Discovering Twistors in Cosmology

D. Baumann, G. Mathys, G. Pimentel and F. Rost

OUTLINE

I. Motivation

II. Embedding Space

III. An Invitation to Twistors

IV. Twistors in Cosmology

VI. Conclusions and Outlook

Paper: [arXiv:2408.02727]
Notes: https://github.com/ddbaumann/twistors

I. MOTIVATION

1.1. Why Twistors in Cosmology?

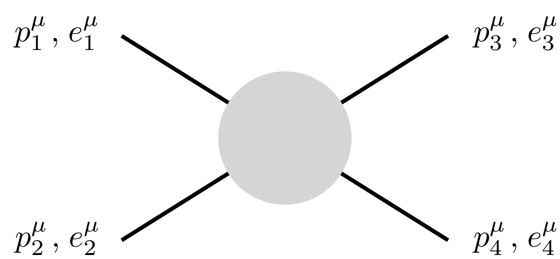
- Correlators of massless fields are **complicated**.
- Amplitudes of massless fields are **simple**:

$$A(1^-2^-3^+\cdots n^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle (n-1)n\rangle\langle n1\rangle}$$

- Reason: We are using inconvenient variables.
- Question: What are the "right" variables?
- Answer: Twistors!

1.2. What Are the Right Variables?

- **Scattering amplitudes**: (for massless particles)



$A(p_i^\mu, e_i^\mu)$

The "right" variables should make manifest:

1. *Lorentz invariance* $\Rightarrow$ $A(p_i\cdot p_j\,,\, e_i\cdot p_j\,,\, e_i\cdot e_j)$
2. *Two helicities* $\Rightarrow$ *Spinor helicity variables*

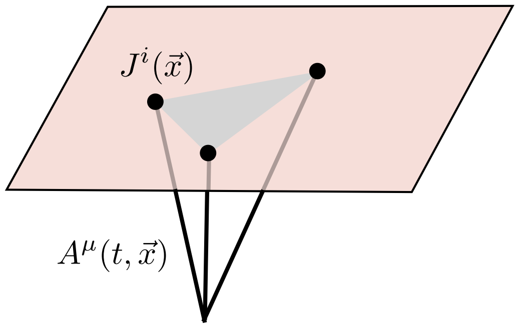
$$p_{\alpha\dot\alpha} = p^\mu(\sigma_\mu)_{\alpha\dot\alpha} = \begin{pmatrix} p_0+p_3 & p_1-ip_2 \\ p_1+ip_2 & p_0-p_3\end{pmatrix} = \lambda_\alpha\tilde\lambda_{\dot\alpha}$$

Given particles i and j, we have:

$$\begin{aligned}\langle ij\rangle &\equiv \epsilon^{\beta\alpha}\lambda_{i,\alpha}\lambda_{j,\beta}\\ [ij] &\equiv \epsilon^{\dot\beta\dot\alpha}\lambda_{i,\dot\alpha}\tilde\lambda_{j,\dot\beta}\end{aligned} \quad\Rightarrow\quad 2p_i\cdot p_j = \langle ij\rangle[ij]$$

Amplitudes take a simple form in terms of the Lorentz-invariant spinor brakets.

- **Cosmological correlators**: (for massless particles in (A)dS)



$$\langle J^i(\vec{x}_1) J^j(\vec{x}_2) J^k(\vec{x}_3)\rangle \equiv f^{ijk}(\vec{x}_1,\vec{x}_2,\vec{x}_3)$$

1. *Conformal invariance* $\quad\Rightarrow\quad$ Ward identity

$$\boxed{\sum_{a=1}^{3} b^l \left[2x_{a,l}\left(1 + x_a^m \frac{\partial}{\partial x_a^m}\right) - x_a^2 \frac{\partial}{\partial x_a^l} + \Sigma_{lm}^{(a)} x_a^m\right] f^{ijk}(\vec{x}_1,\vec{x}_2,\vec{x}_3) = 0} \tag{1}$$

+ translations, rotations, dilatations

2. *Current conservation* $\quad\Rightarrow\quad \int d^dx\, J_i A^i, \quad$ with $\boxed{\partial_i J^i = 0}$

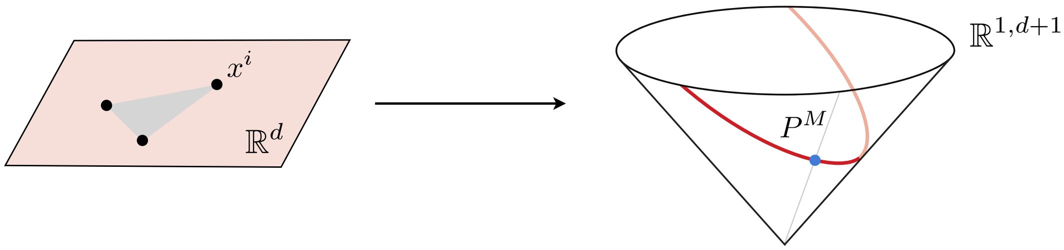
$$\boxed{\frac{\partial}{\partial x_1^i} f^{ijk}(\vec{x}_1,\vec{x}_2,\vec{x}_3) = 0} \quad \text{for} \quad \vec{x}_1 \neq \vec{x}_{2,3}\,. \tag{2}$$

Goal: Find variables that make both (1) and (2) manifest.

- **Step 1**: Trivialize (1) $\Rightarrow$ **Embedding Space**
- **Step 2**: Trivialize (2) $\Rightarrow$ **Twistors**

II. EMBEDDING SPACE

Conformal algebra on $\mathbb{R}^d$ is isomorphic to the algebra of Lorentz transformations on $\mathbb{R}^{1,d+1}$. By defining a suitable embedding of $\mathbb{R}^d$ into $\mathbb{R}^{1,d+1}$, the d-dimensional conformal transformations can be uplifted to $(d+2)$-dimensional Lorentz transformations.



- *Projective null cone* ($\mathbb{M}_{d+2}$):

$$P^2 = 0\,,$$
$$P^M \sim \rho P^M\,, \quad \text{where} \quad \rho \in \mathbb{R}\,.$$

- *Euclidean section* ($\mathbb{E}_d$):

$$(P^+, P^-, P^i) = (1, |\vec{x}|^2, x^i)\,, \quad \text{where} \quad P^{\pm} \equiv P^0 \pm P^{d+1}\,.$$

$\Rightarrow$ Lorentz on $\mathbb{M}_{d+2}$ = conformal on $\mathbb{E}_d$.

- *Tensors*:

Spin-S fields are described by symmetric tensors on $\mathbb{M}_{d+2}$, i.e. $O_{M_1 \dots M_S}(P)$. Write in index-free form:

$$O(P, W) = W^{M_1} \cdots W^{M_S} O_{M_1 \cdots M_S}(P)\,,$$

where $W^2 = P \cdot W = 0$.

- *Scaling*:

Fields of dimension Δ and spin S satisfy

$$O(\rho P, \alpha W) = \rho^{-\Delta} \alpha^S O(P, W)\,.$$

$\Rightarrow$ Scaling on $\mathbb{M}_{d+2}$ = conformal on $\mathbb{E}_d$.

- *Conformal correlators*:

  Lorentz invariance plus scaling gives

$$
\begin{aligned}
f(P_i, W_j) &\equiv \langle O(P_1, W_1) O(P_2, W_2) O(P_3, W_3) \rangle \\
&= \sum_n c_n F_n(P_i \cdot P_j, P_i \cdot W_j, W_i \cdot W_j)\,.
\end{aligned} \tag{*}
$$

  See notes for specific examples.

- *Current conservation*:

  Find the right linear combination in (*) that satisfies

$$
\frac{\partial}{\partial P_1^{M_1}} D^{M_1} f = 0\,, \quad D_M \equiv \left(1 + 2W \cdot \frac{\partial}{\partial W}\right) \frac{\partial}{\partial W^M} - W_M \frac{\partial^2}{\partial W \cdot \partial W}\,.
$$

  See notes for specific examples.

Problem: Why two steps? $\Rightarrow$ **Solution**: Twistors!

III. AN INVITATION TO TWISTORS

3.1. Twistors in d=2

Laplace equation in $\mathbb{R}^2$:

$$\nabla^2 f(x,y) = (\partial_x^2 + \partial_y^2) f(x,y) = 0 \quad \Leftarrow \quad \partial_z \partial_{\bar z} f = 0$$

Most general solution:

$$f(x,y) = \underset{\substack{\uparrow \\ \text{holomorphic}}}{F(z)} + \underset{\substack{\uparrow \\ \text{anti-holomorphic}}}{G(\bar z)}\,, \quad \text{where} \quad z \equiv x + iy \quad \text{and} \quad \bar z = x - iy\,.$$

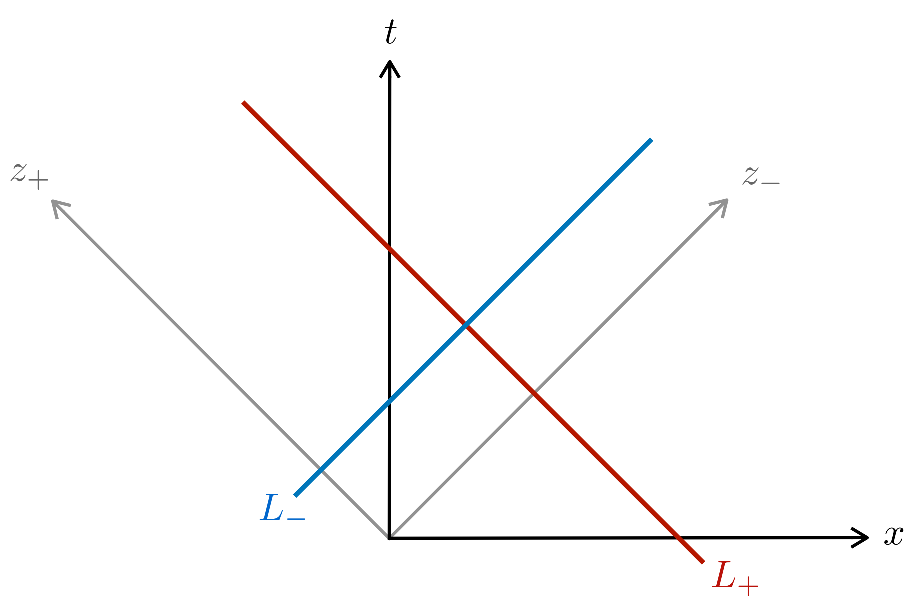
Klein–Gordon equation in $\mathbb{R}^{1,1}$:

$$\Box f(t,x) = \left(-\partial_t^2 + \partial_x^2\right) f(t,x) = 0$$

Most general solution:

$$f(t,x) = \underset{\substack{\uparrow \\ \text{holomorphic}}}{F(z_+)} + \underset{\substack{\uparrow \\ \text{anti-holomorphic}}}{G(z_-)}\,, \quad \text{where} \quad z_\pm \equiv t \pm x \in \mathbb{R}\,.$$

Twistors are **null lines** ($z_\pm = \text{const}$):



3.2. Twistors in d=3

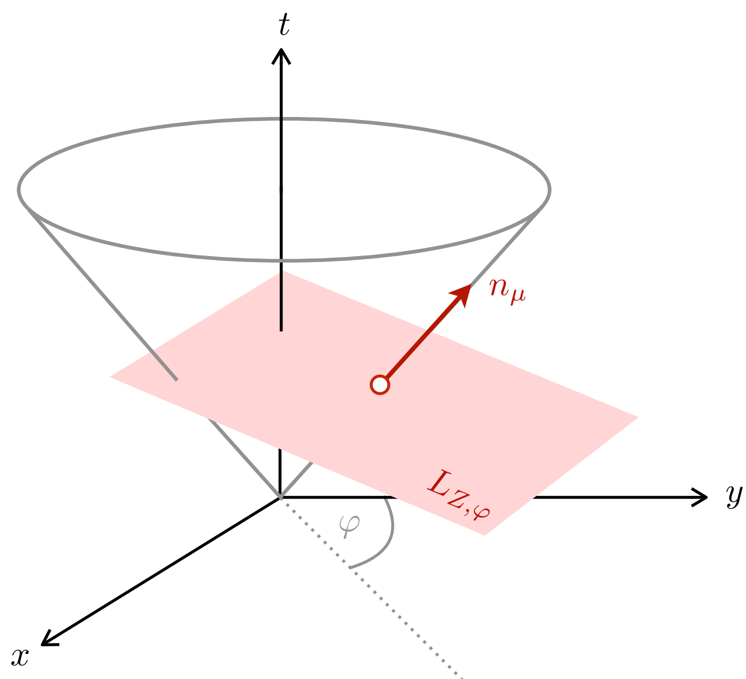
Klein–Gordon equation in $\mathbb{R}^{1,2}$:

$$\Box f(t,x,y) = \left(-\partial_t^2 + \partial_x^2 + \partial_x^2\right) f(t,x,y) = 0\,.$$

Solution:

$$f(t,x,y) = \underset{\substack{\uparrow\\ \text{holomorphic}}}{F(Z)} = F(t + x\sin\varphi + y\cos\varphi)\,, \quad \text{where} \quad \varphi \in \mathbb{R}\,.$$

Twistors are **null planes** (Z =const):



To remove dependence on φ, we integrate over it:

$$\begin{aligned} f(t,x,y) &= \frac{1}{2}\int_{-\pi}^{\pi} d\varphi F(Z) \\ &= \int D\pi\, F(\pi^a x_{ab}\pi^b)\,, \quad \text{where} \quad x_{ab} \equiv \begin{pmatrix} -t+x & -y \\ -y & -t-x \end{pmatrix} \end{aligned}$$

- Projective measure: $D\pi \equiv d\pi^a \pi^b \epsilon_{ab}$
- Projective invariance: $F(rZ) = r^{-2}F(Z)$
- Parameterization: $\pi^a \equiv (1, \tan(\varphi/2))$

3.3. Conserved Currents

Spin-1 current in $\mathbb{R}^{1,2}$:

$$\partial_\mu J^\mu = 0$$

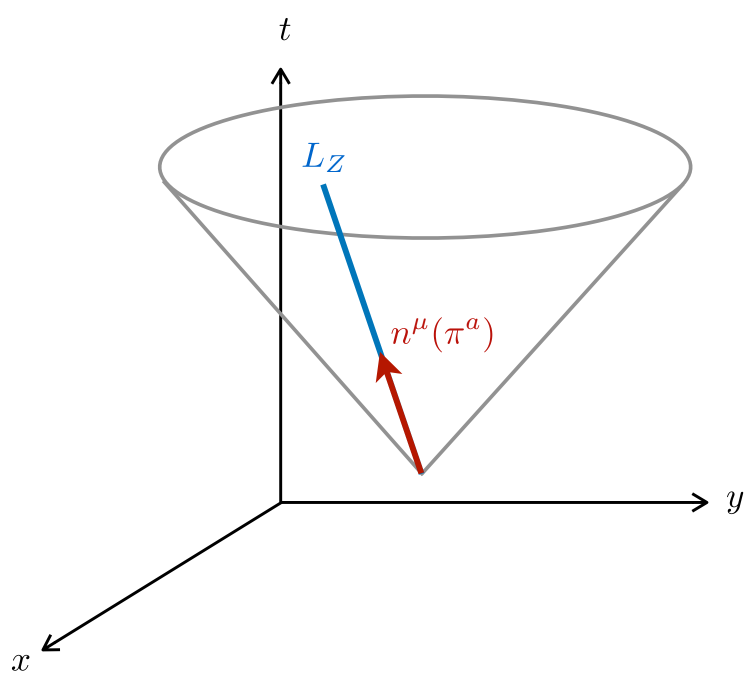
To solve this, let

$$J^\mu = (J^0, J^1, J^2) \quad \Rightarrow \quad J_{ab} = \begin{pmatrix} J^1 - J^0 & -J^2 \\ -J^2 & -J^0 - J^1 \end{pmatrix},$$

so that

$$\boxed{J_{ab}(x^\mu) = \pi_a \pi_b F(x_{ab}\pi^b)} \quad \Rightarrow \quad \frac{\partial}{\partial x^{ab}} J^{ab} \propto \pi_b \pi^b = \epsilon^{ab}\pi_a\pi_b = 0\,.$$

Twistors are **null lines** ($x_{ab}\pi^b = \mu_a \in \mathbb{R}$):



Define

$$Z^A \equiv (\pi^a, \mu_a) \quad \Rightarrow \quad J^{ab}(x^\mu) = \int D\pi\, \pi^a \pi^b F(Z^A)\Big|_{x_{ab}\pi^b = \mu_a}$$

- Projective measure: $D\pi \equiv d\pi^a \pi^b \epsilon_{ab}$
- Projective invariance: $F(rZ) = r^{-4} F(Z)$

IV. TWISTORS IN COSMOLOGY

4.1. A Road Map

The following logic will lead us to twistors:

$$x^i \quad \Rightarrow \quad P^M \quad \Rightarrow \quad P^{AB} \equiv P^M(\Gamma_M)^{AB} = \epsilon^{ab}\Lambda_a^A\Lambda_b^B \quad \Rightarrow \quad Z^A = \pi^a\Lambda_a^A$$

Correlators of conserved currents = holomorphic integrals in twistor space.

4.2. Spinors in Embedding Space

Let

$$P^{AB} \equiv P^M(\Gamma_M)^{AB} = \begin{pmatrix} 0 & -P^3-P^4 & P^2 & P^0+P^1 \\ P^3+P^4 & 0 & P^1-P^0 & -P^2 \\ -P^2 & P^0-P^1 & 0 & P^4-P^3 \\ -P^0-P^1 & P^2 & P^3-P^4 & 0 \end{pmatrix},$$

where $A, B = 1,2,3,4$.

Since $P^2 = 0$, the matrix P^{AB} has rank 2, and thus can be written as

$$P^{AB} = \epsilon^{ab}\Lambda_a^A\Lambda_b^B = \Lambda_1^A\Lambda_2^B - \Lambda_2^A\Lambda_1^B\,.$$

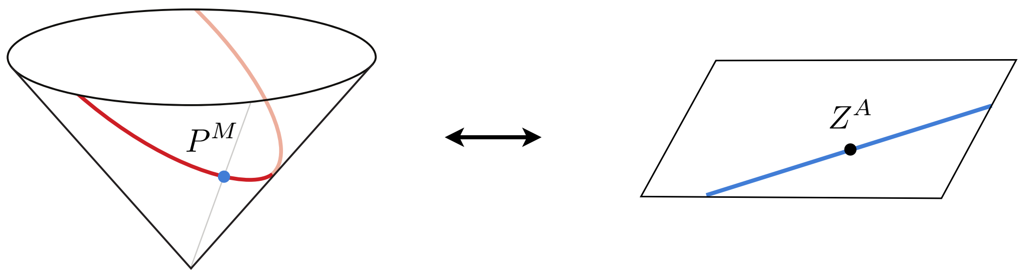
- Little group: $\Lambda_a^A \to V_a{}^b\Lambda_b^A$ implies $P^{AB} \to \det(V_a{}^b)P^{AB} \sim P^{AB}$
- Massless Dirac: $P_A{}^B\Lambda_a^A = 0$

4.3. Holomorphicity in Twistor Space

- 2d CFTs: Currents depend only on λ (holomorphic) *or* $\tilde{\lambda}$ (anti-holomorphic).
- 3d CFTs: Currents depend on Λ_1^A *and* Λ_2^A!

$\Rightarrow$ **Solution**: Currents are holomorphic functions $F(Z^A)$ in twistor space:

$$\boxed{Z^A \equiv \pi^a \Lambda_a^A} \, .$$



4.4. Currents as Twistor Integrals

Integrate over the arbitrary parameters π^a:

$$\boxed{J^{a_1 a_2 \cdots a_{2S}}(\Lambda_a) = \int D\pi \, \pi^{a_1} \pi^{a_2} \cdots \pi^{a_{2S}} F(\pi^a \Lambda_a)} \, .$$

- Conserved: $\partial_{a_1 a_2} J^{a_1 a_2 \cdots a_{2S}} = \int D\pi \, (\pi^{a_1} \pi_{a_1}) \cdots \partial_A F = 0$
- Projective invariance: $F(rZ) = r^{-2(S+1)} F(Z) \;\; \Rightarrow \;\; \Delta = S + 1$
- Index-free form:

$$\begin{aligned} J \equiv \zeta_{a_1} \cdots \zeta_{a_{2S}} J^{a_1 \cdots a_{2S}}(\Lambda_a) &= \int D\pi \, (\zeta_c \pi^c)^{2S} \, F(\pi^a \Lambda_a) \\ &= \int DZ \, (\Upsilon \cdot Z)^{2S} \, F(Z) \, , \end{aligned}$$

where $\Upsilon \cdot Z = \Upsilon^A \Omega_{AB} Z^B$, with $\Omega_{AB} = \begin{pmatrix} 0 & 1_{2\times 2} \\ -1_{2\times 2} & 0 \end{pmatrix}$.

- Alternative representation in *dual twistor space*:

$$\tilde{J} = \int DW \left(\Upsilon \cdot \frac{\partial}{\partial W} \right)^{2S} \tilde{F}(W) \, ,$$

with $W_A = \pi^a (\Omega_{AB} \Lambda_a^B)$.

4.5. Spinning Correlators in Twistor Space

So far: Single current.

Now: Correlators of multiple currents.

Example: Three-point functions of spin-S currents:

$$\langle J_1 J_2 J_3 \rangle = \left[\prod_{j=1}^{3} \int DZ_j \, (\Upsilon_j \cdot Z_j)^{2S} \right] F(Z_1, Z_2, Z_3) \tag{1}$$

$$\langle \tilde{J}_1 \tilde{J}_2 \tilde{J}_3 \rangle = \left[\prod_{j=1}^{3} \int DW_i \left(\Upsilon_j \cdot \frac{\partial}{\partial W_j} \right)^{2S} \right] \tilde{F}(W_1, W_2, W_3) \tag{2}$$

- Conservation manifest.
- Conformal symmetry: $F = F(Z_i \cdot Z_j)$ and $\tilde{F} = \tilde{F}(W_i \cdot W_j)$
- Scaling:

$$\begin{aligned} F(r_i Z_i) &= (r_1 r_2 r_3)^{-2S-2} F(Z_i) \,, \\ \tilde{F}(r_i W_i) &= (r_1 r_2 r_3)^{2S-2} \tilde{F}(W_i) \,, \end{aligned}$$

  This leads to

$$\begin{aligned} F(Z_i \cdot Z_j) &= \int \frac{d^3 c_{ij}}{(2\pi)^3} \exp\left(-i c_{12} Z_1 \cdot Z_2 + \text{cyclic}\right) A(c_{ij}) \,, \\ \tilde{F}(W_i \cdot W_j) &= \int \frac{d^3 c_{ij}}{(2\pi)^3} \exp\left(-i c_{12} W_1 \cdot W_2 + \text{cyclic}\right) \tilde{A}(c_{ij}) \,, \end{aligned}$$

  where

$$A(c_{ij}) = (c_{12} c_{23} c_{31})^S \quad \text{and} \quad \tilde{A}(c_{ij}) = \frac{1}{(c_{12} c_{23} c_{31})^S} \,.$$

  $\Rightarrow$ Remarkably **simple**!

- Integrals (1) and (2) give known results in **embedding space**:

$$\begin{aligned} \langle J_1 J_2 J_3 \rangle &= \{ \ F^3 \ , \ R^3 \ , \ \ldots \ \} \\ \langle \tilde{J}_1 \tilde{J}_2 \tilde{J}_3 \rangle &= \{ \ \text{YM} \ , \ \text{GR} \ , \ \ldots \ \} \end{aligned}$$

- Half-Fourier transform:

$$Z_i^A \equiv \begin{pmatrix} \lambda_i^\alpha \\ \mu_{i,\dot\alpha} \end{pmatrix} \quad \Rightarrow \quad g(\lambda_i, \tilde\lambda_i) \equiv \int d^2\mu_i \exp(i\tilde\lambda_i \cdot \mu_i)\, F(Z_i)\,,$$
$$W_{i,A} \equiv \begin{pmatrix} \tilde\mu_{i,\alpha} \\ \tilde\lambda_i^{\dot\alpha} \end{pmatrix} \quad \Rightarrow \quad \tilde g(\lambda_i, \tilde\lambda_i) \equiv \int d^2\tilde\mu_i \exp(-i\lambda_i \cdot \tilde\mu_i)\, \tilde F(W_i)\,.$$

This gives known results in **Fourier space**:

$$g(\lambda_i, \tilde\lambda_i) = \{\ F^3\ ,\ R^3\ ,\ \dots\ \}$$
$$\tilde g(\lambda_i, \tilde\lambda_i) = \mathrm{Disc}\,\{\ \mathrm{YM}\ ,\ \mathrm{GR}\ ,\ \dots\ \}$$

V. CONCLUSIONS AND OUTLOOK

In this work:

- Found variables that make all kinematic constraints manifest.
- Exposed a hidden simplicity of three-point functions.

Next steps:

- Extend to higher points
- Recursion relations
- Cosmological Parke–Taylor
- Positive geometry
