Fundamental Concepts of Analysis

Week 1 Notes (b)

shaozewxy

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7. Integers, Rationals and Exponents

Summary

1. Define integers from positive integers.

2. Define rational numbers from integers.

3. Define integer exponnts from rational numbers.

   Show $\mathbf{R} \neq \mathbf{Q}$ using exponents and $\sqrt{2}$.

4. *Define n-th root using upper bound. (Now the context has expanded to real numbers).

5. Expand n-th root to all real bases. (**Negative bases don't have even n-th root**)

6. Define exponents with rational exponents.

   First nonnegative base with positive exponent, corresponding to n-th root of nonnegative real.

Then all base with positive odd exponent, corresponding to odd n-th root of any real.

Then expand to negative exponents.

Then expand to any rationals.

7. *Density of rational numbers.

8. Density of irrational numbers using density of rational numbers.

Deifnition of integers

The set of **integers**, denoted $\mathbf{Z}$ is the set

$$\{0\} \cup \mathbf{P} \cup -\mathbf{P}$$

where $-\mathbf{P} = \{-n | n \in \mathbf{P}\}$.

Integers is a group satisfying axioms 1 to 5.

Definition of rational numbers

The set of **rational numbers**, denoted $\mathbf{Q}$ is the set

$$\left\{ \frac{p}{q} | p, q \in \mathbf{Z}, q \neq 0 \right\}$$

The rational numbers is a field satisfying axioms 1 to 11.

Definition of integer exponents

Given $x \in \mathbf{R}$, define

$$x^1 = x, x^{n+1} = x \cdot x^n, n \in \mathbf{P}$$

We can then expand this defintion to all integers:

$$x^0 = 1, x^{-1} = \frac{1}{x^n}, n \in \mathbf{P}$$

Now it is important to show that $\mathbf{R} \neq \mathbf{Q}$.

7.4 Square root of 2

$\nexists r \in \mathbf{Q}$ such that $r^2 = w$.

Proof:

Suppose $\exists r = \frac{p}{q} \in \mathbf{Q}$ such that $r^2 = 2$.

Then assume that not both p, q are even, since if they are, we can keep divide both by 2 until they are not both even.

Then we have $(\frac{p}{q})^2 = \frac{p^2}{q^2} = 2$.

Therefore $p^2 = 2q^2 \to p$ is even, i.e. $\exists k \in \mathbf{Z}$ such that $p = 2k$.

Therefore $2q^2 = (2k)^2 = 4k^2 \to q^2 = 2k^2$, i.e. q is also even, contradiction.

We then prove that such a root exists in $\mathbf{R}$.

7.5 Existence of nth root

Suppose a non-negative integer and $n \in \mathbf{P}$, then $\exists b \geq 0 \in \mathbf{R}$ such taht $b^n = a$.

Proof:

We define

$$X = \{x | x^n \leq a, x \geq 0\} \tag{1}$$

We NTS that there exists a least upper bound for X:

First clearly X is not empty since $0 \in X$.

Then clearly X is bounded above since $a + 1$ would be an upper bound.

Therefore X has a least upper bound, $b = lub\ X$, we claim that $b^n = a$.

Now either $b^n = a$ or $b^n < a, b^n > a$.

We WTS that $b^n < a$ is not possible:

Suppose $b^n < a \to \exists \epsilon > 0 \in \mathbf{R}, \epsilon = a - b^n$.

Then we just need to construct a $b + r$ such that $(b + r)^n < a$ and this will result in a contradiction since $b + r \in X, b + r > b$.

$$\begin{aligned}(b+r)^n &= \sum_{k=0}^{n} \begin{pmatrix} n \\ \\ k \end{pmatrix} b^k r^{n-k} \\ &= b^n + \sum_{k=0}^{n-1} \begin{pmatrix} n \\ \\ k \end{pmatrix} b^k r^{n-k}\end{aligned}$$

Therefore, we just need to pick r so that for each

$$\begin{pmatrix} n \\ \\ k \end{pmatrix} b^k r^{n-k} < \frac{\epsilon}{n}$$

To do so, pick $r = \frac{1}{p}$ where $p \in \mathbf{P}$.

This is to pick a $p \in \mathbf{P}$ such that $p^{n-k} >$ some real number. Since $\mathbf{P}$ is not bounded above, we know that such p exists.

Therefore we have that $b + \frac{1}{p} \in X, b + \frac{1}{p} > b$, which is impossible.

Therefore $b^n < a$ is impossible.

The case against $b^n > a$ can be similarly proven.

We can then expand the definition to when a is negative.

7.6 Existence of nth root expanded

If $a \in \mathbf{R}$ and $n \in \mathbf{P}$ is odd, then $\exists b \in \mathbf{R}$ such that $b^n = a$.

Proof:

Suppose a is positive, then we have proven this in 7.5.

Suppose a is negative, then again we know that by 7.5 $\exists c \in \mathbf{R}$ such that $c^n = |a|$.

Then let $b = -c \to b^n = (-1 \cdot c)^n = -c^n = -|a| = a$.

We can then expand the two results above to a definiton of rational exponents of real numbers.

Definition of rational exponents of real numbers

Nonnegative base with positive rational exponent

Suppose $x \in \mathbf{R}$ is nonnegative and $n\mathbf{P}$, define $x^{1/n}$ to be the nonnegative real number y such that $y^n = x$.

Real base with positive odd rational exponent

Suppose $x \in \mathbf{R}$ and $n \in \mathbf{P}$ is odd, then define $x^{1/n}$ to be the real number such that $y^n = x$.

Real base with negative rational exponent

Suppose $x \in \mathbf{R}$ and $n \in \mathbf{P}$, define

$$x^{-1/n} = \frac{1}{x^{1/n}}$$

If $x \in \mathbf{R}$ and $r = p/q \in \mathbf{Q}$ with p/q the lowest term of r, then define

$$x^r = (x^{1/q})^p$$

In these definitions, there doesn't exists a case where $x \in \mathbf{R}$ is negative whil $n \in \mathbf{P}$ is even, this is because $\forall y \in \mathbf{R}, y^n \geq 0 \neq x$, i.e $\nexists (-x)^{1/2k}$.

We then show that there are many rational numbers and real numbers:

7.8 Existence of rational numbers

Suppose $a, b \in \mathbf{R}$ and $a < b$, then $\exists r \in \mathbf{Q}$ such that $a < r < b$.

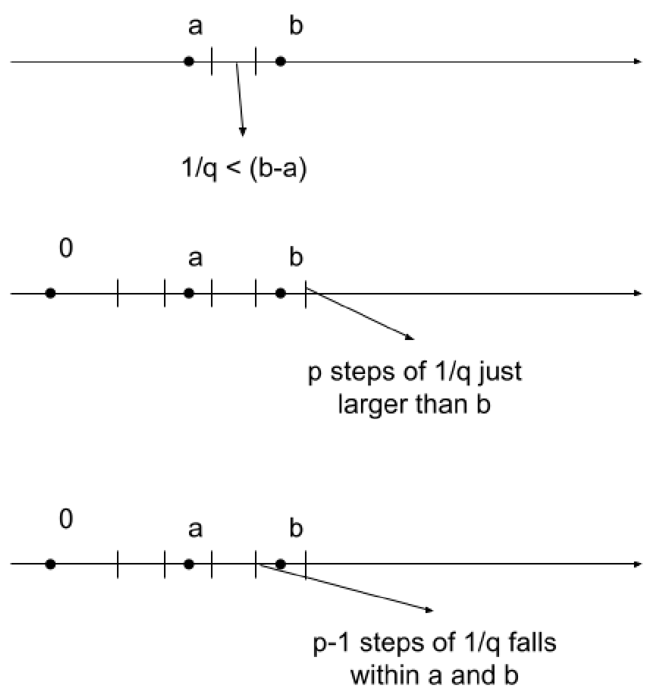
Proof:

Summarize:

We first construct a rational unit length $1/q$ so that it is less then the delta between a, b.

Then we start from 0 and walk towards b using this unit length until we are barely larger than b.

Then we walk back one unit length, then we are bound to be in between a, b.



7.9 Sum with irrational number is irrational

Sum of a rational number and an irrational number is irrational.

Proof:

Suppose s rational and t irrational and $s + t$ rational. Then $t = s + t + (-t)$ is still rational. Contradiction.

7.10 Existence of irrational numbers

Suppose $a, b \in \mathbf{R}$ with $a < b$, then $\exists s$ irrational such that $a < s < b$.

Proof:

By 7.8 $\exists t \in \mathbf{Q}$ such that $a - \sqrt{2} < t < b - \sqrt{2}$, then $a < t + \sqrt{2} < b$.

By 7.9 $t + \sqrt{2}$ is irrational.