Calculus

Notes on 2023/06/9

shaozewxy

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1 Five big theorems (Ch 1.6)

Algebra is all about eqaulity, calculus is all about inequality.

1.1 Convergent subsequence

The first theorem is **a sequence in a compact set has a convergent subsequence**.

1.1.1 Basic concepts

$X \subset \mathbb{R}^n$ is **bounded** if it is contained in a ball in $\mathbb{R}^n$ centered at origin, i.e.

$$X \subset B_R(0)$$

$C \subset \mathbb{R}^n$ is **compact** if it is closed and bounded.

1.1.2 Compact set contains convergent subsequences

$C \subset \mathbb{R}^n$ a compact set contains a sequence $i \mapsto x_i$, then x_i contains a subsequence $j \mapsto (x_i)_j$ whose limit is in C.

Basic idea: Because the regoin is bouned, can infinitely divide into smaller regions where one region contains infinitely many points from x_i. Find an element $(x_i)j$ from each such region to create a subsequence, it will be convergent.

Proof:

Now we need to show that the $(x_i)_j$ created above is convergent in C:

Divide each dimension by 10, so that each element in the same region will have the decimal places and so on.

Then we construct x with the decimal place of each picked region, it is clear that $(x_i)_j \to x$.

1.1.3 Difficulty in finding actual box

Define $x_m = \sin 10^m$ to be contained in $[-1, 1]$, a compact space. Which box contains infinitely many points from x_m?

Dividing $[-1, 1]$ into $[-1, 0), [0, 1), 1$, either $[-1, 0)$ or $[0, 1)$ could work. Say it is $[0, 1)$.

This means that $10^m = k \cdot 2\pi$, where the decimal place of $k < 5$(so that $\sin k \cdot 2\pi > 0$).

This can be turned into $k = \frac{1}{2\pi} \cdot 10^m$, i.e. we are asking if there are infinitely many $0, 1, 2, 3, 4$ in $\frac{1}{2\pi}$.

The last conclusion is very hard to prove, therefore picking an actual box is very difficult.

1.2 Bounded functions in compact set

To put formerly, given $C \subset \mathbb{R}^n$, and $f : C \to \mathbb{R}$ continuous, then $\exists a, b \in C, \forall x \in C, f(x) \geq f(b), f(x) \leq f(a)$.

Basic idea: If f is unbouned, then can create an ever increasing sequence. However then some subsequence would have to be convergent to a limit, i.e.

all large indices correspond to a small ball around this limit, meaning f is not continuous around this limit.

Proof:

Suppose f is unbounded, then we can create a sequence $i \mapsto x_i$ where $f(x_i) > i$, i.e. ever increasing sequence.

Since $x_i \subset C$ a compact set, we know that $\exists j \to (x_i)_j$ such that $(x_i)_j \to b \in C$.

Since f continous at b, we know that $\forall \epsilon > 0, \exists \delta > 0$ such that $|x - b| < \delta \to |f(x) - f(b)| < \epsilon$.

Also because $(x_i)_j \to b$, we know for such $\delta, \exists N$ such that $\forall j > N, |(x_i)_j - b| < \delta$.

This means that $\forall j > N, |f((x_i)_j) - f(b)| < \epsilon$, which is a contradiction.

Now we have shown that f is bounded. Then we NTS that f achieves its maximum.

Denote $\sup f = M$, WTS $\exists a \in C, f(a) = M$.

By the definition of supremum, $\exists i \mapsto x_i$ such that $f(x_i) \to M$.

Then simply find a convergent subsequence $j \mapsto (x_i)_j$ that converges to $(x_i)_j \to a \in C$, then we know that $f((x_i)_j) \to f(a) = M$.

1.3 Uniform continuity in compact set

Given $X \subset \mathbb{R}^n$ a compact set, $f : X \to \mathbb{R}$ continuous, then f uniformly continuous on X.

Basic idea: Given ϵ, non-uniform indicates for any δ, exists some x, y such that $|x - y| < \delta, |f(x) - f(y)| > \epsilon$. Create two sequences x_i, y_i from these, then extract converging subsequences. The subsequences converge to the same point, because all the time the distance of $f(x)$ and $f(y)$ is larger than ϵ, this contradicts that continuity.

Proof:

Suppose f is not uniformly continuous, this means that given $\epsilon, \forall \delta, \exists x, y \in C, |x-y| < delta, |f(x) - f(y)| > \epsilon$.

Create $i \mapsto x_i, y_i$ by $|x_i - y_i| < \frac{1}{i}, |f(x_i) - f(y_i)| > \epsilon$. Find the converging subsequence $(x_i)_j, (y_i)_j$. It is clear that $(x_i)j, (y_i)_k$ converge to the same point a.

Because f continuous at a, for the same $\epsilon, \exists \delta > 0$ such that $|x-a| < \delta \Rightarrow |f(x) - f(a)| < \frac{\epsilon}{2}$.

Then for efficiently large M such that $\frac{1}{M} < \delta$, we know that given $j > M, |f((x_i)_j) - f((y_i)j)| > \epsilon$ and $|(x_i)_j - (y_i)_j| < \frac{1}{j} < \delta$, i.e. $|f((x_i)_j) - f(a)| < \frac{\epsilon}{2}, |f((y_i)_j) - f(a)| < \frac{\epsilon}{2}$, contradiction.

1.4 Mean value theorem

The first result is that **the derivative at maximum or minimum of a function is zero**.

This is easily proven by looking that the derivative from bot sides, one side ≥ 0, one side ≤ 0. Therefore the derivative $= 0$.

The **mean value theorem** states that given $f : [a,b] \to \mathbb{R}$ is continuous, and differentiable on (a,b), then $\exists c \in (a,b)$ such that $f'(c) = \frac{f(a)-f(b)}{a-b}$.

Basic idea: This is saying that $\exists c \in (a,b), f'(c) - \frac{f(a)-f(b)}{a-b} = 0$. Look at this value as the derivative of the difference between f and a linear function from a to b.

Proof:

Define function $g : [a,b] \to \mathbb{R}$ by $g(x) = f(a) + \frac{x-a}{b-a} f(b) - f(a)$.

Then $f-g$ is clearly continuous and $(f-g)(a) = (f-g)(b) = 0$. Assuming $f-g$ is not 0 everywhere, then there is a maxima c of $f-g$, therefore $(f-g)'(c) = 0 = f'(c) - g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$.

1.5 Fundamental theorem of algebra

Basic idea:

1. Find a global minimum for $|p(z)|$. While $|p(z)|$ is not necessarily in a compact set, we find a circle $B_R(0)$ such that any point outside is larger than the minimum z_0 in the circle. Thus the local minimum in the circl will be global.

2. Divide the components of $p(z)$ into three parts, M, a fixed value $p(z_0)$, R, lowest term with non-zero coefficients, and L, the rest. Both R and L is controlled by $u = \rho(\cos\theta + i\sin\theta)$.

3. Find a θ that positions R between 0 and M, then find a ρ such that $|L| < |R| < |M|$, thus $|p(u)| < |p(z+0)|$, contradiction.

Proof:

1. Global minimum. Need to find a circle $B_R(0)$ such that $\forall z \notin B_R(0), |p(z)| > |p(0)|$. Then a minimum $z_0 \in B_R(0)$ will be globally minimum.

Observe $p(z) = z^k + a_{k-1}z^{k-1} + ... + a_0$, the first term z^k will domintate $p(z)$ as z grows larger and therefore we will be able to find such $B_R(0)$.

Take $A = \max\{|a_{k-1}|, ..., |a_0|\}$,

$$\begin{aligned}
|p(z)| &= |z^k + a_{k-1}z^{k-1} + ... + a_0| \\
&\geq |z^k| - (|a_{k-1}z^{k-1}| + |a_0|) \\
&\geq |z^k| - kA|z|^{k-1} = |z|^{k-1}(|z| - kA)
\end{aligned}$$

Therefore, take $z \geq \max\{(k+1)A, 1\}$, we have that $|p(z)| \geq A|z|^{k-1} \geq |a_0| = |p(0)|$.

i.e. the minimum $z_0 \in B_R(0)$ is globally minimum.

2. NTS $|p(z_0)| = 0$.

Take $z = z_0 + u$, then

$$
\begin{aligned}
p(z) &= z^k + a_{k-1}z^{k-1} + ... + a_0 \\
&= (z_0 + u)^k + a_{k-1}(z_0 + u)^{k-1} + ... + a_0 \\
&= u^k + b_{k-1}u^{k-1} + ... + b_0 = q(u)
\end{aligned}
$$

$q(u)$ has three parts, $F = b_0 = p(z_0)$ is a fixed value, $R = b_j u^j$ is the lowest term with non-zero coefficients, and $L = (b_{j+1}u^{j+1} + ... + u^k)$. Both R and L are controlled by $u = \rho(\cos\theta + i\sin\theta)$.

The general idea is that as u gets smaller, L gets smaller much quicker than does R.

We know that $M = F+R = b_0+b_j\rho^j(\cos j\theta+i\sin j\theta)$, this says that regardless of ρ, M revolves around $F = b_0$ as θ changes.

Therefore we can find a θ that places M between F and 0.

Then fixing such θ, find a ρ small enough so that $|L| \leq |R|$. This way $D = q(u) = F + R + L$ will be closer to 0 than $q(0) = b_0$, i.e. $|q(u)| < |b_0|$, contradictory to the fact that $|b_0|$ is the global minimum of $p(z)$.

Such u is easy to find:

Take $A = \max\{b_{j+1}, ..., 1\}$

$$
\begin{aligned}
|R| - |L| &= |b_j u^j| - |b_{j+1}u^{j+1} + ... + u^k| \\
&\geq |b_j u^j| - (k-j)A|u|^{j+1} \\
&= |u|^j(b_j u - (k-j)A)
\end{aligned}
$$

Then take u to be sufficiently small, we have that $|R| \geq |L|$.

Corollary of Fundamental Algebra Theorem

Given $p(z) = (z - c_1)^{k_1} ... (z - c_m)^{k_m}$ where $k_1 + ... + k_m = k$, k_j is called the **multiplicity** of rott c_j.

Any complex $p(z)$ with degree k can be factored into $p(z) = (z - c_1)^{k_1} ... (z - c_m)^{k_m}$ with $k_1 + ... + k_m = k$.

Proof:

Denote $\tilde{p}$ to be the monic polynomial with the highest degree that divides p and is the product of 1 degree polynomials, $\tilde{p}$ has largest degree $\tilde{k}$.

We can then write $p(z) = \tilde{p}(z)q(z)$, with $q(z)$ having $k - \tilde{k}$ degrees.

NTS that $\tilde{k} = k$:

Suppose $\tilde{k} < k$, then $\exists c$ such that $q(c) = 0$, then we can write $q(z) = (z - c)\tilde{q}(z)$.

I.e., we can write $p(z) = ((z - c)\tilde{p}(z))\tilde{q}(z)$, so $\tilde{p}$ is not such polynomial with highest degrees, contradiction.

Therefore $\tilde{k} = k$.

This corollary can be limited to real polynomials with a looser condition:

Any real polynomial can be factored as polynomials of 1 or 2 degrees.

Basic idea: Do the same thing as the complex case, if the root is complex, then transform it into a 2-degree real polynomial.

Proof:

Define $\tilde{p}$ similar as above, **but real**.

Then $p(z) = \tilde{p}(z)q(z)$ and suppose $\tilde{k} < k$, then $\exists c$ such that $q(c) = 0$.

Now if c is real then we are done. So suppose c is complex.

Since q is a polynomial, given $c = a + bi, \overline{c} = a - bi, (\overline{c})^2 = (a - bi)(a - bi) = a^2 - b^2 - 2abi = \overline{c^2}$. Therefore $q(\overline{c}) = \overline{q(c)} = \overline{0} = 0$.

Now we can say that $q(z) = (z - c)(z - \overline{c})\tilde{q}(z) = (z^2 - 2az + a^2 + b^2)\tilde{q}(z)$, therefore $p(z) = ((z^2 - 2az + a^2 + b^2)\tilde{p}(z))\tilde{q}(z)$, contradiction.