Fundamental Concepts of Analysis

Week 1 Notes (b)

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5 Least-Upper-Bound Axiom

5.1 Definition of lower/upper bound

A nonempty subset $X \subseteq \mathbf{R}$ is said to be **bounded above(below)** if there exists a real number a such that $\forall x \in X, x \leq a(x \geq a)$. The number a is called an **upper(lower) bound** for X.

5.2 Definition of least-upper(greatest-lower) bound

Let $X \subseteq \mathbf{R}$ be nonempty.

A number $a \in \mathbf{R}$ is said to be a **least upper bound** for X if

(i) a is an upper bound for X.

(ii) If b is an upper bound for X, then $a \leq b$.

The **greatest lower bound** is defined similarly.

Also, the (ii) part can be stated as:

If $b < a$, then b is not an upper bound for X.

Using 5.1, the statement above is equivalent to (ii') If $b < a$, then $\exists x \in X, x > b$

Sometimes when proving least upper bound, using (ii') is more convenient.

5.3 Least upper bound is unique

Let $X \subseteq \mathbf{R}$. If a, b are least upper bounds of X, then $a = b$.

Proof:

Suppose $a < b$, then by definition $\exists a < b$ such that a is an upper bound, therefore b is not an upper bound. Contradiction.

We can then denote the least-upper(greatest-lower) bound of a set X by

$$lub\ X = sup\ X, glb\ X = inf\ X$$

Axiom 13

A nonempty subset of real numbers bounded above has a least upper bound.

This axiom is essentially saying there are no holes in the real line.

5.4 Conditions for GLB

A nonempty subset of real numbers bounded below has a greatest lower bound.

Proof:

Denote the set $X \subseteq \mathbf{R}$ which is bounded below. Then we create Y as the set of real numbers that are lower bounds of X.

Since X non-empty, $\exists x \in X$ such that x is an upper bound of Y.

Therefore Y bounded above and $\exists y = lub\ Y$.

We claim that $a = lub\ Y = glb\ X$:

First NTS a is a lower bound of X:

Suppose $\exists x \in X$ such that $x < a$, then since $a = lub\ Y$ we know that $\exists y \in Y$ such that $y > x$ but since y is a lower bound of X this is a contradiction.

Therefore no such x exists, and thus a is a lower bound of X.

Then NTS a is the greaster upper bound of X:

$\forall b > a$ clearly $b \notin Y$ and therefore b not a lower bound. Therefore a is the greatest lower bound.

6 Set of Positive Integers

6.1 Definition of sucessor set

A subset $X \subseteq \mathbf{R}$ is said to be a **successor set**

(i) If $1 \in X$.

(ii) If $n \in X$, then $n + 1 \in X$.

Since $\mathbf{R}$ itself is a successor set, we know that successor set exists.

6.2 Intersection of successor sets

If $\mathscr{A}$ is a nonempty collection of successor sets, then $\cap\mathscr{A}$ is a successor set.

Proof:

For (i), $\forall A \in \mathscr{A}$, since $1 \in A \to 1 \in \cap\mathscr{A}$.

For (ii), $\forall n \in \cap\mathscr{A}$, since $\forall A \in \mathscr{A}, n \in A \to n + 1 \in A$, we know that $n + 1 \in \cap\mathscr{A}$.

6.3 Definiton of positive integers

The set $\mathbf{P}$ of **positive integers** is the intersection of the family of all successor sets.

The meaning of the above defintion is that $\mathbf{P}$ is the smallest successor set:

If X is a successor set, then $\mathbf{P} \subset \mathbf{X}$.

6.4 Mathematical Induction

Suppose that for each positive integer n we have a statement $S(n)$ and that

(i) $S(1)$ is true.

(ii) If $S(n)$ is true, then $S(n+1)$ is true.

Then $S(n)$ is true for every positive integer n.

Proof:

Let $G = \{n \in \mathbf{P} | S(n) \text{ is true}\}$, then $G \subseteq \mathbf{P}$.

But by definition $1 \in G$ and $n \in G \to n+1 \in G$, therefore G a successor set and $\mathbf{P} \subseteq G$.

Therefore $G = \mathbf{P}$.

Here are two examples of the induction:

6.5 Positive integer ≥ 1

If n is a positive integer, then $n \geq 1$.

Proof:

For (i), clearly $1 \geq 1$.

For (ii), suppose $n \geq 1$, then $n+1 \geq n \geq 1$.

Therefore this is true for all positive integer n.

6.6 Sum of positive integers are positive integers

If $m, n \in \mathbf{P}$ then $m+n \in \mathbf{P}$.

Proof:

Define $S(m)$ to be "$\forall n \in \mathbf{P}, m+n \in \mathbf{P}$".

For (i), it is by definiton that $\forall n \in \mathbf{P}, 1+n \in \mathbf{P}$.

For (ii), $\forall n \in \mathbf{P} m+1+n = (m+n)+1$, since $m+n \in \mathbf{P} \to (m+n)+1 \in \mathbf{P}$.

Therefore this is true for all positive integer m.

We then prove the well-ordering Theorem

6.7

If $n \in \mathbf{P}$, then either $n - 1 = 0$ or $n - 1 \in \mathbf{P}$.

The proof of this is obvious.

6.8

If $m, n \in \mathbf{P}$ and $m < n$, then $n - m \in \mathbf{P}$.

Proof:

Let $S(m)$ be the statement above.

For (i), if $1 < n$ then $n - 1$ by 6.7 either $n - 1 = 0$ or $n - 1 \in \mathbf{P}$. Since $n \neq 1$, $n - 1 \in \mathbf{P}$.

For (ii), if $m + 1 < n$, then $n - (m + 1) = n - m - 1$ by 6.7 and the fact that $n - m \in \mathbf{P}$, either $n - m - 1 = 0$ or $n - m - 1 \in \mathbf{P}$. Since $n \neq m + 1$, we know that $n - m - 1 \in \mathbf{P}$.

THereofre this is true for all positive integers.

6.9

Let n be a positive integer. No positive integer m satisfies $n < m < n + 1$

Proof:

By 6.8 we know that $m - n \in \mathbf{P}$, but since $m < n + 1 \to m - n < 1$. This contradicts with the fact that $m - n \in \mathbf{P}$ should be ≥ 1.

Therefore no such m exists.

6.10 Well-Ordering Theorem

If X is a non-void subset of positive integers, then X contains a least element $a \in X$ such that $\forall x \in X, a \leq x$.

Proof:

We let the statement $S(n)$ be that "If $n \in X$, then X contains a least element".

For (i), if $1 \in X$ then 1 is the least element by 6.5.

For (ii), if $S(n)$ is true, i.e. $n \in X \rightarrow X$ contains a least element, then suppose $n+1 \in X$, WTS that X has a least element.

Now $X \cup \{n\}$ has a least element m. If $m \in X$ then we are done, therefore assume $m \notin X \rightarrow m = n$.

This means that $\forall x \in X, n \leq x$. Therefore by 6.9 we have $\forall x \in X, n+1 \leq x$, therefore $n+1$ is the least element of X.

6.11

The set of positive integers is not bounded above.

Proof:

If $\mathbf{P}$ is bouned above, then $lub\ \mathbf{P} = a$.

We know that $a - 1 < a$ is not an upper bound of $\mathbf{P}$ and therefore $\exists n \in \mathbf{P}$ such that $a - 1 < n \rightarrow a < n + 1 \in \mathbf{R}$. Contradiction.

The ordering of positive integers also implies some ordering of positive real numbers:

6.12 Archimedean ordering of positive real numbers

If a, b positive real numbers, then $\exists n \in \mathbf{P}$ such that $a < nb$.

Proof:

Since $\mathbf{P}$ is not bounded above, then $\exists n \in \mathbf{P}$ such that $a/b < n$.

Therefore $a < nb$.