Linear Algebra Done Right

Week 3 Notes (a)

shaozewxy

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3.F Duality

Definition of Dual Space and Dual Map

3.92 Definition of Linear Functional

A **linear functional** on V is a linear map from V to $\mathbf{F}$, i.e. the set of $\mathcal{L}(V, \mathbf{F})$

3.93 Examples of Linear Functionals

- Define $\phi : \mathbf{R}^3 \to \mathbf{R}$ by $\phi(x, y, z) = 4x - 5y + 2z$. Then ϕ is a linear function on $\mathbf{R}^3$.
- Fix $(c_1, ..., c_n) \in \mathbf{F}^n$. Define $\phi : \mathbf{F}^n \to F$ by

  $$\phi(x_1, ..., x_n) = c_1 x_1 + ... + c_n x_n$$

  Then ϕ is a linear functional on $\mathbf{F}^n$.
- Define $\phi : \mathcal{P}(\mathbf{R}) \to \mathbf{R}$ by $\phi(p) = 3p''(5) + 7p(4)$. Then ϕ is a linear functional on $\mathcal{P}(\mathbf{R})$.

- Define $\phi : \mathcal{P}(\mathbf{R}) \to \mathbf{R}$ by $\phi(p) = \int_0^1 p(x)dx$. Then ϕ is a linear functional on $\mathcal{P}(\mathbf{R})$.

3.94 Definition of dual space

The **dual space** of V denoted V' is just the vector space of all linear functional on V, i.e. $V' = \mathcal{L}(V, \mathbf{F})$.

3.95 Dual space and the orginal space have same dimension.

Suppose V is finite dimensional, then V' is also finite dimensional and $dim\ V = dim\ V'$.

Proof:

This comes from the fact that

$$dim\ \mathcal{L}(V, \mathbf{F}) = dim\ V \times dim\ \mathbf{F} = dim\ V \qquad \text{using 3.61}$$

3.96 Definition of dual basis

Given $v_1, ..., v_n$ a basis of V, then the **dual basis** of $v_1, ..., v_n$ is the list $\phi_1, ..., \phi_n$ of elements in V', where each ϕ_j is the linear functional on V such that

$$\phi_j(v_k) = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases}$$

3.98 Dual basis is a basis of dual space

Proof:

First NTS $\phi_1, ..., \phi_n$ are linearly independent:

Suppose $\exists \phi = \sum_{i=1}^n a_i \phi_i = 0$, then this means that $\phi(v_j) = a_j \phi_j(v_j) = a_j = 0$, i.e. all $a_i = 0$, therefore $\phi_1, ..., \phi_n$ are linearly independent.

Then NTS $\phi_1, ..., \phi_n$ span V':

Given $\phi \in V'$, we claim that

$$\phi = \sum_{i=1}^{n} \phi(v_i)\phi_i$$

$\forall v = \sum_{i=1}^{n} a_i v_i, \phi(v) = \sum_{i=1}^{n} a_i \phi(v_i)$, similarly,

$$\begin{aligned}
\left(\sum_{i=1}^{n} \phi(v_i)\phi_i\right)\left(\sum_{i=1}^{n} a_i v_i\right) &= \sum_{i=1}^{n} \phi(v_i)\phi_i \left(\sum_{i=1}^{n} a_i v_i\right) \\
&= \sum_{i=1}^{n} a_i \phi(v_i)\phi_i(v_i) \\
&= \sum_{i=1}^{n} a_i \phi(v_i) = \phi(v)
\end{aligned}$$

Therefore $\phi_1, ..., \phi_n$ span V'.

Therefore $\phi_1, ..., \phi_n$ is a basis for V'.

3.99 Definition of dual map

Given $T \in \mathcal{L}(V, W)$, then the **dual map** of T is the linear map $T' \in \mathcal{L}(V, W)$ defined by

$$T'(\phi) = \phi \circ T$$

3.101 Algebraic Properties of Dual Maps

- $\forall S, T \in \mathcal{L}(V, W)(S + T)' = S' + T'$
- $\forall \lambda \in \mathbf{F}, T \in \mathcal{L}(V, W), (\lambda T)' = \lambda T'$
- $\forall T \in \mathcal{L}(U, W), S \in \mathcal{L}(V, W), (ST)' = T'S'$

Properties of Dual Maps

3.102 Definition of annihilator

Given $U \subset V$, the **annihilator** of U, denoted U^0 is defined by

$$U^0 = \{\phi \in V' : \forall u \in U, \phi(u) = 0\}$$

3.103 Example of Annihilator

Given $U \subset \mathcal{P}(\mathbf{R})$ consisting of all polynomial multiples of x^2. Then ϕ defined by

$$\phi(p) = p'(0)$$

is $\in U^0$.

3.104 Example of Annihilator

$e_1, ..., e_5$ the standard basis of $\mathbf{R}^5$, and $\phi_1, ..., \phi_5$ the dual basis of $(\mathbf{R}^5)'$. Suppose

$$U = span(e_1, e_2) = \{(x_1, x_2, 0, 0, 0) \in \mathbf{R}^5 : x_1, x_2 \in \mathbf{R}\}$$

Show that $U^0 = span(\phi_3, \phi_4, \phi_5)$. **Proof:**

First NTS that $span(\phi_3, \phi_4, \phi_5) \subseteq U^0$:

Given $\phi = a_3\phi_3 + a_4\phi_4 + a_5\phi_5, \phi(x_1, x_2, 0, 0, 0) = 0 \to span(\phi_3, \phi_4, \phi_5) \subseteq U^0$.

Then NTS $U^0 \subseteq span(\phi_3, \phi_4, \phi_5)$:

Given $\phi = a_1\phi_1 + a_2\phi_3 + a_3\phi_3 + a_4\phi_4 + a_5\phi_5 \in U^0, \phi(x_1, x_2, 0, 0, 0) = a_1x_1 + a_2x_2 = 0 \forall x_1, x_2 \to a_1 = a_2 = 0$. Therefore $\phi \in span(\phi_3, \phi_4, \phi_5)$, i.e. $U^0 \subseteq span(\phi_3, \phi_4, \phi_5)$

Therefore $U^0 = span(\phi_3, \phi_4, \phi_5)$

3.105 Annihilator is a subspace

The proof for this is easy to verify.

3.106 Dimension of the annihilator

Given V finite dimensional and $U < V$, then we have

$$dim\ U + dim\ U^0 = dim\ V$$

Proof:

The first proof would be to create a basis for U and extend it to a basis of V, then prove that the dual basis of the extension is a basis for U^0.

The second prrof is as follows:

Let $i \in \mathbf{L}(U, V)$ be the inclusion mapping defined by $i(u) = u$.

Then $i' \in \mathbf{L}(V', U')$. We have

$$dim\ range\ i' + dim\ null\ i' = dim\ V' = dim\ V$$

Then we only NTS $range\ i' = U'$ and $null\ i' = U^0$:

WTS $range\ i' = U'$:

Given $\psi \in U'$, then we can extend this to $\phi \in V'$ such that $\forall u \in U, \phi(u) = \psi(u)$. Then clearly $i'(\phi) = \phi \circ i = \psi$.

Therefore $range\ i' = U'$.

WTS $null\ i' = U^0$:

This is basically the definition. Given $\phi \in U^0$, then $\forall u \in U, \phi \circ i(u) = \phi(u) = 0$, i.e. $\phi \circ i(u) = 0 \to \phi \in null\ i'$.

Given $\phi \in null\ i'$, then $\forall u \in U, \phi(u) = \phi \circ i(u) = 0$, therefore $\phi \in U^0$.

Therefore $null\ i' = U^0$.

Therefore $dim\ U + dim\ U^0 = dim\ V$.

3.107 Null space of T'

Given V, W both finite-dimensional and $T \in \mathcal{L}(V, W)$, then

(a) $null\ T' = (range\ T)^0$

(b) $dim\ null\ T' = dim\ null\ T + dim\ W - dim\ V$

Proof:

For a:

First we NTS $null\ T' \subseteq (range\ T)^0$:

Suppose $\phi \in null\ T'$, then we know that $\phi \circ T = 0$.

Therefore, $\forall w \in W$ such that $\exists v \in V, Tv = w$, we know that $\phi(w) = \phi(Tv) = \phi \circ T(v) = 0$, i.e. $\phi \in (range\ T)^0$.

Therefore $null\ T \subseteq (range\ T)^0$.

Then we NTS $(range\ T)^0 \subseteq null\ T'$:

Given $\phi \in (range\ T)^0$, then denote $\psi = \phi \circ T$, we have $\forall v \in V, \psi(v) = \phi \circ Tv = 0$, therefore $(range\ T)^0 \subseteq null\ T'$.

Therefore $null\ T' = (range\ T)^0$.

For b:

With $null\ T' = (range\ T)^0$, we have that

$$\begin{aligned} dim\ null\ T' &= dim\ (range\ T)^0 \\ &= dim\ W - dim\ range\ T \\ &= dim\ W - (dim\ V - dim\ null\ T) \\ &= dim\ W - dim\ V + dim\ null\ T \end{aligned}$$

3.108 T surjective iff T' injective

Proof:

T surjective $\iff$ $dim\ (range\ T)^0 = 0$ $\iff$ $dim\ null\ T' = 0$ $\iff$ T'

injective.

3.109 Range of T'

Given V, W both finite-dimensional and $T \in \mathcal{L}(V,W)$. Then

(a) $dim\ range\ T' = dim\ range\ T$

(b) $range\ T' = (null\ T)^0$

Proof:

(a) We have

$$\begin{aligned} dim\ range\ T' &= dim\ W - dim\ null\ T' \\ &= dim\ W - (dim\ W - dim\ V + null\ T) \\ &= dim\ range\ T \end{aligned}$$

(b) First we NTS $range\ T' \subseteq (null\ T)^0$:

Given $\phi \in range\ T', \exists \psi \in W', \psi \circ T = \phi$.

Therefore $\forall v \in null\ T, \phi(v) = \psi \circ Tv = \psi(0) = 0$.

Therefore we have shown that $range\ T' \subseteq (null\ T)^0$.

Then we NTS $dim\ range\ T' = dim\ (null\ T)^0$:

$$\begin{aligned} dim\ range\ T' &= dim\ W' - dim\ null\ T' \\ &= dim\ W - (dim\ W - dim\ V + dim\ null\ T) \\ &= dim\ V - dim\ null\ T \\ &= dim\ (null\ T)^0 \end{aligned}$$

3.110 T injective iff T' surjective

Proof:

$$
\begin{aligned}
T\text{injective} &\iff \textit{dim null } T = 0 \\
&\iff \textit{dim range } T = V \\
&\iff \textit{dim range } T' = \textit{dim range } T = V \\
&\iff T'\text{surjective}
\end{aligned}
$$

Matrix of a Dual Map

3.113 Transpose of product

Given A an $m \cdot n$ matrix and C an $n \cdot p$ matrix, then

$$(AC)^t = C^t A^t$$

3.104 Matrix of T' is the transpose of matrix of T

Given $T \in \mathcal{L}(V, W)$, then $\mathcal{M}(T') = (\mathcal{M} * (T))^t$.

Proof:

Denote $v_1, ..., v_n$ a basis for V, $v'_1, ..., v'_n$ the corresponding basis for V'. Similarly $w_1, ..., w_m$ a basis for W, $w'_1, ..., w'_n$ the corresponding basis for W'.

We denote $\mathcal{M}(T) = A, \mathcal{M}(T') = C$. Then we have

$$\begin{aligned}
w_j'(Tv_i) &= w_j'\left(\sum_{r=1}^{m} A_{ri} w_r\right) \\
&= \left(\sum_{r=1}^{m} A_{ri} w_j'(w_r)\right) \\
&= A_{ji} \\
&= w_j' \circ T(v_i) \\
&= T'(w_j')v_i \\
&= \left(\sum_{r=1}^{n} C_{rj} v_r'(v_i)\right) \\
&= C_{ij}
\end{aligned}$$

Here the main equality is $w_j'(Tv_i) = T'w_j'(v_i)$. Using two ways to perform this calculation leads to the equality of the two entries of matrices $\mathcal{M}(T)$ and $\mathcal{M}(T')$.

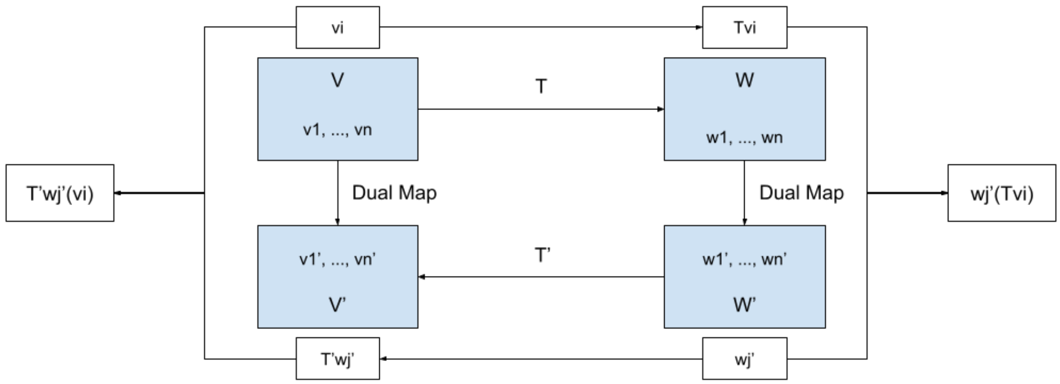
Tv_i selects the i^{th} column of $\mathcal{M}(T)$ and similarly, $T'w_j'$ selects the j^{th} column of $\mathcal{M}(T')$.

Then applying w_j' to Tv_i will result in 0 for all rows $r \neq j$, leaving only $\mathcal{M}(T)_{ji} \cdot w_j'(w_j) = \mathcal{M}(T)_{ji}$.

Simlarly, applying $T'w_j'$ to v_i will result in 0 for all rows $r \neq i$, leaving only $\mathcal{M}(T')_{ij} \cdot v_i'(v_i) = \mathcal{M}(T')_{ij}$.

Therefore $\mathcal{M}(T)_{ji} = \mathcal{M}(T')_{ij} \rightarrow \mathcal{M}(T)^t = \mathcal{M}(T')$.

Check the below graph for a better understanding:



Rank of a Matrix

3.115 Definition of rank

Given A an $m \times n$ matrix with entries in $\mathbf{F}$.

- The **row rank** of A is the dimension of the span of rows of A in $\mathbf{F}^{1,n}$.
- The **column rank** of A is the dimension of the span of the columns of A in $\mathbf{F}^{m,1}$.

3.117 T and rank of M(T)

Given V, W finite-dimensional and $T \in \mathcal{L}(V, W)$. Then $dim\ range\ T =$ column rank of $\mathcal{M}(T)$.

Row rank equals to column rank

Proof:

Given $A \in \mathbf{F}^{m,n}$, then

$$\begin{aligned}
\text{columnrankof} A &= \text{columnrankof} \mathcal{M}(T) \\
&= dim\ range\ T \\
&= dim\ range\ T' \\
&= \text{columnrankof} \mathcal{M}(T') \\
&= \text{columnrankof} A^t \\
&= \text{rowrankofA}
\end{aligned}$$

3.119 Definition of rank

The **rank** of a matrix $A \in \mathbf{F}^{m,n}$ is the column rank of A.