Linear Algebra Done Right

Week 4 Notes (b)

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September 2022

5.C Eigenspaces and Diagonal Matrices

Definition of eigenspace

Given $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$. The **eigenspace** of T corresponding to λ denoted $E(\lambda, T)$ is defined by

$$E(\lambda, T) = null(T - \lambda I)$$

This is to say, $E(\lambda, T)$ is the set of eigenvectors of T corresponding to λ.

5.38 Sum of eigenspaces is a direct sum

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$ with $\lambda_1, ..., \lambda_m$ are distinct eigenvalues of T. Then

$$E(\lambda_1, T) + ... + E(\lambda_m, T)$$

is a direct sum. Furthermore,

$$dim\ E(\lambda_1, T) + ... + dim\ E(\lambda_m, T) \leq dim\ V$$

Proof:

To prove $E(\lambda_i, T)$ are linearly independent, suppose $\exists v_1, ..., v_m$ such that $v_1 + ... + v_m = 0$, but since they are eigenvectors, it is impossible due to 5.10. Then $\sum dim\ E(\lambda_i, T) = dim(\bigoplus E(\lambda_i, T)) \leq dim\ V$

Defintion of diagonalizable

$T \in \mathcal{L}(V)$ is called **diagonalizable** if T has a diagonal matrix w.r.t. some basis of V.

Example of diagonalizable operator

Define $T \in \mathcal{T}(\mathbf{R}^2)$ by

$$T(x, y) = (41x + 7y, -20x + 74y)$$

T with basis $(1, 4)(7, 5)$ has a matrix

$$\begin{pmatrix} 69 & 0 \\ 0 & 46 \end{pmatrix}$$

Given $(x, y) \in \mathbb{R}^2$, we have

$$\begin{pmatrix} 1 & 7 & x \\ 4 & 5 & y \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & \frac{7y-5x}{23} \\ 0 & 1 & \frac{4x-y}{23} \end{pmatrix}$$

Then

$$\begin{pmatrix} 1 & 7 \\ 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 69 & 0 \\ 0 & 46 \end{pmatrix} \cdot \begin{pmatrix} \frac{7y-5x}{23} \\ \frac{4x-y}{23} \end{pmatrix} = \begin{pmatrix} 41x + 7y \\ -20x + 74y \end{pmatrix}$$

5.41 Conditions for diagonalizability

Given V finite dimensional with $T \in \mathcal{L}(V)$. Let $\lambda_1, ..., \lambda_m$ denote the distinct eigenvalues of T. Then the following are equivalent:

(a) T diagonalizable.

(b) V has a basis consisting of eigenvectors of T.

(c) There exists 1-dimensional subspaces $U_1, ..., U_n$ of V each invariant under T such that

$$V = U_1 \oplus ... \oplus U_n$$

(d) $V = \bigoplus E(\lambda_i, T)$

(e) $dim\ V = \sum dim\ E(\lambda_i, T)$

Proof:

First it is clear that $a \iff b$.

Now $b \to c$ since each U_i can just be $span(v_i)$.

Similarly, suppose c holds, picking $v_i \neq 0$ from U_i creates a basis of V. Therefore $c \to b$.

We have now proven that $a \iff b \iff c$. Next WTS that $b \to d \to e \to b$:

Suppose b holds: then $\forall v \in V, v = \sum a_i v_i$. Since $E(\lambda_i, T)$ are linearly independent, we know that $V = \bigoplus E(\lambda_i, T)$. THerefore $b \to d$.

Suppose d holds: then obviously $\sum dim\ E(\lambda_i, T) = dim\ \bigoplus E(\lambda_i, T) = dim\ V$. THerefore $d \to e$.

Suppose e holds: then for each $E(\lambda_i, T)$, pick $dim\ E(\lambda_i, T)$ independent vectors from it. This forms a basis of V since the result is still independent and the size equals to $dim\ V$. Obviously this basis consist of eigenvectors. Therefore $e \to b$.

Now every operator is diagonalizable even in complex space.

Example of un-diagonalizable operator

The operator $T \in \mathcal{L}(\mathbf{C}^2)$ defined by

$$T(w,z) = (z,0)$$

is not diagonalizable.

Proof:

Suppose $Tv = T(w,z) = \lambda v = (\lambda w, \lambda z)$, then we have

$$\begin{cases} \lambda z = 0 \\ \lambda w = z \end{cases}$$

If $\lambda \neq 0$, then $z = w = 0$.

5.44 Enough eigenvalues implies diagonalizability

If $T \in \mathcal{L}(V)$ has $dim\ V$ distinct eigenvalues then T diagonalizable.

This is obvious since the corresponding eigenvectors form a basis of V.

5.45 Example of finding diagonal matrices

Define $T \in \mathcal{L}(\mathbf{F}^3)$ by

$$T(x,y,z) = (2x+y, 5y+z, 8z)$$

Then it has a basis of eigenvectors.

The matrix w.r.t. standard basis is upper triangular, therefore we have

$$\begin{cases} 2x = 2x + y \\ 2y = 5y + 3z \\ 2z = 8z \end{cases} \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 5x = 2x + y \\ 5y = 5y + 3z \\ 5z = 8z \end{cases} \Rightarrow \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{cases} 8x = 2x + y \\ 8y = 5y + 3z \\ 8z = 8z \end{cases} \Rightarrow \begin{pmatrix} 1 \\ 6 \\ 6 \end{pmatrix}$$

These are a basis of eigenvectors for V.