Calculus

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1 Geometry of $\mathbb{R}^n$ (Ch 1.4)

Algebra is all about eqaulity, calculus is all about inequality.

1.1 Determinants and Traces

Currently we only define determinants and traces in $\mathbb{R}^2$.

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}, \det A = a_1 b_2 - b_1 a_2, \mathrm{tr} A = a_1 + b_2$$

Geometric interpretation of determinants

$$\forall \mathbf{a}, \mathbf{b} \in \mathbb{R}^2, \det[\mathbf{a}, \mathbf{b}] = \text{Area of parallelogram by} \mathbf{a}, \mathbf{b}$$

Proof:

We know that $\sin\theta = \sqrt{1-\cos^2\theta}$, and $\cos\theta = \frac{\mathbf{ab}}{|\mathbf{a}||\mathbf{b}|}$, therefore,

$$\begin{aligned}
\sin\theta &= \sqrt{1-\left(\frac{\mathbf{ab}}{|\mathbf{a}||\mathbf{b}|}\right)^2} \\
&= \sqrt{1-\frac{(a_1b_1+a_2b_2)^2}{(a_1^2+a_2^2)(b_1^2+b_2^2)}} \\
&= \sqrt{\frac{a_1^2b_1^2+a_1^2b_2^2+a_2^2b_1^2+a_2^2b_2^2-a_1^2b_1^2-a_2^2b_2^2-2a_1a_2b_1b_2}{a_1^2b_1^2+a_1^2b_2^2+a_2^2b_1^2+a_2^2b_2^2}} \\
&= \sqrt{\frac{(a_2b_1-a_1b_2)^2}{(|\mathbf{a}||\mathbf{b}|)^2}} \\
&= \frac{|\det[\mathbf{a},\mathbf{b}]|}{|\mathbf{a}||\mathbf{b}|}
\end{aligned}$$

i.e., $|\det[\mathbf{a},\mathbf{b}]| = \sin\theta \cdot |\mathbf{a}||\mathbf{b}|$

Given a standard $\mathbb{R}^2$, $\det[\mathbf{a},\mathbf{b}]$ is positive $\iff$ **a** counterclk-wise from **b**.

Proof:

Define **c** by rotating **a** by $\frac{\pi}{2}$, i.e. $\mathbf{c} = \begin{bmatrix} -a_2 \\ a_1 \end{bmatrix}$, then we have $\mathbf{bc} = a_1b_2 - a_2b_1 = \det[\mathbf{a},\mathbf{b}]$.

$$\det[\mathbf{a},\mathbf{b}] > 0 \Rightarrow \theta < \frac{\pi}{2} \Rightarrow \theta + \frac{\pi}{2} < \pi$$

i.e., **a** counterclk-wise from **b**.

Determinants and cross products in $\mathbb{R}^3$

Determinants in $\mathbb{R}^3$

$$\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = a_1 \det\begin{bmatrix} b_2 & c_2 \\ b_3 & c_3 \end{bmatrix} - a_2 \det\begin{bmatrix} b_1 & c_1 \\ b_3 & c_3 \end{bmatrix} + a_3 \det\begin{bmatrix} b_2 & c_2 \\ b_3 & c_3 \end{bmatrix}$$

Cross product in $\mathbb{R}^3$

$$\forall \mathbf{a}, \mathbf{b} \in \mathbf{R}^3, \mathbf{a} \times \mathbf{b} = \text{components of } \det \begin{bmatrix} 1 & a_1 & b_1 \\ 1 & a_2 & b_2 \\ 1 & a_3 & b_3 \end{bmatrix} = \begin{bmatrix} \det \begin{bmatrix} a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} \\ -\det \begin{bmatrix} a_1 & b_1 \\ a_3 & b_3 \end{bmatrix} \\ \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \end{bmatrix}$$

From the definition we can easily see that

$$\det[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

Geometric interpretation of cross product

1. $\mathbf{a} \times \mathbf{b}$ orthogonal to the plane spanned by $\mathbf{a}, \mathbf{b}$.

2. $|\mathbf{a} \times \mathbf{b}|$ is the area of the parallelogram between $\mathbf{a}, \mathbf{b}$.

3. $\mathbf{a}, \mathbf{b}$ no colinear $\iff \det[\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}] > 0$

Proof:

The first one is easy to verify by multiplying $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$ and so on.

For 2, we use $\sin\theta = \sqrt{1-\cos^2\theta}$:

$$\begin{aligned}
\sin\theta &= \sqrt{1-\cos^2\theta} \\
&= \sqrt{1-\left(\frac{\mathbf{ab}}{|\mathbf{a}||\mathbf{b}|}\right)^2} \\
&= \sqrt{\frac{(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)-(a_1b_1+a_2b_2+a_3b_3)^2}{(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)}} \\
&= \sqrt{\frac{a_1^2b_2^2+a_1^2b_3^2+a_2^2b_1^2+a_2^2b_3^2+a_3^2b_1^2+a_3^2b_2^2-2a_1a_2b_1b_2-2a_1a_3b_1b_3-2a_2a_3b_2b_3}{(|\mathbf{a}||\mathbf{b}|)^{\mathbf{2}}}} \\
&= \sqrt{\frac{(a_1b_2-a_2b_1)^2+(a_1b_3-a_3b_1)^2+(a_2b_3-a_3b_2)^2}{(|\mathbf{a}||\mathbf{b}|)^{\mathbf{2}}}}
\end{aligned}$$

i.e. Area of parallelogram $= |\mathbf{a}||\mathbf{b}|\sin\theta = |\mathbf{a}\times\mathbf{b}|$