NOTES ON FLUID MECHANICS

INTERMEDIATE LEVEL

BY

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Contents

Introduction

Just a compilation of notes on different aspects of fluid dynamics I have collected over the years.

The final push was from teaching the course Fluid Dynamics on the International Master of Nuclear Fusion, course 2018. I realized much material was scattered all over the place, on notes, articles, blog entries . . .

I may upload these notes to a collaborative site (e.g. github) so that other people may contribute. As of today, I am the sole author.

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Part I

Ideal fluids

Chapter 1

Continuity of mass

1.1 The concept of particle

1.2 The continuity equation

1.3 Incompressibility

Chapter 2

Euler's equations

2.1 Material derivative

2.2 Pressure forces

2.3 The momentum equation

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + \rho \mathbf{g} \tag{2.1}$$

2.4 Bernouilli's principle

The momentum equation 2.1 is still quite daunting due to its nonlinear term in the convective derivative.

In order to make progress, it was Lamb's idea in 1895 to use the following identity for the vector product of any vector field and its curl:

$$\mathbf{u} \times (\nabla \times \mathbf{u}) = \frac{1}{2}\nabla u^2 - \mathbf{u}(\nabla \mathbf{u})$$

(in fact, the identity is somewhat more general, see exercise 1).

Introducing the name "vorticity" for the curl of the velocity field,

$$\omega = \nabla \times \mathbf{u},$$

we may write the momentum equation as

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\nabla u^2 - \mathbf{u} \times \omega = -\frac{1}{\rho}\nabla p + \mathbf{g}.$$

Our goal is to gather as many terms as we can inside a gradient, for reasons that will become clear soon. First, the gravitational acceleration is, by definition, $\mathbf{g} = -\nabla \varphi$, where φ is the gravitational potential energy (gz if the

z axis is the vertical). Now, the pressure gradient over the density may be related to the gradient of pressure over density, thus:

$$\nabla\left(\frac{p}{\rho}\right)=\frac{\nabla p}{\rho}-\frac{\nabla p}{\rho^2}\nabla\rho$$

Therefore,

$$\frac{\partial\mathbf{u}}{\partial t}+\nabla\left(\frac{1}{2}u^2+\frac{p}{\rho}+\varphi\right)=\mathbf{u}\times\boldsymbol{\omega}+\frac{\nabla p}{\rho^2}\nabla\rho$$

Now, in steady flow we would have

$$\nabla\left(\frac{1}{2}u^2+\frac{p}{\rho}+\varphi\right)=\mathbf{u}\times\boldsymbol{\omega}+\frac{\nabla p}{\rho^2}\nabla\rho.$$

The term involving the vector product of the vorticity and the velocity is perpendicular to both. Hence, it vanishes upon a scalar multiplication with $\mathbf{u}$:

$$\mathbf{u}\cdot\nabla\left(\frac{1}{2}u^2+\frac{p}{\rho}+\varphi\right)=\frac{\nabla p}{\rho^2}\mathbf{u}\cdot\nabla\rho.$$

In steady flow, the continuity equation is

$$\mathbf{u}\cdot\nabla\rho+\rho\nabla\cdot\mathbf{u}=0,$$

hence if the flow is incompressible (technically, divergence-free, as explained in section **??**), $\mathbf{u}\cdot\nabla\rho$. I.e. the density does not change along streamlines. Then,

$$\mathbf{u}\cdot\nabla\left(\frac{1}{2}u^2+\frac{p}{\rho}+\varphi\right)=0,$$

which states the fact that the quantity

$$h=\frac{1}{2}u^2+\frac{p}{\rho}+\varphi$$

is constant along a given streamline. This result is known as Bernoulli's principle, and applies only to ideal, steady, incompressible flow. (There is a variant of it that applies to unsteady flow, as we will see in section **??**.) This combination is called "the head" and is customarily used in elementary applications of this result. Some of its direct applications are: the Venturi effect (by which the pressure decreases in zones with higher velocities), slow drainage of containers, syphons ...

We will also see that in some cases, like in potential flow, the velocity field may be found independently of the pressure. This principle then yields the corresponding pressure from the velocity.

2.5 Exercises

1.) Prove that

$$\mathbf{u}\times(\nabla\times\mathbf{q})=\nabla_{\mathbf{q}}(\mathbf{u}\cdot\mathbf{q})-\mathbf{u}\cdot\nabla\mathbf{q},$$

for any two vector fields. By $\nabla_{\mathbf{q}}$ it is meant that the gradient is applied only on $\mathbf{q}$ (Feyman's subscript notation). Hint: the curl of a field is associated with a tensor with elements:

$$\overline{(\nabla\times\mathbf{q})_{ij}} = \left(\frac{\partial q_j}{\partial x_i} - \frac{\partial q_i}{\partial x_j}\right) \qquad \overline{(\nabla\times\mathbf{q})_{ij}} = \sum_k \epsilon_{ijk}(\nabla\times\mathbf{q})_k.$$

In the latter, ϵ is a rank-three tensor, the totally antisymmetric tensor (also known as the Levi-Civita symbol). If this seems mistifying, all that it is expressing is that e.g. $(\nabla\times\mathbf{q})_z = \partial q_x/\partial y - \partial q_y/\partial x$, et cætera.

This tensor appears also in the vector product:

$$(\mathbf{u}\times\mathbf{q})_i = \sum_{j,k}\epsilon_{ijk}u_j q_k.$$

These two results make the demonstration quite simple. We will also be using these expressions in section **??**, when we talk about rotations of a particle.

2.6 Dimensionless variables

A procedure to gain insight into a physical problem is to try to cast the different magnitudes into dimensionless (or, "reduced") ones. For example, if there is a relevant length scale L, all lengths may be rescaled according to it:

$$x^* = \frac{x}{L} \quad y^* = \frac{y}{L} \quad z^* = \frac{z}{L},$$

where an asterisk marks a dimensionless magnitude. We can also write it in vector notation: $\mathbf{r}^* = \mathbf{r}/L$.

As an example, in some problems this is the only relevant scale, and the movement is driven by gravity, whose accelaration is g. In such cases, the time scale will be given by the only combination of L and g:

$$t_0 \sim \sqrt{\frac{L}{g}}.$$

This is actually the correct result for the period of a simple pendulum, but for a numerical factor of 2π with no dimensions: $T = 2\pi t_0$. No equations have been solved (or even written down) in order to arrive to this result. Notice also that for larger displacements of the pendulum, the amplitude of the motion is another length, which complicates the analysis.

In many fluid problems there is a well-defined velocity u_0 that sets the velocity values (e.g. the upstream velocity in flows around objects). If this is the case,

$$\mathbf{u}^* = \frac{\mathbf{u}}{u_0} \qquad t^* = \frac{t}{L/u_0},$$

so the velocity and length set the time scale. If we apply this to our Euler equation,

$$\rho\frac{d\mathbf{u}}{dt} = -\nabla p + \rho\mathbf{g} \qquad \rho\frac{u_0}{L/u_0}\frac{d\mathbf{u}^*}{dt^*} = -\nabla p + \rho\mathbf{g}.$$

Notice that the ∇ operator can also be cast into dimensionless form. For example, its x component is

$$\nabla_x = \frac{\partial}{\partial x} = \frac{\partial}{\partial(x^* L)} = \frac{1}{L}\frac{\partial}{\partial x^*},$$

so we may define

$$\nabla^* = \frac{1}{L}\nabla.$$

The Euler equation then reads,

$$\rho \frac{u_0}{L/u_0}\frac{d\mathbf{u}^*}{dt^*} = -\frac{1}{L}\nabla^* p + \rho\mathbf{g}.$$

Usually, a reference value ρ_0 for the density is often known, so that $\rho^* = \rho/\rho_0$, and

$$\rho_0\rho^* \frac{u_0}{L/u_0}\frac{d\mathbf{u}^*}{dt^*} = -\frac{1}{L}\nabla^* p + \rho\mathbf{g}.$$

Now, multiplying throughout by $L/(\rho_0 u_0^2)$, and supposing for simplicity that the density is constant,

$$\rho^*\frac{d\mathbf{u}^*}{dt^*} = -\nabla^*\frac{p}{\rho_0 u_0^2} + \frac{L}{u_0^2}\mathbf{g} \qquad \to \qquad \rho^*\frac{d\mathbf{u}^*}{dt^*} = -\nabla^* p^* + \rho^*\mathbf{g}^* \tag{2.2}$$

We have therefore found that the dimensionless pressure and gravity acceleration are given by

$$p^* = \frac{p}{\rho_0 u_0^2} \qquad \mathbf{g}^* = \frac{L}{u_0^2}\mathbf{g}.$$

The reduced pressure was to be expected, given the Bernoulli expressions involving ρu^2. The reduced gravity is directly related to Froude's number, which is historically defined as

$$\mathrm{Fr} = \frac{\sqrt{gL}}{u_0}.$$

Therefore, $\mathbf{g}^* = \mathrm{Fr}^2(\mathbf{g}/g)$, where vector $(\mathbf{g}/g)$ is the unit vector pointing in whichever direction the gravity points to in our problem (usually, $-y$ or $-z$.)

It is easy to check that the continuity equation can likewise be cast into reduced form:

$$\frac{d\rho^*}{dt^*} + (\nabla^*\mathbf{u}^*) = 0.$$

Chapter 3

Hydrostatics

3.1 Incompressible fluids

3.2 Compressible fluids

Chapter 4

Sound waves

4.1 Linearization

4.2 Mach's number and compressibility

Chapter 5

Potential flow

If the flow is initially irrotational, it will remain so due to Lagrange's theorem. This means it must be the gradient of a scalar function, which is called the velocity potential:

$$\mathbf{u} = \nabla\phi.$$

This is similar to the relationship between electric field and electric potential (or the gravitational equivalent), with the important practical difference that no minus sign is used. This means that the velocity points from regions of low potential to regions of high potential, in the direction of steepest ascent.

If the flow is also incompressible, $\nabla \cdot \mathbf{u}$, it follows that the potential must be a solution of Laplace's equation:

$$\nabla^2\phi = 0.$$

5.1 Flow past a cylinder

Let us consider the 2D flow past a cylinder. The velocity field must approach a constant value far away from the obstacle, which we will take as the horizontal direction: $u_x = u_0$. The traverse direction is y, and there is no dependence on z. This value fixes one of the boundary conditions for the potential:

$$\phi \to u_0 x \qquad \text{away from cylinder,}$$

plus a non-important constant, which we will take as 0, fixing ϕ to be zero at the origin (which coincides with the cylinder axis.)

The other boundary condition is related to the presence of the cylinder. The least that the velocity field must satisfy is that the flow does not trespass the surface of the cylinder. This means its normal component should vanish there:

$$\mathbf{un} = 0 \qquad \text{at the cylinder,}$$

where $\mathbf{n}$ is the normal vector at the surface of the cylinder, pointing outside. Given our choice of origin, this vector is the unit radial vector, $\mathbf{n} = \mathbf{e}_r$.

It turns out this condition is the *only one* needed to complete the problem — in the positive sense that we will be able to find explicit solutions to the problem, but also in the negative sense that more physical boundary conditions cannot be accomodated. In particular, it is not possible to impose the often-used "no-slip" boundary condition, in which the velocity would be zero at the surface (not only its normal component, but its tangential components too).

Given the latter condition is somewhat more complex than the first one, it is best to switch to polar coordinates. In Annex **??** we can find the expression for the gradient:

$$u_r = \frac{\partial \phi}{\partial r} \tag{5.1}$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \tag{5.2}$$

$$\tag{5.3}$$

Our boundary conditions for the potential are

$$\phi(r \to \infty) = u_0 r \cos(\theta) \tag{5.4}$$

$$\left.\frac{\partial \phi}{\partial r}\right|_{r=R} = 0. \tag{5.5}$$

Let us try a function

$$\phi = u_0 r \cos(\theta) + f(r) g(\theta).$$

The boundary conditions imply that $g(r)$ should vanish as r gets large. Also, the angular function g must clearly be $\cos\theta$. Otherwise, it will be unable to cancel the $u_0 r \cos(\theta)$ term. Lastly, it must itself be a solution of the Laplace equation. With the Laplacian operator in Annex **??**,

$$\nabla^2 (f \cos(theta)) = \frac{\cos(theta)}{r}\frac{\partial r f}{\partial r} - \frac{f \cos(theta)}{r^2} = 0$$

This equation has solutions $f = r$ (which we already knew), and $f = 1/r$. Of course, a prefactor may be added, so our guess is now

$$\phi = \left(u_0 r + \frac{A}{r}\right) \cos(\theta).$$

Now, in order its radial derivative vanish at $r = R$,

$$\left(u_0 - \frac{A}{R^2}\right) \cos(\theta) = 0,$$

which gives $A = u_0 R^2$ The solution can then be written as

$$\phi = u_0 r \left[1 + \left(\frac{R}{r}\right)^2\right] \cos(\theta).$$

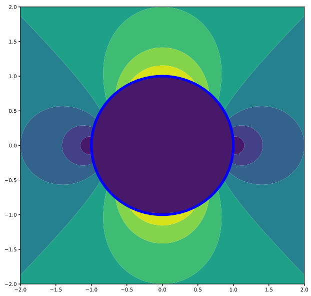


Figure 5.1:

5.1.1 Velocity field

The velocity results immediately from this expression. From the expressions above:

$$u_r = u_0 \left[1 - \left(\frac{R}{r} \right)^2 \right] \cos(\theta) \tag{5.6}$$

$$u_\theta = u_0 \left[1 + \left(\frac{R}{r} \right)^2 \right] \sin(\theta). \tag{5.7}$$

Notice this field has a clear up-down, and left-right symmetry, which was already present in the potential. Also, there are two points, at $r = R$ and $\theta = 0$ and π in which the velocity is null. We said it was impossible to make the velocity equal to zero all over the surface of the cylinder, but it turns out it may be so at some points. These are call stagnation points, because the liquid particles may be traped there for a long time.

Also, the velocity modulus is

$$u = \sqrt{u_r^2 + u_\theta^2},$$

which may be readily computed, see Fig. 5.1, where it is shown that the velocity has maximum values at point at the surface of the cylinder, at $\theta = \pm\pi/2$. At these surface, the radial velocity vanishes and the velocity is readily obtained:

$$u = |u_\theta| = 2u_0 \sin(\theta),$$

so the maximum speed is twice the current velocity.

The pressure field can be obtained from Bernoulli's principle:

$$p = p_0 + \frac{\rho}{2} \left(u_0^2 - u^2 \right),$$

and is plotted in Fig. 5.3. This Figure also shows the potential field, to stress the difference between the two. The velocity field does not simply "goes from

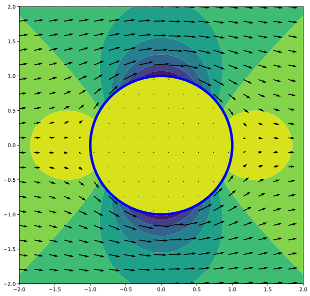


Figure 5.2:

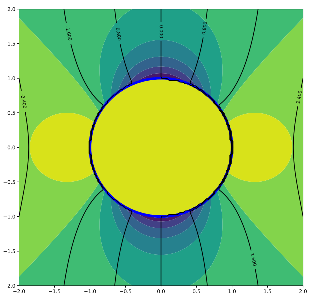


Figure 5.3:

high to low pressures", except in those areas in which the two fields happen to have similar gradients.

The pressure field has a striking left-right symmetry. This of course applied to its value at the surface. Since the pressure is the only force this fluid may exert upon the cylinder, the consequence is that the net force on it is exactly zero. The push it experiences towards the right is exactly cancelled by the push back towards the left. Therefore, the drag, which in this case is the net horizontal force, is zero.

This striking result is known as d'Alembert's paradox, and marked a historical chasm between theoretical fluid mechanics and applied hydraulics. Indeed, the applied community could accept that the push back may exist. Indeed, it had been hypothesized to be solely responsible for objects moving even when not given a force in Aristotelian physics. The effect is real, and is used regularly by cyclist that may take advantage from it, by staying close to the back of a moving vehicle. However, it is not acceptable to accept that it should be exactly equal to the drag.

We now know that the crucial ingredient is the viscosity, which manifests mathematically in a different boundary condition, such as no-slip. This way, a real cylinder is dragged with the flow. We will show later that it is easy to obtain the drag on a sphere in the limit of high viscosity (a cylinder is not solvable due to another paradox, as we will see.)

In case some mathematical justification is required, the pressure at the surface on the cylinder is

$$p = p_0 + \frac{1}{2}u_0^2\left(1 - 4\sin^2(\theta)\right),$$

Now, the drag is given by the net force in the x direction. The (vector) force due to pressure is:

$$\mathbf{F} = \int_0^H \int_0^{2\pi} p(\theta)\mathbf{n}Rd\theta dz,$$

where $p(\theta)\mathbf{n}\mathbf{e}_x Rd\theta dz$ is the pressure force on a differential surface of area $Rd\theta dz$. Now, the drag will be its projection on the horizontal axis:

$$D = \mathbf{F}\cdot\mathbf{e}_x = \int_0^H \int_0^{2\pi} p(\theta)\cos(\theta)Rd\theta dz.$$

The integral is zero, since it consists of two terms. The first one involves constant term. Of course, a constant pressure exerts no net force, and mathematically:

$$\int_0^{2\pi} C\cos(\theta)d\theta = C\int_0^{2\pi}\cos(\theta)d\theta = 0.$$

The other term involves a sin^2 term. Now, this integral is also zero:

$$\int_0^{2\pi}\cos(\theta)\sin^2(\theta)d\theta = 0.$$

qThe reason is that $\sin^2$ may be written as another constant term and a term proportional to $\cos(2\theta)$. (To be precise, $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$.) Cosines form an orthogonal basis: an integral of two different cosines over one period will be zero. Unless they are the same cosine, when the integral is π. To be precise, for integer m and n:

$$\frac{1}{2\pi}\int_0^{2\pi}\cos(m\theta)\cos(n\theta)d\theta = \begin{cases} 0 & \text{if } m \neq n \\ \frac{1}{2} & \text{if } m = n \end{cases} .$$

(It is more elegant to divide by 2π to express the integral as a mean value.) In fact, the previous result about the constant pressure is a particular case in which $m = 0$.

It is also interesting that such an integral involving a sine and a cosine is always zero. This shows that the lift is always null:

$$L = \mathbf{F}\cdot\mathbf{e}_x = \int_0^H\int_0^{2\pi} p(\theta)\sin(\theta)Rd\theta dz = 0$$

5.1.2 Streamlines

For steady 2D flow, it is very convenient to introduce the stream function ψ. It is defined such that its contour lines produce the streamlines. Upon some reflection this means that its gradient should always be perpendicular to the velocity:

$$\mathbf{u}\cdot\nabla\psi = 0.$$

This is at variance with the potential, whose gradient is parallel to the velocity (it is indeed, the velocity itself). Thus ϕ and ψ are orthogonal functions, in the sense that their gradients are.

It is easy to check that if

$$\mathbf{u} = \nabla\psi\times\mathbf{e}_z,$$

the resulting velocity has only x and y components (being perpendicular to $\mathbf{e}_z$), as it must. Also, $\nabla\psi$ will fit the requirements (since its gradient will be both perpendicular to $\mathbf{e}_z$, and to $\mathbf{u}$).

This leads to the identities, in cylindrical coordinates

$$\mathbf{u}_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \tag{5.8}$$

$$\mathbf{u}_\theta = -\frac{\partial\psi}{\partial r}. \tag{5.9}$$

The ψ field is then found to be

$$\psi = u_0\left(r - \frac{R^2}{r}\right)\sin\theta$$

The contour lines of this field are shown in Fig. 5.4.

Interestingly, the ψ field is also a solution of Laplace's equation, $\nabla^2\psi = 0$. There is also the fact that we may write

$$\phi = u_0\Re\left(z + \frac{R^2}{z}\right),$$

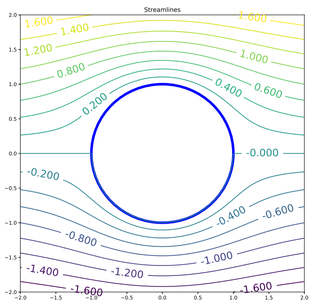


Figure 5.4:

where $z = re^{i\theta}$ is the complex number related to the r and θ coordinates. Then,

$$\psi = u_0 \Im \left(z + \frac{R^2}{z} \right).$$

I.e. both fields are the real and imaginary part of the same complex function:

$$f(z) = u_0 \left(z + \frac{R^2}{z} \right)$$

This is no coincidence, but a consequence of Cauchy-Riemann equations: the real and imaginary part of a complex function are harmonic (i.e. they satisfy Laplace's equation) and orthogonal. To be more precise, the complex function must be analytic. The later requirement is pretty general: it means the function must be single-valued and and must have a derivative everywhere. The reader may worry about the origin and the $1/z$ function. This is, however, outside our domain, which ends when $r > R$. However, the fact that our domain has this "hole" in it has some consequences, as we will see next.

5.1.3 Circulation and lift

It turns out that, despite our previous claims of having found the solution to the problem, this is not a unique solution[1]. The most general one is given by

$$f(z) = u_0 \left(z + \frac{R^2}{z} \right) + \frac{i\Gamma}{2\pi} \log z \qquad \phi = \Re(f) \quad \psi = \Im(f).$$

I.e. we may add an additional term which is also analytic. The $i\Gamma/(2\pi)$ factor is chosen for convenience, as will become clear. Both ϕ and ψ are still harmonic and orthogonal, since $\log(z)$ is analytic. Notice $i\log(z) = i\log(r) - \theta$, so ϕ contains a term that is just the polar angle, while ψ will include a $\log(r)$ term.

[1] This is because our domain has a hole in it. Technically, it is not simply-connected. This means there is an additional parameter to fix the most general solution. This is called the "winding number", and is basically this section's Γ.

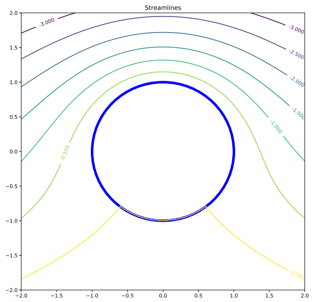


Figure 5.5:

The resulting velocity field still complies with the boundary conditions, since the resulting velocities:

$$u_r = u_0 \left[1 - \left(\frac{R}{r}\right)^2\right] \cos(\theta) \tag{5.10}$$

$$u_\theta = u_0 \left[1 + \left(\frac{R}{r}\right)^2\right] \sin(\theta) + \frac{\Gamma}{2\pi r} \tag{5.11}$$

have an additional term which dies away far from the cylinder, while the no-trespass condition at the surface is still respected (since the radial velocity does not change at all).

The name of Γ is "circulation", since

$$\Gamma = \oint_C \mathbf{u} \cdot d\mathbf{l}$$

for any contour around the cylinder [2].

At the surface of the cylinder,

$$u = u_\theta = u_0 2 \sin(\theta) + \frac{\Gamma}{2\pi R}.$$

The stagnation points then move away from their positions, to points given by

$$\sin(\theta_{\text{st}}) = \frac{\Gamma}{4\pi R u_0}.$$

This has two solutions, of course, as long as the right-hand side is between -1 and 1. For values below or above, the points coalesce into a single stagnation point that is outside the cylinder.

This new velocity field does not solve d'Alambert's paradox, but it does provide a lift, since the pressure at the surface is

$$p = p_0 + \frac{\rho}{2}\left(u_0^2 - \left[2u_0 \sin(\theta) - \frac{\Gamma}{2\pi R}\right]^2\right) = \dots 2\rho u_0 \frac{\Gamma}{2\pi R} \sin\theta,$$

[2] This makes the velocity field non-conservative, which is again allowed due to the domain being not simply-connected.

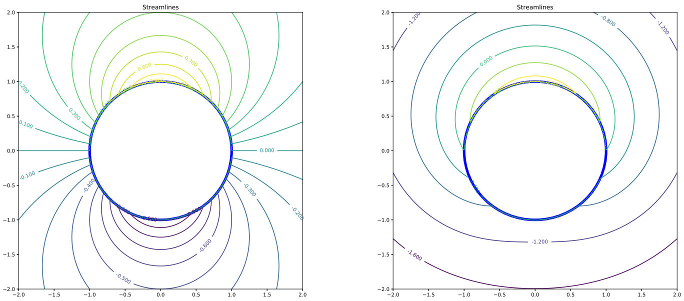


Figure 5.6:

where we single out the only term that can produce a contribution to the lift.

Now,

$$L = \mathbf{F} \cdot \mathbf{e}_x = \int_0^H \int_0^{2\pi} p(\theta) \sin(\theta) R d\theta dz = 2L\rho u_0 R \frac{\Gamma}{2\pi R} \int_0^{2\pi} \sin^2(\theta) d\theta = H\rho u_0 \Gamma.$$

Therefore, the lift per unit length is $L/H = \rho u_0 \Gamma$. It is interesting that heavier fluids, high speeds and high Γ produce higher lift forces.

It turns out that this result for the lift is quite general, at least for these kind of potential flows. It is known as the Kutta-Joukowski Lift Theorem, and has been used for decades in aeronautics. The idea is to map our solution for a cylinder to another, more wing-like, shape, by some conformal transformations. These transformations, or "mappings", preserve the harmonicity of functions — hence the transformed ϕ and ψ will still be solutions of Laplace's equation. Historically, the best known mappings are the Joukowsky Kŕmán–Trefftz transforms.

For a plane wing, it makes sense to relate circulation and lift, since this captures, in a way, the fact that the velocity is higher at the top of the wing, lower below, so the net circulation is non zero. (Of course, this cannot hold right at the surface of an actual wing, since the velocity there will be zero. But this can be measured somewhat further away, beyond a boundary layer.) For a cylinder, this makes little sense, but this picture is sometimes used to explain the Magnus effect. This causes a rotating cylinder to bend as it moves across a fluid. This is used (for spheres) in sports, such as tennis, golf and football. Also, for cylinder-like rotors in "rotor ships," which use the Magnus effect for propulsion.

As an aside, if we move along with the plane, or ship, we will not see those velocity fields. The horizontal component should be avoided in every equation, and the resulting velocity looks different. This is shown in Fig. , for the flow past a cylinder with and without circulation. It is seen that, for an observoer moving with the cylinder, the surrounding fluid is pushed from the fore, and moved around towards the aft (where it pushes our vehicle, magically making our journey a costless one!).

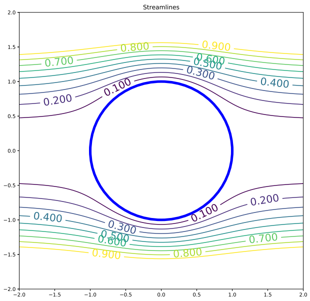


Figure 5.7: Streamlines of the potential flow past sphere

5.1.4 Exercises

The potential flow past a sphere of radius R is given by the potential function

$$\phi(r,z) = u_0 r \left[1 + \frac{1}{2}\left(\frac{R}{r})^3\right)\right]\cos\theta.$$

Spherical coordinates are used: r is the distance to the origin (which coincides with the centre of the sphere), and θ is the angle with the z axis. Flow far away from the sphere is along the $+z$ direction, with magnitude u_0.

1. Demonstrate this fact.

2. Compute the velocity components u_r and u_θ.

3. Show that the stream function is given by (see Fig. **??**).

$$\psi(r,z) = \frac{1}{2}u_0 r^2 \left[1 - \left(\frac{R}{r})^3\right)\right]\sin^2\theta.$$

4. Compute the stream function for the flow as seen when moving with the sphere (see Fig. **??**).

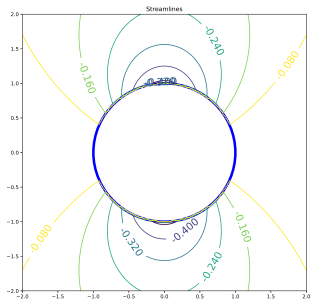


Figure 5.8:

Chapter 6

Gravity waves

> [Water waves] that are easily seen by everyone and which are usually used as an example of waves in elementary courses [...] are the worst possible example [...]; they have all the complications that waves can have. (Richard Feynman, *The Feynman Lectures on Physics*)

6.1 Gravity waves

Large waves, like those at the ocean, are driven by gravity. An initial perturbation on the water surface, like an elevation, will cause a rise in gravitational potenntial. As this energy is transferred to kinetic energy, the water around it moves. Similarly to a pendulum, an area below the mean surface will develop, which will be filled with surrounding liquid. Thus, the perturbation travels away from the initial disturbance.

The mathematical treatment of this problem is, in general, very involved. This is mostly due to the presence of a free surface: the water-air interface. Mathematically, its location is described by a height function η:

$$z = \eta \qquad \eta = \eta(x, y, t)$$

(if no overhangs are present). The fluid velocity is to be solved for a domain between this surface at the ocean bottom, which is supposed to be flat and at a height $z = -h$ (thus, $z = 0$ is taken to be the surface when no waves occur.)

We have the following boundary conditions for the velocity:

$$u_z(z = -h) = 0 \tag{6.1}$$

$$u_z(z = 0) = \frac{\partial \eta}{\partial t} \tag{6.2}$$

The first one is the usual condition at a solid wall, and the last one is called the "kinematic condition", expressing the fact that if the surface moves up or down, the fluid must do the same in order to be "stuck to it".

We will assume an incompressible, irrotational fluid. The fist assumption is a very reasonable one, and the second one turns out to be not so bad, since for small waves there is little vorticity creation (it mainly occurs at the bottom and on breaking waves). The assumption of negligible viscosity is likewise not so bad in this case.

Let us assume a wave train that only depends on the x component (i.e. waves are very long in the y direction):

$$\eta = a\cos(kx - \omega t).$$

The amplitude a, is taken to be small (in the sense $a \ll \lambda$, and $a \ll h$).

Given the assumptions above, the flow may be treated as a potential flow, and moreover the only relevant coordinates will be x and z. However, the fields will be time-dependent in this problem, e.g. $\phi = \phi(x, z, t)$.

The kinematic condition implies

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=\eta} = \frac{\partial\eta}{\partial t}.$$

This is still a difficult expression, so we will approximate it by

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=0} = \frac{\partial\eta}{\partial t}.$$

(The difference between the two spactial derivatives may be quatified by a Fourier expansion, in which the neglected terms are seen to be higher order in a).

Let us consider the ansatz

$$\phi = g(z)\sin(kx - \omega t).$$

The kinematic condition then translates into $g'(0) = a\omega$.

The bottom condition clearly means

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=-h} = g'(-h) = 0.$$

All together, we are looking for a function with these boundary conditions:

$$g'(0) = a\omega \tag{6.3}$$
$$g'(-h) = 0. \tag{6.4}$$

However, the fact that Laplace equation must be satisfied places a strong condition on the kind of function $g(z)$ may be:

$$\nabla^2\phi = 0 \qquad \to \qquad g''(z)\sin(kx - \omega t) - g(z)k^2\sin(kx - \omega t) = 0,$$

which meanse

$$g''(z) = k^2 g(z).$$

Therefore, $g(z)$ must be an exponential function, with the decay constant being exactly equal to k, the wave number. In general, the solution is a linear combination of two exponentials. Instead of that, we may write another less conventional expression using hyperbolic functions. This is equivalent, since these functions are themselves linear combinaions of exponentials. Moreover, we choose to center them at $x = -h$:

$$g(z) = a_1 \cosh(k(z+h)) + a_2 \sinh(k(z+h)).$$

By inspection, we realize $a_2 = 0$, since the $\sinh(k(z+h))$ function was the "wrong" behavior at $z = -h$: it has a slope, while the boundary condition implies it should not.

The kinetic boundary condition implies

$$a_1 = \frac{a\omega}{k \sinh(hk)}.$$

This is indeed the only value for which $g'(0) = a\omega$. The potential is then

$$\phi = \frac{a\omega \cosh(k(z+h))}{k \sinh(hk)} \sin(kx - \omega t).$$

From it, the velocity components may be found. These will have a term featuring $\sin(kx - \omega t)$, a common expression for a traveling wave. However, the theory is still incomplete since ω and k are not unrelated in a physical wave, but coupled by the phase velocity $c = \omega/k$. Remember how in sound waves this velocity was related to fluid compressibility (to be precise, the square root of the variation of pressure with density). In this case, such relationship is as yet missing. Also notice that gravity has played no role whatsoever, despite claims at the beginning of this chapter about this factor being the driving force behind this process.

In order to find the missing link, let us remember our previous expression for the Bernoulli principle, Eq. **??**, which can be written as

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \left[\frac{1}{2} u^2 + \frac{p}{\rho} + \varphi \right] = \frac{p}{\rho} \nabla \cdot \mathbf{u}.$$

In our case the fluid is incompressible, so the right hand side vanishes. Also, in potential flow we may write the whole equation compactly:

$$\nabla \cdot \left[\frac{\partial \phi}{\partial t} + \frac{1}{2} u^2 + \frac{p}{\rho} + \varphi \right] = 0.$$

This means the whole term on which the divergence operator acts must be constant in space, but not in time in general:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} u^2 + \frac{p}{\rho} + \varphi = f(t).$$

This is the less-known unsteady Bernoulli principle.

In this particular case, since boundary conditions do not change in time (in fact, they do, but we are taking the surface to be close to $z = 0$), any function $f(t)$ may be incorporated into the potential by making the following trick:

$$\phi' = \phi + \int_{t_0}^{t} f(t') dt',$$

hence we may just take $f(t) = 0$ in this particular problem.

If we examine the "head" term, $\partial\phi/\partial t + \frac{1}{2}u^2 + \frac{p}{\rho} + \varphi$ at the surface, we notice that the pressure should be constant (equal to atmospheric pressure, as in hydrostatics). The u^2 may be neglected within our assumptions. Last, but not least, gravity finally appears in the potential, $\varphi = g\eta$. This leaves us with

$$\left.\frac{\partial\phi}{\partial t}\right|_{z=0} + g\eta = 0,$$

often called the "dynamic condition".

With our previous result, this means:

$$\phi = -\frac{a\omega^2 \cosh(k(z+h))}{k \sinh(hk)} \cos(kx - \omega t) + ga\cos(kx - \omega t) = 0.$$

The result is the following dispersion relation:

$$\omega^2 = gk\tanh(kh)$$

This is a surprising result, despite being obviously dimensionally correct, whith the two lengths in the problem, $\lambda = 2\pi/k$ and h providing the relevant lenght scales.

We may distinguish two limits.

6.1.1 Shallow water

If the bottom is small, compared to the wavelength λ, then $kh = 2\pi h/\lambda$ is a small number, and we may approximate $\tanh(kh) \approx kh$. Then,

$$\omega^2 = ghk^2 \qquad \to \qquad \omega = k\sqrt{gh}.$$

In this limit, the phase velocity is independent of velocity, which means the process is non-dispersive:

$$c = \frac{\omega}{k} = \sqrt{gh}.$$

The group velocity is of course equal to the phase velocity:

$$c_g = \frac{d\omega}{dk} = \sqrt{gh} = c.$$

The relevant length in this limit is the depth h, which provides the correct dimensions to get a velocity.

Notice "shallow" is defined in relation to the wavelength. A seismic wave may be excited at the deep sea by a geological process that may involve tenths or hundreds of kilometers, much larger than the usual sea depth, which is some few kilometers (its mean value is about 3.8 km.) The sea is then "shallow", and the speed at which the perturbation travels will be about

$$c \approx \sqrt{9.8\,\mathrm{m\,s^{-2}} \times (4 \times 10^3\,\mathrm{m})} \approx 200\,\mathrm{m\,s^{-1}} \approx 700\,\mathrm{km\,h^{-1}}$$

This is an astonishing speed, which is nevertheless in agreement with measurements of these dramatic events.

6.1.2 Deep water

If the bottom is deep, compared to the wavelength λ, then $kh = 2\pi h/\lambda$ is a large number, and its hyperbolic tangent is very nearly 1. In this limit,

$$\omega^2 \approx gk \qquad \to \qquad \omega = \sqrt{gk}.$$

These waves are highly dispersive, since their phase velocity depends on the wavelength:

$$c = \frac{\omega}{k} = \sqrt{g/k} = \sqrt{g\lambda/(2\pi)}.$$

The group velocity is now different from the phase velocity. In fact, it is one half that value:

$$2\log\omega = \log k + \log g \qquad \to \qquad \frac{2}{\omega}\frac{d\omega}{dk} = \frac{1}{k} \qquad \to \qquad c_g = \frac{c}{2}$$

In deep waters, the only two length scales we may think of are the wave amplitude a and their wavength λ. As we expect the amplitude to play no role (as long as it is small, $a \ll \lambda$), we are left with the wavelength to provide the length scale.

Notice that waves high high wavelenghts travel faster. This explains the groundsell phenomenon, by which waves generated by a far away storm may arrive quickly at some shore. These waves may precede the storm by a long time, if the storm does arrive at all. The velocity grows larger and larger with the wavelength, but as we have seen in the previous section, as the wavelength becomes comparable to the depth there will be a crossover to some finite velocity.

Chapter 7

Work and energy

Let us find the work done on a fluid particle.

The First Law of thermodynamics tells us this change is due to work (no heat exchange is considered here):

$$dE = dW$$

On the x direction, this work will be given by the distance travelled by the left wall in the x direction times the pressure force:

$$dW_x(\text{left}) = u_x \Delta t p A.$$

This is work done on the particle (if u_x is positive), hence its sign. There will be a similar contribution from the right wall, so the total work given by the x direction will be

$$dW_x = u_x \Delta t p A - u_x' \Delta t p' A.$$

By expanding in a Taylor series, this may be written as

$$dW_x = -\frac{\partial p u_x}{\partial x} \Delta x A \Delta t.$$

Now, adding the other three components we find

$$\frac{dW}{dt} = -V \nabla \cdot (p\mathbf{u}).$$

The right hand side may be expanded

$$\nabla \cdot (p\mathbf{u}) = p \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla p.$$

Recalling that, from the Euler equation **??** ,

$$\nabla p = \rho \mathbf{g} - \rho \frac{d\mathbf{u}}{dt},$$

we may write

$$\frac{1}{V} \frac{dW}{dt} = -p \nabla \cdot \mathbf{u} + \rho \mathbf{u} \cdot \frac{d\mathbf{u}}{dt} - \rho \mathbf{g} \cdot \mathbf{u}.$$

Now, multiplying by $1/\rho$:

$$\frac{1}{V}\frac{d\epsilon}{dt} = \mathbf{u} \cdot \frac{d\mathbf{u}}{dt} - \mathbf{g} \cdot \mathbf{u} - \frac{p}{\rho}\nabla \cdot \mathbf{u}.$$

Where we have defined the specific energy $\epsilon = E/M$.

This may be written as

$$\frac{d}{dt}\left[\epsilon - \frac{1}{2}u^2 + \mathbf{g} \cdot \mathbf{u}\right] = -\frac{p}{\rho}\nabla \cdot \mathbf{u},$$

or

$$\frac{d}{dt}\left[\epsilon - \frac{1}{2}u^2 - \varphi\right] = -\frac{p}{\rho}\nabla \cdot \mathbf{u},$$

This is clearly a law for

$$e = \epsilon - \frac{1}{2}u^2 - \varphi,$$

where we define e, the specific internal energy as the total energy minus kinetic energy, minus gravitational potential energy. In other words,

$$\epsilon = e + \frac{1}{2}u^2 + \varphi.$$

Notice also that in our steady Bernoulli principle, Eq. **??** we had

$$\mathbf{u} \cdot \nabla \left[\frac{1}{2}u^2 + \frac{p}{\rho} + \varphi\right] = \frac{p}{\rho}\nabla \cdot \mathbf{u},$$

while in the steady state,

$$\mathbf{u} \cdot \nabla e = -\frac{p}{\rho}\nabla \cdot \mathbf{u}.$$

Combining both we have the steady state Bernoulli equation for a compressible fluid:

$$\mathbf{u} \cdot \nabla \left[e + \frac{1}{2}u^2 + \frac{p}{\rho} + \varphi\right] = 0,$$

which tells us the combination $e + \frac{1}{2}u^2 + \frac{p}{\rho} + \varphi = \epsilon + \frac{p}{\rho}$ is constant along streamlines. Notice that, by the definition of enthalpy,

$$H = E + PV,$$

the specific enthalpy is

$$h = H/M = e + p/\rho,$$

so the Bernoulli principle claims that the total specific enthalpy ($h' = h + \frac{1}{2}u^2 + \varphi$) is constant along streamlines.

We may also find an equation for the enthalpy. In Eq. **??** above, the right hand side maybe changed using the continuity equation:

$$-\rho\nabla \cdot \mathbf{u} = \frac{d\rho}{dt}.$$

which means

$$-\frac{p}{\rho}\nabla \cdot \mathbf{u} = \frac{p}{\rho^2}\frac{d\rho}{dt}$$

But

$$\frac{d(p/\rho)}{dt} = \frac{1}{\rho}\frac{dp}{dt} - \frac{p}{\rho^2}\frac{d\rho}{dt}.$$

Hence,

$$\frac{d(e+p/\rho)}{dt} = \frac{1}{\rho}\frac{dp}{dt},$$

or

$$\frac{dh}{dt} = \frac{1}{\rho}\frac{dp}{dt}.$$

Changes in the enthalpy are therefore tied to changes in pressure.

7.0.1 Heat flux

In the case that some heat flux is present, the First Law of thermodynamics reads:

$$dE = dW + dQ.$$

The influx of heat to the particle, dQ, will enter our equations from a vector heat flux $\mathbf{q}$, with units of energy/(area $\times$ time). For example, the heat influx due to transfer in the x direction will be

$$dQ_x = \mathbf{q}dtdydz - \mathbf{q}(x+dx)dtdydz \approx -V\,dt\,\frac{\partial \mathbf{q}}{\partial x},$$

where in the last approximation a Taylor expansion has been employed, as should be customary by now.

Altogether, the total heat flux rate is

$$\frac{dQ}{dt} = -V\nabla \cdot \mathbf{q},$$

and the final equation for the change of specific internal energy of an ideal fluid is

$$\rho\frac{de}{dt} = -p\nabla \cdot \mathbf{u} - \nabla \cdot \mathbf{q}.$$

The heat flux must be either due to an external source, or due to heat diffusion. In the latter case, a well-known theory is Fourier's law, by which the flow is due to a temperature gradient:

$$\mathbf{q} = -k\nabla T,$$

where k is the heat diffusion coefficient. The temperature does not appear in the energy equation, but a common assumption is that

$$e = cT,$$

where c is the specific heat, taken as constant. Then,

$$\rho c\frac{dT}{dt} = -p\nabla \cdot \mathbf{u} + k\nabla^2 T.$$

For an incompressible fluid,

$$\frac{dT}{dt} = \frac{k}{c\rho}\nabla^2 T = \alpha \nabla^2 T,$$

where $\alpha = k/(c\rho)$, a constant if ρ is. This is the convective Fourier heat equation (it is not the most common Fourier law, due to the derivative being convective, and not just partial.)

Part II

Real fluids

Chapter 8

Navier-Stokes equations

8.1 Kinematics of a particle

Let us focus again on a fluid particle, as we did on **??**, but now focusing on how the particle itself distorts as a consequence of a velocity field.

All possible distorsions of a particle will be a combination of the following:

1. Translation
2. Rotation
3. Shear
4. Dilation

A translation is just the motion of its center of mass from one place to another, and for a small time is given simply by $\mathbf{u}dt$. The other motions are more complicated, since they involve spatial derivatives of the velocity. They must: for a constant velocity field translation is the only mode that occurs.

Rotation

We will refer to particle in figure 5.8, with vertices A, B, and C. Vertex D plays no role — also, it is sufficient to focus on the face that is portrait, even if the shape of a particle is supposed to be a cube. It is straightfoward to include the other faces, as we will see.

After a small time d the particle has distorted, so that the vertices are now at positions A′, B′, C′, and D′.

Let us call α the angle between the x axis and the A′-B′ edge, with the usual counter-clockwise convention as positive. Similarly, β is the angle between the A′-C′ edge and the y axis, with the same convention. It is obvious that a net rotation takes place if e.g. both angles are positive. If, on the other hand, they are equal in magnitude but differ in sign, no rotation takes place. This makes it reasonable to define the rotation as the average of both angles:

$$d\Omega_z = \frac{1}{2}\left(\alpha + \beta\right).$$

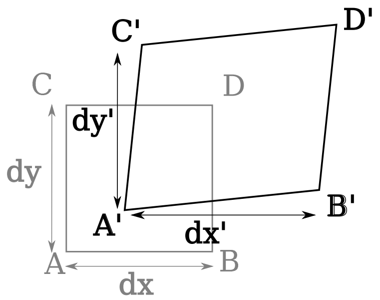


Figure 8.1:

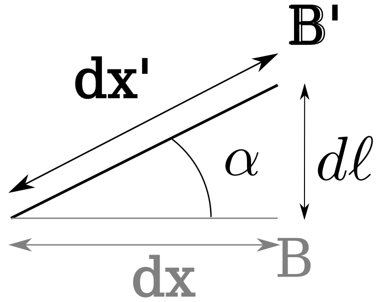


Figure 8.2:

Now, angle α will always be very small as dt gets very tiny. Hence, we may approximate it by its tangent:

$$\alpha \approx \frac{d\ell}{dx'}$$

For a small time, we have the following:

$$dx' = x(B') - x(A') \approx (dx + v_x(B)dt) - v_x(A)dt \approx dx + \left(v_x(A) + \frac{\partial v_x}{\partial x}dx \right) dt - v_x(A)dt = dx + \frac{\partial v_x}{\partial x}d$$

where we have taken the origin to coincide with the position of A. The partial derivative is supposed to be evaluated at A, but in the limit as dx goes to zero it is just "at the particle". Similarly, the opposing side is:

$$d\ell = y(B') - y(B) \approx v_y(B)dt \approx \left(v_y(A) + \frac{\partial v_y}{\partial x}dx \right) dt = \frac{\partial v_y}{\partial x}dxdt,$$

.

Therefore, to first order in dt:

$$\alpha \approx \frac{\partial v_y}{\partial x}dt.$$

In other words, the rate of change of the angle is

$$\frac{d\alpha}{dt} = \frac{\partial v_y}{\partial x}.$$

Notice the cross derivative: what is relevant is the change of the vertical component of the velocity with the horizontal coordinate.

A similar calculation for the other angle reveals

$$\frac{d\beta}{dt} = -\frac{\partial v_x}{\partial y}.$$

Taking all

$$\frac{d\Omega_z}{dt} = \frac{1}{2}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right).$$

This may sound familiar to the reader, since the curl in Cartesian coordinates has a z component with exactly the same expression, but the factor of $1/2$. This analysis may be carried out for rotations about the other two Cartesian axes, with the end result that

$$\frac{d\mathbf{\Omega}}{dt} = \frac{1}{2}\boldsymbol{\omega}.$$

The curl is therefore twice the rate of rotation of a fluid particle.

An example

Let us consider a uniform circular motion about the origin:

$$\mathbf{r} = \begin{cases} x = r\cos(\omega_0 t) \\ y = r\sin(\omega_0 t). \end{cases}$$

The velocity field is

$$\mathbf{u} = \begin{cases} u_x = -r\omega_0 \sin(\omega_0 t) = -\omega_0 y \\ u_y = r\omega_0 \cos(\omega_0 t) = \omega_0 x. \end{cases}$$

If we compute the curl of this field, its only component is the z one:

$$\omega_z = \left(\frac{\partial(\omega_0 x)}{\partial x} - \frac{\partial(-\omega_0 y)}{\partial y}\right) = 2\omega_0.$$

Therefore, the curl indeed is twice the angular velocity.

Strain

8.2 Stress tensor

Let us return again to our particle in order to analyze its movement when stress forces are applied onto its walls. The pressure force may be consider a special case of the latter, but stress forces may also have a shear component.

Thus, the horizontal force will be given by, in part, by the contributions due to the walls at the left, back, and bottom

$$dF_x|_{\mathrm{l,bk,bm}} = -\tau_{xx} dy\, dz - \tau_{yx} dx\, dz - \tau_{zx} dx\, dy.$$

The compression stress τ_{xx} is therefore quite similar to the pressure (see (Eq **??**), and in fact will be seen to include a $-p$ term. The minus sign appears because normal stresses are historically defined as positive if pointing outside the particle (i.e. in the direction of the normal vector). In addittion, two shear stress forces appear. One of them, $\tau_{yx} dx\, dz$ is the horizontal shear force on the back wall ($dx\, dz$ actually equals $dx\, dy$, but it is clearer to write it this way). Similarly, $\tau_{zx} dx\, dy$ is the horizontal stress force at the bottom wall.

Notice the convention for minus signs. First, the normal stress is $\tau_{xx} \parallel \mathbf{n}$. From it, $\tau_{xy} \parallel \mathbf{e}_z \times \tau_{xx}$, $\tau_{xz} \parallel \tau_{xx} \times \mathbf{e}_y$, respecting the (x, y, z) cyclic order.

To get the whole horizontal force, the contributions from the other three walls must be added up:

$$dF_x|_{\mathrm{r,ft,up}} = \tau_{xx}(x + dx, y, z) dy\, dz + \tau_{yx}(x, y + dy, z) dx\, dz + \tau_{zx}(x, y, z + dz) dx\, dy.$$

This time, the sign convention is positive. It the stresses are the same on those faces, the resultant force will then be zero (those forces may exert a net torque, but not a force). In general, they will be different on those faces, as made explitic on their arguments.

Expanding in Taylor series, we get the following net horizontal force:

$$dF_x = \left(\frac{\partial \tau_{xx}}{\partial x} dx \right) dy\, dz + \left(\frac{\partial \tau_{yx}}{\partial y} dy \right) dx\, dz + \left(\frac{\partial \tau_{zx}}{\partial z} dz \right) dx\, dy.$$

The volumetric horizontal force is then,

$$f_x = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}.$$

With integer notation for Cartesian coordinates:

$$df_1 = \sum_{j=1,2,3} \frac{\partial \tau_{j1}}{\partial x_j}.$$

In general, for any component we will have:

$$df_i = \sum_{j=1,2,3} \frac{\partial \tau_{ji}}{\partial x_j}.$$

In order to use vector notation, we have to introduce the divergence of a tensor:

$$\nabla \cdot \tau = \sum_{j=1,2,3} \frac{\partial}{\partial x_j} \tau_{ji},$$

which results in a vector:

$$d\mathbf{f} = \nabla \cdot \tau.$$

The tensor τ has components τ_{ij}. There is also the matrix notation, by which

$$\nabla \cdot \tau = \nabla^t \tau,$$

where ∇^t is a transposed (row) vector operator, multiplying matrix τ from the left.

The resulting Navier-Stokes equation is then,

$$\rho \frac{d\mathbf{u}}{dt} = \nabla \cdot \tau + \rho \mathbf{g}.$$

8.2.1 Newtonian fluids

The previous equation is still too general, and a connection between stress and strain is still needed. Here we consider the case in which there is a linear relationship between both, which involves the coefficient of viscosity.

To begin with, let us consider a simple case in which a fluid is confined between two planes. One of them moves sideways with a certain speed u_0, while the other is kept fixed. After a certain transient, some force is needed in order to keep this shearing. The simplest expression is

$$F = \mu A \frac{u_0}{L}.$$

The force is proportional to the area and to the velocity difference between the planes. It is also inversely proportional to their separation, L (this fact being the least obvious). Finally, a constant of proportionality is given by μ, the viscosity coefficient, or simply "the viscosity". This constant may vary with temperature, density, pressure, but the point with Newtonian fluids is that it does not vary with the velocity field (or its derivatives). Later, in section **??**, this flow will be solved as a solution of the Navier-Stokes equations, the Couette flow. There, it will be shown that the velocity is everywhere in the direction of the force exerted on the upper plane, let us call it x, and varies linearly between the planes, in the y direction. Therefore, the only components of the strain rate tensor are $\epsilon_{xy} = \epsilon_{yx} = u_0/(2L)$. We therefore have

$$\tau_{xy} = \mu \epsilon_{xy}.$$

With these in mind, let us look for a general relationship between τ and ϵ. This is much easier if we go to the principal strain axes. These are the coordinates on which the strain rate is diagonal. Such coordinate system always exist, since the strain rate tensor is symmetric. Notice that in these system strains are not due to shear, only to dilations.

A simple example would be to consider the flow $u_x = u_0 y/L$ (again, Couette flow). In this case,

$$\epsilon = \begin{pmatrix} 0 & u_0/(2L) \\ u_0/(2L) & 0 \end{pmatrix}.$$

It is easy to find the two eigenvalues and associated eigenvectors of this matrix:

$$\begin{aligned} \lambda_1 &= u_0/(2L) & \mathbf{v}_1 &= (1/\sqrt{2}) \begin{pmatrix} 1 & 1 \end{pmatrix}^t \\ \lambda_1 &= -u_0/(2L) & \mathbf{v}_2 &= (1/\sqrt{2}) \begin{pmatrix} 1 & -1 \end{pmatrix}^t \end{aligned}$$

Notice the first eigenvalue correnspond to a dilation along the $x = y$ diagonal, while the second is a compression along the $x = -y$ one.

The diagonal strain rate matrix is then:

$$\tilde{\epsilon} = \begin{pmatrix} u_0/(2L) & 0 \\ 0 & -u_0/(2L) \end{pmatrix}.$$

It is crucial to realize that the stress tensor is also diagonal in this coordinate system. Otherwise, a pure dilation in one of the principal directions would cause a shear stress in some other direction. This does not mean, however, that the two tensors are simply proportional. Instead, we may posit, for the top-most element

$$\tilde{\tau}_{xx} = -p + C_1 \tilde{\epsilon}_{xx} + C_2 \tilde{\epsilon}_{yy} + C_3 \tilde{\epsilon}_{y}.$$

We have added a $-p$ term that has to be there even when there is no movement. This is needed, since a diagonal stress tensor $\tau = -p\mathbb{1}$ produces the $-\nabla p$ term of hydrostatics (in Couette flow, p is constant, and this term is not needed.)

If isotropy is assumed, there should be no distinction between traverse directions y and z. Therefore, $C_2 = C_3$, and

$$\tilde{\tau}_{xx} = -p + C_1 \tilde{\epsilon}_{xx} + C_2(\tilde{\epsilon}_{yy} + \tilde{\epsilon}_{y}) = -p + K\tilde{\epsilon}_{xx} + C_2(\tilde{\epsilon}_{xx} + \tilde{\epsilon}_{yy} + \tilde{\epsilon}_{y}),$$

where the constant $K = C_1 - C_2$. Notice the C_2 term is the divergence of the velocity. But, it is also the trace of the strain tensor, a quantity which is invariant under change of basis. We can now write:

$$\tilde{\tau}_{xx} = -p + C_2 \nabla \cdot \mathbf{u} K \tilde{\epsilon}_{xx}.$$

There will be similar expressions for $\tilde{\tau}_{yy}$ and $\tilde{\tau}_{zz}$, but in them the coefficients must be the same — otherwise isotropy will be violated. Taking all together,

$$\tilde{\tau} = K\tilde{\epsilon} + (-p + C_2 \nabla \cdot \mathbf{u})\mathbb{1}.$$

Now, we may go back to the original Cartesian system and find

$$\tau = K\epsilon + (-p + C_2 \nabla \cdot \mathbf{u})\mathbb{1}.$$

The stress tensor is then also symmetric, a fact that is required in order the particle be torsion-free (remember the fact that rotations have no stress associated.)

A comparison with our previous result reveals $K = 2\mu$. The constant C_2 is called, in the theory of elasticity "the second Lamé coefficient", and receives the symbol λ. Then,

$$\tau = 2\mu\epsilon + (-p + \lambda \nabla \cdot \mathbf{u})\mathbb{1}.$$

To make this explicit, this means that diagonal terms have the form

$$\tau_{ii} = 2\mu\epsilon_{ii} - p + \lambda \nabla \cdot \mathbf{u},$$

while off-diagonal terms are

$$\tau_{ij} = 2\mu\epsilon_{ij}$$

(Why not always work in the system of principal axes? The answer is simple: principal axes vary from one particle to another, since they are defined by local values of velocity derivatives. The Cartesian coordinate system, or any such (cylindrical, polar...) is the same for all particles.)

The off-diagonal terms have a neat expression when the strain rate tensor is written in term of velocity derivatives:

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

The diagonal ones, however, are somewhat puzzling:

$$\tau_{ii} = -p + 2\mu \frac{\partial u_i}{\partial x_i} + \lambda \nabla \cdot \mathbf{u}.$$

The pressure term is natural, but there are two extra dynamical terms.

Let us define a mechanical pressure as minus one-third times the trace of the stress tensor:

$$\bar{p} = -\frac{1}{3} \mathrm{Tr} \tau = -p + (2\mu + \lambda) \nabla \cdot \mathbf{u}.$$

The result is that the mechanical pressure, defined in such a way, is different from the thermodynamic pressure in an incompressible fluid. There are several ways out of this puzzling result. One of them is to assume simply that the fluid is incompressible. This is of course entirely correct, but would limit the applicability of the theory to incompressible problems.

Another approach is to boldly assume $2\mu + \lambda = 0$. This step was taken by Stokes, and defines a "Stokesian fluid". On the other hand, there is no clear evidence of a real fluid that may satisfy such a relationship. Indeed, the few measures of λ have show positive values (while μ, as should be clear, is always positive). We should then keep in mind that in some flows when compressibility is important, mechanical pressure may differ from thermodynamical one.

The Navier-Stokes equation for Newtonian liquids is finally:

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + 2\nabla \cdot (\mu \epsilon) + \nabla [\lambda (\nabla \cdot \mathbf{u})] + \rho \mathbf{g}.$$

8.2.2 Incompressible, athermal case

If the fluid is incompressible, μ may still have a dependence on temperature. The temperature has its own equation, to be explained below, and the viscosity does depend on it quite strongly, as is evident when heating up oil. In problems where this dependence may be avoided, the equation above reduces to

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + 2\mu \nabla \cdot \epsilon + \rho \mathbf{g}.$$

Moreover, in the divergence of the stress tensor, the i-th term will be

$$(\nabla \cdot \epsilon)_i = \sum_{j=1,2,3} \frac{\partial}{\partial x_j} \tau_{ji} = \frac{1}{2} \sum_{j=1,2,3} \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) =$$

$$= \frac{1}{2} \left(\sum_{j=1,2,3} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial x_j} + \underbrace{\sum_{j=1,2,3} \frac{\partial}{\partial x_i} \frac{\partial u_j}{\partial x_j}}_{=0} \right) = \frac{1}{2} \sum_{j=1,2,3} \frac{\partial^2 u_i}{\partial x_j^2},$$

since the underbraced term is zero due to incompressibility:

$$\sum_{j=1,2,3} \frac{\partial}{\partial x_j} \frac{\partial u_j}{\partial x_i} = \frac{\partial}{\partial x_i} \sum_{j=1,2,3} \frac{\partial u_j}{\partial x_j} = 0.$$

The final momentum equation for an incompressible, athermal fluid, is then

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{g}.$$

This equation, in this particular form, is the beginning of a vast ammount of research in physics and applied mathematics.

8.3 Dimensionless variables: the Reynolds number

For the simplest, athermal incompressible case, the term due to viscosity

$$\mu \nabla^2 \mathbf{u} = \mu \frac{u_0}{L^2} \nabla^{*2} \mathbf{u}^*,$$

where we cast the variables into reduced form, as explained in section 2.6.

Recall that in order to arrive to 2.2, the whole equation was multipled by $L/(\rho_0 u_0^2)$. If we do that to our momentum equation, the result is

$$\rho^* \frac{d\mathbf{u}^*}{dt^*} = -\nabla^* p^* + \rho^* \mathbf{g}^* + \frac{\mu}{\rho_0 L u_0} \nabla^{*2} \mathbf{u}^*.$$

The term $\mu/(\rho_0 L u_0)$ must be dimensionless (as can be easily checked). It then represents a reduced viscosity, and should be taken as such: a number that defines whether viscosity is important or not in a given context.

Historically, however, it is its inverse that has a name, the Reynolds' number:

$$\mathbf{Re} = \frac{\rho_0 L u_0}{\mu}. \tag{8.1}$$

This number is therefore large when viscosity is small, and small when it is large. It may also be defined as the ratio of inertial forces and viscous forces:

$$\mathbf{Re} = \frac{\rho_0 u_0^2}{\mu u_0 / L}. \tag{8.2}$$

Indeed, in the numerator $\rho_0 u_0^2$ is the typical strength of the pressure, and in the denominator $\mu u_0 / L$ is the typical strength of viscous stress forces.

Chapter 9

The energy equation

In addition to continuity and momentum, there is an additional Navier-Stokes equation for the energy.

It contains the previous expression for ideal fluid, plus a term expressing energy dissipation by viscosity.

We must re-evaluate the work done on each of the faces of the particles due to stresses. For example, on the left wall the energy due to stress forces is

$$dW(\text{left}) = -(u_x dt)\tau_{xx} dydz - (u_y dt)\tau_{xy} dydz - (u_z dt)\tau_{xz} dydz.$$

Each of the stresses on this wall does work only in its direction of motion: τ_{xx} is compression and will feature a $-p$ term, which τ_{xy} and τ_{xz} produce shear forces. The minus sign stem from the sign convention, since on this wall shear stresses have directions opposed to the Cartesian axes. Similarly,

$$dW(\text{right}) = (u'_x dt)\tau'_{xx} dydz + (u'_y dt)\tau'_{xy} dydz + (u'_z dt)\tau'_{xz} dydz,$$

where the primed values mean those fields may be different from the left wall. Expanding in Fourier series, and adding everything up,

$$dW(\text{left, right}) = \frac{\partial u_x \tau_{xx}}{\partial x} dtdxdydz + \frac{\partial u_y \tau_{xy}}{\partial x} dtdxdydz + \frac{\partial u_z \tau_{xz}}{\partial x} dtdxdydz,$$

or, we find for the power

$$\frac{dW(\text{left, right})}{dt} = dV \frac{\partial}{\partial x}\left(u_x \tau_{xx} + u_y \tau_{xy} + u_z \tau_{xz}\right) = dV \frac{\partial}{\partial x} \sum_j u_j \tau_{xj}$$

Adding the other four walls, we have:

$$\frac{dW}{dt} = dV \sum_i \frac{\partial}{\partial x_i} \sum_j u_j \tau_{ij}.$$

Since the stress tensor is symmetric, we may write the latter as

$$\frac{dW}{dt} = dV \nabla \cdot (\tau \cdot \mathbf{u}),$$

Now, the energy equation is, from the First Law:

$$\frac{dE}{dt} = \frac{dW}{dt} + \frac{dQ}{dt} = dV\nabla \cdot (\tau \cdot \mathbf{u}) - dV\nabla \cdot \mathbf{q}.$$

(The second term, due to heat flux, does not change from the inviscid case.)

Dividing by the mass of the particle,

$$\rho \frac{d\epsilon}{dt} = \nabla \cdot (\tau \cdot \mathbf{u}) - \nabla \cdot \mathbf{q}. \tag{9.1}$$

(In this equation, $\epsilon = E/M$, as in **??**, not the strain rate.)

The term $\nabla \cdot (\tau \cdot \mathbf{u})$ may be written applying the chain rule carefully:

$$\nabla \cdot (\tau \cdot \mathbf{u}) = \tau : \nabla \mathbf{u} + \mathbf{u} \cdot (\nabla \cdot \tau), \tag{9.2}$$

where ":" means a total reduction of two tensors, $a : b = \sum_{ij} a_{ij} b_{ij}$, and $\nabla \mathbf{u}$ is the tensor with components $\partial u_i / \partial x_j$ (as introduced in the Euler equation **??**).

We now just follow the steps already employed when deriving the energy equation for an inviscid fluid (Eqs **??**). The $\nabla \cdot \tau$ appears in the general Navier-Stokes momentum equation **??**:

$$\nabla \cdot \tau = \rho \left(\frac{d\mathbf{u}}{dt} - \mathbf{g} \right).$$

Therefore,

$$\mathbf{u} \cdot (\nabla \cdot \tau) = \rho \left[\frac{1}{2} \frac{du^2}{dt} - \mathbf{g} \cdot \mathbf{u} \right] = \rho \frac{d(u^2/2 - \mathbf{g} \cdot \mathbf{r})}{dt}.$$

The conclusion is then that the energy equation 9.1 may be expressed as a law for the specific internal energy:

$$\rho \frac{de}{dt} = \tau : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}.$$

This looks more similar to **??** for an ideal fluid if we split the stress tensor into the pressure diagonal and the rest:

$$\tau = \tau' - p\mathbb{1} \qquad \rightarrow \qquad \tau' : \nabla \mathbf{u} = \tau' : \nabla \mathbf{u} - p \nabla \cdot \mathbf{u},$$

hence

$$\rho \frac{de}{dt} = -p \nabla \cdot \mathbf{u} - \nabla \cdot \mathbf{q} + \Phi.$$

The term Φ collects the result of viscosity and is termed the "dissipation function":

$$\Phi = \tau' : \nabla \mathbf{u}.$$

This term should always be positive if the Second Law is to hold: viscosity can only subtract energy from the system, never add it. Up until now our derivation has been generic. For a Newtonian fluid, however, one may go a bit further, and write:

$$\Phi = \mu \left[2 \sum_i \epsilon_{ii}^2 + 4(\epsilon_{12}^2 + \epsilon_{13}^2 + \epsilon_{23}^2) \right] + \lambda \nabla \cdot \mathbf{u}^2. \tag{9.3}$$

This looks deceptively positive unconditionally. However, there is no reason λ should be positive. It is a simple exercise to show that the conditions this term be positive are:

$$\mu \geq 0 \qquad 3\lambda + 2\mu \geq 0. \tag{9.4}$$

TODO: exercise on this

The first one comes as a relief, since a μ would be quite unphysical. The second limits the value of λ to regions equal to, or above, $-(2/3)\mu$. This latter term is precisely zero for "Stokesian" fluids, as is obvious (since the corresponding term does not appear at all in the stress tensor for these hypothetical fluids.)

9.1 Exercises

1. Check identity 9.2. Hint: use element notation.

2. Show that the conditions in 9.4 are indeed needed in order Φ in 9.3 be always positive. (Hint: look for "postitive-definite quadratic form". The expression for Φ can be expressed in such a way, and three conditions are obtained for this positiveness. However, one of them is $2\mu - \lambda \geq 0$, which is less restrictive than the other two taken together, so only the two conditions quoted remain.)

Chapter 10

Simple solutions to the NS equations

10.1 Couette flow

As a simple solution, let us derive the flow that was given as an example in our derivation of section **??**. A plane moves in the x direction, parallel of a fixed plane, and separated a distance L from it. The velocity field is supposed to depend only on y and have reached a steady state. (Notice that these assumptions restrict our solution space to a very limited choice. Since the equation are known not to comply with unicity, there may be other solutions, as indeed there are.)

While this particular geometry may seem artificial, the original Couette aparatus used a fluid between two coaxial cylinders. It is quite easy to assemble and is one of the first accurate viscometers. This flat geometry may be thought of as the limit of a thin fluid layer between the curved surfaces.

In this flow, the total derivative in the momentum equation is zero. The partial derivative is zero in steady state, and the non-linear term also is, since $\mathbf{u}\nabla\mathbf{u}$ is

$$u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} = 0.$$

Also, $\nabla \cdot \mathbf{u} = 0$, so the flow will always be incompressible. The pressure then is constant, since it does not need to "ensure" incompressibility.

The equation reduces then to

$$\mu \frac{\partial^2 u_x}{\partial y^2} = 0.$$

The viscosity is then seen to be of no importance (other than it is needed to reach a steady state, as shown below). The solution is simply a linear function of y. The particular shape of the function is given by the boundary conditions. If we take the usual no-slip conditions, the velocity matches the velocity at the walls:

$$u_x(y = 0) = 0 \qquad u_x(y = L) = u_0.$$

Therefore,

$$u_x = u_0 \frac{y}{L}.$$

These are called Dirichlet boundary conditions, since they fix the value of the field.

This is the solution assumed in previous sections, when introducing the viscosity coefficient. Even if the latter does not appear in the velocity field, the stress tensor has only one independent component:

$$\epsilon_{xy} = \epsilon_{yx} = \mu \frac{u_0}{L}.$$

The tensor is also constant throughout the fluid.

This has physical importance, since both plates will feel a total stress force

$$F = A\epsilon_{xy} = \mu A \frac{u_0}{L}.$$

This force must be maintained on the moving plate in order to keep the steady flow (the fixed one must be anchored, and should resist the same force in order not to be dragged along). Energy must then be provided to the system, which is dissipated by viscosity. The power into the system will be

$$F u_0 = \mu A \frac{u_0^2}{L} = \mu V \left(\frac{u_0}{L}\right)^2,$$

where $V = AL$ is the total fluid volume.

Lastly, let us consider the volumetric flux:

$$Q = \int_A v_x = \int_0^H dz \int_0^L dy v_x(y) = H u_0 \int_0^L \frac{y}{L} dy = HLu_0 \int_0^1 y' dy' = \frac{1}{2} A u_0.$$

The mean velocity is defined as Q/A. Therefore,

$$\bar{u} = \frac{1}{2} u_0,$$

so the solution may be written as

$$u_x = 2\bar{u} \frac{y}{L}.$$

10.1.1 Start-up of Couette flow

It is not too difficult to solve the Navier-Stokes equations for non-steady Couette flow. In this case, the partial time derivative will not be absent, and

$$\frac{\partial u_x}{\partial t} = \nu \frac{\partial^2 u_x}{\partial y^2}.$$

Boundary conditions are as above, and let us consider the initial fluid is at rest.

It is easer to work out the solution by substrating the known steady state:

$$u_x(y,t) = u(y,t) + u_0 \frac{y}{L} \qquad u(y,t) = u_x(y,t) - u_0 \frac{y}{L}$$

The reason is that the boundary conditions for the velocity field about the steady state are homegeneous:

$$u(y=0,t) = u(y=L,t) = 0$$

Then, one may use the standard technique of separation of variables:

$$u(y,t) = Y(y)T(t).$$

The equation of motion then reads

$$YT' = \nu TY'' \qquad \rightarrow \qquad \frac{T'}{\nu T} = \frac{Y''}{Y} = -c$$

The last equality follows because functions of two independent variables can only be equal if constant.

Therefore

$$Y'' = -cY,$$

whose n-th solution, given the boundary conditions, is

$$Y_n = A_n \sin(n\pi y/L),$$

where n is an integer greater than zero. The constant must then be $c = (n\pi/h)^2$. The corresponding time function is then,

$$T_n = \exp(-c_n t) = \exp(-n^2\nu(\pi/L)^2 t).$$

In general, the solution will be a combination of all possible modes:

$$u_x(y,t) = u_0\frac{y}{L} + \sum_n A_n \exp(-n^2\nu(\pi/L)^2 t)\sin(n\pi y/h),$$

where it is clearly seen that mode n decays in a time that has an n^2/ν dependence. The longest-lived one is then $n = 1$, and modes with shortest wavelength decay quadratically faster. Also, the viscosity sets the time-scale of the process: high viscosity means shorter relaxation times. In the limit of no viscosity all times diverge, since a fluid without viscosity is unable to transmit the stress produced by the moving plane.

The A_n are given by the initial condition:

$$u_x(y,t=0) = u_0\frac{y}{L} + \sum_n A_n \sin(n\pi y/h) = 0,$$

which is a standard exercise in Fourier series analysis. The result may be shown to be

$$u_x(y,t=0)/u_0 = \frac{y}{L} + \frac{2}{\pi}\sum_n \frac{(-1)^n}{n}\sin(n\pi y/h).$$

This can also be written as

$$u_x(y,t=0)/u_0 = \frac{y}{L} - \frac{2}{\pi}\sum_n \frac{1}{n}\sin(n\pi(1-y/h)),$$

where the last sine term is always starts with a positive slope close to $y = h$. The negative sign of every term in the expansion means that all of them are trying very hard in order to push down the final steady-state linear solution toward the initial one, which is null.

10.1.2 Temperature

For the steady state, the temperature equation reduces to

$$0 = kc_p \frac{\partial^2 T}{\partial y^2} + \Phi.$$

The dissipation function in this case is simply

$$\Phi = \mu \left(\frac{\partial u_y}{\partial x} \right)^2 = 4\mu \bar{u}^2 \left(\frac{1}{L} \right)^2.$$

The equation for the energy is therefore

$$0 = \kappa \frac{\partial^2 T}{\partial y^2} + 4\frac{\nu \bar{u}^2}{L^2},$$

where $\kappa = kc_p/\rho$. The boundary conditions needed may be the temperature at the two walls:

$$T(y=0) = T_0 \qquad T(y=L) = T_1,$$

also known as the "no-jump" temperature conditions. The fluid is supposed to have the same temperature as the walls under this framework. Others may be easily explored, such as fixed energy influx, which translate into conditions for the temperature derivatives at the walls (also known as Neumann boundary conditions). If the derivative is null, one has an adiabatic wall (aka homegeneous Neumann).

Before solving the equation, let us cast it into dimensionless form, by reducing the temperature by its value on one wall: $T^* = T/T_0$. Similarly, $y^* = y/L$. Then,

$$0 = \kappa \frac{T_0}{L^2} \frac{\partial^2 T^*}{\partial y^{*2}} + 4\frac{\nu \bar{u}^2}{L^2},$$

or

$$\frac{\partial^2 T^*}{\partial y^{*2}} = -4\frac{\nu \bar{u}^2}{\kappa T_0} = -4\mathrm{Br},$$

where we define the important Brickman number:

$$\mathrm{Br} = \frac{\nu \bar{u}^2}{\kappa T_0}.$$

The number measures the importance of viscous dissipation over temperature dissipation.

The final solution is:

$$T^* = 1 + \frac{T_1 - T_0}{T_0} y^* + \frac{1}{2}\mathrm{Br}\left(1 - y^{*2}\right).$$

The first two terms ensure the boundary conditions are satisfied, and would be the only ones pressent if there were no viscous dissipation. The latter term provides the needed second derivative, and vanishes at the walls.

10.2 Poiseuille flow

10.2.1 Planar flow

As with Couette flow, let us assume the only component of the velocity field is $u_x(y)$, a function of y only.

The steady 2D Navier-Stokes equations read

$$0 = -\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u_x}{\partial y^2} \tag{10.1}$$

$$0 = -\frac{\partial p}{\partial y}. \tag{10.2}$$

where p is actually p/ρ. The second one stablishes that p is a function of x only. But in the first one, its derivative is related to a second derivative of a funcion of y only. It follows that both equations should equal some constant:

$$\frac{\partial p}{\partial x} = \nu \frac{\partial^2 u_x}{\partial y^2} = -c.$$

I.e. $p = -cx$, plus a constant pressure which makes no difference. Notice the minus sign: we consider a pressure drop in the x direction if $c > 0$.

For the velocity, we must solve for

$$\frac{\partial^2 u_x}{\partial y^2} = -\frac{c}{\nu},$$

given the boundary conditions $u_x(y = 0) = u_x(y = L) = 0$.

The solution is

$$u_x = \frac{c}{2\nu} y(L - y)$$

The flow is

$$Q = \frac{HLcL^2}{12\nu},$$

and the mean velocity,

$$\bar{u} = \frac{cL^2}{12\nu},$$

Which let us write, more elegantly,

$$u_x = 6\bar{u}\, \frac{y}{L} \left(1 - \frac{y}{L}\right).$$

10.2.2 Temperature

As in Couette planar flow, **??**, the temperature equation reduces to

$$0 = \kappa \frac{\partial^2 T}{\partial y^2} + \nu \left(\frac{\partial u_y}{\partial x}\right)^2,$$

or

$$0 = \kappa \frac{\partial^2 T}{\partial y^2} + 36\nu\bar{u}^2 \left[\frac{1}{L}\left(1 - 2\frac{y}{L}\right)\right]^2.$$

Casting it into dimensionless form,

$$0 = \kappa \frac{T_0}{L^2} \frac{\partial^2 T^*}{\partial y^{*2}} + 36 \frac{\nu \bar{u}^2}{L^2} (1 - 2y^*)^2 ,$$

or

$$\frac{\partial^2 T^*}{\partial y^{*2}} = -36 \mathrm{Br} (1 - 2y^*)^2 ,$$

where the Brickman number is again as in Eq. **??**

$$\mathrm{Br} = \frac{\nu \bar{u}^2}{\kappa T_0}.$$

If we define $s = 2y^* - 1$, then

$$4 \frac{\partial^2 T^*}{\partial s^2} = -36 \mathrm{Br} s^2,$$

or

$$\frac{\partial^2 T^*}{\partial s^2} = -9 \mathrm{Br} s^2.$$

The final solution is then:

$$T^* = 1 + \frac{T_1 - T_0}{T_0} y^* + \frac{3}{4} \mathrm{Br} \left(1 - s^4 \right) .$$

The second term is seen to vanish at the two walls (where $s = \pm 1$), while providing the correct second derivative. In terms of y^*,

$$T^* = 1 + \frac{T_1 - T_0}{T_0} y^* + \frac{3}{4} \mathrm{Br} \left[1 - (2y^* - 1)^4 \right] .$$

10.2.3 Flow in circular pipes

The solution is

$$u_x = \frac{c}{4\nu} \left(R^2 - r^2 \right) .$$

The flow is

$$Q = \frac{c \pi R^4}{8\nu},$$

and the mean velocity,

$$\bar{u} = \frac{c R^2}{8\nu},$$

10.2.4 Temperature

At variance with planar flows, **??**, the temperature equation must be written in polar coordinates:

$$0 = \kappa \frac{1}{r} \frac{d}{dr} \left[r \frac{dT}{dr} \right] + \nu \left(\frac{du_z}{dr} \right)^2 ,$$

or

$$0 = \kappa \frac{1}{r} \frac{d}{dr} \left[r \frac{dT}{dr} \right] + 16\nu \bar{u}^2 \frac{r^2}{R^4}$$

Casting it into dimensionless form,

$$0 = \kappa T_\mathrm{w} \frac{1}{R^2} \frac{1}{r^*} \frac{d}{dr^*} \left[r^* \frac{dT^*}{dr^*} \right] + 16\nu \bar{u}^2 \frac{1}{R^1} r^{*2},$$

or

$$\frac{1}{r^*} \frac{d}{dr^*} \left[r^* \frac{dT^*}{dr^*} \right] = -16 \mathrm{Br} r^{*2},$$

where the Brickman number is as in Eq. **??**. In order to solve it, we change it to

$$\frac{d}{dr^*} \left[r^* \frac{dT^*}{dr^*} \right] = -16 \mathrm{Br} r^{*3},$$

or

$$\frac{dT^*}{dr^*} = -4 \mathrm{Br} r^{*3} + \frac{c}{r^*}.$$

Then,

$$T^* = -\mathrm{Br} r^{*4} + c \log(r^*) + d.$$

The *log* term has a singularity at $r^* = 0$, so $c = 0$. The other constant has to be fixed in order to comply with the boundary condition $T^*(r^* = 1) = 1$. The final answer is

$$T^* = 1 + \mathrm{Br} \left(1 - r^{*4} \right),$$

or, bringing back the units for length and temperature:

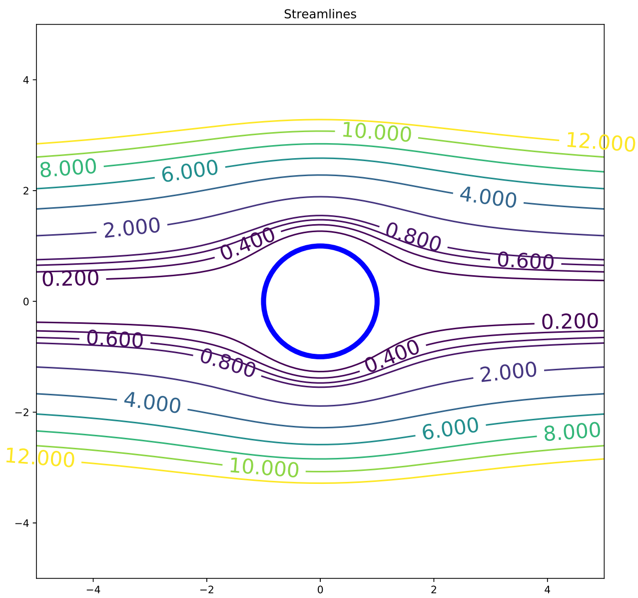
$$T = T_\mathrm{w} + \mathrm{Br} \left[1 - \left(\frac{r}{R} \right)^4 \right],$$

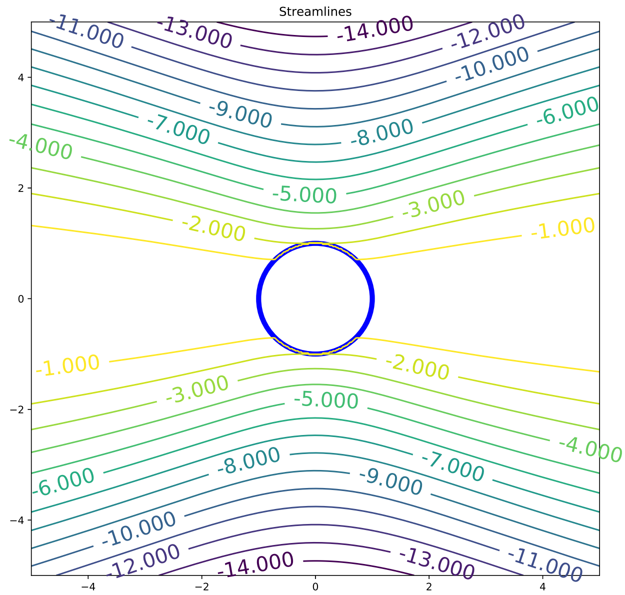
10.3 Exercises

Prove the expression for the unsteady Couette flow

Chapter 11

The overdamped limit




Streamlines
-14.000
-13.000
-12.000
-11.000
-10.000
-9.000
-8.000
-7.000
-6.000
-5.000
-4.000
-3.000
-2.000
-1.000
−4
−2
0
2
4

Chapter 12

The viscous boundary layer

12.1 Stagnation flow

In order to provide a glimse into the difficulties that are encountered as soon as one ventures into slightly more complicated problems, let us work out a solution of Navier-Stokes equations for a simple stagnation situation.

The potential flow solution for a stady-state incompressible 2D stagnation flow pattern close to a flat wall is given by the stream function

$$\psi = Bxy,$$

from which,

$$\begin{cases} u_x = \frac{\partial \psi}{\partial y} = Bx \\ u_y = -\frac{\partial \psi}{\partial x} = -By. \end{cases}$$

The parameter B has units of inverse time, and is given in practical situations by u_0/L, where u_0 is a relevant upstream velocity, and L a relevant size.

This flow pattern looks roughly correct, but of course, at the wall ($y = 0$) the fluid flows freely along the wall. This means the boundary conditions are of the "slip" kind, instead of the more realistic "no-slip" kind.

In order to find a correct flow, let us us the ansatz, originally investigated by Hiemenz in 1911[1].

$$\psi = Bxf(y),$$

where f is a function of y only. This is basically a separation of variables, also guessing a linear dependence on x.

The velocity is now,

$$\begin{cases} u_x = Bxf' \\ u_y = -Bf. \end{cases}$$

The correct no-slip condition then implies $f(0) = f'(0) = 0$. We will also require $f'(\infty) \to 1$ in order to recover our previous, potential flow, solution.

[1]Hiemenz was a student of Prandtl, the founder of the theory of viscous boundary layers.

Now, the steady 2D Navier-Stokes equations read

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = -\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right) \tag{12.1}$$

$$u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} = -\frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} \right). \tag{12.2}$$

(Here, p is not the true pressure, but the "kinematic pressure", p/ρ. For convenience, a ρ factor is assimilated into it, in the same way that $\nu = \mu/\rho$.)

The x equation then is reduced to

$$BxfBf' + (-Bf)Bxf'' = -\frac{\partial p}{\partial x} + \nu Bxf''',$$

which may be written as

$$Bff'' - B(f')^2 + \nu f''' = \frac{1}{Bx}\frac{\partial p}{\partial x}.$$

Now, the left part of the equation is a function of y only. This means the pressure can only have this form:

$$p(x,y) = Cx^2 + h(y),$$

with a constant C and a function of h to be determined later. Moreover, as y gets large we want to recover the potential solution. In this limit,

$$f \to a + y \qquad f' \to 1 \qquad f'' \to 0 \qquad f''' \to 0,$$

so the equation in this limit is

$$-B \to \frac{2Cx}{Bx},$$

which means $C = -B/2$, to be used later when solving for the pressure.

We must then solve

$$B\left(ff'' + 1 - (f')^2\right) + \nu f''' = 0.$$

Now, a lot may be learned from the shape of an equation without even solving it. Let us look for a similarity transformation of the form

$$f(y) = bg(ay),$$

so that equation for f may be cast as an equation for g with no parameters. With this prescription,

$$f' = bag' \qquad f'' = ba^2g'' \qquad f''' = ba^3g''',$$

So then the equation reads

$$B\left(b^2a^2gg'' + 1 - b^2a^2(g')^2\right) + \nu ba^3g''' = 0.$$

Because of the "1" in the parenthesis, it is clear $b = 1/a$, so then,

$$B\left(gg'' + 1 - (g')^2\right) + \nu a^2 g''' = 0.$$

Therefore, if $a = \sqrt{B/\nu}$,

$$gg'' + 1 - (g')^2 + g''' = 0, \tag{12.3}$$

with no parameters, as we wanted. This means that, whichever solution we find, our f is going to be given by

$$f(y) = \sqrt{\frac{\nu}{B}} g\left(\sqrt{\frac{B}{\nu}} y\right) = \sqrt{\frac{\nu}{B}} g\left(\frac{y}{\sqrt{\nu/B}}\right) = \ell g\left(\frac{y}{\ell}\right).$$

Clearly, $\ell = \sqrt{\nu/B}$ sets the scale of variation of the flow away from its potential solution.

Notice that the velocities will be

$$\begin{cases} u_x = Bxg'(y/\ell) = B\ell\frac{x}{\ell}g'(y/\ell) \\ u_y = -B\ell g(y/\ell), \end{cases}$$

so that the velocity scale is set by $B\ell = \sqrt{B\nu}$.

Our task is then to integrate the non-linear ODE 12.3, subject to these boundary conditions:

$$g(0) = 0 \qquad g'(0) = 0 \qquad g''(x \to \infty) \to 0.$$

If we had a condition on $g''(0)$, the problem would be a straight-forward exercise in integration. However, we have instead a condition on the other side of the integration domain, which makes this problem somewhat harder. The technique should then be a "shooting method", in which $g''(0)$ is adjusted until a vanishing small of g'' far away is found. The procedure may be made systematic, but we can also fiddle a bit with the parameters in order to find a reasoable approach. It is quite easy to arrive at $g''(0) \approx 1.234$, as in the jupyter python notebook at Supplementary Material.

In Figure 12.1 left, the function g and its first and second derivatives are plotted. With these, it is easy to plot the resulting streamlines, Figure 12.2 right. In the latter, the potential streamlines are shown in the left. It is apparent how the flow is "moved upwards" due to viscous effects near the wall. This displacement is readily quanfified by the asympotic behavour of g, as shown in Figure 12.1 left. A reasonable approximation is

$$g(y) \approx -0.64 + y.$$

This provides an estimate of the boundary layer thickness as given by

$$\delta = 0.64\ell = 0.64\sqrt{\nu/B},$$

which then increases as the root square of ν, and decreases as, basically the root square of the velocity far away from the wall (through B).

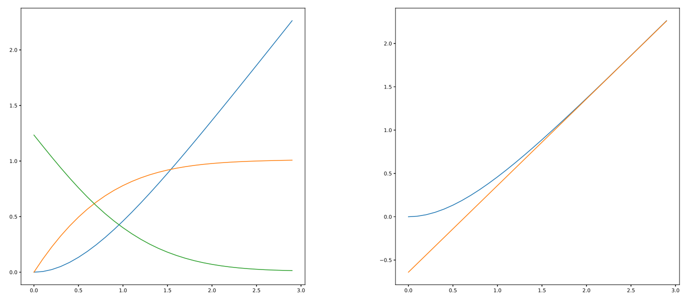


Figure 12.1:

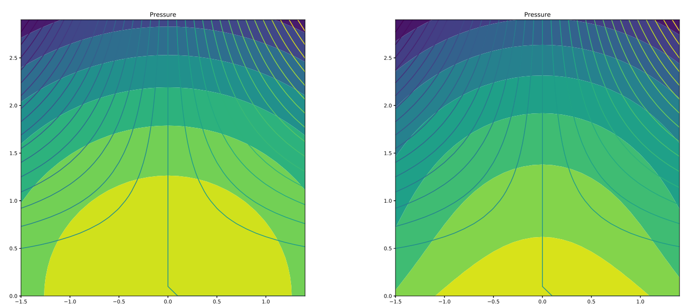


Figure 12.2:

The pressure was partly known already, $p = -(B^2/2)x^2 + h(y)$. The other Navier-Stokes equation, which we have still not used, reads

$$BfBf' = -\frac{\partial p}{\partial y} - \nu B f''.$$

This means

$$h' = -B^2 f f' - \nu B f'',$$

which may be easily integrated :

$$h = -B^2 \frac{1}{2} f^2 - \nu B f'.$$

So, finally:

$$p = -\frac{1}{2}\left[(Bx)^2 + (Bf)^2\right] - \nu B f' = -\frac{1}{2}\left[(u_x/f')^2 + u_y^2\right] - \nu B f',$$

where in the last equality it is shown how the pressure is not so different from the potential solution, which is simply given by the Bernoulli principle:

$$p_{\mathrm{pot}} = -\frac{1}{2}\left[u_x^2 + u_y^2\right] = -\frac{1}{2}\left[(Bx)^2 + (-By)^2\right].$$

An additional, viscous term appears, and it is not u_X which features, but rather u_x/f', which does tend to the u_x away from the wall. Pressures are also included in Fig. 12.2. To make a more quantitative comparison, isobars are plotted in Fig. 12.3. It is interesting that viscosity causes pressure to be "pushed" against the wall, flattening the isobars (and, interestingly, making them not normal to the wall). While, far away from the wall, they approach the same circular shape, but are displaced by the layer thickness, $\approx 0.64\ell$.

Another interesting feature of the flow is its vortixity, which is readily computed from

$$\omega_z = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} = 0 - Bxf''(y) = -Bx^* g''(y^*).$$

Hence the wall is seen to induce vorticity close to it, with a sign change on both sides on the $x = 0$ symmetry plane. The shear stress is given by a similar expression,

$$\tau_{xy} = \mu\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) = \mu B x f''(y) = \mu B x^* g''(y^*).$$

In particular, the wall stress is

$$\tau_{\mathrm{w}} := \tau_{xy}(y=0) = \mu B x^* g''(0) \approx 1.234 \mu B x^*$$

The (horizontal) skin friction coefficient may be defined, for this problem, as

$$C_{\mathrm{f}} := \frac{2\tau_{\mathrm{w}}}{\rho (Bx)^2}.$$

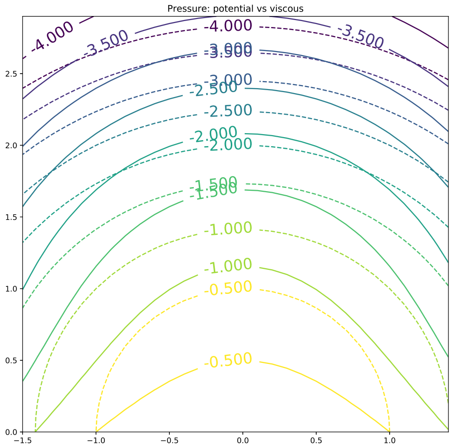


Figure 12.3:

The denominator features the horizontal velocity away from the wall, Bx. Then,

$$C_\mathrm{f} := \frac{2\sqrt{B\nu}g''(0)}{Bx} =: \frac{2g''(0)}{\sqrt{\mathrm{Re}_x}},$$

where the local Reynolds number is defined as

$$\mathrm{Re}_x = \frac{(Bx)x}{\nu}.$$

I.e. a Reynolds number where the typical velocity is the horizontal velocity far from the wall, Bx, and the distance is that to the impact point, x. A dependence of the friction coefficient with the inverse square root of a local Reynolds number is a common feature of laminar boundary layers.

Chapter 13

Turbulence

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