Abstract Algebra

Week 3 Notes (b)

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Quotient Groups: Definitions and Examples

The study of **quotient groups** are essentially the study of **fibres** of ϕ i.e. the set of elements mapping to the same element under ϕ.

Basic properties of homomorphisms and fibres

Definition of quotient group

Given $\phi : G \to H$ be a homomorphism with kernel K. The **quptient group G/K** is the group whose element elements are fibres of ϕ with operation defined as:

$$X = \overline{a}, Y = \overline{b}, XY = \overline{ab}$$

We can also define a quotient group without spelling out the ϕ explicitly.

Properties of fibres and kernels

Given $\phi : G \to H$ a homomorphism with kernel K. Suppose $X \in G/K$ is a fibre above a, i.e. $X = \phi^{-1}(a)$. Then

1. $\forall u \in X, X = \{uk | k \in K\}$.

2. $\forall u \in X, X = \{ku | k \in K\}$.

This is saying that if K is a kernel, then $uK = Ku$. Proof for this is obvious.

The groups defined above can be generalized to any subgroup in G:

Definition of coset

Given $N \leq G, g \in G$, let

$$gN = \{gn | n \in N\}, Ng = \{ng | n \in N\}$$

these are called **left coset and right coset** respectively. Any element of a coset is called a **representative** of that coset.

With the definiton, we can see that **fibres of homomorphisms** are just **left cosets of kernels**.

This enables us to define quotient groups without spelling out the homomorphisms explicitly as long as we know that K is a kernel:

Operations of cosets

Given G a group and K a kernel of some homomorphism from G. Then the set of left cosets of K with operations defined by

$$uK \circ vK = (uv)K$$

forms a group G/K.

Proof:

Given $X, Y \in G/K$ with $X = \phi^{-1}(a), Y = \phi^{-1}(a)$, we denote that $Z = XY = \phi^{-1}(ab)$. We NTS that $\forall u \in X, v \in Y, uv \in Z$:

Since $u \in X, v \in Y$, we know that $\phi(u) = a, \phi(v) = b$. Then $\phi(uv) = \phi(u)\phi(v) = ab \to uv \in Z = \phi^{-1}(ab)$.

Also NTS show that $\forall z = uvk \in Z = (uv)K, \exists u' \in X, v' \in Y$ such that $z = u'v'$:

Since $z \in Z$, we know that $\phi(z) = ab$. Then choose $u \in X = uK, z = u \circ (u^{-1}z)$, we have

$\phi(u)\phi(u^{-1}z) = a \circ \phi(u^{-1}z) = ab \to \phi(u^{-1}z) = b \to u^{-1}z \in Y$.

THerefore we have shown that $Z = uvK = XY = (uK)(vK)$.

Here we note that the multiplication of cosets is **independent of the choice of representations**.

We can therefore denote a coset $X = uK = \overline{u}$, then multiplication can just be written as $\overline{uv}$.

Next we show that G/K is well defined $\iff$ K is a kernel. Such a subgroup K is also called a **normal subgroup**.

Cosets form a partition

Given $N \leq G$, the set of left cosets of N form a partition of G. Furthermore $\forall u, v \in G, uN = vN \iff v^{-1}n \in N$.

Proof:

First $\forall g \in G, g \in gN$ since $1 \in N$. Therefore

$$G = \bigcup_{g \in G} gN$$

Then NTS that $uN \neq vN \to uN \cap vN = \emptyset$:

Suppose $\exists x \in uN, vN$, we claim that $uN = vN$:

Denote $x = un = vm$ for some $n, m \in N$, then we have $u = vmn^{-1}$ with $mn^{-1} \in N$.

Therefore $\forall ut \in uN, ut = vmn^{-1}t \in vN$, i.e. $uN \subseteq vN$.

We can similarly prove $vN \subseteq uN$.

Therefore $uN = vN$.

Conditions for coset multiplication

Given $N \leq G$.

1. The multiplication

$$uN \cdot vN = (uv)N$$

   is well defined $\iff \forall g \in G, n \in N, gng^{-1} \in N$.

2. If the operation is well defined, then left cosets of N is a group. With identity $1N$ and $(gN)^{-1} = g^{-1}N$.

Proof:

Suppose $\forall g \in G, n \in N, gng^{-1} \in N$, WTS $uN \cdot vN = (uv)N$ is well defined:

Given $u' = un \in uN, v' = vm \in vN$, NTS $u'v' \in (uv)N$:

$u'v' = unvm = uv \cdot v^{-1}nvm$. Since $\forall g \in G, n \in N, gng^{-1} \in N$, we know that $\exists n' \in N$ such that $n' = v^{-1}nv$.

Therefore $u'v' = uv \cdot n'm \in (uv)N$.

Then suppose $uN \cdot vN$ well defined, WTS $\forall g \in G, n \in N, gng^{-1} \in N$:

Here given $g \in G$, we have $gN \cdot (g^{-1}N) = (gg^{-1})N = 1N$.

This means that $\forall n \in N$, since $gn \in gN, g^{-1} \in g^{-1}N$, we know that $gng^{-1} \in N$, i.e. $\exists n' \in N$ such that $1 \cdot n' = n' = gng^{-1}$.

This complete the proof for 1. The proof for 2 is easy.

Definition of conjugate and normal

The element gng^{-1} is called the **conjugate** of $n \in N$ by g.

The set $gNg^{-1} = \{gng^{-1} | n \in N\}$ is called the **conjugate** of N by g.

The element g is said to **normalize** N if $gNg^{-1} = N$.

$N \leq G$ is calle **normal** if every element normalizes N, written $N \trianglelefteq G$.

Note that the structure of G is preserved in G/N.

Conditions for normal subgroups

Given $N \leq G$. The following are equivalent:

1. $N \trianglelefteq G$
2. $N_G(N) = G$
3. $\forall g \in G, gN = Ng$
4. The coset multiplication for N make the cosets into a group.
5. $\forall g \in G, gNg^{-1} \subseteq N$.

Proof of this is done throughout this chapter.

In determining the normality of N, using generators can avoid a lot of computations since proving **normality for each of the generators suffices to show normality of the whole group**.

Now we prove the relation between kernel and normal subgroups:

Kernel and normal subgroups

$N \leq G$ is normal $\iff$ it is a kernel of some homomorphism.

Proof:

Suppose N is a kernel of ϕ, WTS $N \trianglelefteq G$:

$\forall g \in G, \phi(gng^{-1}) = \phi(g)\phi(n)\phi(g^{-1}) = \phi(g)\phi(g^{-1}) = 1 \to gng^{-1} \in N$ the kernel of ϕ.

Therefore $N_G(N) = G \to N \trianglelefteq G$.

Suppose $N \trianglelefteq G$, WTS $\exists\phi$ such that N is the kernel of ϕ:

Just use

$$\forall g \in G, \phi(g) = gN$$

The homomorphism defined above is special:

Definition of natural projection

Given $N \trianglelefteq G$, then homomorphism $\pi : G \to G/N$ defined by $\pi(g) = gN$ is called the **natural projection** of G onto G/N.

The normalizer of a subgroup $N \leq G$ is a measure of how close N is to being normal.