Linear Algebra Done Right

Week 3 Notes (c)

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June 2022

3.E Products and Quotients of Vector Space

Products of Vector Spaces

3.71 Definition of product of vector spaces

Given $V_1, ..., V_m$ are vector spaces over $\mathbf{F}$.

- The **product** $V_1 \times ... \times V_m$ is defined by

$$V_1 \times ... \times V_m = \{(v_1, ..., v_m) : v_1 \in V_1, ..., v_m \in V_m\}$$

- Addition of $V_1, ..., V_m$ is defined by

$$(u_1, ..., u_m) + (v_1, ..., v_m) = (u_1 + v_1, ..., u_m + v_m)$$

- Scalar multiplication is defined by

$$\lambda(v_1, ..., v_m) = (\lambda v_1, ..., \lambda v_m)$$

3.73 Product of vector spaces is a vector space

The proof of this is obvious.

3.76 Dimension of a product is the sum of dimensions

Given $V_1, ..., V_m$ finite-dimension vector spaces, then $V_1 \times ... \times V_m$ is also finite-dimensional add

$$dim(V_1 \times ... \times V_m) = dim\ V_1 + ... + dim\ V_m$$

To prove this, just create a basis where all entris all 0 except for a basis vector from on of the vector space.
This is obviously a basis and the dimension is just the sum of dimensions of all the vector spaces.

Products and Direct Sums

3.77 Product and direct sums

Given $U_1, ..., U_m$ subspaces of V. Define map $\Gamma : U_1 \times ... \times U_m \to U_1 + ... + U_m$ by

$$\Gamma(u_1, ..., u_m) = u_1 + ... + u_m$$

Then $U_1 + ... + U_m$ is a direct sum $\iff \Gamma$ is injective.

Proof:

This is basically saying that $\nexists (u_1, ..., u_m) \in U_1 \times ... \times U_m$ such that $u_1 + ... + u_m = 0$, therefore making $U_1 + ... + U_m$ a direct sum.

Since Γ is naturally surjective, therefore we can say it is a direct sum $\iff \Gamma$ is invertible. Therefore, the below result

3.78 Condition for direct sum

Given V finite-dimensional and $U_1, ..., U_m$ subspaces of V. Then $U_1 + ... + U_m$ is a direct sum $\iff$

$$dim(U_1 + ... + U_m) = dim\ U_1 + ... + dim\ U_m$$

Quotient of Vector Spaces

3.79 Definition of v+U

Given $v \in V, U \leq V$. Then $v + U$ is a subset of V defined by

$$v + U = \{v + u : u \in U\}$$

3.81 Definition of affine subset and parallel

- An **affine subset** of V of the form $v + U$ for some $v \in V, U \leq V$.
- For $v \in V, U \leq V$, the affine subset $v + U$ is said to be **parallel** to U.

3.83 Definition of quotient space, V/U

Given $U \leq V$. THen the quotient space V/U is the set of all affine subsets of V parallel U, i.e.

$$V/U = \{v + U : v \in V\}$$

Next we try to show that V/U is a vector space.

3.85 Two affine subsets are equal or disjoint

Given $U \leq V, v, w \in V$, the following are equivalent:

(a) $v - w \in U$

(b) $v+U=w+U$

(c) $(v+U)\cap(w+U)\neq\emptyset$

Proof:

Suppose a is true, then we WTS b is also true:

$\forall u'$ such that $\exists u\in U, v+u=u'$, we have that

$$v+u=w+v-w+u=w+(v-w+u)\in w+U$$

Therefore $v+U\subseteq w+U$, similarly $w+U\subseteq v+U$.

Thereofre $v+U=w+U$.

Now obviously that $b\to c$.

We only NTS $c\to a$:

Suppose $\exists u'\in v+U\cap w+U$ such that $\exists u_1,u_2\in U, v+u_1=u'=w+u_2$, then we have $v-w=u_2-u_1\in U$. Proving a.

With the result above we can define the operations on V/U.

3.86 Definition of addition and scalar multiplication on V/U

Given $U\leq V$. Then **addition** and **scalar multiplication** are defined on V/U by

$$(v+U)+(w+U)=(v+w)+U$$
$$\lambda(v+U)=(\lambda v+U)$$

3.87 Quotient space is a vector space

Proof:

We need to show that the addition and multiplication above are well-defined.

For addition, suppose $v+U=v'+U, w+U=w'+U$. we NTS $(v+U)+(w+U)=(v+w)+U=(v'+U)+(w'+U)=(v'+w')+U$:

Since $v + U = v' + U, \rightarrow v' - v \in U$, similarly $w' - w \in U$.

Therefore we have

$$v' - v + w' - w = (v' + w') - (v + w) \in U$$

Therefore $(v' + w') + U = (v + w) + U$.

The scalar multiplication can be seen as λ times of $+(v + U)$, therefore is already proven.

Definition of quotient map, π

Given $U \leq V$, the **quotient map** $\pi : V \rightarrow V/U$ is defined by

$$\forall v \in V, \pi(v) = v + U$$

3.89 Dimension of quotient space

Given $U \leq V$, then

$$dim\ V/U = dim\ V - dim\ U$$

Proof:

$$\begin{aligned} dim\ V/U &= dim\ range\ \pi \\ &= dim\ V - dim\ null\ \pi \\ &= dim\ V - dim\ U \end{aligned}$$

3.90 Definition of $\tilde{T}$

Given $T \in \mathcal{L}(V, W)$, define $\tilde{T} : V/(null\ T) \rightarrow W$ by

$$\tilde{T}(v + null\ T) = Tv$$

NTS that the definition makes sense:

Given $v_1, v_2 \in V$ such that $v_1 + null\ T = v_2 + null\ T$, then:

$$\tilde{T}v_1 - \tilde{T}v_2 = Tv_1 - Tv_2 = T(v_1 - v_2)$$

Now since $v_1 + null\ T = v_2 + null\ T, \rightarrow v_1 - v_2 \in null\ T$.

Therefoore $\tilde{T}v_1 - \tilde{T}v_2 = T(v_1 - v_2) = 0$.

3.91 Null space and range of $\tilde{T}$

Given $T \in \mathcal{L}(V, W)$, then we have:

(a) $\tilde{T}$ is injective

(b) $range\ \tilde{T} = range\ T$

(c) $V/(null\ T)$ is isomorphic to $range\ T$

Proof:

(a) Given $\tilde{a}, \tilde{b} \in V/U$ such that $\tilde{T}(\tilde{a}) = \tilde{T}(\tilde{b})$, then we have $Ta = Tb \rightarrow a - b \in null\ T \rightarrow \tilde{a} = \tilde{b}$.

(b) This is obvious.

(c) This comes from a and b.