Fundamental Concepts of Analysis

Week 1 Notes (e)

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September 2022

Summary

1. 16.2 Establish bounded and monotone equals convergence.

2. 16.3 16.4 Use the 16.2 to prove convergence without finding the limit.

3. 16.5 16.6 Use 16.2 to prove $(1+1/n)^n$ converges and define the limit to be e.

4. 16.7 Another example of using 16.2 to prove $n^{1/n}$ converges.

16 Monotone Sequences and e

Previously to prove that a sequence $\{a_n\}$ converges, we need to know the limit L.

Here we find a method to prove convergence without knowing the limit beforehand.

Definition of monotonicity

Given $\{a_n\}$, we say that $\{a_n\}$ is **increasing/decreasing** if $a_n \leq a_{n+1}(a_n \geq a_{n+1})$. In either case, we say that $\{a_n\}$ is **monotone**.

In the case where the eqaulity is not necessary, we say $\{a_n\}$ is **strictly increasing(decreasing)**, and also **strictly monotone**.

16.2 Monotone sequense is convergent iff bounded

Proof:

We prove the case for **increasing** since the decreasing will be similar.

Suppose $\{a_n\}$ is increasing and convergent then obviously it is bounded.

Therefore only NTS that if $\{a_n\}$ is increasing and bounded, then $\{a_n\}$ convergent:

We denote $X = \{x|x \text{ upper bound of } a_n\}$. Since $\{a_n\}$ bounded above, so we know that X not empty and X is trivially bounded below by a_1.

Therefore we claim that $L = \inf X$ exists, and it is the limit of $\{a_n\}$.

Suppose $\exists \epsilon > 0$ such that $\forall N \in \mathbf{P}, \exists n > N \in \mathbf{P}$ such that $|L - a_n| = L - a_n > \epsilon$.

Then we can see that $\forall n \in \mathbf{P}$, take $N = n$, then $\exists n' > N = n \in \mathbf{P}$ such that $L - a_{n'} > \epsilon$. Since $\{a_n\}$ increasing, and $n' > n$, we have $a_n < a_{n'} < L - \epsilon$.

This shows that $L - \epsilon$ is an upper bound of $\{a_n\}$ and contradiction, therefore no such ϵ exists.

Therefore we can also prove convergence by showing a sequence is bounded and monotone.

16.3

IF $|a| < 1$, then $\lim_{n\to\infty} a^n = 0$.

Proof:

First suppose $0 < a < 1$, then we know that a^n is bounded above by 1 and below by 0. Therefore a^n is bounded and decreasing.

Therefore $L = \lim_{n\to\infty} a^n$ exsits.

We have that

$$L = \lim_{n\to\infty} a^n = \lim_{n\to\infty} a^{n+1} = a \lim_{n\to\infty} a^n = aL$$

Therefore if $L \neq 0$, we have $a = 1$, which is a contradiction.

We conclude that $L = 0$.

For the case where $-1 < a < 0$, we take the subsequence of $a^{2n} = (a^2)^n$, which becomes the first case.

16.4

If $a > 0$, then $\lim_{n\to\infty} a^{1/n} = 1$.

Proof:

Suppose $a \geq 1$, then $\{a^{1/n}\}$ is decreasing and bounded below by 1. Therefore $L = \lim_{n\to\infty} a^{1/n}$ exists.

We know that

$$\lim_{n\to\infty} a^{2/n} = \lim_{n\to\infty} a^{1/n} \cdot a^{1/n} = (\lim_{n\to\infty} a^{1/n})^2 = L^2$$

Now $\{a^{2/2n}\}$ is a subsequence of $\{a^{2/n}\}$ and therefore has limit L^2.

But $\{a^{2/2n}\} = \{a^{1/n}\}$ by definition, therefore $L = L^2$. Since $L \neq 0$, we know that $L = 1$.

For the case where $0 < a < 1$, we have

$$\begin{aligned}
\lim_{n\to\infty} a^{1/n} &= \lim_{n\to\infty} \left(\frac{1}{\frac{1}{a}}\right)^{1/n} \\
&= \frac{1}{\lim_{n\to\infty}(\frac{1}{a})^{1/n}} \\
&= \frac{1}{1} = 1
\end{aligned}$$

Given these two examples of proving convergence before calculating the limit, we can look at how to define e.

16.5

Given $0 \le a < b$, we have

$$\frac{b^{n+1} - a^{n+1}}{b - a} < (n+1)b^n$$

Proof:

We know that

$$(b^{n+1} - a^{n+1}) = (b - a)\left(\sum_{i=0}^{n} b^i a^{n-i}\right)$$

Therefore,

$$\frac{b^{n+1} - a^{n+1}}{b - a} = \sum_{i=0}^{n} b^i a^{n-i} < (n+1)b^n$$

16.6 Defining e

The sequence $\{(1 + 1/n)^n\}$ is increasing and convergent. The limit is denote e.

Proof:

We denote the sequence $\{x_n\}$ with $b = 1 + 1/n, a = 1 + 1/(n+1)$.

We reorganize the inequality to

$$b^n(b - (n+1)(b - a)) < a^{n+1}$$

We denote $X = (b - (n+1)(b-a))$ the cross step term.

In this case $X = (1 + 1/n - 1/n) = 1$.

Therefroe $b^n = x_n < a^{n+1} = x_{n+1}$, i.e $\{x_n\}$ increasing.

Now we NTS it is bounde above:

Let $a = 1, b = 1 + 1/2n$, then $X = 1/2$, then

$$\left(1 + \frac{1}{2n}\right)^n X < 1$$
$$\left(1 + \frac{1}{2n}\right)^n < 2$$
$$\left(1 + \frac{1}{2n}\right)^{2n} < 4$$

Since $\{x_n\}$ is increasin, we know that $\forall n \in \mathbf{P}, x_n = (1 + 1/n)^n < x_{2n} = (1 + 1/2n)^{2n} = 4$.

Therefore $\{x_n\}$ is bouned above and converges.

16.7

The sequence $\{n^{1/n}\}_{n=3}^{\infty}$ is decreasing and $\lim_{n\to\infty} n^{1/n} = 1$.

Proof:

First we NTS $\{n^{1/n}\}$ decreasing.

$$\begin{aligned}
(n+1)^{1/(n+1)} \leq n^{1/n} &\iff \\
((n+1)^{1/(n+1)})^{n(n+1)} \leq (n^{1/n})^{n(n+1)} &\iff \\
(n+1)^n \leq n^{n+1} &\iff \\
\left(\frac{n+1}{n}\right)^n \leq n &\iff \\
\left(1 + \frac{1}{n}\right)^n \leq n &
\end{aligned}$$

From 16.6 we know that $\left(1 + \frac{1}{n}\right)^n \leq 4$ therefore for $n \geq 4 \in \mathbf{P}, \{n^{1/n}\}$ is decreasing.

Since it is also bounded below by 1, we know $L = \lim_{n\to\infty} n^{1/n}$ exists.

Now we have

$$\begin{aligned} L &= \lim_{n\to\infty} (2n)^{1/2n} \\ &= \lim_{n\to\infty} 2^{1/2n} n^{1/2n} \\ &= \lim_{n\to\infty} 2^{1/2n} \left(\lim_{n\to\infty} n^{1/n}\right)^2 \\ &= 1 \cdot L^2 \end{aligned}$$

Therefore since $L \neq 0$, we know that $L = 1$.