Linear Algebra Done Right

HW 4

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4.1

(Axler 3E.13)

Suppose U is a subspace of V and $v_1 + U, ..., v_m + U$ is a basis of V/U and $u_1, ..., u_n$ is a basis of U. Prove that $v_1, ..., v_m, u_1, ..., u_n$ is a basis of V.

Solution:

First we NTS that $v_1, ..., v_m, u_1, ..., u_n$ span V:

Given $v \in V, \pi(v) = v + U = \sum_{i=1}^{n} a_i(v_i + U) = (\sum_{i=1}^{n} a_i v_i) + U$.

This means $\exists u \in U$ such that $v = \sum_{i=1}^{n} a_i v_i + u = \sum_{i=1}^{n} a_i v_i + \sum_{j=1}^{m} b_j u_j$.

Therefore we have shown that $v_1, ..., v_m, u_1, ..., u_n$ span V.

Next we NTS that $v_1, ..., v_m, u_1, ..., u_n$ independent:

Suppose $\exists \sum_{i=1}^{n} a_i v_i + \sum_{j=1}^{m} b_j u_j = 0 \to \sum_{i=1}^{n} a_i v_i + u = 0 \to \sum_{i=1}^{n} a_i v_i \in U$, this means

$$\sum_{i=1}^{n} a_i \left(v_i + U\right) = 0$$

Contradiction. Therefore $v_1, ..., v_m, u_1, ..., u_n$ independent.

Theerefore we have shown $v_1, ..., v_m, u_1, ..., u_n$ a basis.

4.2

(Axler 3F.7)

Suppose m is a positive integer. Show that the dual basis of the basis $1, x, ..., x^m$ of $\mathcal{P}_m(\mathbf{R})$ is $\phi_0, \phi_1, ..., \phi_m$ where

$$\phi_j(p) = \frac{p^{(j)}(0)}{j!}$$

Solution:

Essentially this is to calculate $\phi_j(x^i)$:

$$\phi_j(x^i) = \begin{cases} x^{i-j}(0) = 0 & i > j \\ 1(0) = 1 & i = j \\ 0(0) = 0 & i < j \end{cases}$$

Therefore we have shown that $\forall j \in \{1, ..., m\}$,

$$\phi_j(x^i) = \begin{cases} 0 & j \neq i \\ 1 & j = i \end{cases}$$

Therefore $\phi_0, ..., \phi_m$ is the dual basis.

4.3

(Axler 3F.8)

Suppose m is a positive integer.

(a) Show that $1, x-5, ..., (x-5)^m$ is a basis of $\mathcal{P}_m(\mathbf{R})$.

(b) What is the dual basis of the basis in part (a)?

Solution:

(a) First $1, x-5, ..., (x-5)^m$ is clearly independent since all coefficients need to be 0 for the polynomial to be 0 for all x.

Then we NTS that $1, x-5, ..., (x-5)^m$ span $\mathcal{P}_m(\mathbf{R})$:

To do this we show that $\forall i \in \{1, ..., m\}, x^i \in span(1, x-5, ..., (x-5)^m)$.

Suppose this is true for all $i \in \{1, ..., n-1\}$, WTS that $x^n \in span(1, x-5, ..., (x-5)^m)$.

First we have $(x-5)^n = x^n + \sum_{i=0}^{n-1} a_i x^i$.

Since $1, x, ..., x^{n-1} \in span(1, x-5, ..., (x-5)^m) \to \sum_{i=0}^{n-1} a_i x^i \in span(1, x-5, ..., (x-5)^m)$.

Therefore $x^n = (x-5)^n - \sum_{i=0}^{n-1} a_i x^i \in span(1, x-5, ..., (x-5)^m)$.

Therefore $1, x, ..., x^m \in span(1, x-5, ..., (x-5)^m)$, thus $span(1, x-5, ..., (x-5)^m) = \mathcal{P}_m(\mathbf{R})$.

THerefore $1, x-5, ..., (x-5)^m$ is a basis of $\mathcal{P}_m(\mathbf{R})$.

(b) Using the same reasoning as 3E.7 the dual basis are $\phi_0, \phi_1, ..., \phi_m$ where

$$\phi_j(p) = \frac{p^{(j)}(5)}{j!}$$

4.4

(Axler 3F.15)

Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $T' = 0 \iff T = 0$.

Solution:

This comes from the fact that $dim\ range\ T = dim\ range\ T'$. Therefore

$$\begin{aligned} T = 0 &\iff dim\ range\ T = 0 \\ &\iff dim\ range\ T' = 0 \\ &\iff T' = 0 \end{aligned}$$

4.5

(Axler 5A.12)

Define $T \in \mathcal{L}(\mathcal{P}_4(\mathbf{R}))$ by $(Tp)(x) = xp'(x)$. Find all eigenvalues and eigenvectors of T.

Suppose $\exists \lambda \in \mathbf{R}, p \in \mathcal{L}(\mathcal{P}_4(\mathbf{R})), Tp = \lambda p$.

Then we have $\lambda p(x) = xp'(x)$.

Denote p as $ax^4 + bx^3 + cx^2 + dx + e$, then $p' = 4ax^3 + 3bx^2 + 2cx + d, xp' = 4ax^4 + 3bx^3 + 2cx^2 + dx$, i.e.

$$\forall x \in \mathbf{R}, \lambda ax^4 + \lambda bx^3 + \lambda cx^2 + \lambda dx + \lambda e = 4ax^4 + 3bx^3 + 2cx^2 + dx$$

In order for $\lambda ax^4 = 4ax^4 \to \lambda = 4, b = 0, c = 0, d = 0, e = 0$, i.e. $p = ax^4$ is an eigenvector of T with eigenvalue 4.

Similarly:

- $p = bx^3$ is an eigenvector of T with eigenvalue 3
- $p = cx^2$ is an eigenvector of T with eigenvalue 2
- $p = dx$ is an eigenvector of T with eigenvalue 1

4.6

(Axler 5A.15)

Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.

(a) Prove that T and $S^{-1}TS$ have the same eigenvalues.

Suppose $\lambda \in \mathbf{F}$ is an eigenvalue of T. Then we know that $T - \lambda I$ is not injective, i.e. $T - \lambda I = 0$ has a non trivial solution.

Then we have

$$S^{-1}TS - \lambda I = S^{-1}TS - \lambda(S^{-1}IS) = S^{-1}TS - S^{-1}\lambda IS = S^{-1}(T - \lambda I)S$$

Therefore we can see that $S^{-1}TS - \lambda I$ is also not injective, i.e. λ is also an eigenvalue of $S^{-1}TS$.

Similarly, if $\lambda \in \mathbf{F}$ is an eigenvalue of $S^{-1}TS$, then $S^{-1}TS - \lambda I = 0$ has a non-trivial solution.

$$T - \lambda I = S(S^{-1}TS)S^{-1} - S(\lambda I)S^{-1} = S(S^{-1}TS - \lambda I)S^{-1}$$

Therefore $T\lambda I$ also has a non-trivial solution and λ is an eigenvalue of T.

This shows that T and $S^{-1}TS$ have the same eigenvalues.

(b) What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$?

From the reasoning in (a), it is easy to see that

- $\forall v$ such that $Tv = \lambda v, \exists S^{-1}vS$ such that

$$S^{-1}TS(S^{-1}vS) = S^{-1}(\lambda v)S = \lambda S^{-1}vS$$

– $\forall v$ such that $S^{-1}TSv = \lambda v, \exists SvS^{-1}$ such that $TSvS^{-1} = \lambda SvS^{-1}$

4.7

(Axler 5A.18)

Show that the operator $T \in \mathcal{L}(\mathbf{C}^\infty)$ defined by

$$T(z_1, z_2, ...) = (0, z_1, z_2, ...)$$

has no eigenvalues.

We show that $\forall v = (z_1, z_2, ...)$ such that $\exists \lambda, Tv = \lambda v, v = 0$.

Suppose k is the first index where $v_k = z_k \neq 0$. Then $Tv = (0, z_1, ...)$ will have $(Tv)_k = z_{k-1} = 0 = \lambda v_k = \lambda z_k \to \lambda = 0$.

Since also $(Tv)_{k+1} = v_k \neq 0 = \lambda v_{k+1} = 0$, which is a contradiction. Therefore no such k exists, i.e. $v = (z_1, z_2, ...) = 0$.

We have shown that no eigenvalues exist for T.

4.8

(Axler 5A.20)

Find all eigenvalues and eigenvectors of the backward shift operator $T \in \mathcal{L}(\mathbf{F}^\infty)$ defined by

$$T(z_1, z_2, z_3, ...) = (z_2, z_3, ...)$$

Suppose $\exists \lambda \in \mathbf{F}, v = (z_1, z_2, z_3, ...) \in \mathbf{F}^\infty$ such that $Tv = \lambda v$.

If $\lambda \neq 0$, then we have

$$\begin{cases} (Tv)_1 = \lambda v_1 = \lambda z_1 = z_2 \\ (Tv)_2 = \lambda v_2 = \lambda z_2 = \lambda^2 z_1 = z_3 \\ (Tv)_3 = \lambda v_3 = \lambda z_3 = \lambda^3 z_1 = z_4 \\ ... \end{cases}$$

Therefore $\forall \lambda \neq 0, v = (\lambda^0 z_1, \lambda^1 z_1, \lambda^2 z_1, ...)$ is an eigenvector with eigenvalue λ.

If $\lambda = 0$, then $Tv = 0 = (z_2, z_3, ...) \rightarrow z_2 = z_3 = ... = 0$.

Therefore $v = (z_1, 0, 0, ...)$ is an eigenvector with eigenvalue 0.

4.9

(Axler 5A.22)

Suppose $T \in \mathcal{L}(V)$ and there exists nonzero vectors v and w in V such that

$$Tv = 3w \text{ and } Tw = 3v$$

Prove that 3 or -3 is an eigenvalue of T.

Now suppose $v + w \neq 0$.

Then we have $u = v + w \neq 0, Tu = T(v + w) = Tv + Tw = 3w + 3v = 3(w + v) = 3u$.

Therefore we see that if $v + w \neq 0$, then 3 is an eigenvalue of T.

Now suppose $v + w = 0$.

This means $w = -v \rightarrow Tv = 3w = -3v$.

Therefore -3 is an eigenvalue of T.

Therefore either 3 or -3 is an eigenvalue of T.

4.10

(Axler 5A.30)

Suppose $T \in \mathcal{L}(\mathbf{R}^3)$ and $-4, 5$ and $\sqrt{7}$ are eigenvalues of T. Prove that there exists $x \in \mathbf{R}^3$ such that $Tx - 9x = (-4, 5, \sqrt{7})$.

Suppose that $T - 9I$ is not injective.

Then this means that 9 is an eigenvalue of T.

However, that would give T 4 eigenvalues, which is larger than the dimension of $V = \mathbf{R}^3$. Contradiction.

Therefore $T - 9I$ is injective, i.e. $(T - 9I)x = (-4, 5, \sqrt{7})$ has a unique solution.

4.11

(Axler 5A.32)

Suppose $\lambda_1, ..., \lambda_n$ is a list of distinct real numbers. Prove that the list $e^{\lambda_1 x}, ..., e^{\lambda_n x}$ is linearly independent in the vector space of real-valued functions on $\mathbf{R}$.

Denote V the vector space of real-valued functions on $\mathbf{R}$. Create $T \in \mathcal{L}(\mathbf{V})$ defined by

$$Tf = f'$$

It is obvious that T is a linear operator.

Now $\forall \lambda, T(e^{\lambda_i x}) = \lambda_i e^{\lambda_i x} \to e^{\lambda_i x}$ is an eigenvector of T with eigenvalue λ_i.

Therefore since $e^{\lambda_1 x}, ..., e^{\lambda_n x}$ is a list of eigenvectors with distinct eigenvalues, then they are linearly independent.

4.12

(Axler 5B.1)

4.13

(Axler 5B.2)

4.14

4.15

4.16