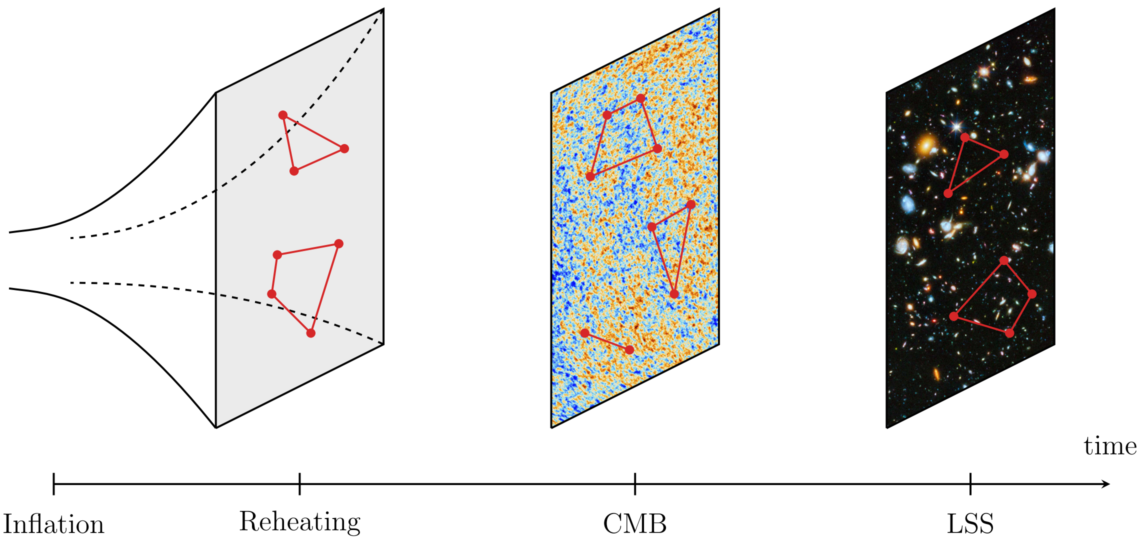
Lectures on Cosmological Correlations

Daniel Baumann



OUTLINE

0. Motivation

I. Wavefunction Approach

II. Cosmological Bootstrap

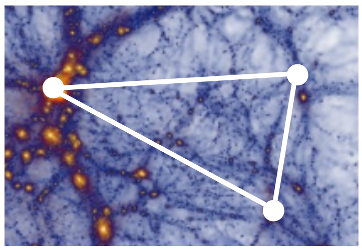
III. A New Twist on Time

Lecture notes and lecture scripts can be found at:
https://github.com/ddbaumann/cosmo-correlators

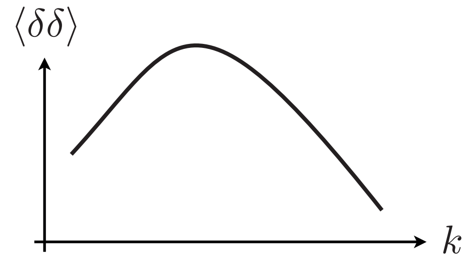
0. MOTIVATION

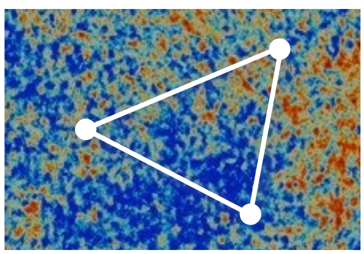
0.1. Practical Motivation

In cosmology, we measure **spatial correlations**:

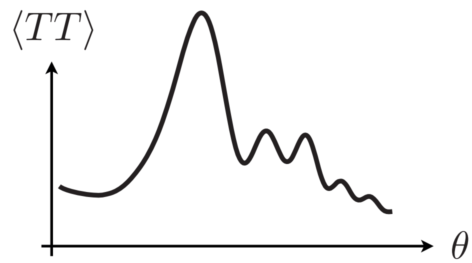


$$\langle \delta\rho(\mathbf{x}_1)\cdots\delta\rho(\mathbf{x}_N)\rangle$$

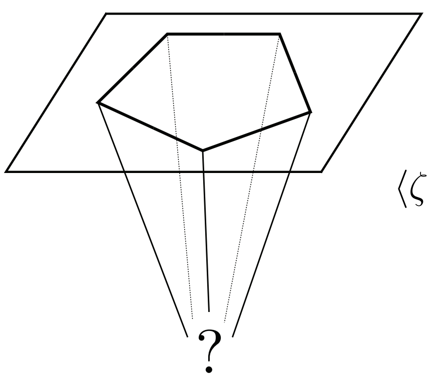




$$\langle \delta T(\theta_1)\cdots\delta T(\theta_N)\rangle$$



These correlations can be traced back to the origin of the hot Big Bang:



$$\langle \zeta(\mathbf{x}_1)\cdots\zeta(\mathbf{x}_N)\rangle$$

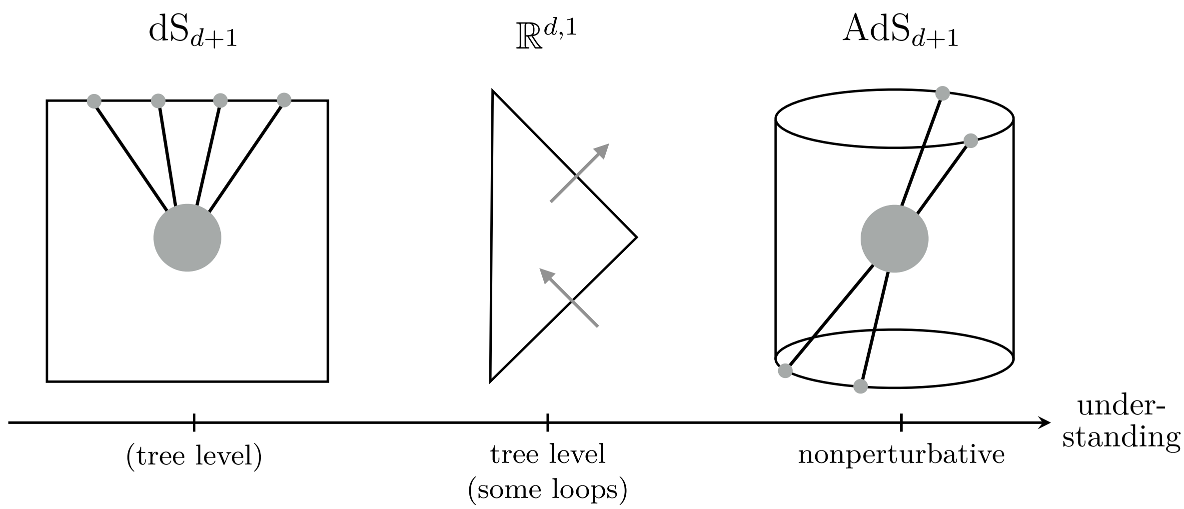
Where did the primordial correlations come from?

- Clue 1: The correlations span superhorizon scales.
- Clue 2: They are scale-invariant.

This suggests that the fluctuations were created **before the hot Big Bang**, during a phase of approximate time-translation invariance (= **inflation**).

0.2. Conceptual Motivation

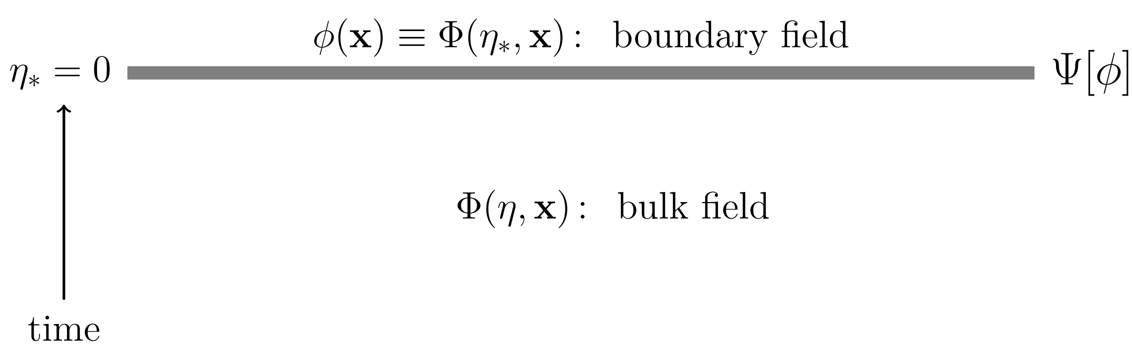
The study of cosmological correlators is also of conceptual interest:



Our understanding of quantum field theory in de Sitter space (cosmology) is still rather underdeveloped $\Rightarrow$ Opportunity for you to make progress!

I. WAVEFUNCTION APPROACH

1.1. Wavefunction of the Universe



The "wavefunction of the universe" is

$$\Psi[\phi] \equiv \langle \phi(\mathbf{x}) | 0 \rangle = \int_{\substack{\Phi(\eta_*)=\phi \\ \Phi(-\infty)=0}} \mathcal{D}\Phi \, e^{iS[\Phi]} \approx e^{iS[\Phi_{\rm cl}]} \, .$$

It defines "boundary correlators"

$$\langle \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_N) \rangle = \int \mathcal{D}\phi \, \phi(\mathbf{x}_1) \cdots \phi(\mathbf{x}_N) \, |\Psi[\phi]|^2 \, .$$

The perturbative expansion of the wavefunction (in momentum space) is

$$\Psi[\phi] \;=\; \exp\left(-\sum_{N=2}^{\infty} \frac{1}{N!} \int \mathrm{d}^3 k_1 \cdots \mathrm{d}^3 k_N \, \Psi_N(\underline{\mathbf{k}}) \, \phi_{\mathbf{k}_1} \cdots \phi_{\mathbf{k}_N} \right) ,$$

where the "wavefunction coefficients" are

$$\Psi_N(\underline{\mathbf{k}}) = (2\pi)^3 \delta_D(\mathbf{k}_1 + \cdots + \mathbf{k}_N) \, \underset{\substack{\uparrow \\ \text{dual operators: } \phi \to O,\ \gamma_{ij} \to T_{ij}}}{\langle O_{\mathbf{k}_1} \cdots O_{\mathbf{k}_N} \rangle'} \, .$$

The relation between correlators and wavefunction coefficients is

$$\begin{aligned}
\langle \phi\phi \rangle &= \frac{1}{2\mathrm{Re}\langle OO \rangle} \, , \\
\langle \phi\phi\phi \rangle &= \frac{2\mathrm{Re}\langle OOO \rangle}{\prod_{n=1}^{3} 2\mathrm{Re}\langle O_n O_n \rangle} \, , \\
\langle \phi\phi\phi\phi \rangle &= \frac{\langle OOOO \rangle}{\langle OO \rangle^4} + \frac{\langle OOX \rangle^3}{\langle XX \rangle \langle OO \rangle^4} \, .
\end{aligned}$$

1.2. Quantum Harmonic Oscillators

In the following, we will study various incarnations of the quantum harmonic oscillator from the wavefunction perspective.

Simple Harmonic Oscillator

Consider

$$S[\Phi] = \int \mathrm{d}t \left(\frac{1}{2}\dot{\Phi}^2 - \frac{1}{2}\omega^2\Phi^2\right) \quad \Rightarrow \quad \ddot{\Phi} + \omega^2\Phi = 0\,.$$

- Classical solution:

$$\Phi_{\rm cl} = \phi e^{i\omega t}\,.$$

- On-shell action:

$$\begin{aligned} S[\Phi_{\rm cl}] &= \int_{t_i}^{t_*} \mathrm{d}t \left[\frac{1}{2}\partial_t(\dot{\Phi}_{\rm cl}\Phi_{\rm cl}) - \frac{1}{2}\Phi_{\rm cl}\underbrace{(\ddot{\Phi}_{\rm cl} + \omega^2\Phi_{\rm cl})}_{=0}\right] \\ &= \frac{1}{2}\dot{\Phi}_{\rm cl}\Phi_{\rm cl}\Big|_{t=t_*} \\ &= \frac{i\omega}{2}\phi^2\,. \end{aligned}$$

- Wavefunction:

$$\Psi[\phi] \approx \exp(iS[\Phi_{\rm cl}]) = \exp\left(-\frac{\omega}{2}\phi^2\right).$$

- Variance of the oscillator amplitude:

$$\boxed{\langle\phi^2\rangle = \frac{1}{2\omega}}\,.$$

Free Fields in Minkowski

In QFT, the same result applies for each Fourier mode:

$$\boxed{\langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle' = \frac{1}{2\omega_k}},$$

where $\omega_k \equiv \sqrt{k^2 + m^2}$.

Time-Dependent Oscillator

Consider

$$S[\Phi] = \int \mathrm{d}t \left(\frac{1}{2} A(t) \dot{\Phi}^2 - \frac{1}{2} B(t) \Phi^2 \right).$$

- Classical solution:

$$\Phi_{\mathrm{cl}} = \phi K(t), \quad \text{with} \quad \begin{array}{l} K(0) = 1 \\ K(-\infty) \sim e^{i\omega t} \end{array}$$

- On-shell action:

$$\begin{aligned} S[\Phi_{\mathrm{cl}}] &= \int_{t_i}^{t_*} \mathrm{d}t \left[\frac{1}{2} \partial_t (A \dot{\Phi}_{\mathrm{cl}} \Phi_{\mathrm{cl}}) - \frac{1}{2} \Phi_{\mathrm{cl}} \underbrace{\left(\partial_t (A \dot{\Phi}_{\mathrm{cl}}) + B \Phi_{\mathrm{cl}} \right)}_{=0} \right] \\ &= \frac{1}{2} A \dot{\Phi}_{\mathrm{cl}} \Phi_{\mathrm{cl}} \Big|_{t=t_*} \\ &= \frac{1}{2} A \phi^2 \partial_t \log K \Big|_{t=t_*}. \end{aligned}$$

- Wavefunction:

$$\Psi[\phi] \approx \exp(i S[\Phi_{\mathrm{cl}}]) = \exp\left(\frac{i}{2} (A \partial_t \log K) \Big|_* \phi^2 \right).$$

- Variance of the oscillator amplitude:

$$|\Psi[\phi]|^2 = \exp\left(-\mathrm{Im}(A \partial_t \log K) \big|_* \phi^2 \right) \implies \boxed{\langle \phi^2 \rangle = \frac{1}{2\, \mathrm{Im}(A\, \partial_t \log K) \big|_*}}.$$

Free Fields in de Sitter

Consider

$$S = \frac{1}{2}\int \mathrm{d}\eta\,\mathrm{d}^3x\, a^2(\eta)\left[(\Phi')^2 - (\nabla\Phi)^2\right] \qquad\qquad a(\eta) \equiv (-H\eta)^{-1}$$

$$= \frac{1}{2}\int \mathrm{d}\eta \frac{\mathrm{d}^3k}{(2\pi)^3}\left[\frac{1}{(H\eta)^2}\Phi'_{\mathbf{k}}\Phi'_{-\mathbf{k}} - \frac{k^2}{(H\eta)^2}\Phi_{\mathbf{k}}\Phi_{-\mathbf{k}}\right].$$

- Classical solution:

$$\Phi_{\rm cl} = \phi K(\eta)\,, \quad \text{with} \quad \begin{array}{l} K(0) = 1 \\ K(-\infty) \sim e^{ik\eta} \end{array}$$

  The function $K(\eta)$ is called the "bulk-to-boundary propagator".

- For a massless field, we have

$$K(\eta) = (1 - ik\eta)e^{ik\eta}\,,$$
$$\log K(\eta) = \log(1 - ik\eta) + ik\eta\,,$$

  and hence

$$\mathrm{Im}(A\partial_\eta \log K)\big|_{\eta=\eta_*} = \frac{1}{(H\eta_*)^2}\,\mathrm{Im}\left(\frac{-ik}{1 - ik\eta_*} + ik\right)$$
$$= \frac{1}{(H\eta_*)^2}\,\mathrm{Im}\left(\frac{k^2\eta + ik^3\eta_*^2}{1 + k^2\eta_*^2}\right) \xrightarrow{\eta_*\to 0} \boxed{\frac{k^3}{H^2}}\,.$$

- The two-point function then is

$$\boxed{\langle\phi_{\mathbf{k}}\phi_{-\mathbf{k}}\rangle' = \frac{H^2}{2k^3}}\,.$$

  This is the famous **scale-invariant spectrum** of inflationary fluctuations.

Anharmonic Oscillator

Consider

$$S[\Phi] = \int \mathrm{d}t \left(\frac{1}{2}\dot{\Phi}^2 - \frac{1}{2}\omega^2\Phi^2 - \frac{1}{3}g\Phi^3 \right) \quad \Rightarrow \quad \ddot{\Phi} + \omega^2\Phi = -g\Phi^2 \,.$$

- Classical solution:

$$\Phi_{\text{cl}}(t) = \phi K(t) + i \int \mathrm{d}t'\, G(t,t') \big(-g\Phi_{\text{cl}}^2(t')\big),$$

  where the "Green's function" is

$$G(t,t') = \frac{1}{2\omega} \left(e^{-i\omega(t-t')}\theta(t-t') + e^{i\omega(t-t')}\theta(t'-t) - e^{i\omega(t+t')} \right).$$

- On-shell action:

$$S[\Phi_{\text{cl}}] = \frac{i\omega}{2}\phi^2 - \frac{g}{3}\int \mathrm{d}t\, \Phi_{\text{cl}}^3(t) - \frac{ig^2}{2}\int \mathrm{d}t\,\mathrm{d}t'\, G(t,t')\,\Phi_{\text{cl}}^2(t')\,\Phi_{\text{cl}}^2(t)\,.$$

  See lecture notes for the derivation.

- Perturbative expansion:

$$\Phi_{\text{cl}}(t) = \Phi^{(0)}(t) + g\Phi^{(1)}(t) + g^2\Phi^{(2)}(t) + \cdots \,.$$

  where

$$\Phi^{(0)}(t) = \phi e^{i\omega t}\,,$$

$$\Phi^{(1)}(t) = i\int \mathrm{d}t'\, G(t,t') \left(-(\Phi^{(0)}(t'))^2 \right) = \frac{\phi^2}{3\omega^2}\left(e^{2i\omega t} - e^{i\omega t} \right).$$

- Wavefunction:

$$\boxed{\Psi[\phi] \approx e^{iS[\Phi_{\text{cl}}]} = \exp\left(-\frac{\omega}{2}\phi^2 - \frac{g}{9\omega}\phi^3 + \frac{g^2}{72\omega^3}\phi^4 + \cdots \right)}\,.$$

From this, we can compute $\langle\phi^3\rangle$, $\langle\phi^4\rangle$, etc.

1.3. Interactions in Field Theory

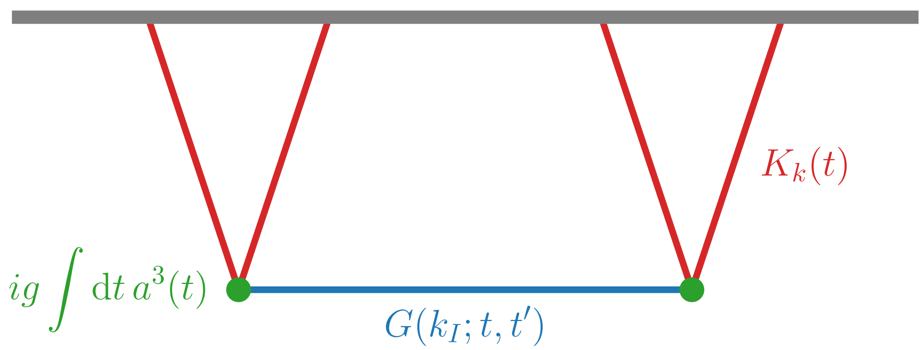
Back to field theory.

$$S[\Phi] = \int \mathrm{d}^4x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} m^2 \Phi^2 - \frac{1}{3} g \Phi^3 \right) .$$

The analysis is similar to that of the anharmonic oscillator ($\Rightarrow$ lecture notes).

In the interest of time, we jump directly to the **Feynman rules**:

- bulk-to-boundary propagator K for every external line
- bulk-to-bulk propagator G for every internal line
- integrate each vertex over time.



Given a mode function $\Phi_{\mathrm{cl}} \equiv f_k(t)$, the bulk-to-boundary and bulk-to-bulk propagators are

$$K_k(t) = \frac{f_k(t)}{f_k(t_*)} ,$$

$$G(k; t, t') = \underbrace{f_k^*(t) f_k(t')\, \theta(t - t') + f_k^*(t') f_k(t)\, \theta(t' - t)}_{= G_F(k; t, t')} - \frac{f_k^*(t_*)}{f_k(t_*)} f_k(t) f_k(t') .$$

1.4. Flat-Space Wavefunction

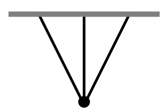
Consider a **massless scalar** Φ is Minkowski space.

- Evaluate correlators at $t_* \equiv 0$.
- The relevant propagators are

$$\begin{aligned} K_k(t) &= e^{ikt}\,, \\ G(k;t,t') &= \frac{1}{2k}\left(e^{-ik(t-t')}\,\theta(t-t') + e^{ik(t-t')}\,\theta(t'-t) - e^{ik(t+t')}\right). \end{aligned}$$

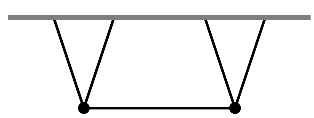
- The three- and four-point wavefunction coefficients in Φ^3 theory are

$$\langle O_1 O_2 O_3 \rangle \equiv$$



$$\begin{aligned} &= ig \int_{-\infty}^{0} \mathrm{d}t\, e^{i(k_1+k_2+k_3)t} \\ &= \frac{g}{(k_1+k_2+k_3)}\,. \end{aligned}$$

$$\langle O_1 O_2 O_3 O_4 \rangle \equiv$$



$$\begin{aligned} &= -g^2 \int_{-\infty}^{0} \mathrm{d}t \int_{-\infty}^{0} \mathrm{d}t'\, e^{ik_{12}t}\, G(k_I;t,t')\, e^{ik_{34}t'}\,, \quad k_{nm} \equiv k_n + k_m \\ &= \frac{g^2}{(k_{12}+k_{34})(k_{12}+k_I)(k_{34}+k_I)}\,. \end{aligned}$$

- More complicated tree graphs can be obtained recursively.
- In the lecture notes, we show that this reproduces the correct in-in correlators.

1.5. De Sitter Wavefunction

Consider a **massive scalar** Φ is de Sitter space.

- Evaluate correlators at $\eta_* \approx 0$.
- Equation of motion of $u \equiv a(\eta)\Phi$ is

$$u'' + \left(k^2 + \frac{m^2/H^2 - 2}{\eta^2} u^2\right) u = 0$$

- The general solution is

$$u_k(\eta) = \sqrt{\frac{\pi}{4}}(-\eta)^{1/2} H_\nu^{(2)}(-k\eta)\,, \quad \text{where} \quad \nu \equiv \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}\,.$$

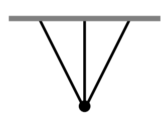
- For $m^2 = 2H^2$, the field is **conformally coupled** and

$$\boxed{u_k(\eta) = \frac{e^{ik\eta}}{\sqrt{2k}}} \quad \Rightarrow \quad \boxed{f_k(\eta) = \frac{u_k(\eta)}{a(\eta)} = (-H\eta)\frac{e^{ik\eta}}{\sqrt{2k}}}\,.$$

  The wavefunction can then be computed analytically.

- The three-point wavefunction coefficient in Φ^3 theory is

$$\langle O_1 O_2 O_3 \rangle \equiv$$



$$= \frac{ig}{f_{k_1}(\eta_*) f_{k_2}(\eta_*) f_{k_3}(\eta_*)} \int_{-\infty}^{\eta_*} \frac{\mathrm{d}\eta}{(H\eta)^4}\, f_{k_1}(\eta) f_{k_2}(\eta) f_{k_3}(\eta)$$

  For generic scalars, the result is Appell F_4.

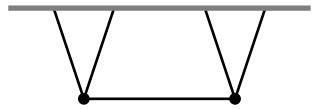
  For conformally coupled scalars, we get

$$\langle O_1 O_2 O_3 \rangle = \frac{ig}{H^4 \eta_*^3} \int_{-\infty}^{\eta_*} \frac{\mathrm{d}\eta}{\eta}\, e^{i(k_1+k_2+k_3)\eta} = \boxed{\frac{ig}{H^4\eta_*^3}\log(iK\eta_*)}\,,$$

  where $K \equiv k_1 + k_2 + k_3$.

- The four-point wavefunction coefficient (of conformally coupled scalars) is

$$\langle O_1 O_2 O_3 O_4 \rangle \equiv$$



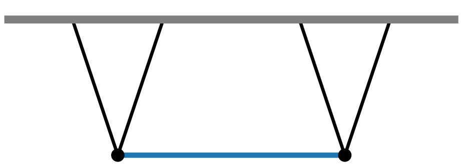
$$
\begin{aligned}
&= -\frac{g^2}{H^6 \eta_*^4} \int_{-\infty}^{0} \frac{\mathrm{d}\eta}{\eta} \int_{-\infty}^{0} \frac{\mathrm{d}\eta'}{\eta'} \, e^{ik_{12}\eta} \, G_{(\text{flat})}(k_I; \eta, \eta') \, e^{ik_{34}\eta'} \, , \\
&= -\frac{g^2}{H^6 \eta_*^4} \int_{k_{12}}^{\infty} \mathrm{d}x \int_{k_{34}}^{\infty} \mathrm{d}y \int_{-\infty}^{0} \mathrm{d}\eta \, \mathrm{d}\eta' \, e^{ix\eta} e^{iy\eta'} G_{(\text{flat})}(k_I; \eta, \eta') \\
&= \frac{1}{H^6 \eta_*^4} \int_{k_{12}}^{\infty} \mathrm{d}x \int_{k_{34}}^{\infty} \mathrm{d}y \, \langle O_1 O_2 O_3 O_4 \rangle_{(\text{flat})}(k_I; x, y) \\
&= \boxed{-\frac{g^2}{H^6 \eta_*^4} \int_{k_{12}}^{\infty} \mathrm{d}x \int_{k_{34}}^{\infty} \mathrm{d}y \, \frac{1}{(x+y)(x+k_I)(y+k_I)}} \, .
\end{aligned}
$$

This integral representation will be important in Lecture 3.

The result can be written in terms of logs and dilogs.

1.6. A Challenge

So far, we have only computed the correlators of conformally coupled scalars. Consider now the exchange of a generic massive scalar:



$$= -g^2 \int \mathrm{d}\eta \int \mathrm{d}\eta' \, \prod \text{Hankel functions}$$

In general, the time integrals cannot be performed analytically.
We need a different approach.
