Wavefunction Approach

Last Time

- We defined the wavefunction approach to compute boundary correlators:

$$\langle \phi(\mathbf{x}_1)\dots\phi(\mathbf{x}_N)\rangle = \int \mathcal{D}\phi\ \phi(\mathbf{x}_1)\dots\phi(\mathbf{x}_N)\,|\Psi[\phi]|^2$$

$$\Psi[\phi] \approx e^{iS[\Phi_{\rm cl}]}$$

- We then used it to study quantum harmonic oscillators:

$$S[\Phi] = \int \mathrm{d}t \left(\frac{1}{2}\dot{\Phi}^2 - \frac{1}{2}\omega^2\Phi^2 - \frac{1}{3}g\Phi^3\right)$$

$$\downarrow$$

$$\Phi_{\rm cl}(t) = \phi\, K(t) + i\int \mathrm{d}t'\, G(t,t')\big(-g\Phi_{\rm cl}^2(t')\big)$$

$$\downarrow$$

$$\Psi[\phi] \approx e^{iS[\Phi_{\rm cl}]} = \exp\left(-\frac{\omega}{2}\phi^2 - \frac{g}{9\omega}\phi^3 + \frac{g^2}{72\omega^3}\phi^4 + \cdots\right)$$

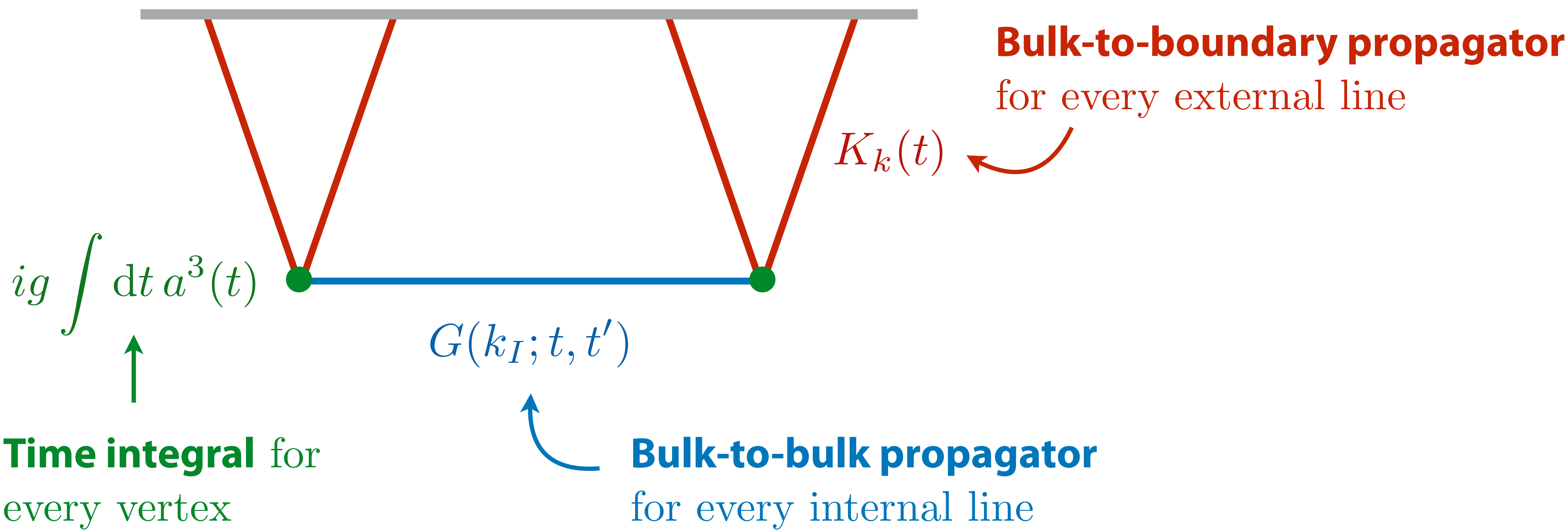
Interacting Fields

Back to field theory:

$$S[\Phi] = \int \mathrm{d}^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}m^2\Phi^2 - \frac{1}{3}g\Phi^3\right)$$

The analysis is similar to that of the anharmonic oscillator (see lecture notes).

In the interest of time, we jump directly to the **Feynman rules**:



Examples

Flat-Space Wavefunction

Consider a massless field in Minkowski:

$$\boxed{S = \int \mathrm{d}^4 x \left(-\frac{1}{2}(\partial\Phi)^2 - \frac{g}{3!}\Phi^3 \right)}$$

The propagators are the same as for the harmonic oscillator:

$$K_k(t) = e^{ikt}$$
$$G(k;t,t') = \frac{1}{2k}\left(e^{-ik(t-t')}\theta(t-t') + e^{ik(t-t')}\theta(t'-t) \; - \; e^{ik(t+t')} \right)$$

We will compute the simplest tree-level correlators in this theory.

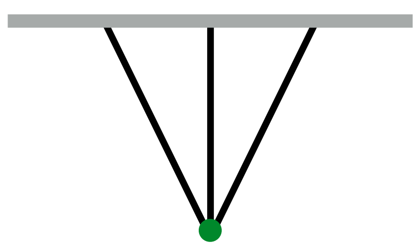
Flat-Space Wavefunction

Consider a massless field in Minkowski:

$$\boxed{S = \int \mathrm{d}^4 x \left(-\frac{1}{2}(\partial\Phi)^2 - \frac{g}{3!}\Phi^3 \right)}$$

The three-point wavefunction coefficient is

$$\langle O_1 O_2 O_3 \rangle =$$



$$= ig \int_{-\infty}^{0} \mathrm{d}t\, K_{k_1}(t) K_{k_2}(t) K_{k_3}(t)$$

$$= ig \int_{-\infty}^{0} \mathrm{d}t\, e^{i(k_1+k_2+k_3)t}$$

$$= \frac{g}{(k_1+k_2+k_3)} \,.$$

We will have more to say about this singularity.

Flat-Space Wavefunction

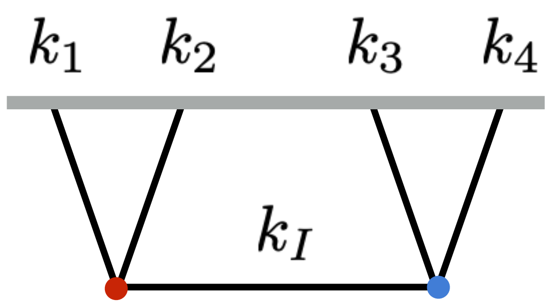
Consider a massless field in Minkowski:

$$\boxed{S = \int \mathrm{d}^4x \left(-\frac{1}{2}(\partial\Phi)^2 - \frac{g}{3!}\Phi^3 \right)}$$

The four-point wavefunction coefficient is

$k_{12} \equiv k_1 + k_2$

$$\langle O_1 O_2 O_3 O_4 \rangle =$$



$$= -g^2 \int_{-\infty}^{0} \mathrm{d}t\, \mathrm{d}t' \; e^{ik_{12}t}\, G(k_I; t, t')\, e^{ik_{34}t'}$$

$$= \frac{g^2}{(k_{12}+k_{34})(k_{12}+k_I)(k_{34}+k_I)}$$

Total energy

Energy entering the left vertex

Energy entering the right vertex

De Sitter Wavefunction

Consider a massive field in de Sitter:

$$\boxed{u'' + \left(k^2 + \frac{m^2/H^2 - 2}{\eta^2}\right) u = 0} \qquad u \equiv a(\eta)\Phi$$

- The general solution is

$$u_k(\eta) = \sqrt{\frac{\pi}{4}}\,(-\eta)^{1/2} H_\nu^{(2)}(-k\eta)\,, \quad \text{where} \quad \nu \equiv \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}\,.$$

- For $m^2 = 2H^2$, the field is **conformally coupled** and

$$u_k(\eta) = \frac{e^{ik\eta}}{\sqrt{2k}} \quad \longrightarrow \quad \boxed{\Phi_k(\eta) = \frac{u_k(\eta)}{a(\eta)} = (-H\eta)\frac{e^{ik\eta}}{\sqrt{2k}}}$$

In this case, the wavefunction can be computed analytically.

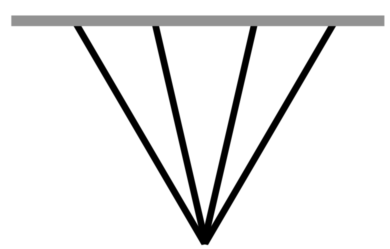
De Sitter Wavefunction

The three-point WF coefficient (for conformal scalars is) is

$$\langle O_1 O_2 O_3 \rangle = \frac{ig}{H^4 \eta_*^3} \int_{-\infty}^{\eta_*} \frac{\mathrm{d}\eta}{\eta}\, e^{i(k_1+k_2+k_3)\eta} = \boxed{\frac{ig}{H^4 \eta_*^3} \log(iK\eta_*)}$$

where $K \equiv k_1 + k_2 + k_3$.

The four-point WF coefficient for a **contact interaction** is

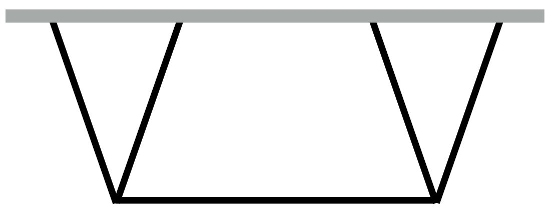
$\langle O_1 O_2 O_3 O_4 \rangle =$ 

$$= \frac{ig}{H^4 \eta_*^4} \int_{-\infty}^{\eta_*} \mathrm{d}\eta\, e^{i(k_1+k_2+k_3+k_4)\eta} = \boxed{\frac{1}{H^4 \eta_*^4} \frac{g}{(k_1+k_2+k_3+k_4)}}$$

Same as in flat space

De Sitter Wavefunction

The four-point WF coefficient corresponding to an **exchange diagram** is

$$\langle O_1 O_2 O_3 O_4 \rangle =$$ 

$$= -\frac{g^2}{H^6 \eta_*^4} \int_{-\infty}^{0} \frac{\mathrm{d}\eta}{\eta} \int_{-\infty}^{0} \frac{\mathrm{d}\eta'}{\eta'} \, e^{ik_{12}\eta} \, G_{(\text{flat})}(k_I; \eta, \eta') \, e^{ik_{34}\eta'}$$

$$= -\frac{g^2}{H^6 \eta_*^4} \int_{k_{12}}^{\infty} \mathrm{d}x \int_{k_{34}}^{\infty} \mathrm{d}y \int_{-\infty}^{0} \mathrm{d}\eta \, \mathrm{d}\eta' \, e^{ix\eta} e^{iy\eta'} G_{(\text{flat})}(k_I; \eta, \eta')$$

$$= \frac{1}{H^6 \eta_*^4} \int_{k_{12}}^{\infty} \mathrm{d}x \int_{k_{34}}^{\infty} \mathrm{d}y \, \langle O_1 O_2 O_3 O_4 \rangle_{(\text{flat})}(k_I; x, y) \, ,$$

$$= \boxed{-\frac{g^2}{H^6 \eta_*^4} \int_{k_{12}}^{\infty} \mathrm{d}x \int_{k_{34}}^{\infty} \mathrm{d}y \, \frac{1}{(x+y)(x+k_I)(y+k_I)}}$$

De Sitter Wavefunction

We see that this four-point function can be written as an integral of the flat-space result:

$$\langle O_1 O_2 O_3 O_4 \rangle = -\frac{g^2}{H^6 \eta_*^4} \int_{k_{12}}^{\infty} \mathrm{d}x \int_{k_{34}}^{\infty} \mathrm{d}y \, \frac{1}{(x+y)(x+k_I)(y+k_I)}$$

This integral can be performed analytically to give

$$\boxed{\begin{aligned}\langle O_1 O_2 O_3 O_4 \rangle = \frac{g^2}{2H^6 \eta_*^4} \frac{1}{k_I} \bigg[&\mathrm{Li}_2\left(\frac{k_{12}-k_I}{E}\right) + \mathrm{Li}_2\left(\frac{k_{34}-k_I}{E}\right) \\ &+ \log\left(\frac{k_{12}+k_I}{E}\right) \log\left(\frac{k_{34}+k_I}{E}\right) - \frac{\pi^2}{6} \bigg]\end{aligned}}$$

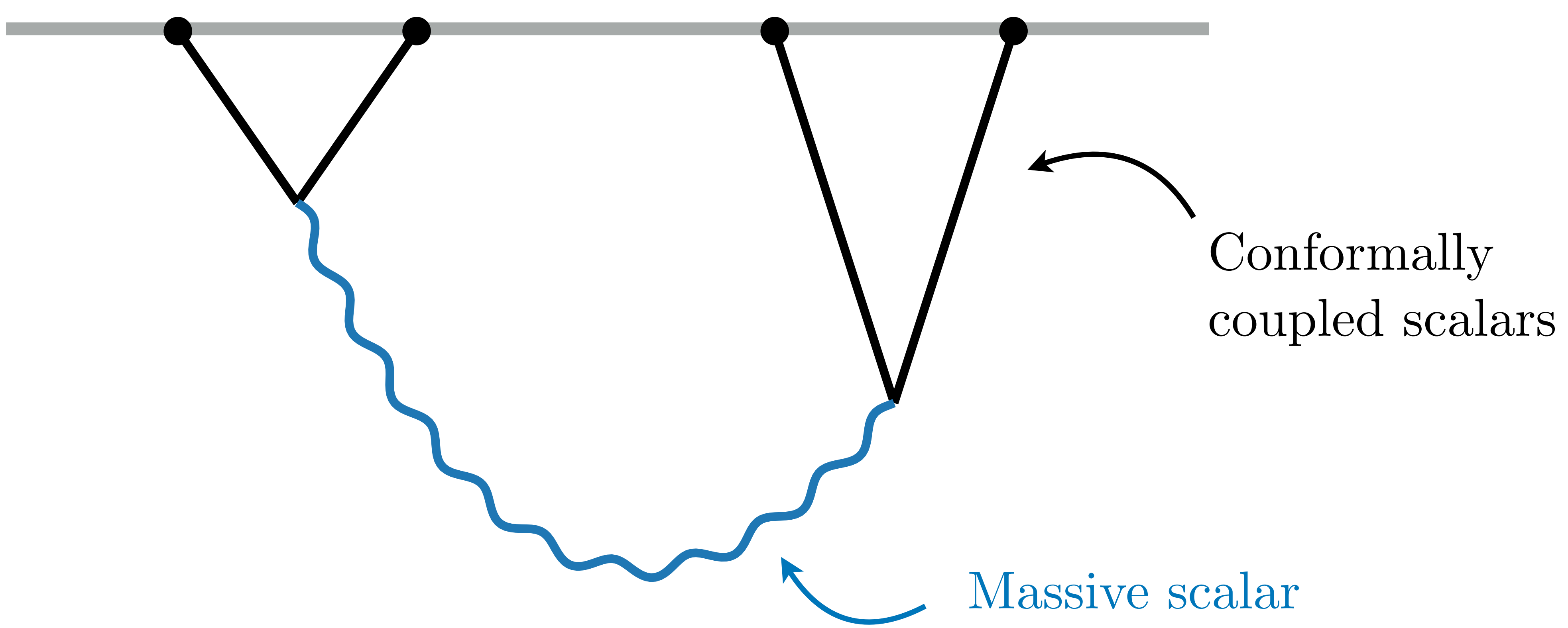
where $E \equiv k_1 + k_2 + k_3 + k_4$ and Li_2 is the dilogarithm.

Arkani-Hamed and Maldacena [2015]

A Challenge

So far, we have only computed the correlators of conformally coupled scalars.
Consider now the **exchange of a generic massive scalar**:

$\langle O_1 O_2 O_3 O_4 \rangle =$

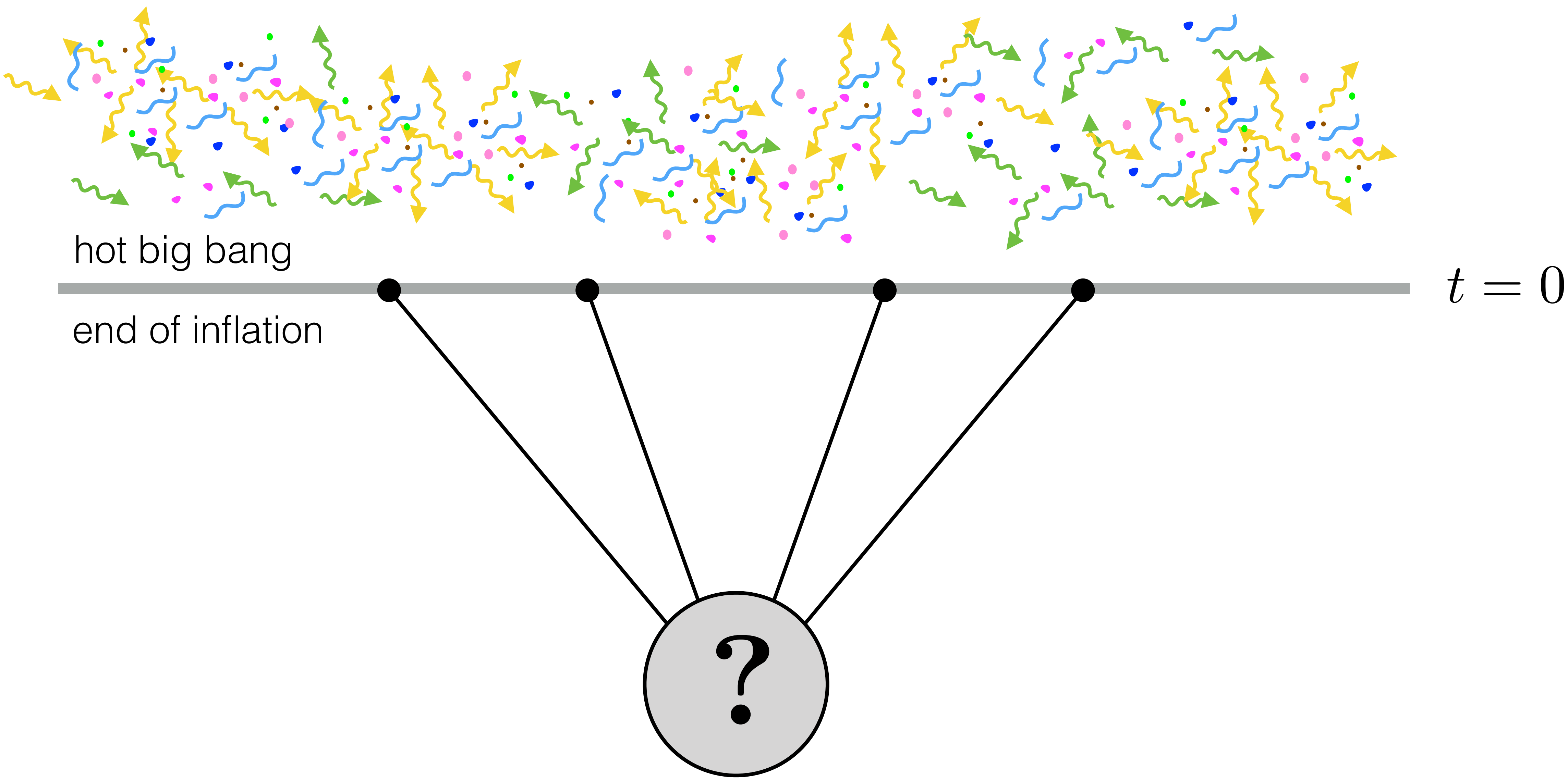


$$= \; -g^2 \int \frac{\mathrm{d}\eta}{\eta^2} \int \frac{\mathrm{d}\eta'}{\eta'^2}\, e^{ik_{12}\eta} e^{ik_{34}\eta'}\, G(k_I;\eta,\eta')$$

In general, the time integrals cannot be performed analytically.
We need a different approach.

Cosmological Bootstrap

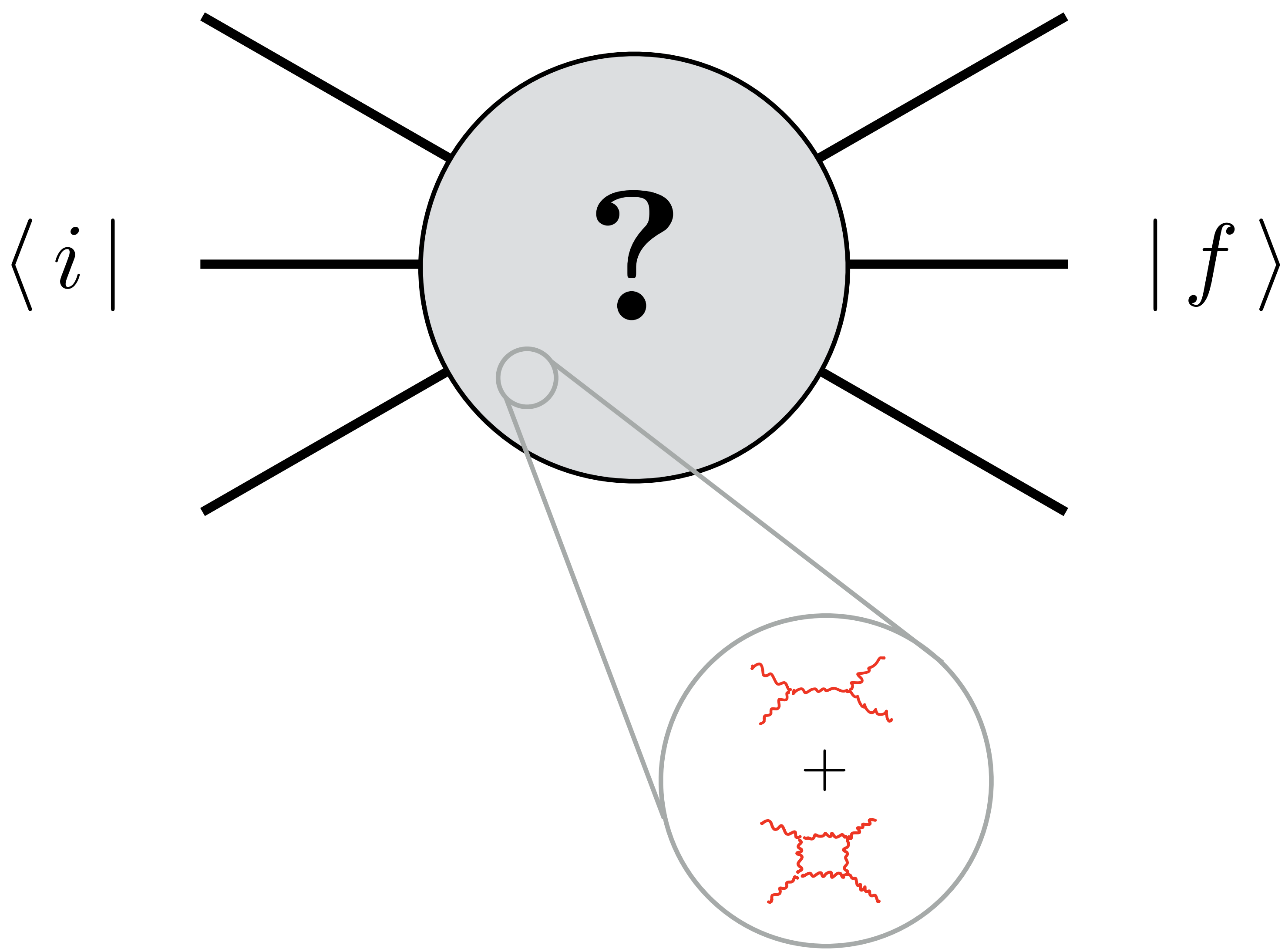
If inflation is correct, then the reheating surface is the future boundary of an approximate **de Sitter spacetime**:



Can we **bootstrap** these boundary correlators directly?

- Conceptual advantage: **focuses directly on observables**.
- Practical advantage: **simplifies calculations**.

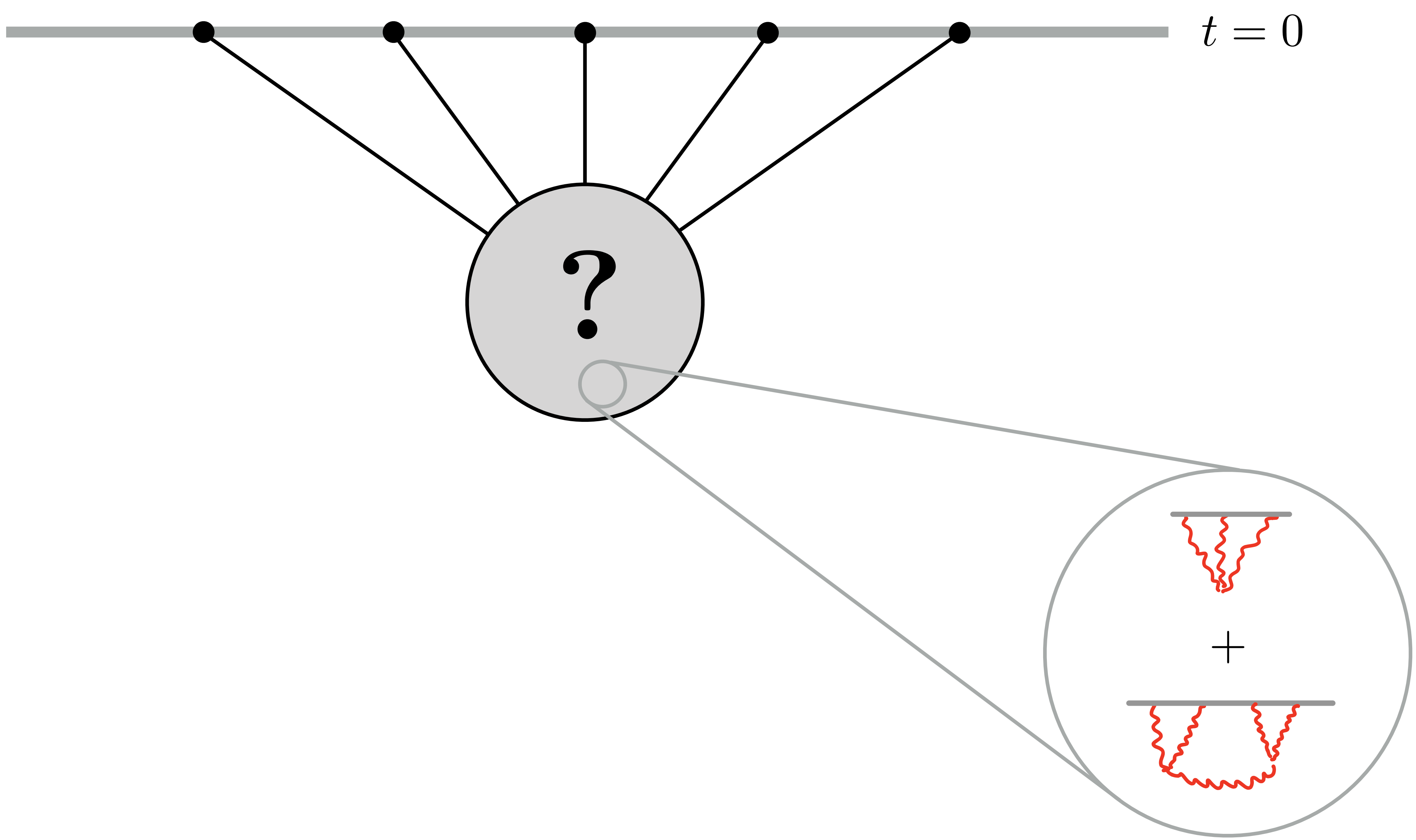
The bootstrap perspective has been very influential for **scattering amplitudes**:



There has be enormous progress in bypassing Feynman diagram expansions to write down on-shell amplitudes directly:

- Practical advantage: **simplifies calculations**.
- Conceptual advantage: **reveals hidden structures**.

I will describe recent progress in developing a similar bootstrap approach for cosmological correlators:



Goal: Develop an understanding of cosmological correlators that parallels our understanding of flat-space scattering amplitudes.

OUTLINE:

The Bootstrap Philosophy	Bootstrapping Tools	Examples

REFERENCES:

DB, Green, Joyce, Pajer, Pimentel, Sleight and Taronna,
Snowmass White Paper: The Cosmological Bootstrap [2203.08121]

DB and Joyce, *Lectures on Cosmological Correlations*

The Bootstrap Philosophy

The Conventional Approach

How we usually make predictions:

Physical Principles $\longrightarrow$ **Models** $\longrightarrow$ **Observables**

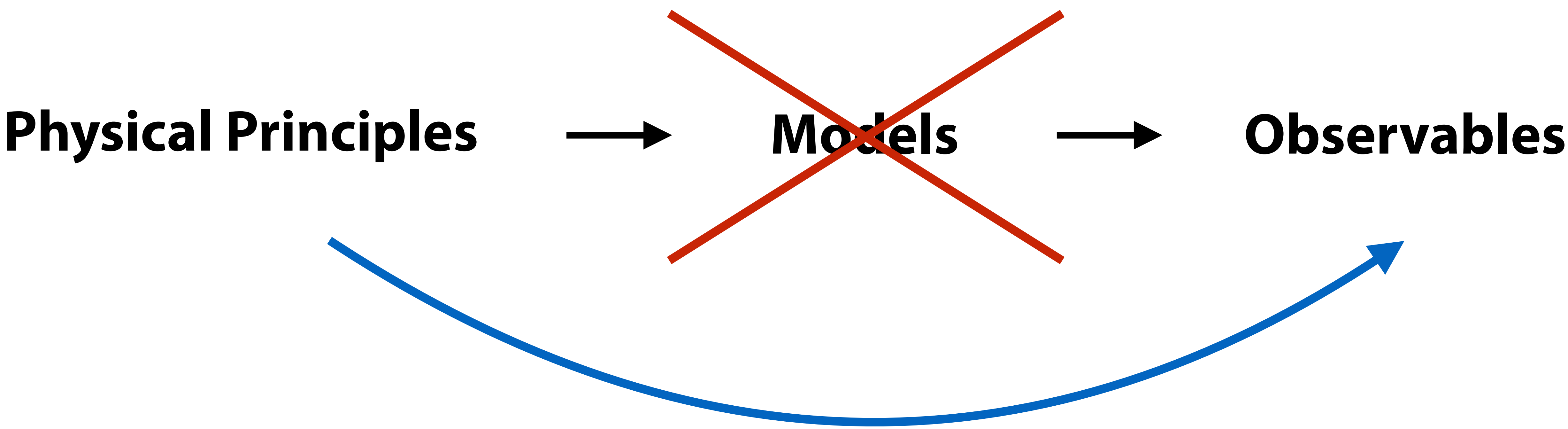
locality, causality,
unitarity, symmetries

Lagrangians
equations of motion
spacetime evolution
Feynman diagrams

Works well if we have a well-established theory (Standard Model, GR, ...) and many observables.

The Bootstrap Method

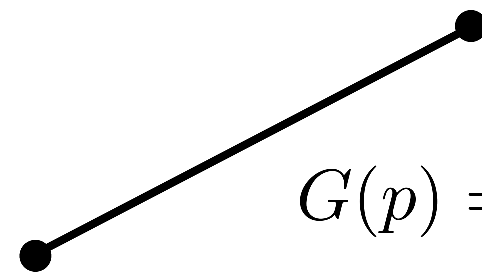
In the bootstrap approach, we cut out the middle man and go directly from fundamental principles to physical observables:



This is particularly relevant when we have many theories (inflation, BSM, ...) and few observables.

S-matrix Bootstrap

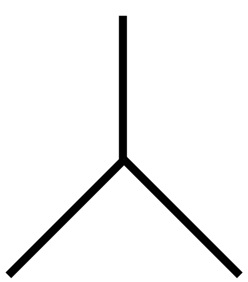
- What stable particles exist?



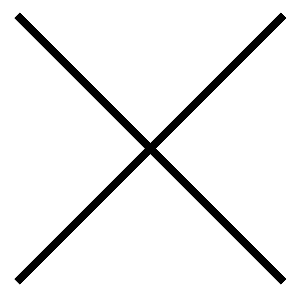
$$G(p) = \frac{1}{p^2 + M^2}$$

fixed by locality and Lorentz symmetry

- What interactions are allowed?

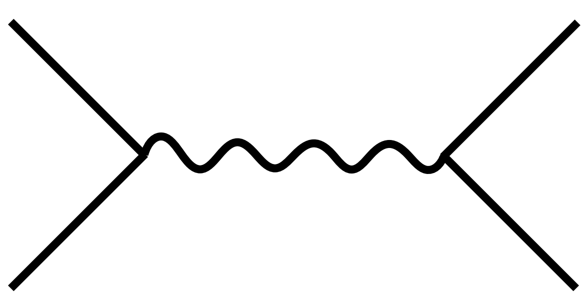


$$= \left(\frac{\langle 12 \rangle^3}{\langle 13 \rangle \langle 23 \rangle} \right)^J$$



$$= \sum_{m,n} a_{nm}\, s^m t^n$$

Lorentz



unitarity

$$= \frac{g^2 P_J(\cos\theta)}{s - M^2}$$

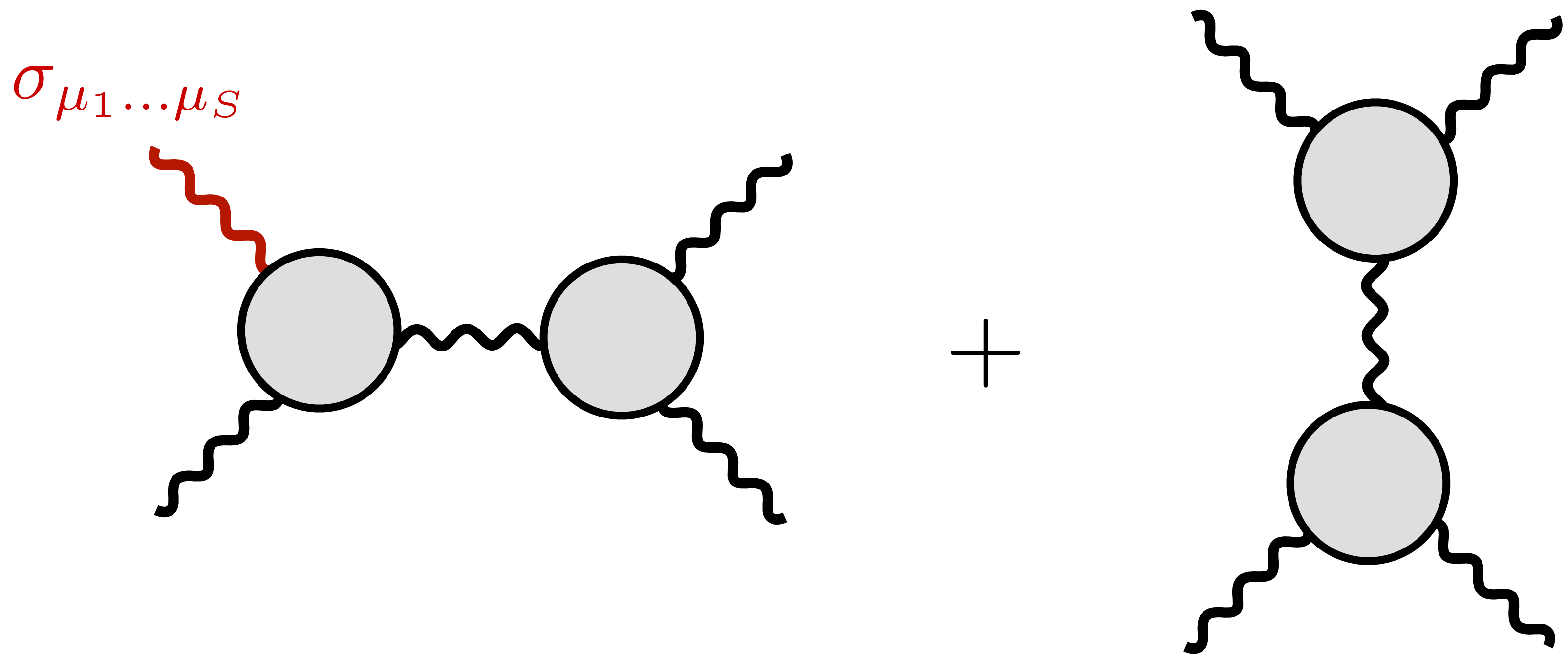
locality

→ No Lagrangian is required to derive this.

→ Basic principles allow only a small menu of possibilities.

S-matrix Bootstrap

Consistent factorization is very constraining for massless particles:



$$+ \quad + \quad \cdots$$

→ Only consistent for spins $S = \{ 0 , \frac{1}{2} , 1 , \frac{3}{2} , 2 \}$

YM SUSY GR

Benincasa and Cachazo [2007]
McGady and Rodina [2013]

A Success Story

The modern amplitudes program has been very successful:

1. New Computational Techniques
   - Recursion relations
   - Generalized unitarity
   - Soft theorems

2. New Mathematical Structures
   - Positive geometries
   - Amplituhedrons
   - Associahedrons

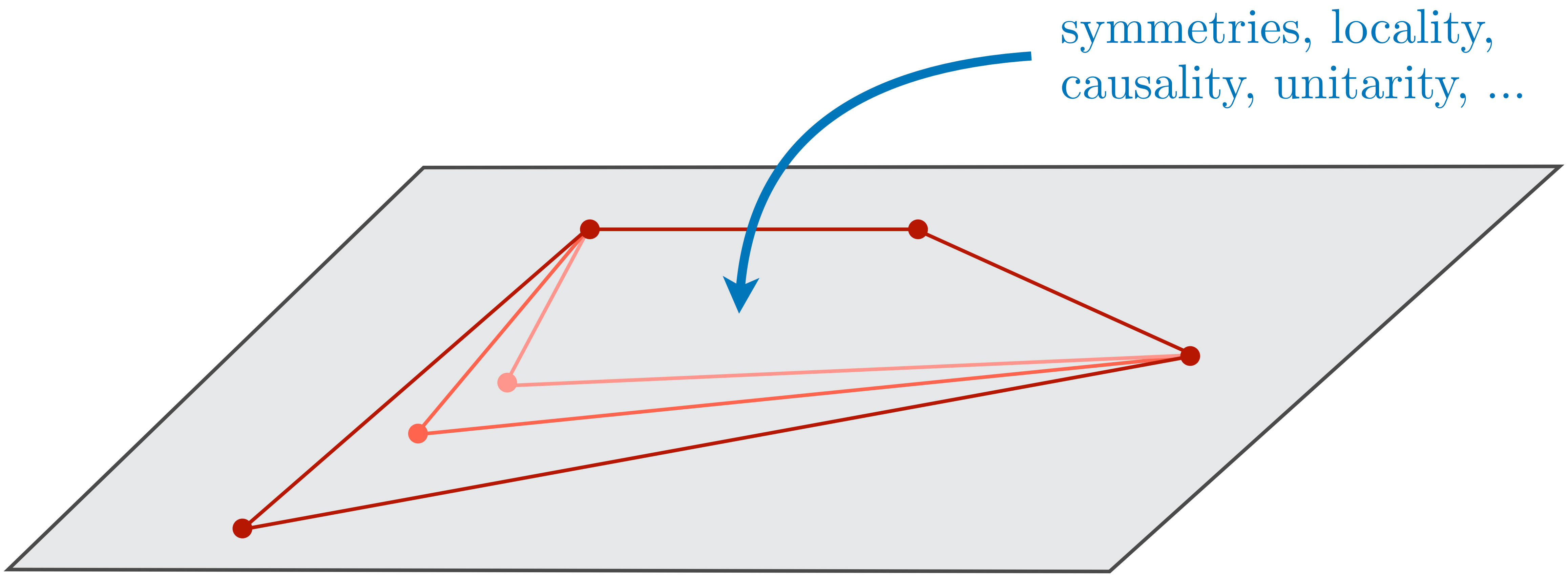
3. New Relations Between Theories
   - Color-kinematics duality
   - BCJ double copy

4. New Constraints on QFTs
   - Positivity bounds
   - EFThedron

5. New Applications
   - Gravitational wave physics
   - Cosmology

Cosmological Bootstrap

Recently, there has been significant progress in **bootstrapping correlators** using physical consistency conditions on the boundary:



In this part of the lectures, I will review these developments.

Bootstrapping Tools

Symmetries	Singularities	Unitarity

Bootstrapping Tools

Symmetries

Singularities

Unitarity

De Sitter Symmetries

The metric of de Sitter is

$$\mathrm{d}s^2 = \frac{1}{H^2\eta^2}\left(-\mathrm{d}\eta^2 + \mathrm{d}\mathbf{x}^2\right)$$

The isometries of the metric are

- Translations: $P_i = \partial_i$
- Rotations: $J_{ij} = x_i\partial_j - x_j\partial_i$
- Dilatations: $D = -\eta\partial_\eta - x^i\partial_i$
- Conformal: $K_i = 2x_i\eta\partial_\eta + \left(2x^j x_i + (\eta^2 - x^2)\delta_i^j\right)\partial_j$

We would like to see how these symmetries act on the boundary fields.

Boundary Fluctuations

At late times, the solution for the bulk field is

$$\Phi(\mathbf{x}, \eta \to 0) = \phi(\mathbf{x})\, \eta^{\Delta} + \bar{\phi}(\mathbf{x})\, \eta^{3-\Delta} \qquad \Delta \equiv \frac{3}{2} - \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}$$

Boundary field profile

Scaling dimension

For light fields, the first term of the asymptotic solution dominates.

The de Sitter isometries then act as

$$\boxed{\begin{aligned} P_i\phi &= \partial_i\phi \\ J_{ij}\phi &= (x_i\partial_j - x_j\partial_i)\phi \\ D\phi &= -(\Delta + x^i\partial_i)\phi \\ K_i\phi &= \left[2x_i\Delta + \left(2x^j x_i - x^2\delta_i^j\right)\partial_j\right]\phi \end{aligned}} \longleftarrow \quad \eta\partial_\eta \mapsto \Delta$$

$\longrightarrow$ Transformations of a primary operator of weight Δ in a **CFT**.

Conformal Ward Identities

The **boundary correlators** are constrained by the conformal symmetry:

$$\sum_{a=1}^{N} \langle \phi_1 \cdots \delta\phi_a \cdots \phi_N \rangle = 0$$

where $\delta\phi_a$ stand for any of the field transformations.

- Translations and rotations require that the correlator is only a function of the separations $|\mathbf{x}_a - \mathbf{x}_b|$.

- Dilatation and conformal invariance imply

$$\boxed{\begin{aligned} 0 &= \sum_{a=1}^{N} \left(\Delta_a + x_a^j \frac{\partial}{\partial x_a^j} \right) \langle \phi_1 \cdots \phi_a \cdots \phi_N \rangle \\ 0 &= \sum_{a=1}^{N} \left(2\Delta_a x_a^i + 2x_a^i x_a^j \frac{\partial}{\partial x_a^j} - x_a^2 \frac{\partial}{\partial x_{a,i}} \right) \langle \phi_1 \cdots \phi_a \cdots \phi_N \rangle \end{aligned}}$$

Conformal Ward Identities

Similar Ward identities can be derived for the **wavefunction coefficients**:

$$\boxed{\begin{aligned} 0 &= \sum_{a=1}^{N}\left((d-\Delta_a) + x_a^j \frac{\partial}{\partial x_a^j}\right)\langle O_1 \cdots O_a \cdots O_N\rangle \\ 0 &= \sum_{a=1}^{N}\left(2(d-\Delta_a)x_a^i + 2x_a^i x_a^j \frac{\partial}{\partial x_a^j} - x_a^2 \frac{\partial}{\partial x_{a,i}}\right)\langle O_1 \cdots O_a \cdots O_N\rangle \end{aligned}}$$

where $\Psi_N \equiv \langle O_1 \cdots O_N\rangle$.

$\longrightarrow$ Same as before, except the dual operators have weight $\bar{\Delta}_a = d - \Delta_a$.

Conformal Ward Identities

In cosmology, we need these Ward identities in **Fourier space**:

$$\boxed{\begin{aligned} \left[-d+\sum_{a=1}^{N} D_a\right]\langle O_1\cdots O_N\rangle'(\underline{\mathbf{k}}) &= 0 \\ \sum_{a=1}^{N} K_a^i\langle O_1\cdots O_N\rangle'(\underline{\mathbf{k}}) &= 0 \end{aligned}}$$

where $D = \bar{\Delta} + k^i\partial_{k^i}$ and $K_i = 2\bar{\Delta}\,\partial_{k^i} - k_i\partial_{k^j}\partial_{k^j} + 2k^j\partial_{k^j}\partial_{k^i}$.

- Easy to make an ansatz that solves the dilatation Ward identity.
- All the juice is in the conformal Ward identity.

Two-Point Functions

Consider the two-point function of **two arbitrary scalar operators**.

- An ansatz that solves the dilatation Ward identity is

$$\langle O_1 O_2 \rangle' \propto k_1^{\bar{\Delta}_1 + \bar{\Delta}_2 - d}$$

- The conformal Ward identity then forces $\bar{\Delta}_1 = \bar{\Delta}_2$, and hence

$$\boxed{\langle O_1 O_2 \rangle' = A\, k_1^{2\bar{\Delta}_1 - d}\, \delta_{\bar{\Delta}_1, \bar{\Delta}_2}}$$

- A massless field $(\bar{\Delta} = 3)$ in $d = 3$ dimensions has

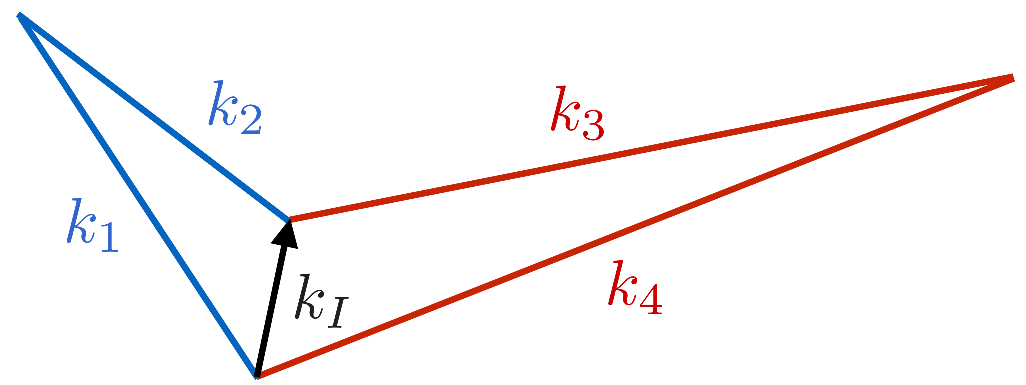
$$\langle \phi_1 \phi_2 \rangle' = \frac{1}{2\,\mathrm{Re}\langle O_1 O_2 \rangle'} \propto \frac{1}{k^3}$$

as expected for a scale-invariant power spectrum.

Four-Point Functions

Consider the four-point function of **conformally coupled scalars** ($\bar{\Delta} = 2$):

$$\langle O_1 O_2 O_3 O_4 \rangle' =$$



$$= \frac{1}{k_I} \hat{F}(u, v)$$

where $u \equiv k_I/(k_1 + k_2)$ and $v \equiv k_I/(k_3 + k_4)$.

- The conformal Ward identity then implies

$$\boxed{(\Delta_u - \Delta_v)\hat{F} = 0}$$

Arkani-Hamed, DB, Lee and Pimentel [2018]

where $\Delta_u \equiv u^2(1 - u^2)\partial_u^2 - 2u^3\partial_u$.

- To solve this equation requires additional physical input.

Bootstrapping Tools

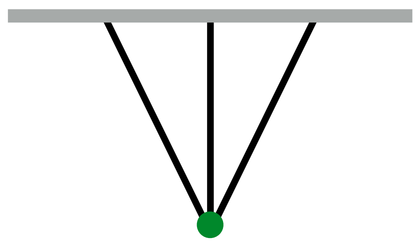
Symmetries	**Singularities**	Unitarity

Much of the physics of scattering
amplitudes is fixed by their **singularities**.

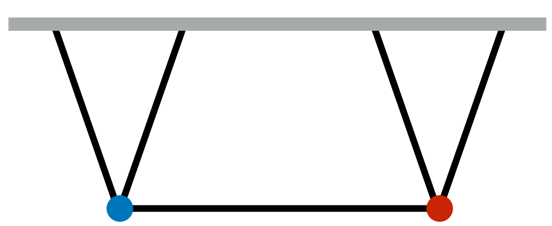
Singularities also play an important role in
constraining the structure of cosmological correlators.

Flat-Space Wavefunction

In the last lecture, we computed wavefunction coefficients in flat space.
Let us look back at some of these results.



$$= \; ig \int_{-\infty}^{0} \mathrm{d}t \; e^{i{\color{green}E}t} = \frac{g}{{\color{green}E}}$$



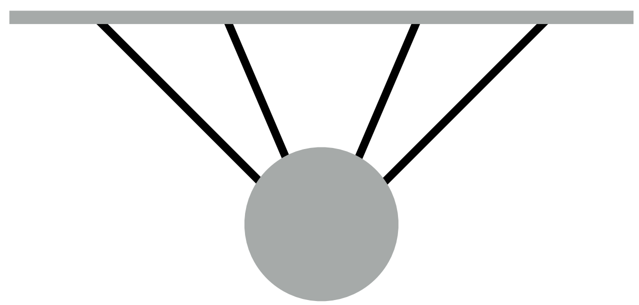
$$= \; -g^2 \int_{-\infty}^{0} \mathrm{d}t_1 \mathrm{d}t_2 \, e^{ik_{12}t_1} \, G_{k_I}(t_1, t_2) \, e^{ik_{34}t_2}$$

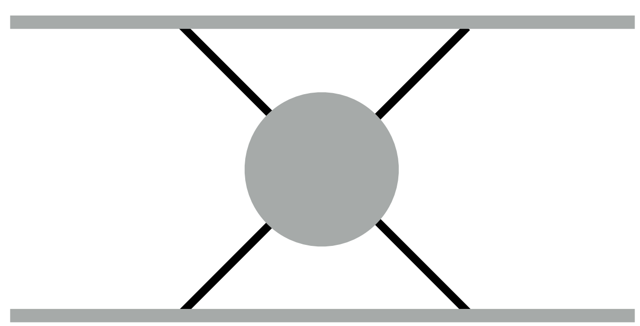
$$= \frac{g^2}{(k_{12}+k_{34}){\color{blue}(k_{12}+k_I)}{\color{red}(k_{34}+k_I)}} \equiv \frac{g^2}{E{\color{blue}E_L}{\color{red}E_R}}$$

$\longrightarrow$ Correlators are singular when energy is conserved.

Total Energy Singularity

Every correlator has a singularity at vanishing total energy:

$\psi_N =$  $\sim \int_{-\infty}^{0} \mathrm{d}t\, e^{iEt} f(t) = \frac{A_N}{E} + \cdots$

$A_N =$  $\sim \int_{-\infty}^{\infty} \mathrm{d}t\, e^{iEt} f(t) = M_N \delta(E)$

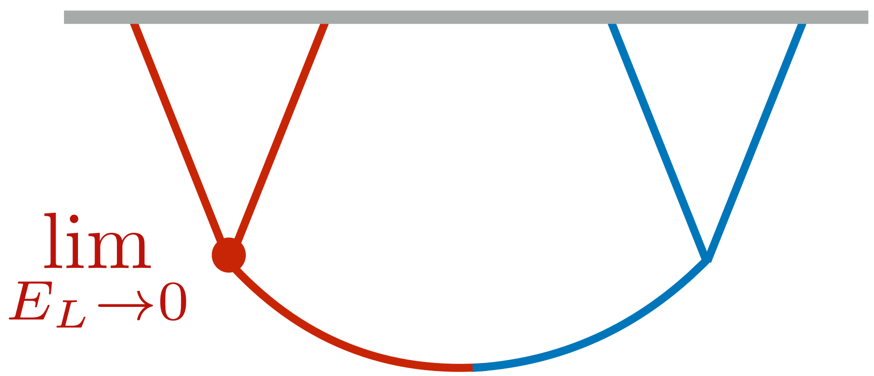
- The singularity arises from the early-time limit of the integration.
- The residue of the total energy singularity is the corresponding amplitude.

Raju [2012]

Maldacena and Pimentel [2011]

Partial Energy Singularities

Exchange diagrams lead to additional singularities:



$$= \frac{A_L \times \tilde{\psi}_R}{E_L} + \cdots$$

- The singularity arises from the early-time limit of the integration, where the **bulk-to-bulk propagator factorizes**:

$$G_{k_I}(t_L, t_R) \xrightarrow{t_L \to -\infty} \frac{e^{ik_I t_L}}{2k_I}\left(e^{-ik_I t_R} - e^{+ik_I t_R}\right)$$

- The correlator factorizes into an **amplitude** and a **shifted correlator**:

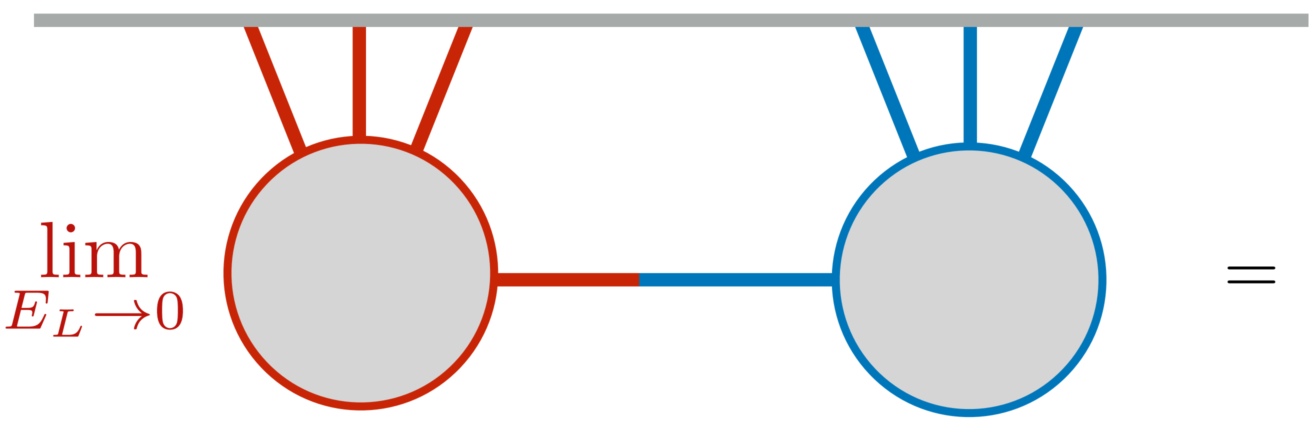
$$\tilde{\psi}_R \equiv \frac{1}{2k_I}\Big(\psi_R(k_{34} - k_I) - \psi_R(k_{34} + k_I)\Big)$$

Partial Energy Singularities

This pattern generalizes to **arbitrary graphs**.

- The wavefunction is singular when the energy flowing into any connected subgraph vanishes:

$$\lim_{E_L \to 0}$$



$$= \frac{A_L \times \tilde{\psi}_R}{E_L} + \cdots$$

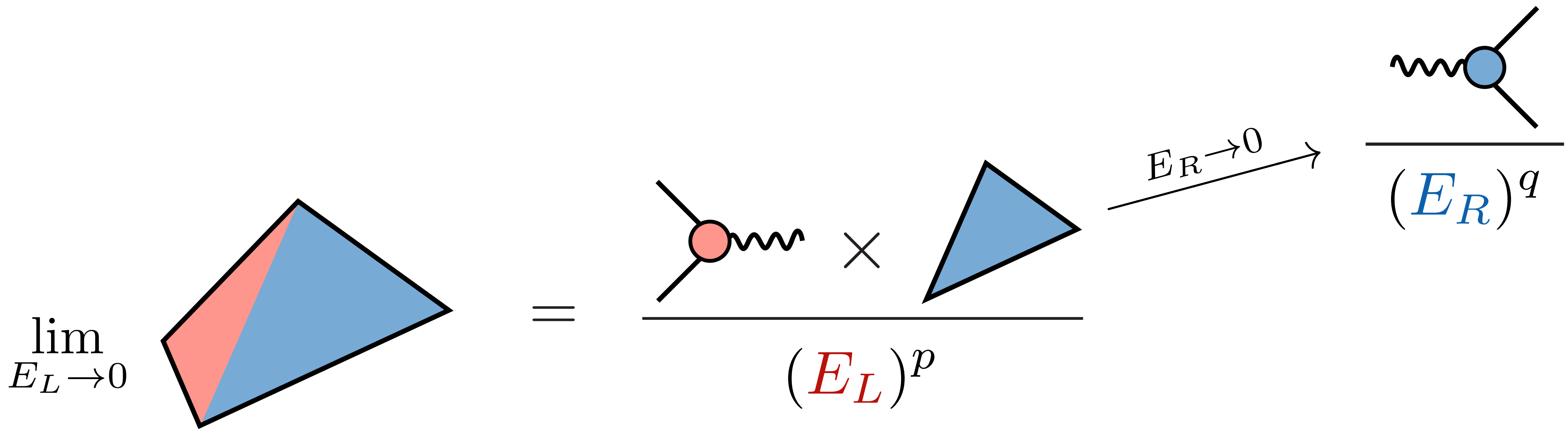
- For **loops**, the poles can become branch points.
- In **de Sitter**, the poles can be higher order.

Arkani-Hamed, Benincasa and Postnikov [2017]
DB, Chen, Duaso Pueyo, Joyce, Lee and Pimentel [2021]
Salcedo, Lee, Melville and Pajer [2022]

Singularities as Input

The singularities of the wavefunction only arise at unphysical kinematics. Nevertheless, they control the structure of cosmological correlators.

- Amplitudes are the **building blocks** of cosmological correlators:



- To construct the full correlator, we need to connect its singularities. We will present two ways of doing this:

1) Ward identities
2) Unitarity

Bootstrapping Tools

Symmetries

Singularities

Unitarity

Unitarity is a fundamental feature of quantum mechanics.

However, the constraints of **bulk unitary** on cosmological correlators were understood only very recently.

Goodhew, Jazayeri and Pajer [2020]
Meltzer and Sivaramakrishnan [2020]

The Optical Theorem

A consistent theory must have a unitary S-matrix:

$$S^\dagger S = 1 \xrightarrow{\quad S = 1 + iT \quad} T - T^\dagger = i T^\dagger T$$

Sandwiching this between the states $\langle f|$ and $|i\rangle$, and inserting a complete set of states, we get

$$\boxed{2\,\mathrm{Im}\,A(i \to f) = \sum_X \int \mathrm{d}\Pi_X \,(2\pi)^4 \delta(p_i - p_X)\, A(i \to X) A^*(f \to X)}$$

The right-hand side can be written in terms of the total cross section, giving the familiar form of the **optical theorem**.

Cutting Rules

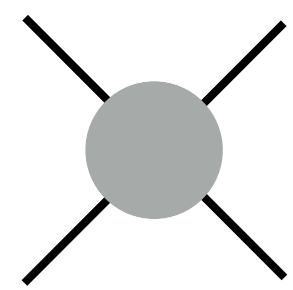
In perturbation theory, the consequences of the optical theorem are captured by the Cutkosky cutting rules.

Exercise: Show that

$$\mathrm{Im}\left(\frac{1}{p^2 - m^2 + i\varepsilon}\right) = -\pi\,\delta(p^2 - m^2)$$

Compute discontinuities of amplitudes by putting internal lines on-shell:

$$\boxed{\mathrm{Disc}[iA] \equiv -2\mathrm{Im}(A) = \sum_{\text{cuts}}}$$



- Especially useful for computing loops.

Cosmological Cutting Rules

Goodhew, Jazayeri, Lee and Pajer [2021]
Melville and Pajer [2021]

Three facts implied by **bulk unitarity**:

1) Coupling constants are real.

2) Propagators are Hermitian analytic: $K^*_{-k^*}(\eta) = K_k(\eta)$

3) Imaginary part of bulk-to-bulk propagator factorizes:

$$\underbrace{\mathrm{Im}\, G_k(\eta, \eta')}_{\text{Cut propagator}} = 2P_k \, \mathrm{Im}\, K_k(\eta) \, \mathrm{Im}\, K_k(\eta')$$

- Using the cut propagators in the computations of the wavefunction leads to simpler objects, because there are no nested integrals anymore.

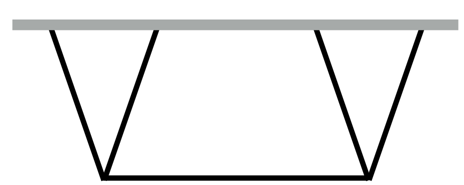
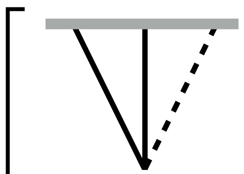
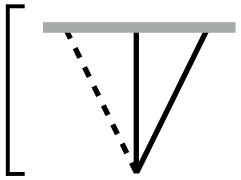
Cosmological Cutting Rules

Goodhew, Jazayeri, Lee and Pajer [2021]
Melville and Pajer [2021]

This gives systematic **cutting rules**:

$$\boxed{\mathrm{Disc}[\psi] \equiv \psi(k_a) - \psi^*(-k_a^*) = \sum \text{all possible cuts}}$$

For example:

Disc[] = $P(k)$ Disc[] Disc[]

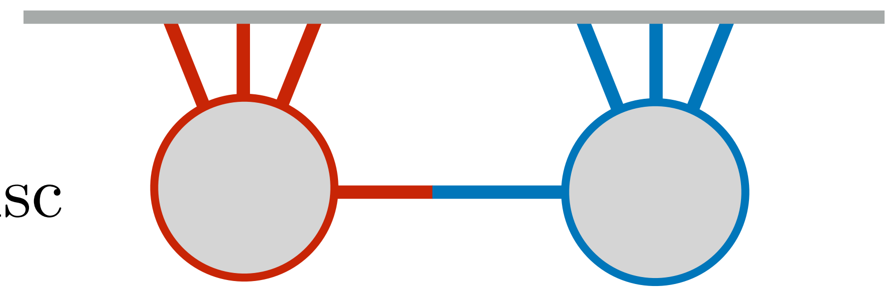
- Already useful at tree level.
- Useful for massless fields, where the discontinuity can be integrated.

Meltzer [2021]

Unitarity as Input

- The discontinuity of a WF coefficient factorizes into lower-point objects.
- This contains slightly more information than the energy singularities:

$$\lim_{E_L \to 0} \ \mathrm{Disc}$$



$$= \lim_{E_L \to 0} \Big[\psi(k_a, k_I) - \psi^*(-k_a^*, k_I) \Big] = \lim_{E_L \to 0} \psi(k_a, k_I)$$

has no singularities in E_L

The cutting rule fixes the entire Laurent series in E_L.

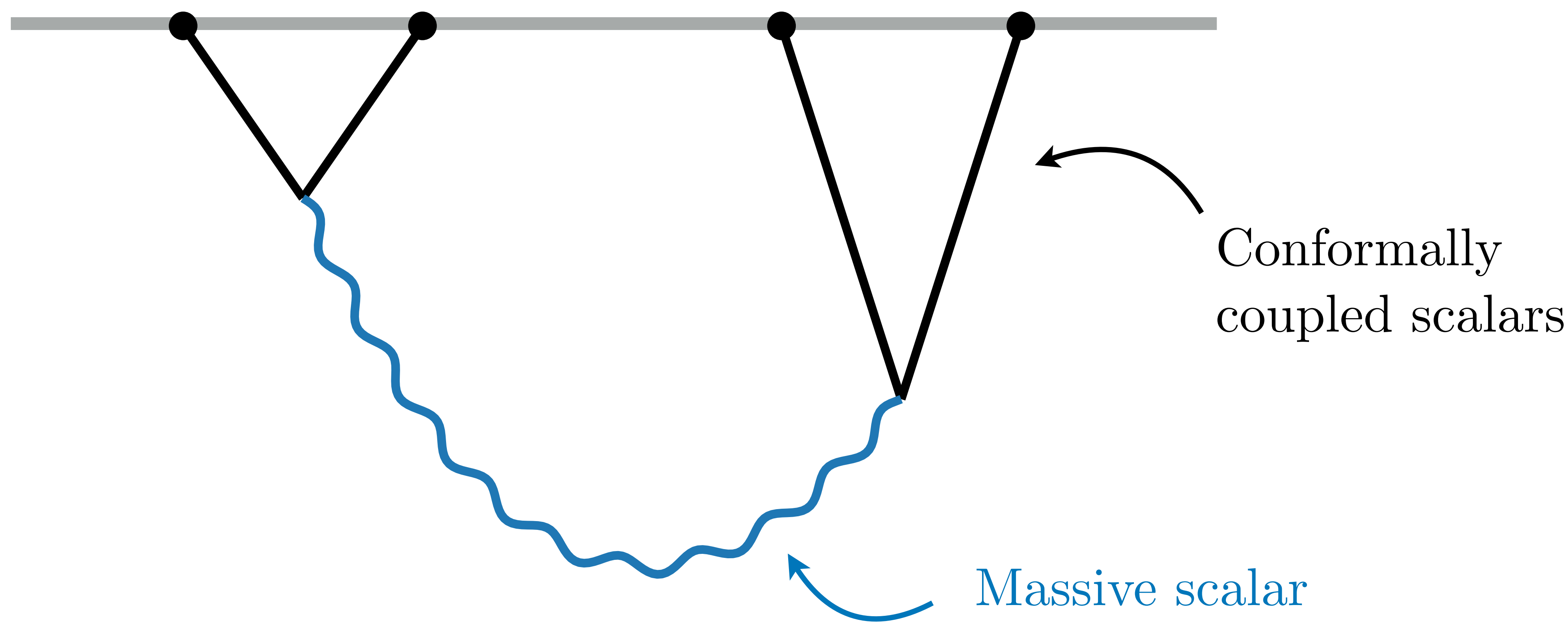
- These unitarity cuts have become an important bootstrapping tool.

Cosmological Collider

Exchange Four-Point Function

We are now ready to return to the challenge of massive particle exchange.

$$\langle O_1 O_2 O_3 O_4 \rangle' =$$



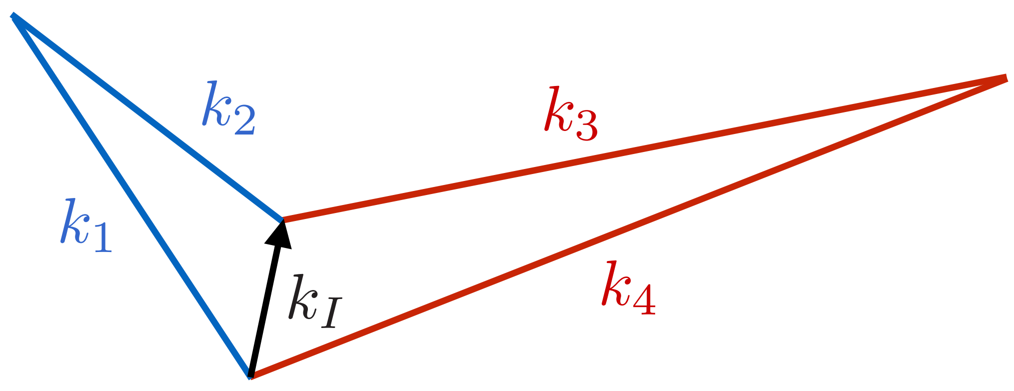
$$= \; -g^2 \int \frac{\mathrm{d}\eta}{\eta^2} \int \frac{\mathrm{d}\eta'}{\eta'^2}\, e^{ik_{12}\eta} e^{ik_{34}\eta'}\, G(k_I; \eta, \eta'')$$

In general, the time integrals cannot be performed analytically.

Conformal Ward Identity

Recall that the s-channel correlator can be written as

$$\langle O_1 O_2 O_3 O_4 \rangle' =$$



$$= \frac{1}{k_I}\hat{F}(u, v)$$

where $u \equiv k_I/(k_1 + k_2)$ and $v \equiv k_I/(k_3 + k_4)$.

We have seen that the **conformal symmetry** implies the following differential equation

$$\boxed{(\Delta_u - \Delta_v)\hat{F} = 0}$$

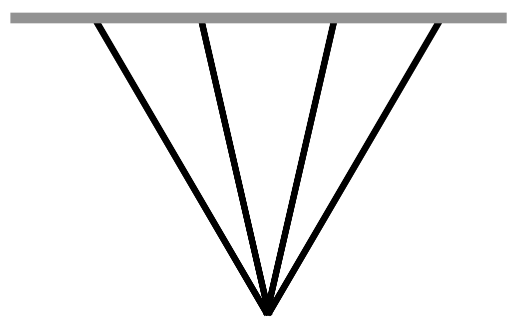
where $\Delta_u \equiv u^2(1 - u^2)\partial_u^2 - 2u^3\partial_u$.

→ We will classify solutions to this equation by their **singularities**.

Contact Solutions

The simplest solutions correspond to **contact interactions**:

$$\hat{F}_c \;=$$



$$= \sum_n \frac{c_n(u,v)}{E^{2n+1}}$$

Fixed by symmetry

Only total energy singularities

- For Φ^4, we have $\hat{F}_{c,0} = \dfrac{k_I}{E} = \dfrac{uv}{u+v}$.
- Higher-derivative bulk interactions, lead to $\hat{F}_{c,n} = \Delta_u^n \hat{F}_{c,0}$.

$$\hat{F}_{c,1} = -2\left(\frac{k_I}{E}\right)^3 \frac{1+uv}{uv} \,,$$

$$\hat{F}_{c,2} = -4\left(\frac{k_I}{E}\right)^5 \frac{u^2+v^2+uv(3u^2+3v^2-4)-6(uv)^2-6(uv)^3}{(uv)^3} \,.$$

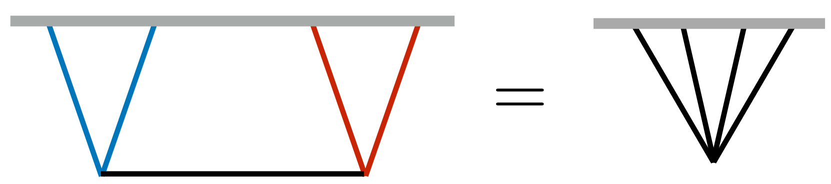
This can get relatively complex, but everything is fixed by symmetry.

Exchange Solutions

For tree exchange, we try

$$\boxed{\begin{aligned}(\Delta_u + M^2)\hat{F}_e = \hat{C}\\ (\Delta_v + M^2)\hat{F}_e = \hat{C}\end{aligned}}$$

$$\begin{matrix}(\Delta_u + M^2)\\ (\Delta_v + M^2)\end{matrix}$$

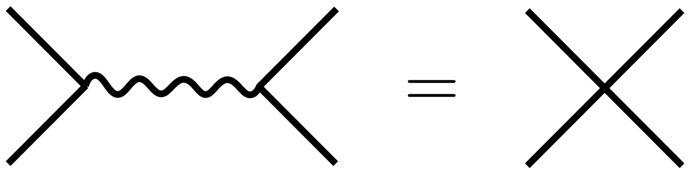


where $\hat{C} = M^2 \hat{F}_c$ is a contact solution and $M^2 \equiv m^2/H^2 - 2$.

For amplitudes, the analog of this is

$$\boxed{(s + m^2)A_e = A_c(s, t)}$$

$$(s + m^2)$$



This is an algebraic relation, while for correlators we have a differential equation.

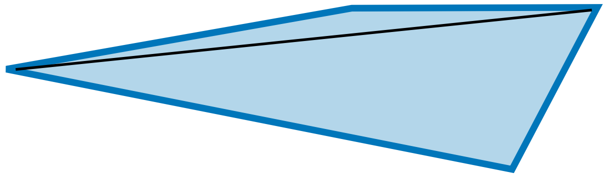
Exchange Solutions

Using the simplest contact interaction as a source, we have

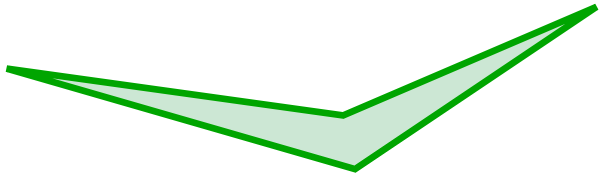
$$\boxed{\left[u^2(1-u^2)\partial_u^2 - 2u^3\partial_u + M^2\right]\hat{F} = \frac{uv}{u+v}}$$

Before solving this equation, let us look at its **singularities**.

- **Flat-space limit**: $\lim_{u\to -v} \hat{F} \propto A_4(u+v)\log(u+v)$
- **Factorization limit**: $\lim_{u,v\to -1} \hat{F} \propto A_3 \log(1+u) \times A_3 \log(1+v)$
- **Folded limit**: $\lim_{u\to +1} \hat{F} \propto \log(1-u)$
- **Collapsed limit**: $\lim_{u\to 0} \hat{F} \propto u^{iM}$



This singularity should be absent in the Bunch-Davies vacuum.



This non-analyticity is a key signature of particle production.

Exchange Solutions

Using the simplest contact interaction as a source, we have

$$\boxed{\left[u^2(1-u^2)\partial_u^2 - 2u^3\partial_u + M^2\right]\hat{F} = \frac{uv}{u+v}}$$

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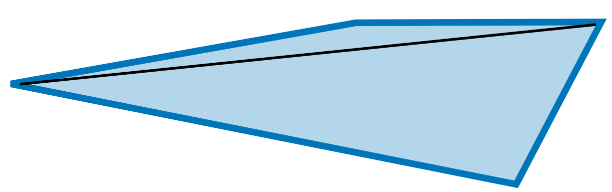
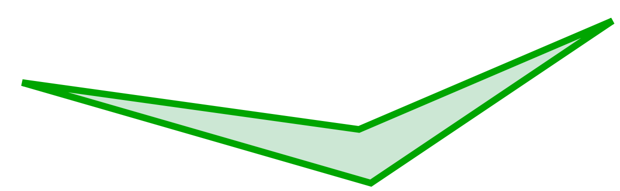




$\longrightarrow$ Factorization and folded limits uniquely fix the solution.
The remaining singularities become consistency checks.

Forced Harmonic Oscillator

In the limit of small internal momentum ($u < v \ll 1$), the equation becomes

$$\boxed{\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} + M^2\right]\hat{F} = \frac{1}{\cosh t}} \qquad t \equiv \log(u/v)$$

whose solution is

$$\boxed{\hat{F} = \underbrace{\sum_n \frac{(-1)^n}{(n+\frac{1}{2})^2 + M^2}\left(\frac{u}{v}\right)^{n+1}}_{\text{EFT expansion}} + \underbrace{\frac{\pi}{\cosh(\pi M)}\frac{\sin(M\log(u/v))}{M}}_{\text{particle production}}}$$

analytic

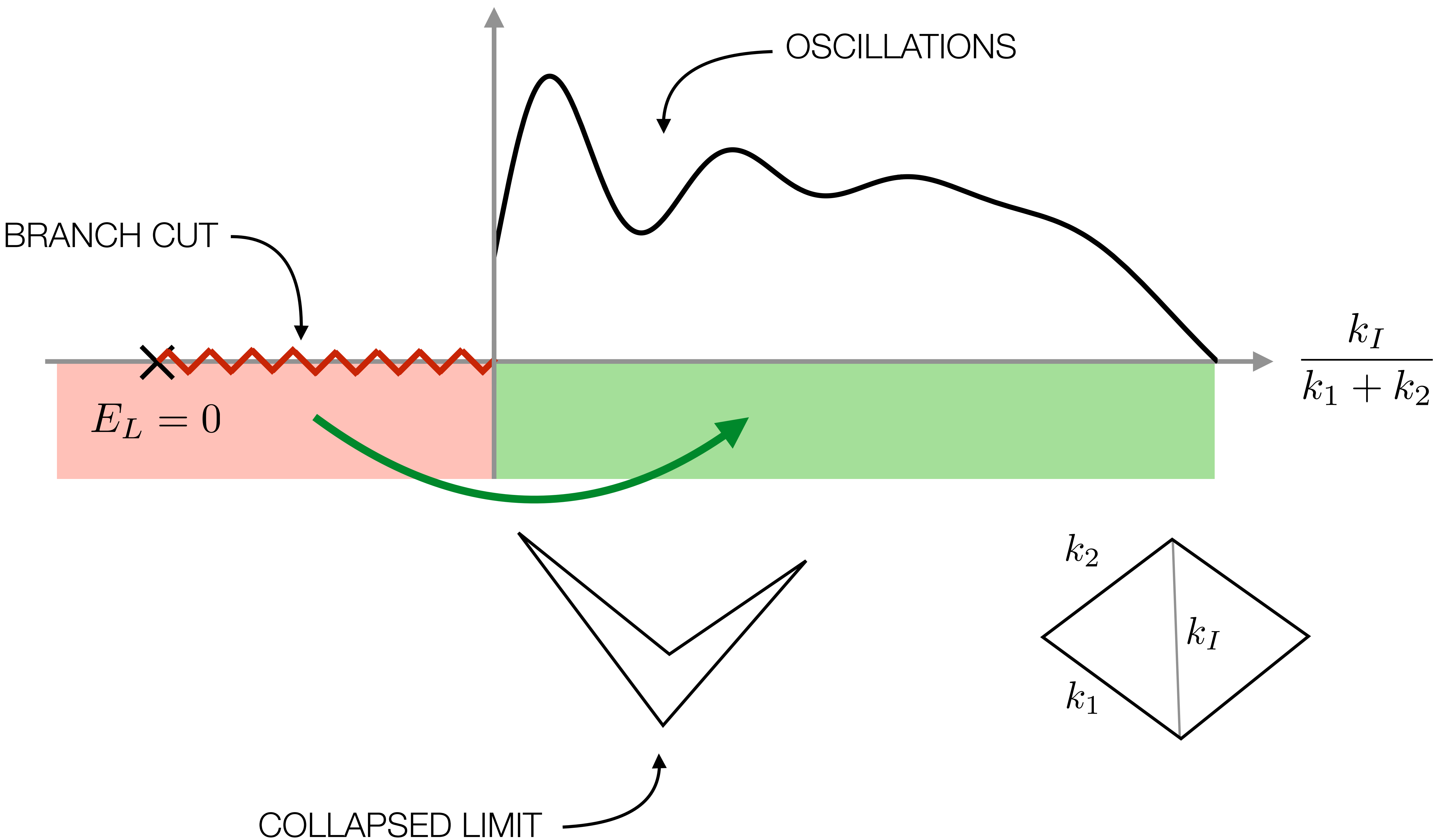
non-analytic

The general solution takes a similar form.

Arkani-Hamed, DB, Lee and Pimentel [2018]

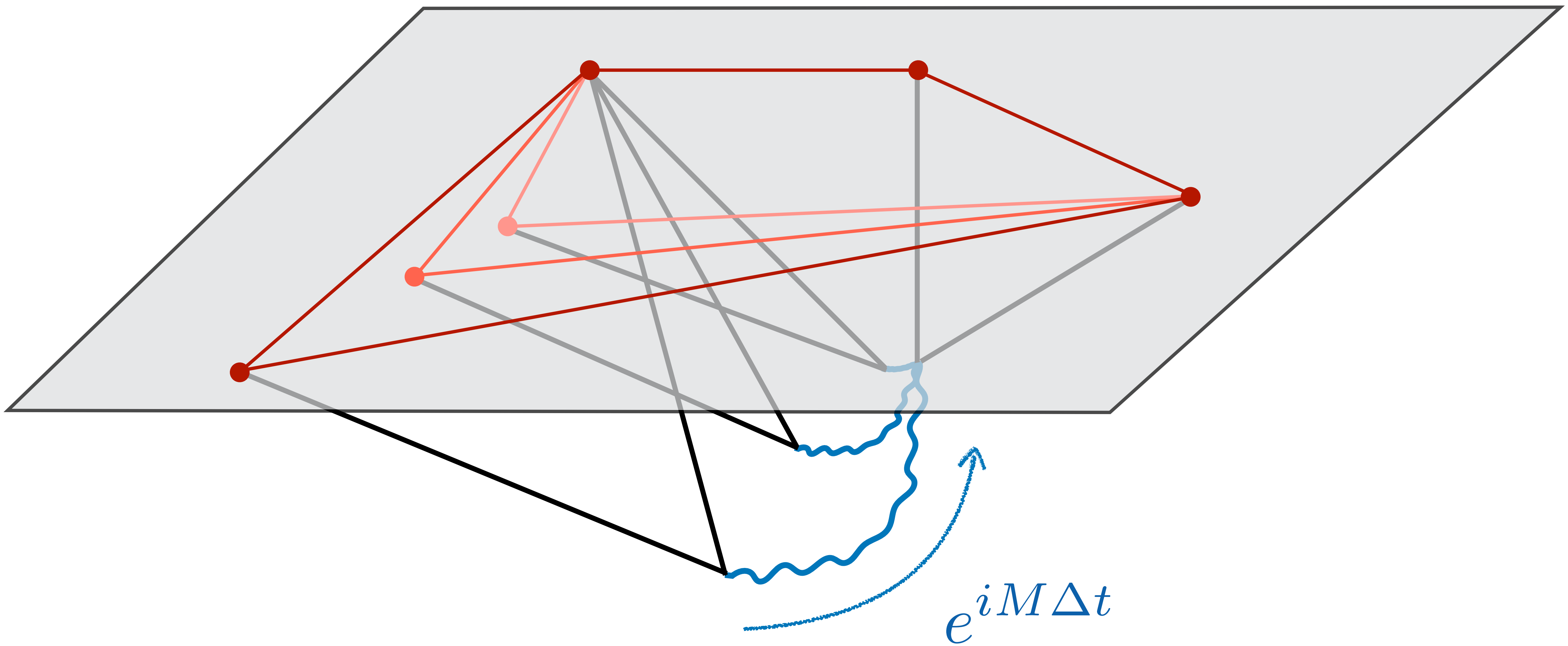
Oscillations

The correlator is forced to have an oscillatory feature in the physical regime:



Particle Production

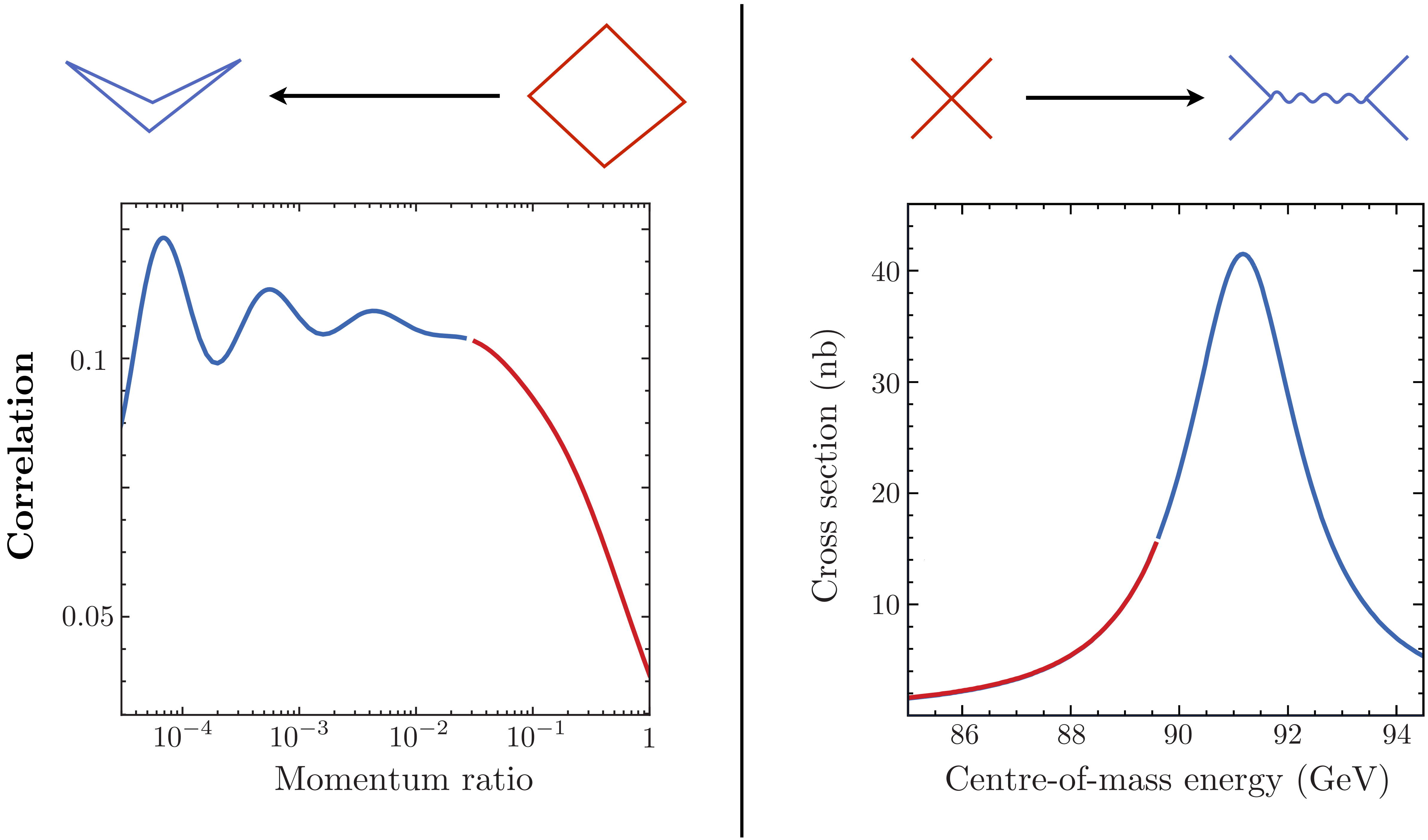
These oscillations reflect the evolution of the massive particles during inflation:



Time-dependent effects have emerged in the solution of the time-independent bootstrap constraints.

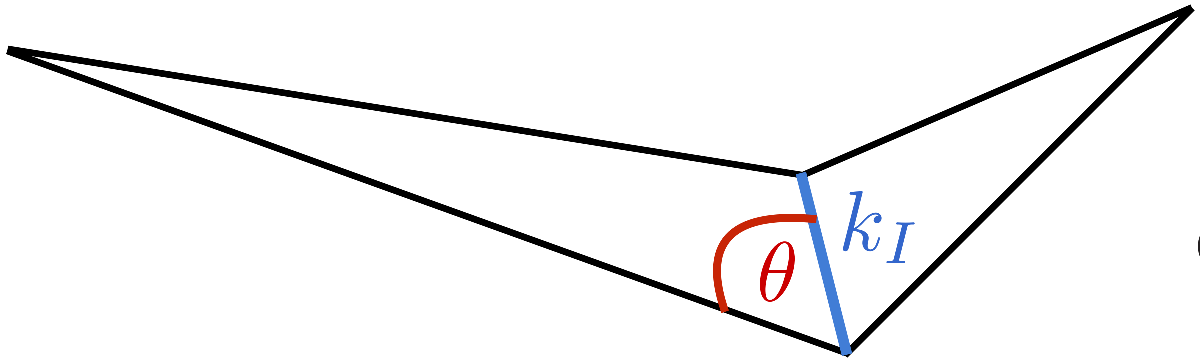
Cosmological Collider Physics

The oscillatory feature is the analog of a **resonance** in collider physics:



Particle Spectroscopy

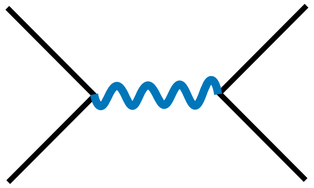
The frequency of the oscillations depends on the **mass** of the particles:



$$\propto \sin[M \log k_I]\, P_J(\cos\theta)$$

The angular dependence of the signal depends on the **spin** of the particles.

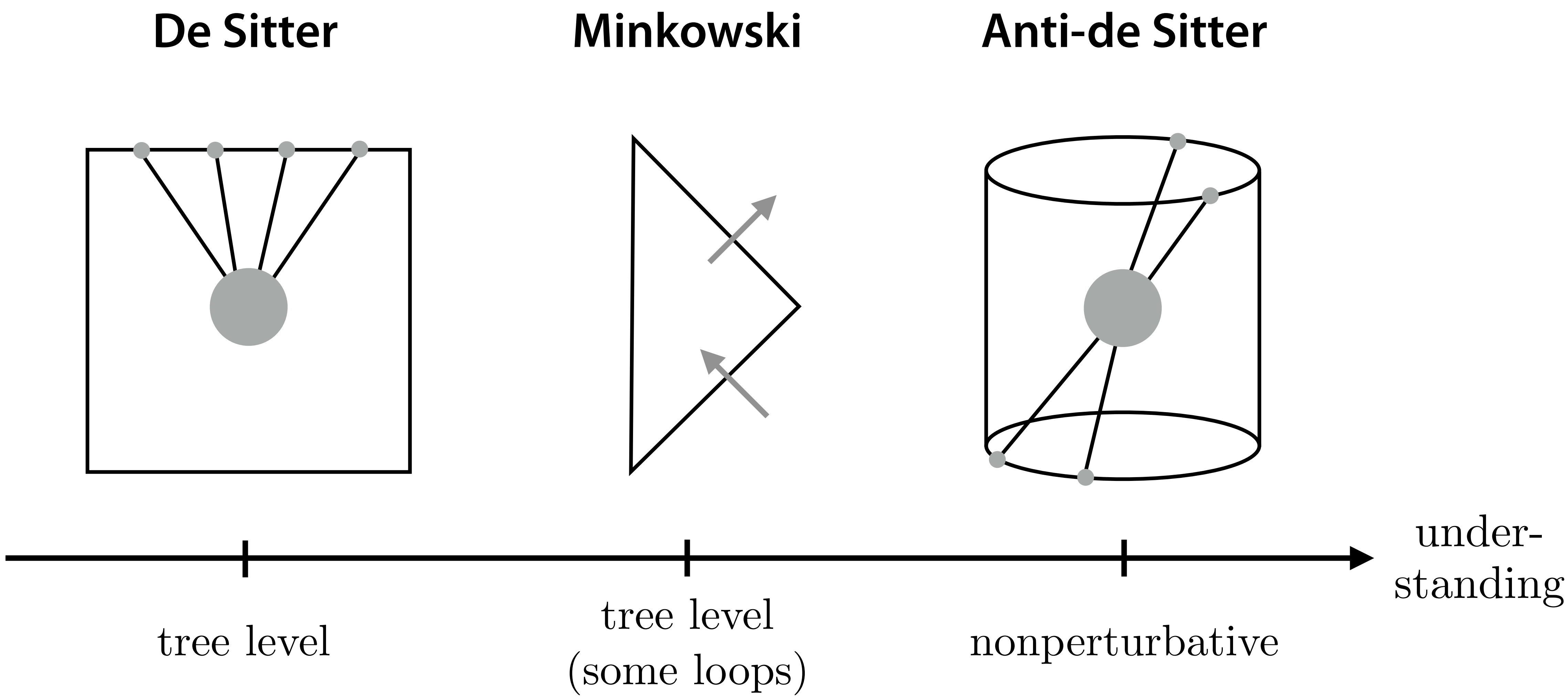
This is very similar to what we do in collider physics:



$$= \frac{g^2}{s - M^2}\, P_J(\cos\theta)$$

Outlook and Speculations

Despite enormous progress in recent years, we are only at the beginning of a systematic exploration of cosmological correlators:



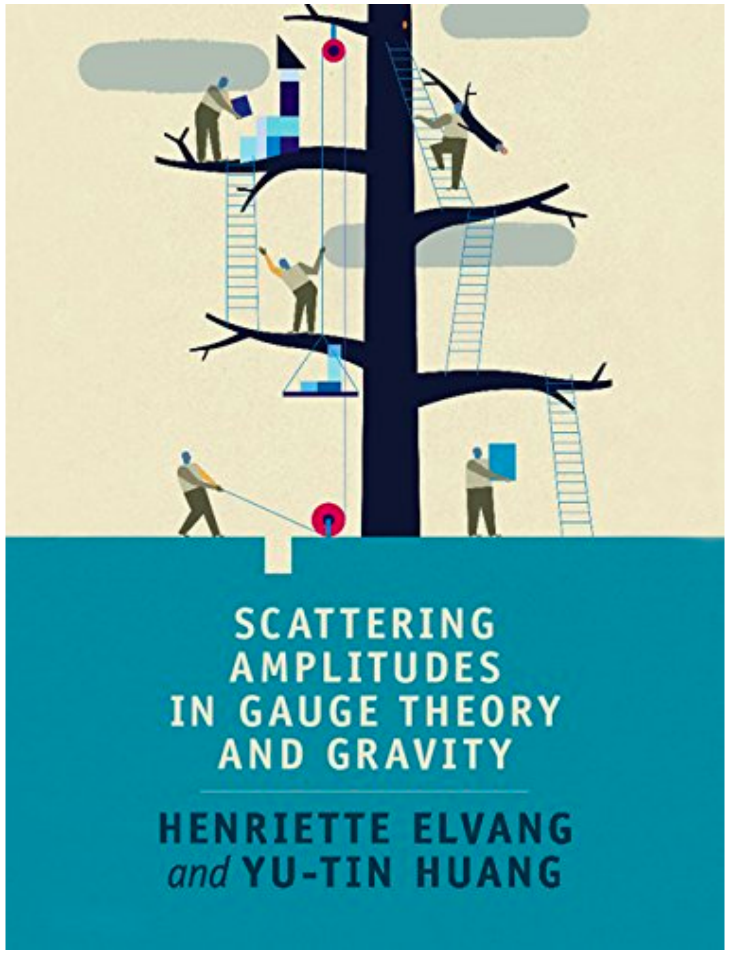
Many fundamental questions still remain unanswered.

Beyond Feynman Diagrams

So far, the cosmological bootstrap has only been applied to individual Feynman diagrams (which by themselves are unphysical).

However, in the **S-matrix bootstrap**, the real magic is found for **on-shell diagrams**:

- Recursion relations
- Generalized unitarity
- Color-kinematics duality
- Hidden positivity
- ...



What is the on-shell formulation of the cosmological bootstrap?
What magic will it reveal?

Beyond Perturbation Theory

So far, the bootstrap constraints (singularities, unitarity cuts, etc.) are only formulated in perturbation theory and implemented only at tree level.

In contrast, we understand the **conformal bootstrap** nonperturbatively:

Recently, some preliminary progress was made to apply unitarity constraints in de Sitter nonperturbatively, but much work remains.

Hogervorst, Penedones and Vaziri [2021]
Di Pietro, Gorbenko and Komatsu [2021]

A nonperturbative bootstrap will be required if we want to understand the UV completion of cosmological correlators.

Beyond Spacetime

In quantum gravity, spacetime is an emergent concept. In cosmology, the notation of “time” breaks down at the Big Bang. What replaces it?

The cosmological bootstrap provides a modest form of emergent time, but there should be a much more radical approach where space and time are outputs, not inputs.

For scattering amplitudes, such a reformulation was achieved in the form of the **amplituhedron**: Arkani-Hamed and Trnka [2013]

$\boxed{\text{What is the analogous object in cosmology? How does the Big Bang arise from it?}}$

Thank you for your attention!