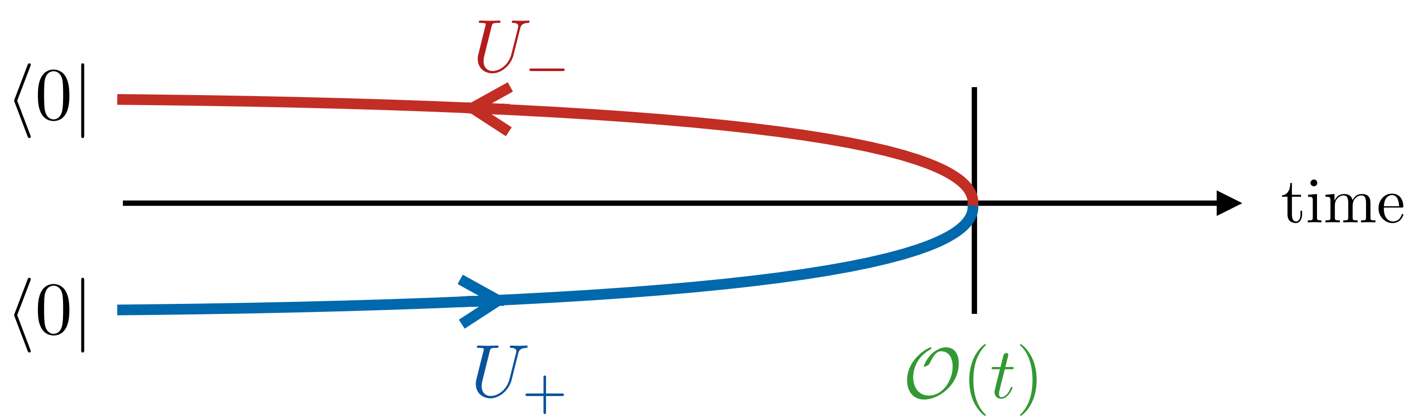
Wavefunction Approach

In-In Formalism

The main observables in cosmology are spatial correlation functions:

$$\langle\Omega|\,\phi(t,\mathbf{x}_1)\phi(t,\mathbf{x}_2)\cdots\phi(t,\mathbf{x}_N)\,|\Omega\rangle$$

These correlators are usually computed in the **in-in formalism**:



which leads to the following master equation

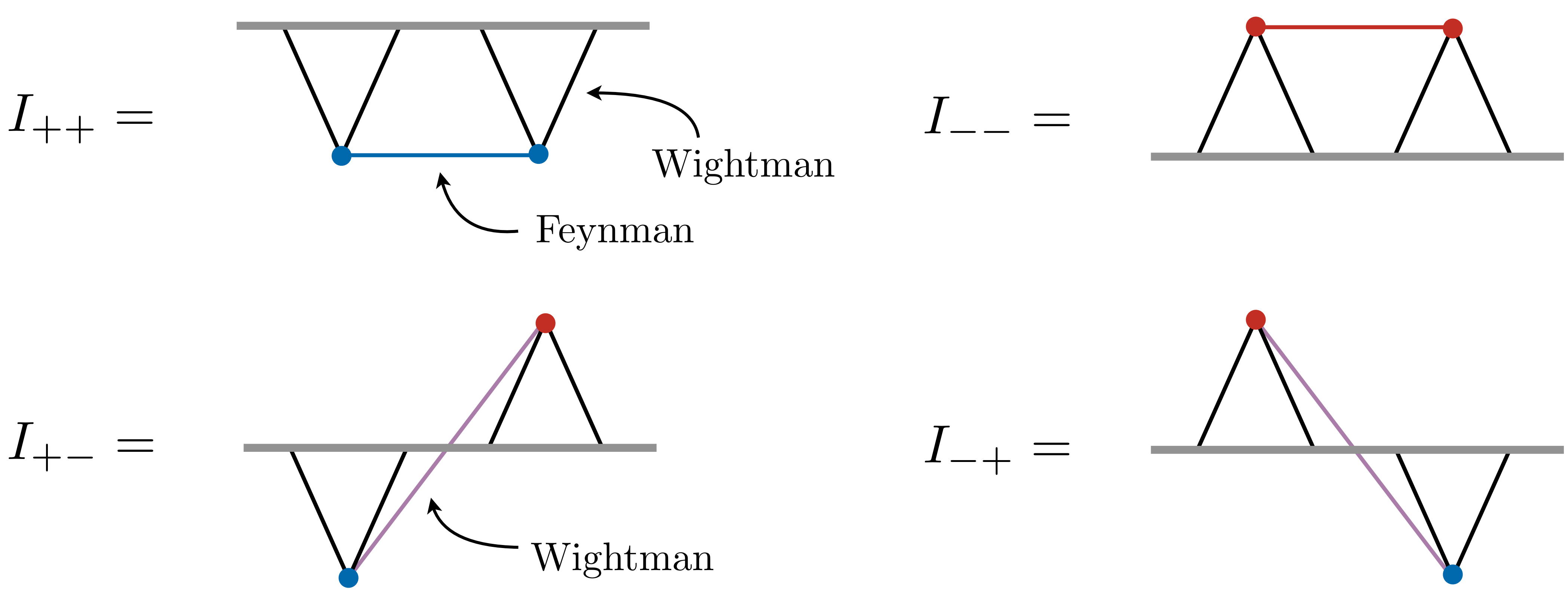
$$\boxed{\langle\mathcal{O}(t)\rangle = \langle 0|\,\bar{T}e^{i\int_{-\infty}^{t}\mathrm{d}t''\,H_{\rm int}(t'')}\,\mathcal{O}(t)\,Te^{-i\int_{-\infty}^{t}\mathrm{d}t'\,H_{\rm int}(t')}|0\rangle}$$

Weinberg [2005]

In-In Formalism

$$\langle \mathcal{O}(t) \rangle = \langle 0 | \, \bar{T} e^{i \int_{-\infty}^{t} \mathrm{d}t'' H_{\rm int}(t'')} \, \mathcal{O}(t) \, T e^{-i \int_{-\infty}^{t} \mathrm{d}t' H_{\rm int}(t')} | 0 \rangle$$

Diagrammatically, this can be written as

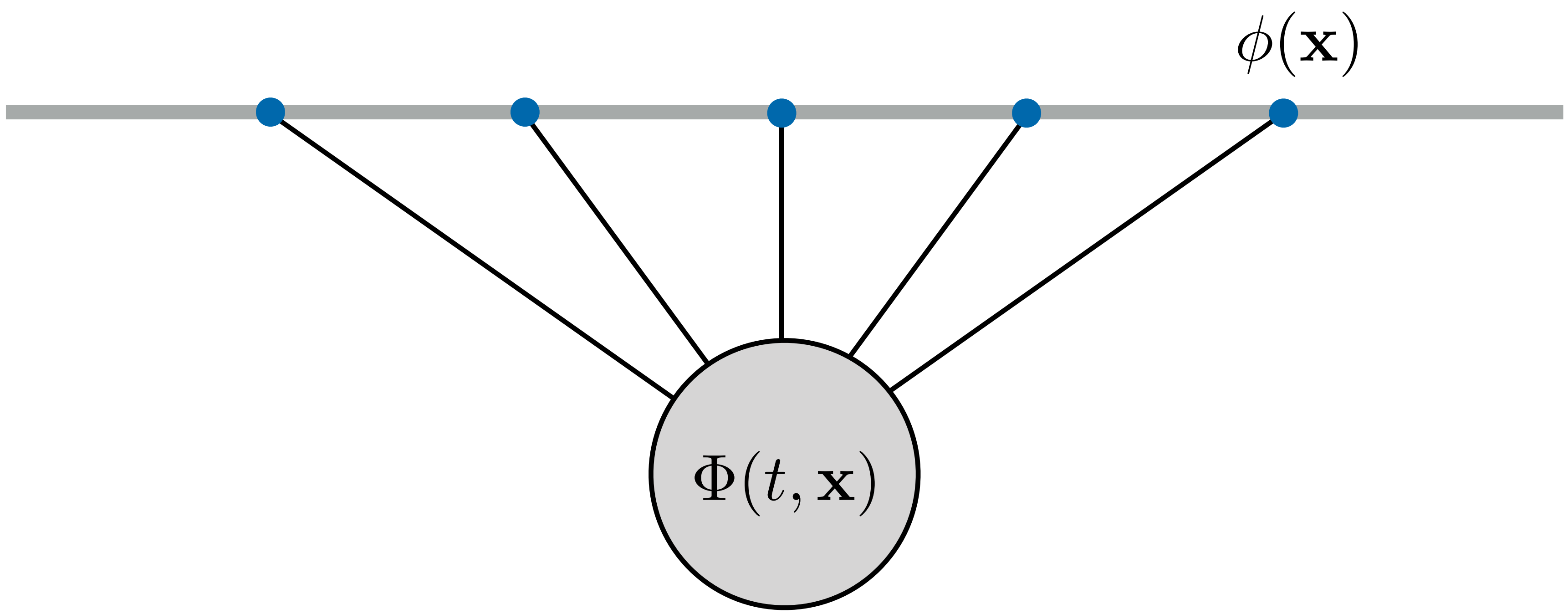


Giddings and Sloth [2010]

In the lecture notes, we apply this to many examples (see Section 3).
Here, we will instead follow the alternative wavefunction approach.

Wavefunction Approach

We first define a **wavefunction** for the late-time fluctuations:



$\Psi[\phi] \equiv \langle \phi | \Omega \rangle$

and then use it to compute the **boundary correlations**

$$\boxed{\langle \phi(\mathbf{x}_1) \ldots \phi(\mathbf{x}_N) \rangle = \int \mathcal{D}\phi \; \phi(\mathbf{x}_1) \ldots \phi(\mathbf{x}_N) \, |\Psi[\phi]|^2}$$

The wavefunction has nicer analytic properties than the correlation functions and it also plays a central role in the cosmological bootstrap.

OUTLINE:

Wavefunction of the Universe	Warmup in QM	Flat-Space Wavefunction	De Sitter Wavefunction

REFERENCE:

DB and Joyce, *Lectures on Cosmological Correlations*

Wavefunction of the Universe

Wavefunction Coefficients

For small fluctuations, we expand the wavefunction as

$$\Psi[\phi] = \exp\left(- \sum_N \int \frac{\mathrm{d}^3 k_1 \dots \mathrm{d}^3 k_N}{(2\pi)^{3N}} \, \Psi_N(\underline{\mathbf{k}}) \, \phi_{\mathbf{k}_1} \dots \phi_{\mathbf{k}_N} \right)$$

Wavefunction coefficient

- By translation invariance, we have

$$\Psi_N(\underline{\mathbf{k}}) = (2\pi)^3 \delta(\mathbf{k}_1 + \cdots + \mathbf{k}_N) \, \psi_N(\underline{\mathbf{k}}) \, .$$

- We will also write

$$\psi_N(\underline{\mathbf{k}}) = \langle O_1 \dots O_N \rangle' \, ,$$

where $O_a \equiv O_{\mathbf{k_a}}$ are **dual operators**.

This will just be notation and (so far) does not have a deeper holographic meaning (as in dS/CFT).

Relation to In-In Correlators

In perturbation theory, the correlation functions

$$\langle \phi_1 \dots \phi_N \rangle = \int \mathcal{D}\phi \; \phi_1 \dots \phi_N \, |\Psi[\phi]|^2$$

are related to the wavefunction coefficients:

$$\boxed{\begin{aligned} \langle \phi_1 \phi_2 \rangle &= \frac{1}{2\,\mathrm{Re}\langle O_1 O_2 \rangle} \\ \langle \phi_1 \phi_2 \phi_3 \rangle &= \frac{\mathrm{Re}\langle O_1 O_2 O_3 \rangle}{4 \prod_{a=1}^{3} \mathrm{Re}\langle O_a O_a \rangle} \\ \langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle &= \frac{\mathrm{Re}\langle O_1 O_2 O_3 O_4 \rangle}{8 \prod_{a=1}^{4} \mathrm{Re}\langle O_a O_a \rangle^4} + \frac{\langle O_1 O_2 X \rangle \langle X O_3 O_4 \rangle + \text{c.c.}}{8\,\mathrm{Re}\langle X X \rangle \prod_{a=1}^{4} \mathrm{Re}\langle O_a O_a \rangle^4} \end{aligned}}$$

Wavefunction coefficients therefore contain the same information, but are easier to compute and have nicer analytic properties.

Computing the Wavefunction

The wavefunction has the following path integral representation:

$$\Psi[\phi] = \int_{\substack{\Phi(0)=\phi \\ \Phi(-\infty)=0}} \mathcal{D}\Phi \, e^{i\underbrace{S[\Phi]}_{\text{Action}}}$$

Boundary conditions: $\Phi(t = -\infty(1-i\varepsilon)) = 0$
selects the Bunch-Davies vacuum.

For tree-level processes, we can evaluate this in a saddle-point approximation:

$$\Psi[\phi] \approx e^{iS[\Phi_{\rm cl}]}$$

Classical solution
with $\Phi_{\rm cl}(0) = \phi$ and $\Phi_{\rm cl}(-\infty) = 0$.

To find the wavefunction, we therefore need to find the classical solution for the bulk field with the correct boundary conditions.

Warmup in Quantum Mechanics

Harmonic Oscillator

Consider our old friend the **simple harmonic oscillator**

$$\boxed{S[\Phi] = \int \mathrm{d}t \left(\frac{1}{2}\dot{\Phi}^2 - \frac{1}{2}\omega^2\Phi^2 \right)}$$

- The classical solution (with the correct boundary conditions) is

$$\Phi_{\rm cl}(t) = \phi\, e^{i\omega t}$$

- The on-shell action becomes

$$S[\Phi_{\rm cl}] = \int_{t_i}^{t_*} \mathrm{d}t \left[\frac{1}{2}\partial_t(\dot{\Phi}_{\rm cl}\Phi_{\rm cl}) - \frac{1}{2}\Phi_{\rm cl}\underbrace{(\ddot{\Phi}_{\rm cl} + \omega^2\Phi_{\rm cl})}_{=0} \right]$$

$$= \frac{1}{2}\dot{\Phi}_{\rm cl}\Phi_{\rm cl}\Big|_{t=t_*}$$

$$= \frac{i\omega}{2}\phi^2$$

Harmonic Oscillator

- The wavefunction then is

Gaussian

$$\Psi[\phi] \approx \exp(iS[\Phi_{\rm cl}]) = \exp\left(-\frac{\omega}{2}\phi^2\right)$$

- The quantum variance of the oscillator therefore is

$$\boxed{\langle\phi^2\rangle = \frac{1}{2\omega}}$$

- In QFT, the same result applies to each Fourier mode

$$\langle\phi_{\mathbf{k}}\phi_{-\mathbf{k}}\rangle' = \frac{1}{2\omega_k}\ ,$$

where $\omega_k = \sqrt{k^2+m^2}$.

Time-Dependent Oscillator

To make this more interesting, consider a **time-dependent oscillator**

$$\boxed{S[\Phi] = \int \mathrm{d}t \left(\frac{1}{2} A(t) \dot{\Phi}^2 - \frac{1}{2} B(t) \Phi^2 \right)}$$

- The classical solution is

$$\Phi_{\mathrm{cl}}(t) = \phi K(t)\,, \quad \text{with} \quad \begin{array}{l} K(0) = 1 \\ K(-\infty) \sim e^{i\omega t} \end{array}$$

- The on-shell action becomes

$$\begin{aligned} S[\Phi_{\mathrm{cl}}] &= \int_{t_i}^{t_*} \mathrm{d}t \left[\frac{1}{2} \partial_t (A \dot{\Phi}_{\mathrm{cl}} \Phi_{\mathrm{cl}}) - \frac{1}{2} \Phi_{\mathrm{cl}} \underbrace{\left(\partial_t (A \dot{\Phi}_{\mathrm{cl}}) + B \Phi_{\mathrm{cl}} \right)}_{=0} \right] \\ &= \frac{1}{2} A \dot{\Phi}_{\mathrm{cl}} \Phi_{\mathrm{cl}} \Big|_{t=t_*} \\ &= \frac{1}{2} A \phi^2 \partial_t \log K \Big|_{t=t_*} \end{aligned}$$

Time-Dependent Oscillator

- The wavefunction then is

$$\Psi[\phi] \approx \exp(iS[\Phi_{\rm cl}]) = \exp\left(\frac{i}{2}(A\partial_t \log K)\Big|_* \phi^2\right),$$

which implies

$$|\Psi[\phi]|^2 = \exp\left(-\mathrm{Im}(A\partial_t \log K)\Big|_* \phi^2\right)$$

$$\Big\downarrow$$

$$\boxed{\langle\phi^2\rangle = \frac{1}{2\,\mathrm{Im}(A\partial_t \log K)\Big|_*}}$$

Free Field in de Sitter

The action of a **massless field in de Sitter** is that of a time-dependent oscillator:

$$
\begin{aligned}
S &= \int \mathrm{d}\eta\, \mathrm{d}^3x\, a^2(\eta)\Big[(\Phi')^2 - (\nabla\Phi)^2\Big] \\
&= \frac{1}{2}\int \mathrm{d}\eta\, \frac{\mathrm{d}^3k}{(2\pi)^3}\left[\frac{1}{(H\eta)^2}\Phi'_{\mathbf{k}}\Phi'_{-\mathbf{k}} - \frac{k^2}{(H\eta)^2}\Phi_{\mathbf{k}}\Phi_{-\mathbf{k}}\right]
\end{aligned}
$$

- The classical solution is

$$
\Phi_{\mathrm{cl}}(\eta) = \phi K(\eta)\,, \quad \text{with} \quad \begin{aligned} K(\eta) &= (1-ik\eta)e^{ik\eta} \\ \log K(\eta) &= \log(1-ik\eta) + ik\eta \end{aligned}
$$

and hence

$$
\begin{aligned}
\mathrm{Im}(A\partial_\eta \log K)\Big|_{\eta=\eta_*} &= \frac{1}{(H\eta_*)^2}\,\mathrm{Im}\left(\frac{-ik}{1-ik\eta_*} + ik\right) \\
&= \frac{1}{(H\eta_*)^2}\,\mathrm{Im}\left(\frac{k^2\eta_* + ik^3\eta_*^2}{1+k^2\eta_*^2}\right) \xrightarrow{\eta_*\to 0} \boxed{\frac{k^3}{H^2}}
\end{aligned}
$$

Free Field in de Sitter

- Using our result for the variance of a time-dependent oscillator, we then get

$$\boxed{\langle \phi_{\mathbf{k}} \phi_{-\mathbf{k}} \rangle' = \frac{H^2}{2k^3}}$$

which we derived in the last lecture using canonical quantization.

- The result for a massive field is derived in the lecture notes.

Anharmonic Oscillator

Consider the following **anharmonic oscillator**

$$\boxed{S[\Phi] = \int \mathrm{d}t \left(\frac{1}{2}\dot{\Phi}^2 - \frac{1}{2}\omega^2\Phi^2 - \frac{1}{3}g\Phi^3 \right)}$$

- The classical equation of motion is

$$\ddot{\Phi}_{\mathrm{cl}} + \omega^2\Phi_{\mathrm{cl}} = -g\Phi_{\mathrm{cl}}^2$$

- A formal solution is

$$\Phi_{\mathrm{cl}}(t) = \phi\, K(t) + i \int \mathrm{d}t'\, G(t,t')\big(-g\Phi_{\mathrm{cl}}^2(t')\big)$$

where

$$K(t) = e^{i\omega t}\,,$$

$$G(t,t') = \frac{1}{2\omega}\Big(\underbrace{e^{-i\omega(t-t')}\theta(t-t') + e^{i\omega(t-t')}\theta(t'-t)}_{\text{Feynman propagator}} \;-\; e^{i\omega(t+t')}\Big)\,.$$

Anharmonic Oscillator

- Deriving the on-shell action is now a bit more subtle:

$$S[\Phi_{\rm cl}] = \int_{t_i}^{t_*} {\rm d}t \left[\frac{1}{2}\partial_t(\Phi_{\rm cl}\dot{\Phi}_{\rm cl}) - \frac{1}{2}\Phi_{\rm cl}(\ddot{\Phi}_{\rm cl} + \omega^2\Phi_{\rm cl}) - \frac{g}{3}\,\Phi_{\rm cl}^3\right]$$
$$= \frac{1}{2}\Phi_{\rm cl}\dot{\Phi}_{\rm cl}\Big|_{t=t_*} + \int {\rm d}t\left[-\frac{1}{2}\Phi_{\rm cl}(\ddot{\Phi}_{\rm cl} + \omega^2\Phi_{\rm cl}) - \frac{g}{3}\,\Phi_{\rm cl}^3\right].$$

Since $\lim_{t\to 0}\partial_t G(t,t') = -ie^{i\omega t'} \neq 0$, the boundary term is

$$\frac{1}{2}\Phi_{\rm cl}\dot{\Phi}_{\rm cl}\Big|_{t=t_*} = \frac{1}{2}\phi\left(i\omega\phi - ig\int {\rm d}t'\,\big(-ie^{i\omega t'}\big)\,\Phi_{\rm cl}^2(t')\right)$$
$$= \frac{i\omega}{2}\phi^2 - \frac{g}{2}\phi\int {\rm d}t'\,e^{i\omega t'}\,\Phi_{\rm cl}^2(t')\,.$$

The action then becomes

$$S[\Phi_{\rm cl}] = \frac{i\omega}{2}\phi^2 - \frac{g}{2}\phi\int {\rm d}t\,e^{i\omega t}\,\Phi_{\rm cl}^2$$
$$+\int {\rm d}t\left[-\frac{1}{2}\left(\phi\, e^{i\omega t} - ig\int {\rm d}t'\,G(t,t')\,\Phi_{\rm cl}^2(t')\right)\left(-g\Phi_{\rm cl}^2(t)\right) - \frac{g}{3}\,\Phi_{\rm cl}^3\right].$$

The terms linear in ϕ cancel.

Anharmonic Oscillator

- The final on-shell action is

$$\boxed{S[\Phi_{\rm cl}] = \frac{i\omega}{2}\phi^2 - \frac{g}{3}\int {\rm d}t\, \Phi_{\rm cl}^3(t) - \frac{ig^2}{2}\int {\rm d}t\, {\rm d}t'\, G(t,t')\, \Phi_{\rm cl}^2(t')\, \Phi_{\rm cl}^2(t)}$$

- To evaluate this, we write the classical solution as $\Phi_{\rm cl}(t) = \sum_n g^n \Phi^{(n)}(t)$, where

$$\Phi^{(0)}(t) = \phi\, e^{i\omega t}\,,$$

$$\Phi^{(1)}(t) = i\int {\rm d}t'\, G(t,t')\left(-\phi^2 e^{2i\omega t'}\right) = \frac{\phi^2}{3\omega^2}\,\left(e^{2i\omega t} - e^{i\omega t}\right).$$

- With this, the wavefunction becomes

$$\boxed{\Psi[\phi] \approx e^{iS[\Phi_{\rm cl}]} = \exp\left(-\frac{\omega}{2}\phi^2 - \frac{g}{9\omega}\phi^3 + \frac{g^2}{72\omega^3}\phi^4 + \cdots\right)}$$

- From this, we can compute $\langle\phi^3\rangle$, $\langle\phi^4\rangle$, etc.

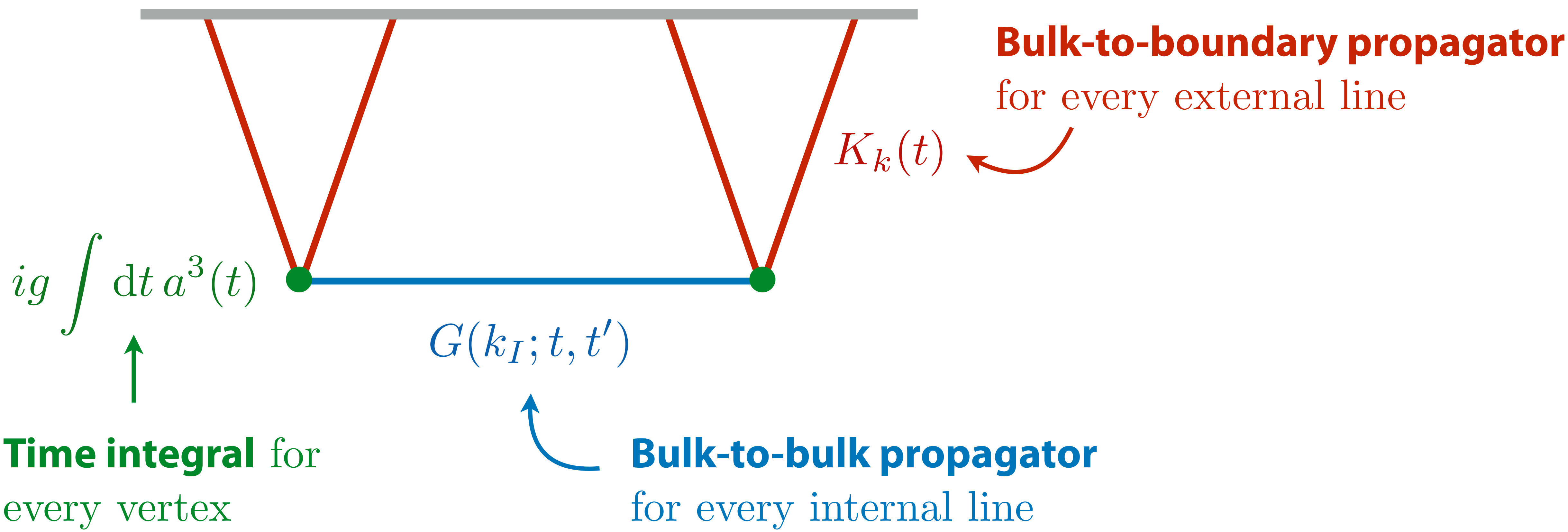
Interacting Fields

Back to field theory:

$$S[\Phi] = \int \mathrm{d}^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}m^2\Phi^2 - \frac{1}{3}g\Phi^3\right)$$

The analysis is similar to that of the anharmonic oscillator (see lecture notes).

In the interest of time, we jump directly to **Feynman rules** for WF coefficients:



Flat-Space Wavefunction

Flat-Space Wavefunction

The wavefunction is an interesting object even in flat space.

- We will study the following simple **toy model**

$$S = \int \mathrm{d}^4 x \left(-\frac{1}{2}(\partial\Phi)^2 - \frac{g}{3!}\Phi^3 \right) .$$

- All correlators can be evaluated at $t_* \equiv 0$ without loss of generality.

- The **propagators** are the same as for the harmonic oscillator:

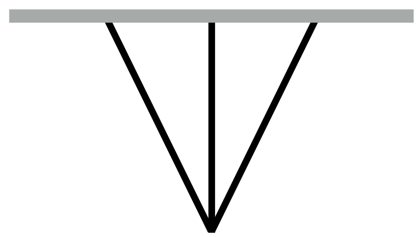
$$\boxed{\begin{aligned} K_k(t) &= e^{ikt} \\ G(k;t,t') &= \frac{1}{2k}\left(e^{-ik(t-t')}\theta(t-t') + e^{ik(t-t')}\theta(t'-t) \; - \; e^{ik(t+t')} \right) \end{aligned}}$$

We will compute the simplest tree-level correlators in this theory.

Contact Diagrams

The three-point wavefunction coefficient is

$$\langle O_1 O_2 O_3 \rangle =$$



$$\begin{aligned} &= ig \int_{-\infty}^{0} \mathrm{d}t\, K_{k_1}(t) K_{k_2}(t) K_{k_3}(t) \\ &= ig \int_{-\infty}^{0} \mathrm{d}t\, e^{i(k_1+k_2+k_3)t} \\ &= \frac{g}{(k_1+k_2+k_3)} \,. \end{aligned}$$

We will have more to say about this singularity.

This is easily generalized to N-point wavefunction coefficients:

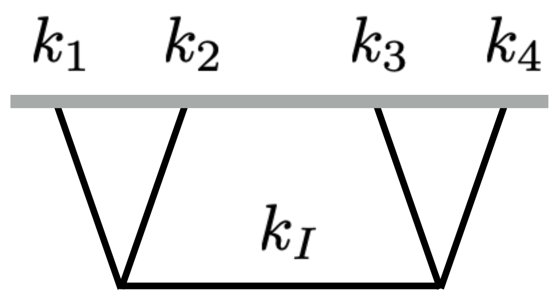
$$\langle O_1 O_2 \dots O_N \rangle = ig \int_{-\infty}^{0} \mathrm{d}t\, e^{i(k_1+k_2+\cdots+k_N)t} = \frac{g}{(k_1+k_2+\cdots+k_N)} \,.$$

Exchange Diagrams

The four-point wavefunction coefficient in Φ^3 theory is

$k_{12} \equiv k_1 + k_2$

$$\langle O_1 O_2 O_3 O_4 \rangle =$$



$$= -g^2 \int_{-\infty}^{0} \mathrm{d}t\, \mathrm{d}t' \; e^{ik_{12}t}\, G(k_I; t, t')\, e^{ik_{34}t'}$$

$$\begin{aligned}
= \; & -\frac{g^2}{2k_I} \int_{-\infty}^{0} \mathrm{d}t \int_{-\infty}^{t} \mathrm{d}t' \, e^{i(k_{12}-k_I)t} e^{i(k_{34}+k_I)t'} \\
& -\frac{g^2}{2k_I} \int_{-\infty}^{0} \mathrm{d}t \int_{-\infty}^{t} \mathrm{d}t' \, e^{i(k_{12}+k_I)t'} e^{i(k_{34}-k_I)t} \\
& +\frac{g^2}{2k_I} \int_{-\infty}^{0} \mathrm{d}t \int_{-\infty}^{0} \mathrm{d}t' \, e^{i(k_{12}+k_I)t} e^{i(k_{34}+k_I)t'}
\end{aligned}$$

$$= \frac{g^2}{2k_I} \left[\frac{1}{(k_{12}+k_{34})(k_{34}+k_I)} + \frac{1}{(k_{12}+k_{34})(k_{12}+k_I)} - \frac{1}{(k_{12}+k_I)(k_{34}+k_I)} \right]$$

$$= \frac{g^2}{(k_{12}+k_{34})(k_{12}+k_I)(k_{34}+k_I)}$$

Exchange Diagrams

The final answer is remarkably simple and has an interesting singularity structure:

$$\langle O_1 O_2 O_3 O_4 \rangle = \frac{g^2}{(k_{12} + k_{34})(k_{12} + k_I)(k_{34} + k_I)}$$

Total energy

Energy entering the left vertex

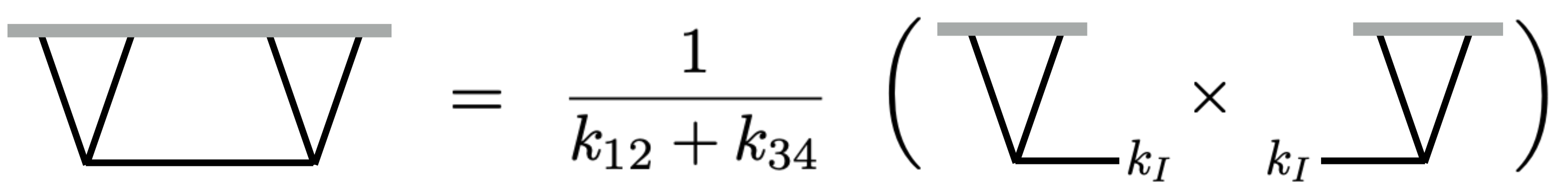
Energy entering the right vertex

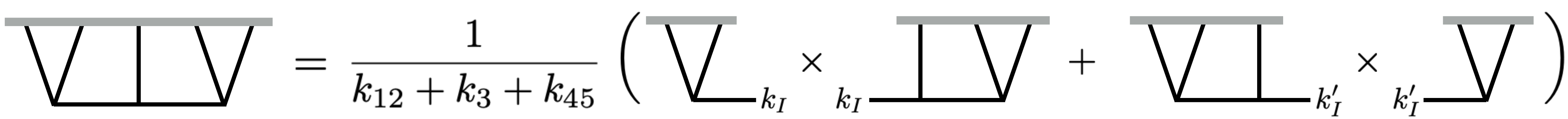
We will have more to say about this later.

Recursion Relation

More complicated graphs can be computed in the same way, but the complexity of the time integrals increases quickly with the number of internal lines.

Fortunately, there is a powerful **recursion relation** that allows complex graphs to be constructed from simpler building blocks:





This provides a simple algorithm to produce the result for any tree graph.

Arkani-Hamed, Benincasa and Postnikov [2017]

De Sitter Wavefunction

Free Field in de Sitter

Consider a **massive scalar** Φ in de Sitter space.

- The action of $u \equiv a(\eta)\Phi$ is

$$S = \frac{1}{2}\int \mathrm{d}\eta\, \mathrm{d}^3x \left[(u')^2 - (\nabla u)^2 - \frac{m^2/H^2 - 2}{\eta^2}u^2\right]$$

- The equation of motion is

$$\boxed{u'' + \left(k^2 + \frac{m^2/H^2 - 2}{\eta^2}\right)u = 0}$$

- The general solution of this equation is

$$u_k(\eta) = \sqrt{\frac{\pi}{4}}(-\eta)^{1/2}H_\nu^{(2)}(-k\eta)\,, \quad \text{where} \quad \nu \equiv \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}\,.$$

Hankel function

Free Field in de Sitter

Consider a **massive scalar** Φ in de Sitter space.

- The action of $u \equiv a(\eta)\Phi$ is

$$S = \frac{1}{2}\int \mathrm{d}\eta\,\mathrm{d}^3x\left[(u')^2 - (\nabla u)^2 - \frac{m^2/H^2 - 2}{\eta^2}u^2\right]$$

- The equation of motion is

$$\boxed{u'' + \left(k^2 + \frac{m^2/H^2 - 2}{\eta^2}\right)u = 0}$$

- For $m^2 = 2H^2$, the field is **conformally coupled** and

$$\boxed{u_k(\eta) = \frac{e^{ik\eta}}{\sqrt{2k}}} \longleftarrow \text{Harmonic oscillator}$$

We will often consider this special case, because the wavefunction can then be computed analytically.

Propagators

For a generic field, with mode function $f_k(\eta) \equiv u_k(\eta)/a(\eta)$, the propagators are

$$\boxed{\begin{aligned} K_k(\eta) &= \frac{f_k(\eta)}{f_k(\eta_*)} \\ G(k;\eta,\eta') &= f_k^*(\eta) f_k(\eta')\,\theta(\eta-\eta') + f_k^*(\eta') f_k(\eta)\,\theta(\eta'-\eta) - \frac{f_k^*(\eta_*)}{f_k(\eta_*)} f_k(\eta) f_k(\eta') \end{aligned}}$$

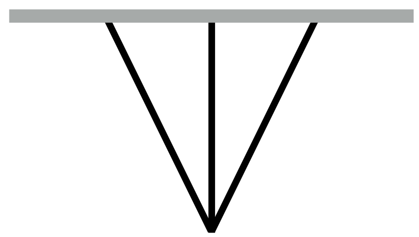
In general, these are given by Hankel functions which leads to complicated integrals in the wavefunction calculations.

Our explicit calculations will therefore be for conformally coupled scalars.
We return to the general case in the cosmological bootstrap.

Contact Diagrams

The **three-point WF coefficient** is

$$\langle O_1 O_2 O_3 \rangle =$$



$$= ig \int_{-\infty}^{0} \mathrm{d}\eta \, a^4(\eta) \, K_{k_1}(\eta) K_{k_2}(\eta) K_{k_3}(\eta)$$

$$= \frac{ig}{f_{k_1}(\eta_*) f_{k_2}(\eta_*) f_{k_3}(\eta_*)} \int_{-\infty}^{\eta_*} \frac{\mathrm{d}\eta}{(H\eta)^4} \, f_{k_1}(\eta) f_{k_2}(\eta) f_{k_3}(\eta)$$

Product of
Hankel functions

$$= \text{Appell } F_4$$

Contact Diagrams

For **conformally coupled scalars**, we get

$$\langle O_1 O_2 O_3 \rangle = \frac{ig}{H^4 \eta_*^3} \int_{-\infty}^{\eta_*} \frac{\mathrm{d}\eta}{\eta}\, e^{i(k_1+k_2+k_3)\eta} = \boxed{\frac{ig}{H^4 \eta_*^3} \log(iK\eta_*)}$$

where $K \equiv k_1 + k_2 + k_3$.

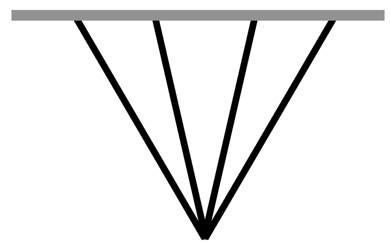
The corresponding **correlator** is

$$\begin{aligned} \langle \phi_1 \phi_2 \phi_3 \rangle &= -\frac{H^6 \eta_*^6}{8 k_1 k_2 k_3} \Big(\langle O_1 O_2 O_3 \rangle + \langle O_1 O_2 O_3 \rangle^* \Big) \\ &= -\frac{g}{8} \frac{H^2 \eta_*^3}{k_1 k_2 k_3} \Big(i \log(iK\eta_*) - i \log(-iK\eta_*) \Big) = \boxed{\frac{\pi}{8}\, g\, \frac{H^2 \eta_*^3}{k_1 k_2 k_3}} \end{aligned}$$

Contact Diagrams

The **four-point WF coefficient** for a contact interaction of conformally coupled scalars is

$$\langle O_1 O_2 O_3 O_4 \rangle =$$



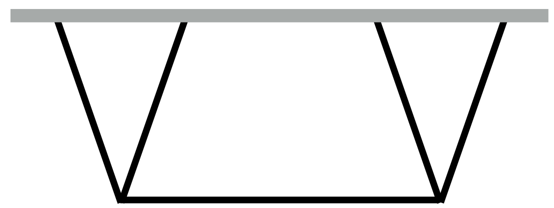
$$= \frac{ig}{H^4 \eta_*^4} \int_{-\infty}^{\eta_*} \mathrm{d}\eta \, e^{i(k_1+k_2+k_3+k_4)\eta} = \boxed{\frac{1}{H^4 \eta_*^4} \frac{g}{(k_1+k_2+k_3+k_4)}}$$

Since the Φ^4 interaction preserves the conformal symmetry of the free theory, this is the same result as in flat space (modulo the prefactor).

Exchange Diagrams

The four-point WF coefficient corresponding to an **exchange diagram** (of conformally coupled scalars) is

$$\langle O_1 O_2 O_3 O_4 \rangle =$$



$$= -\frac{g^2}{H^6 \eta_*^4} \int_{-\infty}^{0} \frac{\mathrm{d}\eta}{\eta} \int_{-\infty}^{0} \frac{\mathrm{d}\eta'}{\eta'} \, e^{ik_{12}\eta} e^{ik_{34}\eta'} \frac{1}{2k_I} \left[e^{-ik_I|\eta-\eta'|} - e^{ik_I(\eta+\eta')} \right]$$

$$= -\frac{g^2}{H^6 \eta_*^4} \int_{k_{12}}^{\infty} \mathrm{d}x \int_{k_{34}}^{\infty} \mathrm{d}y \int_{-\infty}^{0} \mathrm{d}\eta \, \mathrm{d}\eta' \, e^{ix\eta} e^{iy\eta'} \frac{1}{2k_I} \left[e^{-ik_I|\eta-\eta'|} - e^{ik_I(\eta+\eta')} \right]$$

$$= \frac{1}{H^6 \eta_*^4} \int_{k_{12}}^{\infty} \mathrm{d}x \int_{k_{34}}^{\infty} \mathrm{d}y \, \langle O_1 O_2 O_3 O_4 \rangle_{(\text{flat})}(x, y, k_I) \,,$$

$$= \boxed{ -\frac{g^2}{H^6 \eta_*^4} \int_{k_{12}}^{\infty} \mathrm{d}x \int_{k_{34}}^{\infty} \mathrm{d}y \, \frac{1}{(x+y)(x+k_I)(y+k_I)} }$$

Exchange Diagrams

We see that this four-point function can be written as an integral of the flat-space result:

$$\langle O_1 O_2 O_3 O_4 \rangle = -\frac{g^2}{H^6 \eta_*^4} \int_{k_{12}}^{\infty} \mathrm{d}x \int_{k_{34}}^{\infty} \mathrm{d}y \, \frac{1}{(x+y)(x+k_I)(y+k_I)}$$

This integral can be performed analytically to give

$$\boxed{\begin{aligned} \langle O_1 O_2 O_3 O_4 \rangle = \frac{g^2}{2H^6 \eta_*^4} \frac{1}{k_I} \Bigg[&\mathrm{Li}_2\left(\frac{k_{12}-k_I}{E}\right) + \mathrm{Li}_2\left(\frac{k_{34}-k_I}{E}\right) \\ &+ \log\left(\frac{k_{12}+k_I}{E}\right) \log\left(\frac{k_{34}+k_I}{E}\right) - \frac{\pi^2}{6} \Bigg] \end{aligned}}$$

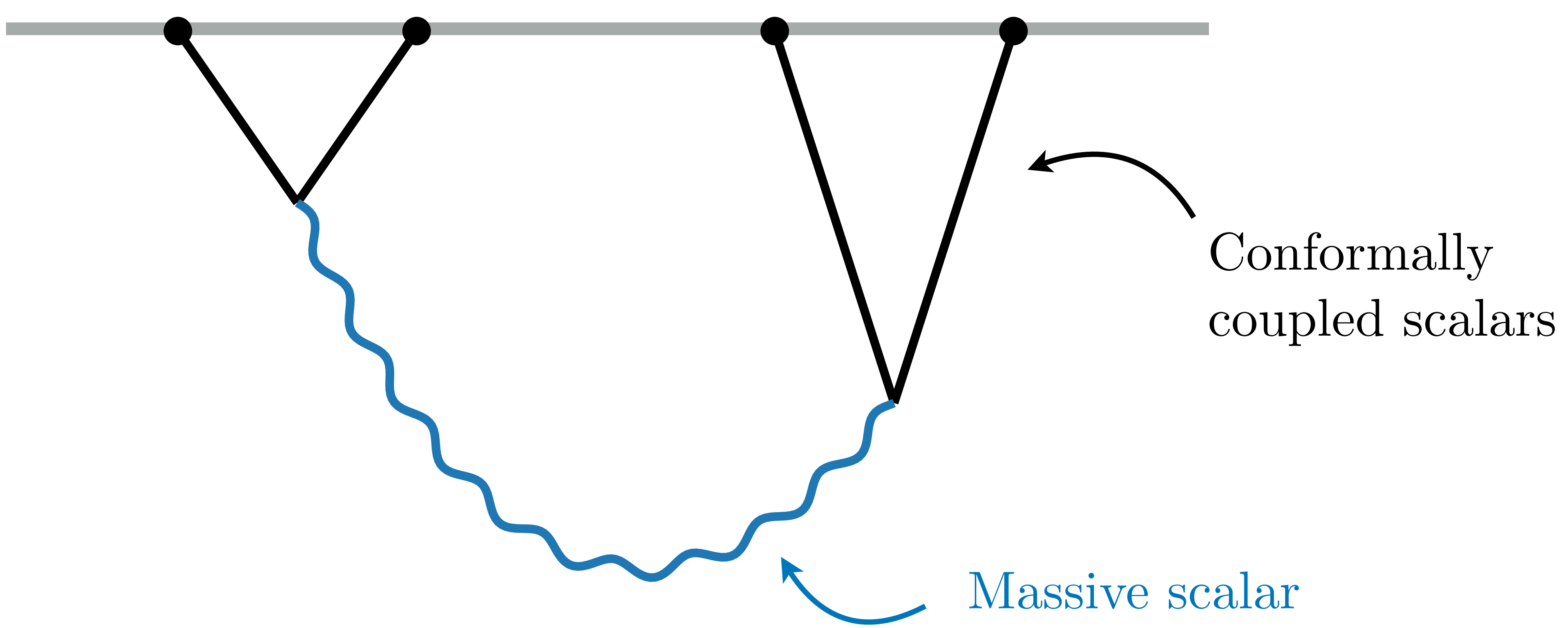
where $E \equiv k_1 + k_2 + k_3 + k_4$ and Li_2 is the dilogarithm.

Arkani-Hamed and Maldacena [2015]

A Challenge

So far, we have only computed the correlators of conformally coupled scalars.
Consider now the **exchange of a generic massive scalar**:

$\langle O_1 O_2 O_3 O_4 \rangle =$



$$= \; -g^2 \int \frac{\mathrm{d}\eta}{\eta^2} \int \frac{\mathrm{d}\eta'}{\eta'^2}\, e^{ik_{12}\eta} e^{ik_{34}\eta'}\, G(k_I; \eta, \eta')$$

In general, the time integrals cannot be performed analytically.
We need a different approach.