Fundamental Concepts of Analysis

Week 1 Notes (a)

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3 Algebraic Axioms of the Real Numbers

3.1 Definition of Binary Operation

A **binary operation** on a set X is a function from $X \times X$ into X.

3.2 Definition of Real Numbers

The **real numbers R** is a set of objects satisfying the Axioms 1 to 13 listed below.

Axiom 1. Closure under addition.

Axiom 2. Associativity under addition.

Axiom 3. Commutativity under addition.

Axiom 4. Existence of an additive identity.

Axiom 5. Existence of additive inverse.

The additive inverse in Axiom 5 can be shown to be unique.

3.3 Theorem: Additive Identity is Unique

Proof:

Suppose $\exists 0' \in \mathbf{R}, \forall x \in \mathbf{R}, x + 0' = x$.

Then $0 + 0' = 0$. Similarly, $0' + 0 = 0'$.

Then from Axiom 3 we know that $0 = 0 + 0' = 0' + 0 = 0'$.

Therefore we have shown that the additive identity is unique.

Some other properties can be proven similarly.

Any mathmatical system that satisfies Axioms 1 to 5 is called an abelian group.

Axiom 6. Closed under multiplication

Axiom 7. Associativity under multiplication

Axiom 8. Commutativity under multiplication

Axiom 9. Existence of an multiplicative identity

Axiom 10. Existence of multiplicative inverse for non-zero real numbers

Axiom 11. Multiplication distributes over addition

$$\forall x, y, z \in \mathbf{R}, x(y + z) = xy + xz, (y + z)x = yx + xz$$

Similar to addition, the identity and inverse of multiplication is unique.

Any mathematical system that satisfies Axioms 1 to 11 is called a field.

3.4 Theorem: Zero Times Anything is Zero

Proof:

$$\forall x \in \mathbf{R}, x \cdot 0 = x \cdot (0 + 0) = x \cdot 0 + x \cdot 0 \qquad \text{Axiom 4, 11}$$

$$x \cdot 0 + [-(x \cdot 0)] = x \cdot 0 + x \cdot 0 + [-(x \cdot 0)] \qquad \text{Axiom 5}$$

$$0 = x \cdot 0 + (x \cdot 0 + [-(x \cdot 0)]) = x \cdot +0 \qquad \text{Axiom 2}$$

Because we have prove that the additive identity is unique, this shows that $x \cdot 0 = 0$.

4 Order Axiom of Real Numbers

Axiom 12. Positive Real Numbers

There is a subset $P \subseteq \mathbf{R}$ called **positive real numbers** satisfying

1. $\forall x, y \in P, x + y, xy \in P$.
2. $\forall x \in \mathbf{R}$, only one of the follow statment is true:

$$x \in P, x = 0, or - x \in P$$

From positive real numbers we can define some notations.

4.1 Definitions of Positive and Negative

Given $x, y \in \mathbf{R}$,

i $-x \in P \to x$ is **negative**.

ii $x > y$ means $x - y$ is positive.

iii $x \geq y$ means $x < y$ or $x = y$.

iv $x < y$ means $y > x$.

v $x \leq y$ means $y \geq x$.

4.2 Theorem: Properties of Orders of Real Numbers

i $1 > 0$

ii $\forall x, y, z \in \mathbf{R}, x > y, y > z \to x > z$

iii $\forall x, y, z \in \mathbf{R}, x > y \to x + z > y + z$

iv $\forall x, y, z \in \mathbf{R}, x > y, z > 0 \to xz > yz$

v $\forall x, y, z \in \mathbf{R}, x > y, z < 0 \to xz < yz$

4.4 Definition of Absolute Value

Given $x \in \mathbf{R}$ we define

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

4.5 Theorem: Properties of Absolute Value

i Let $\epsilon > 0$. Then $|x| < \epsilon \iff -\epsilon < x < epsilon, |x| \leq \epsilon \iff -\epsilon \leq x \leq \epsilon$

ii $\forall x \in \mathbf{R}, x \leq |x|$

iii $\forall x, y \in \mathbf{R}, |xy| = |x||y|$

iv $\forall x, y \in \mathbf{R}, |x + y| \leq |x| + |y|$