Linear Algebra Done Right

Week 3 Notes (a)

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5.A Invariant Subspaces

Suppose we can decompose V into the direct sum of subspaces:

$$V = U_1 \oplus U_2 \oplus ... \oplus U_n$$

Then we can study T restricted to U_i for $i \in \{1, ..., n\}$ respectively. However, $T|_U$ might not map U to itself. Therefore, the concept of invariant subspace is useful.

5.2 Definition of invariant subspace

Given $T \in \mathcal{L}(V)$, a subspace U of V is called **invariant** if $\forall u \in U, Tu \in U$.
In other words, U is invariant iff $T|_U$ is an operator on U.

5.3 5.4 Some examples of invariant subspaces

1. $\{0\}$ is invariant under T:
   $T(0) = 0 \in \{0\}$

2. V is invariant under T:

   $\forall v \in V, Tv \in V$

3. *null* T is invariant under T:

   $\forall v \in null\ T, Tv = 0 \in null\ T$

4. *range* T is invariant under T:

   $\forall v \in range\ T, Tv \in range\ T$

5. Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbf{R}))$ is defined by

   $$Tp = p'$$

   then $\mathcal{P}_4(\mathbf{R})$ is a invariant under T because $\forall p \in \mathcal{P}_4(\mathbf{R}), Tp = p' \in \mathcal{P}_3(\mathbf{R}) \subseteq \mathcal{P}_4(\mathbf{R})$

Eigenvalues and Eigenvectors

The class of invariant subspaces with dimension 1 is of special interests.

Given $v \in V$, we create $U = \{\lambda v : \lambda \in \mathbf{F}\} = span(v)$, then U is a 1-dimensional subspace of V. If U is invariant under T, then $Tv \in U \to \exists \lambda \in \mathbf{F}, Tv = \lambda v$.

Conversely, if $Tv = \lambda v$, then $span(v)$ is a 1-dimensional invariant subspace under T.

We thus give the defintion of eigenvalue and eigenvectors:

5.5 Defintion of eigenvalue

Given $T \in \mathcal{L}(V), \lambda \in \mathbf{F}$ is called **eigenvalue** if $\exists v \neq 0 \in V, Tv = \lambda v$.

5.6 Conditions for eigenvalue

Given V finite-dimensional, $T \in \mathcal{L}(V), \lambda \in \mathbf{F}$, then the following are equivalent:

(a) λ is an eigenvalue of T.

(b) $T - \lambda I$ is not injective.

(c) $T - \lambda I$ is not surjective.

(d) $T - \lambda I$ is not invertible.

Proof:

$a \iff b$: Because $Tv = \lambda v \rightarrow Tv - \lambda v = 0 \rightarrow (T - \lambda I)v = 0$

Therefore, $a \iff b$

It follows that $b \iff c \iff d$ since T is a linear operator.

5.7 Definition of eigenvector

Given $T \in \mathcal{L}(V)$. $v \neq 0 \in V$ is an **eigenvector** if $\exists \lambda \in \mathbf{F}, Tv = \lambda v$.

5.8 Example of eigenvector

Given $T \in \mathcal{L}(\mathbf{F}^2)$ defined by

$$T(w, z) = (-z, w)$$

(a) If $\mathbf{F} = \mathbf{R}$, then suppose $\exists v = (w, z) \in \mathbf{F}, Tv = \lambda v$, we have

$$\begin{cases} w = -\lambda z \\ z = \lambda w \end{cases} \tag{5.9}$$

Therefore $w = -\lambda^2 w$.

Since $\mathbf{F} = \mathbf{R}$, $w = 0 = z$.

Therefore, no eigenvector exists for T when $\mathbf{F} = \mathbf{R}$

(b) Following the logic from above, we know that when $\lambda = i$, then

$$-\lambda^2 w = -i^2 w = w$$

and $z = \lambda w = iw$.

Therefore T has eigenvalue i and eigenvector $v = (w, iw)$ when $\mathbf{F} = \mathbf{C}$

Properties of eigenvectors

5.10 Distinct eigenvectors are independent

Given $T \in \mathcal{L}(V)$, suppose $\lambda_1, ..., \lambda_m$ are distinct eigenvalues and $v_1, ..., v_m$ are corresponding eigenvectors. Then $v_1, ..., v_m$ linearly independent.

Proof:

Suppose $v_1, v_2, ...$ are all linearly independent until v_k. Then we have

$$v_k \in span(v_1, ..., v_{k-1}) \tag{5.11}$$

and therefore,

$$\sum_{i=1}^{k-1} a_i v_i = v_k \tag{5.12}$$

Since also $Tv_k = \lambda_k v_k$,

$$Tv_k = T\left(\sum_{i=1}^{k-1} a_i v_i\right) = \sum_{i=1}^{k-1} a_i \lambda_i v_i = \lambda_k v_k = \sum_{i=1}^{k} a_i \lambda_k v_i$$

This means that

$$\sum_{i=1}^{k-1} a_i \lambda_i v_i = \sum_{i=1}^{k-1} a_i \lambda_k v_i \Rightarrow \sum_{i=1}^{k-1} a_i (\lambda_i - \lambda_k) v_i = 0$$

Which is contradiction to the fact that $v_1, v_2, ...$ are linearly independent until v_k.

Therefore no such v_k exists, i.e. $v_1, ..., v_k$ are linearly independent.

5.13 Number of eigenvalues is smaller than dimension

Proof:

This naturally follows from the statment above since $v_1, ..., v_m$ are linearly independent, there can only be at most $dim\ V$ linearly independent vectors.

Restriction and Quotient Operators

5.14 Definition of $T|_U$ and T/U

Given $T \in \mathcal{L}(V)$ and U invariant in V under T.

- The **restirction operator** $T|_U$ is defined as

$$\forall u \in U, T|_U(u) = Tu$$

- The **quotient operator** T/U is defined as

$$\forall v \in V, (T/U)(v + U) = Tv + U$$

For the two defintions above, the fact that U is invariant under T is very important.

$T|_U$ only maps U to U when U is invariant.

For T/U to make sense, we need to show that $\forall, v, w \in V, v + U = w + U \Rightarrow Tv + U = Tw + U$:

$v + U = w + U \rightarrow \exists u \in U, u + v = w$. Therefore, we have

$$Tw = T(v + u) = Tv + Tu$$

Since U invariant under $T, Tu \in U$, i.e. $\exists u' \in U, Tv + u' = Tw$. This shows that $Tv + U = Tw + U$.

5.15 Limitations of restriction and quotient operators

Sometimes $T|_U$ and T/U do not provide useful information about T. See the example below:

Given $T \in \mathcal{L}(\mathbf{F}^2)$ defined by

$$T(x, y) = (y, 0)$$

and define $U = \{(x, 0) | x \in \mathbf{F}\}$, we observe the following facts:

(a) U invariant under T, and $T|_U$ is 0:
$\forall (x, 0) \in U, T(x, 0) = (0, 0) \in U$, therefore, U invariant under T.
This also proves that $T|_U = 0$.

(b) $\nexists$ a subspace $W \subseteq \mathbf{F}^2$ such that W invariant and $F^2 = U \oplus W$:
Suppose $\exists W$ that satisfies the requirements. Then we know that since $\mathbf{F}^2 = W \oplus U, \forall (x, y) \in W, x \neq 0 \iff y \neq 0$, because otherwise (x, y) would be of form $(x, 0), x \neq 0 \rightarrow (x, y) \in U$, contradiction to the fact that U, W independent.
Therefore given $(x, y) \in W, T(x, y) = (y, 0) \in U \rightarrow T(x, y) \notin W$. Therefore W not invariant under T.

(c) T/U is 0:
$\forall v = (x, y) \in \mathbf{F}^2, T/U(v + U) = T(x, y) + U = (y, 0) + U$
Now obviously $T(x, y) = (y, 0) \in U \rightarrow (y, 0) + U = U$, i.e. $\forall v \in \mathbf{F}^2, (T/U)(v + U) = U$.
Therefore $T/U = 0$.

The example above shows that while U is not $\{0\}$, the restriction and quotient on U doesn't provide any meaningful information of T.