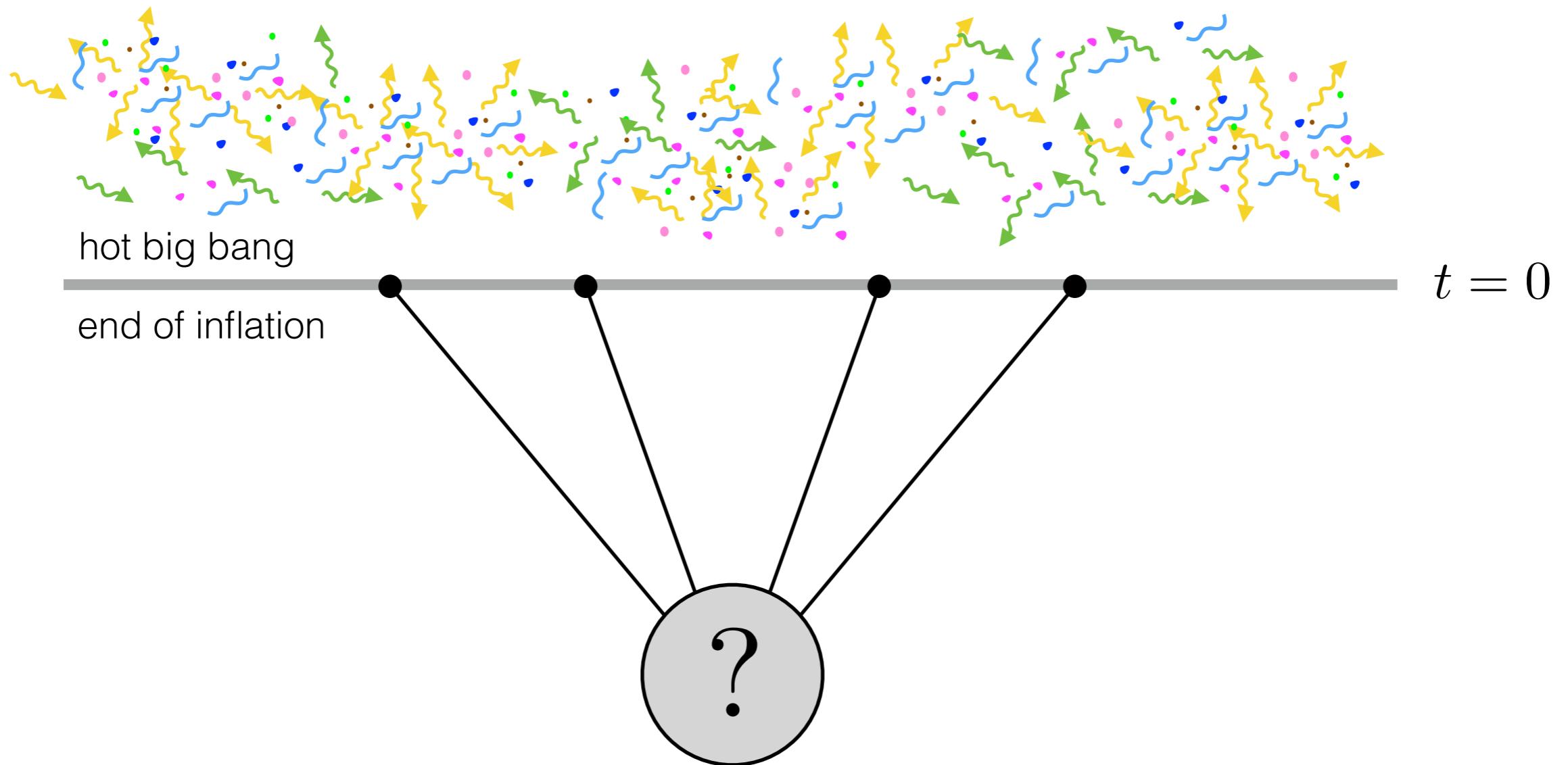


Cosmological Bootstrap

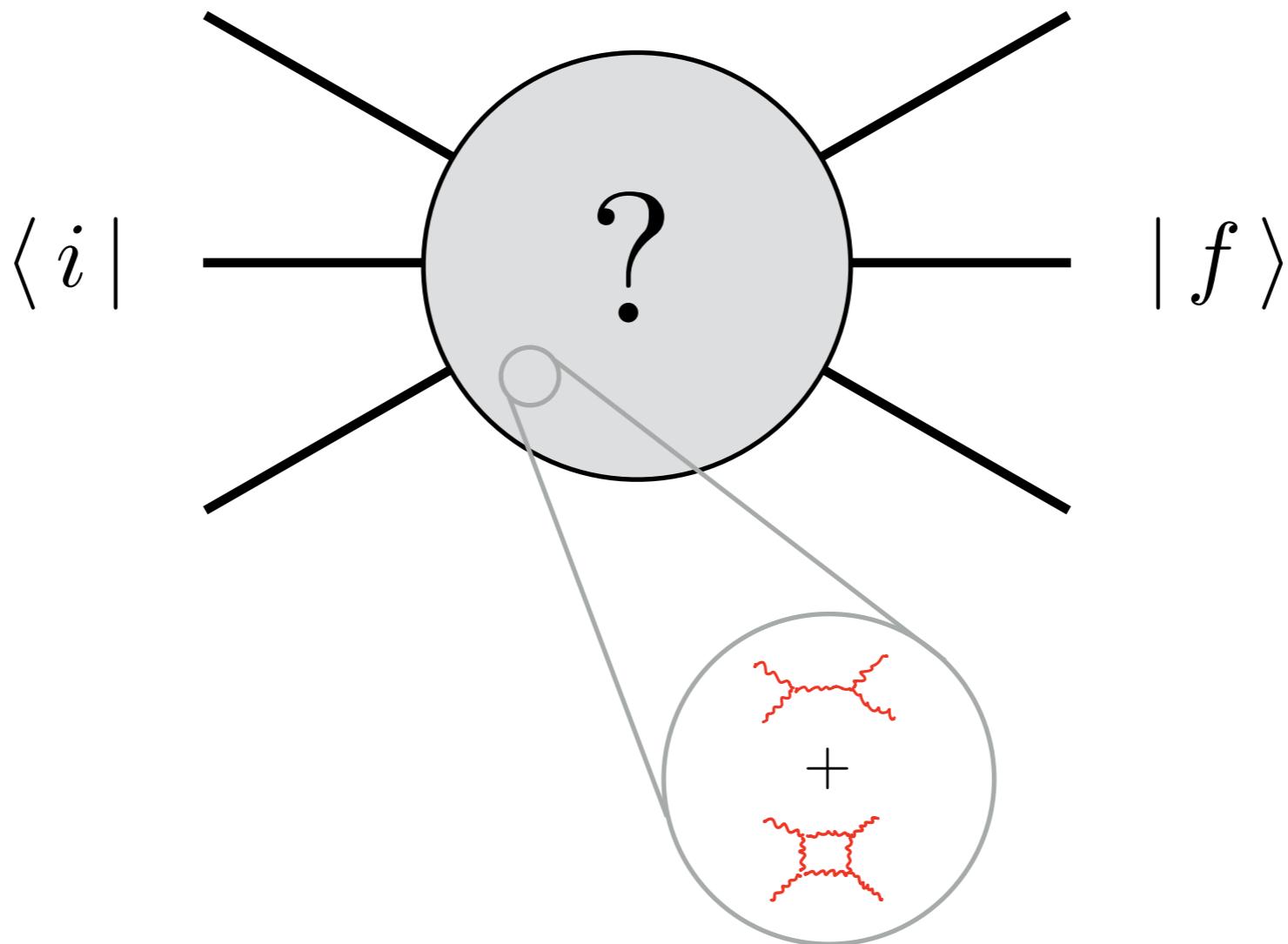
If inflation is correct, then the reheating surface is the future boundary of an approximate **de Sitter spacetime**:



Can we **bootstrap** these boundary correlators directly?

- Conceptual advantage: **focuses directly on observables**.
- Practical advantage: **simplifies calculations**.

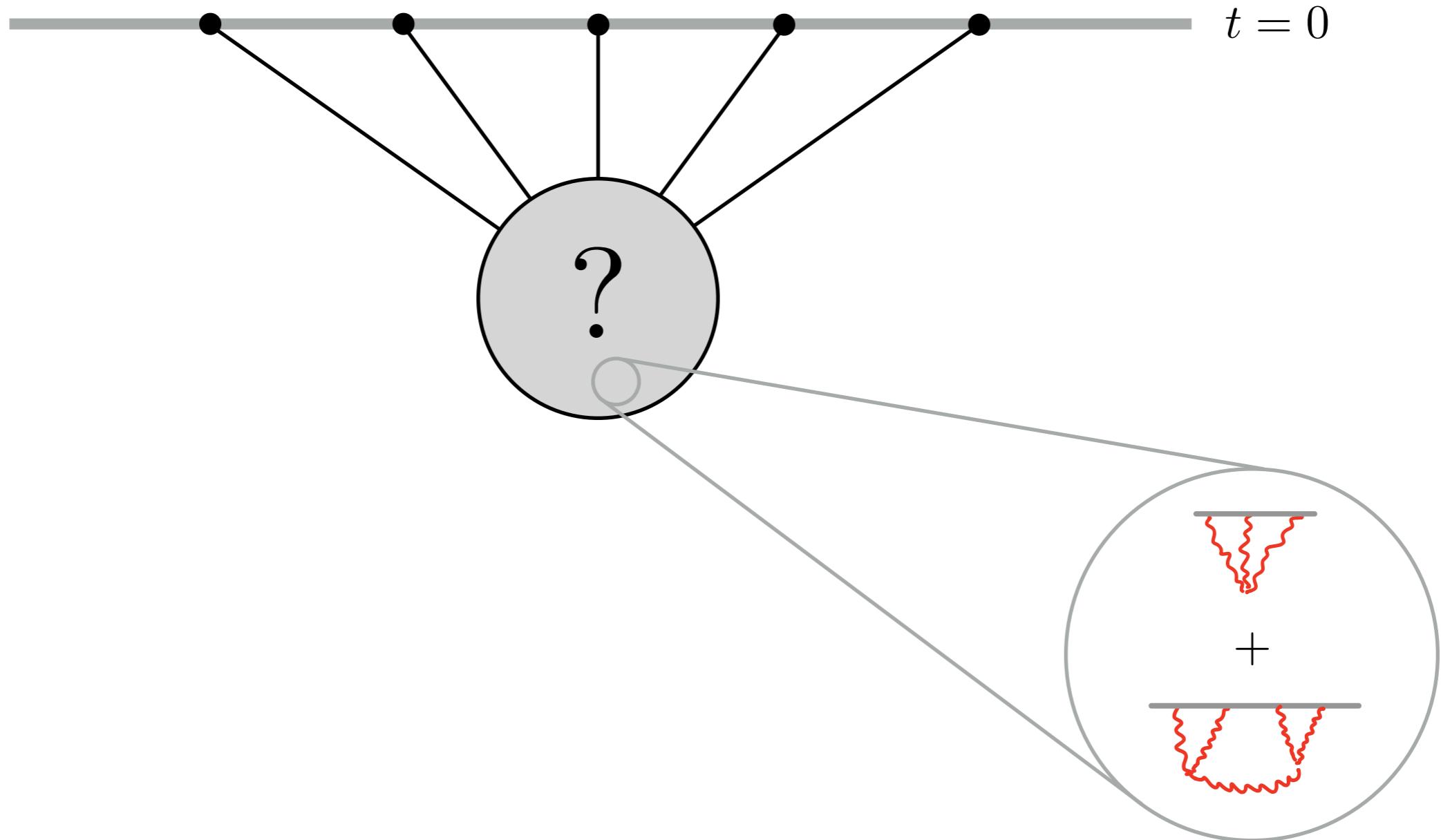
The bootstrap perspective has been very influential for **scattering amplitudes**:



There has been enormous progress in bypassing Feynman diagram expansions to write down on-shell amplitudes directly:

- Practical advantage: **simplifies calculations**.
- Conceptual advantage: **reveals hidden structures**.

I will describe recent progress in developing a similar bootstrap approach for cosmological correlators:



Goal: Develop an understanding of cosmological correlators that parallels our understanding of flat-space scattering amplitudes.

OUTLINE:

The Bootstrap
Philosophy

Bootstrapping
Tools

Examples

REFERENCES:

DB, Green, Joyce, Pajer, Pimentel, Sleight and Taronna,
Snowmass White Paper: The Cosmological Bootstrap [2203.08121]

DB and Joyce, *Lectures on Cosmological Correlations*

The Bootstrap Philosophy

The Conventional Approach

How we usually make predictions:

Physical Principles → **Models** → **Observables**

locality, causality,
unitarity, symmetries

Lagrangians
equations of motion
spacetime evolution
Feynman diagrams

Works well if we have a well-established theory (Standard Model, GR, ...)
and many observables.

The Conventional Approach

Although conceptually straightforward, this has some drawbacks:

- It involves unphysical gauge degrees of freedom.
- Relation between Lagrangians and observables is many-to-one.
- Even at tree level, the computation of cosmological correlators involves complicated time integrals:

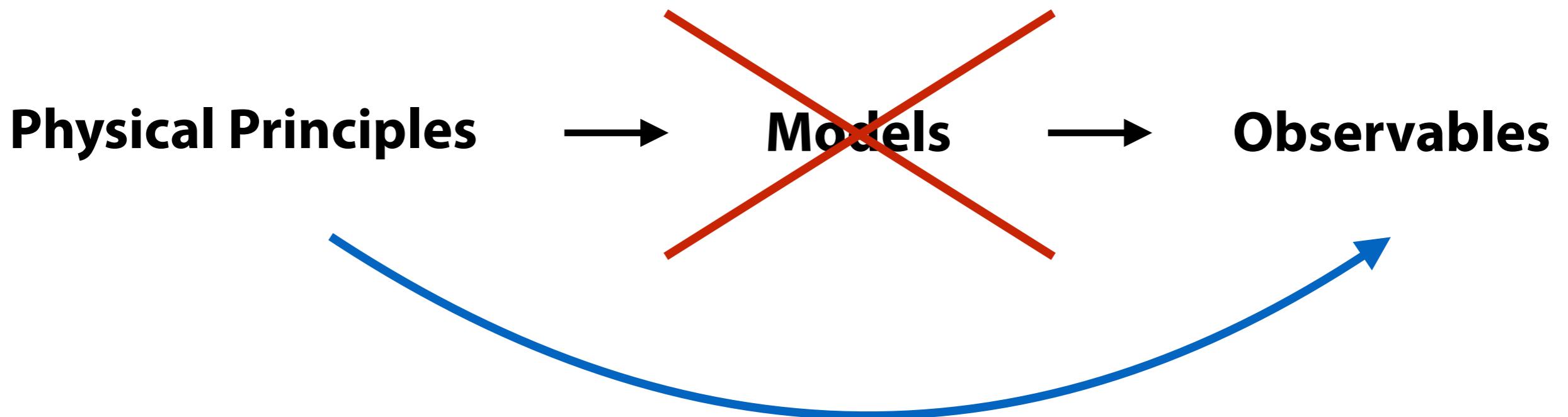
$$\psi_N = \sum_{\text{Diagrams}} \left(\prod_{\text{Vertices}} \int dt \right) \left(\begin{array}{c} \text{External} \\ \text{propagators} \end{array} \right) \left(\begin{array}{c} \text{Internal} \\ \text{propagators} \end{array} \right) \left(\begin{array}{c} \text{Vertex} \\ \text{factors} \end{array} \right)$$

Hard to compute!

- Fundamental principles (e.g. locality and unitarity) are obscured.
- To derive nonperturbative correlators and constraints from the UV completion, we need a deeper understanding.

The Bootstrap Method

In the bootstrap approach, we cut out the middle man and go directly from fundamental principles to physical observables:



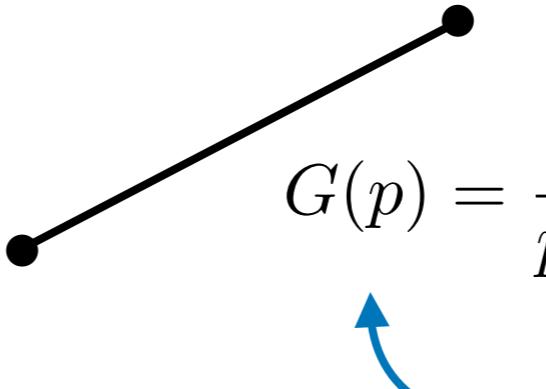
This is particularly relevant when we have many theories (inflation, BSM, ...) and few observables.

S-matrix Bootstrap

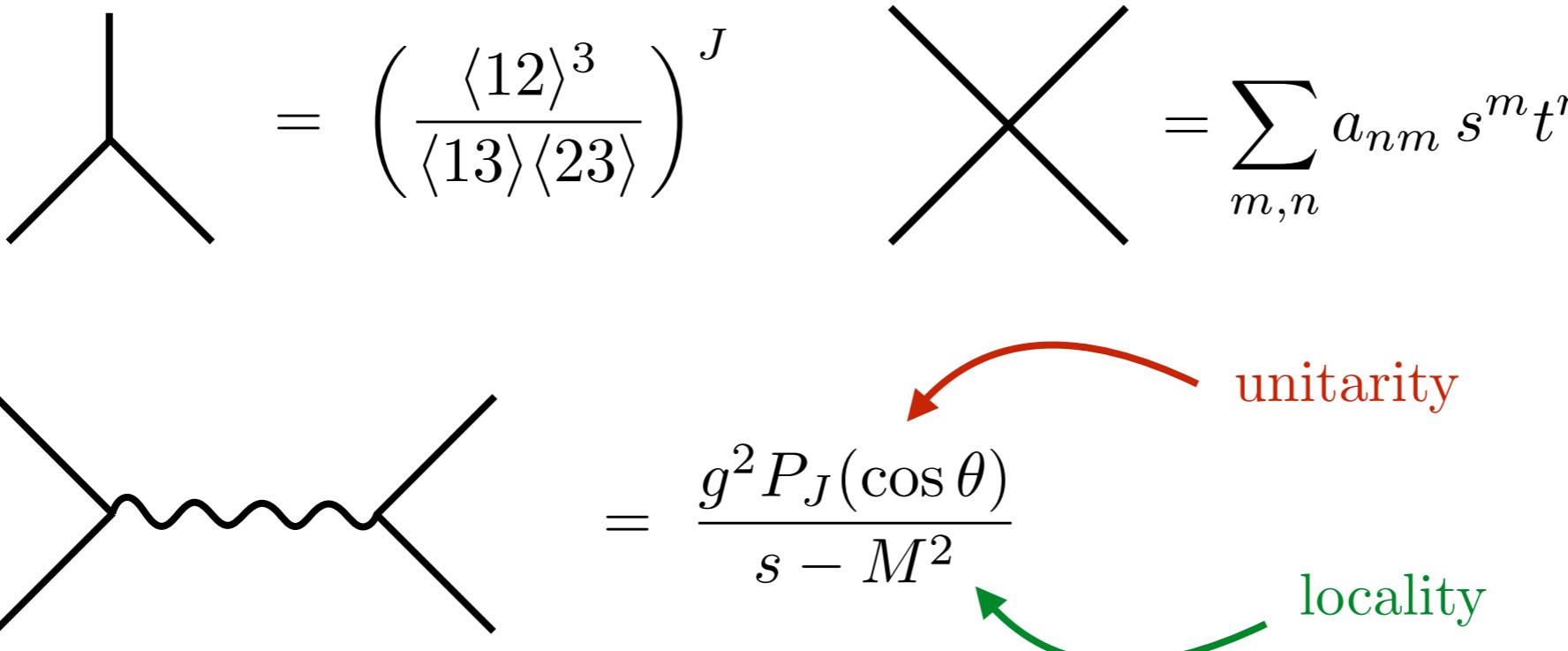
- What stable particles exist?

$$G(p) = \frac{1}{p^2 + M^2}$$

fixed by locality and
Lorentz symmetry



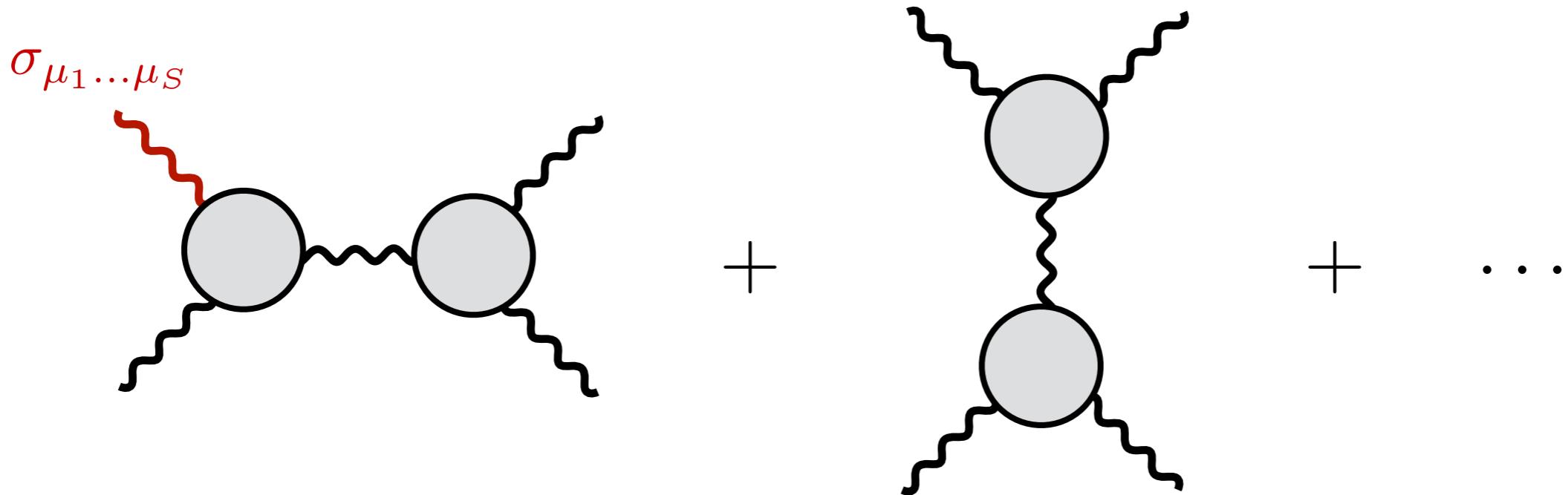
- What interactions are allowed?

$$\begin{array}{c} \text{Y vertex} \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \end{array} = \left(\frac{\langle 12 \rangle^3}{\langle 13 \rangle \langle 23 \rangle} \right)^J$$
$$\begin{array}{c} \text{X vertex} \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \end{array} = \sum_{m,n} a_{nm} s^m t^n \quad \xleftarrow{\text{Lorentz}}$$
$$\begin{array}{c} \text{Wavy line vertex} \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \end{array} = \frac{g^2 P_J(\cos \theta)}{s - M^2} \quad \begin{array}{l} \text{unitarity} \\ \curvearrowleft \\ \text{locality} \end{array}$$


- No Lagrangian is required to derive this.
- Basic principles allow only a small menu of possibilities.

S-matrix Bootstrap

Consistent factorization is very constraining for massless particles:



→ Only consistent for spins

$$S = \{ 0, \frac{1}{2}, 1, \frac{3}{2}, 2 \}$$



A Success Story

The modern amplitudes program has been very successful:

1. New Computational Techniques

- Recursion relations
- Generalized unitarity
- Soft theorems

2. New Mathematical Structures

- Positive geometries
- Amplituhedrons
- Associahedrons

3. New Relations Between Theories

- Color-kinematics duality
- BCJ double copy

4. New Constraints on QFTs

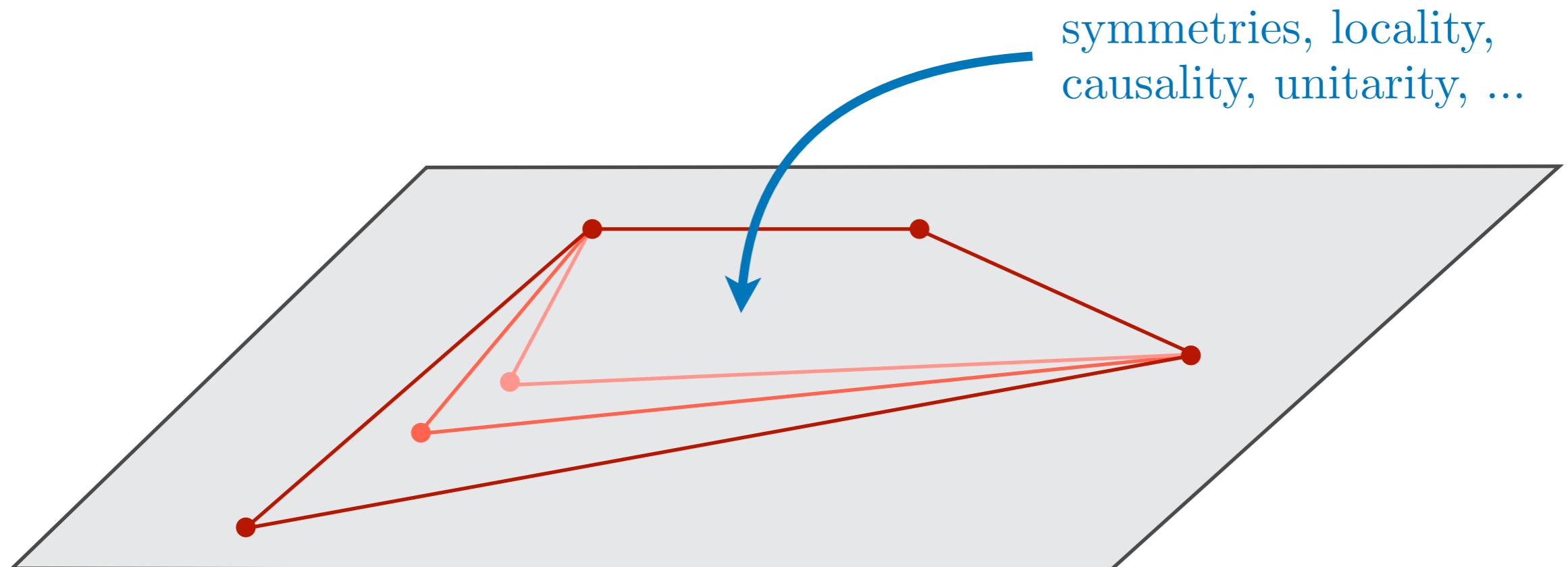
- Positivity bounds
- EFThedron

5. New Applications

- Gravitational wave physics
- Cosmology

Cosmological Bootstrap

Recently, there has been significant progress in **bootstrapping correlators** using physical consistency conditions on the boundary:



In this part of the lectures, I will review these developments.

Bootstrapping Tools

Symmetries

Singularities

Unitarity

Bootstrapping Tools

Symmetries

Singularities

Unitarity

De Sitter Symmetries

D -dimensional de Sitter space can be represented as an **hyperboloid** in $(D+1)$ -dimensional Minkowski space:

$$\eta_{AB} X^A X^B = -X_0^2 + X_1^2 + \cdots + X_D^2 = H^{-2}$$


de Sitter radius

The symmetries of de Sitter then correspond to higher-dimensional **Lorentz transformations**:

$$SO(D, 1) : [J_{AB}, J_{CD}] = \eta_{BC} J_{AD} - \eta_{AC} J_{BD} + \eta_{AD} J_{BC} - \eta_{BD} J_{AC}$$

In a specific time slicing, we define

$$\begin{aligned}\hat{D} &\equiv J_{D0} & \xleftarrow{\hspace{1cm}} & \text{Dilatations} \\ P_i &\equiv J_{Di} - J_{0i} & \xleftarrow{\hspace{1cm}} & \text{Translations} \\ K_i &\equiv J_{Di} + J_{0i} & \xleftarrow{\hspace{1cm}} & \text{Boosts} \\ J_{ij} & & \xleftarrow{\hspace{1cm}} & \text{Rotations}\end{aligned}$$

De Sitter Symmetries

D -dimensional de Sitter space can be represented as an **hyperboloid** in $(D+1)$ -dimensional Minkowski space:

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de Sitter radius

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$$SO(D, 1) : [J_{AB}, J_{CD}] = \eta_{BC} J_{AD} - \eta_{AC} J_{BD} + \eta_{AD} J_{BC} - \eta_{BD} J_{AC}$$

The de Sitter algebra then becomes

$$\begin{aligned}[J_{ij}, J_{kl}] &= \delta_{jk} J_{il} - \delta_{ik} J_{jl} + \delta_{il} J_{jk} - \delta_{jl} J_{ik} , & [\hat{D}, P_i] &= P_i , \\ [J_{ij}, K_l] &= \delta_{jl} K_i - \delta_{il} K_j , & [\hat{D}, K_i] &= -K_i , \\ [J_{ij}, P_l] &= \delta_{jl} P_i - \delta_{il} P_j , & [K_i, P_j] &= 2\delta_{ij} \hat{D} - 2J_{ij} .\end{aligned}$$

De Sitter Symmetries

We will work in the flat-slicing, where the de Sitter metric is

$$ds^2 = \frac{1}{H^2\eta^2} (-d\eta^2 + d\mathbf{x}^2)$$

and the symmetry generators (Killing vectors) become

$$\begin{aligned} P_i &= \partial_i & \hat{D} &= -\eta\partial_\eta - x^i\partial_i \\ J_{ij} &= x_i\partial_j - x_j\partial_i & K_i &= 2x_i\eta\partial_\eta + \left(2x^jx_i + (\eta^2 - x^2)\delta_i^j\right)\partial_j \end{aligned}$$

- **Dilatations** and **boosts** correspond to the following finite transformations:

$$\begin{aligned} \eta &\mapsto \lambda\eta \\ \mathbf{x} &\mapsto \lambda\mathbf{x} \end{aligned}$$

$$\begin{aligned} \eta &\mapsto \eta/\Omega \\ \mathbf{x} &\mapsto [\mathbf{x} + (x^2 - \eta^2)\mathbf{b}]/\Omega \end{aligned}$$

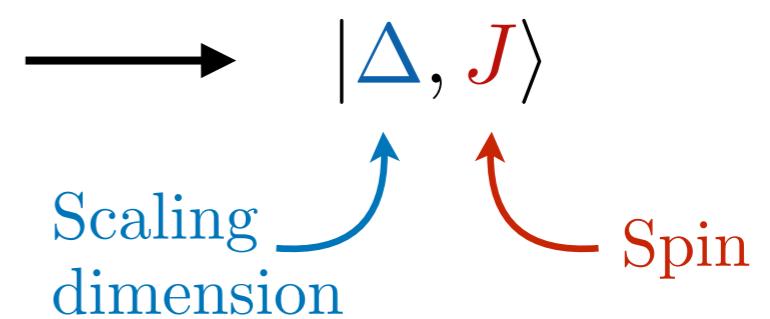
$$\text{where } \Omega \equiv 1 + 2(\mathbf{b} \cdot \mathbf{x}) + b^2(x^2 - \eta^2)$$

- At late times, the boosts become special conformal transformations (**SCTs**).
- Late-time correlators are therefore constrained by **conformal symmetry**.

Unitarity Representations

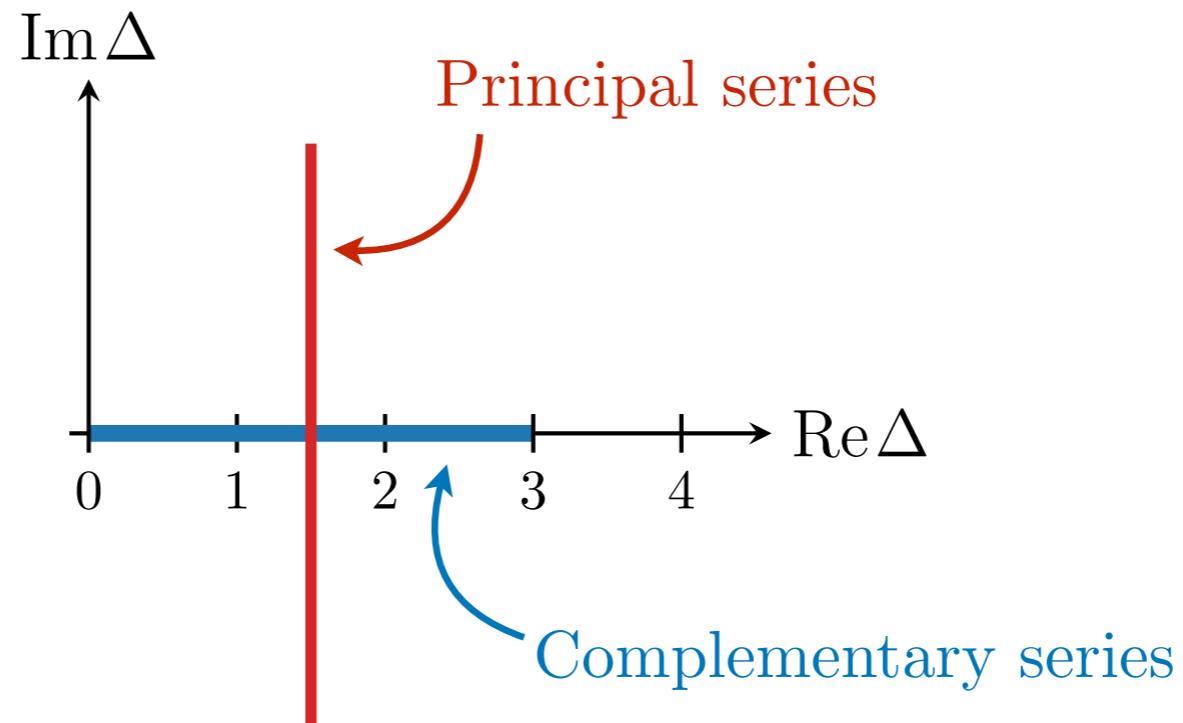
Physical states are described by unitary representations of the de Sitter group. These representations are labelled by the eigenvalues of the Casimir operators:

$$\mathcal{C}_2 \equiv \frac{1}{2} J^{AB} J_{AB} = \hat{D}(d - \hat{D}) + P^i K_i + \frac{1}{2} J^{ij} J_{ij}$$
$$\mathcal{C}_4 \equiv J_{AB} J^{BC} J_{CD} J^{DA}$$



Unitarity puts constraints on the allowed scaling dimensions.

For scalars, we have



Bulk Fields

These representations can be realized by bulk fields.

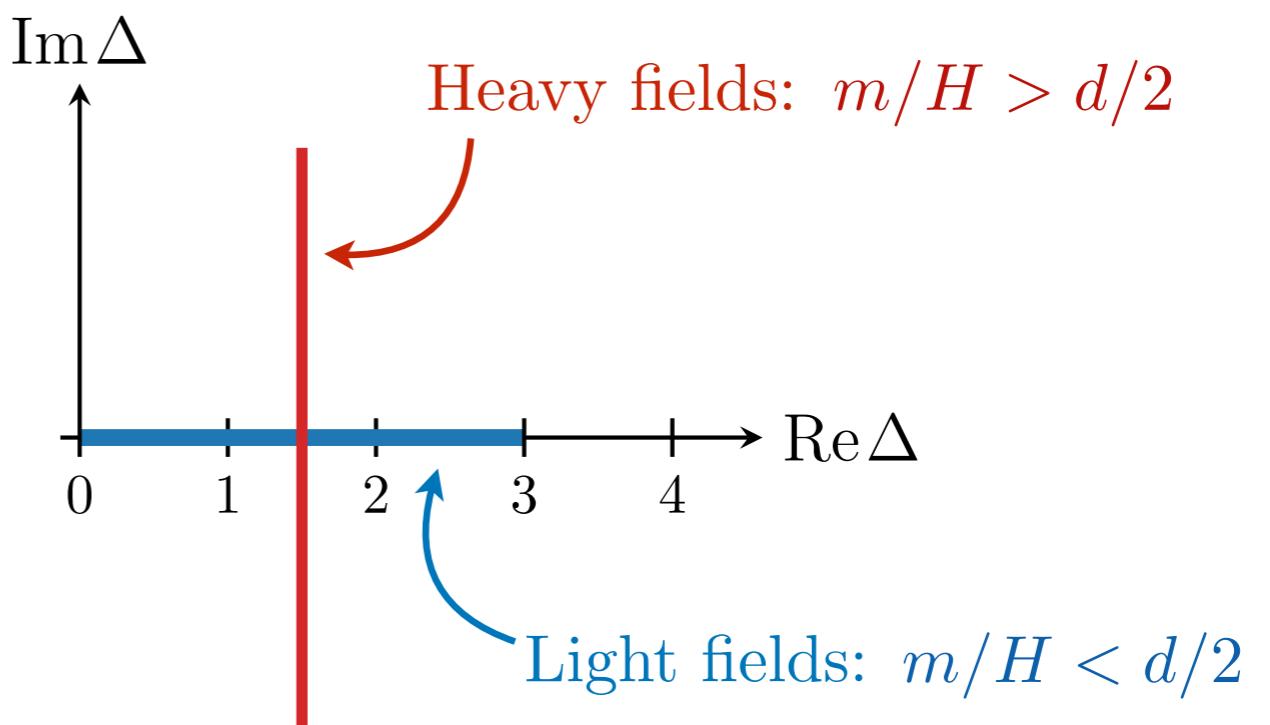
Consider a massive scalar field. Its equation of motion can be written as an eigenvalue equation for the quadratic Casimir:

$$\mathcal{C}_2 \Phi = -\eta^2 \left(\frac{\partial^2}{\partial \eta^2} - \frac{d-1}{\eta} \frac{\partial}{\partial \eta} - \nabla^2 \right) \Phi = \frac{m^2}{H^2} \Phi$$

Using $\mathcal{C}_2 = \Delta(d - \Delta)$, we get

$$\Delta \equiv \frac{d}{2} - \sqrt{\frac{d^2}{4} - \frac{m^2}{H^2}}$$

and $\bar{\Delta} = d - \Delta$.



Boundary Fluctuations

Our main interest are the fluctuations on the late-time boundary.

At late times, the solution for the bulk field is

$$\Phi(\mathbf{x}, \eta \rightarrow 0) = \phi(\mathbf{x}) \eta^\Delta + \bar{\phi}(\mathbf{x}) \eta^{d-\Delta}$$

↗
Boundary field profile

For light fields, the first term of the asymptotic solution dominates.

The de Sitter isometries then act as

$$P_i \phi = \partial_i \phi$$
$$J_{ij} \phi = (x_i \partial_j - x_j \partial_i) \phi$$
$$D \phi = -(\Delta + x^i \partial_i) \phi$$
$$K_i \phi = \left[2x_i \Delta + \left(2x^j x_i - x^2 \delta_i^j \right) \partial_j \right] \phi$$

$\longleftarrow \quad \eta \partial_\eta \mapsto \Delta$

→ Transformations of a primary operator of weight Δ in a **CFT**.

Conformal Ward Identities

The **boundary correlators** are constrained by the conformal symmetry:

$$\sum_{a=1}^N \langle \phi_1 \cdots \delta\phi_a \cdots \phi_N \rangle = 0$$

where $\delta\phi_a$ stand for any of the field transformations.

- Translations and rotations require that the correlator is only a function of the separations $|\mathbf{x}_a - \mathbf{x}_b|$.
- Dilatation and SCT invariance imply

$$0 = \sum_{a=1}^N \left(\Delta_a + x_a^j \frac{\partial}{\partial x_a^j} \right) \langle \phi_1 \cdots \phi_a \cdots \phi_N \rangle$$
$$0 = \sum_{a=1}^N \left(2\Delta_a x_a^i + 2x_a^i x_a^j \frac{\partial}{\partial x_a^j} - x_a^2 \frac{\partial}{\partial x_{a,i}} \right) \langle \phi_1 \cdots \phi_a \cdots \phi_N \rangle$$

Conformal Ward Identities

$$0 = \sum_{a=1}^N \left(\Delta_a + x_a^j \frac{\partial}{\partial x_a^j} \right) \langle \phi_1 \cdots \phi_a \cdots \phi_N \rangle$$
$$0 = \sum_{a=1}^N \left(2\Delta_a x_a^i + 2x_a^i x_a^j \frac{\partial}{\partial x_a^j} - x_a^2 \frac{\partial}{\partial x_{a,i}} \right) \langle \phi_1 \cdots \phi_a \cdots \phi_N \rangle$$

Exercise: Show that for two- and three-point functions the conformal Ward identities are solved by

$$\langle \phi_1 \phi_2 \rangle = \frac{1}{x_{12}^{2\Delta_1}} \delta_{\Delta_1, \Delta_2}$$

$$\langle \phi_1 \phi_2 \phi_3 \rangle = \frac{c_{123}}{x_{12}^{\Delta_t - 2\Delta_3} x_{23}^{\Delta_t - 2\Delta_1} x_{31}^{\Delta_t - 2\Delta_2}}$$

where $\phi_a \equiv \phi_a(\mathbf{x}_a)$, $x_{ab} \equiv |\mathbf{x}_a - \mathbf{x}_b|$ and $\Delta_t \equiv \sum \Delta_a$.

Conformal Ward Identities

Similar Ward identities can be derived for the **wavefunction coefficients**.

Consider

$$\Psi[\phi] = \exp \left(\cdots + \int d^d x_1 \cdots d^d x_N \Psi_N(\underline{\mathbf{x}}) \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_N) + \cdots \right)$$

↓

$$\delta \Psi \supset \sum_{a=1}^N \int d^d x_1 \cdots d^d x_N \Psi_N(\underline{\mathbf{x}}) \phi(\mathbf{x}_1) \cdots \delta \phi(\mathbf{x}_a) \cdots \phi(\mathbf{x}_N)$$

Integrate by parts, so that the generator of the transformation acts on the wavefunction coefficient. For example, for the dilatation, we have

$$\begin{aligned} \delta \Psi &\supset \sum_{a=1}^N \int d^d x_1 \cdots d^d x_N \Psi_N(\underline{\mathbf{x}}) \phi(\mathbf{x}_1) \cdots \left(-\Delta_a - x_a^j \partial_{x_a^j} \right) \phi(\mathbf{x}_a) \cdots \phi(\mathbf{x}_N) \\ &= \sum_{a=1}^N \int d^d x_1 \cdots d^d x_N \left[\left((d - \Delta_a) + x_a^j \partial_{x_a^j} \right) \Psi_N(\underline{\mathbf{x}}) \right] \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_N) \end{aligned}$$

Conformal Ward Identities

Under dilatations, the wavefunction coefficients therefore transform as

$$\delta\Psi_N(\underline{\mathbf{x}}) = \sum_{a=1}^N \left((d - \Delta_a) + x_a^j \partial_{x_a^j} \right) \Psi_N(\underline{\mathbf{x}})$$

A similar derivation applies to SCTs.

Imposing $\delta\Psi_N = 0$ leads to the conformal Ward identity for WF coefficients:

$$0 = \sum_{a=1}^N \left((d - \Delta_a) + x_a^j \frac{\partial}{\partial x_a^j} \right) \langle O_1 \cdots O_a \cdots O_N \rangle$$

$$0 = \sum_{a=1}^N \left(2(d - \Delta_a)x_a^i + 2x_a^i x_a^j \frac{\partial}{\partial x_a^j} - x_a^2 \frac{\partial}{\partial x_{a,i}} \right) \langle O_1 \cdots O_a \cdots O_N \rangle$$

where $\Psi_N \equiv \langle O_1 \cdots O_N \rangle$.

→ Same as before, except the dual operators have weight $\bar{\Delta}_a = d - \Delta_a$.

Conformal Ward Identities

In cosmology, we need these Ward identities in **Fourier space**.

Although the correlators themselves are hard to Fourier transform, the symmetry generators are not. This gives

$$\hat{D}O_{\mathbf{k}} = [\bar{\Delta} + k^i \partial_{k^i}] O_{\mathbf{k}}$$
$$K_i O_{\mathbf{k}} = [2\bar{\Delta} \partial_{k^i} - k_i \partial_{k^j} \partial_{k^j} + 2k^j \partial_{k^j} \partial_{k^i}] O_{\mathbf{k}}$$

Since $\langle O_1 \cdots O_N \rangle = (2\pi)^d \delta^{(d)}(\mathbf{k}_1 + \cdots + \mathbf{k}_N) \langle O_1 \cdots O_N \rangle'$, we need to pay attention to the momentum derivatives hitting the delta function.

This gives an extra term in the dilatation Ward identity, so that

$$\left[-d + \sum_{a=1}^N \hat{D}_a \right] \langle O_1 \cdots O_N \rangle'(\underline{\mathbf{k}}) = 0$$
$$\sum_{a=1}^N K_a^i \langle O_1 \cdots O_N \rangle'(\underline{\mathbf{k}}) = 0$$

Conformal Ward Identities

$$\left[-d + \sum_{a=1}^N \hat{D}_a \right] \langle O_1 \cdots O_N \rangle'(\underline{\mathbf{k}}) = 0$$

$$\sum_{a=1}^N K_a^i \langle O_1 \cdots O_N \rangle'(\underline{\mathbf{k}}) = 0$$

- An N -point function has scaling dimension $\bar{\Delta}_t \equiv \sum_a \bar{\Delta}_a$ in position space.
- The momentum-space correlator then has dimension $\bar{\Delta}_t - d \times N$.
- After stripping off the delta function, the dimension is $\bar{\Delta}_t - d(N - 1)$.

This fixes the overall momentum scaling of the correlator, allowing us to make an ansatz that automatically solves the dilatation Ward identity.

All the juice is then in the SCT Ward identity.

Two-Point Functions

Consider the two-point function of **two arbitrary scalar operators**.

- An ansatz that solves the dilatation Ward identity is

$$\langle O_1 O_2 \rangle' \propto k_1^{\bar{\Delta}_1 + \bar{\Delta}_2 - d}$$

- The SCT Ward identity then forces $\bar{\Delta}_1 = \bar{\Delta}_2$, and hence

$$\langle O_1 O_2 \rangle' = A k_1^{2\bar{\Delta}_1 - d} \delta_{\bar{\Delta}_1, \bar{\Delta}_2}$$

- A massless field ($\bar{\Delta} = 3$) in $d = 3$ dimensions has

$$\langle \phi_1 \phi_2 \rangle' = \frac{1}{2 \operatorname{Re} \langle O_1 O_2 \rangle'} \propto \frac{1}{k^3}$$

as expected for a scale-invariant power spectrum.

Three-Point Functions

Consider the three-point function of **three arbitrary scalar operators**.

- An ansatz that solves the dilatation Ward identity is

$$\langle O_1 O_2 O_3 \rangle' = k_3^{\bar{\Delta}_1 + \bar{\Delta}_2 + \bar{\Delta}_3 - 2d} \hat{G}(p, q)$$

where $p \equiv k_1/k_3$ and $q \equiv k_2/k_3$.

- The SCT Ward identity then implies two differential equations for $\hat{G}(p, q)$, which are solved by

$$\langle O_1 O_2 O_3 \rangle' = \text{Appell } F_4$$

$$\propto \int_0^\infty dx x^{\frac{d}{2}-1} K_{\nu_1}(k_1 x) K_{\nu_2}(k_2 x) K_{\nu_3}(k_3 x)$$

Three-Point Functions

It will be useful to work out more explicitly the case of **two conformally coupled scalars** and **a generic scalar** in 3 dimensions:

$$\langle O_1 O_2 X_3 \rangle' = k_3^{\bar{\Delta}-2} \hat{G}(u, w)$$

where $u \equiv k_3/(k_1 + k_2)$ and $w \equiv k_3/(k_1 - k_2)$.

- The SCT Ward identity then implies $\hat{G}(u, w) = \hat{G}(u)$, with

$$\boxed{\left[\Delta_u + \left(\mu^2 + \frac{1}{4} \right) \right] \hat{G}(u) = 0}$$

where $\Delta_u \equiv u^2(1-u^2)\partial_u^2 - 2u^3\partial_u$ and $\mu^2 + \frac{1}{4} = -(\bar{\Delta} - 2)(\bar{\Delta} - 1)$.

- This is solved by $\hat{G}_+(u) \propto P_{i\mu-\frac{1}{2}}^0(u^{-1})$ and $\hat{G}_-(u) \propto P_{i\mu-\frac{1}{2}}^0(-u^{-1})$.



associated Legendre

Four-Point Functions

Finally, we consider the four-point function of **conformally coupled scalars**:

$$\langle O_1 O_2 O_3 O_4 \rangle' = \begin{array}{c} \text{Diagram showing a four-point function with four external legs meeting at a central point labeled } k_I. \text{ The top-left leg is blue and labeled } k_1, \text{ the top-right leg is red and labeled } k_2, \text{ the bottom-right leg is red and labeled } k_3, \text{ and the bottom-left leg is red and labeled } k_4. \end{array} = \frac{1}{k_I} \hat{F}(u, v)$$

where $u \equiv k_I/(k_1 + k_2)$ and $v \equiv k_I/(k_3 + k_4)$.

- The SCT Ward identity then implies

$$(\Delta_u - \Delta_v) \hat{F} = 0$$

Arkani-Hamed, DB, Lee
and Pimentel [2018]

where $\Delta_u \equiv u^2(1-u^2)\partial_u^2 - 2u^3\partial_u$.

- To solve this equation requires additional physical input.

Bootstrapping Tools

Symmetries

Singularities

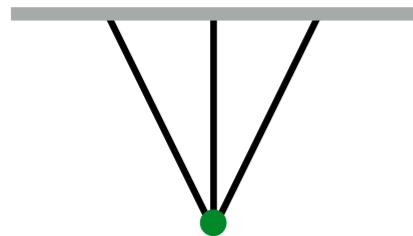
Unitarity

Much of the physics of scattering amplitudes is fixed by their **singularities**.

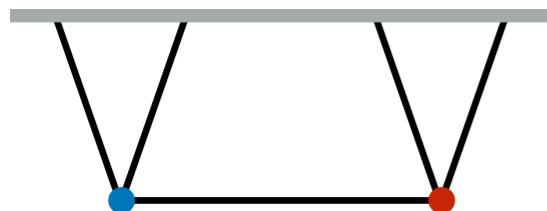
Singularities also play an important role in constraining the structure of cosmological correlators.

Flat-Space Wavefunction

In the last lecture, we computed wavefunction coefficients in flat space.
Let us look back at some of these results.



$$= ig \int_{-\infty}^0 dt e^{i\textcolor{green}{E}t} = \frac{g}{E}$$



$$= -g^2 \int_{-\infty}^0 dt_1 dt_2 e^{ik_{12}t_1} G_{k_I}(t_1, t_2) e^{ik_{34}t_2}$$

$$= \frac{g^2}{(k_{12} + k_{34})(\textcolor{blue}{k}_{12} + k_I)(k_{34} + k_I)} \equiv \frac{g^2}{E \textcolor{blue}{E}_L E_R}$$

- Correlators are singular when energy is conserved.

Total Energy Singularity

Every correlator has a singularity at vanishing total energy:

$$\psi_N = \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} \quad \sim \int_{-\infty}^0 dt e^{iEt} f(t) = \frac{A_N}{E} + \dots$$

$$A_N = \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} \quad \sim \int_{-\infty}^{\infty} dt e^{iEt} f(t) = M_N \delta(E)$$

- The residue of the total energy singularity is the corresponding amplitude.

Total Energy Singularity

- This is easy to confirm for the tree-exchange diagram:

$$\lim_{E \rightarrow 0} \frac{g^2}{EE_L E_R} = \frac{1}{E} \frac{g^2}{(k_{12} + k_I)(-k_{12} + k_I)} = -\frac{1}{E} \frac{g^2}{s} = \frac{A_4}{E}$$

where $s \equiv k_{12}^2 - k_I^2$ is the Mandelstam invariant.

- The total energy singularity arises from the early-time limit of the integration (where the boundary is infinitely far away):

$$\psi_N \sim \int_{-\infty}^0 dt e^{iEt} f(t) = \frac{A_N}{E} + \dots$$

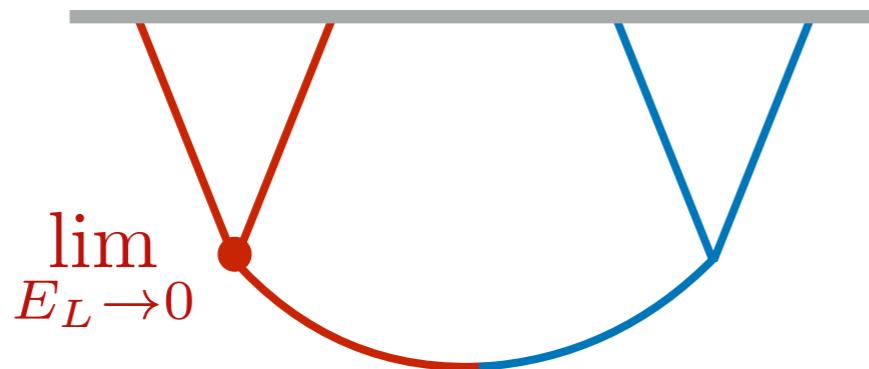
This explains intuitively why this singularity reproduces flat-space physics.

- The singularity can also be seen in the flat-space recursion relation for the wavefunction.

Arkani-Hamed, Benincasa and Postnikov [2017]

Partial Energy Singularities

Exchange diagrams lead to additional singularities:


$$= \frac{A_L \times \tilde{\psi}_R}{E_L} + \dots$$

- The singularity arises from the early-time limit of the integration, where the **bulk-to-bulk propagator factorizes**:

$$G_{k_I}(t_L, t_R) \xrightarrow{t_L \rightarrow -\infty} \frac{e^{ik_I t_L}}{2k_I} \left(e^{-ik_I t_R} - e^{+ik_I t_R} \right)$$

- On the pole, the correlator factorizes into a product of an **amplitude** and a **shifted correlator**:

$$\tilde{\psi}_R \equiv \frac{1}{2k_I} \left(\psi_R(k_{34} - k_I) - \psi_R(k_{34} + k_I) \right)$$

Partial Energy Singularities

This pattern generalizes to **arbitrary graphs**.

- The wavefunction is singular when the energy flowing into any connected subgraph vanishes:

$$\lim_{E_L \rightarrow 0} \text{Diagram} = \frac{A_L \times \tilde{\psi}_R}{E_L} + \dots$$

- For loops, the poles can become branch points.

Arkani-Hamed, Benincasa and Postnikov [2017]
DB, Chen, Duaso Pueyo, Joyce, Lee and Pimentel [2021]
Salcedo, Lee, Melville and Pajer [2022]

Generalization to De Sitter

The wavefunction in **de Sitter** also has singularities when the energy flowing into any connected subgraphs vanishes:

$$\lim_{E_L \rightarrow 0} \text{Diagram} = \frac{A_L \times \tilde{\psi}_R}{E_L^P} + \dots$$

The diagram shows two circular nodes connected by a horizontal blue line. The left node is shaded grey and has two red lines extending upwards from its top. The right node is also shaded grey and has two blue lines extending upwards from its top. A horizontal grey bar is positioned above the nodes.

- These singularities again arise from the early-time limit of the integration, where the bulk-to-bulk propagator factorizes.
- In these singular limits, the correlator factorizes into a product of an **amplitude** and a **shifted correlator**:

$$\tilde{\psi}_R \equiv P(k_I) \left(\psi_R(k_a, -k_I) - \psi_R(k_a, k_I) \right)$$

A blue bracket encloses the term $P(k_I) \left(\psi_R(k_a, -k_I) - \psi_R(k_a, k_I) \right)$. A blue arrow points from this bracket to the text "Power spectrum of exchanged field" below it.

Power spectrum of
exchanged field

Singularities as Input

The singularities of the wavefunction only arise at unphysical kinematics. Nevertheless, they control the structure of cosmological correlators.

- Amplitudes are the **building blocks** of cosmological correlators:

$$\lim_{E_L \rightarrow 0} \text{Diagram} = \frac{\text{Red loop} \times \text{Blue triangle}}{(E_L)^p} \xrightarrow{E_R \rightarrow 0} \frac{\text{Blue loop}}{(E_R)^q}$$

The diagram illustrates the decomposition of a complex Feynman-like diagram into simpler components. On the left, a red shaded loop and a blue shaded triangle are shown. A horizontal line labeled $(E_L)^p$ connects them. An equals sign follows. To the right, the red loop is shown separately with a wavy line, and the blue triangle is shown separately. A horizontal line labeled $(E_R)^q$ connects the two components. An arrow above the line indicates the limit $E_R \rightarrow 0$. Finally, a blue shaded loop is shown with a wavy line attached to it.

- To construct the full correlator, we need to connect its singularities. We will present two ways of doing this:

- 1) **Ward identities**
- 2) **Unitarity**

Bootstrapping Tools

Symmetries

Singularities

Unitarity

Unitarity is a fundamental feature of quantum mechanics.

However, the constraints of **bulk unitary** on cosmological correlators were understood only very recently.

Goodhew, Jazayeri and Pajer [2020]
Meltzer and Sivaramakrishnan [2020]

S-matrix Optical Theorem

A consistent theory must have a unitary S-matrix: $S^\dagger S = 1$

Writing $S = 1 + iT$, this implies

$$T - T^\dagger = iT^\dagger T$$

Sandwiching this between the states $\langle f |$ and $| i \rangle$, and inserting a complete set of states, we get

$$\langle f | T | i \rangle - \langle i | T | f \rangle^* = i \sum_X \int d\Pi_X \langle f | T^\dagger | X \rangle \langle X | T | i \rangle$$

Using $\langle f | T | i \rangle \equiv (2\pi)^4 \delta(p_i - p_f) A(i \rightarrow f)$, we obtain

$$2 \text{Im } A(i \rightarrow f) = \sum_X \int d\Pi_X (2\pi)^4 \delta(p_i - p_X) A(i \rightarrow X) A^*(f \rightarrow X)$$

The right-hand side can be written in terms of the total cross section, giving the familiar form of the **optical theorem**.

Cosmological Optical Theorem

Recently, an analogous result was derived for cosmological correlators.

Recall that $\Psi[\Phi] = \langle \Phi | \hat{U} | 0 \rangle$, where

Goodhew, Jazayeri and Pajer [2020]

$$\hat{U} = T \exp \left(-i \int_{-\infty}^{\eta_*} d\eta \hat{H}_{\text{int}} \right)$$

Probabilities, $|\Psi(\eta)|^2$, are conserved if $U^\dagger U = 1$.

Writing $\hat{U} = 1 + \hat{V}$, this implies

$$\hat{V} + \hat{V}^\dagger = -\hat{V}^\dagger \hat{V}$$

Sandwiching this between $\langle \Phi |$ and $| 0 \rangle$, and inserting a complete set of states, we get

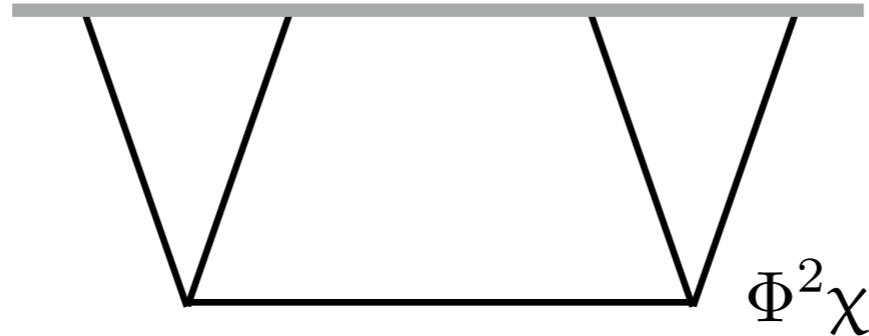
$$\langle \Phi | \hat{V} | 0 \rangle + \langle \Phi | \hat{V}^\dagger | 0 \rangle = - \int \mathcal{D}X \langle \Phi | \hat{V}^\dagger | X \rangle \langle X | \hat{V} | 0 \rangle$$

**Cosmological
Optical Theorem**

To use this, we need to relate the matrix elements to wavefunction coefficients.

Example: Four-Point Exchange

As a concrete example, we consider the **four-point exchange diagram**:



To extract a constraint on the WF coefficient from the COT, we let $|\Phi\rangle$ be the momentum eigenstate $|\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4\rangle \equiv |\{\mathbf{k}_a\}\rangle$:

$$\langle\{\mathbf{k}_a\}|\hat{V}|0\rangle + \langle\{\mathbf{k}_a\}|\hat{V}^\dagger|0\rangle = - \sum_X \int \prod_{b \in X} \frac{d^3\mathbf{q}_b}{(2\pi)^3} \langle\{\mathbf{k}_a\}|\hat{V}^\dagger|\{\mathbf{q}_b\}\rangle \langle\{\mathbf{q}_b\}|\hat{V}|0\rangle$$

where $\hat{V} = -i \int_{-\infty}^{\eta_*} d\eta \hat{H}_{\text{int}}(\eta) - \int_{-\infty}^{\eta_*} d\eta \int_{-\infty}^{\eta} d\eta' \hat{H}_{\text{int}}(\eta) \hat{H}_{\text{int}}(\eta') + \dots$

Our task is to relate the matrix elements in this expression to WF coefficients.

Example: Four-Point Exchange

$$\langle \{\mathbf{k}_a\} | \hat{V} | 0 \rangle + \langle \{\mathbf{k}_a\} | \hat{V}^\dagger | 0 \rangle = - \sum_X \int \prod_{b \in X} \frac{d^3 \mathbf{q}_b}{(2\pi)^3} \langle \{\mathbf{k}_a\} | \hat{V}^\dagger | \{\mathbf{q}_b\} \rangle \langle \{\mathbf{q}_b\} | \hat{V} | 0 \rangle$$

- Substituting the mode expansions

$$\hat{\Phi}_{\mathbf{k}}(\eta) = f_k^*(\eta) \hat{a}_{\mathbf{k}} + f_k(\eta) \hat{a}_{-\mathbf{k}}$$

$$\hat{\chi}_{\mathbf{k}}(\eta) = \chi_k^*(\eta) \hat{b}_{\mathbf{k}} + \chi_k(\eta) \hat{b}_{-\mathbf{k}}$$

the **first term** on the left-hand side becomes

$$\langle \{\mathbf{k}_a\} | \hat{V} | 0 \rangle = -g^2 \int d\eta d\eta' a^4(\eta) a^4(\eta') G_F(k_I; \eta, \eta') \prod_{a=1}^4 f_{k_a}(\eta)$$

where G_F is the Feynman propagator.

Example: Four-Point Exchange

$$\langle \{\mathbf{k}_a\} | \hat{V} | 0 \rangle + \langle \{\mathbf{k}_a\} | \hat{V}^\dagger | 0 \rangle = - \sum_X \int \prod_{b \in X} \frac{d^3 \mathbf{q}_b}{(2\pi)^3} \langle \{\mathbf{k}_a\} | \hat{V}^\dagger | \{\mathbf{q}_b\} \rangle \langle \{\mathbf{q}_b\} | \hat{V} | 0 \rangle$$

Substituting

$$G_F(k; \eta, \eta') = G(k; \eta, \eta') + \frac{\chi_k^*(\eta_*)}{\chi_k(\eta_*)} \chi_k(\eta) \chi_k(\eta')$$

we get

$$\langle \{\mathbf{k}_a\} | \hat{V} | 0 \rangle = \left(\psi_4(k_a, k_I) - \psi_3(k_{12}, k_I) \psi_3(k_{34}, k_I) P_\chi(k_I) \right) \left(\prod_{a=1}^4 f_{k_a}(\eta_*) \right).$$

- A similar analysis for the **second term** leads to

$$\langle \{\mathbf{k}_a\} | \hat{V}^\dagger | 0 \rangle = \left([\psi_4(-k_a, k_I)]^* - \psi_3(k_{12}, -k_I) \psi_3(k_{34}, -k_I) P_\chi(k_I) \right) \left(\prod_{a=1}^4 f_{k_a}(\eta_*) \right)$$

Analytic continuation

Example: Four-Point Exchange

$$\langle \{\mathbf{k}_a\} | \hat{V} | 0 \rangle + \langle \{\mathbf{k}_a\} | \hat{V}^\dagger | 0 \rangle = - \sum_X \int \prod_{b \in X} \frac{d^3 \mathbf{q}_b}{(2\pi)^3} \langle \{\mathbf{k}_a\} | \hat{V}^\dagger | \{\mathbf{q}_b\} \rangle \langle \{\mathbf{q}_b\} | \hat{V} | 0 \rangle$$

- Finally, the **third term** leads to

$$\text{RHS} = [\psi_3(k_{12}, -k_I) \psi_3(k_{34}, k_I) + \psi_3(k_{12}, k_I) \psi_3(k_{34}, -k_I)] P_\chi(k_I) \left(\prod_{a=1}^4 f_{k_a}(\eta_*) \right)$$

- Putting it all together, we get

$$\psi_4(k_a, k_I) + [\psi_4(-k_a, k_I)]^* = -P_\chi(k_I) \tilde{\psi}_3(k_{12}, k_I) \tilde{\psi}_3(k_{34}, k_I)$$

where $\tilde{\psi}_3(k_{12}, k_I) \equiv \psi_3(k_{12}, k_I) - \psi_3(k_{12}, -k_I)$.

Cutting Rules

For more complicated graphs, the explicit evaluation of the COT quickly becomes cumbersome. A simpler alternative are **cutting rules**.

We start with some analytic properties of the propagators defining the perturbative wavefunction:

- Both propagators decay at past infinity, which we can enforce by giving the norm k a negative imaginary part: $k^2 = re^{i\theta}$, with $\theta \in (-2\pi, 0)$.
- A useful concept is the **discontinuity** of a function

$$\text{Disc}[f(k)] \equiv f(k) - f(e^{-i\pi}k)$$

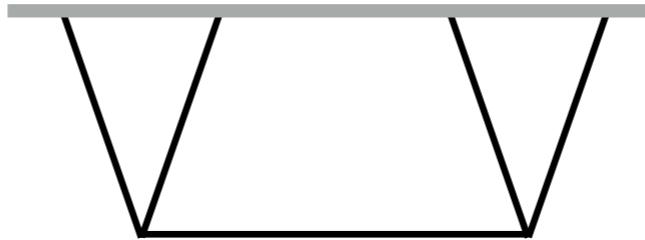
- The discontinuity of the **bulk-to-bulk propagator** is

$$\text{Disc}[G(k; \eta, \eta')] = -P(k) \text{Disc}[K(k, \eta)] \times \text{Disc}[K(k, \eta')]$$

Cutting Rules

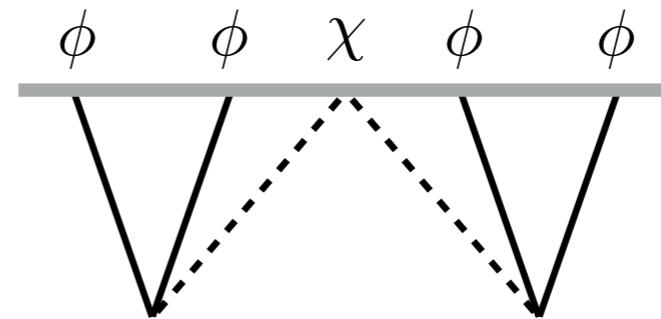
Consider, the four-point exchange diagram

$$\psi_4(k_a, k_I) =$$



Taking the discontinuity (with respect to k_I) corresponds to cutting the internal line:

$$\text{Disc}_{k_I} [\psi_4(k_a, k_I)] = P_\chi(k)$$



Hence, we find

$$\text{Disc}_{k_I} [\psi_4(k_a, k_I)] = P_\chi(k_I) \text{Disc}_{k_I} [\psi_3(k_{12}, k_I)] \times \text{Disc}_{k_I} [\psi_3(k_{34}, k_I)]$$

which is equivalent to the result derived from the COT.

Cutting Rules

This procedure generalizes to systematic **cutting rules**:

$$\text{Disc}[\psi_N] = \sum \text{all possible cuts}$$

These are the analog of the Cutkosky cutting rules for the S-matrix.

More details can be found in the lecture notes and in the relevant literature.

Goodhew, Jazayeri and Pajer [2020]

Meltzer and Sivaramakrishnan [2020]

Caspedes, Davis and Melville [2020]

Goodhew, Jazayeri, Lee and Pajer [2021]

DB, Chen, Duaso Pueyo, Joyce, Lee and Pimentel [2021]

Melville and Pajer [2021]

Sleight and Taronna [2021]

Jazayeri, Pajer and Stefanyszyn [2021]

Meltzer [2021]

Benincasa [2022]

Unitarity as Input

- The discontinuity of a WF coefficient factorizes into lower-point objects.
- This contains slightly more information than the energy singularities:

$$\lim_{E_L \rightarrow 0} \text{Disc} \quad \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} \quad \text{---} \quad \text{---}$$
$$= \lim_{E_L \rightarrow 0} \left[\psi_N(k_a, k_I) - \psi_N(k_a, -k_I) \right] = \lim_{E_L \rightarrow 0} \psi_N(k_a, k_I)$$

has no singularities in E_L

The cutting rule fixes the entire Laurent series in E_L .

- These unitarity cuts have become an important bootstrapping tool.

Examples

Massive
Exchange

Dispersive
Integrals

FRW
Correlators

Examples

**Massive
Exchange**

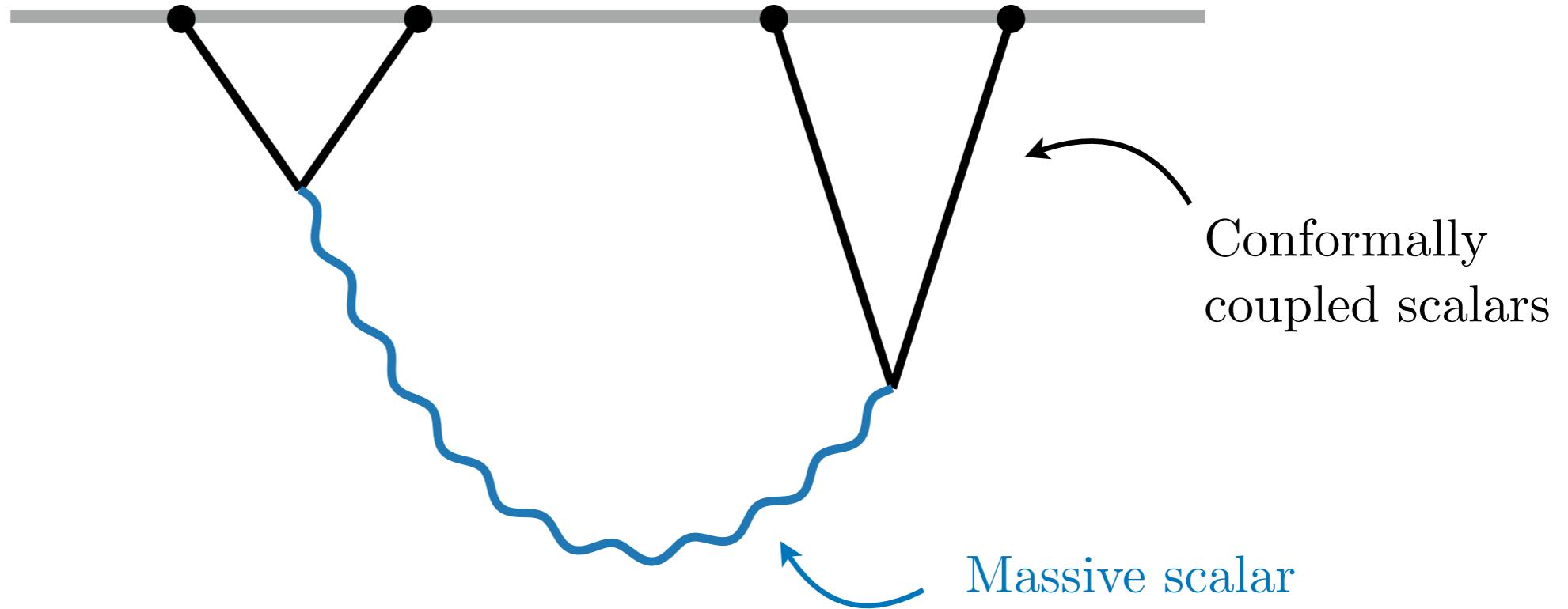
Dispersive
Integrals

FRW
Correlators

Exchange Four-Point Function

We are now ready to return to the challenge of massive particle exchange.

$$\langle O_1 O_2 O_3 O_4 \rangle' =$$



$$= -g^2 \int \frac{d\eta}{\eta^2} \int \frac{d\eta'}{\eta'^2} e^{ik_{12}\eta} e^{ik_{34}\eta'} G(k_I; \eta, \eta'')$$

In general, the time integrals cannot be performed analytically.

Conformal Ward Identity

Recall that the s-channel correlator can be written as

$$\langle O_1 O_2 O_3 O_4 \rangle' = \begin{array}{c} \text{Diagram of an s-channel correlator} \\ \text{Four external legs meeting at a central point labeled } k_I. \\ \text{Blue legs: } k_1, k_2 \\ \text{Red legs: } k_3, k_4 \end{array} = \frac{1}{k_I} \hat{F}(u, v)$$

where $u \equiv k_I/(k_1 + k_2)$ and $v \equiv k_I/(k_3 + k_4)$.

We have seen that the **conformal symmetry** implies the following differential equation

$$(\Delta_u - \Delta_v) \hat{F} = 0$$

where $\Delta_u \equiv u^2(1-u^2)\partial_u^2 - 2u^3\partial_u$.

→ We will classify solutions to this equation by their **singularities**.

Contact Solutions

The simplest solutions correspond to **contact interactions**:

$$\hat{F}_c = \begin{array}{c} \text{---} \\ \backslash \quad / \\ \backslash \quad / \\ \backslash \quad / \\ \backslash \quad / \end{array} = \sum_n \frac{c_n(u, v)}{E^{2n+1}}$$

Fixed by symmetry

Only total energy
singularities

- For Φ^4 , we have $\hat{F}_{c,0} = \frac{k_I}{E} = \frac{uv}{u+v}$.
- Higher-derivative bulk interactions, lead to $\hat{F}_{c,n} = \Delta_u^n \hat{F}_{c,0}$.

$$\hat{F}_{c,1} = -2 \left(\frac{k_I}{E} \right)^3 \frac{1+uv}{uv},$$

$$\hat{F}_{c,2} = -4 \left(\frac{k_I}{E} \right)^5 \frac{u^2 + v^2 + uv(3u^2 + 3v^2 - 4) - 6(uv)^2 - 6(uv)^3}{(uv)^3}.$$

This can get relatively complex, but everything is fixed by symmetry.

Exchange Solutions

For tree exchange, we try

$$\begin{aligned}(\Delta_u + M^2)\hat{F}_e &= \hat{C} \\ (\Delta_v + M^2)\hat{F}_e &= \hat{C}\end{aligned}$$

$$(\Delta_u + M^2) \quad \text{V} \quad (\Delta_v + M^2) \quad \text{V} = \quad \text{VV}$$

where $\hat{C} = M^2 \hat{F}_c$ is a contact solution and $M^2 \equiv m^2/H^2 - 2$.

For amplitudes, the analog of this is

$$(s + m^2)A_e = A_c(s, t)$$

$$(s + m^2) \quad \text{X} \quad \text{wavy} \quad \text{X} = \quad \text{X}$$

This is an algebraic relation, while for correlators we have a differential equation.

Exchange Solutions

Using the simplest contact interaction as a source, we have

$$\left[u^2(1-u^2)\partial_u^2 - \underline{2u^3\partial_u} + M^2 \right] \hat{F} = \underline{\frac{uv}{u+v}}$$

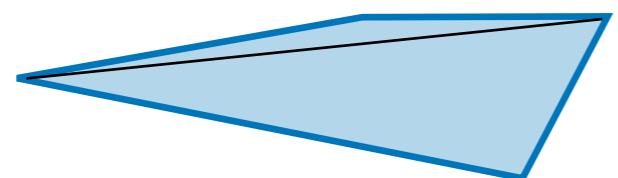
Before solving this equation, let us look at its **singularities**.

- **Flat-space limit:** $\lim_{u \rightarrow -v} \hat{F} \propto A_4(u+v) \log(u+v)$

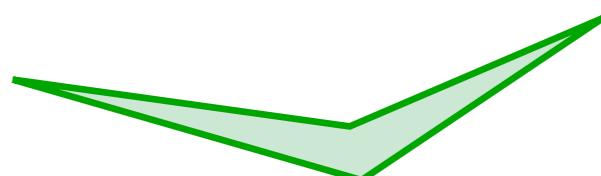
- **Factorization limit:** $\lim_{u,v \rightarrow -1} \hat{F} \propto A_3 \log(1+u) \times A_3 \log(1+v)$

- **Folded limit:** $\lim_{u \rightarrow +1} \hat{F} \propto \log(1-u)$

- **Collapsed limit:** $\lim_{u \rightarrow 0} \hat{F} \propto u^{iM}$



This singularity should be absent in the [Bunch-Davies vacuum](#).



This non-analyticity is a key signature of [particle production](#).

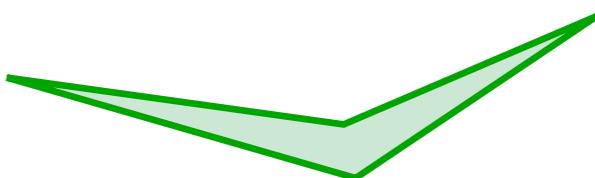
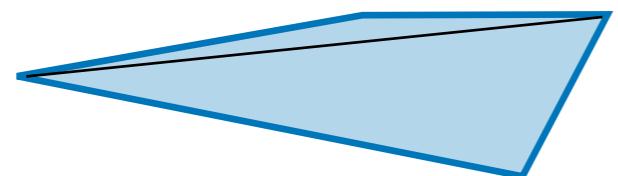
Exchange Solutions

Using the simplest contact interaction as a source, we have

$$\left[u^2(1-u^2)\partial_u^2 - \underline{2u^3\partial_u} + M^2 \right] \hat{F} = \underline{\frac{uv}{u+v}}$$

Before solving this equation, let us look at its **singularities**.

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- **Folded limit:** $\lim_{u \rightarrow +1} \hat{F} \propto \log(1-u)$
- **Collapsed limit:** $\lim_{u \rightarrow 0} \hat{F} \propto u^{iM}$



Factorization and folded limits uniquely fix the solution.
The remaining singularities become consistency checks.

Forced Harmonic Oscillator

In the limit of small internal momentum ($u < v \ll 1$), the equation becomes

$$\left[\frac{d^2}{dt^2} + M^2 \right] \hat{F} = \frac{1}{\cosh t}$$

$$t \equiv \log(u/v)$$

whose solution is

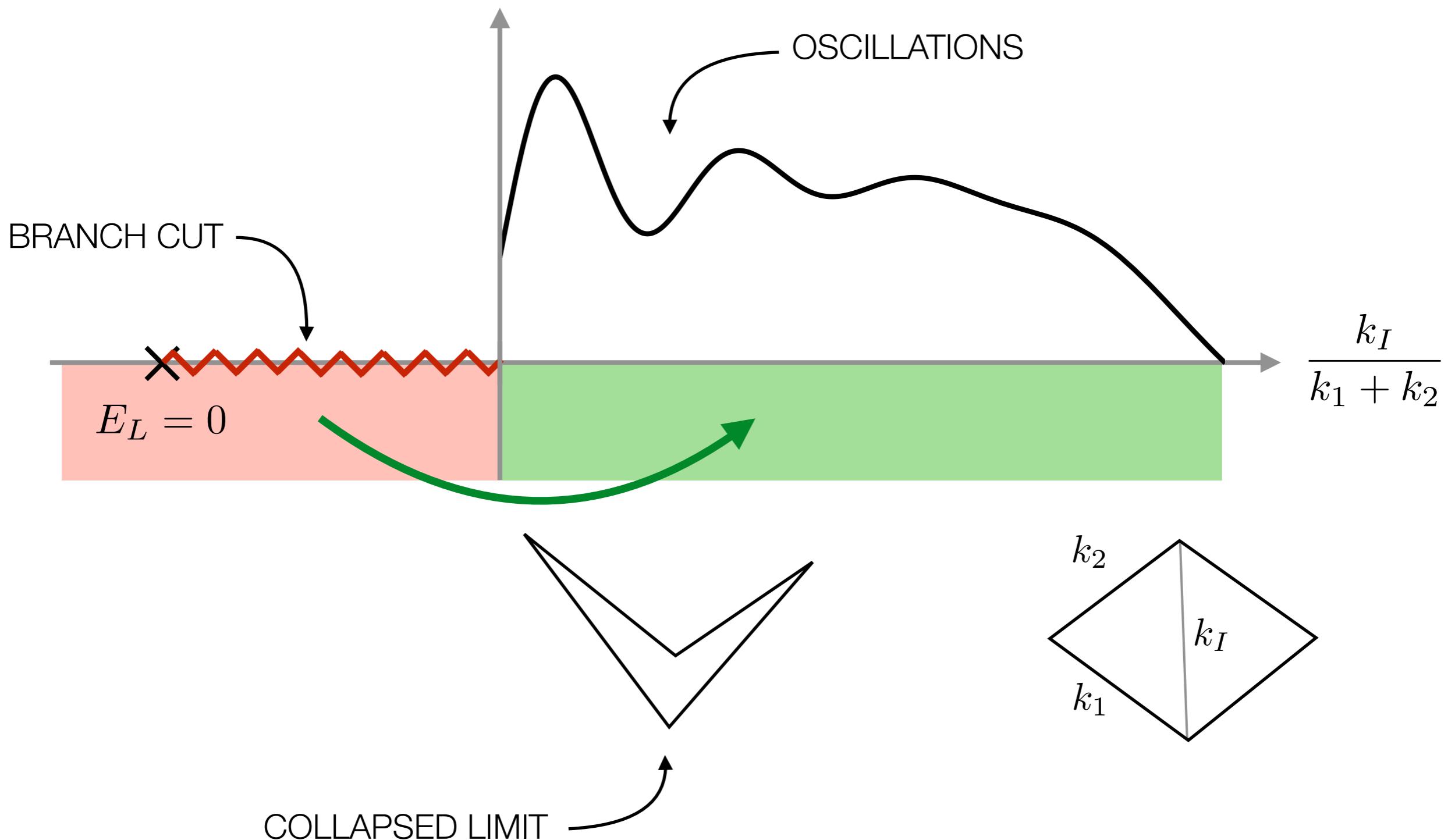
$$\hat{F} = \sum_n \frac{(-1)^n}{(n + \frac{1}{2})^2 + M^2} \left(\frac{u}{v}\right)^{n+1} + \frac{\pi}{\cosh(\pi M)} \frac{\sin(M \log(u/v))}{M}$$

The general solution takes a similar form.

Arkani-Hamed, DB, Lee and Pimentel [2018]

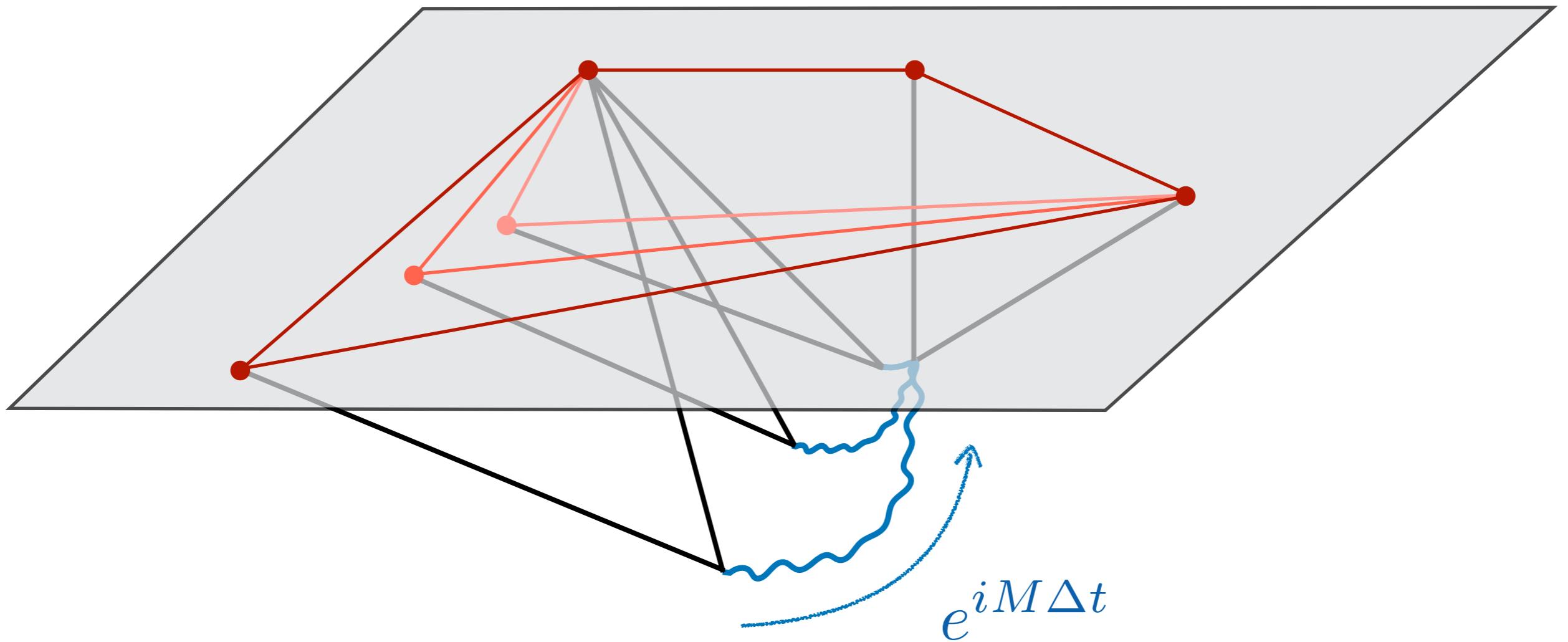
Oscillations

The correlator is forced to have an oscillatory feature in the physical regime:



Particle Production

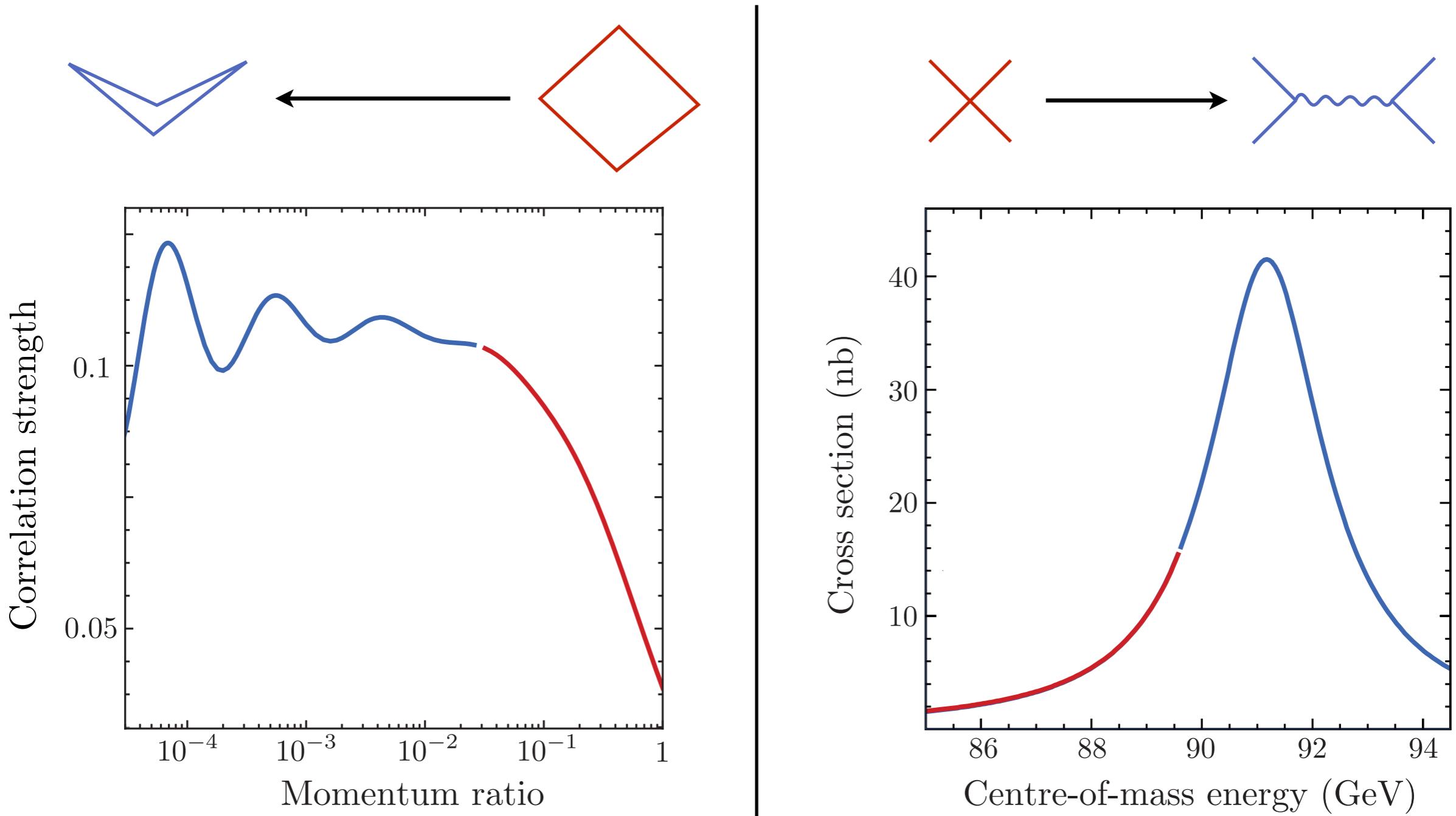
These oscillations reflect the evolution of the massive particles during inflation:



Time-dependent effects have emerged in the solution of the time-independent bootstrap constraints.

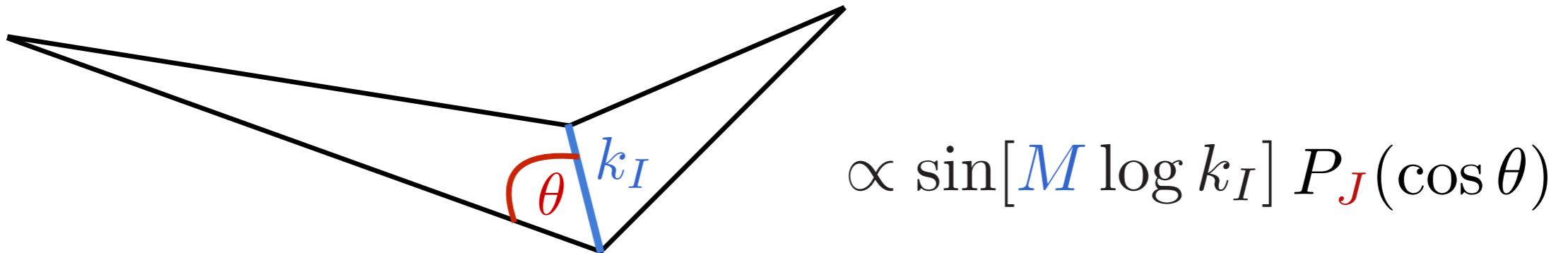
Cosmological Collider Physics

The oscillatory feature is the analog of a **resonance** in collider physics:



Particle Spectroscopy

The frequency of the oscillations depends on the **mass** of the particles:



The angular dependence of the signal depends on the **spin** of the particles.

This is very similar to what we do in collider physics:

$$\text{collider diagram} = \frac{g^2}{s - M^2} P_J(\cos \theta)$$

Examples

Massive
Exchange

**Dispersive
Integrals**

FRW
Correlators

We have seen that the discontinuity of a wavefunction coefficient is related to lower-point objects:

$$\text{Disc}_{k_I} = P(k)$$

For rational correlators, we can “integrate” the discontinuity to get the full correlator.

Dispersive Integral

By causality, the **bulk-to-bulk propagator** is an analytic function of k^2 that decays at large k . It can therefore be written as

$$G(k; \eta, \eta') = \frac{1}{2\pi i} \int_0^\infty \frac{dq^2}{q^2 - k^2 + i\varepsilon} \text{Disc}[G(k; \eta, \eta')]$$

This can be used to integrate the discontinuities of tree-level WF coefficients. As an example, we consider the four-point exchange diagram:

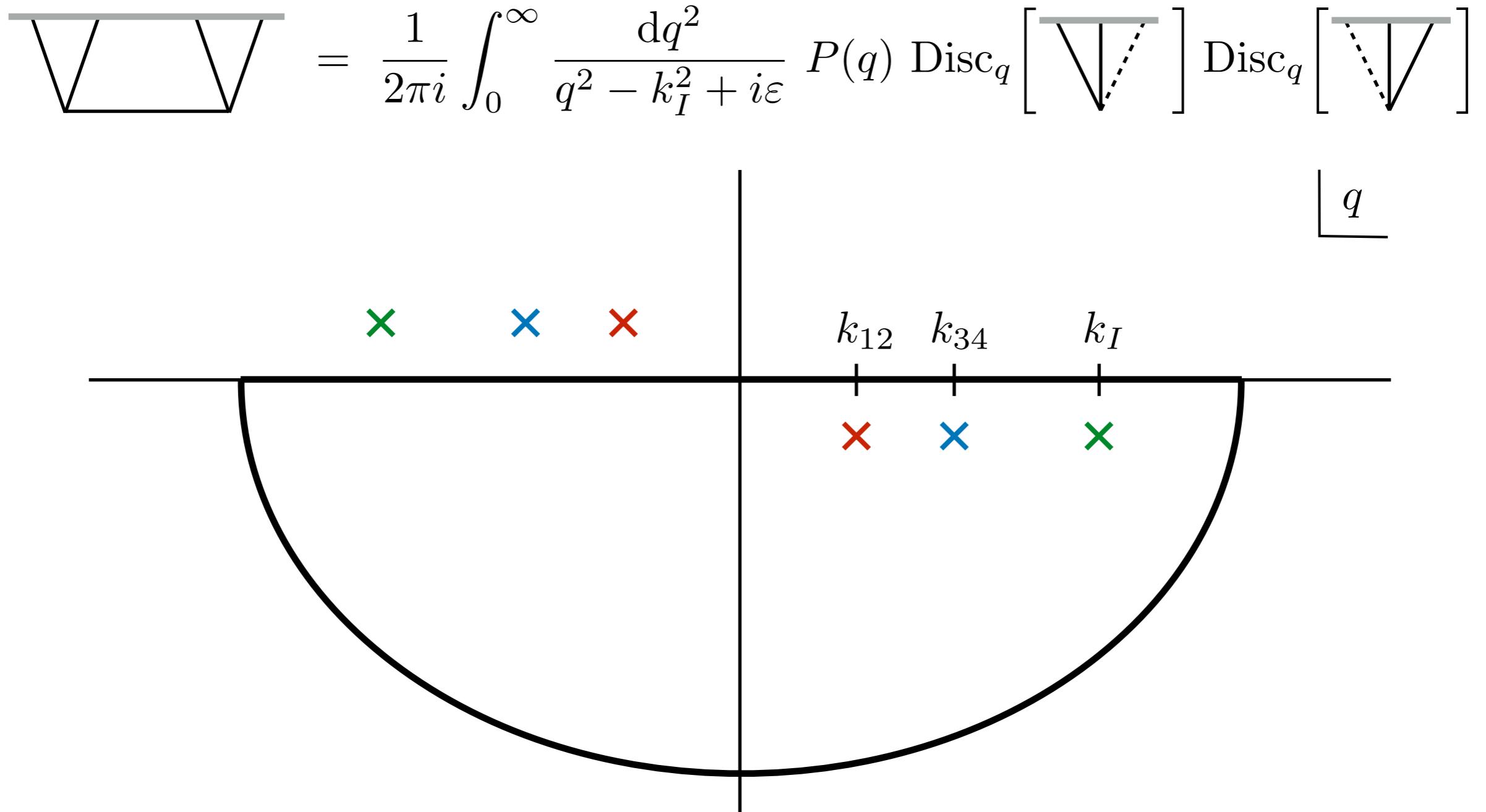

$$= \frac{1}{2\pi i} \int_0^\infty \frac{dq^2}{q^2 - k_I^2 + i\varepsilon} \text{Disc}_q \left[\begin{array}{c} \text{Feynman diagram} \\ \text{with two internal lines and two external lines} \end{array} \right]$$

Meltzer [2021]

Often this **dispersive integral** is easier to compute than the bulk time integrals.

Dispersive Integral

Using the unitarity cut, this becomes



The full correlator can then be written as a sum over residues, using only three-point data.

Conformally Coupled Scalar

As an example, let us reproduce the result for conformally coupled scalars.

- Input: three-point function

$$\overline{\nabla} = i \log(k_1 + k_2 + k_3)$$

- Discontinuity:

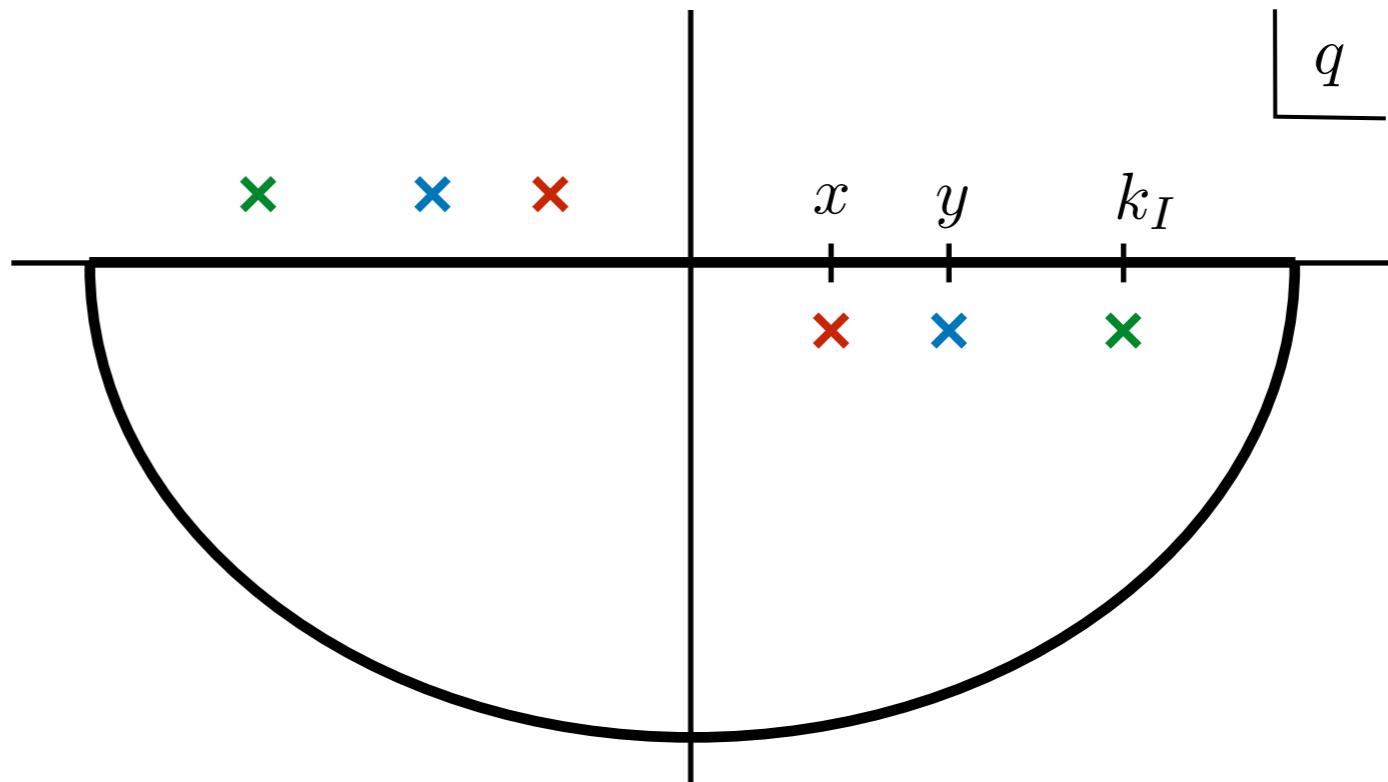
$$\begin{aligned} \text{Disc}_q \left[\begin{array}{c} \overline{\nabla} \\ \backslash \quad / \end{array} \right] &= P(q) \text{Disc}_q \left[\begin{array}{c} \overline{\nabla} \\ \backslash \quad \backslash \end{array} \right] \text{Disc}_q \left[\begin{array}{c} \backslash \quad / \\ \backslash \quad / \end{array} \right] \\ &= \frac{1}{2q} \log \left(\frac{k_{12} - q}{k_{12} + q} \right) \log \left(\frac{k_{34} - q}{k_{34} + q} \right) \\ &= 4q \int_{k_{12}}^{\infty} \frac{dx}{x^2 - q^2} \int_{k_{34}}^{\infty} \frac{dy}{y^2 - q^2} \end{aligned}$$

Conformally Coupled Scalar

As an example, let us reproduce the result for conformally coupled scalars.

- Dispersive integral

$$\begin{array}{c} \text{Diagram of a dispersive integral} \\ \text{Two parallel horizontal lines with a central gap} \end{array} = \int_{k_{12}}^{\infty} dx \int_{k_{34}}^{\infty} dy \int_{-\infty}^{\infty} \frac{dq}{\pi i} \frac{q^2}{q^2 - k_I^2} \frac{1}{x^2 - q^2} \frac{1}{y^2 - q^2}$$



$$= \int_{k_{12}}^{\infty} dx \int_{k_{34}}^{\infty} dy \frac{1}{(x+y)(x+k_I)(y+k_I)}$$

Correct!

EFT of Inflation

One of the important features of using unitarity cuts to bootstrap correlators is that it doesn't rely the interactions being approximately boost-invariant.

Exercise: Consider $\dot{\pi}^3$ in the EFT of inflation.

- The three-point wavefunction coefficient is

$$\overline{\nabla} = \frac{(k_1 k_2 k_3)^2}{(k_1 + k_2 + k_3)^3}$$

- Using the dispersive integral, show that the exchange 4pt function is

$$\begin{aligned}\overline{\nabla} \overline{\nabla} &= \frac{(k_1 k_2 k_3 k_4)^2}{E^5 E_L^3 E_R^3} \left[k_I^2 \left(6E_L^2 E_R^2 + 3E E_L E_R (E_L + E_R) \right. \right. \\ &\quad \left. \left. + E^2 (E_L + E_R)^2 + E^3 (E_L + E_R) + E^4 \right) - 6E_L^3 E_R^3 \right]\end{aligned}$$

Examples

Massive
Exchange

Dispersive
Integrals

FRW
Correlators

We have seen that cosmological correlators satisfy interesting differential equations in kinematic space:

$$\left[u^2(1-u^2)\partial_u^2 - 2u^3\partial_u + \left(\mu^2 + \frac{1}{4}\right) \right] \overline{\text{V}} = \overline{\text{V}}$$

We derived these equations from conformal symmetry, but is there a deeper reason for their existence?

What classes of correlation functions can arise from consistent cosmological evolution? Surprisingly little is known about this in a systematic fashion.

In an upcoming paper, we will answer this question for conformal scalars in a power-law FRW background.

Arkani-Hamed, DB, Hillman, Joyce,
Lee and Pimentel [to appear]

Conformal Scalars in dS

We have seen that the four-point function of conformal scalars in de Sitter space can be written as an integral of the flat-space result:

$$\psi \equiv \langle O_1 O_2 O_3 O_4 \rangle \propto \int_{k_{12}}^{\infty} dx_1 \int_{k_{34}}^{\infty} dx_2 \frac{1}{(x_1 + x_2)(x_1 + k_I)(x_2 + k_I)}$$



$$\psi = \int_0^{\infty} dx_1 dx_2 \frac{1}{(X_1 + X_2 + x_1 + x_2)(X_1 + x_1 + 1)(X_2 + x_2 + 1)}$$

where $X_1 \equiv k_{12}/k_I$ and $X_2 \equiv k_{34}/k_I$.



Conformal Scalars in FRW

An interesting generalization of this integral is

$$\psi = \int_0^\infty dx_1 dx_2 \frac{(x_1 x_2)^\varepsilon}{(X_1 + X_2 + x_1 + x_2)(X_1 + x_1 + 1)(X_2 + x_2 + 1)}$$

 Twist

This describes the four-point function of conformal scalars in FRW backgrounds with scalar factor

$$a(\eta) = \left(\frac{\eta}{\eta_0}\right)^{-(1+\varepsilon)}$$

Similar integrals arise for **loop amplitudes** in dimensional regularization.

In that case, powerful mathematics can be used to evaluate these integrals:

- Twisted cohomology
- Canonical differential equations

We will show that a similar approach applies to our FRW correlators.

Family of Integrals

To attack the problem, we introduce a **family of integrals** with the same singularities (divisors):

$$I_{n_1 n_2 n_3 n_4 n_5} \equiv \int (\textcolor{red}{x}_1 \textcolor{blue}{x}_2)^\varepsilon \Omega, \quad \Omega \equiv \frac{dx_1 \wedge dx_2}{\textcolor{red}{L}_1^{n_1} \textcolor{blue}{L}_2^{n_2} \textcolor{green}{L}_3^{n_3} \textcolor{purple}{L}_4^{n_4} \textcolor{orange}{L}_5^{n_5}},$$

where

$$\begin{aligned} \textcolor{red}{L}_1 &\equiv x_1, & \textcolor{green}{L}_3 &\equiv X_1 + x_1 + 1, \\ \textcolor{blue}{L}_2 &\equiv x_2, & \textcolor{purple}{L}_4 &\equiv X_2 + x_2 + 1, & \textcolor{orange}{L}_5 &\equiv X_1 + X_2 + x_1 + x_2. \end{aligned}$$

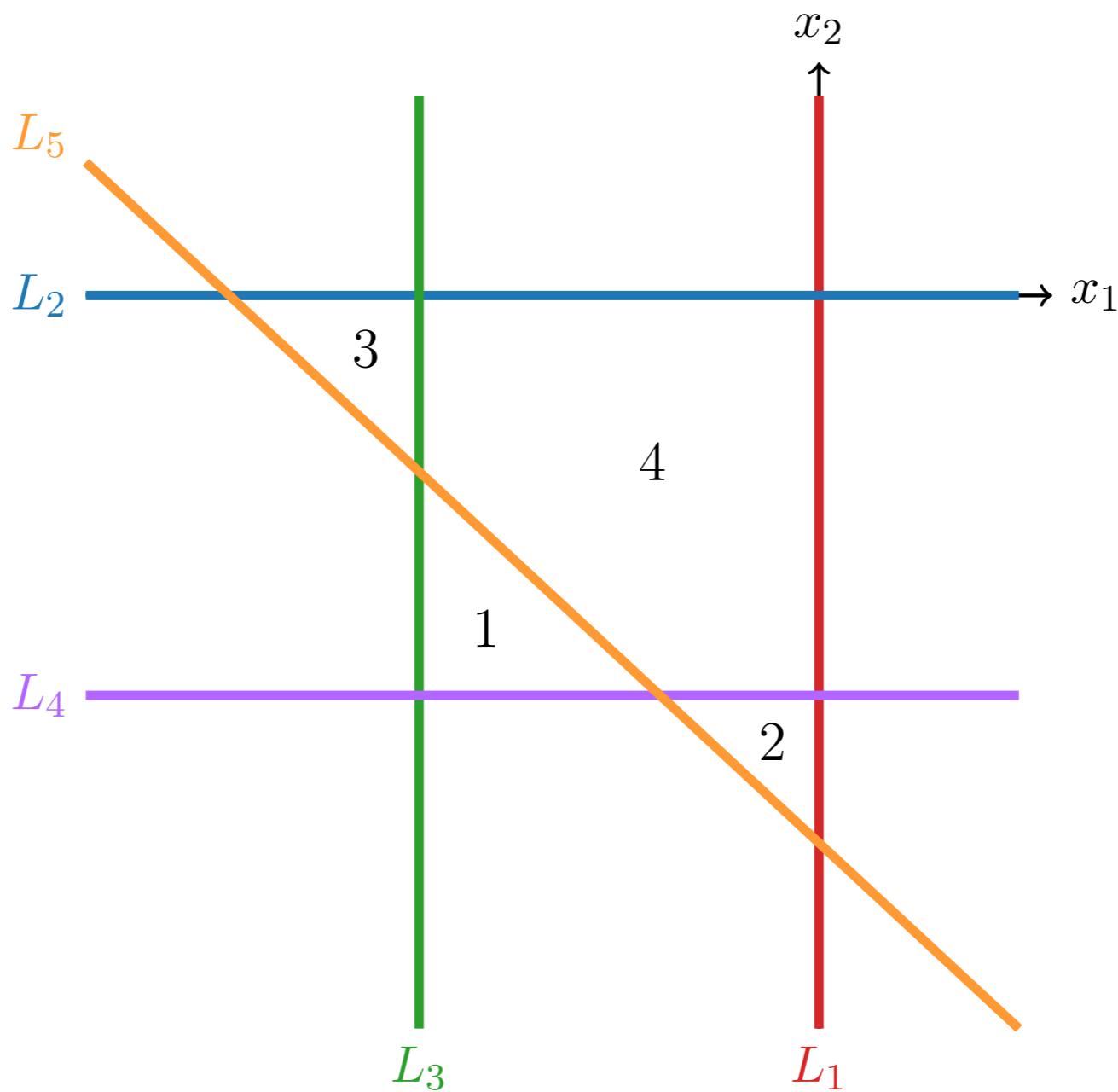
- Not all of these integrals are independent because of integration-by-parts and partial fraction identities.
- How many independent **master integrals** are there?
→ The answer is provided by a beautiful fact of **twisted cohomology**.

Aomoto [1975]

Mastrolia and Mizera [2018]

A Beautiful Fact

The number of independent master integrals is equal to the number of **bounded regions** defined by the divisors of the integrand.



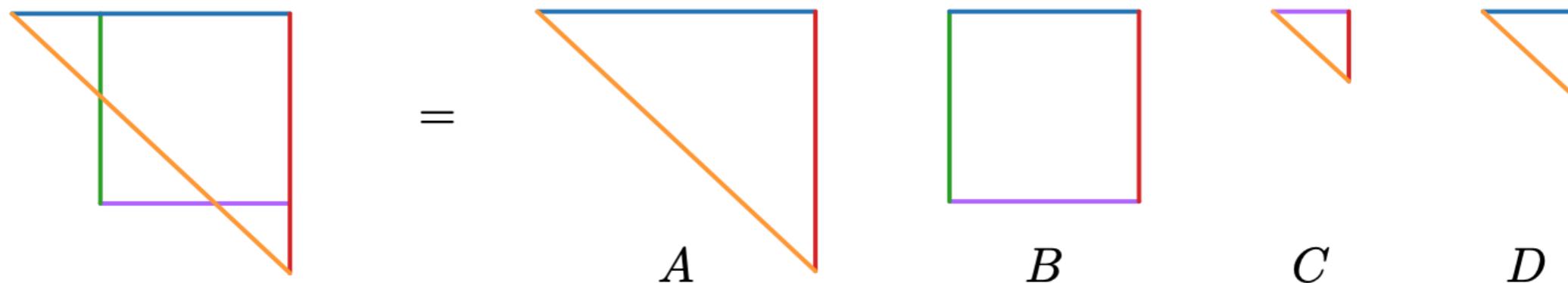
Master Integrals

The **vector space** of independent master integrals is four-dimensional:

$$\mathcal{I} \equiv \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \int (x_1 x_2)^\varepsilon \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ \Omega_4 \end{bmatrix}$$

 **Canonical forms**

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$$\begin{aligned} \Omega_A &= (X_1 + X_2) \frac{dx_1 \wedge dx_2}{L_1 L_2 L_5}, & \Omega_C &= -(X_1 - 1) \frac{dx_1 \wedge dx_2}{L_1 L_4 L_5}, \\ \Omega_B &= -(X_1 + 1)(X_2 + 1) \frac{dx_1 \wedge dx_2}{L_1 L_2 L_3 L_4}, & \Omega_D &= -(X_2 - 1) \frac{dx_1 \wedge dx_2}{L_2 L_3 L_5}. \end{aligned}$$

Divide and Conquer

Our original integral of interest is

$$\psi(X_1, X_2) = \int (x_1 x_2)^\varepsilon (\Omega_A + \Omega_B + \Omega_C + \Omega_D)$$

i.e. it is a sum of all basis integrals. Each basis integral is easier to evaluate!

Key idea:

The basis integral forms a finite dimensional vector space, so taking derivatives with respect to X_1 and X_2 must lead to coupled differential equations:

$$d\mathcal{I} = \varepsilon A \mathcal{I}$$

Exterior derivative 

$$d \equiv \sum_i \partial_{X_i} dX_i$$

 4×4 matrix

Solving this differential equation turns out to be easy.

Differential Equation

$$d\mathcal{I} = \varepsilon A \mathcal{I}$$

Explicitly, we find

$$\begin{aligned} A = & \alpha_1 d \log(X_1 + X_2) + \alpha_2 d \log(X_1 + 1) + \alpha_3 d \log(X_2 + 1) \\ & + \alpha_4 d \log(X_1 - 1) + \alpha_5 d \log(X_2 - 1) \end{aligned}$$

with

$$\alpha_1 \equiv \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \alpha_2 \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad \alpha_3 \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\alpha_4 \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \alpha_5 \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The Solution

More abstractly, we have

$$A = \begin{bmatrix} & & \\ & \text{decoupled} & \\ \text{coupled} & & \end{bmatrix}$$

This leads to **two types of functions:**

$$\left. \begin{aligned} I_A &= (X_1 + X_2)^{2\varepsilon} \\ I_B &= (1 + X_1)^\varepsilon (1 + X_2)^\varepsilon \end{aligned} \right] \text{Power laws}$$

$$\left. \begin{aligned} I_C &= (X_1 + X_2)^{2\varepsilon} {}_2F_1 \left[\begin{array}{c|c} 1, \varepsilon & 1 - X_1 \\ 1 - \varepsilon & 1 + X_2 \end{array} \right] \\ I_D &= I_C(X_1 \leftrightarrow X_2) \end{aligned} \right] \text{Hypergeometric}$$

In the de Sitter limit, these become the logs and dialogs we found earlier.

The Solution

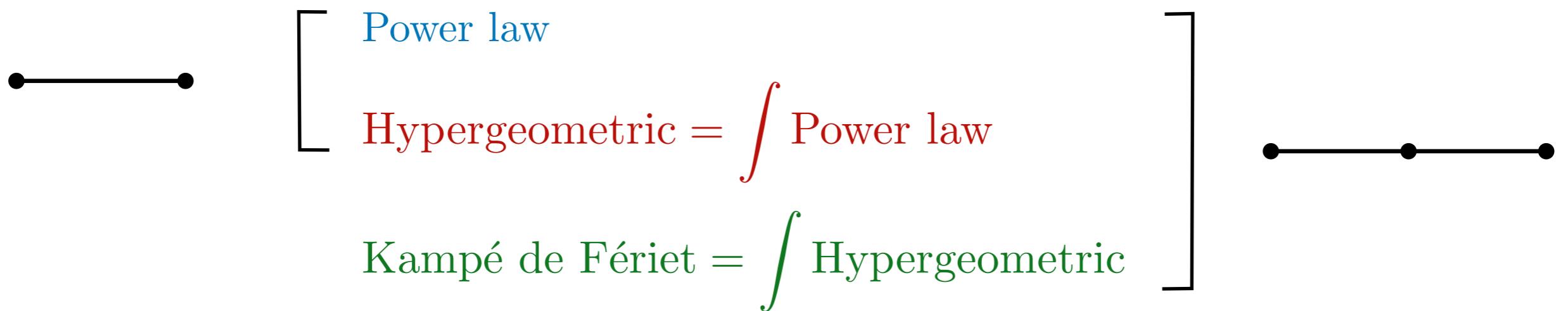
Imposing appropriate boundary conditions (i.e. singularities), the solution for our original integral is

$$\begin{aligned}\psi &= c_1(\varepsilon) (1 + X_1)^\varepsilon (1 + X_2)^\varepsilon \\ &\quad c_2(\varepsilon) (X_1 + X_2)^{2\varepsilon} \left(1 - {}_2F_1 \left[\begin{array}{c|c} 1, \varepsilon & \\ 1 - \varepsilon & \end{array} \middle| \frac{1 - X_2}{1 + X_1} \right] - (X_1 \leftrightarrow X_2) \right)\end{aligned}$$

Arkani-Hamed, DB, Hillman, Joyce,
Lee and Pimentel [to appear]

Space of Functions

The number of special functions depends on the number of bulk vertices, and the function have an interesting iterative structure:



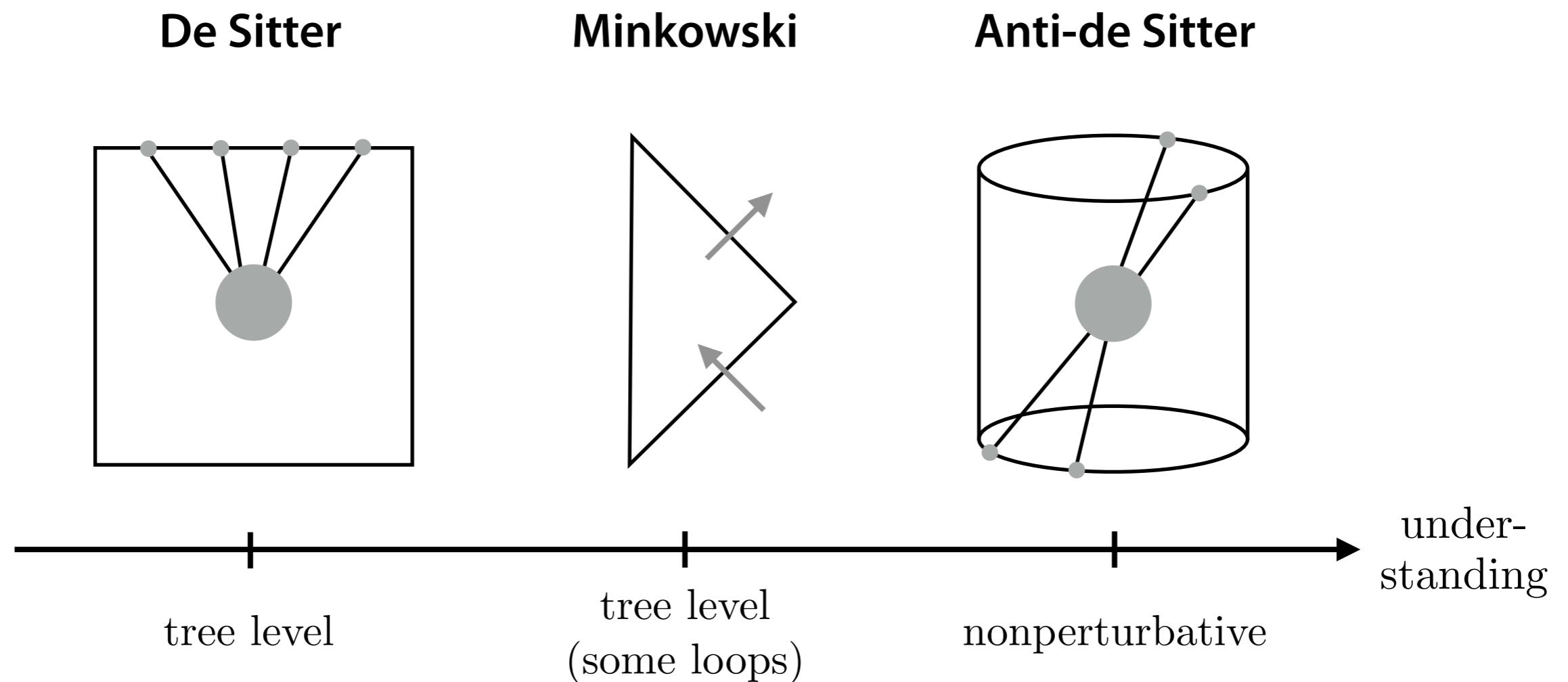
This structure is hidden in bulk perturbation theory.

→ Just the beginning of a systematic exploration of these mathematical structures living inside cosmological correlators.

Arkani-Hamed, DB, Hillman, Joyce,
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Outlook and Speculations

Despite enormous progress in recent years, we are only at the beginning of a systematic exploration of cosmological correlators:



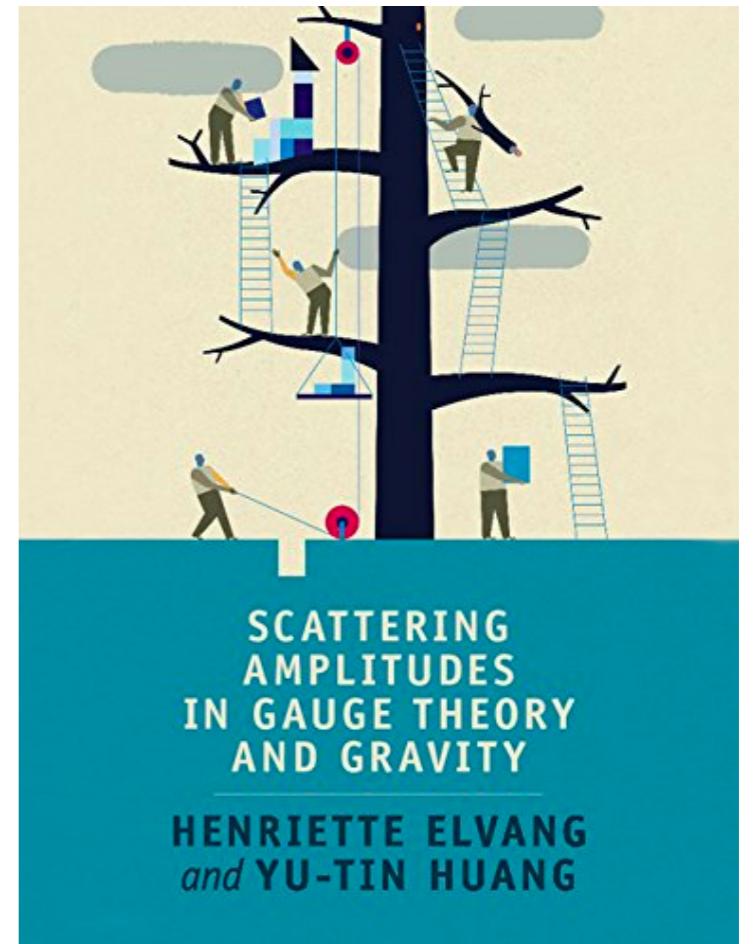
Many fundamental questions still remain unanswered.

Beyond Feynman Diagrams

So far, the cosmological bootstrap has only been applied to individual Feynman diagrams (which by themselves are unphysical).

However, in the **S-matrix bootstrap**, the real magic is found for **on-shell diagrams**:

- Recursion relations
- Generalized unitarity
- Color-kinematics duality
- Hidden positivity
- ...



What is the on-shell formulation of the cosmological bootstrap?
What magic will it reveal?

Beyond Perturbation Theory

So far, the bootstrap constraints (singularities, unitarity cuts, etc.) are only formulated in perturbation theory and implemented only at tree level.

In contrast, we understand the **conformal bootstrap** nonperturbatively:



Recently, some preliminary progress was made to apply unitarity constraints in de Sitter nonperturbatively, but much work remains.

Hogervorst, Penedones and Vaziri [2021]
Di Pietro, Gorbenko and Komatsu [2021]

A nonperturbative bootstrap will be required if we want to understand the UV completion of cosmological correlators.

Beyond Spacetime

In quantum gravity, spacetime is an emergent concept. In cosmology, the notation of “time” breaks down at the Big Bang. What replaces it?

The cosmological bootstrap provides a modest form of emergent time, but there should be a much more radical approach where space and time are outputs, not inputs.

For scattering amplitudes, such a reformulation was achieved in the form of the **amplituhedron**: Arkani-Hamed and Trnka [2013]



What is the analogous object in cosmology?
How does the Big Bang arise from it?



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Thank you for your attention!