Calculus

Notes on 2023/03/28

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March 2023

1 Geometry of $\mathbb{R}^n$ (Ch 1.4)

Algebra is all about eqaulity, calculus is all about inequality.

1.1 Interpretation of dot product

$$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}| \cos(\theta) \tag{1.4.3}$$

Proof:

This comes from the **cosine law**.

Since we know $\mathbf{x}$, $\mathbf{y}$, $\mathbf{x} - \mathbf{y}$ form a triangle, therefore:

$$|\mathbf{x} - \mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2|\mathbf{x}||\mathbf{y}| \cos \alpha$$

Since also,

$$|\mathbf{x} - \mathbf{y}|^2 = (\mathbf{x} - \mathbf{y})^2$$

We have

$$|\mathbf{x}|^2 + |\mathbf{y}|^2 - 2\mathbf{x}\mathbf{y} = |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2|\mathbf{x}||\mathbf{y}| \cos \alpha$$

$$\mathbf{x}\mathbf{y} = |\mathbf{x}||\mathbf{y}| \cos \alpha$$

1.2 Angle between vectors in $\mathbb{R}^n > 3$

We want to define angles between $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n, n > 3$ with

$$\alpha = \arccos \frac{\mathbf{xy}}{|\mathbf{x}||\mathbf{y}|}$$

However we cannot guarantee that $\frac{\mathbf{xy}}{|\mathbf{x}||\mathbf{y}|} \in [-1, 1]$. Hence we need to first prove the following:

Schwartz's Inequality

$$\forall \mathbf{v}, \mathbf{w} \in \mathbb{R}^n, |\mathbf{vw}| \leq |\mathbf{v}||\mathbf{w}| \tag{1.4.5}$$

Proof:

Suppose $\mathbf{v}, \mathbf{w} \neq 0$, then consider $f(t) = |\mathbf{v} + t\mathbf{w}|^2$:

$$f(t) = |\mathbf{w}|^2 t^2 + 2\mathbf{vw}t + |\mathbf{v}|^2$$

It is obvious that $f(t) \geq 0$, this means that

$$\begin{aligned} (2\mathbf{vw})^2 - 4|\mathbf{v}|^2|\mathbf{w}|^2 < 0 \\ 4|\mathbf{vw}|^2 < 4(|\mathbf{v}||\mathbf{w}|)^2 \end{aligned}$$

With Schwartz's Inequality, we have proven that

$$\forall \mathbf{v}, \mathbf{w} \in \mathbb{R}^n, -1 \leq \frac{\mathbf{vw}}{|\mathbf{v}||\mathbf{w}|} \leq 1$$

And now we can define the angle between any two vectors as:

$$\alpha = \arccos \frac{\mathbf{vw}}{|\mathbf{v}||\mathbf{w}|}$$

1.2.1 Application of Schwartz's Inequality

$$\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n, |\mathbf{x}+\mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}| \tag{1.4.9}$$

Proof:

For the LHS, we have

$$\begin{aligned} |\mathbf{x}+\mathbf{y}|^2 &= (\mathbf{x}+\mathbf{y})^2 \\ &= |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\mathbf{x}\mathbf{y} \end{aligned}$$

For the RHS, we have

$$(|\mathbf{x}| + |\mathbf{y}|)^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2|\mathbf{x}||\mathbf{y}|$$

Since from Schwartz's Inequality we know that $2\mathbf{x}\mathbf{y} \leq 2|x||y|$, therefore,

$$|\mathbf{x}+\mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\mathbf{x}\mathbf{y} \leq (|\mathbf{x}| + |\mathbf{y}|)^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2|\mathbf{x}||\mathbf{y}|$$

1.3 Geometry of Matrices

First we need to define **the length of a matrix:**

$$\forall A \in \mathbb{R}^{n\times m}, |A|^2 = \sum_{i,j} a_{ij}^2 \tag{1.4.10}$$

Then we can define the **product involving matrices:**

$$\forall A \in \mathbb{R}^{n\times m}, B \in \mathbb{R}^{m\times k}, b \in \mathbb{R}^m |A\mathbf{b}| \leq |A||\mathbf{b}|, |AB| \leq |A||B| \tag{1.4.11}$$

Proof:

Using $A_{i\_}$ to denote the vector consisting of $[a_{i1}, a_{i2}, ..., a_{im}]$, we have:

$$|A\mathbf{b}|^2 = \sum_{1 \le i \le n} (A_{i\_}\mathbf{b})^2$$

while

$$(|A||b|)^2 = \sum_{1 \le i \le n} |A_{i\_}|^2 |\mathbf{b}|^2$$

Since from Schwartz's Inequality we know that $(A_{i\_}b)^2 \le |A_{i\_}|^2 |\mathbf{b}|^2$, we have proven that

$$|A\mathbf{b}|^2 \le (|A||\mathbf{b}|)^2$$

For $|AB|^2$, we know that

$$|AB|^2 = \sum_{1 \le j \le k} |AB_{\text{-}j}|^2$$

And since for each $|AB_{\text{-}j}|^2 \le (|A||B_{\text{-}j}|)^2$ (proven previously), we know that

$$|A|^2 |B|^2 = |A|^2 \sum_{1 \le j \le k} B_{\text{-}j}^2 \ge |AB|^2$$