Abstract Algebra

Week 3 Notes (c)

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3.2 More on Cosets and Lagrange's Theorem

Lagrange's Theorem

If G is a finite group and $H \leq G$, then the order of H divides the order of G, i.e. $|H|\big||G|$ and the number of cosets of H in G is $\frac{|G|}{|H|}$.

Proof:

Let $|H| = n$ and number of left cosets of H be k.

We know that cosets of H partition G. Now we NTS that $\forall g \in G, |gH| = |H|$:

Define a map $H \to gH$ by $h \rightarrowtail gh$. This map is by definition surjective.

Then from cancellation law we can see that $h_1 \neq h_2 \in H \to gh_1 \neq gh_2$, i.e. the map is also injective and therefore bijective.

Therefore we conclude that $\forall g \in G, |gH| = |H|$.

This then shows that $|G| = \sum_{g \in G} |gH| = kn$.

Definition of index

Given G a group (could be infinite) and $H \leq G$, the number of left cosets of H in G is called the **index** of H in G and denoted $|G : H|$.

If G is finite then $|G:H| = \frac{|G|}{|H|}$. If G is infinite then the index could be either finite or infinite.

Now we look at some consequences of Lagrange's Theorem.

Order of elements divides the group

Given G a group and $x \in G$, then $|x|\big||G|$. In particular $\forall x \in G, x^{|G|} = 1$.

This is pretty obvious since $|x| = |\langle x \rangle|$.

Prime order group is cyclic

If G a group of prime order p, then G cyclic and $G \simeq Z_p$.

This is also pretty obvious.

Examples of Lagrange's Theorem

1. Given $H = \langle (1\ 2\ 3) \rangle \leq S_3$, we show that $H \trianglelefteq S_3$:

   This is essentially saying the $N_G(H) = G$.

   We know that $H \leq N_G(H) \leq G$, so $|H| = 3\big||N_G(H)|\big||G| = 6$, i.e. $|N_G(H)| = 3$ or 6.

   But since $(1\ 2)(1\ 2\ 3)(2\ 1) = (1\ 3\ 2) \in \langle (1\ 2\ 3) \rangle$.

   This means that $(1\ 2)$ conjugates a generator to another generator, therefore $|N_G(H)| \neq 3 \rightarrow |N_G(H)| = 6 \rightarrow N_G(H) = G$.

2. Given G a group with $H \leq G, |G:H| = 2$, show that $H \trianglelefteq G$:

   Given $g \in G \notin H$, we know that $gH = G - H$ since gH and H are the only two left cosets of H.

   For similar reasoning, $Hg = G - H = gH$. Therefore $H \trianglelefteq G$.

   This shows that $\langle i \rangle, \langle j \rangle, \langle k \rangle \trianglelefteq Q_8$ and $\langle s, r^2 \rangle, \langle r \rangle, \langle sr, r^2 \rangle \trianglelefteq D_8$.

3. The normal relationship is not transitive.

For example:

$$\langle s \rangle \trianglelefteq \langle s, r^2 \rangle \trianglelefteq D_8$$

But $\langle s \rangle$ is not normal in D_8.