Linear Algebra Done Right

HW 3

shaozewxy

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3.10

Question 2. Let V and W be finite dimensional, and consider $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, U)$.

a) Prove that $\dim(\operatorname{range} ST) \leq \dim(\operatorname{range} T)$.

b) Prove that $\dim(\operatorname{range} ST) = \dim(\operatorname{range} T)$ if and only if

$$\operatorname{range} T + \operatorname{null} S = \operatorname{range} T \oplus \operatorname{null} S.$$

c) Prove that $\dim(\operatorname{null} ST) \leq \dim(\operatorname{null} S) + \dim(\operatorname{null} T)$.

d) Challenge problem: Can you give some description (in terms of conditions on T, S, V, etc.) of when we get equality in the previous part, i.e. $\dim(\operatorname{null} ST) = \dim(\operatorname{null} S) + \dim(\operatorname{null} T)$?

a.

$$dim(range\ ST) = dim\ V - dim\ null\ ST$$

$$dim(range\ T) = dim\ V - dim\ null\ T$$

We just NTS that $dim\ null\ ST > dim\ null\ T$.

To do so, WTS that $null\ T \subseteq null\ ST$:

This is obvious since $\forall v \in null\ T, ST(v) = S(Tv) = S(0) = 0 \rightarrow v \in null\ ST$.

Therefore we have shown that $dim(range\ ST) \leq dim(range\ T)$.

b.

From a. we know that the equality is achieved $\iff$ $dim\, null\, T = dim\, null\, ST$, i.e. $null\ T = null\ ST$.

Now suppose $dim(range\ ST) = dim(range\ T)$, WTS $range\ T + null\ S = range\ T \oplus null\ S$.

Since we know $null\ T = null\ ST$, and obviously $range\ T$ independent with $null\ T$, we conclude that $range\ T$ independent with $null\ ST$ and therefore $range\ T + null\ S = range\ T \oplus null\ S$.

Then suppose $range\ T + null\ S = range\ T \oplus null\ S$, WTS $dim(range\ ST) = dim(range\ T)$:

Suppose $\exists v \in null\ ST - null\ T$, then this means $Tv = w \neq 0 \in null\ S$, i.e. $w \neq 0 \in null\ S \cap range\ T$, contradiction with the fact that $range\ T$ independent with $null\ S$. Therefore no such v exists and thus $null\ ST = null\ T \Rightarrow dim(range\ ST) = dim(range\ T)$.

c.

Given $v \in V$, if $v \in null\ T$, then $Tv = 0 \Rightarrow ST(v) = 0 \Rightarrow v \in null\ ST$. Therefore we know that $null\ S \subseteq null\ ST$.

Then we create $F \in \mathcal{L}(T^{-1}(null\ S), null\ S)$ defined as $Fv = Tv$.

By definition $T^{-1}(null\ S) = null\ ST$.

Obviously $null\ F = null\ T$.

Therefore we can conclude that

$$dim\ range\ F + dim\ null\ F = dim(T^{-1}(null\ S))$$

$$dim\ range\ F + dim\ null\ T = dim\ null\ ST$$

$$dim\ null\ ST \leq dim\ null\ S + dim\ null\ T$$

d.

From c. we see that the equality is achived when $dim\ range\ F = dim\ null\ S$, i.e.

$$null\ S \subseteq range\ T$$