Fundamental of Analysis

Week 1 HW (Axiom Proof)

shaozewxy

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Important Exercises from Section 3

3.1

Prove that the additive inverse of axiom 5 is unique.

We know $\forall x, \exists x^{-1}$ such that $x + x^{-1} = 0$.

Then assume that $\exists x'$ such that $x + x' = 0$. Then we have

$$\begin{aligned} x + x^{-1} &= 0 \\ x + x^{-1} + x' &= 0 + x' = x' \\ (x^{-1} + x) + x' &= x' \\ x^{-1} + (x + x') &= x' \\ x^{-1} + 0 &= x^{-1} = x' \end{aligned}$$

3.3

Prove that $-(-x) = x$ for all $x \in \mathbf{R}$.

Given $x \in \mathbf{R}$, we have

$$x + -x = 0$$

$$x + -x + -(-x) = 0 + -(-x) = -(-x)$$

$$x + (-x + -(-x)) = -(-x)$$

$$x + 0 = x = -(-x)$$

3.4

Prove that $-(x+y) = -x - y$ for all $x, y \in \mathbf{R}$.

$\forall x, y \in \mathbf{R}$, we have

$$(x+y) - (x+y) = 0$$

$$(x+y) - (x+y) - x - y = 0 - x - y = -x - y$$

$$((x+y) - x - y) - (x+y) = -x - y$$

$$0 - (x+y) = -x - y$$

$$-(x+y) = -x - y$$

3.5

Let $x, y \in \mathbf{R}$. Prove that $xy = 0$ if and only if $x = 0$ or $y = 0$.

Suppose $x \neq 0, y \neq 0, xy = 0$.

Then according to Axiom 10, since $x \neq 0, y \neq 0, \rightarrow \exists x^{-1}, y^{-1}$ such that $xx^{-1} = yy^{-1} = 1$. Then we have

$$xy = 0$$

$$xyy^{-1} = x(yy^{-1}) = x = 0 \cdot y^{-1} = 0$$

$$x = 0$$

Which is contradiction since $x \neq 0$.

Therefore we can see that it is impossible when not $x = 0$ and $y = 0$ for $xy = 0$.

3.6

Let $x, y \in \mathbf{R}$. Prove that if $xy = xz$ and $x \neq 0$, then $y = z$.

$\forall x, y, z \in \mathbf{R}$ such that $xy = xz, x \neq 0$, we have

Since $x \neq 0, \to \exists x^{-1} \in \mathbf{R}$ such that $xx^{-1} = 1$.

Therefore,

$$\begin{aligned} xy &= xz \\ x^{-1}(xy) &= x^{-1}(xz) \\ (x^{-1}x)y &= (x^{-1}x)z \\ y &= z \end{aligned}$$

3.7

Prove that $-(xy) = x(-y) = (-x)y$ for all $x, y \in \mathbf{R}$.

$\forall x, y \in \mathbf{R}$

$$\begin{aligned} -xy + xy &= 0 \\ -xy + xy + x(-y) &= 0 + x(-y) = x(-y) \\ -xy + x(y - y) &= x(-y) \\ -xy + 0 &= -xy = x(-y) \end{aligned}$$

The case for $-xy = (-x)y$ can be similarly prove.

3.8

Prove that $(-1)x = -x$ for all $x \in \mathbf{R}$.

$\forall x \in \mathbf{R}$, we have

$$\begin{aligned} 0 &= 0 \\ x \cdot 0 &= 0 \\ x \cdot (1 + (-1)) &= 0 \\ x + (-1)x &= 0 \\ x + (-1)x + (-x) &= -x \\ (x + (-x)) + (-1)x &= -x \\ 0 + (-1)x = (-1)x &= -x \end{aligned}$$

Proof of Theorem 4.2

i

Prove that $1 > 0$.

We try to prove that $1 - 0 = 1 \in P$:

Since $1 \neq 0$, we know that either $1 \in P$ or $-1 \in P$.

Suppose for a contradiction that $-1 \in P$, then we have $-1 \cdot -1 \in P$.

However, $-1 \cdot -1 = -(-1) = 1 \notin P$. Contradiction. Therefore $-1 \notin P$, i.e. $1 \in P$.

ii

Prove that if $x > y$ and $y > z$, then $x > z$ for all $x, y, z \in \mathbf{R}$.

$\forall x, y, z \in \mathbf{R}$ such that $x > y, y > z$, we have

$x - y \in P$ and $y - z \in P$, therefore we have

$$(x - y) + (y - z) = x - y + y - z = x - z \in P$$

Therefore $x > z$.

iii

Prove that if $x > y$, then $x + z > y + z$ for all $x, y, z \in \mathbf{R}$.

$\forall x, y, z \in \mathbf{R}$ such that $x > y$, we have

$$x - y = x - y$$
$$x - y + 0 = x - y + (z - z) = x - y$$
$$x - y + z - z = (x + z) - y - z = (x + z) - (y + z) = (x - y)$$

Therefore $x - y \in P \to (x + z) - (y + z) \in P$, i.e. $x + z > y + z$.

iv

If $x > y$ and $z > 0$, then $xz > yz$ for all $x, y, z \in \mathbf{R}$.

$\forall x, y, z \in \mathbf{R}$ such that $x > y, z > 0$, we have

$$x - y \in P, z - 0 = z \in P$$

Therefore $(x - y)z = xz - yz \in P$, i.e. $xz > yz$.

v

If $x > y$ and $z < 0$, then $xz < yz$ for all $x, y, z \in \mathbf{R}$.

$\forall x, y, z \in \mathbf{R}$, such that $x > y, z < 0$, we have

$$x - y \in P, 0 - z = -z \in P$$

Therefore $(x - y)(-z) = -xz + yz = yz - xz \in P$, i.e. $xz < yz$

Proof of Theorem 4.5

i

Prove that let $\epsilon > 0$, then $|x| < -\epsilon$ if and only if $-\epsilon < x < \epsilon$ and $|x| \leq \epsilon$ if and only if $-\epsilon \leq x \leq \epsilon$.

Suppose $x \geq 0$, and $|x| < \epsilon$.

Then we have that $x \geq 0$ and $\epsilon > 0$, therefore $-\epsilon < 0$ and therefore $x > -\epsilon$.

Since $|x| = x$ and $|x| < \epsilon$, we have $x < \epsilon$.

Therefore $-\epsilon < x < \epsilon$.

Suppose $x < 0$ and $|x| < \epsilon$.

Then we have $|x| = -x$.

Since $|x| = -x < \epsilon$ and $-1 < 0$, we have

$$-1 \cdot -x > -1 \cdot \epsilon$$
$$x > -\epsilon$$

Since $x < 0$ and $0 < \epsilon$, we have $x < \epsilon$.

Therefore $-\epsilon < x < \epsilon$.

For the case where $|x| \leq \epsilon$, we consider two cases:

- $|x| \neq \epsilon$: this is just the case above where $|x| < \epsilon$, which is already prove.

- $|x| = \epsilon$: then either $x = \epsilon$ or $-x = \epsilon$.

  If $x = \epsilon$, then $-\epsilon \leq x \leq \epsilon$ is true by nature.

  If $-x = \epsilon$, then $x = -\epsilon$, therefore $-\epsilon \leq x$ is true.

  Since $1 > 0, 0 > -1$, we have $1 > -1$.

  Since $1 > -1$ and $\epsilon > 0$, we have $1 \cdot \epsilon > -1 \cdot \epsilon$, i.e. $x = -\epsilon < \epsilon$.

  Therefore $-\epsilon \leq x \leq \epsilon$.

ii

Prove that $x \leq |x|$ for all $x \in \mathbf{R}$.

$\forall x \in \mathbf{R}$, we have:

If $x \geq 0$, then $|x| = x$ and therefore $x \leq |x|$ is true.

If $x < 0$, then $|x| = -x > 0$, since $|x| > 0$ and $x < 0$, we have $x < |x|$, which makes $x \leq |x|$ true.

Therefore $x \leq |x|$ is always true.

iii

Prove that $|xy| = |x||y|$ for all $x, y \in \mathbf{R}$.

Prove by discussing 3 situations.

iv

Prove that $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbf{R}$.

If $x + y \geq 0$, then $|x + y| = x + y \leq |x| + |y|$.

If $x + y < 0$, then $|x + y| = -x - y \leq |-x| + |-y| = |x| + |y|$.

Therefore $|x + y| \leq |x| + |y|$.