'') \\ &= -g^2 \int Dt' Dt'' \frac{W_{k_1}(t_*, t')}{|f_{k_1}(t_*)|^2} \frac{W_{k_2}(t_*, t')}{|f_{k_2}(t_*)|^2} \left[G_F(k_I; t', t'') - \frac{W_{k_I}(t_*, t') W_{k_I}(t_*, t'')}{|f_{k_I}(t_*)|^2} \right] \\ &\quad \times \frac{W_{k_3}(t_*, t'')}{|f_{k_3}(t_*)|^2} \frac{W_{k_4}(t_*, t'')}{|f_{k_4}(t_*)|^2}. \end{aligned} \quad (\text{E.164})$$

The term involving the Feynman propagator is equal to $16 \prod_{n=1}^4 \langle O_n O_n \rangle' \times I_{++}$. Similarly, $\langle O_1 O_2 O_3 O_4 \rangle'^*$ will contain a term equal to $16 \prod_{n=1}^4 \langle O_n O_n \rangle' \times I_{--}$. The first line in (4.56) therefore becomes

$$I_1 \equiv \frac{2 \operatorname{Re} \langle O_1 O_2 O_3 O_4 \rangle'}{16 \prod_{n=1}^4 \langle O_n O_n \rangle'} = I_{++} + I_{--} + 2 \operatorname{Re}(\Delta I_1), \quad (\text{E.165})$$

where

$$\Delta I_1 \equiv g^2 \int Dt' Dt'' W_{k_1}(t_* t') W_{k_2}(t_*, t') \frac{W_{k_I}(t_*, t') W_{k_I}(t_*, t'')}{|f_{k_I}(t_*)|^2} W_{k_3}(t_*, t'') W_{k_4}(t_*, t''). \quad (\text{E.166})$$

To reproduce the correct in-in result, the extra term ΔI_1 must cancel against the last line of (4.56).

Up to one complex conjugation, the derivation of the last line of (4.56) is the same as that of the second line; cf. (E.161). We find that

$$\begin{aligned} I_3 \equiv \frac{\langle O_1 O_2 O_I \rangle' \langle O_I O_3 O_4 \rangle' + c.c.}{32 \langle O_I O_I \rangle' \prod_{n=1}^4 \langle O_n O_n \rangle'} &= -g^2 \int Dt' Dt'' W_{k_1}(t_*, t') W_{k_2}(t_*, t') \\ &\quad \times \frac{W_{k_I}(t_*, t') W_{k_I}(t_*, t'')}{|f_{k_I}(t_*)|^2} W_{k_3}(t_*, t'') W_{k_4}(t_*, t'') + c.c. \\ &= -2 \operatorname{Re}(\Delta I_1). \end{aligned} \quad (\text{E.167})$$

The sum of all contributions therefore indeed reproduces the correct in-in result:

$$\begin{aligned} \langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle' &= I_1 + I_2 + I_3 \\ &= (I_{++} + I_{--} + 2 \operatorname{Re}(\Delta I_1)) + (I_{+-} + I_{-+}) - 2 \operatorname{Re}(\Delta I_1) \\ &= I_{++} + I_{--} + I_{+-} + I_{-+}. \end{aligned} \quad (\text{E.168})$$

Note that the nontrivial cancellation between the terms required the correct form of the bulk-to-bulk propagator.

Solution to Exercise 4.8

For three-site chain, there are two edges to cut:

$$E \xrightarrow{k_I} k'_I = \text{Diagram with edge } k_I + \text{Diagram with edge } k'_I \quad (\text{E.169})$$

where $E \equiv k_{12} + k_3 + k_{34}$ is the total energy entering the graph. The terms on the right-hand side are then products of the one- and two-site graphs:

$$\text{Diagram with edge } k_I = \frac{1}{(k_{123} + k'_I) E_L E_M} \times \frac{1}{E_R}, \quad (\text{E.170})$$

$$\text{Diagram with edge } k'_I = \frac{1}{E_L} \times \frac{1}{(k_{345} + k_I) E_M E_R}, \quad (\text{E.171})$$

where $E_L \equiv k_{12} + k_I$, $E_M \equiv k_3 + k_I + k'_I$ and $E_R \equiv k_{45} + k'_I$. Combining these expressions, we get

$$\boxed{\langle O_1 O_2 O_3 O_4 O_5 \rangle' = \frac{1}{E} \frac{1}{E_L E_M E_R} \left[\frac{1}{k_{123} + k'_I} + \frac{1}{k_{345} + k_I} \right]}, \quad (\text{E.172})$$

as required.

Solution to Exercise 4.9

Using the Feynman rules, the three-point wavefunction coefficient of two conformally coupled scalars and a generic massive scalar is

$$\langle O_1 O_2 X_3 \rangle' = \frac{ig}{f_{k_1}(\eta_*) f_{k_2}(\eta_*) \chi_{k_3}(\eta_*)} \int_{-\infty}^{\eta_*} \frac{d\eta}{(H\eta)^4} f_{k_1}(\eta) f_{k_2}(\eta) \chi_{k_3}(\eta), \quad (\text{E.173})$$

where the relevant mode functions are

$$f_k(\eta) = H(-\eta) \frac{e^{ik\eta}}{\sqrt{2k}}, \quad (\text{E.174})$$

$$\chi_k(\eta) = H \sqrt{\frac{\pi}{4}} e^{-\frac{i\pi}{4}(1+2\nu)} (-\eta)^{3/2} H_\nu^{(2)}(-k\eta). \quad (\text{E.175})$$

Equation (E.173) can then be written as

$$\langle O_1 O_2 X_3 \rangle' = \frac{ig}{(-\eta_*)^{7/2} H_\nu^{(2)}(-k_3\eta_*)} \frac{1}{H^4} \int_{-\infty}^{\eta_*} d\eta (-\eta)^{-1/2} e^{ik_{12}\eta} H_\nu^{(2)}(-k_3\eta). \quad (\text{E.176})$$

where we have dropped the factor of $e^{-ik_{12}\eta_*}$ (which becomes unity in the limit $\eta_* \rightarrow 0$). For the Hankel function, we then use the integral representation

$$H_\nu^{(2)}(x) = \frac{i}{\sqrt{\pi}} \frac{2^{\nu+1} e^{i\nu\pi}}{\Gamma[\nu + \frac{1}{2}]} x^\nu e^{-ix} \int_0^\infty ds e^{-2ixs} s^{\nu - \frac{1}{2}} (1+s)^{\nu - \frac{1}{2}}. \quad (\text{E.177})$$

Substituting this into (E.176), we get

$$\begin{aligned} \langle O_1 O_2 X_3 \rangle' &= -\frac{2g}{\sqrt{\pi} H^4} \frac{e^{i\nu\pi}}{\Gamma[\nu + \frac{1}{2}]} \frac{(2k_3)^\nu}{(-\eta_*)^{7/2} H_\nu^{(2)}(-k_3\eta_*)} \\ &\quad \times \int_0^\infty ds s^{\nu-\frac{1}{2}} (1+s)^{\nu-\frac{1}{2}} \int_{-\infty}^{\eta_*} d\eta e^{i[k_{12}+(1+2s)k_3]\eta} (-\eta)^{\nu-\frac{1}{2}}. \end{aligned} \quad (\text{E.178})$$

The integrals in the second line have already appeared in Exercise 3.4; cf. (E.70). Following the same steps, we find

$$\begin{aligned} \langle O_1 O_2 X_3 \rangle' &= -\sqrt{\frac{2}{\pi}} \frac{g}{H^4} e^{i\frac{\pi}{2}\nu} \frac{(-ik_3\eta_*)^{-1/2}}{(-\eta_*)^3 H_\nu^{(2)}(-k_3\eta_*)} \\ &\quad \times \Gamma[\frac{1}{2} - \nu] \Gamma[\frac{1}{2} + \nu] {}_2F_1 \left[\begin{matrix} \frac{1}{2} + \nu, \frac{1}{2} - \nu \\ 1 \end{matrix} \middle| \frac{u-1}{2u} \right], \end{aligned} \quad (\text{E.179})$$

where $u \equiv k_3/k_{12}$. To evaluate this at late times, $\eta_* \rightarrow 0$, we use the following limit of the Hankel function

$$H_\nu^{(2)}(x) \xrightarrow{x \rightarrow 0} \frac{i}{\pi} \left(\Gamma(\nu) \left(\frac{x}{2} \right)^{-\nu} + e^{i\pi\nu} \Gamma(-\nu) \left(\frac{x}{2} \right)^\nu \right). \quad (\text{E.180})$$

Assuming a small mass, $m < \frac{3}{2}H$, the parameter ν is real and positive. The first term in (E.180) then dominates at late times. In addition, we use the identities

$$\Gamma[\frac{1}{2} - \nu] \Gamma[\frac{1}{2} + \nu] = \frac{\pi}{\cos(\pi\nu)}, \quad (\text{E.181})$$

$$\Gamma[\nu] \Gamma[1 - \nu] = \frac{\pi}{\sin(\pi\nu)}. \quad (\text{E.182})$$

Equation (E.179) then becomes

$$\langle O_1 O_2 X_3 \rangle' = \frac{\sqrt{\pi}}{2^{\nu-\frac{1}{2}}} \frac{ig}{H^4} \frac{\Gamma[1-\nu]}{\cot(\pi\nu)} \frac{(-ik_3\eta_*)^{\nu-\frac{1}{2}}}{(-\eta_*)^3} {}_2F_1 \left[\begin{matrix} \frac{1}{2} + \nu, \frac{1}{2} - \nu \\ 1 \end{matrix} \middle| \frac{u-1}{2u} \right], \quad (\text{E.183})$$

as expected.

Solution to Exercise 4.10

Start from the action of a conformally coupled scalar with a polynomial interaction

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{2} (\partial\Phi)^2 - H^2 \Phi^2 - \frac{g}{n!} \Phi^n \right). \quad (\text{E.184})$$

The bulk-to-boundary propagator for this field is given by

$$K_k(\eta) = \frac{\eta}{\eta_*} e^{ik\eta}. \quad (\text{E.185})$$

Using the Feynman rules, the wavefunction coefficient of interest is

$$\begin{aligned}
\langle O_1 \cdots O_n \rangle' &\equiv \text{Feynman diagram} \\
&= -ig \int_{-\infty}^{\eta_*} \frac{d\eta}{(H\eta)^4} K_{k_1}(\eta) \cdots K_{k_n}(\eta) \\
&= -\frac{ig}{H^4 \eta_*^n} \int_{-\infty}^{\eta_*} d\eta \eta^{n-4} e^{iK\eta}, \tag{E.186}
\end{aligned}$$

where we have defined $K \equiv k_1 + \cdots + k_n$. Using (E.112), we can perform the integral and get

$$\boxed{\langle O_1 \cdots O_n \rangle'_c = -i^n \frac{\Gamma[n-3]}{n!} \frac{g}{H^4 \eta_*^n} \frac{1}{K^{n-3}}}, \tag{E.187}$$

where we have multiplied by the normalization factor (4.80) (in this case $1/n!$) in order to turn the Feynman diagram into a true wavefunction coefficient. Notice that the Gamma function diverges as $n \rightarrow 3$, but being careful about this divergence, we can extract the log behavior of the conformally coupled scalar three-point function. Here, we will specialize to $n \geq 4$.

We can repeat the calculation using the in-in formalism. The same Feynman diagram corresponds to

$$\begin{aligned}
I_+ &= -ig f_{k_1}^*(\eta_*) \cdots f_{k_n}^*(\eta_*) \int_{-\infty}^{\eta_*} \frac{d\eta}{(H\eta)^4} f_{k_1}(\eta) \cdots f_{k_n}(\eta) \\
&= -ig \frac{H^{2n-4} \eta_*^n}{2^n k_1 \cdots k_n} \int_{-\infty}^{\eta_*} d\eta \eta^{n-4} e^{iK\eta} \\
&= -i^n g \Gamma[n-3] \frac{H^{2n-4} \eta_*^n}{2^n k_1 \cdots k_n} \frac{1}{K^{n-3}}. \tag{E.188}
\end{aligned}$$

Adding this together with $I_- = I_+^*$, we get

$$\boxed{\langle \phi_1 \cdots \phi_n \rangle'_c = -2 \cos\left(\frac{n\pi}{2}\right) \Gamma[n-3] \frac{g H^{2n-4} \eta_*^n}{2^n k_1 \cdots k_4} \frac{1}{K^{n-3}}}, \tag{E.189}$$

where for n odd, the relative phases of I_+ and I_- cause the in-in correlator to vanish. We can also see this from the wavefunction perspective where both $\langle O_1 \cdots O_n \rangle'_c$ and $\langle O_1 \cdots O_n \rangle'^*_c$ contribute to the correlator, but these two contributions cancel for $n \geq 4$ odd. This is the same type of cancellation seen in Exercise 3.6, but now as a function of n rather than D . (See [125] for more examples.)

In order to match the wavefunction and in-in calculations, we need the conformally coupled power spectrum

$$\langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle' = \frac{H^2 \eta_*^2}{2k}. \tag{E.190}$$

The difference between the in-in correlator and $\langle O_1 \cdots O_n \rangle'_c + \langle O_1 \cdots O_n \rangle'^*_c$ is therefore just factors of the power spectrum and combinatoric factors from the Gaussian integrals.

Solution to Exercise 4.11

The relevant wavefunction coefficients are

$$\langle O_1 O_1 \rangle' = \frac{k_1}{H^2 \eta_*^2}, \quad (\text{E.191})$$

$$\langle O_1 O_2 O_3 \rangle' = \frac{i g}{H^4 \eta_*^3} \log(i K \eta_*), \quad (\text{E.192})$$

$$\begin{aligned} \langle O_1 O_2 O_3 O_4 \rangle' &= \frac{g^2}{2 H^6 \eta_*^4 k_I} \left[\text{Li}_2\left(\frac{E - E_L}{E}\right) + \text{Li}_2\left(\frac{E - E_R}{E}\right) \right. \\ &\quad \left. + \log\left(\frac{E_L}{E}\right) \log\left(\frac{E_R}{E}\right) - \frac{\pi^2}{6} \right], \end{aligned} \quad (\text{E.193})$$

and the four-point correlator (in the s -channel) is given by

$$\begin{aligned} \langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle' &= \frac{1}{8 \prod_{n=1}^4 \langle O_n O_n \rangle'} \left(\langle O_1 O_2 O_3 O_4 \rangle' + \frac{\langle O_1 O_2 O_I \rangle' \langle O_I O_3 O_4 \rangle'^* + \text{c.c.}}{4 \langle O_I O_I \rangle'} \right. \\ &\quad \left. + \frac{\langle O_1 O_2 O_I \rangle' \langle O_I O_3 O_4 \rangle' + \text{c.c.}}{4 \langle O_I O_I \rangle'} \right). \end{aligned} \quad (\text{E.194})$$

The connected component of the correlator simply is

$$\begin{aligned} \langle O_1 O_2 O_3 O_4 \rangle'_{(\text{c})} &\equiv \frac{H^8 \eta_*^8}{8 k_1 k_2 k_3 k_4} \langle O_1 O_2 O_3 O_4 \rangle' \\ &= \frac{g^2 H^2 \eta_*^4}{16 k_1 k_2 k_3 k_4 k_I} \left[\text{Li}_2\left(\frac{E - E_L}{E}\right) + \text{Li}_2\left(\frac{E - E_R}{E}\right) \right. \\ &\quad \left. + \log\left(\frac{E_L}{E}\right) \log\left(\frac{E_R}{E}\right) - \frac{\pi^2}{6} \right], \end{aligned} \quad (\text{E.195})$$

which is the same as (2 times) $I_{++}^{(\text{c})}$ in equation (3.81). The disconnected component is

$$\begin{aligned} \langle O_1 O_2 O_3 O_4 \rangle'_{(\text{d})} &\equiv \frac{H^{10} \eta_*^{10}}{32 k_1 k_2 k_3 k_4 k_I} \left(\langle O_1 O_2 O_I \rangle' \langle O_I O_3 O_4 \rangle'^* + \langle O_1 O_2 O_I \rangle' \langle O_I O_3 O_4 \rangle' + \text{c.c.} \right) \\ &= \boxed{\frac{g^2 H^2 \eta_*^4}{32 k_1 k_2 k_3 k_4 k_I} \left(\log(i E_L \eta_*) \log(-i E_R \eta_*) - \log(i E_L \eta_*) \log(i E_R \eta_*) + \text{c.c.} \right)}. \end{aligned} \quad (\text{E.196})$$

We recognize the first term as I_{+-} in (3.83) and the second term as $I_{++}^{(\text{d})}$ in (3.82). As we have seen in (3.85), these contributions combine with their complex conjugates into

$$\boxed{\langle O_1 O_2 O_3 O_4 \rangle'_{(\text{d})} = \frac{g^2 H^2 \eta_*^4}{32 k_1 k_2 k_3 k_4 k_I} \pi^2}. \quad (\text{E.197})$$

The sum of the connected and disconnected contributions then correctly reproduces the in-in correlator (3.86).

Solution to Exercise 4.12

Start from the action

$$S = \int d^4x a^3(t) \left(\frac{1}{2} \dot{\Phi}^2 - \frac{1}{2a^2(t)} (\nabla\Phi)^2 + \frac{1}{2} \dot{\chi}^2 - \frac{1}{2a^2(t)} (\nabla\chi)^2 - H^2 \chi^2 - \frac{g}{2} \dot{\Phi}^2 \chi \right), \quad (\text{E.198})$$

where the interaction breaks the de Sitter symmetries. Writing this in terms of conformal time, we get

$$S = \int d\eta d^3x a^2(\eta) \left(\frac{1}{2} \Phi'^2 - \frac{1}{2} (\nabla\Phi)^2 + \frac{1}{2} \chi'^2 - \frac{1}{2} (\nabla\chi)^2 - a^2 H^2 \chi^2 - \frac{g}{2} \Phi'^2 \chi \right). \quad (\text{E.199})$$

The bulk-to-boundary propagator for a massless field and its derivative are

$$K_k(\eta) = (1 - ik\eta) e^{ik\eta}, \quad (\text{E.200})$$

$$K'_k(\eta) = k^2 \eta e^{ik\eta}. \quad (\text{E.201})$$

The bulk-to-bulk propagator for a conformally coupled scalar can be written simply in terms of the flat-space propagator:

$$G_{(\text{cc})}(k_I, \eta', \eta'') = H^2 \eta' \eta'' G_{(\text{flat})}(k_I, \eta', \eta''), \quad (\text{E.202})$$

where $G_{(\text{flat})}$ is given by (4.84). Applying the Feynman rules, we then see that the four-point wavefunction coefficient for the massless scalar is

$$\begin{aligned} \langle O_1 O_2 O_3 O_4 \rangle' &= -\frac{g^2}{H^2} k_1^2 k_2^2 k_3^2 k_4^2 \int d\eta' d\eta'' \eta' \eta'' e^{ik_{12}\eta'} e^{ik_{34}\eta''} G_{(\text{flat})}(k_I, \eta', \eta'') \\ &= \frac{g^2}{H^2} k_1^2 k_2^2 k_3^2 k_4^2 \partial_{k_{12}} \partial_{k_{34}} \int d\eta' d\eta'' e^{ik_{12}\eta'} e^{ik_{34}\eta''} G_{(\text{flat})}(k_I, \eta', \eta''). \end{aligned} \quad (\text{E.203})$$

Notice that this can be written as energy derivatives of the flat-space result (4.89), so that

$$\langle O_1 O_2 O_3 O_4 \rangle' = -\frac{g^2}{H^2} k_1^2 k_2^2 k_3^2 k_4^2 \partial_{k_{12}} \partial_{k_{34}} \frac{1}{(k_{12} + k_{34})(k_{12} + k_I)(k_{34} + k_I)}. \quad (\text{E.204})$$

Explicitly taking the derivatives, we obtain

$$\langle O_1 O_2 O_3 O_4 \rangle' = -\frac{g^2}{H^2} k_1^2 k_2^2 k_3^2 k_4^2 \frac{2E_L E_R + E(E_L + E_R) + E^2}{E^3 E_L^2 E_R^2}, \quad (\text{E.205})$$

where we have defined $E \equiv k_{12} + k_{34}$, $E_L \equiv k_{12} + k_I$, and $E_R \equiv k_{34} + k_I$. This strategy of relating de Sitter processes to simpler flat-space ones is quite useful for de Sitter-violating interactions involving massless fields. See [47, 126], for more examples.

Solution to Exercise 4.13

From the definition $u \equiv k_I/k_{12}$, we have

$$\partial_{k_{12}} = -\frac{u^2}{k_I} \partial_u, \quad (\text{E.206})$$

$$\partial_{k_{12}}^2 = -\frac{u^2}{k_I} \partial_u \left(-\frac{u^2}{k_I} \partial_u \right) = \frac{1}{k_I^2} (u^4 \partial_u^2 + 2u^3 \partial_u), \quad (\text{E.207})$$

and hence

$$(k_{12}^2 - k_I^2) \partial_{k_{12}}^2 = (1 - u^2) k_{12}^2 \partial_{k_{12}}^2 = (1 - u^2) (u^2 \partial_u^2 + 2u \partial_u), \quad (\text{E.208})$$

$$2k_{12} \partial_{k_{12}} = -2u \partial_u. \quad (\text{E.209})$$

Moreover, we have

$$\frac{1}{E} = \frac{1}{k_{12} + k_{34}} = \frac{1}{k_I(u^{-1} + v^{-1})} = \frac{1}{k_I} \frac{uv}{u+v}. \quad (\text{E.210})$$

Substituting these expressions in (4.127), we get

$$\boxed{\left(u^2(1-u^2)\partial_u^2 - 2u^3\partial_u + \frac{m^2}{H^2} - 2 \right) \hat{F} = g^2 H^2 \frac{uv}{u+v}}, \quad (\text{E.211})$$

as required.

Solution to Exercise 4.14

The integral form of the three-point wavefunction coefficient of two conformally coupled scalars and a generic massive scalar is

$$G \equiv \langle O_1 O_2 X_3 \rangle' = \frac{ig}{f_{k_1}(\eta_*) f_{k_2}(\eta_*)} \int_{-\infty}^{\eta_*} \frac{d\eta}{(H\eta)^4} f_{k_1}(\eta) f_{k_2}(\eta) \mathcal{K}_\nu(k_3\eta) \quad (\text{E.212})$$

$$= \frac{ig}{H^4 \eta_*^2} \int_{-\infty}^{\eta_*} \frac{d\eta}{\eta^2} e^{i(k_1+k_2)\eta} \mathcal{K}_\nu(k_3\eta), \quad (\text{E.213})$$

where $\mathcal{K}_\nu(k_3\eta)$ is the bulk-to-boundary propagator of the massive field. The propagator satisfies the following equation of motion

$$\left(\eta^2 \frac{\partial^2}{\partial \eta^2} - 2\eta \frac{\partial}{\partial \eta} + k_3^2 \eta^2 + \frac{m^2}{H^2} \right) \mathcal{K}_\nu(k_3\eta) = 0. \quad (\text{E.214})$$

This implies that

$$\frac{ig}{H^4 \eta_*^2} \int_{-\infty}^{\eta_*} \frac{d\eta}{\eta^2} e^{ik_{12}\eta} \left(\eta^2 \frac{\partial^2}{\partial \eta^2} - 2\eta \frac{\partial}{\partial \eta} + k_3^2 \eta^2 + \frac{m^2}{H^2} \right) \mathcal{K}_\nu(k_3\eta) = 0. \quad (\text{E.215})$$

As in the four-point example shown in the text, we integrate the time derivatives by parts and trade factors of η for $-i\partial_{k_{12}}$ to obtain [DB: What about boundary terms, since K_ν doesn't vanish on the boundary?] [AJ: not sure, I think maybe the un-normalized $\eta^{3/2} H^{(2)}$ does vanish on the boundary for principal series fields?]

$$\begin{aligned} 0 &= \frac{ig}{H^4 \eta_*^2} \int_{-\infty}^0 \frac{d\eta}{\eta^2} \left((k_{12}^2 - k_3^2) \partial_{k_{12}}^2 + 2k_{12} \partial_{k_{12}} + \frac{m^2}{H^2} - 2 \right) e^{ik_{12}\eta} \mathcal{K}_\nu(k_3\eta) \\ &= \left((k_{12}^2 - k_3^2) \partial_{k_{12}}^2 + 2k_{12} \partial_{k_{12}} + \frac{m^2}{H^2} - 2 \right) G, \end{aligned} \quad (\text{E.216})$$

where, in the second line, we have pulled the differential operator outside the integral which we then recognize as (E.213). Defining $u \equiv k_3/k_{12}$, we have (see Exercise 4.13)

$$\partial_{k_{12}} = -\frac{u^2}{k_3} \partial_u, \quad \partial_{k_{12}}^2 = \frac{1}{k_3^2} (u^4 \partial_u^2 + 2u^3 \partial_u), \quad (\text{E.217})$$

so that

$$\boxed{\left[(1 - u^2) u^2 \partial_u^2 - 2u^3 \partial_u + \left(\mu^2 + \frac{1}{4} \right) \right] G = 0}, \quad (\text{E.218})$$

where we have defined $\nu = i\mu$.

Solution to Exercise 5.1

[DB: ADD: discuss residues / factorization on each pole.]

Solution to Exercise 5.2

[DB: ADD]

Solution to Exercise 5.3

The generators of the $(D + 1)$ -dimensional Lorentz transformations are

$$J_{AB} = X_A \partial_B - X_B \partial_A. \quad (\text{E.219})$$

The corresponding Killing vector is

$$K_A \equiv \omega_{AB} X^B, \quad (\text{E.220})$$

with ω_{AB} an anti-symmetric matrix. It is straightforward to check that this satisfies the Killing equation $\partial_{(A} K_{B)} = 0$.

The flat slicing corresponds to the embedding

$$\begin{aligned} X^0 &= \frac{L}{2(-\eta)} (1 - \eta^2 + x^2), \\ X^i &= \frac{Lx^i}{(-\eta)}, \\ X^D &= \frac{L}{2(-\eta)} (1 + \eta^2 - x^2), \end{aligned} \quad (\text{E.221})$$

where $L \in (0, \infty)$, $\eta \in (-\infty, 0)$ and $x^i \in (-\infty, \infty)$. It is straightforward to check that the embedding coordinates satisfy $\eta_{AB} X^A X^B = H^{-2}$, if we define $L \equiv H^{-1}$; i.e. they parameterize the dS hyperboloid with Hubble parameter H . In fact, these coordinates foliate half the dS hyperboloid with flat slices, and in these coordinates the $(D + 1)$ -dimensional Minkowski metric takes the form

$$ds^2 = d\rho^2 + (LH)^2 ds_{\text{dS}_D}^2, \quad (\text{E.222})$$

where the dS metric in the inflationary slicing is given by

$$ds_{\text{dS}_D}^2 = \frac{1}{H^2 \eta^2} (-d\eta^2 + \delta_{ij} dx^i dx^j). \quad (\text{E.223})$$

By inverting the embedding (E.221), we can express the ambient space derivatives in terms of intrinsic coordinates [127]

$$\begin{aligned}\frac{\partial}{\partial X^0} &= \frac{1}{2(-\eta)} (-1 + \eta^2 - x^2) \partial_L + \frac{1}{2L} (1 + \eta^2 + x^2) \partial_\eta + \frac{\eta}{L} x^i \partial_i, \\ \frac{\partial}{\partial X^i} &= \frac{x^i}{(-\eta)} \partial_L - \frac{x^i}{L} \partial_\eta - \frac{\eta}{L} \partial_i, \\ \frac{\partial}{\partial X^D} &= \frac{1}{2(-\eta)} (1 + \eta^2 - x^2) \partial_L + \frac{1}{2L} (-1 + \eta^2 + x^2) \partial_\eta + \frac{\eta}{L} x^i \partial_i.\end{aligned}\tag{E.224}$$

Equations (E.221) and (E.224) can then be used to translate ambient space formulae into (A)dS formula in the flat slicing.

It is natural to define

$$\begin{aligned}\hat{D} &\equiv J_{D0}, \\ P_i &\equiv J_{Di} - J_{0i}, \\ K_i &\equiv J_{Di} + J_{0i}.\end{aligned}\tag{E.225}$$

Using (E.221) and (E.224), we then obtain

$$\begin{aligned}J_{ij} &= x_i \partial_j - x_j \partial_i, \\ P_i &= \partial_i, \\ \hat{D} &= -\eta \partial_\eta - x^i \partial_i, \\ K_i &= 2x_i \eta \partial_\eta + \left(2x^j x_i + (\eta^2 - x^2) \delta_i^j \right) \partial_j.\end{aligned}\tag{E.226}$$

Notice that all L dependence has dropped out, which is a manifestation of the fact that these transformations are parallel to the dS hyperboloid. One can check that these do indeed generate an $\text{so}(D, 1)$ algebra.

Solution to Exercise 5.4

[DB: ADD]

Solution to Exercise 5.5

[DB: ADD]

Solution to Exercise 5.6

We begin by collecting some basic formulas. Differentiating a momentum vector gives

$$\frac{\partial k_n^i}{\partial k_n} = \frac{\partial k_n}{\partial k_n^i} = \hat{k}_n^i, \quad \frac{\partial k_n^i}{\partial k_n^j} = \delta^{ij}, \quad \frac{\partial \hat{k}_n^i}{\partial k_n^j} = \frac{\delta^{ij} - \hat{k}_n^i \hat{k}_n^j}{k_n}, \quad \frac{\partial \hat{k}_n^i}{\partial k_n} = 0.\tag{E.227}$$

In addition, it will be useful to define the variable $s \equiv |\mathbf{k}_1 + \mathbf{k}_2|$ and $t \equiv |\mathbf{k}_2 + \mathbf{k}_3|$ (physically, these are the exchange energies in the s - and t -channels). The derivatives of the s, t variables are

$$\begin{aligned}\frac{\partial s}{\partial k_1^i} &= \frac{\partial s}{\partial k_2^i} = \frac{k_{12}^i}{s}, \quad \frac{\partial s}{\partial k_3^i} = \frac{\partial s}{\partial k_4^i} = 0, \\ \frac{\partial t}{\partial k_2^i} &= \frac{\partial t}{\partial k_3^i} = \frac{k_{23}^i}{t}, \quad \frac{\partial t}{\partial k_1^i} = \frac{\partial t}{\partial k_4^i} = 0,\end{aligned}\tag{E.228}$$

where $k_{nm}^i \equiv k_n^i + k_m^i$. Using the chain rule, the first derivatives with respect to the momentum vectors can be replaced by

$$\begin{aligned}\frac{\partial}{\partial k_1^i} &= \frac{k_1^i}{k_1} \frac{\partial}{\partial k_1} + \frac{k_{12}^i}{s} \frac{\partial}{\partial s}, \quad \frac{\partial}{\partial k_3^i} = \frac{k_3^i}{k_3} \frac{\partial}{\partial k_3} + \frac{k_{23}^i}{t} \frac{\partial}{\partial t}, \\ \frac{\partial}{\partial k_2^i} &= \frac{k_2^i}{k_2} \frac{\partial}{\partial k_2} + \frac{k_{12}^i}{s} \frac{\partial}{\partial s} + \frac{k_{23}^i}{t} \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial k_4^i} = \frac{k_4^i}{k_4} \frac{\partial}{\partial k_4}.\end{aligned}\tag{E.229}$$

Moreover, the second derivatives are

$$\begin{aligned}\frac{\partial^2}{\partial k_1^i \partial k_1^j} &= \hat{k}_1^i \hat{k}_1^j \frac{\partial^2}{\partial k_1^2} + \frac{\delta^{ij} - \hat{k}_1^i \hat{k}_1^j}{k_1} \frac{\partial}{\partial k_1} + \frac{k_1^i k_{12}^j + k_{12}^i k_1^j}{sk_1} \frac{\partial^2}{\partial s \partial k_1} + \frac{\delta^{ij} - \hat{s}^{ij}}{s} \frac{\partial}{\partial s} + \hat{s}^{ij} \frac{\partial^2}{\partial s^2}, \\ \frac{\partial^2}{\partial k_2^i \partial k_2^j} &= \hat{k}_2^i \hat{k}_2^j \frac{\partial^2}{\partial k_2^2} + \frac{\delta^{ij} - \hat{k}_2^i \hat{k}_2^j}{k_2} \frac{\partial}{\partial k_2} + \frac{k_{23}^i k_{12}^j + k_{12}^i k_{23}^j}{st} \frac{\partial^2}{\partial s \partial t} \\ &\quad + \frac{\delta^{ij} - \hat{s}^{ij}}{s} \frac{\partial}{\partial s} + \frac{k_2^i k_{12}^j + k_{12}^i k_2^j}{sk_2} \frac{\partial^2}{\partial s \partial k_2} + \hat{s}^{ij} \frac{\partial^2}{\partial s^2} \\ &\quad + \frac{\delta^{ij} - \hat{t}^{ij}}{t} \frac{\partial}{\partial t} + \frac{k_2^i k_{23}^j + k_{23}^i k_2^j}{tk_2} \frac{\partial^2}{\partial t \partial k_2} + \hat{t}^{ij} \frac{\partial^2}{\partial t^2}, \\ \frac{\partial^2}{\partial k_3^i \partial k_3^j} &= \hat{k}_3^i \hat{k}_3^j \frac{\partial^2}{\partial k_3^2} + \frac{\delta^{ij} - \hat{k}_3^i \hat{k}_3^j}{k_3} \frac{\partial}{\partial k_3} + \frac{k_3^i k_{23}^j + k_{23}^i k_3^j}{tk_3} \frac{\partial^2}{\partial t \partial k_3} + \frac{\delta^{ij} - \hat{t}^{ij}}{t} \frac{\partial}{\partial t} + \hat{t}^{ij} \frac{\partial^2}{\partial t^2}, \\ \frac{\partial^2}{\partial k_4^i \partial k_4^j} &= \hat{k}_4^i \hat{k}_4^j \frac{\partial^2}{\partial k_4^2} + \frac{\delta^{ij} - \hat{k}_4^i \hat{k}_4^j}{k_4} \frac{\partial}{\partial k_4},\end{aligned}\tag{E.230}$$

where we have defined

$$\hat{s}^{ij} \equiv \frac{(k_1^i + k_2^i)(k_1^j + k_2^j)}{s^2}, \quad \hat{t}^{ij} \equiv \frac{(k_2^i + k_3^i)(k_2^j + k_3^j)}{t^2}.\tag{E.231}$$

Below, we write down the constraint equations arising from dilatations and SCTs.

In the previous exercise, we have shown that the conformal Ward identities in Fourier space are

$$\left[9 - \sum_{n=1}^4 \left(\Delta_n - k_n^j \frac{\partial}{\partial k_n^j} \right) \right] F = 0,\tag{E.232}$$

$$\sum_{n=1}^4 \left[k_n^i \frac{\partial^2}{\partial k_n^j \partial k_n^i} - 2k_n^j \frac{\partial^2}{\partial k_n^j \partial k_n^i} + 2(\Delta_n - 3) \frac{\partial}{\partial k_n^i} \right] F = 0.\tag{E.233}$$

We will now massage them into a nicer form.

Dilatation Expanding the dilatation Ward identity (E.232) gives

$$0 = (9 - \Delta_t)F + k_1 \frac{\partial F}{\partial k_1} + k_2 \frac{\partial F}{\partial k_2} + k_3 \frac{\partial F}{\partial k_3} + k_4 \frac{\partial F}{\partial k_4} + s \frac{\partial F}{\partial s} + t \frac{\partial F}{\partial t}.\tag{E.234}$$

where $\Delta_t \equiv \sum_{n=1}^4 \Delta_n$ denotes the total conformal weight. Differentiating this with respect to s and t gives

$$-\frac{\partial^2 F}{\partial s^2} = \frac{10 - \Delta_t}{s} \frac{\partial F}{\partial s} + \frac{k_1}{s} \frac{\partial^2 F}{\partial s \partial k_1} + \frac{k_2}{s} \frac{\partial^2 F}{\partial s \partial k_2} + \frac{k_3}{s} \frac{\partial^2 F}{\partial s \partial k_3} + \frac{k_4}{s} \frac{\partial^2 F}{\partial s \partial k_4} + \frac{t}{s} \frac{\partial^2 F}{\partial s \partial t}, \quad (\text{E.235})$$

$$-\frac{\partial^2 F}{\partial t^2} = \frac{10 - \Delta_t}{t} \frac{\partial F}{\partial t} + \frac{k_1}{t} \frac{\partial^2 F}{\partial t \partial k_1} + \frac{k_2}{t} \frac{\partial^2 F}{\partial t \partial k_2} + \frac{k_3}{t} \frac{\partial^2 F}{\partial t \partial k_3} + \frac{k_4}{t} \frac{\partial^2 F}{\partial t \partial k_4} + \frac{s}{t} \frac{\partial^2 F}{\partial s \partial t}. \quad (\text{E.236})$$

We will substitute these derivatives in our derivation of the SCT constraints.

Linear term We first consider the first-order term in (E.233). We have

$$\begin{aligned} \sum_{n=1}^4 2(\Delta_n - 3) \frac{\partial F}{\partial k_n^i} &= \textcolor{red}{k}_1^i \left[2(\Delta_1 - 3) \frac{1}{k_1} \frac{\partial F}{\partial k_1} + 2(\Delta_{12} - 6) \frac{1}{s} \frac{\partial F}{\partial s} \right] \\ &\quad + \textcolor{orange}{k}_2^i \left[2(\Delta_2 - 3) \frac{1}{k_2} \frac{\partial F}{\partial k_2} + 2(\Delta_{12} - 6) \frac{1}{s} \frac{\partial F}{\partial s} + 2(\Delta_{23} - 6) \frac{1}{t} \frac{\partial F}{\partial t} \right] \\ &\quad + \textcolor{green}{k}_3^i \left[2(\Delta_3 - 3) \frac{1}{k_3} \frac{\partial F}{\partial k_3} + 2(\Delta_{23} - 6) \frac{1}{t} \frac{\partial F}{\partial t} \right] \\ &\quad + \textcolor{blue}{k}_4^i \left[2(\Delta_4 - 3) \frac{1}{k_4} \frac{\partial F}{\partial k_4} \right], \end{aligned} \quad (\text{E.237})$$

where $\Delta_{nm} \equiv \Delta_n + \Delta_m$.

Cross term There are two second-order terms in (E.233). The cross term is given by

$$\begin{aligned} \sum_{n=1}^4 k_n^j \frac{\partial^2 F}{\partial k_n^i \partial k_n^j} &= \textcolor{red}{k}_1^i \left[\frac{\partial^2}{\partial k_1^2} + \frac{s_1^2 + k_1^2}{sk_1} \frac{\partial^2 F}{\partial s \partial k_1} + \frac{t_2^2}{st} \frac{\partial^2 F}{\partial s \partial t} + \frac{k_2}{s} \frac{\partial^2 F}{\partial s \partial k_2} + \frac{\partial^2 F}{\partial s^2} \right] \\ &\quad + \textcolor{orange}{k}_2^i \left[\frac{k_1^2}{sk_1} \frac{\partial^2 F}{\partial s \partial k_1} + \frac{\partial^2 F}{\partial s^2} + \frac{\partial^2 F}{\partial k_2^2} + \frac{s_2^2 + t_2^2}{st} \frac{\partial^2 F}{\partial s \partial t} + \frac{s_2^2 + k_2^2}{sk_2} \frac{\partial^2 F}{\partial s \partial k_2} + \frac{t_2^2 + k_2^2}{tk_2} \frac{\partial^2 F}{\partial t \partial k_2} \right. \\ &\quad \left. + \frac{k_3^3}{tk_3} \frac{\partial^2 F}{\partial t \partial k_3} + \frac{\partial^2 F}{\partial t^2} \right] \\ &\quad + \textcolor{green}{k}_3^i \left[\frac{\partial^2 F}{\partial k_3^2} + \frac{t_3^2 + k_3^2}{tk_3} \frac{\partial^2 F}{\partial t \partial k_3} + \frac{s_2^2}{st} \frac{\partial^2 F}{\partial s \partial t} + \frac{k_2}{t} \frac{\partial^2 F}{\partial t \partial k_2} + \frac{\partial^2 F}{\partial t^2} \right] \\ &\quad + \textcolor{blue}{k}_4^i \left[\frac{\partial^2 F}{\partial k_4^2} \right], \end{aligned} \quad (\text{E.238})$$

where we have defined

$$s_1^2 \equiv k_1^i(k_1^i + k_2^i), \quad s_2^2 \equiv k_2^i(k_1^i + k_2^i), \quad t_2^2 \equiv k_2^i(k_2^i + k_3^i), \quad t_3^2 \equiv k_3^i(k_2^i + k_3^i). \quad (\text{E.239})$$

Quadratic term Lastly, we consider the ‘‘Laplacian’’ term in (E.233). This is given by

$$\begin{aligned}
\sum_{n=1}^4 k_n^i \frac{\partial^2 F}{\partial k_n^j \partial k_n^j} &= \textcolor{red}{k}_1^i \left[\frac{\partial^2 F}{\partial k_1^2} + \frac{2}{k_1} \frac{\partial F}{\partial k_1} + \frac{2s_1^2}{sk_1} \frac{\partial^2 F}{\partial s \partial k_1} + \frac{2}{s} \frac{\partial F}{\partial s} + \frac{\partial^2 F}{\partial s^2} \right] \\
&\quad + \textcolor{orange}{k}_2^i \left[\frac{\partial^2 F}{\partial k_2^2} + \frac{2}{k_2} \frac{\partial F}{\partial k_2} + \frac{2s_2^2 + 2t_2^2 + 2k_1^i k_3^i - 2k_2^2}{st} \frac{\partial^2 F}{\partial s \partial t} + \frac{2}{s} \frac{\partial F}{\partial s} + \frac{2s_2^2}{sk_2} \frac{\partial^2 F}{\partial s \partial k_2} \right. \\
&\quad \left. + \frac{\partial^2 F}{\partial s^2} + \frac{2}{t} \frac{\partial F}{\partial t} + \frac{2t_2^2}{tk_2} \frac{\partial^2 F}{\partial t \partial k_2} + \frac{\partial^2 F}{\partial t^2} \right] \\
&\quad + \textcolor{green}{k}_3^i \left[\frac{\partial^2 F}{\partial k_3^2} + \frac{2}{k_3} \frac{\partial F}{\partial k_3} + \frac{2t_3^2}{tk_3} \frac{\partial^2 F}{\partial t \partial k_3} + \frac{2}{t} \frac{\partial F}{\partial t} + \frac{\partial^2 F}{\partial t^2} \right] \\
&\quad + \textcolor{blue}{k}_4^i \left[\frac{\partial^2 F}{\partial k_4^2} + \frac{2}{k_4} \frac{\partial F}{\partial k_4} \right]. \tag{E.240}
\end{aligned}$$

Total We now sum over the contributions from the terms above. This gives

$$\begin{aligned}
\sum_{n=1}^4 K_n^i F &= \textcolor{red}{k}_1^i \left[\frac{2(\Delta_1 - 2)}{k_1} \frac{\partial F}{\partial k_1} + \frac{2(\Delta_{12} - 5)}{s} \frac{\partial F}{\partial s} - \frac{\partial^2 F}{\partial k_1^2} - \frac{2k_1}{s} \frac{\partial^2 F}{\partial s \partial k_1} - \frac{2t_2^2}{st} \frac{\partial^2 F}{\partial s \partial t} \right. \\
&\quad \left. - \frac{2k_2}{s} \frac{\partial^2 F}{\partial s \partial k_2} - \frac{\partial^2 F}{\partial s^2} \right] \\
&\quad + \textcolor{orange}{k}_2^i \left[\frac{2(\Delta_2 - 2)}{k_2} \frac{\partial F}{\partial k_2} + \frac{2(\Delta_{12} - 5)}{s} \frac{\partial F}{\partial s} + \frac{2(\Delta_{23} - 5)}{t} \frac{\partial F}{\partial t} - \frac{\partial^2 F}{\partial k_2^2} - \frac{\partial^2 F}{\partial s^2} - \frac{\partial^2 F}{\partial t^2} \right. \\
&\quad \left. - \frac{2k_1}{s} \frac{\partial^2 F}{\partial s \partial k_1} - \frac{2k_2}{s} \frac{\partial^2 F}{\partial s \partial k_2} - \frac{2k_2}{t} \frac{\partial^2 F}{\partial t \partial k_2} - \frac{2k_3}{t} \frac{\partial^2 F}{\partial t \partial k_3} + \frac{2(k_1^i k_3^i - k_2^2)}{st} \frac{\partial^2 F}{\partial s \partial t} \right] \\
&\quad + \textcolor{green}{k}_3^i \left[\frac{2(\Delta_3 - 2)}{k_3} \frac{\partial F}{\partial k_3} + \frac{2(\Delta_{23} - 5)}{t} \frac{\partial F}{\partial t} - \frac{\partial^2 F}{\partial k_3^2} - \frac{2k_3}{t} \frac{\partial^2 F}{\partial t \partial k_3} - \frac{2s_2^2}{st} \frac{\partial^2 F}{\partial s \partial t} \right. \\
&\quad \left. - \frac{2k_2}{t} \frac{\partial^2 F}{\partial t \partial k_2} - \frac{\partial^2 F}{\partial t^2} \right] \\
&\quad + \textcolor{blue}{k}_4^i \left[\frac{2(\Delta_4 - 2)}{k_4} \frac{\partial F}{\partial k_4} - \frac{\partial^2 F}{\partial k_4^2} \right]. \tag{E.241}
\end{aligned}$$

Because of the asymmetric definitions of s and t in terms of the momenta k_n , the result is not manifestly symmetric under cyclic permutations. We can use momentum conservation to bring these into a more symmetric form. Moreover, to make the resulting expression symmetric between s and t , we use the constraints (E.235) and (E.236). As a consequence of momentum conservation, we can drop all terms that are equal in each square brackets. These include terms such as $\partial^2 F / \partial s^2$, $\partial^2 F / \partial t^2$ and $\partial F / \partial s$, $\partial F / \partial t$.

We see that the final result is of the form

$$\sum_{n=1}^4 k_n^i D_n F = 0, \tag{E.242}$$

where we have defined

$$D_1 F \equiv \left[\frac{\partial^2}{\partial k_1^2} + \frac{1}{s} \frac{\partial}{\partial s} \left(k_1 \frac{\partial}{\partial k_1} + k_2 \frac{\partial}{\partial k_2} \right) + \frac{1}{t} \frac{\partial}{\partial t} \left(k_1 \frac{\partial}{\partial k_1} + k_4 \frac{\partial}{\partial k_4} \right) - \frac{k_3^2}{st} \frac{\partial^2}{\partial s \partial t} \right. \\ \left. - \frac{2(\Delta_1 - 2)}{k_1} \frac{\partial}{\partial k_1} + \frac{\Delta_1 + \Delta_2}{s} \frac{\partial}{\partial s} + \frac{\Delta_1 + \Delta_4}{t} \frac{\partial}{\partial t} \right] F, \quad (\text{E.243})$$

and the rest are given by the cyclic permutation of the indices (remembering that $t \rightarrow s$ under a cyclic shift). The only nontrivial way to satisfy (E.242) is to demand that all $D_n F$ are equal to each other, so that $\sum_n k_n^i D_n F$ vanishes as a consequence of momentum conservation.⁷⁰ To satisfy the SCT constraint, we can therefore pick any three of the six conditions

$$(D_n - D_m) F = 0, \quad (\text{E.244})$$

for $n, m = 1, \dots, 4$.

An object of particular interest is the four-point function of conformally coupled scalars, mediated by the tree-level exchange of massive scalars. In that case, the s -channel contribution⁷¹ can be written as $F = s^{-1} \hat{F}(u, v)$, an ansatz which automatically satisfies the equations $D_{12} F = 0$ and $D_{34} F = 0$, where $D_{nm} \equiv D_n - D_m$. The remaining conformal invariance equation $D_{13} F = 0$ becomes

$$\boxed{(\Delta_u - \Delta_v) \hat{F} = 0}, \quad (\text{E.245})$$

where we have introduced the differential operator

$$\Delta_u \equiv u^2(1-u^2)\partial_u^2 - 2u^3\partial_u. \quad (\text{E.246})$$

This completes the exercise!

Solution to Exercise 5.7

We start from the seed solution

$$\hat{F}_{c,0} = \frac{uv}{u+v}, \quad (\text{E.247})$$

and act on it with the operator

$$\Delta_u \equiv u^2(1-u^2)\partial_u^2 - 2u^3\partial_u. \quad (\text{E.248})$$

This gives

$$\hat{F}_{c,1} = \Delta_u \hat{F}_{c,0} = \boxed{-2 \left(\frac{uv}{u+v} \right)^3 \frac{1+uv}{uv}}, \quad (\text{E.249})$$

$$\hat{F}_{c,2} = \Delta_u^2 \hat{F}_{c,0} = \boxed{-4 \left(\frac{uv}{u+v} \right)^5 \frac{u^2 + v^2 + uv(3u^2 + 3v^2 - 4) - 6(uv)^2 - 6(uv)^3}{(uv)^3}}, \quad (\text{E.250})$$

⁷⁰If all momenta are collinear, then perhaps other possibilities are allowed. However, in that case the four-point function would have to vanish in any non-collinear configuration, which we rule out on the basis of continuity.

⁷¹It suffices to impose conformal invariance of a single channel to fix the whole four-point function. To see this, note that the correlator $\langle OOO'O' \rangle$ with $\Delta_O = \Delta_{O'}$, which is conformally invariant, coincides with the s -channel of $\langle OOOO \rangle$. The other channels can be included by cyclic permutations, e.g. by replacing u with $|\mathbf{k}_2 + \mathbf{k}_3|/(k_2 + k_3)$ and $|\mathbf{k}_2 + \mathbf{k}_4|/(k_2 + k_4)$ for the t - and u -channels, respectively. Finally, for contact interactions, “ s -channel” refers to a specific permutation of the external momenta.

as required.

Solution to Exercise 5.8

Consider equation (5.122):

$$\left[\Delta_u + \left(\mu^2 + \frac{1}{4} \right) \right] \hat{F}_e = \hat{C}, \quad (\text{E.251})$$

$$\left[\Delta_v + \left(\mu^2 + \frac{1}{4} \right) \right] \hat{F}_e = \hat{C}. \quad (\text{E.252})$$

Acting on (E.251) with Δ_v and on (E.252) with Δ_u , we get

$$\Delta_v \left[\Delta_u + \left(\mu^2 + \frac{1}{4} \right) \right] \hat{F}_e = \Delta_v \hat{C}, \quad (\text{E.253})$$

$$\Delta_u \left[\Delta_v + \left(\mu^2 + \frac{1}{4} \right) \right] \hat{F}_e = \Delta_u \hat{C}. \quad (\text{E.254})$$

Subtracting these equations from each other gives

$$(\Delta_u \Delta_v - \Delta_v \Delta_u) \hat{F}_e + \left(\mu^2 + \frac{1}{4} \right) (\Delta_u - \Delta_v) \hat{F}_e = (\Delta_u - \Delta_v) \hat{C}. \quad (\text{E.255})$$

The first term on the left-hand side vanishes, because \hat{F}_e is symmetric under exchange of u and v , while the second term vanishes by conformal invariance. Hence, we find that

$$\boxed{(\Delta_u - \Delta_v) \hat{C} = 0}, \quad (\text{E.256})$$

i.e. the source term \hat{C} has to be conformally invariant.

Solution to Exercise 5.9

The differential equation for the four-point exchange diagram of a conformally coupled scalar is

$$\left[u^2(1-u^2) \partial_u^2 - 2u^3 \partial_u \right] \hat{F}_e = \hat{F}_c. \quad (\text{E.257})$$

In terms of $u \equiv k_I/k_{12}$ and $v \equiv k_I/k_{34}$, we have

$$\begin{aligned} \hat{F}_e \equiv k_I \langle O_1 O_2 O_3 O_4 \rangle'_e &= \frac{g^2}{2H^6 \eta_*^4} \left[\text{Li}_2 \left(\frac{v(1-u)}{u+v} \right) + \text{Li}_2 \left(\frac{u(1-v)}{u+v} \right) \right. \\ &\quad \left. + \log \left(\frac{v(1+u)}{u+v} \right) \log \left(\frac{u(1+v)}{u+v} \right) - \frac{\pi^2}{6} \right]. \end{aligned} \quad (\text{E.258})$$

Substituting this into the left-hand side of (E.257) gives

$$\left[u^2(1-u^2) \partial_u^2 - 2u^3 \partial_u \right] \hat{F}_e = \frac{g^2}{H^6 \eta_*^4} \frac{uv}{u+v} = \frac{g^2}{H^6 \eta_*^4} \frac{k_I}{E}, \quad (\text{E.259})$$

which equals $\hat{F}_c \equiv k_I \langle O_1 O_2 O_3 O_4 \rangle'_c$ if we set $\lambda = g^2/H^2$. This confirms that our previous solutions for the contact and exchange four-point functions of a conformally coupled scalar satisfy the conformal Ward identity.

Solution to Exercise 5.10

Consider equation (5.133):

$$\left[\xi^2 \partial_\xi^2 + \left(\mu^2 + \frac{1}{4} \right) \right] \tilde{F} = \frac{\xi}{1+\xi}, \quad (\text{E.260})$$

where $\tilde{F} \equiv \sqrt{\xi} f$. Around $\xi = 0$, we write the particular solution of this equation as a series solution

$$\tilde{F}_<(\xi) = \sum_{n=1}^{\infty} c_n \xi^n. \quad (\text{E.261})$$

Substituting this into (E.260), we get

$$\sum_{n=1}^{\infty} c_n \left[n(n-1) + \left(\mu^2 + \frac{1}{4} \right) \right] \xi^n = \frac{\xi}{1+\xi}. \quad (\text{E.262})$$

Multiplying through by $(1 + \xi)$ gives

$$c_1 \left(\mu^2 + \frac{1}{4} \right) \xi + \sum_{n=1}^{\infty} \left(c_{n+1} \left[\left(n + \frac{1}{2} \right)^2 + \mu^2 \right] + c_n \left[\left(n - \frac{1}{2} \right)^2 + \mu^2 \right] \right) \xi^{n+1} = \xi, \quad (\text{E.263})$$

which implies that

$$c_1 = \left(\mu^2 + \frac{1}{4} \right)^{-1} \quad \text{and} \quad c_{n+1} = -\frac{(n - \frac{1}{2})^2 + \mu^2}{(n + \frac{1}{2})^2 + \mu^2} c_n. \quad (\text{E.264})$$

The solution for $\tilde{F}_<$ then is

$$\tilde{F}_< = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\xi^n}{(n - \frac{1}{2})^2 + \mu^2}, \quad (\text{E.265})$$

and $f_< = \tilde{F}_</\sqrt{\xi}$ can be written as

$$f_<(\xi) = \sqrt{\xi} \sum_{n=0}^{\infty} (-1)^n \frac{\xi^n}{(n + \frac{1}{2})^2 + \mu^2}, \quad (\text{E.266})$$

as required.

Solution to Exercise 5.11

Let $x = 1/\xi$. Using

$$\partial_\xi = \frac{dx}{d\xi} \partial_x = -\frac{1}{\xi^2} \partial_x = -x^2 \partial_x, \quad (\text{E.267})$$

$$\partial_\xi^2 = -x^2 \partial_x (-x^2 \partial_x) = x^2 (x^2 \partial_x^2 + 2x \partial_x), \quad (\text{E.268})$$

we find that equation (E.260) becomes

$$\left[x^2 \partial_x^2 + 2x \partial_x + \left(\mu^2 + \frac{1}{4} \right) \right] \tilde{F} = \frac{1}{1+x}. \quad (\text{E.269})$$

Around $x = 0$, we write the particular solution of this equation as a series solution

$$\tilde{F}_>(x) = \sum_{n=0}^{\infty} c_n x^n. \quad (\text{E.270})$$

Substituting this into (E.269), we get

$$\sum_{n=0}^{\infty} c_n \left[n(n-1) + 2n + \left(\mu^2 + \frac{1}{4} \right) \right] x^n = \frac{1}{1+x}. \quad (\text{E.271})$$

Multiplying through by $(1+x)$, this implies that

$$c_0 \left(\mu^2 + \frac{1}{4} \right) + \sum_{n=1}^{\infty} \left(c_n \left[\left(n + \frac{1}{2} \right)^2 + \mu^2 \right] + c_{n-1} \left[\left(n - \frac{1}{2} \right)^2 + \mu^2 \right] \right) x^n = 1. \quad (\text{E.272})$$

Hence, we have $c_0 = (\mu^2 + \frac{1}{4})^{-1}$ and

$$c_{n+1} = -\frac{(n - \frac{1}{2})^2 + \mu^2}{(n + \frac{1}{2})^2 + \mu^2} c_n, \quad \text{for } n > 0. \quad (\text{E.273})$$

The solution for $f_> = \tilde{F}_>/\sqrt{\xi}$ can then be written as

$$f_>(\xi) = \frac{1}{\sqrt{\xi}} \sum_{n=0}^{\infty} (-1)^n \frac{\xi^{-n}}{(n + \frac{1}{2})^2 + \mu^2}, \quad (\text{E.274})$$

as required.

Solution to Exercise 5.12

We start from the differential equation (5.127):

$$\left[u^2(1-u^2)\partial_u^2 - 2u^3\partial_u + \left(\mu^2 + \frac{1}{4} \right) \right] \hat{F} = \frac{uv}{u+v}. \quad (\text{E.275})$$

For $u < v$, we seek a power series solution of the form

$$\hat{F}_<(u, v) = \sum_{m,n=0}^{\infty} c_{mn} u^{m+1} (u/v)^n. \quad (\text{E.276})$$

This form of the series solution is motivated by the series expansion of the source term $uv/(u+v)$. Plugging this ansatz into the differential equation, we find that the series coefficients obey the following recursive relation:

$$c_{0n} = \frac{(-1)^n}{(n + \frac{1}{2})^2 + \mu^2}, \quad c_{1n} = 0, \quad c_{m+2,n} = \frac{(m+n+2)(m+n+1)}{(m+n+\frac{5}{2})^2 + \mu^2} c_{mn}, \quad (\text{E.277})$$

where the condition $c_{1n} = 0$ implies that c_{mn} vanishes for all odd m . Redefining $2m \rightarrow m$ and solving the recursive relation, we find

$$\hat{F}_<(u, v) = \sum_{m,n=0}^{\infty} c_{mn} u^{2m+1} (u/v)^n, \quad (\text{E.278})$$

with

$$c_{mn} = \frac{(-1)^n(n+1)(n+2)\cdots(n+2m)}{[(n+\frac{1}{2})^2+\mu^2][(n+\frac{5}{2})^2+\mu^2]\cdots[(n+\frac{1}{2}+2m)^2+\mu^2]}. \quad (\text{E.279})$$

Note that the series solution (5.141) is the unique particular solution that is regular at the origin, since both homogeneous solutions are non-analytic at $u = 0$.

Solution to Exercise 5.13

In Exercise 5.12, we found the solution $\hat{F}_<(u, v)$ for $u < v$; cf. equation (E.278). The full solution is symmetric under the exchange $u \leftrightarrow v$. For $u > v$, the solution therefore is

$$\hat{F}_>(u, v) = \hat{F}_<(v, u) = \sum_{m,n=0}^{\infty} c_{mn} v^{2m+1} (v/u)^n, \quad (\text{E.280})$$

where c_{mn} are same coefficients as in (E.279). It is straightforward to check that (E.280) solves (E.275).

As before, the difference between the two particular solutions must be a homogeneous solution, so we can write

$$\hat{F}_<(u, v) - \hat{F}_>(u, v) = \sum_{\pm} A_{\pm}(v; \mu) \hat{F}_{\pm}(u), \quad (\text{E.281})$$

where the functions \hat{F}_{\pm} are given by (5.128). Evaluating this at $u = v$ gives $A_{\pm}(v; \mu) = \mp a(v; \mu) \hat{F}_{\mp}(v)$, for some function $a(v; \mu)$. Matching the u -derivative at $u = v$ fixes the function in terms of the Wronskian of the homogeneous solutions, namely

$$\begin{aligned} \left(\partial_u \hat{F}_< - \partial_u \hat{F}_> \right) \Big|_{u \rightarrow v} &= a(v; \mu) \left(\hat{F}_+(v) \hat{F}'_-(v) - \hat{F}'_+(v) \hat{F}_-(v) \right) \\ &= \frac{a(v; \mu)}{1 - v^2} = a(v; \mu) \times \sum_{m=0}^{\infty} v^{2m}. \end{aligned} \quad (\text{E.282})$$

It remains to evaluate the left-hand side

$$\left(\partial_u \hat{F}_< - \partial_u \hat{F}_> \right) \Big|_{u \rightarrow v} = \sum_{m,n=0}^{\infty} (2m+2n+1) c_{mn} v^{2m}. \quad (\text{E.283})$$

Somewhat remarkably, the sum over n above is m -independent; we have the identity

$$\sum_{n=0}^{\infty} (2m+2n+1) c_{mn} = \sum_{n=0}^{\infty} \frac{(-1)^n (2m+2n+1)(n+1)\cdots(n+2m)}{[(n+\frac{1}{2})^2+\mu^2]\cdots[(n+\frac{1}{2}+2m)^2+\mu^2]} = \frac{\pi}{\cosh \pi \mu}. \quad (\text{E.284})$$

Once again, this identity can be established by matching the residues on both sides. For instance, for $m = 1$, only the terms in the sum with $n = k-2, k$ have a residue as $\mu \rightarrow \pm i(k + \frac{1}{2})$; the residue is

$$-\frac{(-1)^k (1+k)(2+k)}{2(1+2k)} + \frac{(-1)^k (k-1)k}{2(1+2k)} = (-1)^k, \quad (\text{E.285})$$

which matches that of the right-hand side. This allows us to fix $a(v; \mu) = \pi / \cosh \pi \mu$ independent of v .

The explicit solution for $\hat{F}_<$ is

$$\hat{F}_<(u, v) = \begin{cases} \sum_{m,n=0}^{\infty} c_{mn} u^{2m+1} (u/v)^n & u \leq v, \\ \sum_{m,n=0}^{\infty} c_{mn} v^{2m+1} (v/u)^n + \frac{\pi}{\cosh \pi \mu} (\hat{F}_+(v) \hat{F}_-(u) - \hat{F}_-(v) \hat{F}_+(u)) & u \geq v, \end{cases} \quad (\text{E.286})$$

[DB: FINISH]

Solution to Exercise 5.14

[DB: ADD]

Solution to Exercise B.1

Solution to Exercise B.2

Solution to Exercise B.3

Solution to Exercise C.1

Solution to Exercise C.2

Solution to Exercise C.3

Solution to Exercise C.4

Solution to Exercise C.5

Solution to Exercise C.6

Solution to Exercise D.1

The metric transforms as

$$\begin{aligned} g_{\mu\nu} \rightarrow g'_{\mu\nu} &= \frac{dx^\alpha}{dx'^\mu} \frac{dx^\beta}{dx'^\nu} g_{\alpha\beta} = (\delta_\mu^\alpha - \partial_\mu \epsilon^\alpha)(\delta_\nu^\beta - \partial_\nu \epsilon^\beta)g_{\alpha\beta} \\ &= g_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu). \end{aligned} \quad (\text{E.287})$$

For a conformal transformation, we require $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$ to be proportional to the metric,

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x) g_{\mu\nu}. \quad (\text{E.288})$$

The factor $f(x)$ is found by taking the trace on both sides:

$$f(x) = \frac{2}{d} \partial \cdot \epsilon. \quad (\text{E.289})$$

Acting with ∂_ρ on (E.288), permuting the indices and taking a linear combination, we find

$$2\partial_\mu \partial_\nu \epsilon_\rho = (-g_{\mu\nu} \partial_\rho + g_{\rho\mu} \partial_\nu + g_{\nu\rho} \partial_\mu)f. \quad (\text{E.290})$$

Contracting with $g^{\mu\nu}$, this becomes

$$2\Box \epsilon_\mu = (2-d)\partial_\mu f. \quad (\text{E.291})$$

Applying ∂_ν to this expression, and using (E.288), we find

$$g_{\mu\nu} \Box f = (2-d)\partial_\mu \partial_\nu f. \quad (\text{E.292})$$

Contracting this with $g^{\mu\nu}$, we obtain $(d-1)\Box f = 0$, so that the previous expression becomes

$$\partial_\mu \partial_\nu f = 0. \quad (\text{E.293})$$

The function $f(x)$ is therefore at most linear in x . Substituting $f(x) = A + B_\mu x^\mu$ into (E.290), we see that $\partial_\mu \partial_\nu \epsilon_\rho$ is constant, which means that ϵ_μ is at most quadratic in x :

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho. \quad (\text{E.294})$$

Since the constraints (E.288) and (E.290) hold for all x , we can treat each power of the transformation separately. Substituting the linear term into (E.288), we find

$$b_{\mu\nu} = \lambda g_{\mu\nu} + r_{\mu\nu}, \quad (\text{E.295})$$

where $r_{\mu\nu} = -r_{\nu\mu}$. Similarly, substituting the quadratic term into (E.290), we get

$$c_{\mu\nu\rho} = g_{\mu\nu}b_\rho + g_{\mu\rho}b_\nu - g_{\nu\rho}b_\mu, \quad b_\mu \equiv \frac{1}{d}g^{\alpha\beta}c_{\alpha\beta\mu}. \quad (\text{E.296})$$

The special conformal transformation then takes the form

$$\begin{aligned} \epsilon_\mu &= c_{\mu\nu\rho}x^\nu x^\rho \\ &= (g_{\mu\nu}b_\rho + g_{\mu\rho}b_\nu - g_{\nu\rho}b_\mu)x^\nu x^\rho \\ &= 2(b \cdot x)x_\mu - x^2 b_\mu, \end{aligned} \quad (\text{E.297})$$

which establishes the result (D.1).

Solution to Exercise D.2

Solution to Exercise D.3

[DB: Four-point function:]

The reason that there are two cross-ratios can be understood as follows:

- Using SCTs, we can move x_4 to infinity.
- Using T , we can move x_1 to zero.
- Using R and D , we can move x_3 to $(1, 0, \dots, 0)$.
- Using R , with x_3 fixed, we can move x_2 to $(x, y, 0, \dots, 0)$.

This leaves two undetermined quantities x and y , giving two independent conformal invariants:

$$u = z\bar{z}, \quad v = (1 - z)(1 - \bar{z}),$$

where $z \equiv x + iy$.

Solution to Exercise D.4

Solution to Exercise D.5

Solution to Exercise D.6

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