Fundamental Concepts of Analysis

Week 1 Notes (d)

shaozewxy

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Summary

A list of basic theorem for sequences.

10 Sequences of Real Numbers

Definition of sequences

Given X a set. A **sequence** of elements of X is a function from $\mathbf{P}$ the set of positive integers to X.

So **a real sequence** is a function from $\mathbf{P} \to \mathbf{R}$.

The notion is $\{a_n\}_{n=1}^{\infty}$ where a denotes the function and a_n is the value of the function at $n \in \mathbf{P}$.

Can also use $a_1, a_2, ...$ or $\{a_n\}$ to denote a sequence.

Definition of limit of a sequence

Given $\{a_n\}$ a sequence, we say that $\{a_n\}$ has **limit** $L \in \mathbf{R}$ if $\forall \epsilon > 0, \exists N \in \mathbf{P}$ such that $n \geq N \to |a_n - L| < \epsilon$.

10.3 Limit is unique

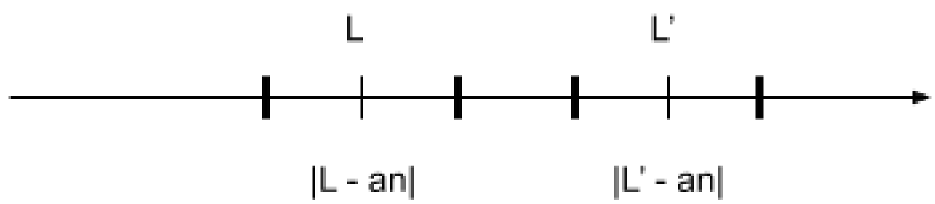
Proof:

Suppose $L \neq L'$ both limits of $\{a_n\}$.

Then choose $\epsilon = |L - L'|/4$, we have that $\exists N_1$ such that $\forall n > N_1 \in \mathbf{P}, |L - a_n| < \epsilon$, similarly N_2 exists for L', we choose N to be the larger of N_1, N_2.

Then choose $n > N$, we have $|L - a_n| < \epsilon, |L' - a_n| < \epsilon$.

However, it is obvious such a_n doesn't exist:



Now we can denote this unique limit of $\{a_n\}$ as $\lim_{n\to\infty} a_n = L$.

Test of non-limit

From the definition we know that $\{a_n\}$ doesn't have a limit L if $\exists \epsilon$ such that there are infinitely many $n \in \mathbf{P}$ such that $|L - a_n| > \epsilon$.

11 Subsequences

Definition of subsequence

Given $\{a_n\}$ a sequence. Define a $f : \mathbf{P} \to \mathbf{P}$ to be strictly increasing. The sequence $\{a_{f(n)}\}_{n=1}^{\infty}$ is called a **subsequence** of the sequence $\{a_n\}$.

11.2 Subsequences share the limit

Given $\{a_n\}$ with limit L, then any subsequence of $\{a_n\}$ also has limit L.

Proof:

$\forall \epsilon > 0$ we know that $\exists N$ such that $\forall n > N \in \mathbf{P}, |a_n - L| < \epsilon$.

Then given an strictly increasing $f : \mathbf{P} \to \mathbf{P}$, we know $\exists N'$ such that $f(N') > N$.

Then we say that with this $N', \forall n \in \mathbf{P} > N', a_{f(n)} = a_{n'}$ for some $n' = f(n) > N' > N, \to |L - a_{f(n)}| < \epsilon$.

12 Algebra of Limits

We say that $\{a_n\}$ is **convergent** if it has a limit and **divergent** if it doesn't have a limit.

12.1 Unchanged sequence converges

The proof of this is obvious.

12.2-12.9 Operation of limits

These operations of convergent sequences reflect to their limits:

Addition, subtraction, multiplication with number, multiplication, division (given it makes sense).

Here the case of division needs to be taken special care of.

Suppose $\lim_{n\to\infty} a_n = L \neq 0$, then there can **only be finitely many** $a_n = 0$.

This comes from the fact that L is the limit of $\{a_n\}$, so as n increases, $\{a_n\}$ will contract into a smaller range centering on L, only finitely many n outside of this range and could be $= 0$.

Then we can define $\frac{1}{a_n}$ as a sequence with limit $\frac{1}{L}$.

13 Bounded Sequences

Definition of bounded

A sequence $\{a_n\}$ is **bounded above(below)** if $\exists M$ such that $\forall n \in \mathbf{P} a_n \leq M (a_n \geq M)$.

From this definition we can see that $\{a_n\}$ is bounded $\iff \exists M \mathbf{P}$ such that $\forall n \in \mathbf{P}, |a_n| \leq M$.

Therefore we have

13.2 Convergent sequences are bounded

13.3 Multiplication of bounded sequeces

Given $\{a_n\}, \{b_n\}$ with $\lim_{n \to \infty} a_n = 0$ and $\{b_n\}$ bounded, then we have $\lim_{n \to \infty} a_n b_n = 0$.