Discovering Twistors in Cosmology

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In these notes, we give an elementary (and hopefully pedagogical) introduction to our work on twistors in cosmology. The presentation is based on a recent paper with Grégoire Mathys and Guilherme Pimentel [1], but adds further background and explanations.

Contents

1 Motivation

1.1 Why Twistors in Cosmology?

In 1986, Parke and Taylor discovered a remarkable formula describing the scattering of n gluons in the maximally helicity violating configuration [2]:

$$A(1^-2^-3^+\cdots n^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle (n-1)n\rangle\langle n1\rangle}\,, \tag{1.1}$$

where the brackets $\langle ij\rangle$ represent the momenta of the external particles in terms of spinor helicity variables. At the time, this result was a stunning simplification of a very complex computation using Feynman diagrams, but subsequently it was shown that the Parke–Taylor formula can be bootstrapped from basic physical consistency requirements, avoiding the crippling complexity of Feynman diagrams. This started a revolution in our understanding of scattering amplitudes, with new on-shell methods allowing more direct computations of amplitudes, which then exposed hidden symmetries [3] and new geometrical objects [4] underlying the physics of particle scattering.

Cosmology has not yet had its Parke–Taylor moment, meaning that the analog of a formula like (1.1) has not been discovered. Part of the problem is that, in the cosmological context, we don't have kinematic variables that describe the sum of Feynman diagrams as a unified physical object. Instead, cosmological correlators are still described in terms of their separation into Feynman diagrams, even for gauge theory and gravity where the individual diagrams themselves aren't even physical. This is against the spirit of the on-shell description of scattering amplitudes, so it isn't surprising that explicit results for gluon and graviton correlators in (A)dS are very complicated (see e.g. [5–8]).

The reason for this surprising complexity of spinning correlators lies in an inconvenient choice of kinematic variables. We will show that formulating these correlators using the language of twistors exposes their hidden simplicity.

1.2 What Are the Right Variables?

To describe massless particles with spin, the "right" variables should make two important properties manifest:

1. *Spacetime symmetries*

   Physical observables should be invariant under the relevant spacetime symmetries and the right variables would make this manifest. For example, scattering amplitudes in asymptotically Minkowski space should satisfy Lorentz symmetry (e.g. a four-particle amplitude of scalars is a function of the Mandelstam variables s, t, u).

   In inflationary cosmology, we are interested in the correlations of fluctuations living on the future boundary of an approximate de Sitter spacetime. If the bulk interactions aren't too strong, these correlations inherit a conformal symmetry from the isometries of the spacetime. The right variables should make this conformal symmetry of the boundary correlators manifest. The same variables may also be useful to describe conformal field theories (CFTs) more generally.

2. *Degrees of freedom*
   Massless particles with spin contain *two* physical degrees of freedom called *helicities*.[1] In contrast, a massive spin-S particle has additional longitudinal polarizations for a total of $2S+1$ degrees of freedom. The challenge often is to find a proper kinematic description that propagates the right number of degrees of freedom. Recall that in quantum field theory, particles are excitations of fields. These fields typically have additional (unphysical) degrees of freedom. For example, the two degrees of freedom of the massless photon are embedded in the four components of the vector potential A^μ. To remove the unwanted extra degrees of freedom, we invent a "gauge symmetry," i.e. an equivalence class $A_\mu \sim A_\mu + \partial_\mu \chi$. Together with a constraint coming from the equation of motion,[2] this projects out the two extra modes. It would be convenient to have a kinematic description that focuses directly on the physical degrees of freedom without introducing anything extra that then has to be removed by a gauge symmetry.

For scattering amplitudes, the right variables that both make Lorentz symmetry manifest and directly describe the two physical helicities of massless spin-S particles are the so-called *spinor helicity variables*. We will introduce them below. We will then explain that the analog of such variables does not exist in cosmology (or, at least, the currently used spinor helicity variables are not as powerful as their flat-space counterparts). Finally, we will show that twistors are the right variables to parameterize massless correlators in (Anti-)de Sitter spacetimes (or, equivalently, CFT correlators of conserved currents).

Massless amplitudes

For scattering amplitudes of massless particles, we have spinor helicity variables as a powerful way to describe the on-shell kinematics. In the following, we briefly review these variables. Experts are invited to skip this part.

The kinematics of a scattering process is described by the four-momenta, $p^\mu = (p^0, \vec{p})$, and the polarization vectors, $e^\mu = (e^0, \vec{e})$, of the particles. For massless particles, it will be convenient to arrange the components of the four-momenta into a two-by-two matrix

$$p_{\alpha\dot{\alpha}} = p_\mu (\sigma^\mu)_{\alpha\dot{\alpha}} = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix}, \tag{1.2}$$

where $\sigma^\mu = (1, \vec{\sigma})$ are the Pauli matrices. The determinant of the above matrix vanishes on-shell, $\det(p) = -p^\mu p_\mu = 0$. The matrix is therefore of rank one and can be written as an outer product of two *spinors*

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \,. \tag{1.3}$$

For real momenta, the matrix $p_{\alpha\dot{\alpha}}$ is Hermitian, implying that $\tilde{\lambda}_{\dot{\alpha}} = \pm(\lambda^*)_{\dot{\alpha}}$, for positive and negative energy states, respectively. It is also often convenient to allow the momenta to be complex, so that λ_α and $\tilde{\lambda}_{\dot{\alpha}}$ are independent spinors.

[1]The helicity of a particle is the projection of its spin onto the direction of momentum.
[2]The equation of motion itself can be derived from the unique ghost-free action of a spin-1 field.

Notice that (1.3) is invariant under the *little group* transformation $\lambda_\alpha \mapsto t\lambda_\alpha$ and $\tilde{\lambda}_{\dot{\alpha}} \mapsto t^{-1}\tilde{\lambda}_{\dot{\alpha}}$. The polarization vectors e^μ, on the other hand, do transform under the little group rescaling, which leads to nontrivial constraints on fixed-helicity amplitudes, $A(1^{h_1}\cdots N^{h_N}) = e^{h_1}_{\mu_1}\cdots e^{h_N}_{\mu_N} A^{\mu_1\cdots\mu_N}$, where $h_i = \pm 1$ are the helicities of each particle.

Given two particles i and j, we can define the following Lorentz-invariant "angle" and "square" brackets:

$$\begin{aligned}\langle ij\rangle &\equiv \epsilon^{\beta\alpha}\lambda_{i,\alpha}\lambda_{j,\beta}\,,\\ [ij] &\equiv \epsilon^{\dot{\beta}\dot{\alpha}}\lambda_{i,\dot{\alpha}}\tilde{\lambda}_{j,\dot{\beta}}\,,\end{aligned} \tag{1.4}$$

where $\epsilon^{\alpha\beta}$ and $\epsilon^{\dot{\alpha}\dot{\beta}}$ are the anti-symmetric Levi-Civita symbols. Note that these brackets are anti-symmetric under the exchange of the particles, $\langle ij\rangle = -\langle ji\rangle$ and $[ij] = -[ji]$, which implies that $\langle ii\rangle = [ii] = 0$. The familiar Mandelstam variables can be expressed in terms of the spinor brackets as

$$s_{ij} \equiv (p_i + p_j)^2 = 2p_i \cdot p_j = \langle ij\rangle[ij]\,. \tag{1.5}$$

In fact, the kinematic data of all massless amplitudes can be written in terms of only these brackets and the scattering amplitudes for massless particles simplify dramatically when written in terms of these spinor helicity variables. An example is given in (1.1).

The use of spinor brackets makes the Lorentz symmetry of the amplitudes manifest. It also describes directly the physical fixed-helicity amplitudes $A(1^{h_1}\cdots n^{h_n})$—rather than the unphysical tensorial amplitudes $A^{\mu_1\cdots\mu_n}$—and hence focuses directly on the relevant degrees of freedom.

Massless correlators

Our interest is in the correlators of massless fields evaluated on the spacelike boundary of de Sitter space (i.e. the time slice at future infinity). Above we explained that massless particles imply a gauge symmetry for the bulk fields. On the boundary, this is reflected in the existence of a conserved current:[3]

$$\partial_{i_1} J^{i_1\ldots i_S} = 0\,, \tag{1.6}$$

where $J^{i_1\cdots i_S}$ is a symmetric rank-S tensor and $\partial_i \equiv \partial/\partial x^i$ is the partial derivative with respect to the boundary coordinates. We will be interested in the correlators of these conserved currents (which are equivalent to the wavefunction coefficients for the boundary value of the gauge field). For example, the three-point function of a spin-1 current is

$$\langle J^i(\vec{x}_1)J^j(\vec{x}_2)J^k(\vec{x}_3)\rangle \equiv \psi^{ijk}(\vec{x}_1,\vec{x}_2,\vec{x}_3)\,. \tag{1.7}$$

Higher-point functions and correlators of higher-spin currents are obvious generalizations of this. In de Sitter space, the functional form of these correlators must satisfy the following two constraints:

[3]Consider, for example, a massless spin-1 field A^μ. On the boundary, it couples to a current J^i via the term $\int d^dx\, J_i A^i$ in the action. In order for the action to be invariant under a gauge transformation, $A^\mu \to A^\mu + \partial^\mu\chi$, the current must be conserved, $\partial_i J^i = 0$, so that $J_iA^i \to J_i(A^i + \partial^i\chi) = J_iA^i - (\partial^iJ_i)\chi = J_iA^i$ after integrating by parts.

1. *Conformal symmetry*
   If the interactions aren't too strong, the boundary correlators inherit a conformal symmetry from the isometries of the bulk spacetime. Mathematically, this is expressed by the following *Ward identities* (illustrated here for the three-point function of a spin-1 current):

$$
\begin{aligned}
&\sum_{a=1}^{3}\left(2+x_a^m\frac{\partial}{\partial x_a^m}\right)\psi^{ijk}(\vec{x}_1,\vec{x}_2,\vec{x}_3)=0\,,\\
&\sum_{a=1}^{3}b^l\left[2x_{a,l}\left(1+x_a^m\frac{\partial}{\partial x_a^m}\right)-x_a^2\frac{\partial}{\partial x_a^l}+\Sigma_{lm}^{(a)}x_a^m\right]\psi^{ijk}(\vec{x}_1,\vec{x}_2,\vec{x}_3)=0\,,
\end{aligned}
\tag{1.8}
$$

   where $\Sigma_{lm}^{(a)}$ are generators of spatial rotations that act on the index i, j, k of ψ^{ijk} for $a=1$, 2, 3, respectively. These two equations describe the invariance under dilatations and special conformal transformations, respectively.

2. *Current conservation*
   The constraint (1.6) implies the *Ward–Takahashi identity* for the correlators. For the case of a spin-1 current, this reads

$$
\frac{\partial}{\partial x_1^i}\psi^{ijk}(\vec{x}_1,\vec{x}_2,\vec{x}_3)=0 \quad \text{for} \quad \vec{x}_1\neq\vec{x}_{2,3}\,. \tag{1.9}
$$

   For coincident points, $\vec{x}_1=\vec{x}_{2,3}$, this constraint has a nontrivial right-hand side.

The right variables should make the properties in (1.8) and (1.9) manifest. However, the variables that are typically used in cosmology (and in conformal field theory) do not have that feature.

Conformal field theories are often expressed in the so-called *embedding space* (introduced by Dirac [9] in 1936). This starts from the observation that the conformal algebra on $\mathbb{R}^d$ is isomorphic to the algebra of Lorentz transformations on $\mathbb{R}^{1,d+1}$. By defining a suitable embedding of $\mathbb{R}^d$ into $\mathbb{R}^{1,d+1}$, the d-dimensional conformal transformations (which are complicated) can be uplifted to $(d+2)$-dimensional Lorentz transformations (which are simple). Going to embedding space therefore helps to find solutions to the conformal Ward identifies like (1.8).

In Section 2, we will provide a review of the embedding-space formalism. Here, we will just give a brief sketch of the general idea. Let P^M, with $M=0,1,\cdots,d+1$, be the coordinates on the higher-dimensional space. We also define a *projective null cone* as

$$
P^2=0\,, \tag{1.10}
$$

$$
P^M\sim\rho P^M\,, \tag{1.11}
$$

where ρ is a rescaling parameter. In addition, the *Euclidean section* is defined by the constraint

$$
(P^+,P^-,P^i)=(1,|\vec{x}|^2,x^i)\,, \tag{1.12}
$$

where $P^\pm\equiv P^0\pm P^{d+1}$ are light-cone coordinates and x^i, with $i=1,\dots,d$, are the coordinates on $\mathbb{R}^d$. Equation (1.12) defines an embedding of the spatial coordinate x^i in the projective null cone of the higher-dimensional space. The embedding function is chosen in such a way the

Lorentz transformations in the ambient space become conformal transformations on the Euclidean section [10]. Defining conformally-invariant correlators then becomes very easy. We simply write all Lorentz-invariant inner products of the embedding-space positions,[4] $P_i \cdot P_j$ and combine them in such a way that the resulting correlator scales as expected when the positions of each field are rescaled. We describe this in more detail in Section 2.

However, while conformal symmetry becomes much simpler to enforce in embedding space, current conservation is now harder to implement [11]. In practice, one first has to define the space of conformally-invariant structures and then impose a differential constraint like (1.9) to select the sub-space of conserved correlators. This two-step procedure is conceptually unsatisfying. It would be much nicer if we could write down an expression for the correlator that from the start makes both conformal symmetry and current conservation manifest. In [1], we showed that this can be achieved in *twistor space*.

1.3 A (Simple) Road Towards Twistors

For our applications to cosmology, we will be interested in CFTs in $d = 3$ dimensions. Although, in these real world applications, the CFTs are defined in $\mathbb{R}^3$ (Euclidean), we will also consider the analytic continuation to $\mathbb{R}^{1,2}$ (Lorentzian). In that case, we write the boundary coordinates as x^μ (instead of x^i). In the following, we will list the steps that naturally leads us to describe the correlators of conserved currents in these CFTs in twistor space.

1. *Spinor variables*

   Since the embedding-space positions P^M satisfy $P^2 = 0$, it is natural to write them in terms of spinors as

$$\not{P}^{AB} = P^M(\Gamma_M)^{AB} = \begin{pmatrix} 0 & -P^3 - P^4 & P^2 & P^0 + P^1 \\ P^3 + P^4 & 0 & P^1 - P^0 & -P^2 \\ -P^2 & P^0 - P^1 & 0 & P^4 - P^3 \\ -P^0 - P^1 & P^2 & P^3 - P^4 & 0 \end{pmatrix} = \epsilon^{ab}\Lambda_a^A \Lambda_b^B \,, \quad (1.13)$$

   where $(\Gamma_M)^{AB}$ are 4×4 gamma matrices and Λ_a^A, with $A = 1, 2, 3, 4$ and $a = 1, 2$, are two four-component spinors.

2. *Twistor coordinates*

   We then define a linear combination of the embedding-space spinors

$$Z^A = \pi^a \Lambda_a^A \,, \quad (1.14)$$

   where $\pi^a \equiv (\pi^1, \pi^2)$ are two real parameters. The variables Z^A are coordinates in *twistor space*. As we will explain, conserved currents are functions $F(Z^A)$ that are *holomorphic* in the twistor coordinates, i.e. they depend on the two embedding-space spinors Λ_a^A only through the combination (1.14).

[4]For fields with spin, we also have polarization vectors W^M, which leads to additional inner products $P_i \cdot W_j$ and $W_i \cdot W_j$.

3. *Currents as twistor integrals*

   The twistor coordinates in (1.14) depend on the arbitrary parameters π^a. We erase the arbitrariness in the choice of these parameters by integrating over them, using the projective measure $D\pi \equiv d\pi^b \pi^c \epsilon_{bc}$. For a spin-$S$ current, this leads to the following twistor integral

   $$J^{a_1 a_2 \cdots a_{2S}}(\Lambda_a) = \int D\pi \, \pi^{a_1} \pi^{a_2} \cdots \pi^{a_{2S}} F(Z) \,, \tag{1.15}$$

   where we represented the current as a symmetric tensor $J^{a_1 a_2 \cdots a_{2S}}(\Lambda_a)$, with $2S$ little group indices. In order for this integral to be well-defined projectively, the function $F(Z)$ must be a homogeneous function of degree $-2(S+1)$, i.e. $F(rZ) = r^{-2(S+1)} F(Z)$.

4. *Correlators in twistor space*

   We can make conservation manifest for each current in a n-point correlator by considering a function $F(Z_i) = F(Z_1, \cdots, Z_n)$ that depends on each of the embedding-space spinors $\Lambda^A_{i,a}$ only through the combinations $Z^A_i \equiv \pi^a_i \Lambda^A_{i,a}$. Conformal symmetry in the 3d position space is automatically satisfied if the integrand depends on the twistors only through their inner products, $F(Z_i \cdot Z_j)$, and we can bootstrap this function by imposing how it scales with respect to each twistor Z^A_i. Finally, to make conservation manifest, we write an integral like (1.15) for each field in the n-point function.

   As a simple illustration of the power of these ideas, consider the three-point function of gravitons (or stress tensors) in twistor space:

   $$\langle T(Z_1) T(Z_2) T(Z_3) \rangle^{\pm} = \int \mathrm{d}c_{ij} \, \exp\left(i \sum_{i,j} c_{ij} Z_i \cdot Z_j \right) (c_{12} c_{23} c_{31})^{\pm 2} \,, \tag{1.16}$$

   where Z_i are the twistor coordinates and c_{ij} are Schwinger parameters. The choice of $-$ or $+$ in the exponent corresponds to Einstein gravity and higher-derivative R^3 interactions, respectively. Note that the Schwinger-parameterized correlator, $A^{\pm}(c_{ij}) = (c_{12} c_{23} c_{31})^{\pm 2}$, is remarkably simple.

In the rest of these notes, we will go through these steps in more details.

2 Embedding Space

Conformal transformations are complicated nonlinear transformations and become particularly involved for spinning operators. At the same time, it is easy to show that the conformal algebra on $\mathbb{R}^d$ is isomorphic to the algebra of Lorentz transformations on $\mathbb{R}^{1,d+1}$. This suggests that a suitable embedding of $\mathbb{R}^d$ into $\mathbb{R}^{1,d+1}$, should make conformal transformations as simple as Lorentz transformations. The embedding-space formalism of conformal field theory goes back to Dirac [9], and has found many powerful applications in the recent CFT literature, e.g. [12, 13].

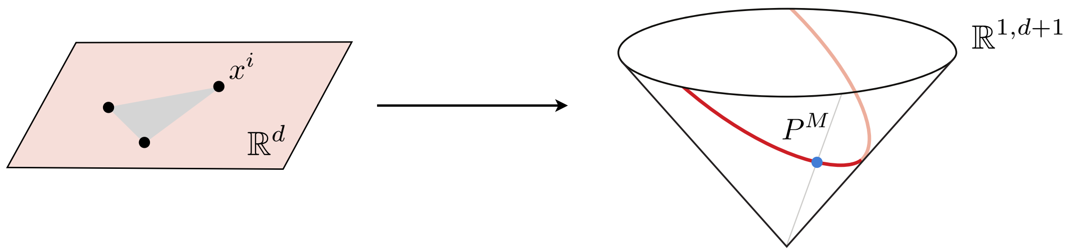

Figure 1: In the embedding-space formalism, we map points on $\mathbb{R}^d$ to a section of a projective null cone in $\mathbb{R}^{1,d+1}$. Lorentz transformations in the ambient space become conformal transformations on section.

2.1 Conformal Symmetry Simplified

In this section, we provide a pedagogical review of CFTs in embedding space, based mostly on the excellent treatment in [10]. Readers who are familiar with the embedding-space formalism (or are satisfied with the brief sketch that we gave above), may skip this part and jump directly to Section 2.3 where we introduce spinors for the embedding-space coordinates.

Projective null cone We begin by describing the embedding of d-dimensional Euclidean space as a slice through a higher-dimensional light-cone. Consider $d+2$ dimensional Minkowski space, with coordinates

$$P^M\,,\ M = 0, 1, \ldots, d+1\,. \tag{2.1}$$

The goal is to find an embedding of $\mathbb{R}^d$ into $\mathbb{R}^{1,d+1}$ on which the Lorentz transformations, $P^M \to L^M{}_N P^N$, become conformal transformations. We first restrict ourselves to points living on the *projective null cone* in the embedding space:

$$P^2 = 0\,, \tag{2.2}$$
$$P^M \sim \rho P^M\,, \tag{2.3}$$

The condition (2.2) is Lorentz invariant and removes one of the coordinates in (2.1). To remove a second coordinate and obtain a d-dimensional subspace, we define a *section* of the light-cone $P^+ \equiv f(P^i)$, where $P^\pm \equiv P^0 \pm P^{d+1}$ are light-cone coordinates and the coordinates P^i are identified with the coordinates x^i on $\mathbb{R}^d$.

Next, we specialize to the case where $f(P^i) = 1$. It can then be shown [10], that Lorentz transformations on the embedding space—combined with the projective equivalence (2.3)—imply conformal transformations on the *Euclidean section*:

$$P^M = (P^+, P^-, P^i) = (1, |\vec{x}|^2, x^i)\,. \tag{2.4}$$

As we will show next, correlators in the d-dimensional Euclidean space are lifted to homogeneous functions on the light-cone of the $(d+2)$-dimensional Minkowski spacetime, where the conformal group acts as the Lorentz group.

Tensors in embedding space Consider a symmetric, traceless and transverse tensor $O_{M_1\ldots M_S}$ defined on the cone $P^2 = 0$. Under the rescaling $P \to \rho P$, the tensor transforms as

$$O_{M_1\ldots M_S}(\rho P) = \rho^{-\Delta} O_{M_1\ldots M_S}(P)\,, \tag{2.5}$$

i.e. it is a homogeneous function of degree $-\Delta$. This implies that the tensor is known everywhere on the cone if it is known on the section (2.4). The corresponding tensor on $\mathbb{R}^d$ is then defined through the following projection

$$O_{i_1 i_2\ldots}(x) = O_{M_1 M_2\ldots}(P)\,\frac{\partial P^{M_1}}{\partial x^{i_1}}\frac{\partial P^{M_2}}{\partial x^{i_2}}\cdots\,, \qquad \frac{\partial P^M}{\partial x^i} = (0, 2x_i, \delta_i^j)\,. \tag{2.6}$$

It is straightforward to show that the scaling transformation (2.5) for $O_{M_1 M_2\ldots}(P)$ implies a conformal transformation for $O_{i_1 i_2\ldots}(x)$.

Contracting the tensors with auxiliary null polarization vectors w^i and W^M, we can write them in index-free notation

$$\begin{aligned} O^{(S)}(x, w) &= w^{i_1}\cdots w^{i_S} O_{i_1\cdots i_S}(x)\,, \\ O^{(S)}(P, W) &= W^{M_1}\cdots W^{M_S} O_{M_1\cdots M_S}(P)\,. \end{aligned} \tag{2.7}$$

In embedding space, any symmetric traceless tensor operator can therefore be written as a homogeneous function $O^{(S)}(P, W)$ of the coordinates $P, W \in \mathbb{R}^{d+1,1}$ such that $P^2 = P\cdot W = W^2 = 0$, with "gauge invariance" under $W \to W + cP$. Together with the scaling (2.5), this gauge invariance removes exactly two components per index from the tensor in embedding space, so its independent components match with those of the tensor on the Euclidean section. Under rescalings of the embedding coordinates and the polarization vectors, we have

$$O^{(S)}(\rho P, \alpha W) = \rho^{-\Delta}\alpha^S O^{(S)}(P, W)\,, \tag{2.8}$$

where Δ, S are the dimension and spin of $O^{(S)}$.

Conformal correlators Conformal correlators in embedding space are simply the most general Lorentz-invariant expressions with the correct scaling behavior according to (2.8). It is convenient to define the following (parity-even) conformally-invariant structures

$$\begin{aligned} P_{ij} &\equiv -P_i\cdot P_j\,, \\ H_{ij} &\equiv -2\big[(W_i\cdot W_j)(P_i\cdot P_j) - (W_i\cdot P_j)(W_j\cdot P_i)\big]\,, \\ V_{i,jk} &\equiv \frac{(W_i\cdot P_j)(P_k\cdot P_i) - (W_i\cdot P_k)(P_j\cdot P_i)}{P_j\cdot P_k}\,, \end{aligned} \tag{2.9}$$

which serve as the basic building blocks for conformal correlators. For instance, the two-point function of a spin-S field is

$$\langle O_1^{(S)} O_2^{(S)} \rangle = \frac{H_{12}^S}{(P_{12})^{\Delta+S}} \,, \tag{2.10}$$

where we have dropped an overall normalization constant. The subscript on $O_i^{(S)}$ denotes both the type of the field and its position, i.e. $O_i^{(S)} = O_i^{(S)}(P_i^M)$. Equation (2.10) is the only object that can be constructed out of P_{12} and H_{12} that has the appropriate rescaling covariance (2.8) for each field. Similarly, the three-point function of two scalars and a spin-S field is

$$\langle O_1 O_2 O_3^{(S)} \rangle = \frac{V_3^S}{P_{12}^{(\Delta_1+\Delta_2-\Delta_3-S)/2} P_{23}^{(\Delta_2+\Delta_3-\Delta_1+S)/2} P_{31}^{(\Delta_3+\Delta_1-\Delta_2+S)/2}} \,, \tag{2.11}$$

where we defined $V_i \equiv V_{i,jk}$ for $\{i,j,k\}$ a cyclic permutation of $\{1,2,3\}$, and dropped an overall constant factor.

There are also examples where more than one structure is consistent with conformal invariance. For instance, the three-point function of identical spin-2 tensors is

$$\langle O_1^{(2)} O_2^{(2)} O_3^{(2)} \rangle = \frac{1}{(P_{12}P_{23}P_{31})^{1+\Delta/2}} \sum_{n=1}^{5} c_n F_n \,, \quad \text{with} \quad F_n \equiv \begin{pmatrix} V_1^2 V_2^2 V_3^2 \\ H_{12} V_1 V_2 V_3^2 + \text{cyclic} \\ H_{12}^2 V_3^2 + \text{cyclic} \\ H_{12} H_{23} V_1 V_3 + \text{cyclic} \\ H_{12} H_{23} H_{31} \end{pmatrix} , \tag{2.12}$$

which has five allowed structures that can be combined with arbitrary weights c_n.[5] As we increase the spin of the fields, the number of allowed structures also increases. For example, the correlator of two identical spin-2 fields and a spin-4 field, $\langle O_1^{(2)} O_2^{(2)} O_3^{(4)} \rangle$, has 10 allowed structures [11].

2.2 Current Conservation

As we describe in Section 1.2, correlators of conserved tensors must satisfy an additional kinematic constraint, which in position space reads

$$\partial_{i_1} J^{i_1 \dots i_S} = 0 \,. \tag{2.14}$$

As long as the scaling dimension of the conserved current takes the value $\Delta = S + d - 2$, this conservation condition can be uplifted to embedding space as [11]

$$\frac{\partial}{\partial P_{M_1}} D_{M_1} J = 0 \,, \quad \text{where} \quad D_M \equiv \left(\frac{1}{2} + W \cdot \frac{\partial}{\partial W} \right) \frac{\partial}{\partial W^M} - \frac{1}{2} W_M \frac{\partial^2}{\partial W \cdot \partial W} \,. \tag{2.15}$$

[5] In $d=3$, the building blocks in (2.9) satisfy the relation

$$-2H_{12}H_{23}H_{31} = (V_1 H_{23} + V_2 H_{31} + V_3 H_{12} + 2V_1V_2V_3)^2 \,, \tag{2.13}$$

which implies that the last structure in (2.12) is not independent from the others. In this case, there are therefore only 4 independent structures for $\langle O_1^{(2)} O_2^{(2)} O_3^{(2)} \rangle$ and 9 independent structures for $\langle O_1^{(2)} O_2^{(2)} O_3^{(4)} \rangle$ [14].

To impose conservation for a certain correlator, we then deduce the linear combinations of the conformally-invariant structures that are compatible with the conservation condition.

Let us specialize to the case $d = 3$, where a conserved spin-S current has dimension $\Delta = S+1$. As an example, consider the case of a conserved spin-1 current. The parity-preserving three-point function is [15–17]

$$\langle J_1^{\tilde{a}} J_2^{\tilde{b}} J_3^{\tilde{c}} \rangle = f^{\tilde{a}\tilde{b}\tilde{c}} \frac{c_1 V_1 V_2 V_3 + c_2 (V_1 H_{23} + V_2 H_{13} + V_3 H_{12})}{(P_{12} P_{13} P_{23})^{3/2}} \,, \tag{2.16}$$

where $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ are color indices and $f^{\tilde{a}\tilde{b}\tilde{c}}$ are the structure constants. We see that there are two independent structures for this correlator that are compatible with both conformal symmetry and current conservation.[6] In the bulk, these two structures are associated to the Yang–Mills three-point vertex ($c_1 = c_2$) and a higher-derivative cubic term F^3 ($c_1 = 5c_2$) [18].

The (parity-even) three-point function of a conserved spin-2 tensor (like the stress tensor) can be any linear combination of the four structures in (2.12) that satisfies the conservation condition (2.15). This reduces the space of allowed structures from four to two:

$$\langle T_1 T_2 T_3 \rangle_{\rm GR} \propto \frac{\big(6 V_1^2 H_{2,3}^2 + 16 V_2 V_3 H_{31} H_{12} + 4 H_{23} V_1^2 V_2 V_3 + \text{cyclic}\big) - 9 V_1^2 V_2^2 V_3^2}{(P_{12} P_{23} P_{31})^{5/2}} \,, \tag{2.17}$$

$$\langle T_1 T_2 T_3 \rangle_{W^3} \propto \frac{\big(2 V_1^2 H_{2,3}^2 - 16 V_2 V_3 H_{31} H_{12} - 52 H_{23} V_1^2 V_2 V_3 + \text{cyclic}\big) - 147 V_1^2 V_2^2 V_3^2}{(P_{12} P_{23} P_{31})^{5/2}} \,. \tag{2.18}$$

In the bulk, these two structures are associated to the three-point vertex of Einstein gravity and a higher-derivative Weyl-cubed term W^3, respectively.

Conceptually, it is somewhat unsatisfying that we first have to write down the larger space of conformally-invariant structures and then find the specific linear combination(s) corresponding to conserved correlators. It would be nicer if we could write down directly the structures where both conformal invariance and conservation are made manifest from the start. In Section 4, we will show that this is possible in twistor space.

2.3 Introducing Spinors

Recall that a four-dimensional vector p^μ that is null (i.e. $p^2 = 0$) can be traded for two spinors. Since the $(d+2)$-dimensional embedding-space positions P^M satisfy $P^2 = 0$, it is natural to write them in terms of two spinors as well. These spinor variables strongly depend on the specific dimension d of the CFT, so we will first present them for $d = 2$ and then consider the case of primary interest $d = 3$.

$\boldsymbol{d = 2}$ Note that P^M is a four-dimensional null vector, that we take in $\mathbb{R}^{1,3}$. Hence, just like in (1.2), we can write

$$P_{\alpha\dot{\alpha}} = P_M (\sigma^M)_{\alpha\dot{\alpha}} = \begin{pmatrix} P_0 + P_3 & P_1 - iP_2 \\ P_1 + iP_2 & P_0 - P_3 \end{pmatrix} = \Lambda_\alpha \tilde{\Lambda}_{\dot{\alpha}} \,. \tag{2.19}$$

[6] In addition, there is one parity-violating structure that we have not displayed.

Again, we simply constructed a 2×2 matrix from the four-vector P^M. Since it has a vanishing determinant $\det(P_{\alpha\dot{\alpha}}) = -P^2 = 0$, we can write it as the tensorial product of two spinors Λ_α and $\tilde{\Lambda}_{\dot{\alpha}}$.

$\boldsymbol{d = 3}$ Next, we consider the case $d = 3$. Instead of CFTs in Euclidean space $\mathbb{R}^3$, for future convenience, we study CFTs in the Lorentzian space $\mathbb{R}^{1,2}$, with coordinates x^μ. The embedding space is then $\mathbb{R}^{2,3}$, with metric $\eta_{MN} = \mathrm{diag}(-1, 1, 1, 1, -1)$. The embedding-space position P^M is a five-dimensional null vector with $M = 0, 1, 2, 3, 4$. As before, it is convenient to maps the components of P^M into a 4×4 matrix:

$$\not{P}^{AB} = P^M (\Gamma_M)^{AB} = \begin{pmatrix} 0 & -P^3 - P^4 & P^2 & P^0 + P^1 \\ P^3 + P^4 & 0 & P^1 - P^0 & -P^2 \\ -P^2 & P^0 - P^1 & 0 & P^4 - P^3 \\ -P^0 - P^1 & P^2 & P^3 - P^4 & 0 \end{pmatrix}, \tag{2.20}$$

where $A, B = 1, 2, 3, 4$. The five 4×4 matrices $(\Gamma_M)^{AB}$ can be read off from the previous equation (e.g. Γ_2 can be obtained by taking $P^2 = 1$ with all other components vanishing).[7] The matrix $\not{P}^{AB}$ is anti-symmetric and Ω-traceless, $\Omega_{AB} \not{P}^{AB} = 0$, where Ω_{AB} is given by

$$\Omega_{AB} \equiv \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{2.22}$$

We use this matrix to raise and lower spinor indices with the following left-multiplication convention

$$U^A = \Omega^{AB} U_B \quad \text{and} \quad U_A = \Omega_{AB} U^B\,, \tag{2.23}$$

where $\Omega^{AB} = -\Omega_{AB}$. Moreover, we will denote contractions of the indices A, B by

$$U \cdot V \equiv U^A V_A\,, \quad \text{and} \quad U \cdot M \cdot V \equiv U^A M_A{}^B V_B\,, \tag{2.24}$$

with the understanding that they are always contracted from upper left to lower right. Finally, when we write a matrix M with no indices, we mean $M = M_A{}^B$ with the first index lowered and the second index raised.

It can be shown that the anti-symmetric matrix $\not{P}^{AB}$ in (2.20) has rank 2 and can therefore be written as

$$\not{P}^{AB} = \Lambda_1^A \Lambda_2^B - \Lambda_2^A \Lambda_1^B \equiv \epsilon^{ab} \Lambda_a^A \Lambda_b^B\,, \tag{2.25}$$

where Λ_1^A and Λ_2^A are two real spinors. The index $a = 1, 2$ in Λ_a^A labels these two spinors and ϵ^{ab} is the Levi-Civita symbol, with $\epsilon^{12} = 1$. These indices a, b are raised and lowered with ϵ^{ab} using the

[7] Note that the matrices $(\Gamma_M)^{AB}$ are simply a basis of the five-dimensional space of Ω-traceless, anti-symmetric 4×4 matrices. They have a deeper mathematical structure though, as these matrices $\Gamma^M = (\Gamma^M)_A{}^B$—with one index lowered following (2.23)—satisfy the anti-commutation relations

$$\{\Gamma^M, \Gamma^N\} = \Gamma^M \cdot \Gamma^N + \Gamma^N \cdot \Gamma^M = 2\eta^{MN} \mathbf{1}_{4\times 4}\,, \tag{2.21}$$

the so-called *Clifford algebra*.

same convention as in (2.23). Note that the right-hand side is automatically an anti-symmetric matrix, and that the Ω-tracelessness of $\not{P}^{AB}$ implies the constraint $\Lambda_1 \cdot \Lambda_2 = 0$. This in turn implies the *massless Dirac equation* for the spinors:

$$\not{P}_A{}^B \Lambda_a^A = 0\,, \tag{2.26}$$

i.e. they belong to the kernel of the transpose of $\not{P}_A{}^B$. We can also see that the spinors $\Lambda_{a,B}$ (with one index lowered) satisfy $\not{P}_A{}^B \Lambda_{a,B} = 0$, i.e. they belong to the kernel of $\not{P}_A{}^B$.

Consider the transformation

$$\Lambda_a^A \mapsto r\, V_a{}^b \Lambda_b^A\,, \tag{2.27}$$

where $r > 0$ and $V_a{}^b$ is a real matrix satisfying $\det(V) = 1$. Using (2.25), this implies $\not{P} \mapsto r^2 \not{P}$, i.e. the transformation only changes the embedding-space position by an overall rescaling, which has no physical importance since $P^M \sim \rho P^M$. The transformations in (2.27) are therefore *little group* transformations, and thus a, b are called little group indices.

Recall that, in Section 2.1, we showed that a spin-S field in embedding space can be written in index-free form by contracting the relevant rank-S tensor with auxiliary null vectors

$$J(P,W) = W^{M_1} \cdots W^{M_S}\, J_{M_1 \cdots M_S}\,, \tag{2.28}$$

where W^M satisfies $W^2 = P \cdot W = 0$. It will be convenient to expand this result in terms of little group-covariant tensors[8]

$$J(P,W) = \zeta^{a_1} \cdots \zeta^{a_{2S}}\, J_{a_1 \cdots a_{2S}}(\Lambda_a)\,, \tag{2.30}$$

where ζ^a are auxiliary vectors that absorb the little group indices of $J_{a_1 \cdots a_{2S}}(\Lambda_a)$.

2.4 Conservation and Holomorphicity

We would like to make manifest that a correlator of conserved tensors satisfies the differential conservation condition (2.15). We will first consider the simpler case of $d = 2$, where conservation arises from holomorphicity, and then discuss the challenge of generalizing this to $d = 3$.

$\boldsymbol{d = 2}$ It is straightforward to bootstrap three-point correlators of conserved tensors, with scaling dimension $\Delta_i = S_i$ [1]:

$$\langle J_1 J_2 J_3 \rangle = \frac{\text{const.}}{\langle 12\rangle^{n_3} \langle 23\rangle^{n_1} \langle 31\rangle^{n_2}}\,, \tag{2.31}$$

$$\langle \bar{J}_1 \bar{J}_2 \bar{J}_3 \rangle = \frac{\text{const.}}{[12]^{\bar{n}_3} [23]^{\bar{n}_1} [31]^{\bar{n}_2}}\,, \tag{2.32}$$

where the exponents are $n_i = \bar{n}_i = S_j + S_k - S_i$. We see that these are either *holomorphic* or *anti-holomorphic* in the sense that they depend only on angle or square brackets, respectively.

[8] The polarization vector W^M can be written

$$\not{W}^{AB} \equiv W^M (\Gamma_M)^{AB} \equiv \Upsilon^A \Upsilon^{*,B} - \Upsilon^{*,A} \Upsilon^B\,, \tag{2.29}$$

where Υ^A is a spinor that satisfies $\not{P}_A{}^B \Upsilon^A = 0$ and $\Upsilon^{*,A}$ is its dual, defined by $\Upsilon^{*,A} \not{P}_A{}^B = \Upsilon^B$. Expanding $\Upsilon^A \equiv \zeta^a \Lambda_a^A$ in the basis of $\ker(\not{P}^T)$, leads to the result in (2.30).

After projecting to the Poincaré slice, this translates into holomorphicity or anti-holomorphicity in physical position space. We thus conclude that, for 2d CFTs, conservation is intimately connected to holomorphicity, and that we can make conservation manifest by taking our correlators to be holomorphic.

$\boldsymbol{d=3}$ How does this generalize to 3d CFTs? Making conservation manifest is not as trivial as before, because the correlators cannot depend on only one of the two spinors Λ_a, with $a = 1, 2$. Instead, it will always depend on both. Hence, correlators cannot be holomorphic in the same sense as for $d = 2$. However, something special must happen for these correlators to be conserved: There should be an analog of holomorphicity for 3d CFTs. We will learn, in the next section, that this hidden holomorphicity can be exposed by writing correlators in twistor space.

3 An Invitation to Twistors

Twistor theory is a rather intimidating subject. However, the elements of twistors that we will need are rather basic and can be illustrated in elementary examples. In this section, we give an introduction to the concept of twistors in the simplest possible setting. We show that twistors naturally arise when we try to solve differential equations with an ansatz that is *holomorphic* in a sense that we will explain.

3.1 Twistors in d=2

As a warmup, we consider the familiar example of two-dimensional space(time). Consider Laplace's equation in $\mathbb{R}^2$:

$$\nabla^2 f(x,y) = (\partial_x^2 + \partial_y^2) f(x,y) = 0\,. \tag{3.1}$$

It is well known that the most general solution to this equation can be written as

$$f(x,y) = F(z) + G(\bar{z})\,, \tag{3.2}$$

where $z \equiv x+iy$ and $\bar{z} = x-iy$. The function $F(z)$ is called *holomorphic* because it only depends on the complex coordinate z and not its conjugate $\bar{z}$. Similarly, $G(\bar{z})$ is called *anti-holomorphic*.

Next, we look at the Klein–Gordon equation in $\mathbb{R}^{1,1}$:

$$\Box f(t,x) = \left(-\partial_t^2 + \partial_x^2\right) f(t,x) = 0\,. \tag{3.3}$$

In this case, it is useful to introduced the light-cone coordinates $z_\pm = t \pm x \in \mathbb{R}$. The most general solution to (3.3) can then be written as

$$f(t,x) = F(z_+) + G(z_-)\,. \tag{3.4}$$

We call $F(z_+)$ a holomorphic function because it only depends on z_+ and not z_-. Similarly, we call $G(z_-)$ anti-holomorphic. This is simply the Lorentzian version of (3.2).[9]

[9]In fact, (3.3) can be obtained from (3.1) by writing $y = it$. In general, the coordinates x and y can be taken to be complex and twistors are then null lines in $\mathbb{C}^2$. Special cases, like $\mathbb{R}^{1,1}$, can be derived from this generalization by imposing certain reality conditions on the complex coordinates.

For fixed z_+, the function $F(z_+)$ is constant along *null lines* parameterize by varying z_-:

$$L_- \equiv \left\{ (t,x) = \left(\frac{z_+ + z_-}{2}, \frac{z_+ - z_-}{2} \right), \quad \forall\, z_- \in \mathbb{R} \right\}. \tag{3.5}$$

Hence, the level sets of the holomorphic function $F(z_+)$ are given by the null lines L_-, and thus we can think of it as a function of the null lines L_- rather than the points (t,x). In this sense, we say that the function $F(z_+)$ is *localized* on the null lines L_-. Similarly, for fixed z_-, the function $G(z_-)$ is constant along null lines parameterize by z_+:

$$L_+ \equiv \left\{ (t,x) = \left(\frac{z_+ + z_-}{2}, \frac{z_+ - z_-}{2} \right), \quad \forall\, z_+ \in \mathbb{R} \right\}. \tag{3.6}$$

These null lines are illustrated in Figure 2. Another name for the null lines of $\mathbb{R}^{1,1}$ is *twistors* and the space of twistors is the space of all null lines in $\mathbb{R}^{1,1}$, which has the topology $\mathbb{R} \times \mathbb{Z}_2$.

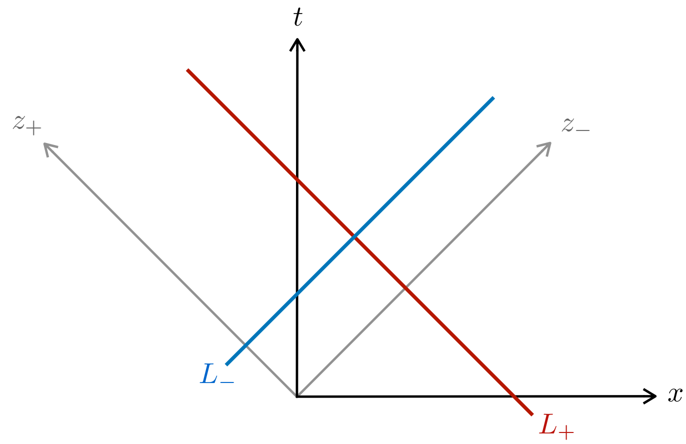

Figure 2: Illustration of twistors in $\mathbb{R}^{1,1}$ as null lines.

3.2 Twistors in d=3

Next, we move to three dimensions. Consider the Klein–Gordon equation in $\mathbb{R}^{1,2}$:

$$\Box f(t,x,y) = \left(-\partial_t^2 + \partial_x^2 + \partial_x^2\right) f(t,x,y) = 0\,. \tag{3.7}$$

A generic solution of this equation is

$$f(t,x,y) = F(Z) = F(t + x\sin\varphi + y\cos\varphi)\,. \tag{3.8}$$

where φ is a constant parameter. This can also be written as $Z \equiv n_\mu x^\mu$, where $n_\mu \equiv (1, \sin\varphi, \cos\varphi)$ is a null vector because $n_\mu n^\mu = 0$. The function $F(Z)$ is holomorphic in the sense that it depends on the coordinates (t,x,y) only through the linear combination in (3.8). Unlike in $\mathbb{R}^{1,1}$, where we had only two null directions $(1,1)$ and $(1,-1)$, the null directions in $\mathbb{R}^{1,2}$ are labelled by an angle φ and therefore form a continuous space S^1. Hence, there is no analogue of an anti-holomorphic function.

For fixed Z, the function $F(Z)$ is constant along *null planes*:

$$L_{Z,\varphi} \equiv \Big\{ (t,x,y) \;\Big|\; t + x \sin\varphi + y\cos\varphi = Z\,, \quad \forall\ (Z,\varphi) \in \mathbb{R} \Big\}\,. \tag{3.9}$$

Geometrically, this corresponds to planes whose normal vector is the null vector n_μ (see Figure 3). It is useful to think of $F(Z)$ as a function of the null planes $L_{Z,\varphi}$ rather than the points (t,x,y). In this sense, we say that the function $F(Z)$ is *localized* on the null planes $L_{Z,\varphi}$. Another name for the null planes of $\mathbb{R}^{1,2}$ is *twistors*[10] and the space of twistors is the space of all null planes in $\mathbb{R}^{1,1}$, which has the topology $\mathbb{R} \times S^1$.

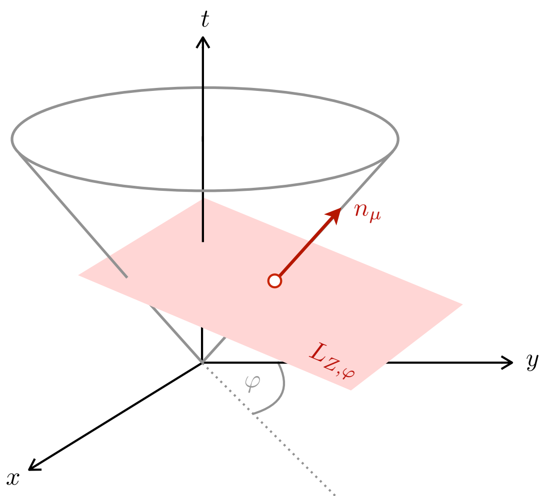

Figure 3: Illustration of twistors in $\mathbb{R}^{1,2}$ as null planes.

One price to pay in this construction is the introduction of the parameter φ. To get a solution that depends only on the coordinates, we integrate over φ as [19]

$$f(t,x,y) = \frac{1}{2}\int_{-\pi}^{\pi} d\varphi\, F(Z)\,. \tag{3.10}$$

Geometrically, this is an integral along all null planes (twistors) $L_{Z,\varphi}$ that pass through the point $x^\mu = (t,x,y)$. Since the point x^μ determines Z, these null planes are parameterized only by the angle φ, and thus we only need to integrate over φ. We can also think of the integral as the analog of the sum over holomorphic and anti-holomorphic functions in (3.4).

The formula (3.10) can be made more projective-looking by writing

$$f(x^\mu) = \int D\pi\, F(\pi^a x_{ab} \pi^b)\,, \quad \text{where} \quad x_{ab} \equiv \begin{pmatrix} x^1 - x^0 & -x^2 \\ -x^2 & -x^0 - x^1 \end{pmatrix}. \tag{3.11}$$

The two components of $\pi^a = (\pi^1, \pi^2)$ are real, and we are therefore integrating over the projective space $\mathbb{RP}^1$ of lines in $\mathbb{R}^2$ that pass through the origin with the measure $D\pi \equiv d\pi^a \pi^b \epsilon_{ab}$. In order for the integral to be invariant under the rescaling $\pi^a \to r\pi^a$, the integrand $F(Z)$ must behave

[10]These twistors for $\mathbb{R}^{1,2}$ are also referred to as *mini-twistors* in the literature [19].

as $F(r^2 Z) = r^{-2}F(Z)$. Parameterizing $\pi^a \equiv (1, \tan(\varphi/2))$, and integrating φ over the interval $(-\pi, \pi)$, we recover the result (3.10). As explained in [19], the integral in (3.11) is a *Penrose transform* that defines a map from twistor space to coordinate space.

3.3 Current Conservation

So far, we have shown that twistors are a convenient way to describe the solutions to the equations of motion of a massless scalar field. In the following, we will demonstrate that twistors also characterize the solutions for conserved currents; cf. (1.6).

For concreteness, we consider a spin-1 current which satisfies $\partial_\mu J^\mu = 0$ in $\mathbb{R}^{1,2}$. Using the chain rule, this constraint can be written as

$$0 = \partial_\mu J^\mu = \frac{\partial x_{ab}}{\partial x^\mu}\frac{\partial}{\partial x_{ab}} J^\mu = \partial^{ab} J_{ab}\,, \tag{3.12}$$

where x_{ab} is given by (3.11), and we defined

$$J_{ab} \equiv \frac{\partial x_{ab}}{\partial x^\mu} J^\mu = \begin{pmatrix} J^1 - J^0 & -J^2 \\ -J^2 & -J^0 - J^1 \end{pmatrix}. \tag{3.13}$$

Since x_{ab} is symmetric, both the operator $\partial^{ab} \equiv \partial/\partial x_{ab}$ and the current J_{ab} are also symmetric two-by-two matrices.

Notice that a generic solution of (3.12) can be written as

$$J_{ab}(x^\mu) = \pi_a \pi_b F(x_{cd}\pi^d)\,. \tag{3.14}$$

It is easy to see that this solves the conservation equation:

$$\begin{aligned} \partial^{ab} J_{ab} &= \pi_a\pi_b \frac{\partial}{\partial x_{ab}} F(x_{cd}\pi^d) = \pi_a\pi_b \frac{\partial(x_{ef}\pi^f)}{\partial x_{ab}} \frac{\partial}{\partial(x_{ef}\pi^f)} F(x_{cd}\pi^d) \\ &= \pi_a\pi_b\pi^b \frac{\partial}{\partial(x_{af}\pi^f)} F(x_{cd}\pi^d) = 0\,, \end{aligned} \tag{3.15}$$

where we used that $\pi_b\pi^b = \epsilon_{ba}\pi^a\pi^b = 0$. We say that this solution is holomorphic in the sense that it depends on x_{ab} only through $x_{ab}\pi^b$.

As before, we identify twistors as the locations where the solution is a constant:

$$x_{ab}\pi^b = \mu_a \in \mathbb{R}\,. \tag{3.16}$$

This constraint is known as the "incidence relation." For fixed π^b and μ_a, the three-dimensional positions x_{ab} that satisfy these two linear and inhomogeneous equations form a one-dimensional line in $\mathbb{R}^{1,2}$:

$$x_{ab}(c) = x^*_{ab} + c\, n_{ab}\,, \tag{3.17}$$

where x^*_{ab} is a specific position that satisfies this equation, c is the parameter of the line and n_{ab} is its direction. Since $x_{ab}\pi^b$ remains constant under the shift $x_{ab} \mapsto x_{ab} + c\pi_a\pi_b$, the direction of the line (3.17) must be $n_{ab} = \pi_a \pi_b$, which corresponds to a vector $n^\mu(\pi_a)$ that is automatically null as

$$n^2 = n_\mu n^\mu = -\frac{1}{2} n_{ab} n^{ab} = -\frac{1}{2} (\pi_a \pi^a)^2 = 0 \,. \tag{3.18}$$

Hence, twistors are *null lines* in $\mathbb{R}^{1,2}$, and twistor space is the space of these null lines (see Figure 4). Note that this is different from the (mini-)twistors in $\mathbb{R}^{1,2}$, which are null planes. This is because the latter are related to holomorphic solutions of the 3d Klein–Gordon equation, while the twistors (null lines) described here are related to holomorphic solutions of the 3d conservation equation.

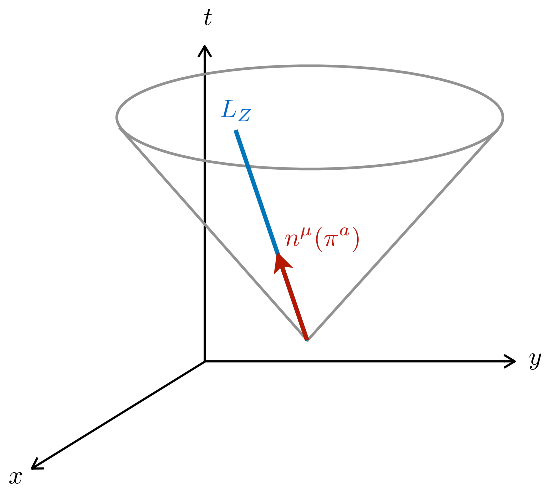

Figure 4: Illustration of twistors in $\mathbb{R}^{1,2}$ as null lines solving the conservation equation.

It is also conventional to define the twistor coordinates as a four-dimensional spinor

$$Z^A \equiv (\pi^a, \mu_a) \,, \tag{3.19}$$

which is projective since the scale of Z^A drops out in (3.16). The most general solution to the conservation equation can then be written as

$$J^{ab}(x^\mu) = \int D\pi \, \pi^a \pi^b F(Z^A)\Big|_{\mu_a = x_{ab}\pi^b} \,, \tag{3.20}$$

where we integrate π^a over $\mathbb{RP}^1$ as in (3.11). Here, we have allowed for an explicit dependence on π^c in the function $F(Z^A) = F(\pi^c, x_{cd}\pi^d)$. Note that the integrand $F(Z^A)$ must scale as $F(rZ^A) = r^{-4} F(Z^A)$ in order for the integral to be invariant under the rescaling $\pi^a \to r\pi^a$. As before, the geometrical interpretation is that we integrate over all twistors that pass through a certain point, which are parameterized by the direction $n_{ab} = \pi_a \pi_b$ of these null lines.

In the next section, we will study how to make conservation manifest in embedding space, so that the constraints from conformal symmetry are trivialized as well. We will see that this is achieve by similar integral formulas for the currents in embedding space.

4 Twistors for Cosmology

We now return to our problem of interest. We will show, in this section, that the correlators of conserved currents in a conformal field theory are most conveniently written as integrals in twistor space, where the integrand is a holomorphic function of the twistor coordinates.

4.1 Holomorphicity in Twistor Space

As we explained in Section 2.4, the correlators of conserved currents in a 3d CFT always depend on *both* embedding-space spinors Λ_1^A and Λ_2^A, and hence they cannot be holomorphic in the same sense as for $d = 2$. Taking into account the lessons we have just learned about holomorphicity in $d = 3$, we consider a function $F(Z^A)$ that depends on the spinors Λ_a^A only through a specific linear combination

$$Z^A \equiv \pi^a \Lambda_a^A \,. \tag{4.1}$$

In order for the spinors Z^A to satisfy the same reality condition as the embedding-space spinors Λ_a^A (i.e. for their components to be purely real), the coefficients π^a must be real. Since the spinors Λ_a^A form a basis of the kernel of (the transpose of) $\not{P}_A{}^B$, we have

$$\not{P}_A{}^B Z^A = 0 \,, \tag{4.2}$$

which defines a map between the null positions P^M in embedding space and the spinors Z^A. This relation is non-local because it maps one position P^M to a whole two-dimensional subspace of spinors. After modding out the scale of these spinors, this subspace becomes a projective one-dimensional subspace, which can be pictorially represented as a line (see Figure 5). The map in (4.2) can also be interpreted as the so-called "incidence relation" between position space and twistor space [20], which in turn allows us to identify the spinor Z^A as a twistor.

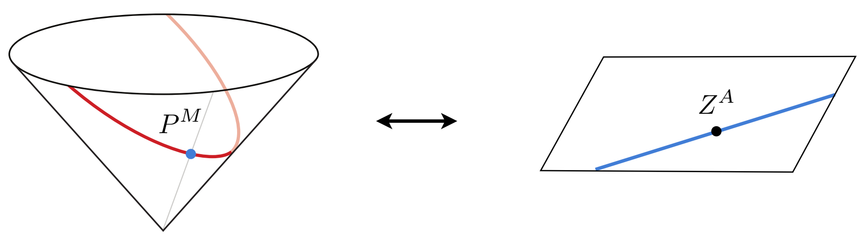

Figure 5: Illustration of the map between embedding space and twistor space defined by the relation (4.2).

4.2 Currents as Twistor Integrals

The twistor coordinates in (4.1) depend on the arbitrary parameters π^a. We erase the arbitrariness in the choice of these parameters by integrating over them, using the projective measure $D\pi \equiv d\pi^b \pi^c \epsilon_{bc}$ as in (3.11). For a conserved spin-S current, the resulting integral transform is

$$J^{a_1 a_2 \cdots a_{2S}}(\Lambda_a^A) = \int D\pi \, \pi^{a_1} \pi^{a_2} \cdots \pi^{a_{2S}} F(Z^A) \,, \tag{4.3}$$

where we used the little group-covariant representation of the current in (2.30). This integral representation makes conservation manifest simply because the integrand $F(Z^A)$ is holomorphic, in the sense that it depends only on the linear combination in (4.1). The proof of conservation is similar to that in (3.15):

$$\partial_{a_1 a_2} J^{a_1 a_2 \cdots a_{2S}} \sim \int D\pi \cdots \pi^{a_1} \frac{\partial F(Z^A)}{\partial \Lambda^{a_1,A}} \sim \int D\pi \cdots \pi^{a_1} \pi_{a_1} \frac{\partial F(Z^A)}{\partial Z^A} = 0\,, \tag{4.4}$$

where we used that $\pi^a \pi_a = 0$, and ignored irrelevant factors to highlight the proof's essence (see [1] for details). This is precisely what we wanted to achieve in Section 2.4: Any tensor written as (4.3) will automatically be a conserved current!

In order for the integral over π_a to be well-defined projectively, the function $F(Z^A)$ must be a homogeneous function of degree $-2(S+1)$, i.e.

$$F(rZ^A) = r^{-2(S+1)} F(Z^A)\,. \tag{4.5}$$

Since $Z^2 \sim \not{P}$, we see that the dimension of the field is therefore fixed to be $\Delta = S+1$. This is really nice. Requiring the integral transform (4.3) to be well-defined has fixed the conformal scaling weight of the field to be precisely that of a conserved current.

Let us recall that the little group-covariant representation of the current in (4.3) can be related to the index-free embedding-space field via

$$J(P, W) = \zeta^{a_1} \cdots \zeta^{a_{2S}} J_{a_1 \cdots a_{2S}}(\Lambda_a^A) = \int D\pi\, (\zeta^a \pi_a)^{2S} F(Z^A)\,, \tag{4.6}$$

where ζ^a are auxiliary vectors that absorb the little group indices, and we plugged in (4.3) in the last step. This integral can also be written only in terms of the twistor Z^A as

$$J(P, \Upsilon^*) = \int DZ\, (\Upsilon^* \cdot Z)^{2S} F(Z^B)\,. \tag{4.7}$$

Let us explain the elements in this formula: First, we are integrating Z^A over the line in twistor space defined by $\not{P}_A{}^B Z^A = 0$. It can be parameterized as $Z^A = \pi^a \Lambda_a^A$, where the measure is $DZ = D\pi$. Second, the spinor Υ^*_A is defined such that $\Upsilon^* \cdot Z = -\Upsilon^*_A Z^A = \zeta^a \pi_a$. Thus, the factor $(\Upsilon^* \cdot Z)^{2S}$ contains the $2S$ spinors Υ^*_A that correspond to the S polarization vectors W^M; cf. (2.29).

Finally, let us mention that there is an alternative integral representation of conserved currents in *dual* twistor space:

$$\tilde{J}(P, \Upsilon^*) = \int DW \left(\Upsilon^* \cdot \frac{\partial}{\partial W}\right)^{2S} \tilde{F}(W_B)\,, \tag{4.8}$$

where the integral is defined over the line given by $\not{P}_A{}^B W_B = 0$. Note that (4.8) is of the same form as (4.7) with $\Upsilon^* \cdot Z$ replaced by $\Upsilon^* \cdot \partial_W$. In order for the integral to be independent of the scale of the dual twistor coordinate, the integrand must scale as $\tilde{F}(rW_A) = r^{2S-2} \tilde{F}(W_A)$. This integral representation is also automatically conserved, and can be related to an integral of the form (4.7) by Fourier-transforming the integrand.

4.3 Spinning Correlators in Twistor Space

So far, we have explained how to trivialize conservation for a single current by writing it as integrals of the form (4.7) or (4.8). However, the observables of interest are correlators of conserved currents defined at multiple points. We can make conservation manifest for each current in a n-point correlator by considering a function $F(Z_i^A) = F(Z_1^A, \cdots, Z_n^A)$ that depends on each of the embedding-space spinors $\Lambda_{i,a}^A$ only through the combinations

$$Z_i^A \equiv \pi_i^a \Lambda_{i,a}^A \,. \tag{4.9}$$

As before, we must integrate over the n twistors Z_i^a to obtain an integral representation of the correlator:

$$\langle J_1 J_2 \cdots J_n \rangle = \left[\prod_{j=1}^{n} \int DZ_j \, (\Upsilon_j^* \cdot Z_j)^{2S_j} \right] F(Z_i^A) \,. \tag{4.10}$$

The conformal symmetry of the correlators can be made manifest if we take the integrand to only depend on products between twistors:

$$F(Z_i^A) = F(Z_i \cdot Z_j) \,, \quad \text{where} \quad Z_i \cdot Z_j = Z_i^A \Omega_{AB} Z_j^B \,. \tag{4.11}$$

The reason is that the double cover of the conformal group $\mathrm{SO}(2,3)$ is the real symplectic group $\mathrm{Sp}(4,\mathbb{R})$. Thus, the latter acts on the twistors as linear transformations $Z_i^A \mapsto Z_i^B M_B{}^A$ that preserve the inner product (4.11) and the reality condition $Z_i^A \in \mathbb{R}$.

Similarly, we can make conservation manifest for a n-point correlator by using the integral (4.8) in dual twistor space:

$$\langle \tilde{J}_1 \tilde{J}_2 \cdots \tilde{J}_n \rangle = \left[\prod_{j=1}^{n} \int DW_i \left(\Upsilon_j^* \cdot \frac{\partial}{\partial W_j} \right)^{2S_j} \right] \tilde{F}(W_{i,A}) \,. \tag{4.12}$$

As before, we can trivialize conformal symmetry by taking $\tilde{F}(W_{i,A})$ to depend only on the inner products $W_i \cdot W_j = -W_{i,A} \Omega^{AB} W_{j,B}$. Of course, we can also consider a mixed ansatz where some integrals are over twistors Z^A and others are over dual twistors W_A [1], but this will not be necessary for our discussion, which focuses on three-point functions.

The upshot of this discussion is that the natural language to describe CFT correlators of conserved currents, with all of their kinematic requirements made manifest, is that of holomorphic functions in twistor space! Every point in embedding space defines a pair of spinors, from which one can build an associated twistor. The resulting holomorphic functions satisfy stringent constraints, essentially from scaling, and can be bootstrapped very easily for two- and three-point functions. We will see that the twistor formulas are extremely simple.

4.4 Bootstrapping Three-Point Functions

The illustrate to power of twistors, let us describe how to bootstrap the three-point functions of conserved currents in twistor space. For concreteness, we take all fields have equal spin $S_i = S$. Under a rescaling of the coordinates, the correlators must behave as

$$F(r_i Z_i^A) = (r_1 r_2 r_3)^{-2S-2} F(Z_i^A) \,, \tag{4.13}$$

$$\tilde{F}(r_i W_{i,A}) = (r_1 r_2 r_3)^{2S-2}\,\tilde{F}(W_{i,A})\,, \tag{4.14}$$

so that the integrals (4.10) and (4.12) are projective. We further impose that these function only depend on products between twistors, i.e. $F(Z_i \cdot Z_j)$ and $\tilde{F}(W_i \cdot W_j)$. This allows us to easily bootstrap the twistor correlators in (4.10) and (4.12) to be[11]

$$F(Z_i \cdot Z_j) = \int \frac{d^3 c_{ij}}{(2\pi)^3} \exp\left(-ic_{12} Z_1 \cdot Z_2 + \text{cyclic}\right) A(c_{ij})\,, \tag{4.15}$$

$$\tilde{F}(W_i \cdot W_j) = \int \frac{d^3 c_{ij}}{(2\pi)^3} \exp\left(-ic_{12} W_1 \cdot W_2 + \text{cyclic}\right) \tilde{A}(c_{ij})\,, \tag{4.16}$$

where

$$A(c_{ij}) = (c_{12}\,c_{23}\,c_{31})^S \quad \text{and} \quad \tilde{A}(c_{ij}) = \frac{1}{(c_{12}\,c_{23}\,c_{31})^S}\,. \tag{4.17}$$

We see that the results take a particularly simple form in terms of the Schwinger parameters c_{ij}.

In [1], we computed explicitly the twistor integrals (4.10) and (4.12) for $n = 3$, with these integrands, and showed that they indeed reproduce the known three-point correlators involving conserved currents in embedding space; cf. Section 2.2.[12] We found that the twistor correlator (4.15) gives the structure arising from *higher-derivative interactions* in the bulk (F^3 for spin 1 and W^3 for spin 2), while the dual twistor correlator (4.16) gives the structure corresponding to the *leading interactions* in the bulk (Yang–Mills for spin 1 and GR for spin 2). We stress that this formalism gives directly the conserved structures without first having to construct the larger space of conformally-invariant structures that may or may not be conserved.

Interestingly, unlike $A(c_{ij})$, the Schwinger-parameterized correlator $\tilde{A}(c_{ij})$ has a non-trivial *denominator*. Since only the latter is associated to leading interactions in the bulk (YM or GR), this is precisely analogous to what happens for three-point scattering amplitudes in flat space expressed in spinor helicity variables. Indeed, despite being purely holomorphic/anti-holomorphic, amplitudes only have denominators for the cases of Yang–Mills or GR, thus signalizing a (very mild) breaking of holomorphicity.

Finally, we notice from (4.17) that, in terms of the Schwinger parameters, the three-point gravity ($S = 2$) correlators $\tilde{A}(c_{ij})$ are the square of the gauge-theory ($S = 1$) correlators. Although this double-copy structure looks rather trivial in twistor space, it does not work like this for correlators in embedding space or in momentum space, meaning that we do *not* get the gravity correlator by squaring the corresponding gauge-theory correlator [21]. The fact that this actually works for the twistor correlators written in Schwinger parameters is just an example of their simplicity compared to their counterparts in embedding and momentum space.

[11] We also impose that the correlators must be even under the composition parity and time reversal, which forbids power-law ansätze for $F(Z_i \cdot Z_j)$ or $\tilde{F}(W_i \cdot W_j)$ [1].

[12] This calculation involves an analytic continuation from totally spacelike configurations to 3d Euclidean position space, see footnote 15 in [1].

5 Back to Momentum Space

The complexity of spinning correlators in Fourier space is somewhat unsatisfying [5, 6, 22], especially when compared to the dramatically simpler results for scattering amplitudes. In this section, we will relate them to our twistor-space correlators via a so-called *half-Fourier transform*, which has been used to map scattering amplitudes in four-dimensional momentum space to twistor space [23–26].

5.1 Half-Fourier Transforms

To define the half-Fourier transform, we first write the twistor coordinates Z_i^A in terms of two-component spinors

$$Z_i^A \equiv \begin{pmatrix} \lambda_i^\alpha \\ \mu_{i,\dot{\alpha}} \end{pmatrix} . \tag{5.1}$$

The half-Fourier transform with respect to μ_i is then defined as

$$g(\lambda_i, \tilde{\lambda}_i) \equiv \int d^2\mu_i \, \exp(i\tilde{\lambda}_i \cdot \mu_i) \, f(Z_i^A) \, , \tag{5.2}$$

where we use the same convention as (2.24) for contractions of the spinor indices, $v \cdot u = v^\alpha u_\alpha$.

The result is a function of the spinors λ_i^α and $\tilde{\lambda}_i^{\dot{\alpha}}$, that can be interpreted as the cosmological spinor-helicity variables described in [22]. These can be defined from a Lorentzian and spacelike three-momenta $k^\mu = (k^0, k^1, k^2)$ and its norm $k = \sqrt{k_\mu k^\mu}$ as[13]

$$k_{\alpha\dot{\alpha}} = \begin{pmatrix} -k^0 - k^1 & k + k^2 \\ -k + k^2 & -k^0 + k^1 \end{pmatrix} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}} \, . \tag{5.3}$$

Hence, the result of the half-Fourier transform lives in Fourier space, and thus this map relates objects in twistor space to their counterparts in momentum space. Let us point out that, unlike for amplitudes, it is allowed to contract dotted and undotted indices with the tensor $\tau^{\alpha\dot{\beta}} = \epsilon^{\alpha\dot{\beta}}$, such that $\tau^{\alpha\dot{\beta}} k_{\alpha\dot{\beta}} = 2k$. Defining $\bar{\lambda}_\alpha = \tau_\alpha{}^{\dot{\alpha}} \tilde{\lambda}_{\dot{\alpha}}$, the correlators can be written in terms of the following spinor brackets: $\langle ij \rangle \equiv \epsilon^{\beta\alpha} \lambda_\alpha^i \lambda_\beta^j$, $\langle \bar{i}\bar{j} \rangle \equiv \epsilon^{\beta\alpha} \bar{\lambda}_\alpha^i \bar{\lambda}_\beta^j$ and $\langle i\bar{j} \rangle \equiv \epsilon^{\beta\alpha} \lambda_\alpha^i \bar{\lambda}_\beta^j$, where the mixed brackets are indeed allowed.

Analogously, to define the dual half-Fourier transform, we first write the dual twistor as

$$W_{i,A} \equiv \begin{pmatrix} \tilde{\mu}_{i,\alpha} \\ \tilde{\lambda}_i^{\dot{\alpha}} \end{pmatrix} , \tag{5.4}$$

and then define the transform with respect to $\tilde{\mu}_i$ as

$$\tilde{g}(\lambda_i, \tilde{\lambda}_i) \equiv \int d^2\tilde{\mu}_i \, \exp(-i\lambda_i \cdot \tilde{\mu}_i) \, \tilde{f}(W_{i,A}) \, . \tag{5.5}$$

This maps objects in dual twistor space to momentum space. Let us now apply these transforms to the equal-spin three-point twistor correlators that we bootstrapped in Section 4.4.

[13] For the map (5.2) to make sense as a Fourier transform, we need the spinors λ_i, $\tilde{\lambda}_i$, μ_i to be real and independent [24]. This corresponds to spacelike momenta in a space $\mathbb{R}^{1,2}$ with Lorentzian signature, so that the matrix in (5.3) is purely real.

5.2 Higher-Derivative Interactions

The twistor correlator

$$F(Z_i \cdot Z_j) = \int \frac{d^3 c_{ij}}{(2\pi)^3} \exp\left(-ic_{12} Z_1 \cdot Z_2 + \text{cyclic}\right) A(c_{ij})\,, \tag{5.6}$$

with $A(c_{ij}) = (c_{12}c_{23}c_{31})^S$, has been mapped via (4.10) to the three-point structures in embedding space arising from higher-derivative interactions in the bulk. In particular, for spins $S = 1$ and $S = 2$ these are given by F^3 and W^3 interactions, respectively. Performing the half-Fourier transform (5.2) of this result, we get

$$G(\lambda_i, \tilde{\lambda}_i) = \left[\prod_{i=1}^{3} \int d^2\mu_i \exp(i\tilde{\lambda}_i \cdot \mu_i)\right] F(Z_j \cdot Z_k) = (2\pi)^3 \delta^{(3)}\big(\vec{k}_1 + \vec{k}_2 + \vec{k}_3\big)\, g(\lambda_i, \tilde{\lambda}_i)\,, \tag{5.7}$$

where the result after stripping off the momentum-conserving delta function is [1]

$$g(\lambda_i, \tilde{\lambda}_i) = \frac{A(c_{ij})}{4}\Big|_{c_{ij} = \langle \bar{i}\bar{j}\rangle/E} = \frac{1}{4}\left(\frac{\langle \bar{1}\bar{2}\rangle\langle \bar{2}\bar{3}\rangle\langle \bar{3}\bar{1}\rangle}{E^3}\right)^S. \tag{5.8}$$

This turns out to be proportional to the all-plus three-point function of spin-S currents coming from higher-derivative interactions, like F^3 or W^3 for spins $S = 1$ and $S = 2$, respectively.

5.3 Leading Interactions

The dual twistor correlator

$$\tilde{F}(W_i \cdot W_j) = \int \frac{d^3 c_{ij}}{(2\pi)^3} \exp\left(-ic_{12} W_1 \cdot W_2 + \text{cyclic}\right) \tilde{A}(c_{ij})\,, \tag{5.9}$$

with $\tilde{A}(c_{ij}) = (c_{12}c_{23}c_{31})^{-S}$, has been mapped via (4.12) to the three-point structures in embedding space that arise from the leading interactions (i.e. YM and GR for spins $S = 1$ and $S = 2$, respectively). Performing the same analysis as before, the half-Fourier transform (5.5) applied to (5.9) gives

$$\tilde{G}(\lambda_i, \tilde{\lambda}_i) = \left[\prod_{i=1}^{3} \int d^2\tilde{\mu}_i \exp(-i\lambda_i \cdot \tilde{\mu}_i)\right] \tilde{F}(W_j \cdot W_k) = (2\pi)^3 \delta^{(3)}\big(\vec{k}_1 + \vec{k}_2 + \vec{k}_3\big)\, \tilde{g}(\lambda_i, \tilde{\lambda}_i)\,, \tag{5.10}$$

where the result after stripping off the momentum-conserving delta function is [1]

$$\tilde{g}(\lambda_i, \tilde{\lambda}_i) = \frac{\tilde{A}(c_{ij})}{4}\Big|_{c_{ij} = \langle ij\rangle/E} = \frac{1}{4}\left(\frac{\langle \bar{1}\bar{2}\rangle\langle \bar{2}\bar{3}\rangle\langle \bar{3}\bar{1}\rangle}{(E-2k_1)(E-2k_2)(E-2k_3)}\right)^S. \tag{5.11}$$

In the second equality, we used that $\langle ij\rangle/E = (E - 2k_l)/\langle \bar{i}\bar{j}\rangle$, with $\{i, j, l\}$ a cyclic permutation of $\{1, 2, 3\}$, which is valid on the support of three-point kinematics [5]. Defining

$$\tilde{g}(\lambda_i, \tilde{\lambda}_i) = \left(\frac{\langle \bar{1}\bar{2}\rangle\langle \bar{2}\bar{3}\rangle\langle \bar{3}\bar{1}\rangle \hat{E}}{k_1 k_2 k_3}\right)^S \frac{\tilde{f}_{[SSS]}(k_i)}{(k_1 k_2 k_3)^{S-1}}\,, \tag{5.12}$$

we can extract the form factor $\tilde{f}_{[SSS]}(k_i)$ from the result (5.11). This turns out to be proportional to the *discontinuity* of the form factor $f_{[SSS]}(k_i)$ that determines the three-point function of spin-S currents coming from leading interactions [6]:

$$\tilde{f}_{[SSS]}(k_i) \propto \mathrm{Disc}_{k_1^2}\,\mathrm{Disc}_{k_2^2}\,\mathrm{Disc}_{k_3^2} f_{[SSS]}(k_i)\,, \tag{5.13}$$

which we checked explicitly up to spin 6.

We can derive the actual form factor $f_{[SSS]}(k_i)$ of the momentum-space correlator by performing the following dispersive integral [27, 28]

$$f_{[SSS]}(k_i) \propto \left[\prod_{i=1}^{3} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\omega_i\, d\omega_i}{\omega_i^2 - k_i^2}\right] \tilde{f}_{[SSS]}(\omega_j)\,. \tag{5.14}$$

It is rather remarkable how these complex form factors arise from the transformation of the extremely simple twistor correlators $A(c_{ij}) = (c_{12}c_{23}c_{31})^{-S}$.

6 Conclusions and Outlook

Although this paper has not yet realized the dream of a Parke–Taylor-like formula for spinning correlators in cosmology, it provides a new avenue towards finding such a structure. In particular, we believe that twistors are the right kinematic variables to expose the hidden simplicity of spinning correlators. Using twistors has allowed us to write integral expressions for these correlators that make both conformal symmetry and current conservation manifest. At the three-point level, the kinematic constraints completely fix the correlators and we have shown that our results in twistor space are consistent with known results in momentum space and embedding space, but arguably they are simpler.

There are a few natural next steps to take in this adventure of using twistors in cosmology:

- First of all, it is important to ask how the twistor representation of spinning correlators can be extended to higher points. While three-point functions are completely fixed by kinematics, determining higher-point functions requires additional dynamical input. For CFTs, these additional constraints come from consistency of the OPE, while for scattering amplitudes one imposes consistent factorization on all poles. Already at four points, these constraints impose severe restrictions on the space of consistent theories [29, 30]. It will be interesting to explore how the space of consistent cosmological correlators is constrained by similar considerations.

- For scattering amplitudes, the construction of higher-point amplitudes can be made systematic through the use of powerful recursion relations [31]. It would be nice to find a similar recursive method for our correlators in twistor space. Indeed, in [25, 26], it was shown that BCFW recursion relations for amplitudes also have a natural representation in twistor space. It will be interesting to see how these insights translate to the cosmological context.

- We would then like to return to the ambitious goal of deriving (or guessing) a Parke–Taylor-like formula for all-multiplicity gluon correlators in de Sitter space. While until recently finding such a formula seemed like a distant dream, the simplicity of our twistor-based approach makes us optimistic that this is now within reach.

- Finally, twistor space can potentially allow us to interpret cosmological correlators as volumes of *positive geometries* in kinematic space, as it happens for scattering amplitudes. In fact, writing amplitudes in twistor space was instrumental to relate them with Grassmanians and positive geometries [32]. In searching for such geometries, an important first step is to elucidate the "best kinematic description" of the observables, and then search for a geometry in this kinematic space. So far, most of the structures found for cosmological correlators either refer to a single diagram or to the combinatorics of the sum of diagrams in scalar theories [33–37]. Twistors open up the possibility of studying spinning theories in cosmology, combining all diagrams in rigid formulas, and thus making the search for positive geometries feasible.

We will return to these tasks in future work.

References

[1] D. Baumann, G. Mathys, G. L. Pimentel, and F. Rost, "A New Twist on Spinning (A)dS Correlators," arXiv:2408.02727 [hep-th].

[2] S. Parke and T. Taylor, "An Amplitude for n Gluon Scattering," *Phys. Rev. Lett.* **56** (1986) 2459.

[3] J. Drummond, J. Henn, G. Korchemsky, and E. Sokatchev, "Dual Superconformal Symmetry of Scattering Amplitudes in N=4 Super-Yang-Mills Theory," *Nucl. Phys. B* **828** (2010) 317–374, arXiv:0807.1095 [hep-th].

[4] N. Arkani-Hamed and J. Trnka, "The Amplituhedron," *JHEP* **10** (2014) 030, arXiv:1312.2007 [hep-th].

[5] D. Baumann, C. Duaso Pueyo, A. Joyce, H. Lee, and G. L. Pimentel, "The Cosmological Bootstrap: Spinning Correlators from Symmetries and Factorization," *SciPost Phys.* **11** (2021) 071.

[6] D. Baumann, W.-M. Chen, C. Duaso Pueyo, A. Joyce, H. Lee, and G. L. Pimentel, "Linking the Singularities of Cosmological Correlators," *JHEP* **09** (2022) 010, arXiv:2106.05294 [hep-th].

[7] S. Albayrak, S. Kharel, and D. Meltzer, "On Duality of Color and Kinematics in (A)dS Momentum Space," *JHEP* **03** (2021) 249, arXiv:2012.10460 [hep-th].

[8] J. Bonifacio, H. Goodhew, A. Joyce, E. Pajer, and D. Stefanyszyn, "The Graviton Four-Point Function in De Sitter Space," *JHEP* **06** (2023) 212, arXiv:2212.07370 [hep-th].

[9] P. A. M. Dirac, "Wave Equations in Conformal Space," *Annals Math.* **37** (1936) 429–442.

[10] S. Rychkov, *EPFL Lectures on Conformal Field Theory in D>= 3 Dimensions*. arXiv:1601.05000 [hep-th].

[11] M. Costa, J. Penedones, D. Poland, and S. Rychkov, "Spinning Conformal Correlators," *JHEP* **11** (2011) 071, arXiv:1107.3554 [hep-th].

[12] M. Costa, J. Penedones, D. Poland, and S. Rychkov, "Spinning Conformal Blocks," *JHEP* **11** (2011) 154, arXiv:1109.6321 [hep-th].

[13] D. Karateev, P. Kravchuk, and D. Simmons-Duffin, "Weight Shifting Operators and Conformal Blocks," *JHEP* **02** (2018) 081, arXiv:1706.07813 [hep-th].

[14] P. Kravchuk and D. Simmons-Duffin, "Counting Conformal Correlators," *JHEP* **02** (2018) 096, arXiv:1612.08987 [hep-th].

[15] S. Giombi, S. Prakash, and X. Yin, "A Note on CFT Correlators in Three Dimensions," *JHEP* **07** (2013) 105, arXiv:1104.4317 [hep-th].

[16] A. Zhiboedov, "A Note on Three-Point Functions of Conserved Currents," arXiv:1206.6370 [hep-th].

[17] H. Osborn and A. Petkou, "Implications of Conformal Invariance in Field Theories for General Dimensions," *Annals of Physics* **231** no. 2, (1994) 311–362.

[18] S. Caron-Huot and Y.-Z. Li, "Helicity Basis for Three-Dimensional Conformal Field Theory," *JHEP* **06** (2021) 041, arXiv:2102.08160 [hep-th].

[19] R. Ward, "Twistors in 2+1 Dimensions," *J. Math. Phys.* **30** no. 10, (1989) 2246.

[20] T. Adamo, "Lectures on Twistor Theory," *PoS* **Modave2017** (2018) 003, arXiv:1712.02196 [hep-th].

[21] H. Lee and X. Wang, "Cosmological Double-Copy Relations," *Phys. Rev. D* **108** no. 6, (2023) L061702, arXiv:2212.11282 [hep-th].

[22] J. Maldacena and G. L. Pimentel, "On Graviton Non-Gaussianities During Inflation," *JHEP* **09** (2011) 045, arXiv:1104.2846 [hep-th].

[23] V. Nair, "A Current Algebra for Some Gauge Theory Amplitudes," *Phys. Lett. B* **214** (1988) 215–218.

[24] E. Witten, "Perturbative Gauge Theory as a String Theory in Twistor Space," *Commun. Math. Phys.* **252** (2004) 189–258, arXiv:hep-th/0312171.

[25] N. Arkani-Hamed, F. Cachazo, C. Cheung, and J. Kaplan, "The S-Matrix in Twistor Space," *JHEP* **03** (2010) 110, arXiv:0903.2110 [hep-th].

[26] L. Mason and D. Skinner, "Scattering Amplitudes and BCFW Recursion in Twistor Space," *JHEP* **01** (2010) 064, arXiv:0903.2083 [hep-th].

[27] D. Meltzer, "Dispersion Formulas in QFTs, CFTs, and Holography," *JHEP* **05** (2021) 098, arXiv:2103.15839 [hep-th].

[28] D. Meltzer, "The Inflationary Wavefunction from Analyticity and Factorization," *JCAP* **12** no. 12, (2021) 018, arXiv:2107.10266 [hep-th].

[29] P. Benincasa and F. Cachazo, "Consistency Conditions on the S-Matrix of Massless Particles," arXiv:0705.4305 [hep-th].

[30] D. McGady and L. Rodina, "Higher-Spin Massless S-matrices in Four Dimensions," *Phys. Rev. D* **90** no. 8, (2014) 084048, arXiv:1311.2938 [hep-th].

[31] R. Britto, F. Cachazo, B. Feng, and E. Witten, "Direct Proof of Tree-Level Recursion Relation in Yang-Mills Theory," *Phys. Rev. Lett.* **94** (2005) 181602, arXiv:hep-th/0501052.

[32] N. Arkani-Hamed, F. Cachazo, C. Cheung, and J. Kaplan, "A Duality For The S Matrix," *JHEP* **03** (2010) 020, arXiv:0907.5418 [hep-th].

[33] N. Arkani-Hamed, P. Benincasa, and A. Postnikov, "Cosmological Polytopes and the Wavefunction of the Universe," arXiv:1709.02813 [hep-th].

[34] P. Benincasa, "Cosmological Polytopes and the Wavefuncton of the Universe for Light States," arXiv:1909.02517 [hep-th].

[35] N. Arkani-Hamed, D. Baumann, A. Hillman, A. Joyce, H. Lee, and G. L. Pimentel, "Kinematic Flow and the Emergence of Time," arXiv:2312.05300 [hep-th].

[36] N. Arkani-Hamed, D. Baumann, A. Hillman, A. Joyce, H. Lee, and G. L. Pimentel, "Differential Equations for Cosmological Correlators," arXiv:2312.05303 [hep-th].

[37] S. De and A. Pokraka, "Cosmology Meets Cohomology," *JHEP* **03** (2024) 156, arXiv:2308.03753 [hep-th].