Linear Algebra Done Right

Week 4 Notes (a)

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5.B Eigenvectors and Upper-Triangular Matrices

Polynomials applied to operators

Operators, unlike more general linear maps, can be raised to power, allowing more theory to be developed.

5.16 Definition of power of operators

Given $T \in \mathcal{L}(V)$ and m is a positive integer.

- T^m is defined by

$$T^m = \underbrace{T...T}_{m \text{ times}}$$

- T^0 is defined to be I.

- If T is invertible with inverse T^{-1}, then T^{-m} is defined by

$$T^{-m} = (T^{-1})^m$$

5.17 Definition of $p(T)$

Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbf{F})$ is a polynomial given by

$$p(z) = a_0 + a_1 z + a_2 z^2 + ... + a_m z^m$$

Then $p(T)$ is the operator defined by

$$p(T) = a_0 I + a_1 T + a_2 T^2 + ... + a_m T^m$$

5.19 Definition of product od polynomials

If $p, q \in \mathcal{P}(\mathbf{F})$, then $pq \in \mathcal{P}(\mathbf{F})$ is the polynomial defined by

$$(pq)(z) = p(z)q(z)$$

5.20 Properties of operator multiplications

Suppose $p, q \in \mathcal{P}(\mathbf{F})$ and $T \in \mathcal{L}(V)$, Then

(a) $(pq)(T) = p(T)q(T)$

(b) $p(T)q(T) = q(T)p(T)$

Proof of this is obvious.

Existence of eigenvalues

5.21 Operators on complex spaces have eigenvalue

Every operator on a finite-dimensional, nonzero, complex vector space has an eigenvalue.

Proof:

Suppose V is a complex vector space with dimension $n > 0$ and $T \in \mathcal{L}(V)$. Choose $v \neq 0 \in V$. Then

$$v, Tv, T^2v, ..., T^nv$$

is not linearly independent since there are $n+1$ of them. Thus there exists $a_0, ..., a_n$ all complex numbers such that

$$a_0v + a_1Tv + ... + a_nT^nv = 0$$

Here $a_1, ..., a_n$ cannot all be 0 since otherwise a_0 will also need to be 0. From this we obtain a polynomial which by the **Fundamental Theorem of ALgebra** has a factorization

$$a_0 + a_1z + ... + a_nz^n = c(z - \lambda_1)....(z - \lambda_m)$$

Here m is not necessarily n since the coefficient a_n may be 0.
We then have

$$\begin{aligned} 0 &= a_0v + a_1Tv + ... + a_nT^nv \\ &= (a_0I + a_1T + ... + a_nT^n)v \\ &= c(T - \lambda_1 I)...(T - \lambda_m I)v \end{aligned}$$

Therefore at least one $T - \lambda_j I$ is not injective and T has an eigenvalue.

Upper-Trianguler Matrices

A central goal of linear algebra is to show that given an operator, there is a basis w.r.t which the matrix is reasonably simple.
In the case of complex vector spaces, we can show that we can make the matrix

of T have 0s everywhere on the first column except for the first place:

Since T definitely has an eigenvector v, we extend from v to form a basis of V and $\mathcal{M}(T)$ will have 0s everywhere on the first column except for the first entry which will be λ, i.e. the eigenvalue for v.

5.24 Definition of diagonal

The **diagonal** of a square matrix consists of the entris along the line from the upper left corner to the bottom right corner.

5.25 Definition of upper-triangular matrix

A matrix is called **upper triangular** if all the entris below the diagonal equal 0.

5.26 Conditions for upper triangular matrix

Suppose $T \in \mathcal{L}(V)$ and $v_1, ..., v_n$ a basis of V. Then the following are equivalent:

(a) the matrix of T w.r.t $v_1, ..., v_n$ is upper triangular;

(b) $Tv_j \in span(v_1, ..., v_j)$;

(c) $span(v_1, ..., v_j)$ is invariant under T.

Proof:

a $\iff$ b and c $\rightarrow$ b are all obvious. Therefore, only NTS b $\rightarrow$ c:

Given j, we know that

$$
\begin{aligned}
Tv_1 \in span(v_1) \subseteq span(v_1, ..., v_j); \\
Tv_2 \in span(v_1, v_2) \subseteq span(v_1, ..., v_j); \\
... \\
Tv_j \in span(v_1, ..., v_j)
\end{aligned}
$$

Thus given $v = a_1v_1 + ... + a_jv_j \in spab(v_1, ..., v_j), Tv \in span(v_1, ..., v_j)$, therefore $span(v_1, ..., v_j)$ is invariant under T.

5.27 Every operator in complex space has upper-triangular matrix

Suppose V is finite-dimensional complex vector space and $T \in \mathcal{L}(V)$, then T has an upper-triangular matrix w.r.t. some basis of V.

Two proofs are presented below. Both use the idea of creating a smaller vector space to use induction on and then expand the basis of the smaller space to a basis of V.

Proof 1:

Here we create a subspace of range of $T - \lambda I$.

We denote the eigenvalue of T as λ.

Then let $U = range\ (T - \lambda I)$.

Clearly $dim\ U < n$ since $(T - \lambda I)v = 0$ therefore $T - \lambda I$ not surjective.

Then we can use induction on $T - \lambda I$ to assume that $T - \lambda I$ has a basis $u_1, ..., u_m$ such that $\mathcal{M}(T - \lambda I)$ is upper-triangular.

Moreover, U is also invariant under T:

$$\forall u \in U, Tu = (T - \lambda I)u + \lambda u$$

Then we extend to get a basis of $V : (u_1, ..., u_m, v_1, ..., v_k)$ and claim that $\mathcal{M}(T)$ under this basis is upper triangular:

Clearly $\forall u_i, Tu_i \in span(u_1, ..., u_i)$. Then for each $v_j, Tv_j = (T - \lambda I)v_j + \lambda v_j = u + \lambda v_j$ for some $u \in U$, therefore $Tv_j \in span(u_1, ..., u_i, v_1, ..., v_j)$.

Therefore $\mathcal{M}(T)$ under this basis is upper-triangular.

Proof 2:

Here we create a subspace of quotient space of span of the eigenvector.

We denote the eigenvector v, and $U = span(v)$.

Clearly $dim\ U = 1$ and thus $dimV/U = n-1$. Therefore, we can use induction on V/U to assume that V/U has a basis $v_2 + U, ..., v_n + U$ such that $\mathcal{M}(T/U)$ under this basis is upper-triangular.

Since v is a basis of U, then $v, v_2, ..., v_n$ is a basis of V. We claim that $\mathcal{M}(T)$ under this basis is upper-traingular:

Since $\mathcal{M}(T/U)$ is upper triangular, we know that $\forall v_i, (T/U)(v_i + U) \in span(v_2 + U, ..., v_i + U)$, i.e.

$$v_i + U = a_2(v_2 + U) + ... + a_i(v_i + U)$$

$$v_i + U = (a_2v_2 + ... + a_iv_i) + U$$

$$\exists u \in U, v_i = a_2v_2 + ... + a_iv_i + u$$

Since $U = span(v)$, this is to say $v_i = a_1v + a_2v_2 + ... + a_nv_n$.

Therefore $v_i \in span(v, v_2, ..., v_i)$ and $\mathcal{M}(T)$ is upper-triangular.

5.30 Invertibility and upper-triangular matrix

Suppose $T \in \mathcal{L}(V)$ has an upper-triangular matrix w.r.t. a basis of V. Then T is invertible $\iff$ all entries on the diagonal of that upper-triangular matrix are non-zero.

Proof:

We use the fact that if the operator is surjective/injective, then it is invertible.

Suppose T has nonzero entries along the diagonal.

Then clearly $v_1 \in range\ T$ since $Tv_1 = \lambda_1 v_1$.

Then suppose $v_1, ..., v_{i-1} \in range\ T$, WTS $v_i \in range\ T$:

We have that $Tv_i = a_1v_1 + ... + a_{i-1}v_{i-1} + \lambda_i v_i$, since $v_1, ..., v_{i-1} \in range\ T$,

$$\exists v' \in V, Tv_i = Tv' + \lambda_i v_i$$

Therefore $v_i \in range\ T$, i.e. T surjective and thus invertible.

Suppose T invertible.

Then clearly $\lambda_1 \neq 0$ since otherwise $Tv_1 = 0$ making T not injective and thus not invertible, contradiction.

Then suppose at λ_i is the first entry on the diagonal that equals 0.

Then we have $Tv_1, ..., Tv_i \in span(v_1, ..., v_{i-1})$, i.e. T maps $v_1, .., v_i$ to $span(v_1, ..., v_{i-1})$.

Therefore T is not injective and thus $\exists v \in span(v_1, ..., v_i)$ such that $Tv = 0$, contradiction. Therefore no such λ_i exists.

5.32 Eigenvalues from upper-triangular matrix

Suppose $T \in \mathcal{L}(V)$ has an upper-triangular matrix w.r.t. a basis of V. Then eigenvalues of T are the entries on the diagonal of that matrix.

Proof:

Denote that matrix $\mathcal{M}(T)$, and the diagonal entris $\lambda_1, ..., \lambda_n$.

Then $T - \lambda I$ will have $\lambda_1 - \lambda, ..., \lambda_n - \lambda$ on the diagonal.

Using 5.31, we know that $T - \lambda I$ is not injective $\iff$ one of the diagonal entry is 0, i.e. $\lambda = \lambda_i$ for some i.