NOTES ON FLUID MECHANICS

INTERMEDIATE LEVEL

BY

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Symbols

$\mathbb{1}$	identity matrix/tensor
$\hat{\mathbf{e}}$	unit vector
$\mathbf{r}$	position vector
t	time
$d\cdot$	ordinary derivative
$\partial\cdot$	partial derivative
$\frac{d\cdot}{dt}$	material time derivative
$\nabla\cdot$	del operator
$\mathbf{u}$	velocity field
u_0	typical velocity scale
L	typical length scale
$\mathbf{g}$	acceleration of gravity
F	force
$\mathbf{f}$	volumetric force
$\boldsymbol{\omega}$	vorticity
$\nabla\cdot\mathbf{u}$	divergence of velocity
$\mathbf{q}$	heat flux
$^{\mathsf{T}}$	matrix/tensor transpose
T	temperature
p	pressure
ρ	mass density
m	mass of a particle
V	volume of a particle
μ	shear (first) viscosity coefficient
λ	second viscosity coefficient
ζ	volume viscosity coefficient
ϵ	strain rate tensor, specific energy, Levi-Civita symbol
τ	shear stress tensor
ϕ	velocity potential, azimuthal angle in cylindrical and spherical coordinates
ψ	stream function
φ	gravitational potential

Ποταμοῖσι τοῖσιν αὐτοῖσιν ἐμβαίνουσιν, ἕτερα καὶ ἕτερα ὕδατα ἐπιρρεῖ
Ever-newer waters flow on those who step into the same rivers.
Heraclitus [4].

Foreword

Just a compilation of notes on different aspects of fluid dynamics I have collected over the years.

The final push was from teaching the course Fluid Dynamics on the International Master of Nuclear Fusion, course 2018. I realized much material was scattered all over the place, on notes, articles, blog entries . . .

I may upload these notes to a collaborative site (e.g. github) so that other people may contribute. As of today, I am the sole author.

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Chapter 1

Introduction

Fluids, by definition, flow.

The main field is the velocity field, and the displacement is secundary ...

1.1 Notation

When labeling coordinates, we will follow this convention:

$$\mathbf{r} = \begin{cases} (x, y, z) & \text{Cartesian} \\ (r, \theta, 0) & \text{Polar} \\ (\rho, \theta, z) & \text{Cylindrical} \\ (r, \phi, \theta) & \text{Spherical.} \end{cases}$$

For spherical coordinates, we stick to the naming convention that is most common in physics, and is specified by ISO standard 80000-2:2009, and earlier in ISO 31-11 (1992). I.e.: ϕ is the azimuthal angle, about the z axis, and θ is the polar angle with the z angle (also called "the co-latitude".)

In cylindrical coordinates, ρ, the distance to the z axis, could be confused with the density, but it always is quite clear to tell them apart. We deviate from the standard (ISO 31-11, to be precise) by using θ for the azimuthal angle for 2D polar and cylindrical coordinates. We are aware that the right symbol is ϕ (or, φ), and we are moving towards implementing the recommendations, but the usage of θ is quite ingrained. For the 2D polar distance to the origin, either r or ρ are valid, depending on whether this 2D case is taken as: $z = 0$ from cylindrical coordinates; or, as $\theta \to \pi/2$ from spherical ones. We choose the former.

The velocity field is represented in many ways. Here, we use bold letters for vectors, and italice letters with subindices for its components, Cartesian or otherwise:

$$\mathbf{u} = \begin{cases} (u_x, u_y, u_z) & \text{Cartesian} \\ (u_r, u_\theta, 0) & \text{Polar} \\ (u_\rho, u_\theta, u_z) & \text{Cylindrical} \\ (u_r, u_\phi, u_\theta) & \text{Spherical.} \end{cases}$$

This convention is applied for any vector field, we just write it down explicitely because the velocity is, by far, the most used field in these notes.

We *do not* use the traditional convention of using different letters for each Cartesian velocity component:

$$\mathbf{u} = (u, v, w) = u\hat{\mathbf{e}}_x + v\hat{\mathbf{e}}_x + w\hat{\mathbf{e}}_x \qquad \text{(not used)}$$

(we do use pointy hats, or “circumflexes”, for unit vectors.)

1.2 Style

In these notes, concepts are introduced *at needed*. This is at variance with other texts, in which there may be an introduction to the general physics of fluids, another for mathematical analysis, and so on. So, for example, the concept of streamlines is not introduced until the potential flow past a cylinder is discussed. In fact, only the simple 2D case is introduced — the more complicated axisymmetric situation is later described, for the flow past a sphere.

The few starred sections refer to “advanced” material, some of which may be the authors’ own calculations (therefore, to be read with care).

Part I

Ideal fluids

Chapter 2

Continuity of mass

2.1 The concept of small regions

In these notes, we will be dealing with regions that are ”infinitesimally small” quite a lot. This is because fluid dynamics are part of what is called continuum mechanics, in which bodies are supposed to be continuous in a mathematical sense. By this me mean, we can pieces of any material as small as we want, so that we may use the mathematical language of calculus. Nevertheless, these regions are large enough so that may be considered continuous, and have defined macroscopic physical parameters: density, mass, volume, temperature, and so on.

This is an excellent approximation, most of the time, since the number of molecules involved in a flow is usually so large, that even for a very small region of it, there are very many molecules. An interesting exercise is to go through an order-of-magnitude calculation of these numbers.

In these notes, we will consider two sort of regions: volume parcels, which are very small regions that are fixed in space, and particles, which are small regions of the moving fluid (thus, in general, changing their positions with time). A parcel is related to an “Eulerian” point of view, in which a flow moves over fixed coordinates — particles are related to the complementary, “Lagrangian,” view of the particle as moving with the flow, as will be discussed in 3.1.1.

2.2 The continuity equation

Let us consider a fixed that is cubic in shape, with one of their corners at the origin and the three incident edges aligned with the Cartesian coordinates. This choice is completely general, given our freedom in defining a region (as long as we make it infinitesimally small at the end), and in origin and axes for coordinates.

There may be a mass influx though the left face of the parcel due to advection (i.e. the existence of a velocity field). For a small time dt, it will be given by

$$dm|_\mathrm{l} = \rho(u_x\, dt)\, dy\, dz.$$

Indeed, all the fluid material that a distance $\delta = u_x\, dt$ away, or closer, will pass through the surface. The whole mass influx is therefore ρ times δdA, where $dA = dydz$ is the cross- sectional area. (Since the parcel is cubic, $dx = dy = dz$, but it is clearer to use different symbols.)

At the right wall, the mass change will be negative (if v_x is positive there):

$$dm|_\mathrm{r} = -\rho'(u'_x\, dt)\, dy\, dz.$$

The mass change due to the horizontal component of the velocity field is therefore

$$dm|_x = \left[\rho' u'_x - \rho u_x\right]\, dt\, dy\, dz.$$

Notice that we have allowed for the possibility that the combination ρu_x may be somewhat different at the left and right walls. Indeed, if they were equal, there would be no net mass change, since the mass that enters would equal that leaving. Nevertheless, since dx is small, we may approximate the value at the right wall in a Taylor series:

$$\rho' u'_x \approx \rho u_x + \frac{\partial \rho u_x}{\partial x} dx.$$

Therefore,

$$dm|_x \approx -\frac{\partial \rho u_x}{\partial x}\, dt\, dx\, dy\, dz.$$

Or,

$$\frac{1}{V} \left.\frac{dm}{dt}\right|_x \approx -\frac{\partial \rho u_x}{\partial x}$$

Since $dm/V = \rho$, the density of the parcel, and V does not change with time, since the parcel does not move, we may write, in the limit of $dt \to 0$,

$$\left.\frac{\partial \rho}{\partial t}\right|_x = -\frac{\partial \rho u_x}{\partial x}.$$

Notice the usage of the ∂ symbol: since the parcel is fixed, with matter entering and leaving, this is the correct symbol.

In order to get the total rate of change in density, we have to add up the other two directions. In doing so, we obtain

`{eq:continuity0}`

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho u_x}{\partial x} - \frac{\partial \rho u_y}{\partial y} - \frac{\partial \rho u_z}{\partial z} = -\nabla \cdot (\rho \mathbf{u}). \tag{2.1}$$

In the last equation the divergence of field $\rho\mathbf{u}$ is seen to appear naturally.

This is the correct expression, which is usually written as

`{eq:continuity}`

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{2.2}$$

A more direct procedure to get it involves integrals, rather than differentials.

By definition, mass is the integral of the density:

$$dm = \int_V \rho d\mathbf{r}$$

However, a mass flux $\rho\mathbf{u}$ may cause changes in it:

$$dm = -\oint_{\partial V} \rho\mathbf{u} \cdot \mathbf{n} dS,$$

where ∂V is the surface around volume V. The minus sign appears because the surface normal, $\mathbf{n}$ points from the inside toward the outside of the surface. We may transform the latter integral into a volume integral from Gauss' theorem:

$$dm = -\int_V \nabla \cdot (\rho\mathbf{u}) dV.$$

Equating both changes,

$$\int_V \rho d\mathbf{r} = -\int_V \nabla \cdot (\rho\mathbf{u}) dV.$$

Since the identity is valid for any volume V, the integrands may be equal. This fact leads to the same expression as before.

2.2.1 Incompressibility

By "incompressibility", it could be meant that variations in ρ (spatial or temporal) are negligible. Continuity then readily implies:

$$\nabla \cdot \mathbf{u} = 0,$$

I.e., the velocity field must be divergence-free — another adjective is "solenoidal", as an analogy with the magnetic field. The latter, as no magnetic charges ("monopoles") have ever been seen, satisfies this equation.

Indeed, this is sometimes the meaning of the term. A technical issue is incompressibility is often defined, rather, as $\nabla \cdot \mathbf{u} = 0$. This does *not* imply that ρ is constant, even though its reverse is true. Indeed, in this case:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla\rho + \rho \underbrace{\nabla \cdot \mathbf{u}}_{=0} = 0$$

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla\rho = 0.$$

As will be discussed in 3.1.1, this means that the total (or "convective") time derivative of ρ is zero. This implies that ρ is constant at every moving particle, or along streamlines.

In order to make the distiction clear, the term "divergence-free" (or "divergenceless") is sometimes used.

Chapter 3

Euler’s equations

3.1 Material derivative

When deriving our equations, it is important to distinguish between a partial time variation and the total one. This is mathematically encapsulated in the expression for the total variation of a function of three Cartesian coordinates plus time:

$$\frac{dA(x,y,z,t)}{dt} = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial y}\frac{dy}{dt} + \frac{\partial A}{\partial z}\frac{dz}{dt}.$$

The Cartesian components are not to be considered as independent of the time. Rather, their time derivatives are precisely the components of the velocity field:

$$\frac{dA(x,y,z,t)}{dt} = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial x}u_x + \frac{\partial A}{\partial y}u_y + \frac{\partial A}{\partial z}u_z.$$

A way to write this expression more concisely is

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \mathbf{u}\cdot\nabla A,$$

where the operator $\mathbf{u}\cdot\nabla$ is

$$\mathbf{u}\cdot\nabla = \sum_i u_i \frac{\partial}{\partial x_i},$$

which indeed looks like a scalar product of vector $\mathbf{u}$ and the vector operator ∇.

This total time derivative received very many names. Among the most popular we find “material”, “Eulerian”, “convective”, “advective” …

In any case, the consequence is that convection by the velocity field adds a new term. This has an important consequence for the change of the velocity field itself, which turns out to be non-linear. Indeed, for component i,

$$\frac{du_i}{dt} = \frac{\partial v_i}{\partial t} + \mathbf{u}\cdot\nabla u_i.$$

The second, non-linear term is called the convective acceleration. When expressing this equality in vector form, it is important not to make the eror of writing

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}(\nabla \cdot \mathbf{u}) \qquad \text{(wrong!)},$$

which would imply that incompressible fluids have no such extra acceleration. The correct expression is

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \qquad \text{(right)}.$$

Another concise, but more obscure, way of writing it is

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot (\nabla \mathbf{u}),$$

where now $\nabla \mathbf{u}$ is to be interpreted as a tensor, with components

`{eq:nabla_u_def}`

$$(\nabla \mathbf{u})_{ij} = \frac{\partial u_j}{\partial x_i}, \tag{3.1}$$

i.e. a direct product of the vector operator ∇ and vector $\mathbf{u}$ (also known as "the gradient of vector $\mathbf{u}$.") The operation $\mathbf{u} \cdot (\nabla \mathbf{u})$ is then similar to a left multiplication of a matrix by a row vector, resulting in a column vector. This sort of tensors will be again considered in Section 9.1. Given these two possible interpretations, the acceleration is often written without any parentheses:

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}.$$

Yet another expression involves the direct product of the velocity vector with itself:

$$(\mathbf{u} \otimes \mathbf{u})_{ij} := u_i u_j,$$

which lets us write

`{eq:Euler_uu}`

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}), \tag{3.2}$$

as may be easily checked. This last expression is useful to write down so-called "conservation form" of the equations, in which one desires an overall ∇ operator acting on several expressions.

3.1.1 Continuity, revisited

The continuity equation, Eq. 2.2 may be written as

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{u} + (\mathbf{u} \cdot \nabla)\rho = 0,$$

or

`{eq:convergence}`

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \mathbf{u}. \tag{3.3}$$

This expression is called the "convergence equation" by Joe Monaghan: if $\nabla \cdot \mathbf{u}$ is the divergence, $-\nabla \cdot \mathbf{u}$ could be called the convergence.

We now see that incompressibility, defined as $\nabla \cdot \mathbf{u} = 0$, implies $d(\rho)/(dt) = 0$. This means that density is "carried with the flow", as explained in next session.

3.1.2 Physical meaning of the material derivative

If we measure the value of a given field at a fixed point in space, its plot as a function of time has a first derivative which, by definition, is the partial time derivative. If however, we measure its value as we move along with the flow, we will obtain its total derivative.

A common example considers temperature, partly because this is a quantity that humans can readily "feel", and which is routinely measured with thermometers. Imagine a thermometer probe inserted at some point along a stream. Its reading are approximately constant since the flow is steady. If a gate is opened upstream at the bottom of a reservoir, colder water will move downstream, and our temperature readings will drop.

Imagine instead a thermometer inserted through a cork flowing with the flow. If a thermometer is released upstream, close to the reservoir, its readings will change little. This is true for the case before the gate opening, and for the case after the gate. In the later case, the reading will be lower, but constantly so. This later reading is approximately the total derivative dT/dt, while the previous is the partial $\partial T/\partial t$.

Notice that we may have $dT/dt = 0$ even if $\partial T/\partial t \neq 0$, if

$$\frac{\partial T}{\partial t} = -\mathbf{u} \cdot \nabla T \qquad \Longrightarrow \qquad \frac{dT}{dt} = 0.$$

The first equation precisely describes the situation in which a change in the local value of T is due only to convection. This is the case that closely resembles the reservour release experiment.

Equivalently, we may have $\partial T/\partial t = 0$, but $dT/dt \neq 0$:

$$\frac{dT}{dt} = \mathbf{u} \cdot \nabla T \qquad \Longrightarrow \qquad \frac{\partial T}{\partial t} = 0,$$

which expresses the fact that a particle moving with the flow may experience changes in temperature even if this field is locally constant. This may apply to a flow in which a temperature gradient is established, from a cold region to a hotter one. Fluid going from one to the other will register this change in temperature even if locally the variation is null.

We stress that this discussion applies to any field, even if they are vector fields (since it applies to each of its components, which are scalar).

3.1.3 The steady state

If the state of the flow is *steady* this means that all fields do not change with time when evaluated at a given point — but, they may still change in space. Mathematically.

$$\frac{\partial A}{\partial t} = 0$$

for all A field (including vector fields, through e.g. their Cartesian components).

In particular, the velocity field will be fixed in time. The paths in space traced by the velocity field are called streamlines. Intuitively, one starts at some point and moves following the velocity field, thus tracing a streamline. Numerically, this is in essence what is computed by CFD software. Mathematically, the equation will be

$$\frac{d\mathbf{r}}{\partial s} = \mathbf{u}(\mathbf{f}(s)),$$

where we use s instead of t to stress the fact that the parameter in this "trajectory" is *not* the physical time, and is only used to trace the line.

It is instructive to consider mass continuity in this case. It reduces to

$$\nabla \cdot (\rho \mathbf{u}) = \rho \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \rho = 0.$$

If the velocity field is, moreover, divergence-free,

$$\mathbf{u} \cdot \nabla \rho = 0.$$

Now, the gradient of a field is a vector field perpendicular to its iso-contours (the contours at which the field has constant values). Since the gradient of the density is perpendicular to the velocity, this equation tell us that the density is *constant* along streamlines. This applies not only to the density, but to any conserved quantity. See Exercise 1 for a concrete simple case.

3.2 External and pressure forces

Let us discuss possible forces acting upon a particle. They may be split into volume, or external, or surface. The first one includes all forces that are due to field that act upon the material of the particle as a whole. The most common ones are gravitational, and electromagnetic. We will not consider the latter ones for simplicity. Its study defines magnetohydrodynamics, and is relevant and interesting, but beyond the scope of these notes. We are left with gravity. If the scale of our problem is within some kilometers around the surface of the Earth, the force of gravity is constant and equal to $dm\mathbf{g}$, where $\mathbf{g}$ is a gravitational acceleration vector. This vector points "down" (usually, the $-z$ or $-y$ direction, depending on our convention), and has a magnitude $g = 9.8\,\mathrm{m\,s^{-1}}$. Its effect is given by Newton's Second law, according to which the net force on a particle with mass dm is

$$\mathbf{F}_{\text{ext}} = dm\mathbf{g},$$

causing a volumetric force

$$\mathbf{f}_{\text{ext}} = \frac{\mathbf{F}_{\text{ext}}}{dV} = \rho \mathbf{g}.$$

Surface forces may in turn may split into compression and stress forces. The latter are absent in an ideal fluid by definition, and we are left only with compression. This effect is caused by the best known surface force field: the pressure. Indeed, a pressure on the left wall of the particle will cause a horizontal force

{eq:pressure_force_on_left}

$$F_{p,x}\big|_1 = p\,dy\,dz. \tag{3.4}$$

The net force is the pressure times the area, $dy\,dz$, and there is a minus sign because by definition pressure acts from outside to the inside. The total horizontal force will also have an influence from the right wall, which has a pressure p' that may be different from p:

$$F_{p,x} = (p - p')dy\,dz.$$

Notice that the force is null if the pressure is equal. Also, thanks to our sign convention, if $p' < p$ we get a net horizontal force, which is in agreement with the concept of a pressure drop causing a force in the direction in which density is lower. Now, expanding in Taylor series,

$$p' \approx p + \frac{\partial p}{\partial x}dx,$$

so

$$F_{p,x} = -\frac{\partial p}{\partial x}dx\,dy\,dz = -\frac{\partial p}{\partial x}dV.$$

I.e., the volumetric force in the horizontal direction will be

$$f_{p,x} = -\frac{\partial p}{\partial x}.$$

Similar expressions will result for the y and z directions. All three may be encapsulated in vector form:

$$\mathbf{f}_p = -\nabla p.$$

I.e. the volumetric pressure force is minus the gradient of the pressure.

The total volumetric force is then

$$\mathbf{f} = -\nabla p + \rho\mathbf{g}.$$

3.3 The momentum equation

A bold and quick way to reach Euler's momentum equation is to equate mass times acceleration with forces, in the spirit of Newton's Second law:

$$dm\frac{d\mathbf{u}}{dt} = V\mathbf{f}.$$

Dividing by the volume,

$$\rho\frac{d\mathbf{u}}{dt} = \mathbf{f}.$$

This is in fact correct, and leads to

$$\rho\frac{d\mathbf{u}}{dt} = -\nabla p + \rho\mathbf{g}. \qquad (3.5)$$ `{eq:Euler_momentum}`

In Cartesian components, the equation reads

$$\rho\frac{du_i}{dt} = -\frac{\partial p}{\partial x_i} + \rho g_i. \qquad (3.6)$$ `{eq:Euler_momentum_C}`

Notice, however that the derivation involves supposing that dm may be taken out of the time derivative. This is because the mass of a moving particle does not change — in Section 9.1.1 we will see how this fact takes us to the correct continuity equation.

3.3.1 Conservation form

It is well known that Newton's second law applied to mass times acceleration, or equivalently, to the variation of linear momentum (since the mass of a particle does not change). The reader may be therefore worried about this law being the right one. In fact, a direct application of the ideas in Section 2.2 would lead to this sort of equation for the change in the i-th component of the linear momentum:

$$\frac{\partial(\rho u_i)}{\partial t} = -\nabla\cdot(\rho u_i \mathbf{u}) - \frac{\partial p}{\partial x_i} + \rho g_i.$$

In it, the change of linear momentum per volume is changed due to volumetric forces, and also due to convection of momentum.

This can also be written in vector form:

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) = -\nabla p + \rho\mathbf{g},$$

which strongly resembles the expression of 3.2. The difference is that now the density ρ is inside the time and space derivatives.

The happy answer is that both are equivalent, thanks to this handy identity, which applies to any field:

$$\frac{\partial(\rho A)}{\partial t} + \nabla\cdot(\rho\mathbf{u}A) = A\underbrace{\left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\right)}_{=0} + \rho\underbrace{\left(\frac{\partial A}{\partial t} + \mathbf{u}\cdot\nabla A\right)}_{=dA/dt},$$

or

`{eq:conserv_to_total}`

$$\frac{\partial(\rho A)}{\partial t} + \nabla\cdot(\rho\mathbf{u}A) = \rho\frac{dA}{dt}. \tag{3.7}$$

This identity permits the conversion from the "conservation" expression of any quantity ρA as a "convective form", $\rho dA/dt$. It is interesting that if A is a constant, contінity is recovered.

Since this identity applies to each of the Cartesian components of the velocity, then it applies to the velocity itself.

We may therefore write Euler's equation in conservation form:

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u}) = -\nabla p + \rho\mathbf{g}.$$

3.4 Vorticity

The momentum equation 3.5 is still quite daunting due to its nonlinear term in the convective derivative.

In order to make progress, it was Lamb's idea in 1895 to use the following identity for the vector product of any vector field and its curl:

`{eq:Lambs_identity}`

$$\mathbf{u}\times(\nabla\times\mathbf{u}) = \frac{1}{2}\nabla u^2 - \mathbf{u}(\nabla\cdot\mathbf{u}) \tag{3.8}$$

(in fact, the identity is somewhat more general, see exercise 2).

Introducing the name “vorticity” for the curl of the velocity field,

$$\boldsymbol{\omega} := \nabla \times \mathbf{u},$$

we may write the momentum equation as

er_momentum_w_vort}

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\nabla u^2 - \mathbf{u} \times \boldsymbol{\omega} = -\frac{1}{\rho}\nabla p + \mathbf{g}. \tag{3.9}$$

Now, recalling that the curl of a gradient is always zero, we may apply the curl operator to the whole equation. If we assume constant density, then

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = 0.$$

In e.g. [17] we find an expression for the curl of a vector product:

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}\ (\nabla \cdot \mathbf{b}) - \mathbf{b}\ (\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}.$$

Hence,

$$\nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = \mathbf{u}\ (\nabla \cdot \boldsymbol{\omega}) - \boldsymbol{\omega}\ (\nabla \cdot \mathbf{u}) + (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\boldsymbol{\omega}.$$

The first term is always zero, since the vorticity is divergence-free (it being the curl of a field). The second is also zero, since we have already assumed constant density (which, according to the discussion in §2.2.1, implies the velocity is divergence-free). In this case,

$$\nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\boldsymbol{\omega}, \tag{3.10}$$

{eq:conv_accel_as_vort}

and we may write

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u}.$$

The left hand side is the advective derivative. Then:

$$\frac{d\boldsymbol{\omega}}{dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u},$$

which means vorticity is “almost” conserved. The righ-hand side represents an advection of velocity by vorticity, which sounds a bit upside-down. This term may be neglected if the flow is slow (since it involves a term quadratic in the velocity), to arrive at $\frac{d\boldsymbol{\omega}}{dt} \approx 0$. We will see, in Section 9.4, how viscosity adds a diffusion term to this equation.

In any case, the equation involves only the velocity and its curl, the pressure having been “curled-away.” Indeed, there is a class of numerical methods, called “vortex methods” in which this simplification is employed. In many applications, however, its usefulness is limited. For example, boundary conditions may be difficult to define for these terms.

There is, however, an important consequence of this equation: if a given flow is curl-free at some instant, it must *remain* so at every other time (both future, and past). This is because the equation is first order in time, and a null right-hand side translates into a null change. We will see later that when viscosity is introduced, a diffusion term appears which is second order in space.

This term is, in many cases, responsible for the generation of vorticity. In the clearest instance, the no-slip boundary condition close to solid walls, by which the velocity must be zero there, creates zones of vorticity generation. That boundary condition cannot be enforced within the Euler framework, because it only features first order spatial derivatives of the velocity.

3.5 Bernouilli's principle

Let us approach Eq. 3.9 in a completely different manner. Instead of getting rid of the gradient terms, let us express as many terms as we can as gradients, for reasons that will become clear soon. First, the gravitational acceleration is, by definition, $\mathbf{g} = -\nabla\varphi$, where φ is the gravitational potential energy (gz if the z axis is the vertical). Now, the pressure gradient over the density may be related to the gradient of pressure over density, thus:

$$\nabla\left(\frac{p}{\rho}\right) = \frac{\nabla p}{\rho} - \frac{p}{\rho^2}\nabla\rho$$

Therefore,

{eq:Bern_unsteady}

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}u^2 + \frac{p}{\rho} + \varphi\right) = \mathbf{u}\times\boldsymbol{\omega} + \frac{p}{\rho^2}\nabla\rho \tag{3.11}$$

Now, in steady flow we would have

$$\nabla\left(\frac{1}{2}u^2 + \frac{p}{\rho} + \varphi\right) = \mathbf{u}\times\boldsymbol{\omega} - \frac{p}{\rho^2}\nabla\rho.$$

The term involving the vector product of the vorticity and the velocity is perpendicular to both. Hence, it vanishes upon a scalar multiplication with $\mathbf{u}$:

$$\mathbf{u}\cdot\nabla\left(\frac{1}{2}u^2 + \frac{p}{\rho} + \varphi\right) = -\frac{p}{\rho^2}\mathbf{u}\cdot\nabla\rho.$$

In steady flow, the continuity equation is

$$\mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = 0,$$

and we may also write

{eq:Bern_u_times}

$$\mathbf{u}\cdot\nabla\left(\frac{1}{2}u^2 + \frac{p}{\rho} + \varphi\right) = \frac{p}{\rho}\nabla\cdot\mathbf{u}. \tag{3.12}$$

Hence, if the flow is incompressible (technically, divergence-free, as explained in section 2.2.1),

$$\mathbf{u}\cdot\nabla\left(\frac{1}{2}u^2 + \frac{p}{\rho} + \varphi\right) = 0,$$

which states the fact that the quantity

$$h = \frac{1}{2}u^2 + \frac{p}{\rho} + \varphi$$

is constant along a given streamline. This result is known as Bernoulli's principle, and applies only to ideal, steady, incompressible flow. (There is a variant of it that applies to unsteady flow, as we will see in section 7.1.) This combination is called "the head" and is customarily used in elementary applications of this result. Some of its direct applications are: the Venturi effect (by which the pressure decreases in zones with higher velocities), slow drainage of containers, syphons . . .

We will also see that in some cases, like in potential flow, the velocity field may be found independently of the pressure. This principle then yields the corresponding pressure from the velocity.

3.6 Dimensionless variables

A procedure to gain insight into a physical problem is to try to cast the different magnitudes into dimensionless (or, "reduced") ones. For example, if there is a relevant length scale L, all lengths may be rescaled according to it:

$$x^* = \frac{x}{L} \quad y^* = \frac{y}{L} \quad z^* = \frac{z}{L},$$

where an asterisk marks a dimensionless magnitude. We can also write it in vector notation: $\mathbf{r}^* = \mathbf{r}/L$.

As an example, in some problems this is the only relevant scale, and the movement is driven by gravity, whose accelaration is g. In such cases, the time scale will be given by the only combination of L and g:

$$t_0 \sim \sqrt{\frac{L}{g}}.$$

This is actually the correct result for the period of a simple pendulum, but for a numerical factor of 2π with no dimensions: $T = 2\pi t_0$. No equations have been solved (or even written down) in order to arrive to this result. Notice also that for larger displacements of the pendulum, the amplitude of the motion is another length, which complicates the analysis.

In many fluid problems there is a well-defined velocity u_0 that sets the velocity values (e.g. the upstream velocity in flows around objects). If this is the case,

$$\mathbf{u}^* = \frac{\mathbf{u}}{u_0} \qquad t^* = \frac{t}{L/u_0},$$

so the velocity and length set the time scale. If we apply this to our Euler equation,

$$\rho\frac{d\mathbf{u}}{dt} = -\nabla p + \rho\mathbf{g} \qquad \rho\frac{u_0}{L/u_0}\frac{d\mathbf{u}^*}{dt^*} = -\nabla p + \rho\mathbf{g}.$$

Notice that the ∇ operator can also be cast into dimensionless form. For example, its x component is

$$\nabla_x = \frac{\partial}{\partial x} = \frac{\partial}{\partial(x^* L)} = \frac{1}{L}\frac{\partial}{\partial x^*},$$

so we may define

$$\nabla^* = \frac{1}{L}\nabla.$$

The Euler equation then reads,

$$\rho \frac{u_0}{L/u_0} \frac{d\mathbf{u}^*}{dt^*} = -\frac{1}{L}\nabla^* p + \rho \mathbf{g}.$$

Usually, a reference value ρ_0 for the density is often known, so that $\rho^* = \rho/\rho_0$, and

$$\rho_0 \rho^* \frac{u_0}{L/u_0} \frac{d\mathbf{u}^*}{dt^*} = -\frac{1}{L}\nabla^* p + \rho \mathbf{g}.$$

Now, multiplying throughout by $L/(\rho_0 u_0^2)$, and supposing for simplicity that the density is constant,

{eq:Euler_just_before_adim}

$$\rho^* \frac{d\mathbf{u}^*}{dt^*} = -\nabla^* \frac{p}{\rho_0 u_0^2} + \frac{L}{u_0^2}\mathbf{g} \qquad \Longrightarrow \qquad \rho^* \frac{d\mathbf{u}^*}{dt^*} = -\nabla^* p^* + \rho^* \mathbf{g}^* \tag{3.13}$$

We have therefore found that the dimensionless pressure and gravity acceleration are given by

$$p^* = \frac{p}{\rho_0 u_0^2} \qquad \mathbf{g}^* = \frac{L}{u_0^2}\mathbf{g}.$$

The reduced pressure was to be expected, given the Bernoulli expressions involving ρu^2. The reduced gravity is directly related to Froude's number, which is historically defined as

$$\mathrm{Fr} = \frac{\sqrt{gL}}{u_0}.$$

Therefore, $\mathbf{g}^* = \mathrm{Fr}^2(\mathbf{g}/g)$, where vector $(\mathbf{g}/g)$ is the unit vector pointing in whichever direction the gravity points to in our problem (usually, $-y$ or $-z$.)

It is easy to check that the continuity equation can likewise be cast into reduced form:

$$\frac{d\rho^*}{dt^*} + (\nabla^* \mathbf{u}^*) = 0.$$

3.7 Exercises

1. Consider a 2D velocity field (y, x). Sketch it, or plot it and guess how the streamlines should look. Next, find a density field that satisfies $\mathbf{u} \cdot \nabla \rho = 0$ (other than the trivial, constant, solution)

2. Prove that

   $$\mathbf{u} \times (\nabla \times \mathbf{q}) = \nabla_{\mathbf{q}}(\mathbf{u} \cdot \mathbf{q}) - \mathbf{u} \cdot \nabla \mathbf{q},$$

   for any two vector fields. By $\nabla_{\mathbf{q}}$ it is meant that the gradient is applied only on $\mathbf{q}$ (Feyman's subscript notation). Hint: the curl of a field is associated with a tensor with elements:

   $$\overline{(\nabla \times \mathbf{q})_{ij}} = \left(\frac{\partial q_j}{\partial x_i} - \frac{\partial q_i}{\partial x_j} \right) \qquad \overline{(\nabla \times \mathbf{q})_{ij}} = \sum_k \epsilon_{ijk} (\nabla \times \mathbf{q})_k.$$

In the latter, ϵ is a rank-three tensor, the totally antisymmetric tensor (also known as the Levi-Civita symbol). If this seems mystifying, all that it is expressing is that e.g. $(\nabla \times \mathbf{q})_z = \partial q_x / \partial y - \partial q_y / \partial x$, et cætera.

This tensor appears also in the vector product:

$$(\mathbf{u} \times \mathbf{q})_i = \sum_{j,k} \epsilon_{ijk} u_j q_k.$$

These two results make the demonstration quite simple. We will also be using these expressions in section 9.1, when we talk about rotations of a particle.

Chapter 4

Hydrostatics

While certainly not as involved as dynamics, the static case of the equations of fluid mechanics is nevertheless interesting. This secion applies even to real fluids (with viscosity), because stresses are constant on them (otherwise, a viscous force would arise that would cause a motion.)

In equilibrium, all time derivatives (both partial, and material) are zero, and the Euler momentum Equation 3.5 reduces to

$$\nabla p = \rho \mathbf{g}. \tag{4.1}$$ `{eq:hidrostatics}`

4.1 Incompressible fluids

The simplest case is when the density is constant. In this case,

$$\nabla p = \rho \mathbf{g} \qquad \Longrightarrow \qquad p = p_0 + \rho \mathbf{g} \cdot \mathbf{r},$$

Since,

$$\nabla (\mathbf{g} \cdot \mathbf{r}) = \nabla (-gz) = -g \hat{\mathbf{e}}_z = \mathbf{g},$$

where we assume the acceleration of gravity points toward negative values of the z coordinate: $\mathbf{g} = -g\hat{\mathbf{e}}_z$.

Of course, the expression for the pressure in terms of $\mathbf{g} \cdot \mathbf{r}$ is general and does not depend on our choice of Cartesian axes. Nevertheless, it is usually written in function of the “depth”, this being $h = -z$ in our choice (in general, the depth is the coordinate at which we go against gravity). This means,

$$p = p_0 + \rho g h.$$

This means that pressure should increase as we go deeper. This can be deduced from much simpler grounds, computing the pressure difference between two parallel sides of a prismatic object immersed in a fluid. This pressure difference translates into a net vertical force that must balance the force due to gravity. The same conclusion is reached. This is the standard way to derive the law of floating bodies (Archimedes’ law), also for the case of partly-floating boadies (icebergs, boats…).

The pressure features a reference pressure which should be provided in order to fix the absolute value of the pressure. This reference is in principle not important. Indeed, in most application "pressure" actually refers to excess pressure over atmospheric pressure, i.e. manometric pressure. Notice also that it is only gradients of the pressure which are relevant, never the value itself. Exceptions arise when thermodynamics is relevant — e.g. a pressure can never be negative.

A usual case in which the pressure is known is at a liquid free surface, most commonly, the water-air surface. There, the pressure is the atmospheric pressure (which depends on the weather), and the liquid must have that same pressure (otherwise the interface is not in mechanical equilibrium). This means that the pressure increases linearly as we dive deeper into the water. The increase is given by $\rho g h$. If we write $\rho g =: p_0/L$, the law becomes

$$p = p_0 \left(1 + \frac{h}{L}\right),$$

so that the pressure increases two-fold at $h = L$, three-fold at $h = 2L$, etc. For water, this value is $L = p_0/(\rho g h) = 10.3\,\mathrm{m}$. This is a well-known result for scuba divers: expect an increase of one atmosphere every 10 m, approximately.

Of course, this simple result is only true for constant density. In actual fact, sea water has density variations, due not so much to compression, or temperature (which is quite constant, at about 4 °C, at which the density is greater), but due to salinity. Nevertheless, if we neglect those factors, we may get an estimation of the greatest pressure attained in oceans of our planet. At the end of the Marianas Trench, $h \approx 11\,\mathrm{km}$, and $p \approx 1060 p_0$, about 1000 atmospheres. This is a good match to the measured value of $1090 p_0$. A refined calculation involving the compressibility of water shows that at this pressure water is about 5% denser than at standard conditions, a measurable change if still small.

4.2 Compressible fluids

If compressibility is taken into account, there will be a relationship between pressure and density at least (in a more realistic picture, temperature also has to be included). We will limit our discussion to one of the most common examples: ideal gases. The expremelly well know relationship may be rewritten in order to include the mass density:

`{eq:ideal_gas_EOS}`

$$pV = nRT \qquad \Longrightarrow \qquad p = \rho \frac{RT}{m}, \tag{4.2}$$

where m is the molar gas. In many cases, the temperature is not constant, but we will assume it to be so for simplicity.

In this case,

$$\nabla p = \rho \mathbf{g} \qquad \Longrightarrow \qquad \frac{\partial p}{\partial z} = -g\rho.$$

Since gravity is parallel to the z axis (in our convention), p can only vary in this direction. The opposite is interesting: a horizontal variation in pressure can never reach equilibrium [7].

With our equation of state,

$$\frac{RT}{m}\frac{\partial \rho}{\partial z} = -g\rho,$$

or

$$\frac{\partial \rho}{\partial z} = -\frac{1}{L}\rho,$$

where the height scale is seen to be given by $L := RT/(gm)$. Since $RT/m = p_0/\rho_0$ at some known values of p_0 and ρ_0, we may find it more convenient to write $p_0/(g\rho_0)$. Of course, our known values are $p_0 = 1010\,\mathrm{hPa}$ and $\rho_0 = 1.22\,\mathrm{kg\,m^{-3}}$, standard air conditions. The result is $L = 8.47\,\mathrm{km}$. This tells us that air has constant density and pressure at human scales (at a given time, we know weather changes it), and variations are observed only at high altitudes. The solution to the equation above is

$$\rho = \rho_0 e^{-x/L}$$
$$p = p_0 e^{-x/L}$$

Even if our approximation of constant temperature is surely wrong, it is not too wrong for the troposphere, where these variations are not strong (remember this is absolute temperature, where a 30 °C change is still around a 10% variation.) We are also ignoring other factors, such as air humidity. In spite of this, we can boldly estimate the pressure and density at the top of Mount Everest to be about $\exp(-8.848/8.47) \approx 35\%$ lower than at sea level (it is quite a coincidence that L is so close to the elevation of this mountain.) Indeed, typical readings indicate pressures of about 400 Pa, with temperatures of -17 °C (not so very far below the standard value), and a relative humidity of 40%. At the highest point at which our approximation is valid, the tropopause, $z \approx 17\,\mathrm{km}$, and we find a reduction of about 13%.

It is noteworthy that for climbers it is the *oxygen* concentration which is important. Since the oxygen molecule has a lower molar weight that the air, the same calculation yields $L \approx 14.5\,\mathrm{km}$, and a O_2 concentration at the top of Mt. Everest of $\exp(-8/14.5) \approx 56\%$ the value at sea level. This is in rough agreement with the value of 40% that is often quoted, the different being probably due to the temperature.

Chapter 5

Sound waves

The existence of pressure and density waves, as solution of the Euler equations, is established in this section.

Our set of equations includes the Euler momentum equation, neglecting gravity

$$\frac{d\mathbf{u}}{\partial t} = -\frac{1}{\rho}\nabla p.$$

Continuity is also important: since we are expecting density waves, this quantity should not be considered to be constant. The full equation is therefore:

$$\nabla \cdot \rho \mathbf{u} + \frac{\partial \rho}{\partial t} = 0$$

Finally, in order to close the equations, we need some relationship between variations in the density and variations in the pressure. This comes from thermodynamics. Indeed, the reference pressure and density are tied by an equation of state

$$p = p(\rho, T).$$

For example, at about $T_0 = 300\,\mathrm{K}$, and a pressure of $p_0 = 1 \times 10^3\,\mathrm{hPa}$, we know $\rho_0 \approx 1.225\,\mathrm{kg}^3\,\mathrm{m}^{-1}$, since those are so-called standard conditions.

5.1 Linearization

Now, we will assume small fluctuations about constant backgrounds. I.e:

$$\mathbf{u} \approx \mathbf{u} \tag{5.1}$$
`{eq:sound_small_u}`

$$\rho \approx \rho_0 + \rho' \tag{5.2}$$
`{eq:sound_small_rho}`

$$p \approx p_0 + p' \approx p_0 + \kappa \rho' \tag{5.3}$$
`{eq:sound_small_p}`

Equation 5.1 means the velocity is to be considered small in some sense, but there is no average drift on top of which our waves may travel. Such a drift may be present if there are currents (water) or winds (air), and it is instructive to derive its effect. We will neglect it here.

In Equation 5.3 implies that a small increase in density should cause a small increase in the pressure. It is hard to assume otherwise, except when close to a phase transition. Mathematically, this is a Taylor expansion:

$$p \approx p_0 + \left.\frac{dp}{d\rho}\right|_{\rho_0} (\rho - \rho_0) \qquad \Longrightarrow \qquad p' \approx \left.\frac{dp}{d\rho}\right|_{\rho_0} \rho'.$$

The proportionality is given by κ, defined as

{eq:sound_kappa}

$$\kappa = \left.\frac{dp}{d\rho}\right|_{\rho_0} \tag{5.4}$$

Then, neglecting second and higher order perturbation terms we may obtain linearized equations, involving only the velocity and the pressure:

{eq:sound_lin_cont}

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho_0}\nabla p' \tag{5.5}$$

$$\rho_0 \kappa \nabla \cdot \text{æ}\mathbf{u} + \frac{\partial p'}{\partial t} = 0 \tag{5.6}$$

The kinematic viscosity is defined as $\nu = \mu/\rho_0$. Notice the convective part of the material derivatives is neglected, since it is of higher order.

If we differentiate with respect to space (by applying ∇) in the first equation, and with respect to time in the second, we can substract the two, and eliminate the velocity term. The following equation for the pressure results:

$$\frac{\partial^2 p'}{\partial t^2} = \kappa \nabla^2 p',$$

which is the wave equation. Its solutions are waves, and their speed, the speed of sound, is $c^2 = \kappa$. Our first impulse should be to check whether our prediction of the speed of sound matches the known value for some standard situations. By far, the best known situation is the speed of sound in air in standar conditions. Since air is, to a good approximation, a perfect gas, and Eq. 4.2 applies.

Then,

$$\kappa = \left.\frac{dp}{d\rho}\right|_{\rho_0} = \frac{RT_0}{m} = \frac{p_0}{\rho_0}.$$

The last equation saves us from checking up the value of R and the molar mass of air (an effective value, since the air is a mixture of gases.)

Our numerical result would then be

$$c = \sqrt{\frac{p_0}{\rho_0}} = \sqrt{\frac{1 \times 10^3\,\mathrm{hPa}}{1.2\,\mathrm{kg\,m^{-3}}}} = 290\,\mathrm{m\,s^{-1}}.$$

This number, while not terribly wrong, is not equal to the speed of sound in standard conditions, which is about $343\,\mathrm{m\,s^{-1}}$. (A mnemonic rule is that it takes about three seconds for the air to cover one kilometer — hence the practice of counting the time in seconds between lighning flash and thunder sound, and divide by three to estimate the distance to a lightning in a storm.)

The mistake implicit in our calculations, which was present in the original derivation by Isaac Newton, was later corrected as our understanding in thermodynamics increased throughout the XIX century. The error is to perform the derivative $dp'/d\rho'$ at fixed temperature. We know now that the speed of sound sets the limit at which a thermodynamic process may be considered isothermal. For sound waves, the process is better modeled as adiabatic.

In adiabatic processes, each fluid particle is unable to tranfer energy to its surroundings, since the wave phenomenon is too fast. Without going into details (which may be found at any standard book on thermodynamics), the well-known law for an adiabatic process is

$$pV^{\gamma} = \mathrm{ct},$$

while in our previous, isothermal derivation, we had assumed $pV = \mathrm{ct}$. The exponent γ is the famous adiabatic exponent, which depends on the ideal gas being monoatomic or diatomic (in the standard temperature range, at least). The air is really neither, being a mixture, but most of its components (N_2 and O_2) are diatomic, so a good approximation to γ is 7/5, the diatomic value.

We can manipulate the adiabatic relationship this way:

$$pV^{\gamma} = \mathrm{ct} \qquad \Longrightarrow \qquad pV^{\gamma} = \mathrm{ct}_2 m^{\gamma} \qquad \Longrightarrow \qquad p = \mathrm{ct}_2 \rho^{\gamma},$$

where ct_2 is some other constant, and we introduce m^{γ} because the mass of a particle, m, is constant (as we will ellaborate again later, see e.g. Section 9.1.1.) By the shortcut of applying logarithms we readily find our κ:

$$\log p = \mathrm{ct}_3 + \gamma \log \rho \qquad \Longrightarrow \qquad \frac{1}{p}\frac{dp}{d\rho} = \gamma \frac{1}{\rho},$$

i.e.

$$\kappa = \gamma \frac{p_0}{\rho_0}.$$

Our new numerical result is the same as before, but with an extra $\sqrt{7/5}$ factor, a 14% increase approximately. Finally:

$$c = \sqrt{\frac{\gamma p_0}{\rho_0}} = \sqrt{\frac{(7/5) 1 \times 10^3\,\mathrm{hPa}}{1.2\,\mathrm{kg\,m^{-3}}}} = 330\,\mathrm{m\,s^{-1}},$$

a value much closer to the real value of $343\,\mathrm{m\,s^{-1}}$.

Let us finish by analyzing a simple sound wave. Our wave is harmonic: it has a well defined frequency f, and therefore angular frequency $\omega = 2\pi f$. Of course, the word "harmonic" stems from our knowlodge of sound waves, hence "harmonic analysis", and so on (but, the equivalent concept in optics is "monochromatic".) These sort of waves sound "pure" (also, quite dull) and may be obtained from tuning forks or, in our days, synthetisers. It is also planar, since it depends only on x — no such waves exist, but this is a rather good approximation for small regions far away from the sound source. Its mathematical expression is

$$\rho = \rho_0 + \rho_A \cos[k(x - \omega/kt)] = \rho_0 + \rho_A \cos(kx - \omega t).$$

It is traveling since, as shown in the first equality, it is a function of the form $f(x - ct)$. Also, $c = \omega/k$, which fixes the relationship $\omega = ck$. Since ω, the angular frequency, is proportional to k, the wave number, this wave is non-dispersive: the speed of sound is the same for all wave numbers. The wave number is related to the wave-length, $k = 2\pi/\lambda$, therefore $f\lambda = c$. The relationship between the length of a wave and its frequency is of course very relevant for musical instrument makers and architecture of buildings with musical interest (concert halls, recording studios.)

The amplitude of the density modulation is ρ_A, and $\rho_A \ll \rho_0$ since modulations are small.

The corresponding pressure modulation is

$$p = p_0 + \rho_A c^2 \cos(kx - \omega t).$$

The number $\rho_A c^2$ is much "larger" than ρ_A, but that should not bother us: a large density modulation of, say $\rho_A = 0.1\,\mathrm{kg}^3\,\mathrm{m}^{-1}$ (which means a modulation of the air density of approximately 10% about its unperturbed value) translates into about 140 hPa, which is a 14% pressure modulation, more or less.

Finally, the velocity may be recovered from 5.5. The solution to it is

$$\mathbf{u} = u_0 \cos(kx - \omega t)\mathbf{e}_x.$$

It is therefore a purely longitudinal wave, since the particles move in the direction of wave propagation, x (at variance with planar light waves in vacuum, which is purely transversal).

The value of u_0 is set by the requirement that $\mathbf{u}$ is indeed a solution:

$$\omega\rho_A - \rho_0 u_0 k = 0 \qquad \Longrightarrow \qquad u_0 = \frac{\rho_A}{\rho_0} c.$$

The velocity of the particles is therefore related to the speed of sound c, but lower than it, since the ratio ρ_A/ρ_0 (of about 0.1 in the example above) must be small for our approximations to remain valid. Also, all three fields, pressure, density, and velocity are in phase. The speed is moving to the right, and the zones with high pressure and density move to the right, while the ones with low pressure and density move to the left. If the waved moved to the left, the sign of u_0 would change, and the situation would be the opposite.

5.2 Mach's number and compressibility

Chapter 6

Potential flow

If the flow is initially irrotational, it will remain so due to Lagrange's theorem. This means it must be the gradient of a scalar function, which is called the velocity potential:

$$\mathbf{u} = \nabla \phi.$$

This is similar to the relationship between electric field and electric potential (or the gravitational equivalent), with the important practical difference that no minus sign is used. This means that the velocity points from regions of low potential to regions of high potential, in the direction of steepest ascent.

If the flow is also incompressible, $\nabla \cdot \mathbf{u}$, it follows that the potential must be a solution of Laplace's equation:

$$\nabla^2 \phi = 0.$$

6.1 Flow past a cylinder

Let us consider the 2D flow past a cylinder. The velocity field must approach a constant value far away from the obstacle, which we will take as the horizontal direction: $u_x = u_0$. The traverse direction is y, and there is no dependence on z. This value fixes one of the boundary conditions for the potential:

$$\phi \to u_0 x \qquad \text{away from cylinder,}$$

plus a non-important constant, which we will take as 0, fixing ϕ to be zero at the origin (which coincides with the cylinder axis.)

The other boundary condition is related to the presence of the cylinder. The least that the velocity field must satisfy is that the flow does not trespass the surface of the cylinder. This means its normal component should vanish there:

$$\mathbf{u} \cdot \mathbf{n} = 0 \qquad \text{at the cylinder,}$$

where $\mathbf{n}$ is the normal vector at the surface of the cylinder, pointing outside. Given our choice of origin, this vector is the unit radial vector, $\mathbf{n} = \hat{\mathbf{e}}_r$.

It turns out this condition is the *only one* needed to complete the problem — in the positive sense that we will be able to find explicit solutions to the problem, but also in the negative sense that more physical boundary conditions cannot be accomodated. In particular, it is not possible to impose the often-used "no-slip" boundary condition, in which the velocity would be zero at the surface (not only its normal component, but its tangential components too).

Given the latter condition is somewhat more complex than the first one, it is best to switch to polar coordinates. In this system, the gradient is:

$$u_r = \frac{\partial \phi}{\partial r} \tag{6.1}$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \tag{6.2}$$

$$\tag{6.3}$$

Our boundary conditions for the potential are

$$\phi(r \to \infty) = u_0 r \cos(\theta) \tag{6.4}$$

$$\left.\frac{\partial \phi}{\partial r}\right|_{r=R} = 0. \tag{6.5}$$

Let us try a function

$$\phi = u_0 r \cos(\theta) + f(r) g(\theta).$$

The boundary conditions imply that $g(r)$ should vanish as r gets large. Also, the angular function g must clearly be $\cos\theta$. Otherwise, it will be unable to cancel the $u_0 r\cos(\theta)$ term. Lastly, it must itself be a solution of the Laplace equation. With the Laplacian operator in polar coordinates:

$$\nabla^2(f\cos(\theta)) = \frac{\cos(\theta)}{r}\frac{\partial r f}{\partial r} - \frac{f\cos(\theta)}{r^2} = 0$$

This equation has solutions $f = r$ (which we already knew), and $f = 1/r$. Of course, a prefactor may be added, so our guess is now

{eq:pot_phi_guess}

$$\phi = \left(u_0 r + \frac{A}{r}\right)\cos(\theta). \tag{6.6}$$

Now, in order its radial derivative vanish at $r = R$,

$$\left(u_0 - \frac{A}{R^2}\right)\cos(\theta) = 0,$$

which gives $A = u_0 R^2$ The solution can then be written as

$$\phi = u_0 r\left[1 + \left(\frac{R}{r}\right)^2\right]\cos(\theta).$$

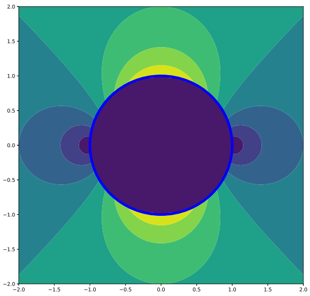

Figure 6.1: Potential flow past a cylinder. Velocity magnitude contours, u/u_0.

6.1.1 Velocity field

The velocity results immediately from this expression. From the expressions above:

$$u_r = u_0 \left[1 - \left(\frac{R}{r}\right)^2\right] \cos(\theta) \tag{6.7}$$

$$u_\theta = -u_0 \left[1 + \left(\frac{R}{r}\right)^2\right] \sin(\theta). \tag{6.8}$$

Notice this field has a clear up-down, and left-right symmetry, which was already present in the potential. Also, there are two points, at $r = R$ and $\theta = 0$ and π in which the velocity is null. We said it was impossible to make the velocity equal to zero all over the surface of the cylinder, but it turns out it may be so at some points. These are call stagnation points, because the liquid particles may be traped there for a long time.

Also, the velocity modulus is

$$u = \sqrt{u_r^2 + u_\theta^2},$$

which may be readily computed as

$$u = u_0^2 \left[1 + \left(\frac{r}{R}\right)^4 - 2\left(\frac{r}{R}\right)^2 \cos(2\theta),\right]$$

where we have used $\cos(2\theta) = \cos^2(\theta) - \sin^2(2\theta)$. In Fig. 6.1 it is shown that the velocity has maximum values at point at the surface of the cylinder, at $\theta = \pm\pi/2$. At these surface, the radial velocity vanishes and the velocity is readily obtained:

$$u = |u_\theta| = 2u_0 \sin(\theta),$$

so the maximum speed is twice the current velocity.

The pressure field can be obtained from Bernoulli's principle:

$$p = p_0 + \frac{\rho}{2}\left(u_0^2 - u^2\right),$$

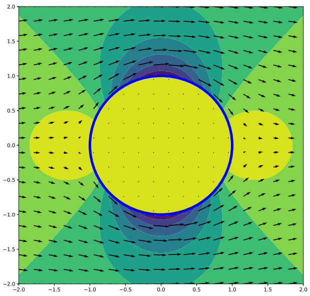

Figure 6.2: Potential flow past a cylinder. Velocity field, $\mathbf{u}/u_0$, and pressure field $(p - p_0)/(\rho u_0^2)$.

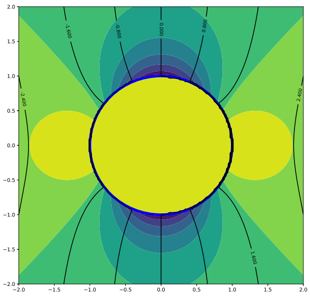

Figure 6.3: Potential flow past a cylinder. Pressure field $(p - p_0)/(\rho u_0^2)$ and potential $\phi/(u_0 R)$.

so it basically resembles the velocity modulus above, but for a non-important constant shift, and multiplication by $-\rho/2$. Isobars are plotted in Fig. 6.2, together with the velocity field. It is clear that the velocity field does not simply “goes from high to low pressures”. To make this point clearer, Figure 6.3 shows isobars and contours of the potential field, to stress the difference between the two, except in those areas in which the two fields happen to have similar gradients.

The pressure field has a striking left-right symmetry. This of course applied to its value at the surface. Since the pressure is the only force this fluid may exert upon the cylinder, the consequence is that the net force on it is exactly zero. The push it experiences towards the right is exactly cancelled by the push back towards the left. Therefore, the drag, which in this case is the net horizontal force, is zero.

This striking result is known as d’Alembert’s paradox, and marked a historical chasm between theoretical fluid mechanics and applied hydraulics. Indeed, the applied community could accept that the push back may exist. Indeed, it had been hypothesized to be solely responsible for objects moving

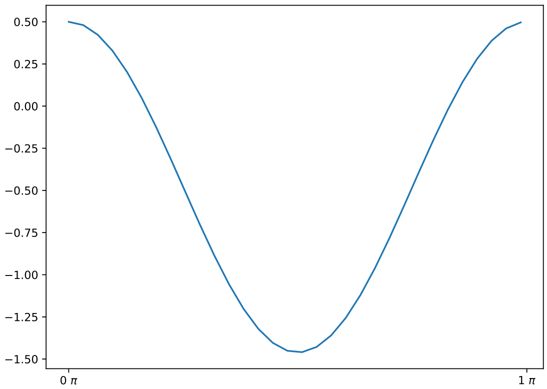

Figure 6.4: Potential flow past a cylinder. Pressure on cylinder surface $(p - p_0)/(\rho u_0^2)$. Position 0 corresponds to the "stern" of the cylinder, position π to the "bow".

even when not given a force in Aristotelian physics. The effect is real, and is used regularly by cyclist that may take advantage from it, by staying close to the back of a moving vehicle. However, it is not acceptable to accept that it should be exactly equal to the drag.

We now know that the crucial ingredient is the viscosity, which manifests mathematically in a different boundary condition, such as no-slip. This way, a real cylinder is dragged with the flow. We will show later that it is easy to obtain the drag on a sphere in the limit of high viscosity (a cylinder is not solvable due to another paradox, as we will see.)

In case some mathematical justification is required, the pressure at the surface on the cylinder is

$$p = p_0 + \rho u_0^2 \left[\frac{1}{2} - \cos(2\theta) \right], \tag{6.9}$$ {eq:press_at_cyl_pot}

as plotted in Figure.

Now, the drag is given by the net force in the x direction. The (vector) force due to pressure is:

$$\mathbf{F} = \int_0^H \int_0^{2\pi} p(\theta) \mathbf{n} R d\theta dz,$$

where $p(\theta)\mathbf{n}\hat{\mathbf{e}}_x R d\theta dz$ is the pressure force on a differential surface of area $R d\theta dz$. Now, the drag will be its projection on the horizontal axis:

$$D = \mathbf{F} \cdot \hat{\mathbf{e}}_x = \int_0^H \int_0^{2\pi} p(\theta) \cos(\theta) R d\theta dz.$$

The integral is zero, since it consists of two terms. The first one involves constant term. Of course, a constant pressure exerts no net force, and mathematically:

$$\int_0^{2\pi} C\cos(\theta)d\theta = C\int_0^{2\pi}\cos(\theta)d\theta = 0.$$

The other term involves a $\cos(2\theta)$ term. Now, this integral is also zero:

$$\int_0^{2\pi}\cos(\theta)\cos(2\theta)d\theta = 0.$$

The reason is that cosines form an orthogonal basis: an integral of two different cosines over one period will be zero — unless they are the same cosine, when the integral is π. To be precise, for integer m and n:

$$\frac{1}{2\pi}\int_0^{2\pi}\cos(m\theta)\cos(n\theta)d\theta = \begin{cases} 0 & \text{if } m \neq n \\ \frac{1}{2} & \text{if } m = n \end{cases} .$$

(It is more elegant to divide by 2π to express the integral as a mean value.) In fact, the previous result about constant pressure is a particular case in which $m = 0$.

It is also interesting that similar integrals involving a sine and a cosine are always zero (i.e. sines are cosines are orthogonal.) This shows that the lift is always null:

$$L = \mathbf{F}\cdot\hat{\mathbf{e}}_x = \int_0^H\int_0^{2\pi} p(\theta)\sin(\theta)Rd\theta dz = 0$$

6.1.2 Streamlines

For steady 2D flow, it is very convenient to introduce the stream function ψ. It is defined such that its contour lines produce the streamlines , i.e. the lines the fluid particles trace as they move. Upon some reflection, this means that its gradient should always be perpendicular to the velocity:

{eq:stream_perp_to_u}

$$\mathbf{u}\cdot\nabla\psi = 0. \tag{6.10}$$

In steady flow, this means $d\psi/dt = 0$, so the value of ψ is carried with the flow. This is at variance with the potential, whose gradient is parallel to the velocity (it is indeed, the velocity itself). Thus ϕ and ψ are orthogonal functions, in the sense that their gradients are.

The streamlines are important to visualize flows, in much the same way as Faraday's field lines are for electromagnetic fields (mathematically, they are equivalent). In computational fluid dynamics, they are customarilly computed and plotted, by "seeding" points and integrating their motion following the velocity field. A fine point is that for unsteady flow, these streamlines are in general different from the actual trajectories of fluid particles, which are called "pathlines" .

For steady flow, it is often convenient to define a vector potential in much the same way as it is done for the magnetic field in electromagnetism:

{eq:vector_potential}

$$\mathbf{u} = \nabla\times\mathbf{A}. \tag{6.11}$$

The resulting velocity field will always be incompressible, since the divergence of a curl is zero.

In the case of 2D, the choice for this vector potential is

{eq:A_psi_2D}

$$\mathbf{A} = \psi(x, y)\hat{\mathbf{e}}_z, \tag{6.12}$$

perpendicular to the plane, but dependent on the in-plane coordinates. With this expression,

$$\begin{aligned} u_x &= \frac{\partial \psi}{\partial y} \\ u_y &= -\frac{\partial \psi}{\partial x}, \end{aligned}$$

and it is easy to check that this stream function indeed traces stream-lines, since it complies with 6.10.

In cylindrical coordinates [18],

$$\begin{aligned} u_r &= \frac{1}{r}\frac{\partial \psi}{\partial \theta} \\ u_\theta &= -\frac{\partial \psi}{\partial r}. \end{aligned}$$

6.1.3 Streamlines around the cylinder

With the latter expression for the stream function, and from our results for the velocity field, the ψ field is found to be

$$\psi = u_0 \left(r - \frac{R^2}{r} \right) \sin\theta. \tag{6.13}$$

{eq:streamlines_past_cylinder}

This method, of course, is working backwards from the known solution. It is also possible to derive the velocity field from the fact that ψ satisfies Laplace's equation, plus the appropriate boundary conditions, in a manner very similar to what has been done for the potential (see Exercise 6.2.2).

The streamlines, which are the contour lines of this field are shown in Fig. 6.5.

6.1.4 Potential function, stream function and complex analysis

Interestingly, the ψ field is also a solution of Laplace's equation, $\nabla^2\psi = 0$, as can be checked explicitely — but also from the identity for the curl of the curl [17]:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}. \tag{6.14}$$

{eq:curl_of_curl}

Then, since the velocity field is curl-free:

$$0 = \nabla \times \mathbf{u} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = -\hat{\mathbf{e}}_z \nabla^2 \psi, \tag{6.15}$$

{eq:if_no_curl_then_stream_har

where we use the fact that the $\mathbf{A}$ field of 6.12 is divergence-free.

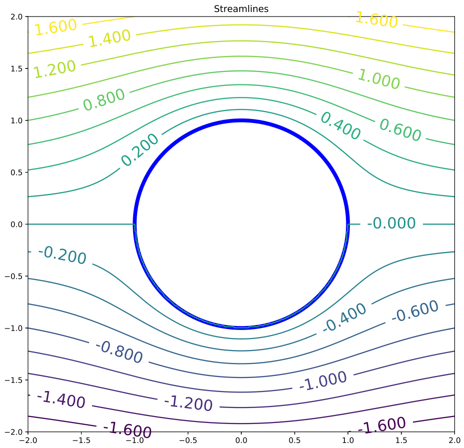

Figure 6.5: Potential flow past cylinder, streamlines as contour lines of function $\psi/(u_0 R)$.

There is also the intriguing fact that we may write

$$\phi = u_0 \Re \left(z + \frac{R^2}{z} \right),$$

where $z = re^{i\theta}$ is the complex number related to the r and θ coordinates. Then,

$$\psi = u_0 \Im \left(z + \frac{R^2}{z} \right).$$

I.e. both fields are the real and imaginary part of the same complex function:

$$f(z) = u_0 \left(z + \frac{R^2}{z} \right)$$

This is no coincidence, but a consequence of Cauchy-Riemann equations: the real and imaginary part of a complex function are harmonic (i.e. they satisfy Laplace's equation) and orthogonal. To be more precise, the complex function must be analytic. The later requirement is pretty general: it means the function must be single-valued and and must have a derivative everywhere. The reader may worry about the origin and the $1/z$ function. This is, however, outside our domain, which ends when $r > R$. However, the fact that our domain has this "hole" in it has some consequences, as we will see next.

6.1.5 Circulation and lift

It turns out that, despite our previous claims of having found the solution to the problem, this is not a unique solution[1]. The most general one is given by

$$f(z) = u_0\left(z + \frac{R^2}{z}\right) + \frac{i\Gamma}{2\pi}\log z \qquad \phi = \Re(f) \quad \psi = \Im(f).$$

I.e. we may add an additional term which is also analytic. The $i\Gamma/(2\pi)$ factor is chosen for convenience, as will become clear. Both ϕ and ψ are still harmonic and orthogonal, since $\log(z)$ is analytic. Notice $i\log(z) = i\log(r) - \theta$, so ϕ contains a term that is just the polar angle, while ψ will include a $\log(r)$ term.

The resulting velocity field still complies with the boundary conditions, since the resulting velocities:

$$u_r = u_0\left[1 - \left(\frac{R}{r}\right)^2\right]\cos(\theta) \tag{6.16}$$

$$u_\theta = u_0\left[1 + \left(\frac{R}{r}\right)^2\right]\sin(\theta) + \frac{\Gamma}{2\pi r} \tag{6.17}$$

{eq:u_theta_cyl_pot_circ}

have an additional term which dies away far from the cylinder, while the no-trespass condition at the surface is still respected (since the radial velocity does not change at all).

The name of Γ is "circulation", since

$$\Gamma = \oint_C \mathbf{u}\cdot d\mathbf{l}$$

for any contour around the cylinder [2].

At the surface of the cylinder,

$$u = u_\theta = u_0 2\sin(\theta) + \frac{\Gamma}{2\pi R}.$$

The stagnation points then move away from their positions, to points given by

$$\sin(\theta_{\mathrm{st}}) = \frac{\Gamma}{4\pi R u_0}.$$

This has two solutions, of course, as long as the right-hand side is between -1 and 1. For values below or above, the points coalesce into a single stagnation point that is outside the cylinder.

This new velocity field does not solve d'Alambert's paradox, but it does provide a lift, since the pressure at the surface is

$$p = p_0 + \frac{\rho}{2}\left(u_0^2 - \left[2u_0\sin(\theta) - \frac{\Gamma}{2\pi R}\right]^2\right) = \ldots 2\rho u_0\frac{\Gamma}{2\pi R}\sin\theta,$$

---

[1] This is because our domain has a hole in it. Technically, it is not simply-connected. This means there is an additional parameter to fix the most general solution. This is called the "winding number", and is basically this section's Γ.

[2] This makes the velocity field non-conservative, which is again allowed due to the domain being not simply-connected.

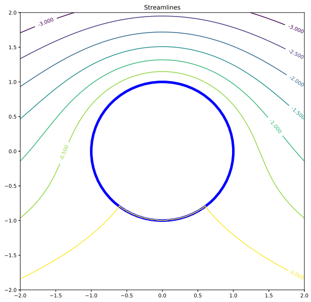

Figure 6.6:

where we single out the only term that can produce a contribution to the lift.

Now,

$$L = \mathbf{F} \cdot \hat{\mathbf{e}}_x = \int_0^H \int_0^{2\pi} p(\theta) \sin(\theta) R d\theta dz = 2L\rho u_0 R \frac{\Gamma}{2\pi R} \int_0^{2\pi} \sin^2(\theta) d\theta = H\rho u_0 \Gamma. \tag{6.18}$$

`{eq:lift_potential_c`

Therefore, the lift per unit length is $L/H = \rho u_0 \Gamma$. It is interesting that heavier fluids, high speeds and high Γ produce higher lift forces.

It turns out that this result for the lift is quite general, at least for these kind of potential flows. It is known as the Kutta-Joukowski Lift Theorem, and has been used for decades in aeronautics. The idea is to map our solution for a cylinder to another, more wing-like, shape, by some conformal transformations. These transformations, or "mappings", preserve the harmonicity of functions — hence the transformed ϕ and ψ will still be solutions of Laplace's equation. Historically, the best known mappings are the Joukowsky Kármán–Trefftz transforms.

For a plane wing, it makes sense to relate circulation and lift, since this captures, in a way, the fact that the velocity is higher at the top of the wing, lower below, so the net circulation is non zero. (Of course, this cannot hold right at the surface of an actual wing, since the velocity there will be zero. But this can be measured somewhat further away, beyond a boundary layer.) For a cylinder, this makes little sense, but this picture is sometimes used to explain the Magnus effect. This causes a rotating cylinder to bend as it moves across a fluid. This is used (for spheres) in sports, such as tennis, golf and football. Also, for cylinder-like rotors in "rotor ships," which use the Magnus effect for propulsion.

We may obtain an expression for the Magnus force by equating the circulation term $\Gamma/(2\pi R)$ for the tangential velocity of the fluid at the surface of the cylinder (Eq. 6.17) with the cylinder surface velocity:

$$\frac{\Gamma}{2\pi R} = \omega_0 R$$

This is a sort of "no-slip" condition — but, incomplete, since the correct condition would be the whole tangential velocity being equal to the cylinder surface

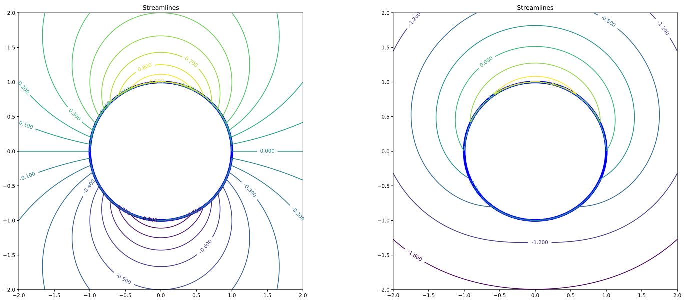

Figure 6.7: Potential flow past cylinder, streamlines as seen from the fluid, contour lines of function $(\psi - u_0 r \sin\theta)/(u_0 R)$.

velocity. This requirement is, again, too strong for an ideal fluid, and is only met when viscosity is introduced.

The expression for the Magnus force is then,

$$\frac{L}{H} = 2\pi\rho u_0 \omega_0 R^2,$$

proportional to both the wind speed u_0 and the angular velocity of the spinning cylinders, ω_0 (of course, also proportional to their length). Since the force is perpendicular to the wind, the Magnus effect works best for wind coming from the sides ("abeam"). Also, this is the "apparent wind": the wind at the cylinders is the vector sum of the true wind and the velocity of the ship. The effect of the latter is to bring the apparent wind closer to the fore.

As an aside, if we move with the flow velocity $\mathbf{u}_0$, we will *not* measure those velocity fields. The far-way term should be avoided in every equation, and the resulting velocity looks different. This is shown in Fig. 6.7, for the flow past a cylinder with and without circulation. It is seen that, for an observer moving with the flow, a cylinder moving to the left pushes the surrounding fluid from the fore, which then moves around towards the aft (where it pushes our vehicle, magically making our journey a cost-less one!).

6.2 Flow past a sphere

With the same assumption of potential flow as in the cylinder, the axisymmetric flow past a sphere may be tackled by using the potential, which should satisfy Laplace's equation, plus these boundary conditions:

$$\phi(r \to \infty) \to u_0 z = u_0 r \cos\theta \qquad u_r(r = R) = \left.\frac{\partial \phi}{\partial r}\right|_{r=R} = 0.$$

Spherical coordinates are used: r is the distance to the origin, θ is the polar angle (angle with the z axis), and the azimuthal angle (angle around the z axis), here absent due to symmetry, is ϕ [3].

---

[3] See 1.1 for naming conventions is spherical coordinates.

Similarly to the cylinder, let us try a function

$$\phi = (u_0 r + f(r))\cos(\theta).$$

By using the expression of the Laplacian in spherical coordinates [18], we soon come to the conclusion that $f(r) = A/r$ in order Laplace's equation be satisfied. This function also vanishes as r gets large, as it should. With the no-trespassing condition for the velocity the value of A may be found out.

The final result is

$$\phi(r,z) = u_0 r \left[1 + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]\cos\theta.$$

From its gradient we get the velocity field:

$$u_r = u_0 \left[1 - \left(\frac{R}{r}\right)^3\right]\cos\theta$$

$$u_\theta = -u_0 \left[1 + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]\sin\theta$$

6.2.1 Streamlines past a sphere

For axisymmetric flows, streamlines may be defined on a plane, since the resulting flow does not depend on the azimuthal angle (the angle around the axis of symmetry, which is z). However, these are not 2D flows, and the equations are more involved.

In this case, the velocity field depends only on the other two coordinates. In spherical coordinates,

$$\mathbf{u} = \mathbf{u}(r,\theta).$$

The vector potential may still be thought of as "perpendicular" to the other two coordinates. In spherical coordinates:

$$\mathbf{A} = A(r,\theta)\hat{\mathbf{e}}_\phi,$$

i.e. it is purely azimuthal. It is also divergence-free, so that is satisfies Laplace's equation for a curl-free flow, by the argument exactly as in Eq. 6.15.

However, the simple choice of $A = \psi$ will *not* result in the correct streamline behavior of Eq. 6.10. We must recall that the contour lines of the streamline function correspond to the velocity stream lines, as encapsulated in the requirement $\mathbf{u}\cdot\nabla\psi = 0$.

By writing the gradient of ψ in spherical coordinates,

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\mathbf{e}}_\theta,$$

and the velocity as the curl of the vector potential in spherical coordinates,

`{eq:u_from_A_spherical}`

$$u_r = (\nabla\times A)_r = \frac{1}{r\sin\theta}\frac{\partial\,(A\sin\theta)}{\partial\theta} \tag{6.19}$$

$$u_\theta = (\nabla\times A)_\theta = -\frac{1}{r}\frac{\partial(rA)}{\partial r}, \tag{6.20}$$

the scalar product of both is

$$\begin{aligned}\mathbf{u}\cdot\nabla\psi =& \frac{1}{r\sin\theta}\frac{\partial\left(A\sin\theta\right)}{\partial\theta}\frac{\partial\psi}{\partial r}- \\ &-\frac{1}{r^2}\frac{\partial(rA)}{\partial r}\frac{\partial\psi}{\partial\theta}.\end{aligned}$$

This is satisfied with the choice

$$\mathbf{A}=\frac{\psi(r,\theta)}{r\sin\theta}\hat{\mathbf{e}}_\phi. \tag{6.21}$$

{eq:Stokes_stream_spherical}

This ψ is called "Stokes stream function", since there are differences with the 2D stream function (traditionally called "Lagrange stream function".)

The curl expressions 6.19 in terms of **A** translate to more symmetrical versions in terms of this ψ:

$$\begin{aligned}u_r &= \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \\ u_\theta &= -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}.\end{aligned} \tag{6.22}$$

{eq:u_from_psi_spherical}

Note that there may be an additional azimuthal u_ϕ velocity component which cannot be obtained from ψ (see Ex. 6.2.2).

For completeness, in cylindrical coordinates the appropriate expression is

$$\mathbf{A}=\frac{\psi(\rho,z)}{\rho}\hat{\mathbf{e}}_\phi,$$

where ρ is the distance to the z axis. Notice this is the same expression as in spherical coordinates, since $\rho=r\sin\theta$. There is a reason for this, as explained in Ref. [1] p. 78.

The resulting velocity field is

$$\begin{aligned}u_\rho &= -\frac{1}{\rho}\frac{\partial\psi}{\partial z} \\ u_z &= \frac{1}{\rho}\frac{\partial\psi}{\partial\rho}\end{aligned}$$

For the flow around a sphere, the resulting stream function is then found to be

$$\psi(r,\theta)=\frac{1}{2}u_0r^2\left[1-\left(\frac{R}{r}\right)^3\right]\sin^2\theta. \tag{6.23}$$

{eq:potential_sphere_stream}

Again, this is found by working backwards from the known solution. See Exercise 6.2.2) to derive the velocity field from ψ.

In Fig. 6.8 these streamlines are plotted. The corresponding streamlines for the flow as seen from the sphere are in Fig. 6.9.

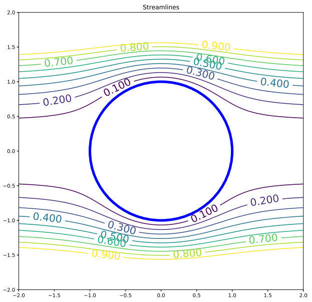

Figure 6.8: Streamlines of the potential flow past sphere

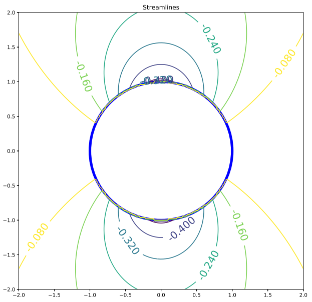

Figure 6.9:

6.2.2 Exercises

- The fact that $\nabla \cdot \mathbf{u} = 0$ implies $\mathbf{u} = \nabla \times A$. If the flow is, moreover, irrotational, $\nabla \times \mathbf{u} = 0$, then the vector potential satisfies Laplace's equation, $\nabla^2 \mathbf{A} = 0$, given identity 6.14. Notice that $\mathbf{A}$ is divergence-free by construction, since it depends on two spherical coordinates, r and θ, while pointing in the other one, ϕ. Solving the Laplace equation for this field, taking into account the appropriate boundary conditions.

- Derive the velocity field for potential flow around a sphere from ψ, solving Laplace equation for this field with the appropriate boundary conditions. Use the the following identity for the Laplacian:

$$\nabla^2 \left(\frac{\psi(r,\theta)}{r\sin\theta} \hat{\mathbf{e}}_\phi \right) = \frac{1}{r\sin\theta} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \left[\frac{\partial^2}{\partial \theta^2} - \cot(\theta) \frac{\partial}{\partial \theta} \right] \right) \psi \hat{\mathbf{e}}_\phi. \qquad (6.24)$$

{eq:psi_eq_from_A}

- Prove the previous identity. Notice that $\mathbf{A}$ is a vector field, so one need to apply the expression for the vector Laplacian of e.g. [18]. The expressions are not so bad since the field is azimuthal only, and dependent on the other two components: $\mathbf{A} = A(r,\theta)\hat{\mathbf{e}}_\phi$.

- Repeat the previous two calculation using cylindrical coordinates. Prove that

$$\nabla^2 \left(\frac{\psi(\rho,z)}{\rho} \hat{\mathbf{e}}_\phi \right) = \frac{1}{\rho} \left(\frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} \right) \psi \hat{\mathbf{e}}_\phi. \qquad (6.25)$$

{eq:psi_eq_from_A_cyl}

- Prove that the velocity may indeed have an azimuthal component for axisymmetric flow. Begin by writing down the expressions for the divergence and the curl in spherical coordinates, and neglecting all variation on the azimuthal angle, allowing for the possibility that the vector potential may have all three components, not just than A_ϕ. This result in a u_ϕ velocity component that can have basically any expression as far as continuity is concerned, since it will always be satisfied.

Chapter 7

Gravity waves

> [Water waves] that are easily seen by everyone and which are usually used as an example of waves in elementary courses [...] are the worst possible example [...]; they have all the complications that waves can have. (Richard Feynman, *The Feynman Lectures on Physics*)

7.1 Gravity waves

Large waves, like those at the ocean, are driven by gravity. An initial perturbation on the water surface, like an elevation, will cause a rise in gravitational potenntial. As this energy is transferred to kinetic energy, the water around it moves. Similarly to a pendulum, an area below the mean surface will develop, which will be filled with surrounding liquid. Thus, the perturbation travels away from the initial disturbance.

The mathematical treatment of this problem is, in general, very involved. This is mostly due to the presence of a free surface: the water-air interface. Mathematically, its location is described by a height function η:

$$z = \eta \qquad \eta = \eta(x, y, t), \tag{7.1}$$

{eq:waves_elevation}

which is valid if no overhangs are present (this is known as a "Monge representation"). The fluid velocity is to be solved for a domain between this surface at the ocean bottom, which is supposed to be flat and at a height $z = -h$ (thus, $z = 0$ is taken to be the surface when no waves occur.)

We have the following boundary conditions for the velocity:

$$u_z(z = -h) = 0 \tag{7.2}$$

{eq:waves_bc}

$$u_z(z = 0) = \frac{\partial \eta}{\partial t} \tag{7.3}$$

{eq:waves_bc2}

The first one is the usual condition at a solid wall, and the last one is called the "kinematic condition", expressing the fact that if the surface moves up or down, the fluid must do the same in order to be "stuck to it".

We will assume an incompressible, irrotational fluid. The fist assumption is a very reasonable one, and the second one turns out to be not so bad, since for small waves there is little vorticity creation (it mainly occurs at the bottom and on breaking waves). The assumption of negligible viscosity is likewise not so bad in this case.

Let us assume a wave train that only depends on the x component (i.e. waves are very long in the y direction):

`{eq:wave_on_x}`

$$\eta = a\cos(kx - \omega t). \tag{7.4}$$

The amplitude a, is taken to be small (in the sense $a \ll \lambda$, and $a \ll h$).

Given the assumptions above, the flow may be treated as a potential flow, and moreover the only relevant coordinates will be x and z. However, the fields will be time-dependent in this problem, e.g. $\phi = \phi(x, z, t)$.

The kinematic condition implies

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=\eta} = \frac{\partial\eta}{\partial t}.$$

This is still a difficult expression, so we will approximate it by

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=0} = \frac{\partial\eta}{\partial t}.$$

(The difference between the two spactial derivatives may be quatified by a Fourier expansion, in which the neglected terms are seen to be higher order in a).

Let us consider the ansatz

$$\phi = g(z)\sin(kx - \omega t).$$

The kinematic condition then translates into $g'(0) = a\omega$.

The bottom condition clearly means

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=-h} = g'(-h) = 0.$$

All together, we are looking for a function with these boundary conditions:

$$g'(0) = a\omega \tag{7.5}$$

$$g'(-h) = 0. \tag{7.6}$$

However, the fact that Laplace equation must be satisfied places a strong condition on the kind of function $g(z)$ may be:

$$\nabla^2\phi = 0 \qquad \Longrightarrow \qquad g''(z)\sin(kx - \omega t) - g(z)k^2\sin(kx - \omega t) = 0,$$

which meanse

$$g''(z) = k^2 g(z).$$

Therefore, $g(z)$ must be an exponential function, with the decay constant being exactly equal to k, the wave number. In general, the solution is a linear combination of two exponentials. Instead of that, we may write another less conventional expression using hyperbolic functions. This is equivalent, since these functions are themselves linear combinaions of exponentials. Moreover, we choose to center them at $x = -h$:

$$g(z) = a_1 \cosh(k(z+h)) + a_2 \sinh(k(z+h)).$$

By inspection, we realize $a_2 = 0$, since the $\sinh(k(z+h))$ function was the "wrong" behavior at $z = -h$: it has a slope, while the boundary condition implies it should not.

The kinetic boundary condition implies

$$a_1 = \frac{a\omega}{k\sinh(hk)}.$$

This is indeed the only value for which $g'(0) = a\omega$. The potential is then

$$\phi = \frac{a\omega\cosh(k(z+h))}{k\sinh(kh)}\sin(kx-\omega t). \tag{7.7}$$ {eq:wave_potential}

From it, the velocity components may be found. These will have a term featuring $\sin(kx-\omega t)$, a common expression for a traveling wave. However, the theory is still incomplete since ω and k are not unrelated in a physical wave, but coupled by the phase velocity $c = \omega/k$. Remember how in sound waves this velocity was related to fluid compressibility (to be precise, the square root of the variation of pressure with density). In this case, such relationship is as yet missing. Also notice that gravity has played no role whatsoever, despite claims at the beginning of this chapter about this factor being the driving force behind this process.

In order to find the missing link, let us remember our previous expression for the Bernoulli principle, Eq. 3.11. With no vorticity, it reads

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\cdot\left[\frac{1}{2}u^2 + \frac{p}{\rho} + \varphi\right] = \frac{p}{\rho^2}\nabla\cdot\mathbf{u}.$$

In our case the fluid is incompressible, so the right hand side vanishes. Also, in potential flow we may write the whole equation compactly:

$$\nabla\cdot\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}u^2 + \frac{p}{\rho} + \varphi\right] = 0.$$

This means the whole term on which the divergence operator acts must be constant in space, but not in time in general:

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}u^2 + \frac{p}{\rho} + \varphi = f(t).$$

This is the less-known unsteady Bernoulli principle.

In this particular case, since boundary conditions do not change in time (in fact, they do, but we are taking the surface to be close to $z = 0$), any function $f(t)$ may be incorporated into the potential by making the following trick:

$$\phi' = \phi + \int_{t_0}^{t} f(t')dt',$$

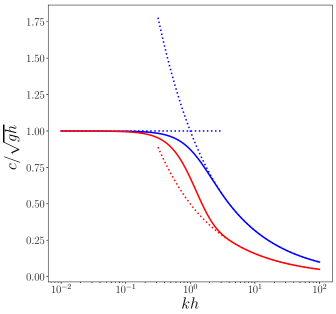

Figure 7.1: Dispersion relationship for gravity waves as a function of reduced wave-vector kh.

hence we may just take $f(t) = 0$ in this particular problem.

If we examine the "head" term, $\partial\phi/\partial t + \frac{1}{2}u^2 + \frac{p}{\rho} + \varphi$ at the surface, we notice that the pressure should be constant (equal to atmospheric pressure, as in hydrostatics). The u^2 may be neglected within our assumptions. Last, but not least, gravity finally appears in the potential, $\varphi = g\eta$. This leaves us with

$$\left.\frac{\partial\phi}{\partial t}\right|_{z=0} + g\eta = 0,$$

often called the "dynamic condition".

With our previous result, this means:

$$\phi = -\frac{a\omega^2 \cosh(k(z+h))}{k \sinh(hk)} \cos(kx - \omega t) + ga\cos(kx - \omega t) = 0.$$

The result is the following dispersion relation:

$$\omega^2 = gk\tanh(kh)$$

This is a surprising result, despite being obviously dimensionally correct, whith the two lengths in the problem, $\lambda = 2\pi/k$ and h providing the relevant lenght scales.

As a result, the phase velocity is then

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tanh(kh)},$$

while the group velocity, after some algebra is found to be

$$c_{\mathrm{gr}} := \frac{d\omega}{dk} = \frac{c}{2}\left(1 + kh\frac{1-\tanh^2(kh)}{\tanh(kh)}\right)$$

These two velocities are shown in 7.1. We may distinguish two limits, which we discuss separately.

7.1.1 Shallow water

If the bottom is small, compared to the wavelength λ, then $kh = 2\pi h/\lambda$ is a small number, and we may approximate $\tanh(kh) \approx kh$. Then,

$$\omega^2 = ghk^2 \qquad \Longrightarrow \qquad \omega = k\sqrt{gh}.$$

In this limit, the phase velocity is independent of velocity, which means the process is non-dispersive:

$$c = \frac{\omega}{k} = \sqrt{gh}.$$

The group velocity is of course equal to the phase velocity:

$$c_{\mathrm{g}} := \frac{d\omega}{dk} = \sqrt{gh} = c.$$

The relevant length in this limit is the depth h, which provides the correct dimensions to get a velocity.

Notice "shallow" is defined in relation to the wavelength. A seismic wave may be excited at the deep sea by a geological process that may involve tenths or hundreds of kilometers, much larger than the usual sea depth, which is some few kilometers (its mean value is about 3.8 km.) The sea is then "shallow", and the speed at which the perturbation travels will be about

$$c \approx \sqrt{9.8\,\mathrm{m\,s^{-2}} \times (4 \times 10^3\,\mathrm{m})} \approx 200\,\mathrm{m\,s^{-1}} \approx 700\,\mathrm{km\,h^{-1}}$$

This is an astonishing speed, which is nevertheless in agreement with measurements of these dramatic events.

7.1.2 Deep water

If the bottom is deep, compared to the wavelength λ, then $kh = 2\pi h/\lambda$ is a large number, and its hyperbolic tangent is very nearly 1. In this limit,

$$\omega^2 \approx gk \qquad \Longrightarrow \qquad \omega = \sqrt{gk}. \tag{7.8}$$ {eq:water_disp_deep}

These waves are highly dispersive, since their phase velocity depends on the wavelength:

$$c = \frac{\omega}{k} = \sqrt{g/k} = \sqrt{g\lambda/(2\pi)}.$$

The group velocity is now different from the phase velocity. In fact, it is one half that value:

$$2\log\omega = \log k + \log g \qquad \Longrightarrow \qquad \frac{2}{\omega}\frac{d\omega}{dk} = \frac{1}{k} \qquad \Longrightarrow \qquad c_{\mathrm{g}} = \frac{c}{2}$$

In deep waters, the only two length scales we may think of are the wave amplitude a and their wavength λ. As we expect the amplitude to play no role (as long as it is small, $a \ll \lambda$), we are left with the wavelength to provide the length scale.

Notice that waves high high wavelenghts travel faster. This explains the groundsell phenomenon, by which waves generated by a far away storm may arrive quickly at some shore. These waves may precede the storm by a long time, if the storm does arrive at all. The velocity grows larger and larger with the wavelength, but as we have seen in the previous section, as the wavelength becomes comparable to the depth there will be a crossover to some finite velocity.

7.2 Gravity waves from energy

There is an alternative derivation of the dispersion relation which is interesting, and will be useful when the role of surface tension is discussed in the next Section. [14]

It begins by considering the energy of the whole system:

$$E := \int (T + U)\, d\mathbf{r},$$

where the integral extends to the whole volume of the fluid, T is the total kinetic energy, and U the potential energy:

$$T = \frac{1}{2} \int (\rho u^2) d\mathbf{r} \tag{7.9}$$

$$U = \int (\rho \varphi) d\mathbf{r} \tag{7.10}$$

For the simple case of gravity, the latter is simply given by

$$U = \int (\rho g z) d\mathbf{r}. \tag{7.11}$$

If the flow is potential, the kinetic energy can be written as

$$T = \frac{1}{2} \rho \int |\nabla \phi|^2 \, d\mathbf{r}, \tag{7.12}$$

a form that is appreciated in physics, where these sort of square gradient theories are widely studied. (They are often called "London-Linzburg-Landau" theories, or a subset of those names, but they begun, it seems, with van der Waals' seminal theory for the liquid-vapor interface.)

In order to add more content ot our previous results, let us here consider both an upper phase ("air") of density ρ' and a lower one ("water") with density ρ (all primed variables refer to the upper phase in this section). They are separated by the interfacial surface at $z = \eta(x, y)$, but we will suppose that the surface fluctuates around a well-defined position, which we will take, for convenience, as $z = 0$. The total potential energy is then

$$U = \iint dxdy \int_{-H}^{\eta} dz (\rho g z) \iint dxdy \int_{\eta}^{H'} dz (\rho' g z),$$

where we suppose the bottom is at $z = -H$ and the air ends at $z = H'$. These limits are not important, since we will consider later the limit in which both are very large.

We may split the two integrals at $z = 0$, the equilibrium surface:

$$\iint dxdy \int_{-H}^{\eta} dz = \iint dxdy \int_{-H}^{0} dz + \iint dxdy \int_{0}^{\eta} dz$$
$$\iint dxdy \int_{\eta}^{H'} dz = \iint dxdy \int_{\eta}^{0} dz \iint dxdy \int_{0}^{H'} dz$$

But, since two of those four integrals have fixed integration limits, they will contribute a constant energy, so we may ignore them and just write

$$U = \iint dxdy \int_{0}^{\eta} \rho gz dz - \iint dxdy \int_{\eta}^{0} \rho' gz dz =$$
$$g(\rho - \rho') \iint dxdy \int_{0}^{\eta} zdz = \frac{1}{2} g(\rho - \rho') \iint dxdy \eta^2.$$

It is natural that the potential energy is due to fluctiations of the surface. However, the kinetic energy, which involves the motion of the upper and lower fluid masses, can also be cast as a surface integral. We may use the divergence (Gauss') theorem for this purpose. In order to express the kinetic energy integrand as a divergence, we may write

$$\nabla \cdot [\phi(\nabla\phi)] = |\nabla\phi|^2 + \phi\nabla^2\phi.$$

But the last term is null due to incompressibility. The same applies to ϕ'. This means the kinetic energy of the denser fluid can be written as

$$T_1 = \frac{1}{2}\rho \int \nabla \cdot [\phi(\nabla\phi)] d\mathbf{r} \approx \frac{1}{2} \iint dxdy \phi(\nabla\phi)\mathbf{e}_z$$

In this expression, we have applied Gauss theorem, and approximated the true normal to the surface with its equilibrium value $\hat{\mathbf{e}}_z$. In other words,

$$T_1 = \rho \frac{1}{2} \iint dxdy \phi \frac{\partial \phi}{\partial z}$$

Adding the upper fluid, the total kinetic energy is

$$T = \frac{1}{2} \iint dxdy \left(\rho\phi \frac{\partial\phi}{\partial z} - \rho'\phi' \frac{\partial\phi'}{\partial z} \right),$$

because the outward-pointing normal of the upper phase is approximated by $-\hat{\mathbf{e}}_z$. The appearance of those partial derivatives in the vertical direction is appealing, because they are to be made equal to the vertical velocity of the surface, according to boundary condition 7.3. Assuming a planar wave as in 7.4, and with our solution for the potential in 7.7, the necessary ingredients are readily found:

$$\frac{\partial\phi}{\partial z} = \frac{\partial\eta}{\partial t} = a\omega \sin(kx - \omega t)$$
$$\phi(0) = \frac{a\omega}{k} \tanh(kh) \sin(kx - \omega t).$$

For the upper phase, the appropriate potential is easily seen to be

$$\phi' = -\frac{a\omega \cosh(k(h'-z))}{k\sinh(kh')}\sin(kx-\omega t),$$

since it must decay in the z direction, satisfy the boundary condition at $z = h'$, and the kinematic condition at the interface.

From now on, we will take the deep-water limit, leaving the general case as an exercise to the reader. In this limit, the potentials approach

$$\phi(0) = -\phi'(0) \to \frac{a\omega}{k}\sin(kx-\omega t),$$

and the kinetic energy is

$$T = \frac{1}{2}(\rho+\rho')\frac{(a\omega)^2}{k}\iint dxdy \sin^2(kx-\omega t).$$

For the same planar wave, the potential energy is

$$U = \frac{1}{2}g(\rho-\rho')a^2\iint dxdy \cos^2(kx-\omega t).$$

Now, the integral may be carried out: the y integration yields L_y, the length of the system in that direction, and on x we may have $N = L_x/\lambda$ total repetitions of the same wave, each one of them we can integrate:

$$\iint dxdy \cos^2(kx-\omega t) = \iint dxdy \sin^2(kx-\omega t) =$$
$$L_y N \iint_0^\lambda dx \cos^2(kx-\omega t) = \frac{1}{2}L_y N\lambda = \frac{1}{2}A,$$

where $A = L_x L_y$ is the total interfacial area.

This means both energies per surface area are

$$\frac{U}{A} = \frac{1}{4}g(\rho-\rho')a^2 \qquad \frac{T}{A} = \frac{1}{4}(\rho+\rho')\frac{(a\omega)^2}{k}.$$

The dispersion relation 7.8 is recovered (for $\rho' = 0$) if both energies are made equal. This is a tipical requirement for oscillatory motion in physics, often called "equipartition" (not to be confused with a similar concept in statistical mechanics). Its most well-known instance is the harmonic oscillator.

A more formal justification comes from the Lagrangian. This quantity (not to be confused with the Lagrangian framework, or the Lagrangian derivative) is defined as

$$L = T - E.$$

In classical mechanics, extremization of this quantity results in the proper equations of motion. In our case,

$$\frac{L}{A} = \frac{1}{4}a^2\left((\rho+\rho')\frac{\omega^2}{k} - g(\rho-\rho')\right).$$

Extremization means that the partial derivatives of the Lagrangian with respect to the parameters must be zero (technically, the Lagrangian is not at a minimum or maximum, but at a saddle point). In our case, the only parameter is the amplitude, a, so

$$\frac{\partial L}{\partial a} \implies (\rho+\rho')\frac{\omega^2}{k} - g(\rho-\rho').$$

This also implies $T = U$.

Hence, our previous deep-water dispersion relation 7.8 is now modified to

$$\omega^2 = gk\frac{\rho-\rho'}{\rho+\rho'}$$

The combination $(\rho-\rho')/(\rho+\rho')$ is called the Atwood number, and is closely 1 when the denser fluid is much heavier than the lighter one.

7.3 Capillary waves

The previous method can readily accomodate capillary waves, also known as “ripples”. These are small waves that are dominated by surface tension, not gravity. This is the dominant force at small scales, and we will get an exact expression for the crossover lenght below which this holds. Again, only the deep-water limit will be considered.. Notice that, while this limit was bound to be violated sooner or later for long gravity waves (when their wave-length begins to be comparable to the water depth), capillary waves will eventually comply with this limit at small wave-lengths.

The surface tension causes a cost in modifying an interface from its equilibrium value:

$$U_{\text{st}} = \sigma(A - A_0),$$

where σ is the surface tension, with units of energy per area.

The area of a distorted surface in the Monge representation is given by

$$A = \iint dxdy\sqrt{1+\left(\frac{\partial\eta}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial y}\right)^2}.$$

When fluctuation are not great, the cumbersome square root may be expanded in Fourier series, and we find:

$$A \approx \iint dxdy\left(1+\frac{1}{2}\left[\left(\frac{\partial\eta}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial y}\right)^2\right]\right) =$$
$$= A_0 + \frac{1}{2}\iint dxdy\left(\left(\frac{\partial\eta}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial y}\right)^2\right)$$

For our perturbation 7.4 we have,

$$U_{\text{st}} = \frac{1}{2}\sigma\iint dxdy\left(\frac{\partial\eta}{\partial x}\right)^2 = \frac{1}{4}\sigma A(ak)^2,$$

where we have performed the same integration as in the previous section.

Our Lagrangian is now

$$\frac{L}{A} = \frac{1}{4}a^2 \left(\sigma k^2 + (\rho + \rho') \frac{\omega^2}{k} - g(\rho - \rho') \right).$$

Its extreme now yields this dispersion relation:

$$\omega^2 = gk \frac{\rho - \rho'}{\rho + \rho'} + \frac{\sigma}{\rho + \rho'} k^3.$$

So we see that the short wave-length regime (high values of k) is indeed dominated by a new k^3 term due to surface tension. The long wave-length regime (low k values) is, on the other hand, unaffected.

In order to analyze this expression, let us introduce reduced units once more,

$$\omega = \omega_0 \omega^* \qquad k = k_0 k^*,$$

where the typical angular frequency and wave vector are found by demanding this simple expression holds:

$$\omega^{*2} = k^* + k^{*3}.$$

After some algebra, we find

{eq:capillary_k0}

$$k_0 = \sqrt{\frac{g(\rho - \rho')}{\sigma}} \tag{7.13}$$

{eq:capillary_om0}

$$\omega_0^2 = g \frac{\rho - \rho'}{\rho + \rho'} k_0 \tag{7.14}$$

Now, the reduced phase velocity is given by

$$c^* := \frac{\omega^*}{k^*} = \sqrt{\frac{1}{k^*} + k^*},$$

so that it goes from a $\sim 1/\sqrt{k}$ growth at small k (this is the gravity regime) to a $\sim \sqrt{k}$ growth at high k (surface tension regime). It therefore has a minimum at some crossover wave-vector k_{cross}, which we may find by

$$\frac{\partial c^*}{\partial k^*} = 0.$$

It is actually easier to differenciate c^{*2}. The value is found at $k^*_{\text{cross}} = 1$, at which

$$c^*_{\text{cross}} = \sqrt{2}.$$

The values of k_{cross} are given by bringing back the dimension factors in 7.13 and 7.14. The velocity is given by the factor

$$c_0 = \frac{\omega_0}{k_0} = \frac{[\sigma g(\rho - \rho')]^{1/4}}{(\rho - \rho')^{1/2}}.$$

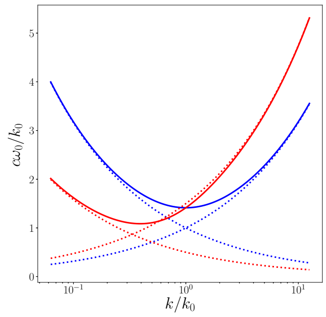

Figure 7.2:

For water and air in standard conditions, $\sigma = 70\,\mathrm{mJ\,m^{-1}}$, and we find

$$k_{\mathrm{cross}} = 374\,\mathrm{rad\,m^{-1}} \implies \lambda_{\mathrm{cross}} = \frac{2\pi}{k_{\mathrm{cross}}} = 1.7\,\mathrm{cm},$$

and

$$c_{\mathrm{cross}} = 23\,\mathrm{cm\,s^{-1}}.$$

This is the scale below wich surface tension is important, and the phase velocity is the lower possible.

The group velocity is defined as

$$c_{\mathrm{gr}} := \frac{\partial \omega}{\partial k}.$$

Notice that, by differenciating $c = \omega/k$ with respect to k it is easy to prove the identity:

$$c_{\mathrm{gr}} = c + k\frac{\partial c}{\partial k}.$$

This clearly shows that the a difference in the values of the two velocities is caused by dispersion: the phase velocity having a dependence on the wave vector, as evident in the dc/dk term. This also shows that at the crossover velocity, where this term vanishes, both velocities are the same. These waves are, then the only non-dispersive ones possible in the deep-water regime.

The expression for the group velocity is, in reduced units,

$$c^*_{\mathrm{gr}} := \frac{1+k^2}{2\sqrt{k+k^3}}.$$

It then goes from a low-k regime in which $c_{\mathrm{gr}} \to c/2$ (gravity) to a high-k regime with $c_{\mathrm{gr}} \to 3c/2$ (surface tension). In the latter regime, the group velocity is 50% faster than the phase velocity. In 7.2 we plot both velocities as functions of the reduced wave-vector. Both are seen to cross at a value of 1. In Figure 7.3 the same is plotted as a function of wave-length. The crossing takes place at the reduced wave-length value of $\lambda^* = 2\pi$.

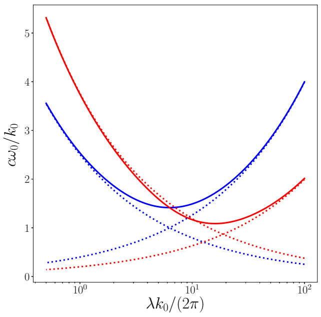

Figure 7.3:

Chapter 8

Work and energy

Let us find the work done on a fluid particle.

The First Law of thermodynamics tells us this change is due to work (no heat exchange is considered here):

$$dE = dW$$

On the x direction, this work will be given by the distance travelled by the left wall in the x direction times the pressure force:

$$dW_x(\text{left}) = u_x \Delta t p A.$$

This is work done on the particle (if u_x is positive), hence its sign. There will be a similar contribution from the right wall, so the total work given by the x direction will be

$$dW_x = u_x \Delta t p A - u_x' \Delta t p' A.$$

By expanding in a Taylor series, this may be written as

$$dW_x = -\frac{\partial p u_x}{\partial x} \Delta x A \Delta t.$$

Now, adding the other three components we find

$$\frac{dW}{dt} = -V \nabla \cdot (p\mathbf{u}).$$

The right hand side may be expanded

$$\nabla \cdot (p\mathbf{u}) = p \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla p.$$

Recalling that, from the Euler equation 3.5 ,

$$\nabla p = \rho \mathbf{g} - \rho \frac{d\mathbf{u}}{dt},$$

we may write

$$\frac{1}{V} \frac{dW}{dt} = -p \nabla \cdot \mathbf{u} + \rho \mathbf{u} \cdot \frac{d\mathbf{u}}{dt} - \rho \mathbf{g} \cdot \mathbf{u}.$$

Now, multiplying by $1/\rho$:

$$\frac{d\epsilon}{dt} = \mathbf{u} \cdot \frac{d\mathbf{u}}{dt} - \mathbf{g} \cdot \mathbf{u} - \frac{p}{\rho} \nabla \cdot \mathbf{u}, \tag{8.1}$$

`{eq:specific_energy`

where we have defined the specific energy $\epsilon = E/M$.

This may be written as

$$\frac{d}{dt}\left[\epsilon - \frac{1}{2}u^2 + \mathbf{g} \cdot \mathbf{r}\right] = -\frac{p}{\rho} \nabla \cdot \mathbf{u},$$

or

$$\frac{d}{dt}\left[\epsilon - \frac{1}{2}u^2 - \varphi\right] = -\frac{p}{\rho} \nabla \cdot \mathbf{u},$$

This is clearly a law for

$$e = \epsilon - \frac{1}{2}u^2 - \varphi,$$

where we define e, the specific internal energy as the total energy minus kinetic energy, minus gravitational potential energy. In other words,

$$\epsilon = e + \frac{1}{2}u^2 + \varphi.$$

The specific internal energy then satisfies a convection equation with a source term due to the divergence of the velocity

`{eq:Euler_energy}`

$$\frac{de}{dt} = -\frac{p}{\rho} \nabla \cdot \mathbf{u}, \tag{8.2}$$

Notice also that in the steady state,

$$\mathbf{u} \cdot \nabla e = -\frac{p}{\rho} \nabla \cdot \mathbf{u}.$$

But, recalling our steady Bernoulli principle, Eq. 3.12, replacing $\frac{p}{\rho} \nabla \cdot \mathbf{u}$, we arrive at the steady state Bernoulli equation for a compressible fluid:

$$\mathbf{u} \cdot \nabla \left[e + \frac{1}{2}u^2 + \frac{p}{\rho} + \varphi\right] = 0,$$

which tells us the combination $e + \frac{1}{2}u^2 + \frac{p}{\rho} + \varphi = \epsilon + \frac{p}{\rho}$ is constant along streamlines. Notice that, by the definition of enthalpy,

$$H = E + PV,$$

the specific enthalpy is

$$h = H/M = e + p/\rho,$$

so the Bernoulli principle claims that the total specific enthalpy ($h' = h + \frac{1}{2}u^2 + \varphi$) is constant along streamlines.

We may also find an equation for the enthalpy. In Eq. 8.2 above, the right hand side may be changed using the continuity equation:

$$-\rho \nabla \cdot \mathbf{u} = \frac{d\rho}{dt}.$$

which means

$$-\frac{p}{\rho}\nabla\cdot\mathbf{u}=\frac{p}{\rho^2}\frac{d\rho}{dt}$$

But

$$\frac{d(p/\rho)}{dt}=\frac{1}{\rho}\frac{dp}{dt}-\frac{p}{\rho^2}\frac{d\rho}{dt}.$$

Hence,

$$\frac{d(e+p/\rho)}{dt}=\frac{1}{\rho}\frac{dp}{dt},$$

or

$$\frac{dh}{dt}=\frac{1}{\rho}\frac{dp}{dt}.$$

Changes in the enthalpy are therefore tied to changes in pressure.

8.0.1 Heat flux

In the case that some heat flux is present, the First Law of thermodynamics reads:

$$dE=dW+dQ.$$

The influx of heat to the particle, dQ, will enter our equations from a vector heat flux $\mathbf{q}$, with units of energy/(area × time). For example, the heat influx due to transfer in the x direction will be

$$dQ_x=\mathbf{q}dtdydz-\mathbf{q}(x+dx)dtdydz\approx -V\,dt\,\frac{\partial\mathbf{q}}{\partial x},$$

where in the last approximation a Taylor expansion has been employed, as should be customary by now.

Altogether, the total heat flux rate is

$$\frac{dQ}{dt}=-V\nabla\cdot\mathbf{q},$$

and the final equation for the rate of change of specific internal energy of an ideal fluid is

$$\rho\frac{de}{dt}=-p\nabla\cdot\mathbf{u}-\nabla\cdot\mathbf{q}. \tag{8.3}$$

{eq:Euler_energy_w_q}

The heat flux must be either due to an external source, or due to heat diffusion. In the latter case, a well-known theory is Fourier's law, by which the flow is due to a temperature gradient:

$$\mathbf{q}=-k\nabla T,$$

where k is the heat diffusion coefficient. The temperature does not appear in the energy equation, but a common assumption is that

$$e=cT,$$

where c is the specific heat, taken as constant. Then,

$$\rho c\frac{dT}{dt}=-p\nabla\cdot\mathbf{u}+k\nabla^2T.$$

For an incompressible fluid,

$$\frac{dT}{dt} = \frac{k}{c\rho}\nabla^2 T = \alpha \nabla^2 T,$$

where $\alpha = k/(c\rho)$, a constant if ρ is. This is the convective Fourier heat equation (it is not the most common Fourier law, due to the derivative being convective, and not just partial.)

Part II

Real fluids

Chapter 9

Navier-Stokes equations

9.1 Kinematics of a particle

Let us focus again on a fluid particle, as we did on Sections 2.2 and 3.2, but now focusing on how the particle itself distorts as a consequence of a velocity field.

All possible distortions of a particle will be a combination of the following:

1. Translation
2. Rotation
3. Shear
4. Dilation

A translation is just the motion of its center of mass from one place to another, and for a small time is given simply by $\mathbf{u}dt$. The other motions are more complicated, since they involve spatial derivatives of the velocity. They must: for a constant velocity field translation is the only motion that occurs.

Rotation

We will refer to particle in figure 6.6, with vertices A, B, and C. Vertex D plays no role — also, it is sufficient to focus on the face that is portrait, even if the shape of a particle is supposed to be a cube. It is straightforward to include the other faces, as we will see.

After a small time d the particle has distorted, so that the vertices are now at positions A′, B′, C′, and D′.

Let us call α the angle between the x axis and the A′-B′ edge, with the usual counter-clockwise convention as positive. Similarly, β is the angle between the A′-C′ edge and the y axis, with the same convention. It is obvious that a net rotation takes place if e.g. both angles are positive. If, on the other hand, they are equal in magnitude but differ in sign, no rotation takes place. This makes it reasonable to define the rotation as the average of both angles:

$$d\Omega_z = \frac{1}{2}\left(\alpha + \beta\right).$$

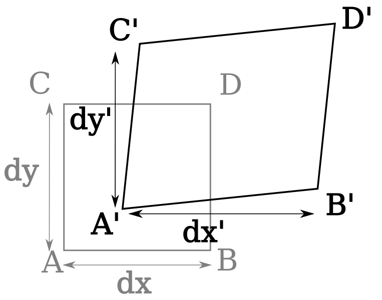

Figure 9.1:

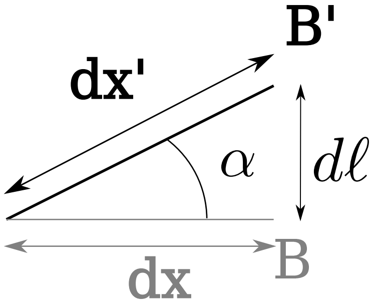

Figure 9.2:

Now, angle α will always be very small as dt gets very tiny. Hence, we may approximate it by its tangent:

$$\alpha \approx \frac{d\ell}{dx'}$$

For a small time, we have the following:

$$dx' = x(B') - x(A') \approx (dx + v_x(B)dt) - v_x(A)dt \approx dx + \left(v_x(A) + \frac{\partial v_x}{\partial x} dx \right) dt - v_x(A)dt = dx + \frac{\partial v_x}{\partial x} d$$

where we have taken the origin to coincide with the position of A. The partial derivative is supposed to be evaluated at A, but in the limit as dx goes to zero it is just "at the particle". Similarly, the opposing side is:

$$\begin{aligned} d\ell &= y(B') - y(A') \approx y(B) + v_y(B)dt - y(A) - v_y(A)dt \\ &= \left(v_y(B) - v_y(A)\right) dt \approx \frac{\partial v_y}{\partial x} dxdt. \end{aligned}$$

Therefore, to first order in dt:

$$\alpha \approx \frac{\partial v_y}{\partial x} dt.$$

In other words, the rate of change of the angle is

$$\frac{d\alpha}{dt} = \frac{\partial v_y}{\partial x}.$$

Notice the cross derivative: what is relevant is the change of the vertical component of the velocity with the horizontal coordinate.

A similar calculation for the other angle reveals

$$\frac{d\beta}{dt} = -\frac{\partial v_x}{\partial y}.$$

Taking all

$$\frac{d\Omega_z}{dt} = \frac{1}{2}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right).$$

This may sound familiar to the reader, since the curl in Cartesian coordinates has a z component with exactly the same expression, but the factor of $1/2$. This analysis may be carried out for rotations about the other two Cartesian axes, with the end result that

$$\frac{d\boldsymbol{\Omega}}{dt} = \frac{1}{2}\boldsymbol{\omega}.$$

The curl is therefore twice the rate of rotation of a fluid particle.

As an example, let us consider a uniform circular motion about the origin:

$$\mathbf{r} = \begin{cases} x = r\cos(\omega_0 t) \\ y = r\sin(\omega_0 t). \end{cases}$$

The velocity field is

$$\mathbf{u} = \begin{cases} u_x = -r\omega_0 \sin(\omega_0 t) = -\omega_0 y \\ u_y = r\omega_0 \cos(\omega_0 t) = \omega_0 x. \end{cases}$$

If we compute the curl of this field, its only component is the z one:

$$\omega_z = \left(\frac{\partial(\omega_0 x)}{\partial x} - \frac{\partial(-\omega_0 y)}{\partial y}\right) = 2\omega_0.$$

Therefore, the curl indeed is twice the angular velocity.

Finally, let us recall, from the definition of the tensor $\nabla\mathbf{u}$ of Eq. 3.1,

$$\frac{d\Omega_{ij}}{dt} = \frac{1}{2}\left[(\nabla\mathbf{u})_{ij} - (\nabla\mathbf{u})_{ji}\right], \tag{9.1}$$

`{eq:dOmega_as_grad_u}`

where the cyclic convention (i, j, k) is implied in order to relate the indices of $d\Omega_{ij}/dt$ and the components $d\Omega_k/dt$ (see Exercise 1 for a definition in terms of the Levi-Civita symbol.)

Strain

Similarly to the definition of rotation, let us define the strain as half the difference between angle β and α:

$$d\epsilon_{xy} = \frac{1}{2}(\alpha - \beta).$$

Indeed, if both are equal we get a pure rotation, and no strain. If, however, they have the same magnitude but opposite sign, we have a pure strain and no rotation.

This definition leads to

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right).$$

Notice the notation: indices xy implay a strain on that plane. In general,

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right).$$

This defines off-diagonal terms of a tensor that is obviously symmetric. All together, three terms in 3D (xy, xz, and yz).

Also, from the definition of the tensor $\nabla \mathbf{u}$ of Eq. 3.1,

{eq:epsilon_as_grad_u}

$$\epsilon_{ij} = \frac{1}{2}\left[(\nabla \mathbf{u})_{ij} + (\nabla \mathbf{u})_{ji}\right], \tag{9.2}$$

so that ϵ is the symmetrized version of the gradient of the velocity.

We may also extend these definitions to the diagonal terms:

$$\epsilon_{ii} = \frac{\partial v_i}{\partial x_i},$$

which are not really strains, as we show next.

Dilation

We have derived before that the change on side dx is

$$dx' \approx dx + \frac{\partial v_x}{\partial x} dx dt,$$

which features the diagonal components of the strain tensor just introduced. In general, for the i Cartesian direction,

$$dx'_i \approx dx_i + \epsilon_{ii} dx_i dt = dx_i \left(1 + \epsilon_{ii} dt\right)$$

If we consider the volume, its new value will be

$$V' = dx' dy' dz' \approx \prod_i dx_i \left(1 + \epsilon_{ii} dt\right) \approx \left(\prod_i dx_i\right)\left(1 + \sum_j \epsilon_{jj} dt\right),$$

where in the last expression we have neglected all higher order terms (dt^2 or higher). Therefore, the new volume of a particle is

$$V' = V\left(1 + \sum_j \epsilon_{jj} dt\right).$$

Its change is

$$V' - V = V \sum_j \epsilon_{jj} dt.$$

In the limit as $dt \to 0$,

$$\frac{dV}{dt} = V \sum_j \epsilon_{jj}.$$

This clearly identifies the diagonal components of the strain tensor as those responsible for the relative rate of change of the volume. Moreover, due to its definition, the sum of all components (i.e. the trace of the tensor) is precisely the divergence of the velocity:

$$\sum_j \epsilon_{jj} = \nabla \cdot \mathbf{u}.$$

We therefore arrive at the expression for the rate of change of a particle:

$$\frac{dV}{dt} = V \nabla \cdot \mathbf{u}.$$

9.1.1 Continuity, re-revisited

This is the expression that was lacking in our derivation in Section 2.2. Indeed, if the mass of a particle is constant, it is trivial to recover the continuity equation for the density. The latter is $\rho = m/V$, therefore,

$$\frac{d\rho}{dt} = \frac{d(m/V)}{dt} = \frac{1}{V} \underbrace{\frac{dm}{dt}}_{=0} - m \frac{1}{V^2} \frac{dV}{dt} = -m \frac{1}{V} \nabla \cdot \mathbf{u} = -\rho \nabla \cdot \mathbf{u},$$

which indeed is the right expression (the "convergence" Equation 3.3.)

This also shows that we are working on an Lagrangian reference frame, so that advective derivatives appear naturally. Indeed, $dm/dt = 0$ for a particle, whereas mass may enter and leave a fixed (Eulerian) zone in space.

9.2 Stress tensor

Let us return again to our particle in order to analyze its movement when stress forces are applied onto its walls. The pressure force may be consider a special case of the latter, but stress forces may also have a shear component.

Thus, the horizontal force will be given by, in part, by the contributions due to the walls at the left, back, and bottom

$$dF_x|_{\mathrm{l,bk,bm}} = \tau_{xx} dy\, dz + \tau_{yx} dx\, dz + \tau_{zx} dx\, dy. \qquad (9.3)$$ `{eq:wall_shear_stress}`

The compression stress τ_{xx} is therefore quite similar to the pressure (see (Eq 3.4), and in fact will be seen to include a $-p$ term. The minus sign appears because normal stresses are historically defined as positive if pointing outside the particle (i.e. in the direction of the normal vector). In addittion, two shear stress forces appear. One of them, $\tau_{yx} dx\, dz$ is the horizontal shear force on the back wall ($dx\,dz$ actually equals $dx\,dy$, but it is clearer to write it this way). Similarly, $\tau_{zx} dx\,dy$ is the horizontal stress force at the bottom wall.

Notice the convention for labeling: τ_{ij} is the stress in the j direction on a face normal to axis i. I.e. the left face has the same convention as the right one, the back as the front, the top as the bottom.

To get the whole horizontal force, the contributions from the other three walls must be considered:

$$dF_x|_{\mathrm{r,ft,up}} = \tau_{xx}(x+dx,y,z)dy\,dz + \tau_{yx}(x,y+dy,z)dx\,dz + \tau_{zx}(x,y,z+dz)dx\,dy.$$

To get the total horizontal force due to stresses, the two contributions must be subtracted:

$$\begin{aligned} dF_x = &\left(\tau_{xx}(x+dx,y,z) - \tau_{xx}\right) dy\,dz + \\ &\left(\tau_{yx}(x,y+dy,z) - \tau_{yx}\right) dx\,dz + \\ &\left(\tau_{zx}(x,y,z+dz) - \tau_{zx}\right) dx\,dy \end{aligned}$$

. Notice that the substraction is in order: if the stresses are the same on those faces, the resultant force will then be zero (those forces may exert a net torque, but not a force). In general, they will be different on those faces, as as been made explitic on their arguments.

The choice is such that an variation in a given direction is assigned a positive sign. This affects later results, in particular the fact that the pressure appears with a negative sign.

Expanding in Taylor series, we get the following net horizontal force:

$$dF_x = \left(\frac{\partial \tau_{xx}}{\partial x} dx\right) dy\,dz + \left(\frac{\partial \tau_{yx}}{\partial y} dy\right) dx\,dz + \left(\frac{\partial \tau_{zx}}{\partial z} dz\right) dx\,dy.$$

The volumetric horizontal force is then,

$$f_x = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}.$$

With integer notation for Cartesian coordinates:

$$f_1 = \sum_{j=1,2,3} \frac{\partial \tau_{j1}}{\partial x_j}.$$

In general, for any component we will have:

$$f_i = \sum_{j=1,2,3} \frac{\partial \tau_{ji}}{\partial x_j}.$$

In order to use vector notation, we have to introduce the divergence of a tensor:

$$\nabla \cdot \tau = \sum_{j=1,2,3} \frac{\partial}{\partial x_j} \tau_{ji},$$

which results in a vector:

$$\mathbf{f} = \nabla \cdot \tau.$$

The tensor τ has components τ_{ij}. There is also the matrix notation, by which

$$\nabla \cdot \tau = \nabla^{\mathsf{T}} \tau,$$

where ∇^{T} is a transposed (row) vector operator, multiplying matrix τ from the left.

The resulting Navier-Stokes momentum equation is then

$$\rho \frac{d\mathbf{u}}{dt} = \nabla \cdot \tau + \rho \mathbf{g}. \tag{9.4}$$ `{eq:NS}`

9.2.1 Newtonian fluids

The previous equation is still too general, and a connection between stress and strain is still needed. Here we consider the case in which there is a linear relationship between both, which involves the coefficient of viscosity.

To begin with, let us consider a simple case in which a fluid is confined between two planes. One of them moves sideways with a certain speed u_0, while the other is kept fixed. After a certain transient, some force is needed in order to keep this shearing. The simplest expression is

$$F = \mu A \frac{u_0}{L}. \tag{9.5}$$ `{eq:Newton_simple_law}`

The force is proportional to the area and to the velocity difference between the planes. It is also inversely proportional to their separation, L (this fact being the least obvious). Finally, a constant of proportionality is given by μ, the viscosity coefficient, or simply "the viscosity". This constant may vary with temperature, density, pressure, but the point with Newtonian fluids is that it does not vary with the velocity field (or its derivatives). Later, in section 11.1, this flow will be solved as a solution of the Navier-Stokes equations, the Couette flow. There, it will be shown that the velocity is everywhere in the direction of the force exerted on the upper plane, let us call it x, and varies linearly between the planes, in the y direction. Therefore, the only components of the strain rate tensor are $\epsilon_{xy} = \epsilon_{yx} = u_0/(2L)$. We therefore have

$$\tau_{xy} = \mu \epsilon_{xy}.$$

With these in mind, let us look for a general relationship between τ and ϵ. This is much easier if we go to the principal strain axes. These are the coordinates on which the strain rate is diagonal. Such coordinate system always exist, since the strain rate tensor is symmetric. Notice that in these system strains are not due to shear, only to dilations.

A simple example would be to consider the flow $u_x = u_0 y/L$ (again, Couette flow). In this case,

$$\epsilon = \begin{pmatrix} 0 & u_0/(2L) \\ u_0/(2L) & 0 \end{pmatrix}.$$

It is easy to find the two eigenvalues and associated eigenvectors of this matrix:

$$\begin{aligned} \lambda_1 &= u_0/(2L) & \mathbf{v}_1 &= (1/\sqrt{2})\begin{pmatrix}1 & 1\end{pmatrix}^{\mathsf{T}} \\ \lambda_2 &= -u_0/(2L) & \mathbf{v}_2 &= (1/\sqrt{2})\begin{pmatrix}1 & -1\end{pmatrix}^{\mathsf{T}} \end{aligned}$$

Notice the first eigenvalue cornespond to a dilation along the $x = y$ diagonal, while the second is a compression along the $x = -y$ one.

The diagonal strain rate matrix is then:

$$\tilde{\epsilon} = \begin{pmatrix} u_0/(2L) & 0 \\ 0 & -u_0/(2L) \end{pmatrix}.$$

It is crucial to realize that the stress tensor is also diagonal in this coordinate system. Otherwise, a pure dilation in one of the principal directions would cause a shear stress in some other direction. This does not mean, however, that the two tensors are simply proportional. Instead, we may posit, for the top-most element

$$\tilde{\tau}_{xx} = -p + C_1\tilde{\epsilon}_{xx} + C_2\tilde{\epsilon}_{yy} + C_3\tilde{\epsilon}_{y}.$$

We have added a $-p$ term that has to be there even when there is no movement. This is needed, since a diagonal stress tensor $\tau = -p\mathbb{1}$ produces the $-\nabla p$ term of hydrostatics (in Couette flow, p is constant, and this term is not needed.) The minus sign of $-p$ stems from the fact that the pressure force points toward the interior of a particle.

If isotropy is assumed, there should be no distinction between traverse directions y and z. Therefore, $C_2 = C_3$, and

$$\tilde{\tau}_{xx} = -p + C_1\tilde{\epsilon}_{xx} + C_2(\tilde{\epsilon}_{yy} + \tilde{\epsilon}_{y}) = -p + K\tilde{\epsilon}_{xx} + C_2(\tilde{\epsilon}_{xx} + \tilde{\epsilon}_{yy} + \tilde{\epsilon}_{y}),$$

where the constant $K = C_1 - C_2$. Notice the C_2 term is the divergence of the velocity. But, it is also the trace of the strain tensor, a quantity which is invariant under change of basis. We can now write:

$$\tilde{\tau}_{xx} = -p + C_2\nabla\cdot\mathbf{u} + K\tilde{\epsilon}_{xx}.$$

There will be similar expressions for $\tilde{\tau}_{yy}$ and $\tilde{\tau}_{zz}$, but in them the coefficients must be the same — otherwise isotropy will be violated. Taking all together,

$$\tilde{\tau} = K\tilde{\epsilon} + (-p + C_2\nabla\cdot\mathbf{u})\mathbb{1}.$$

Now, we may go back to the original Cartesian system and find

$$\tau = K\epsilon + (-p + C_2\nabla\cdot\mathbf{u})\mathbb{1}.$$

The stress tensor is then also symmetric, a fact that is required in order the particle be torsion-free (remember the fact that rotations have no stress associated.)

A comparison with our previous result reveals $K = 2\mu$. The constant C_2 is called, in the theory of elasticity "the second Lamé coefficient", and receives the symbol λ (it is also called the "second viscosity coefficient", μ being the first.) Then,

$$\tau = 2\mu\epsilon + (-p + \lambda\nabla\cdot\mathbf{u})\mathbb{1}.$$

To make this explicit, this means that diagonal terms have the form

$$\tau_{ii} = 2\mu\epsilon_{ii} - p + \lambda\nabla\cdot\mathbf{u}, \tag{9.6}$$ {eq:tau_diagonal}

while off-diagonal terms are

$$\tau_{ij} = 2\mu\epsilon_{ij} \qquad j \neq i \tag{9.7}$$ {eq:tau_off_diagonal}

(Why not always work in the system of principal axes? The answer is simple: principal axes vary from one particle to another, since they are defined by local values of velocity derivatives. The Cartesian coordinate system, or any such (cylindrical, polar…) is the same for all particles.)

The off-diagonal terms have a neat expression when the strain rate tensor is written in term of velocity derivatives:

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

The diagonal ones, however, are somewhat puzzling:

$$\tau_{ii} = -p + 2\mu\frac{\partial u_i}{\partial x_i} + \lambda\nabla\cdot\mathbf{u}.$$

The pressure term is natural, but there are two extra dynamical terms.

Let us define a mechanical pressure as minus one-third times the trace of the stress tensor:

$$\bar{p} = -\frac{1}{3}\operatorname{Tr}\tau = p - \left(\frac{2}{3}\mu + \lambda\right)\nabla\cdot\mathbf{u}.$$

Defining volume viscosity as

$$\eta := \frac{2}{3}\mu + \lambda, \tag{9.8}$$ {eq:vol_visc_definition}

we may write the mechanical pressure as

$$\bar{p} := p - \eta\nabla\cdot\mathbf{u}. \tag{9.9}$$ {eq:mech_p_definition}

The result is that the mechanical pressure, defined this a way, is *different* from the thermodynamic pressure in an incompressible fluid. There are several ways out of this puzzling result. One of them is to assume simply that the fluid is incompressible. This is of course entirely correct, but would limit the applicability of the theory to incompressible problems.

Another approach is to boldly assume $(2/3)\mu + \lambda = 0$. This step was taken by Stokes, and defines a "Stokesian fluid". On the other hand, there is no clear evidence of a real fluid that may satisfy such a relationship. Indeed, the few measures of λ have show positive values (while μ, as should be clear, is always positive). We should then keep in mind that in some flows when compressibility is important, mechanical pressure may differ from thermodynamical one. One such example is the attenuation of sound waves, explained in Section 11.4.

The Navier-Stokes equation for Newtonian liquids is finally:

{eq:NS_Newtonian}

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + 2\nabla \cdot (\mu \epsilon) + \nabla [\lambda (\nabla \cdot \mathbf{u})] + \rho \mathbf{g}. \tag{9.10}$$

Pure shear and compression

Recalling the definition of the strain tensor ϵ in Eq. 9.2, the viscous force may be written as

$$\mathbf{f}_\text{v} = \nabla \cdot \left[\mu \left(\nabla \mathbf{u} + \nabla \mathbf{u}^\mathsf{T} \right) \right] + \nabla \cdot [\lambda (\nabla \cdot \mathbf{u}) \mathbb{1}] \, ,$$

where $\nabla \cdot [A \mathbb{1}]$ is just another way to write ∇A, for any scalar field A.

If the volume viscosity of Eq. 9.8 is introduced, the equation may be rearranged to read

:pure_stress_pure_compression}

$$\mathbf{f}_\text{v} = \nabla \cdot \left[\mu \left(\nabla \mathbf{u} + \nabla \mathbf{u}^\mathsf{T} - \frac{2}{3} (\nabla \cdot \mathbf{u}) \mathbb{1} \right) \right] + \nabla \cdot [\eta (\nabla \cdot \mathbf{u}) \mathbb{1}] \, . \tag{9.11}$$

This expression is rather elegant, since the term multiplied by μ describes pure shear flow, with no compression effects, and the term with η, pure compression.

This is easily demonstrated, since

$$\operatorname{Tr} \nabla \mathbf{u} = \nabla \cdot \mathbf{u} \qquad \Longrightarrow \qquad \operatorname{Tr} \left(\nabla \mathbf{u} - \frac{1}{3} (\nabla \cdot \mathbf{u}) \mathbb{1} \right) = 0.$$

The same goes for $\nabla \mathbf{u}^\mathsf{T}$, hence the $2/3$ factor.

9.2.2 Particular instances

We here consider instancies in which Eq. 9.10 for Newtonian liquid is further simplified.

Athermal case

It may happen that the viscosity coefficients do not vary, when the variations on the other fields are not too great. In particular, viscosity depends on temperature quite strongly, as is evident when heating up oil. The temperature has its own equation, to be explained in the next chapter. In this case, Eq. 9.10 may be written as

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + 2\mu \nabla \cdot \epsilon + \lambda \nabla (\nabla \cdot \mathbf{u})) + \rho \mathbf{g}.$$

Now, $\nabla \cdot \nabla \mathbf{u} = \nabla^2 \mathbf{u}$, as can be demonstrated since the i-th component of the resulting vector is

$$(\nabla \cdot \nabla \mathbf{u})_i = \sum_{j=1,2,3} \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial x_j} = \sum_{j=1,2,3} \frac{\partial^2}{\partial x_j^2} u_i = \nabla^2 u_i.$$

However, $\nabla \cdot \nabla \mathbf{u}^\mathsf{T} = \nabla(\nabla \cdot \mathbf{u})$:

$$(\nabla \cdot \nabla \mathbf{u}^\mathsf{T})_i = \sum_{j=1,2,3} \frac{\partial}{\partial x_j} \frac{\partial u_j}{\partial x_i} = \frac{\partial}{\partial x_i} \sum_{j=1,2,3} \frac{\partial u_j}{\partial x_j} = (\nabla(\nabla \cdot \mathbf{u}))_i.$$

This means the NS equation can be written, in this limit, as

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + \mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla(\nabla \cdot \mathbf{u})) + \rho \mathbf{g}. \qquad (9.12)$$

{eq:NS_const_viscs}

If the mechanical pressure (9.9) is used instead,

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla \bar{p} + \mu \nabla^2 \mathbf{u} + \frac{1}{3} \mu \nabla(\nabla \cdot \mathbf{u})) + \rho \mathbf{g}, \qquad (9.13)$$

{eq:NS_const_viscs_mech_press}

where one the shear viscosity appears (the other one being swept under the rug, in the mechanical pressure.)

Incompressible, athermal case

If, in addition to having constant viscosity coefficients, the flow is incompressible, the terms with $\nabla \cdot \mathbf{u}$ in the previous section may be neglected.

The final momentum equation for an incompressible, athermal fluid, is then

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{g}. \qquad (9.14)$$

{eq:NS_usual}

This equation, in this particular form, is the beginning of a vast ammount of research in physics and applied mathematics.

9.3 Dimensionless variables: the Reynolds number

For the simplest, athermal incompressible case, the term due to viscosity

$$\mu \nabla^2 \mathbf{u} = \mu \frac{u_0}{L^2} \nabla^{*2} \mathbf{u}^*,$$

where we cast the variables into reduced form, as explained in section 3.6.

Recall that in order to arrive to 3.13, the whole equation was multipled by $L/(\rho_0 u_0^2)$. If we do that to our momentum equation, the result is

$$\rho^* \frac{d\mathbf{u}^*}{dt^*} = -\nabla^* p^* + \rho^* \mathbf{g}^* + \frac{\mu}{\rho_0 L u_0} \nabla^{*2} \mathbf{u}^*.$$

The term $\mu/(\rho_0 L u_0)$ must be dimensionless (as can be easily checked). It then represents a reduced viscosity, and should be taken as such: a number that defines whether viscosity is important or not in a given context.

Historically, however, it is its inverse that has a name, the Reynolds' number:

$$\mathbf{Re} = \frac{\rho_0 L u_0}{\mu}. \tag{9.15}$$

This number is therefore large when viscosity is small, and small when it is large. It may also be defined as the ratio of inertial forces and viscous forces:

$$\mathbf{Re} = \frac{\rho_0 u_0^2}{\mu u_0 / L}. \tag{9.16}$$

Indeed, in the numerator $\rho_0 u_0^2$ is the typical strength of the pressure, and in the denominator $\mu u_0 / L$ is the typical strength of viscous stress forces.

In many mathematical contexts, all dimensions are forfeited, and the momentum equation is simply written as

{eq:NS_usual_reduced}

$$\frac{d\mathbf{u}}{dt} = -\nabla p + \frac{1}{\mathbf{Re}} \nabla^2 \mathbf{u} + \mathbf{g}. \tag{9.17}$$

9.4 Vorticity

Let us begin applyy the method in Sec. 9.4 to our momentum equation for a Newtonian fluid undergoing athermal incompressible flow,9.14, in order to obtain an equation for the vorticity and velocity alone.

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\nabla \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{g},$$

(where "p" is actually "p/ρ", as customary in incompressible flow), we may begin modifying the convective acceleration using Lamb's identity 3.8:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\nabla(u^2) - \mathbf{u} \times \boldsymbol{\omega} = -\nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{g}.$$

The viscous term may be written as $\nabla^2 \mathbf{u} = -\nabla \times \boldsymbol{\omega}$, from the expression for the curl of a curl 6.14 for incompressible flows. Taking the curl of this expression, all the terms that are gradients of fields vanish, and we get

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = -\nu \nabla \times \nabla \times \boldsymbol{\omega}.$$

Now, the curl of a vector product may be simplified as in Section 3.4, $\nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = -\mathbf{u}\nabla\boldsymbol{\omega} + \boldsymbol{\omega}\nabla\mathbf{u}$ (again, valid for incompressible flow). The double curl can again be cast as minus the Laplacian, since the curl is always divergence-free.

Therefore,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u}\nabla\boldsymbol{\omega} = \boldsymbol{\omega}\nabla\mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}.$$

Finally,

$$\frac{d\boldsymbol{\omega}}{dt} = \boldsymbol{\omega}\nabla\mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}.$$

This is clearly the Euler expression for the evolution of vorticity, times an extra, diffusion term. The constant of diffusivity is ν, the same as velocity. It may seem that little is changed from that situation. The reality is much different: it is actually the boundary conditions which are usually sources of vorticity, which is then carried onto the rest of the fluid by a combination of advecion and diffusion. The ratio of these two processes depend of course on the particular fluid (through ν) and the flow features.

9.5 Exercises

1. Cast Eq. 9.1 as
$$\frac{d\Omega_k}{dt} = \frac{1}{2}\epsilon_{ijk}\left[(\nabla \mathbf{u})_{ij} - (\nabla \mathbf{u})_{ji}\right],$$
where summation over repeated indices is implicit (Einstein's convention) , and ϵ is the Levi-Civita symbol, see Exercise 2 of Section 3.7.

2. Check that $\mu/(\rho_0 L u_0)$ is indeed dimensionless.

Chapter 10

The energy equation

In addition to continuity and momentum, there is an additional Navier-Stokes equation for the energy.

It contains the previous expression for ideal fluid, plus a term expressing energy dissipation by viscosity.

We must re-evaluate the work done on each of the faces of the particles due to stresses. For example, on the left wall the energy due to stress forces is

$$dW(\text{left}) = -(u_x dt)\tau_{xx} dydz - (u_y dt)\tau_{xy} dydz - (u_z dt)\tau_{xz} dydz.$$

Each of the stresses on this wall does work only in its direction of motion: τ_{xx} is compression and will feature a $-p$ term, which τ_{xy} and τ_{xz} produce shear forces. The minus sign stem from the sign convention, since on this wall shear stresses have directions opposed to the Cartesian axes. Similarly,

$$dW(\text{right}) = (u'_x dt)\tau'_{xx} dydz + (u'_y dt)\tau'_{xy} dydz + (u'_z dt)\tau'_{xz} dydz,$$

where the primed values mean those fields may be different from the left wall. Expanding in Fourier series, and adding everything up,

$$dW(\text{left, right}) = \frac{\partial u_x \tau_{xx}}{\partial x} dtdxdydz + \frac{\partial u_y \tau_{xy}}{\partial x} dtdxdydz + \frac{\partial u_z \tau_{xz}}{\partial x} dtdxdydz,$$

or, we find for the power

$$\frac{dW(\text{left, right})}{dt} = dV \frac{\partial}{\partial x} \left(u_x \tau_{xx} + u_y \tau_{xy} + u_z \tau_{xz} \right) = dV \frac{\partial}{\partial x} \sum_j u_j \tau_{xj}$$

Adding the other four walls, we have:

$$\frac{dW}{dt} = dV \sum_i \frac{\partial}{\partial x_i} \sum_j u_j \tau_{ij}.$$

Since the stress tensor is symmetric, we may write the latter as

$$\frac{dW}{dt} = dV \nabla \cdot (\tau \cdot \mathbf{u}),$$

Now, the energy equation is, from the First Law:

$$\frac{dE}{dt} = \frac{dW}{dt} + \frac{dQ}{dt} = dV\nabla \cdot (\tau \cdot \mathbf{u}) - dV\nabla \cdot \mathbf{q}.$$

(The second term, due to heat flux, does not change from the inviscid case.)

Dividing by the mass of the particle,

{eq:NS_en_2}

$$\rho\frac{d\epsilon}{dt} = \nabla \cdot (\tau \cdot \mathbf{u}) - \nabla \cdot \mathbf{q}. \tag{10.1}$$

(In this equation, $\epsilon = E/M$, as in 8.1, not the strain rate.)

The term $\nabla \cdot (\tau \cdot \mathbf{u})$ may be written applying the chain rule carefully:

{eq:nabla_tau_u}

$$\nabla \cdot (\tau \cdot \mathbf{u}) = \tau : \nabla\mathbf{u} + \mathbf{u} \cdot (\nabla \cdot \tau), \tag{10.2}$$

where ":" means a total reduction of two tensors, $a : b = \sum_{ij} a_{ij}b_{ij}$, and $\nabla\mathbf{u}$ is the tensor with components $\partial u_i/\partial x_j$ (as introduced in Equation 3.1 in the context of convective acceleration.)

We now just follow the steps already employed when deriving the energy equation for an inviscid fluid (Eqs 8.2 and 8.3). The $\nabla \cdot \tau$ appears in the general Navier-Stokes momentum equation 9.4:

$$\nabla \cdot \tau = \rho\left(\frac{d\mathbf{u}}{dt} - \mathbf{g}\right).$$

Therefore,

$$\mathbf{u} \cdot (\nabla \cdot \tau) = \rho\left[\frac{1}{2}\frac{du^2}{dt} - \mathbf{g} \cdot \mathbf{u}\right] = \rho\frac{d(u^2/2 - \mathbf{g} \cdot \mathbf{r})}{dt}.$$

The conclusion is then that the energy equation 10.1 may be expressed as a law for the specific internal energy:

$$\rho\frac{de}{dt} = \tau : \nabla\mathbf{u} - \nabla \cdot \mathbf{q}.$$

This looks more similar to 8.3 for an ideal fluid if we split the stress tensor into the pressure diagonal and the rest:

$$\tau = \tau' - p\mathbb{1} \qquad \Longrightarrow \qquad \tau' : \nabla\mathbf{u} = \tau' : \nabla\mathbf{u} - p\nabla \cdot \mathbf{u},$$

hence

$$\rho\frac{de}{dt} = -p\nabla \cdot \mathbf{u} - \nabla \cdot \mathbf{q} + \Phi.$$

The term Φ collects the result of viscosity and is termed the "dissipation function":

$$\Phi = \tau' : \nabla\mathbf{u}.$$

This term should always be positive if the Second Law is to hold: viscosity can only subtract energy from the system, never add it. Up until now our derivation has been generic. For a Newtonian fluid, however, one may go a bit further, and write:

{eq:Phi_Newtonian}

$$\Phi = \mu\left[2\sum_i \epsilon_{ii}^2 + 4(\epsilon_{12}^2 + \epsilon_{13}^2 + \epsilon_{23}^2)\right] + \lambda\nabla\cdot\mathbf{u}^2. \tag{10.3}$$

This looks deceptively positive unconditionally. However, there is no reason λ should be positive. It is a simple exercise to show that the conditions this term be positive are:

{eq:mu_lambda_cond}

$$\mu \geq 0 \qquad 3\lambda + 2\mu \geq 0. \tag{10.4}$$

TODO: exercise on this

The first one comes as a relief, since a μ would be quite unphysical. The second limits the value of λ to regions equal to, or above, $-(2/3)\mu$. This latter term is precisely zero for "Stokesian" fluids, as is obvious (since the corresponding term does not appear at all in the stress tensor for these hypothetical fluids.)

10.1 Exercises

1. Check identity 10.2. Hint: use element notation.
2. Show that the conditions in 10.4 are indeed needed in order Φ in 10.3 be always positive. (Hint: look for "postitive-definite quadratic form". The expression for Φ can be expressed in such a way, and three conditions are obtained for this positiveness. However, one of them is $2\mu - \lambda \geq 0$, which is less restrictive than the other two taken together, so only the two conditions quoted remain.)

Chapter 11

Simple solutions to the NS equations

11.1 Couette flow

As a simple solution, let us derive the flow that was given as an example in our derivation of section 9.2.1. A plane moves in the x direction, parallel of a fixed plane, and separated a distance L from it. The velocity field is supposed to depend only on y and have reached a steady state. (Notice that these assumptions restrict our solution space to a very limited choice. Since the equation are known not to comply with unicity, there may be other solutions, as indeed there are.)

While this particular geometry may seem artificial, the original Couette aparatus used a fluid between two coaxial cylinders. It is quite easy to assemble and is one of the first accurate viscometers. This flat geometry may be thought of as the limit of a thin fluid layer between the curved surfaces.

In this flow, the total derivative in the momentum equation is zero. The partial derivative is zero in steady state, and the non-linear term also is, since $\mathbf{u}\nabla\mathbf{u}$ is

$$u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} = 0.$$

Also, $\nabla \cdot \mathbf{u} = 0$, so the flow will always be incompressible. The pressure then is constant, since it does not need to "ensure" incompressibility.

The equation reduces then to

$$\mu \frac{\partial^2 u_x}{\partial y^2} = 0.$$

The viscosity is then seen to be of no importance (other than it is needed to reach a steady state, as shown below). The solution is simply a linear function of y. The particular shape of the function is given by the boundary conditions. If we take the usual no-slip conditions, the velocity matches the velocity at the walls:

$$u_x(y = 0) = 0 \qquad u_x(y = L) = u_0.$$

Therefore,

$$u_x = u_0 \frac{y}{L}.$$

These are called Dirichlet boundary conditions, since they fix the value of the field.

This is the solution assumed in previous sections, when introducing the viscosity coefficient. Even if the latter does not appear in the velocity field, the stress tensor has only one independent component:

$$\epsilon_{xy} = \epsilon_{yx} = \mu \frac{u_0}{L}.$$

The tensor is also constant throughout the fluid.

This has physical importance, since both plates will feel a total stress force

`{eq:Couette_force}`

$$F = A\epsilon_{xy} = \mu A \frac{u_0}{L}. \tag{11.1}$$

This force must be maintained on the moving plate in order to keep the steady flow (the fixed one must be anchored, and should resist the same force in order not to be dragged along). Energy must then be provided to the system, which is dissipated by viscosity. The power into the system will be

$$F u_0 = \mu A \frac{u_0^2}{L} = \mu V \left(\frac{u_0}{L}\right)^2,$$

where $V = AL$ is the total fluid volume.

Lastly, let us consider the volumetric flux:

$$Q = \int_A v_x = \int_0^H dz \int_0^L dy v_x(y) = H u_0 \int_0^L \frac{y}{L} dy = HL u_0 \int_0^1 y' dy' = \frac{1}{2} A u_0.$$

The mean velocity is defined as Q/A. Therefore,

$$\bar{u} = \frac{1}{2} u_0,$$

so the solution may be written as

$$u_x = 2\bar{u} \frac{y}{L}.$$

11.1.1 Start-up of Couette flow

It is not too difficult to solve the Navier-Stokes equations for non-steady Couette flow. In this case, the partial time derivative will not be absent, and

$$\frac{\partial u_x}{\partial t} = \nu \frac{\partial^2 u_x}{\partial y^2}.$$

Boundary conditions are as above, and let us consider the initial fluid is at rest.

It is easer to work out the solution by substrating the known steady state:

$$u_x(y,t) = u(y,t) + u_0 \frac{y}{L} \qquad u(y,t) = u_x(y,t) - u_0 \frac{y}{L}$$

The reason is that the boundary conditions for the velocity field about the steady state are homegeneous:

$$u(y=0,t) = u(y=L,t) = 0$$

Then, one may use the standard technique of separation of variables:

$$u(y,t) = Y(y)T(t).$$

The equation of motion then reads

$$YT' = \nu T Y'' \qquad \Longrightarrow \qquad \frac{T'}{\nu T} = \frac{Y''}{Y} = -c$$

The last equality follows because functions of two independent variables can only be equal if constant.

Therefore

$$Y'' = -cY,$$

whose n-th solution, given the boundary conditions, is

$$Y_n = A_n \sin(n\pi y/L),$$

where n is an integer greater than zero. The constant must then be $c = (n\pi/h)^2$. The corresponding time function is then,

$$T_n = \exp(-c_n t) = \exp(-n^2\nu(\pi/L)^2 t).$$

In general, the solution will be a combination of all possible modes:

$$u_x(y,t) = u_0 \frac{y}{L} + \sum_n A_n \exp(-n^2\nu(\pi/L)^2 t)\sin(n\pi y/h),$$

where it is clearly seen that mode n decays in a time that has an n^2/ν dependence. The longest-lived one is then $n = 1$, and modes with shortest wavelength decay quadratically faster. Also, the viscosity sets the time-scale of the process: high viscosity means shorter relaxation times. In the limit of no viscosity all times diverge, since a fluid without viscosity is unable to transmit the stress produced by the moving plane.

The A_n are given by the initial condition:

$$u_x(y,t=0) = u_0 \frac{y}{L} + \sum_n A_n \sin(n\pi y/h) = 0,$$

which is a standard exercise in Fourier series analysis. The result may be shown to be

$$u_x(y,t=0)/u_0 = \frac{y}{L} + \frac{2}{\pi}\sum_n \frac{(-1)^n}{n}\sin(n\pi y/h).$$

This can also be written as

$$u_x(y,t=0)/u_0 = \frac{y}{L} - \frac{2}{\pi}\sum_n \frac{1}{n}\sin(n\pi(1-y/h)),$$

where the last sine term is always starts with a positive slope close to $y = h$. The negative sign of every term in the expansion means that all of them are trying very hard in order to push down the final steady-state linear solution toward the initial one, which is null.

11.1.2 Temperature

For the steady state, the temperature equation reduces to

$$0 = k\frac{\partial^2 T}{\partial y^2} + \Phi.$$

The dissipation function in this case is simply

$$\Phi = \mu \left(\frac{\partial u_y}{\partial x}\right)^2 = 4\mu\bar{u}^2 \left(\frac{1}{L}\right)^2.$$

The equation for the energy is therefore

$$0 = k\frac{\partial^2 T}{\partial y^2} + 4\frac{\mu\bar{u}^2}{L^2},$$

where k is the thermal conductivity (units of power / (length × temperature), the whole equation has units of power / volume .) The boundary conditions needed may be the temperature at the two walls:

$$T(y=0) = T_0 \qquad T(y=L) = T_1,$$

also known as the "no-jump" temperature conditions. The fluid is supposed to have the same temperature as the walls under this framework. Others may be easily explored, such as fixed energy influx, which translate into conditions for the temperature derivatives at the walls (also known as Neumann boundary conditions). If the derivative is null, one has an adiabatic wall (aka homegeneous Neumann).

Before solving the equation, let us cast it into dimensionless form, by reducing the temperature by its value on one wall: $T^* = T/T_0$. Similarly, $y^* = y/L$. Then,

$$0 = k\frac{T_0}{L^2}\frac{\partial^2 T^*}{\partial y^{*2}} + 4\frac{\mu\bar{u}^2}{L^2},$$

or

$$\frac{\partial^2 T^*}{\partial y^{*2}} = -4\frac{\mu\bar{u}^2}{kT_0} = -4\mathrm{Br},$$

where we define the important Brinkman number:

{eq:Brinkman}

$$\mathrm{Br} = \frac{\mu\bar{u}^2}{kT_0}. \tag{11.2}$$

The number measures the importance of viscous dissipation over temperature dissipation.

The final solution is:

$$T^* = 1 + \frac{T_1 - T_0}{T_0}y^* + 2\mathrm{Br}y^*\left(1 - y^*\right).$$

The first two terms ensure the boundary conditions are satisfied, and would be the only ones pressent if there were no viscous dissipation. The latter term provides the needed second derivative, and vanishes at the walls.

11.2 Poiseuille flow

11.2.1 Planar flow

As with Couette flow, let us assume the only component of the velocity field is $u_x(y)$, a function of y only.

The steady 2D Navier-Stokes equations read

$$0 = -\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u_x}{\partial y^2} \tag{11.3}$$

$$0 = -\frac{\partial p}{\partial y}. \tag{11.4}$$

where p is actually p/ρ. The second one stablishes that p is a function of x only. But in the first one, its derivative is related to a second derivative of a funcion of y only. It follows that both equations should equal some constant:

$$\frac{\partial p}{\partial x} = \nu \frac{\partial^2 u_x}{\partial y^2} = -c.$$

I.e. $p = -cx$, plus a constant pressure which makes no difference. Notice the minus sign: we consider a pressure drop in the x direction if $c > 0$.

For the velocity, we must solve for

$$\frac{\partial^2 u_x}{\partial y^2} = -\frac{c}{\nu},$$

given the boundary conditions $u_x(y = 0) = u_x(y = L) = 0$.

The solution is

$$u_x = \frac{c}{2\nu} y(L - y) \tag{11.5}$$

{eq:Poiseuille_u}

The flow is

$$Q = \frac{HLcL^2}{12\nu},$$

and the mean velocity,

$$\bar{u} = \frac{cL^2}{12\nu},$$

Which let us write, more elegantly,

$$u_x = 6\bar{u}\, \frac{y}{L} \left(1 - \frac{y}{L}\right).$$

11.2.2 Temperature

As in Couette planar flow, 11.1, the temperature equation reduces to

$$0 = k \frac{\partial^2 T}{\partial y^2} + \mu \left(\frac{\partial u_y}{\partial x} \right)^2,$$

or

$$0 = k \frac{\partial^2 T}{\partial y^2} + 36 \mu \bar{u}^2 \left[\frac{1}{L} \left(1 - 2\frac{y}{L}\right) \right]^2.$$

Casting it into dimensionless form,

$$0 = k\frac{T_0}{L^2}\frac{\partial^2 T^*}{\partial y^{*2}} + 36\frac{\mu\bar{u}^2}{L^2}\left(1-2y^*\right)^2,$$

or

$$\frac{\partial^2 T^*}{\partial y^{*2}} = -36\text{Br}\left(1-2y^*\right)^2,$$

where the Brinkman number is again as in Eq. 11.2

$$\text{Br} = \frac{\mu\bar{u}^2}{\kappa T_0}.$$

If we define $s = 2y^* - 1$, then

$$4\frac{\partial^2 T^*}{\partial s^2} = -36\text{Br}s^2,$$

or

$$\frac{\partial^2 T^*}{\partial s^2} = -9\text{Br}s^2.$$

The final solution is then:

$$T^* = 1 + \frac{T_1 - T_0}{T_0}y^* + \frac{3}{4}\text{Br}\left(1 - s^4\right).$$

The second term is seen to vanish at the two walls (where $s = \pm 1$), while providing the correct second derivative. In terms of y^*,

$$T^* = 1 + \frac{T_1 - T_0}{T_0}y^* + \frac{3}{4}\text{Br}\left[1 - \left(2y^* - 1\right)^4\right].$$

11.2.3 Flow in circular pipes

The solution is

$$u_x = \frac{c}{4\nu}\left(R^2 - r^2\right).$$

The flow is

$$Q = \frac{c\pi R^4}{8\nu},$$

and the mean velocity,

$$\bar{u} = \frac{cR^2}{8\nu},$$

11.2.4 Temperature

At variance with planar flows, the temperature equation must be written in polar coordinates:

$$0 = k\frac{1}{r}\frac{d}{dr}\left[r\frac{dT}{dr}\right] + \mu\left(\frac{du_z}{dr}\right)^2,$$

or

$$0 = k\frac{1}{r}\frac{d}{dr}\left[r\frac{dT}{dr}\right] + 16\mu\bar{u}^2\frac{r^2}{R^4}$$

Casting it into dimensionless form,

$$0 = kT_\mathrm{w} \frac{1}{R^2} \frac{1}{r^*} \frac{d}{dr^*} \left[r^* \frac{dT^*}{dr^*} \right] + 16\mu\bar{u}^2 \frac{1}{R^1} r^{*2},$$

or

$$\frac{1}{r^*} \frac{d}{dr^*} \left[r^* \frac{dT^*}{dr^*} \right] = -16\mathrm{Br} r^{*2},$$

where the Brinkman number is as in Eq. 11.2. In order to solve it, we change it to

$$\frac{d}{dr^*} \left[r^* \frac{dT^*}{dr^*} \right] = -16\mathrm{Br} r^{*3},$$

or

$$\frac{dT^*}{dr^*} = -4\mathrm{Br} r^{*3} + \frac{c}{r^*}.$$

Then,

$$T^* = -\mathrm{Br} r^{*4} + c \log(r^*) + d.$$

The *log* term has a singularity at $r^* = 0$, so $c = 0$. The other constant has to be fixed in order to comply with the boundary condition $T^*(r^* = 1) = 1$. The final answer is

$$T^* = 1 + \mathrm{Br} \left(1 - r^{*4} \right),$$

or, bringing back the units for length and temperature:

$$T = T_\mathrm{w} + \mathrm{Br} \left[1 - \left(\frac{r}{R} \right)^4 \right],$$

11.3 Taylor-Green vortices

The Taylor-Green vortex sheet is a solution to the 2D Navier-Stokes equations for an incompressible Newtonian fluid that describes a periodic array of vortices. The vortex pattern repeats itself in the x and$y$ directions with a periodic length L:

$$\begin{aligned} u_x &= f(t) u_0 \sin kx \cos ky \\ u_y &= -f(t) u_0 \cos kx \sin ky, \end{aligned}$$

where $k = 2\pi/L$, and the function$f(t)$ is

$$f(t) = \exp(-2\nu k^2 t),$$

so that the decay time of the vortices due to viscosity is given by $\tau = 1/(2\nu k^2)$. The maximum modulus of the velocity field at time zero is u_0.

The pressure field is given by

$$p = \frac{\rho u_0^2}{4} f(t)^2 \left(\cos(2kx) + \cos(2ky) \right).$$

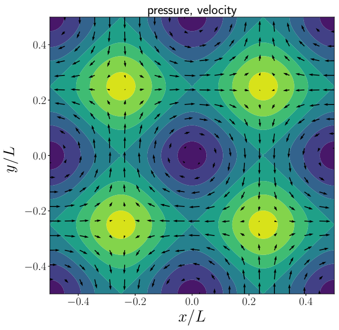

Figure 11.1: Taylor-Green vortices. Contour of iso-pressure (isobars), velocity as arrows

Hence the vortices go around zones of low pressure, either clockwise or counter-clockwise (see Figure 11.1.)

Insertion of these two fields into the Navier-Stokes equation shows that indeed this is a solution. It is interesting that the pressure gradient term exactly cancels the convective one, while the viscosity term cancels the partial derivative. That means that in the inviscid limit the vortices will never decay.

The vorticity field is given by

$$\omega = 2f(t)\sin(kx)\sin(ky)$$

and the stream function is just $\psi = \omega/2$. Notice the vorticity satisfies the convection-diffusion equation

$$\frac{d\omega}{dt} = \nu\nabla^2\omega.$$

11.3.1 Reduced units

Let us introduce the dimensionless time, built from time, maximum initial velocity, and typical length L (another choice would be $L/2$, which is the actual length of a vortex)

$$t^* = tu_0/L.$$

Notice that $t^* = 1$ is the time a fluid particle would need to travel a distance L.

Function f can be written as

$$f(t) = \exp(-2\nu(2\pi/L)^2(Lt^*/L)) = \exp(-8\pi^2 t^*/\mathrm{Re}),$$

where the Reynolds number appears naturally:

$$\mathrm{Re} := Lu_0/\nu.$$

The decay time is then seen to be $\tau^* = \mathrm{Re}/(8\pi^2)$ in reduced units.

11.4 Attenuation of sound waves*

This section contains a simple discussion of the role of viscosity in the attenuation of sound waves. Original work is by Stokes, 1845.

Let us start with the Navier-Stokes momentum equation, which will be linearized by disregarding high-order deviations from equilibrium values. The two viscosity coefficients always multiply the velocity terms. Therefore, they are at to be taken at their equilibrium values, since any departure would entail higher order terms. This points out 9.12 as our starting point — even if this case is not athermal, the fact that the variation of the viscosity coefficients is neglected leads to the same equation.

After linearizing, the two relevant equations are continuity and momentum:

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{u} = 0 \qquad (11.6)$$

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -c^2 \nabla \rho' + \mu \nabla^2 \mathbf{u} + (\mu + \lambda) \nabla (\nabla \cdot \mathbf{u}). \qquad (11.7)$$

{eq:NS_Newtonian_const_visc}

The same assumptions as those in Eqs. 5.1-5.3 have been made. Also, we write $\kappa = c^2$ for the proportinality of pressure modulations with density modulations, where c is the sound speed when viscosity is neglected. The subscript "$_0$" is not written for the viscosity coefficients for simplicity.

Eq. 11.7 may seem different from previous expressions, but it is the correct one when μ is constant, but incompressibility does not hold. In this case, the term in 9.10 is not zero, but rather gives rise to a $(\mu + \lambda)\nabla(\nabla \cdot \mathbf{u})$ term.

Our systems is not as simple as the one in Section 5, where the continuity and momentum equations where equally simple. There, we could easily eliminate either the velocity or the density from our equations in order to get a single equation for the other field. Now, the continuity equation is the same, but the one for momentum one is much more involved. This leaves us with eliminating the density as the only viable scheme.

If we apply the gradient operator to continuity, and differentiate with respect to space in momentum, we arrive to this equation for the velocity only:

$$\frac{\partial^2}{\partial t^2} \mathbf{u} = c^2 \nabla \cdot (\nabla \cdot \mathbf{u}) + \nu \nabla^2 \frac{\partial \mathbf{u}}{\partial t} + (\zeta - \nu) \nabla (\nabla \cdot \frac{\partial \mathbf{u}}{\partial t}), \qquad (11.8)$$

{eq:sound_att_Fourier}

where $\nu = \mu / \rho_0$, the usual kinematic viscosity, and we define, for future convenience, an additional kinematic viscosity $\zeta = 2\nu + \lambda / \rho_0$.

Clearly, this reduces to our previous sound wave equation if there was no viscosity.

Let us try a harmonic wave solution of the form

$$\mathbf{u} = \Re \left[\mathbf{u}_0 e^{i(\beta x - \omega t)} \right]. \qquad (11.9)$$

{eq:wave_form_visc}

Now $\mathbf{u}_0$ is a fixed amplitude and polarization vector, which may be parallel to $\hat{\mathbf{e}}$ in a longitudinal sound wave, but may also be perpendicular to it, for a traverse wave. In general, of course, a wave may be a combination of both.

Notice the usage of complex exponentials, which is common in wave physics. Of course, the actual solution is real, so usually the real part of the complex solution is kept (sometimes it is the imaginary, which is equally valid). The flexibility comes partly from the fact that $\mathbf{u}_0$ could be complex, which would represent a phase. In our case, this is not important, but rather the possibility that β may be complex. In this case, its values encapsulates both a real wave number and an spatial decay. Indeed, if

$$\beta = k + \alpha i,$$

then

$$\mathbf{u} = \mathbf{u}_0 e^{-\alpha x} e^{i(kx - \omega t)},$$

which clearly identifies $\omega = 2\pi/\lambda$ as the (real) wave-number for a wavelenght λ, and $\alpha = 1/\ell$, as a sound attenuation coefficient, with ℓ the attenuation length.

Now, the time derivative is

$$\frac{\partial \mathbf{u}}{\partial t} = -i\omega \mathbf{u},$$

and the second derivative,

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = -\omega^2 \mathbf{u}.$$

But, it is quite interesting that the two second space derivatives are not the same. The Laplacian is

$$\nabla^2 \mathbf{u} = -\beta^2 \mathbf{u},$$

while the gradient of the divergence is

$$\nabla(\nabla \cdot \mathbf{u}) = -\beta^2 u_x \hat{\mathbf{e}}_x,$$

where $u_x = \mathbf{u} \cdot \hat{\mathbf{e}}_x$ is the longitudinal component of the sound wave. The latter then acts only on longitudinal waves: tranverse waves are inherently divergence-free, since depend on one coordinate but point toward a perpendicular direction (as in Couette and Poiseuille flows). Let us consider them separately.

11.4.1 Longitudinal waves

In this case, insertion of the wave form 11.9 into Eq. 11.7 yields

$$-\omega^2 = -c^2\beta^2 + i\zeta\omega\beta^2. \tag{11.10}$$

The expression includes the volume bulk viscosity, as seems fitting for a longitudinal disturbance, which involves compression. The appearance of the shear viscosity may come as a surprise [1] The tensor μ term in Eq. 9.11, despite being traceless, nevertheless has a non-zero divergence, as can be easily

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[1] To be precise, the fact that η, the volume viscosity, is not the only viscosity coefficient that appears, this beign a bit obscured by our usage of ζ instead of μ and η.

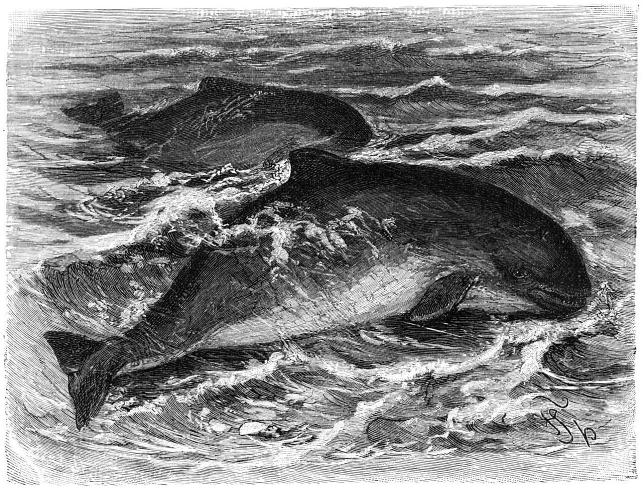

Figure 11.2: Harbor porpoises. From [11].

checked. This means this wave does not consist of pure compression, in spite of its appearance: it is also a shear wave (see Exercises 1 and 2.)

If viscosities were negligible, the solution is just

$$\omega = c\beta,$$

the usual dispersion relation for sound of Sec. 5. In general, though:

$$\beta^2 = \frac{\omega^2}{c^2 - i\zeta\omega} = \frac{\omega^2}{c^2}\frac{1}{1 - i\omega/\omega_c}, \tag{11.11}$$

{eq:waves_att_dispersion}

where we define the crossover angular frequency

$$\omega_c := \frac{c^2}{\zeta}.$$

This is the frequency above which the viscosity begins to be relevant. Numerical values can be found in Table 11.1. The frequency is really high for the human hearing range, which goes up to about 20 kHz for humans. The record in the animal kingdom is held by porpoises (11.2), but it is only about 160 kHz. Ultrasonic cleaners used in dentistry operate at 40 kHz. Medical ultrasound for imaging goes as high as 16 MHz. Only acoustic microscopy [15] reaches a few GHz, matching our estimate for air.

It is somewhat tedious to find the real and imaginary parts of β from Eq. 11.11 (the general solution is shown in Figure 11.3.) The resulting expressions are, moreover, not very iluminating except at low and high frequencies (with respect to ω_c). We find it better then to study these two limits from approximations to Eq. 11.11, and leave the general solution as Exercise 3, for the interested reader.

	c (m/s)	ν (m^2/s)	λ (m^2/s)	f_c (GHz)
water (15°C)	1480	1.13×10^{-6}	3.1×10^{-6}	68
air	340	1.48×10^{-5}	5×10^{-6} ?	0.5

Table 11.1: Numerical values for two important substances. Values with question marks are speculative, since in Cramer[3] air is said to have a negligible bulk viscosity. . . but in a graph this quantity's value is seen to be about half the shear viscosity value, and air is mostly nitrogen.

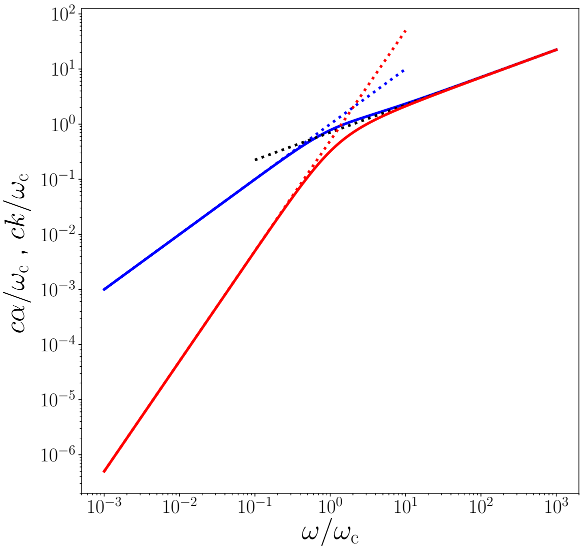

Figure 11.3: Plot of reduced wave-vector, kc/ω_c (blue), and attenuation coefficient, $\alpha c/\omega_c$ (red), as functions of reduced frecuency ω/ω_c. Dotted lines are assymptotic regimes at low and high frequencies.

Low frequencies

At frequencies much below the crossover frequency, we may expand the term in the denominator in a Taylor series, then again for the square root. The end result is

$$\beta = \frac{\omega}{c}\left(1 + i\frac{\omega}{2\omega_c}\right).$$

Therefore, the wave number is

$$\beta = \frac{\omega}{c},$$

as if there were no viscosity (this corresponds to the dotted blue line at low frequencies in Figure 11.3).

The attenuation coefficient is

$$\alpha = \frac{\omega^2}{2c\omega_c} = \frac{\zeta\omega^2}{2c^3}.$$

It therefore grows as the square of the frequency (red dotted blue line at low frequencies in Figure 11.3). This agrees is called "Stokes' law of sound" . Notice that the alternative form is often found [16]:

$$\alpha = \frac{\omega^2}{\rho_0 c^3}\left(\frac{2}{3}\mu + \frac{1}{2}\eta\right),$$

with $\eta = (2/3)\mu + \lambda$ the volume compressiblity.

This means that for that the highest sound in a porpoise's range, at 160 kHz the attenuation length would be about 8 km (in water, of course). This may be important for the long-range communication of these animals. For medical ultrasound at 16 MHz this length is $\approx$ 80 cm, with water values (a fair approximation for the human body.) This can have an impact for human tissues, in the sense that in a distance of some tens of centimeters the energy of the ultrasound waves is dissipated inside the body of the patient, and may cause unwanted heating.

High frequencies

At frequencies much higher than the crossover, we may neglect the "1" in the denominator of 11.11, to obtain

$$\beta^2 = \frac{i\omega\omega_c}{c^2}.$$

If we substitute the crossover frequency:

$$\beta^2 = \frac{\omega}{i\zeta} = \frac{i\omega}{\zeta}$$

Recalling $i = e^{(pi/2)i}$, the solution is

$$\beta = \sqrt{\frac{\omega}{\zeta}} e^{(\pi/4)i}.$$

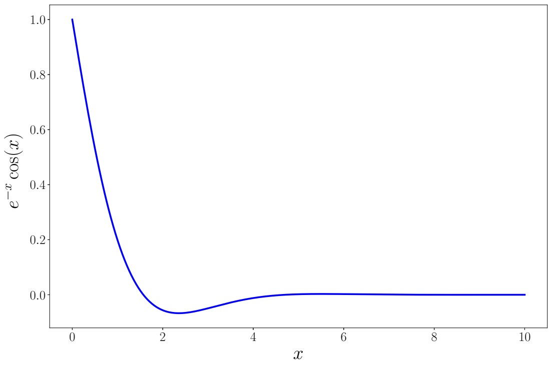

Figure 11.4: Graph of the function $e^{-x}\cos(x)$

Therefore

$$k = \alpha = \sqrt{\frac{\omega}{2\zeta}}.$$

This is an interesting result, since the attenuation coefficient is equal to the wave number (dotted black line at high frequencies in Figure 11.3).

The attenuation length is

$$\ell = \frac{1}{\alpha} = \sqrt{\frac{2\zeta}{\omega}}$$

The function looks therefore like a simple function

$$g(x) = f(x/\ell) \qquad f(x) = e^{-x}\cos(x).$$

In Figure 11.4, it is seen to decay very fast, with just one maximum of minimum of importance.

At the crossover frequency

A fast check on the above approximations is to see what happens when the frequency is exactly the crossover frequency. At this point, the growth of the attenuation coefficient as the square of frequency crosses over to a growth as the square root. This would mean a bend in a log-log plot, between two straight lines with different slopes. This is indeed apparent in Figure 11.3.

The extrapolation of the low frequency expression yields

$$\alpha_c \approx \frac{\omega_c}{2c},$$

whereas the high frequency expression yields

$$\alpha_c \approx \frac{\omega_c}{\sqrt{2}c},$$

which are similar.

The exact expression can be shown to be, after some involved algebra,

$$\alpha_c = \frac{\sin(\pi/8)\omega_c}{(2^{1/4}c},$$

a value just a bit below the other two. This means the approximations remain quite fair up to the limit of their respective ranges.

By the way, for water this value corresponds to an attenuation length of about 10 nm, which is a really short length, in the atomic range. For air, it is about 0.3 µm, the size of a very small cell.

11.4.2 Transverse waves

We tend to think sound waves are longitudinal. We here discuss how they may have a transverse component, but it is dampened at al frequencies.

If we write Eq. 11.8 for the y or z Cartesian component, we find

$$-\omega^2 = i\nu\omega\beta^2,$$

or

$$\beta^2 = \frac{i\omega}{\nu}.$$

Only ν is involved, which is sensible: transverse waves do involve only shear, not compression (at variance with longitudinal waves, which involve both.)

The expression is very similar to the one for longitudinal waves at high frequencies, only this one is valid at *all* frequencies. Its solution is

$$k = \alpha = \sqrt{\frac{\omega}{2\nu}}.$$

Figure 11.4 still applies to these waves, with the difference that the decay length is

$$\ell = \frac{1}{\alpha} = \sqrt{\frac{2\nu}{\omega}}$$

For frequencies of ordinary "sound", i.e. audible frequencies, that length is quite small. With the numerical data in the Table, that length is only about 0.7 mm for air, at a very low frequency of 10 Hz (just below the lower hearing threshold), and will decrease further as the inverse of the square root of the frequency.

11.5 Exercises

1. Write down the pure-shear stress tensor (see also Eq. 9.11):

$$\tau_{\mathrm{ps}} := \mu\left[\left(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathsf{T}} - \frac{2}{3}(\nabla\cdot\mathbf{u})\mathbb{1}\right)\right]$$

for our linear wave 11.9 in the case it is longitudinal. Show that it is traceless (which it must, by definition), but non-zero. This latter fact indicates that our wave is not purely compressive in nature.

2. Show that the volumetric force due to the previous pure-stress is

$$\mu \nabla \cdot \tau_{\mathrm{ps}} = -\frac{4}{3}\mu\beta^2 \mathbf{u}$$

3. Find the general solution of Eq. 11.11 for the real and imaginary parts of β. Hint: use reduced quantities $\beta' = c\beta/\omega$, $\omega' = \omega/\omega_{\mathrm{c}}$, in terms of which the equation reads simpler:

$$\beta'^2 = \frac{1}{1 - i\omega'}.$$

Then, write $\beta' = k' + i\alpha'$, find its square, and equate real and complex parts on both sides of the equation. The solution is

$$2\alpha'^2 = \frac{1}{\sqrt{1+\omega'^2}} - \frac{1}{1+\omega'^2}$$
$$2k'^2 = \frac{1}{\sqrt{1+\omega'^2}} + \frac{1}{1+\omega'^2}$$

11.6 Exercises

Prove the expression for the unsteady Couette flow

Chapter 12

The overdamped limit

The low Reynolds number is called the creeping, or over-damped regime. Let us generalize slightly the momentum equation Eq 9.14 to a general volumetric external force $\mathbf{f}$:

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f}.$$

We may again cast the equation into reduced form, to get

$$\rho^* \frac{d\mathbf{u}^*}{dt^*} = -\nabla^* p^* + \frac{1}{\mathrm{Re}} \nabla^{*2} \mathbf{u}^* . + \mathbf{f}^*,$$

where $\mathbf{f}^* = L/(\rho_0 u_0^2)\mathbf{f}$.

This may be trivially written as

$$\mathrm{Re}\, \rho^* \frac{d\mathbf{u}^*}{dt^*} = -\mathrm{Re}\, \nabla^* p^* + \nabla^2 \mathbf{u}^* + \mathrm{Re}\, \mathbf{f}^*.$$

As Re is very small, the left-hand side will vanish. However, there is no reason for the pressure and force terms on the right to do so. Indeed, the combination $\mathrm{Re}\, p^*$ may be written as $L/(\mu u_0)p$. That is, in this limit it is not the combination ρu_0^2 which sets the pressure scale, but rather $\mu u_0/L$, as seems fitting for a regime dominated by viscosity (recall Newton's law, 9.5). The same goes for the term $\mathrm{Re}\, \mathbf{f}^*$, in which the force is seen to be scaled by $\mu u_0/L^2$. We may use double asterisks are used for this new reduced pressure and force:

$$p^{**} = \left(\frac{L}{\mu u_0} \right) p$$

$$\mathbf{f}^{**} = \left(\frac{L^2}{\mu u_0} \right) \mathbf{f}$$

The equation will tend to

$$0 = -\nabla^* p^{**} + \nabla^2 \mathbf{u}^* + \mathbf{f}^{**}.$$

All time derivatives are gone from the equation! The only time variation that is sometimes considered is the case in which $\mathbf{f}$ is a explicit function of time. In this case, the velocity field adapts instantaneously following the changes in external force. Coming from previous chapters, devoted to cases where inertial effects are not negligible, this may come as a surprise — nevertheless, this is the correct description of a regime which is important in fields such as microfluidics and the biological physics of small systems (from small insects down to large molecules). Mathematically, the resulting equation is now linear in the velocity, which makes its treatment simpler.

An objection could be raised regarding why the pressure gradient and the external force terms are kept in the equation, despite being multiplied by the Reynolds number. The answer is that they are not in fact negligible, as an alternative scaling is employed. This is discussed in the next section.

12.0.1 Kolmogorov flow

In order to get an idea of the features of the flow (typical speed, time scale, Reynolds number), we may gain some insight from the solution from Kolmogorov flow.

This is an exact solution of the momentum equations when there is a periodic applied force in the x direction that depends on y only

$$\mathbf{f} = f_0 \cos\left(\frac{2\pi y}{L}\right)\mathbf{e}_x.$$

In this case, the typical length-scale L_0 and velocity u_0 are set by the force. The former is clearly $L_0 = L$, but the velocity is actually part of the solution. It is therefore more convenient to introduce an alternative way to cast the momentum equation in non-dimensional form. First, dividing by f_0 and going to dimensionless spacial coordinates:

{eq:Navier-Stokes_nondim1}

$$\frac{\rho}{f_0}\frac{d\mathbf{u}}{dt} = \frac{\mu}{f_0 L^2}(\nabla^*)^2\mathbf{u} - \frac{1}{f_0 L}\nabla^* p + \mathbf{f}^*(\mathbf{r}^*). \tag{12.1}$$

The viscous term sets the velocity scale:

{eq:reduced_u0}

$$u_0 = (f_0 L^2)/\mu. \tag{12.2}$$

On the left hand side of 12.1, the total derivative limits the setting of the time scale: $t_0 = L/u_0$. This means the equation may be written as

$$\frac{\rho L^3 f_0}{\mu^2}\frac{d\mathbf{u}^*}{dt^*} = (\nabla^*)^2\mathbf{u}^* - \nabla^* p^* + \mathbf{f}^*(\mathbf{r}^*),$$

or

{eq:Navier-Stokes_nondim2}

$$\mathrm{Re}\frac{d\mathbf{u}^*}{dt^*} = (\nabla^*)^2\mathbf{u}^* - \nabla^* p^* + \mathbf{f}^*(\mathbf{r}^*), \tag{12.3}$$

where we again find the Reynolds number, given as

$$\mathrm{Re} = \frac{\rho L^3 f_0}{\mu^2}.$$

This definition looks rather different from the standard one, $\mathrm{Re} = \rho L u_0/\mu$, but upon insertion of the typical velocity both are seen to coincide.

Notice that the reduced response time is given by the Reynolds number. This is a sort of "double-reduced" time, given by $t^{**} = t^{*}/\mathrm{Re}$. A high Reynolds number therefore means a long reduced time to respond, while a low Reynolds one the response is very rapid, and the equilibrium solution is reached very fast. Also, the pressure is reduced as $p^{*} = p/(f_0 L)$.

Let us apply this scaling to the Kolmogorov flow. Assuming that the resulting velocity, as the driving force, only has an x component varying on y (as for planar Couette and Poiseuille flows),

$$\rho \frac{\partial u_x}{\partial t} = \mu \frac{\partial u_x}{\partial y^2} - \nabla p + f_0 \cos(2\pi y/L). \tag{12.4}$$

`{eq:Kolmo_orig}`

As in previous flow patterns, we may set the pressure to some constant value. Therefore, in reduced units (dropping the askterisks for clarity),

$$\mathrm{Re} \frac{\partial u_x}{\partial t} = \frac{\partial u_x}{\partial y^2} u + \cos(2\pi y).$$

The solution is easily found:

$$u_x = \frac{1}{(2\pi)^2} \left(1 - e^{-(2\pi)^2 t/Re}\right) \cos(2\pi y).$$

Putting the scales back in, we find

$$u = \frac{f_0 L^2}{(2\pi)^2 \mu} \left(1 - e^{-(2\pi)^2 \mu t/(\rho L^2)}\right) \cos(2\pi y/L),$$

which indeed can be checked to be the solution to the original Equation 12.4.

12.1 General solution

In Equation 12.3, we may disregard completely the left hand side if the Reynolds number is very low. This leads to Stokes' equation :

$$\mu \nabla^2 \mathbf{u} - \nabla p = -\mathbf{f}(\mathbf{r}). \tag{12.5}$$

`{eq:Stokes}`

As mentioned, this is a *linear* equation for the velocity. We may get a general solution to the equation that gives us the velocity field for any external field, by using Fourier techniques.

In Fourier space all the fields are given as

$$\phi_{\mathbf{q}} = \int d\mathbf{r}\, \phi(\mathbf{r}) e^{-i\mathbf{q}\cdot\mathbf{r}} \tag{12.6}$$

`{eq:F_r_to_q}`

$$\phi(\mathbf{r}) = \frac{1}{(2\pi)^d} \int d\mathbf{q}\, \phi_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}}. \tag{12.7}$$

`{eq:F_q_to_r}`

We follow the convention that is usual in physics, in which Fourier components are indicated by an index — also, no tilde ("wiggly") hat is used for them for simplicity. Thus, $\phi_{\mathbf{q}}$ instead of $\tilde{\phi}(\mathbf{q})$.

Stokes' equation for Cartesian coordinate i reads [2]

$$-\mu q^2 \mathbf{u}_{\mathbf{q},i} - i\mathbf{q}_i p_{\mathbf{q}} = -\mathbf{f}_{\mathbf{q},i},$$

or

$$\mathbf{u}_{\mathbf{q},i} = \frac{1}{\mu q^2}\left[-i\mathbf{q}_i p_{\mathbf{q}} + \mathbf{f}_{\mathbf{q},i}\right]. \tag{12.8}$$

{eq:u_Fourier0}

Now, the pressure is fixed by incompressibility. The condition $\nabla \cdot \mathbf{u} = 0$, reads in Fourier space

$$\sum_i \mathbf{q}_i \cdot \mathbf{u}_{\mathbf{q},i}.$$

Multiplying Equation 12.8 by $\mathbf{q}_i$ and adding up the d equations:

$$\sum_i \mathbf{q}_i \mathbf{u}_{\mathbf{q},i} = \frac{1}{\mu q^2}\left[-iq^2 p_{\mathbf{q}} + q_i \mathbf{f}_{\mathbf{q},i}\right] = 0.$$

Therefore,

$$-q^2 p_{\mathbf{q}} = iq_i \mathbf{f}_{\mathbf{q},i}.$$

This looks like the Fourier version of a Poisson pressure equation:

$$\nabla^2 p = \nabla \cdot \mathbf{f}.$$

This is no coincidence, since in 12.5 we may use the identity $\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u} - \nabla \times \nabla \times \mathbf{u}$. The first term must be zero for an incompressible fluid, and applying $\nabla\cdot$ we may get rid of the first one too (since a rotational field is divergence free). The Poisson equation results. It is interesting that much the same equation is found in computational methods that enforce incompressibility by projection.

The pressure is then

{eq:p_Fourier0}

$$p_{\mathbf{q}} = -i\frac{\sum_i q_i \mathbf{f}_{\mathbf{q},i}}{q^2}. \tag{12.9}$$

With this result we may write Equation 12.8 as

{eq:u_sol_Fourier}

$$\mathbf{u}_{\mathbf{q},i} = \frac{1}{\mu q^2}\sum_j \left[\delta_{ij} - \frac{\mathbf{q}_i \mathbf{q}_j}{q^2}\right]\mathbf{f}_{\mathbf{q},j} =: \sum_j T_{ij}\mathbf{f}_{\mathbf{q},j}, \tag{12.10}$$

where the last equation the Oseen tensor is defined:

$$T_{\mathbf{q},ij} := \frac{1}{\eta q^2}\left[\delta_{ij} - \frac{\mathbf{q}_i \mathbf{q}_j}{q^2}\right].$$

In 3D, Equation 12.9 may be inverted back to real space to yield [9].

$$p(\mathbf{r}) = \frac{\mathbf{f}\cdot\mathbf{r}}{4\pi r^3}.$$

Oseen's tensor can also be inverted:

$$T_{ij}(\mathbf{r}) := \frac{1}{8\pi\eta r}\left(\delta_{ij} + \frac{\mathbf{r}_i \mathbf{r}_j}{r^2}\right)$$

But notice Equation 12.10 is a convolution in real space:

{eq:u_sol_real}
$$\mathbf{u}_i(\mathbf{r}) = \sum_j \int d\mathbf{r}' T_{ij}(\mathbf{r}' - \mathbf{r})\mathbf{f}_j(\mathbf{r}'). \tag{12.11}$$

In 2D, on the other hand, the Fourier expressions cannot be inverted. This is the famous "Stokes paradox": there are no solutions to his equation for steady translational motion in 2D (it also applies to 2D problems in 3D, such as the motion of an infinite cylinder).

12.1.1 Monolayers and membranes

In their influential 1975 article Saffman and Delbrück [10] discussed several ways out of this paradox, and found out the most satisfactory way was to take into account the viscosity of the surrounding liquid, $\mu_\mathbf{f}$.

In [8], Lubensky and Goldstein proposed a modification of the original equation for 2D liquids. This are indeed feasible in a lab, as Langmuir monolayers , in which a one molecule thick layer of lipids forms between water and air. Also, in membranes, where a 2D bilayer lipid is surrounded on both sides by water (these are more involved to study, and may be prone to curve). The equation of motion, in the creeping regime is

$$\mu\nabla^2\mathbf{u} - \nabla p = -\mathbf{f}(\mathbf{r}) - \mathbf{f}'(\mathbf{r}), \tag{12.12}$$
{eq:Stokes_2D}

where all variables are two-dimensional: μ is the 2D shear viscosity, p the 2D pressure, and $\mathbf{f}$ a force per area. All variables have their physical dimensions changed from the 3D case (and the rest of this book), but for the position $\mathbf{r}$ and the velocity $\mathbf{u}$ — both are in-plane, but their units do not change: length and length per time.

Notice that the fluid is taken to be Newtonian, which may not be such a good approximation for molecules as complex as lipids. In addition, the 2D flow is incompressible:

$$\nabla \cdot \mathbf{u} = 0$$

The solvent produces an additional term $\mathbf{f}'(\mathbf{r})$, to be added to any external force that causes the flow (which may be an external forcing due to the motion of an immersed body, or perhaps due to a concentration gradient [5]. Of course, this surrounding flow is caused by the 2D fluid, so this will be a sort of self-interaction effect, as we will see.

The solvent itself satisfies another momentum equation. It the solvent is taken to be also in the creeping regime:

$$\mu'\nabla'^2\mathbf{u}' - \nabla' p' = 0.$$

In this section, primes refer to volume properties of the solvent (also for ∇', which is our usual, 3D, del operator). In this equation μ' is the ordinary solvent shear viscosity, often simply water. It is also taken to be Newtonian, an approximation whose validity depends on the nature of the solvent. This flow is also incompressible:

$$\nabla' \cdot \mathbf{u}' = 0.$$

The reader may wonder how the 2D fluid affects the 3D one which is underneath (this is the case of monolayers, while in membranes there are two such regions, above and below.) The answer is, through the boundary condition at the surface:

$$\mathbf{u}'(z=0)=\mathbf{u}.$$

In addition to this condition, another one must be set. Usually, disturbances die out quite fast (as we will see below), so the deep-water limit is a good approximation:

$$\mathbf{u}'(z\to-\infty)=0.$$

A more general condition $\mathbf{u}'(z=-H)=0$ is also easy, but somewhat obscures the discussion.

Let us consider the 2D Fourier transform of our in-plane velocity field, $\mathbf{u}_\mathbf{q}$. The wave-vector $\mathbf{q}$ is also a 2D vector. Let us consider this ansatz for the solvent velocity field:

$$\mathbf{u}'=\frac{1}{(2\pi)^2}\int d\mathbf{q}\,\mathbf{u}_\mathbf{q}e^{i\mathbf{q}\cdot\mathbf{r}}e^{qz}.$$

Notice that this is a half-Fourier inverse transform, only in q_x and q_y, but not on q_z. What we have here is a dilution of each of the 2D Fourier components $\mathbf{u}_\mathbf{q}$ as we go deeper into the solvent, each decaying as e^{qz} (remember, z is negative!). This means the sharper features will wash out first, and the longest-wavelength features will be the ones that will extend deeper.

Notice that this velocity field is laminar: it is always parallel to the 2D layer (i.e. it has no z component). It automatically satisfies incompressibility. It is also harmonic: $\nabla^2\mathbf{u}'=0$ (for this, it is imperative that the decay is precisely e^{qz}. These two facts imply that the pressure field is not needed, and we may take it to be constant.

Lastly, the force this velocity field will exert on the planar interface can only be a wall shear stress. Recalling our expressions for the shear stress on walls, e.g. Eq. 9.3, the force per unit surface on the 2D fluid will be

$$f_x=-\tau_{zx}\qquad f_y=-\tau_{zy}\qquad f_z=-\tau_{zz},$$

since we would be considering only forces on the top wall of the cube of Fig. **??**. The minus sign arises because these forces are the reaction to the forces exerted by the wall upon the particles closeby. (If this is confusing, recall that in Poiseuille flow the force upon the walls of Eq 11.1 of course goes *against* the flow.)

Since our fluid is Newtonian, Eqs. 9.6 and 9.7 yield

$$\tau_{zz}=-p'\qquad \tau_{zx}=\mu'\frac{\partial u_x'}{\partial z}\qquad \tau_{zy}=\mu'\frac{\partial u_y'}{\partial z}$$

where many terms have vanished because $\mathbf{u}'$ does not have a z component. The second viscosity of Eq. 9.6 does not appear since the flow is incompressible. The first equation represents a push from the constant pressure underneath (recall that minus sign again), but in an experimental situation this is balanced by either some external pressure in the case of monolayers (most simply, atmospheric pressure), or by an equal pressure on the other side for the case of membranes.

We may combine the two latter expressions to write down a vectorial equation for the force per unit area on the surface due to the solvent:

$$\mathbf{f}' = -\mu' \left.\frac{\partial \mathbf{u}'}{\partial z}\right|_{z=0}.$$

With our previous ansatz,

$$\mathbf{f}' = -\frac{1}{(2\pi)^2}\mu' \int d\mathbf{q}\, q \mathbf{u}_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}}.$$

This is a very interesting result. It is clear that the 2D Fourier transform of the solvent force is

$$\mathbf{f}'_{\mathbf{q}} = \mu' q \mathbf{u}_{\mathbf{q}}.$$

Therefore, if we cast our original 2D momentum equation 12.12 into Fourier space:

$$-\mu q^2 \mathbf{u}_{\mathbf{q}} - i\mathbf{q} p_{\mathbf{q}} = -\mathbf{f}_{\mathbf{q}} - \mathbf{f}'_{\mathbf{q}},$$

we have

$$-(\mu q^2 + \mu' q)\mathbf{u}_{\mathbf{q}} - i\mathbf{q} p_{\mathbf{q}} = -\mathbf{f}_{\mathbf{q}}.$$

We may now repeat the steps by which we obtained 12.10, with the result

$$\mathbf{u}_i(\mathbf{q}) = \sum_j T_{ij} \mathbf{f}_j(\mathbf{q}),$$

but now the Oseen tensor is re-defined:

$$T_{ij}(\mathbf{q}) := \frac{1}{\mu q^2 + \mu' q}\left[\delta_{ij} - \frac{\mathbf{q}_i \mathbf{q}_j}{q^2}\right].$$

The denominator is usually written as

$$\mu q^2 + \mu' q =: \mu(q^2 + q/\xi),$$

where $\xi := \mu'/\mu$ is the Saffman and Delbrück length . (Remember the two viscosities have different dimensions, so their ratio is indeed a length). This length separates two regimes. One of them is the high-q, short wave-length regime, when $q \gg 1/\xi$. In this case the effect of the solvent is negligible, as we recover our previous 2D result. The other is the low-q, high wave-length regime, when $q \ll 1/\xi$. In this case the $q\xi$ term dominates, and the tensor switches over to a $1/q$ increase which makes it Fourier invertible. Stokes' paradox is therefore removed! Sadly, the resulting expressions cannot be analitically expressed in general (they can in those two limits.)

For membranes the discussion is exactly the same, with two mirroring fluid zones above and below the membrane, both of them affecting the membrane. The only difference is therefore that in this case

$$T_{ij}(\mathbf{q}) = \frac{1}{\mu q^2 + 2\mu' q}\left[\delta_{ij} - \frac{\mathbf{q}_i \mathbf{q}_j}{q^2}\right],$$

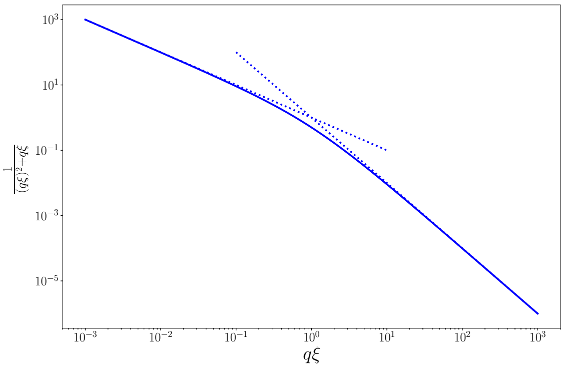

Figure 12.1: Plot of the $1/(q^2+q/\xi) = \frac{\xi^2}{(q\xi)^2+(q\xi)}$ function

and the Saffman and Delbrück length is $\xi = 2\mu'/\mu$.

In Figure 12.1 we plot the function $1/(q^2+q/\xi)$ in log-log scale. At high q (short distances), the denominator approaches $\eta_{\mathrm{m}}q^2$, which is the prediction from the standard theory as just described. At low q (long distances), on the other hand, the denominator approaches $2\eta_{\mathrm{f}}q$, which will cause a convergent real-space Oseen tensor with magnitude given by the viscosity of the surrounding liquid.

12.2 Creeping flow past a sphere

Unfortunately, the creeping flow past a cylinder cannot be solved, a fact known as "Stokes paradox". This may have been anticipated by dimensional analysis. Indeed, the drag force on a sphere of radius R is expected to be given by

{eq:drag_sphere_guess}

$$D \sim \mu u_0 R, \tag{12.13}$$

which has the right combination of units to yield a net force. It also makes sense, being proportional to the viscosity, velocity, and radius. (One may have expected R^2, the projected area, but this does not work dimensionally.) It is indeed correct, a fact that will be proved in this section, together with the missing numerical prefactor.

If we write something similar for the drag force on a cylinder, the answer should come as a force per length in the perpendicular direction (as e.g. the lift for our cylinder with circulation, Eq. 6.18). I.e.

{eq:drag_cyl_guess}

$$\frac{D}{L} \sim \mu u_0. \tag{12.14}$$

But this makes little sense, as all dependence on R is lost — and there is not clear candidate to provide another length parameter. See, nevertheless Section 12.4 for a "dirty way" around this problem.

To solve for the sphere, we begin with the Stokes equation with no forces:

{eq:Stokes_no_f}
$$\mu\nabla^2\mathbf{u} - \nabla p = 0. \tag{12.15}$$

We may intruduce the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$, and recall the expression for the curl of the curl used previously, Eq. 6.14,

$$\nabla \times \boldsymbol{\omega} = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2\mathbf{u}.$$

The first term is null due to incompressibility. Hence,

$$\mu\nabla \times \boldsymbol{\omega} = -\nabla p. \tag{12.16}$$

{eq:creeping_p_vort}

If we take the divergence of this expression we find that the pressure is harmonic:

$$\nabla^2 p = 0$$

If, on the other hand, we take the curl, we find

$$\nabla \times (\nabla \times \boldsymbol{\omega}) = 0$$

This is again, the curl of a curl, so Eq. 6.14 may be again used:

$$\nabla \times (\nabla \times \boldsymbol{\omega}) = \nabla(\nabla \cdot \boldsymbol{\omega}) - \nabla^2\boldsymbol{\omega}.$$

But the vorticity is always divergence-free, it being the curl of another field. Therefore vorticity is also harmonic!:

$$\nabla^2\boldsymbol{\omega} = 0.$$

Our method of solution will make use of the stream function. Introducing our vector potential of Eq. 6.11

$$\boldsymbol{\omega} = \nabla \times (\nabla \times \mathbf{A}).$$

Using, for the third time, Eq. 6.14,

$$\boldsymbol{\omega} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}.$$

If we make sure $\mathbf{A}$ is divergence-free, then

$$\nabla^2\left(\nabla^2\mathbf{A}\right) = 0 \implies \nabla^4\mathbf{A} = 0. \tag{12.17}$$

{eq:bi-laplacian}

The operator ∇^4 is called the bi-harmonic operator, or bi-Laplacian.

This is a daunting equation, but we may make use of our experience when dealing with potential axysimmetric problems in spherical coordinates. Once again, the vector potential will be purely azimuthal, and related with the stream function ψ as in 6.21.

The identity of Eq. 6.24 leads to an equation of the form:

$$\nabla^2\mathbf{A} = \frac{1}{r\sin\theta}\mathcal{L}[\psi]\,\hat{\mathbf{e}}_\phi,$$

where the operator $\mathcal{L}$ is defined as

$$\mathcal{L} := \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\left[\frac{\partial^2}{\partial \theta^2} - \cot(\theta)\frac{\partial}{\partial \theta}\right].$$

Now, something very interesting happens: this result is relatively simple due to the way ψ and $\mathbf{A}$ are related in 6.21, in particular $\frac{\psi}{r\sin\theta}$ being the only component, purely azimuthal, and dependent on r and θ. But, our result for $\nabla^2\mathbf{A}$ complies with this very same conditions! This means, that by defining the function $g := \mathcal{L}\psi$, we get

$$\nabla^2\mathbf{A} := \frac{1}{r\sin\theta} g(r,\theta)\hat{\mathbf{e}}_\phi,$$

and we know the result of another application of the Laplacian is going to be

$$\nabla^2(\nabla^2\mathbf{A}) = \frac{1}{r\sin\theta}\mathcal{L}g\hat{\mathbf{e}}_\phi,$$

or

$$\nabla^4\mathbf{A} = \frac{1}{r\sin\theta}\mathcal{L}^2\psi\hat{\mathbf{e}}_\phi.$$

Then, the bi-Laplacian equation for $\mathbf{A}$ translates into

$$\mathcal{L}^2\psi = 0.$$

This still looks quite impressive, but it turns out that simple separation of variables yields the answer. First, our boundary conditions must be

$$\psi(r\to\infty,\theta) \to \frac{1}{2}u_0 r^2\sin^2\theta \qquad \left.\frac{\partial\psi}{\partial\theta}\right|_{r=R} = \left.\frac{\partial\psi}{\partial r}\right|_{r=R} = 0$$

The first one reproduces the correct uniform field in the z direction away from the sphere (see the potential stream function, Eq. 6.23.) The other two are no-slip boundary conditions at the sphere surface.

Let us try this simple ansatz: $\psi = f(r)\sin^2\theta$. Moreover, let us find out which power-laws satisfy the bi-Laplacian equation by exploring $f(r) \sim r^a$.

If we try $\psi = r^2\sin^2\theta$ we find

$$\mathcal{L}\psi = [a(a-1)-2]\, r^{a-2}\sin^2\theta,$$

so the possible exponent values for $\mathcal{L}\psi = 0$ are $a = 2$ and $a = -1$ (this basically solves the potential flow problem.)

Notice that, once again, the outcome contains a function that is very similar to the input. This means that, applying twice the differential operator we will get

$$\mathcal{L}^2\psi = [a(a-1)-2]\,[(a-2)(a-3)-2]\, r^{a-4}\sin^2\theta,$$

so the possible exponent values for $\mathcal{L}^2\psi = 0$ are $a = 2$ and $a = -1$, as before, plus $a = 4$ and $a = 1$.

The solution we try will be a combination of all four posibilities — but the $a = 4$ we can discard right away because it is too high a divergence. Let us write

$$\psi = u_0\left[\frac{1}{2}r^2 + \frac{A}{r} + Br\right]\sin^2\theta.$$

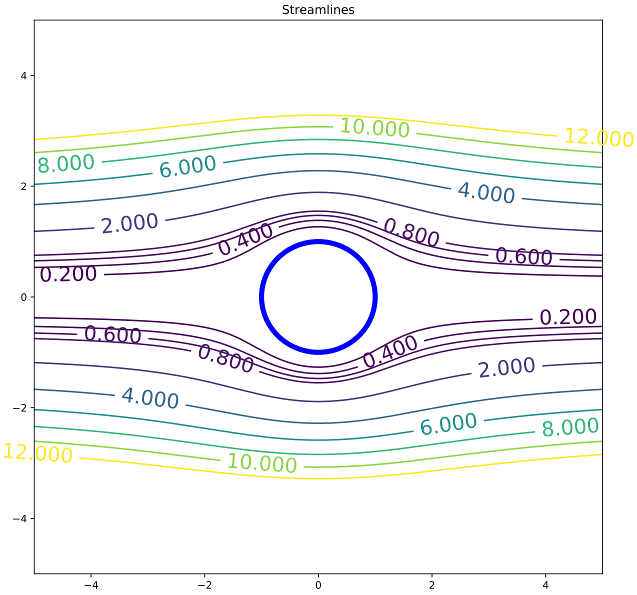

Figure 12.2: Creeping flow past a shpere: streamlines.

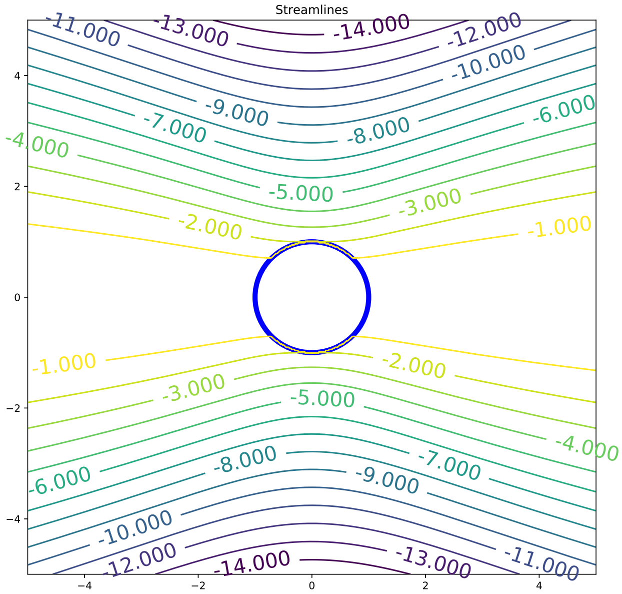
Streamlines
-11.000
-13.000
-14.000
-12.000
-10.000
-9.000
-7.000
-8.000
-6.000
-4.000
-5.000
-3.000
-2.000
-1.000
-1.000
-2.000
-3.000
-5.000
-4.000
-6.000
-8.000
-7.000
-9.000
-10.000
-12.000
-14.000
-13.000
-11.000
4
2
0
−2
−4
−4
−2
0
2
4

Figure 12.3:

Then our boundary conditions at $r = R$ imply

$$\left.\frac{\partial\psi}{\partial r}\right|_{r=R} = 0 \implies R - \frac{A}{R^2} + B = 0$$
$$\left.\frac{\partial\psi}{\partial\theta}\right|_{r=R} = 0 \implies \frac{1}{2}R^2 - \frac{A}{R} + BR = 0,$$

from which we may obtain A and B, and finally write the solution

$$\psi = \frac{1}{2}u_0 R^2 \left[\left(\frac{r}{R}\right)^2 + \frac{1}{2}\frac{R}{r} - \frac{3}{2}\frac{r}{R}\right]\sin^2\theta.$$

The components of the velocity are readily found from 6.22,

$$u_r = u_0\left[1 + \frac{1}{2}\left(\frac{R}{r}\right)^3 - \frac{3}{2}\frac{R}{r}\right]\cos\theta$$
$$u_\theta = -u_0\left[1 - \frac{1}{4}\left(\frac{R}{r}\right)^3 - \frac{3}{4}\frac{R}{r}\right]\sin\theta$$

The pressure may be found from 12.15, but it is easier to obtain it from the vorticity, using 12.16. The latter is may be computed, resulting in a rather simple expression:

$$\boldsymbol{\omega} = -\frac{3}{2}\frac{\mu u_0 R\sin\theta}{r^2}\mathbf{e}_\phi.$$

Taking its curl, and focusing on its r component, the pressure field is obtained as

$$p = p_0 - \frac{3}{2}\frac{\mu u_0}{R}\left(\frac{R}{r}\right)^2\cos\theta,$$

where p_0 is the pressure far from the sphere (of course, the ϕ component yields the same result, as can be checked). Notice this pressure field is harmonic, as discussed above. At the sphere surface,

$$p(r = R) = p_0 - \frac{3}{2}\frac{\mu u_0}{R}\cos\theta,$$

with a high-pressure zone at the fore of the sphere (the point facing the upstream direction, $\theta = \pi$), and a lower pressure at the aft ($\theta = 0$). This clearly asymmetric pressure distribution results in a net drag on the sphere

$$D_p = -\int_S p\cos\theta dA = -2\pi R^2\int_0^\pi d\theta\sin\theta p(\theta)\cos\theta = +3\pi\mu u_0 R\int_0^\pi d\theta\sin\theta\cos^2\theta.$$

The minus sign comes from the fact that we want the pressure upon the sphere, which by reaction is opposite the pressure upon the fluid. The last integral is immediate: the primitive is $-(1/3)\cos^3\theta$, so the definite integral provides a $2/3$ factor. Finally,

$$D_p = 2\pi\mu u_0 R.$$

This agrees with our guess of Eq. 12.13, based on dimensional considerations.

However, this is the only part of the drag force due to pressure. There will be an additional shear drag, which may be calculated from the shear stress. The stress tensor has diagonal elements τ_{rr} and $\tau_{\theta\theta}$ that vanish at the sphere (except for the presure part). For example,

$$\tau_{rr} = \mu u_0 \left(\frac{3R}{2r^2} - \frac{3R^3}{2r^4} \right) \cos\theta - p$$

The relevant elements of the tensor are the two (equal) off-diagonal ones that correspond to shear (see [12], Eq. B17, p. 584):

{eq:tau_r_th}

$$\tau_{r\theta} = \mu \left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + r \frac{\partial}{\partial r}\left(\frac{u_\theta}{r} \right) \right). \tag{12.18}$$

This can be computed (in fact, the only terms that give a contribution are the $1/r$ ones) to obtain

$$\begin{aligned} \tau_{r\theta} = & \frac{\mu u_0 \cos\theta}{R} \frac{R}{r} \left[1 + \frac{1}{2}\left(\frac{R}{r}\right)^3 - \frac{3}{2}\left(\frac{R}{r}\right)\frac{R}{r} \right] + \\ & \frac{\mu u_0 \sin\theta r}{R} \frac{r}{R} \left[\left(\frac{R}{r}\right)^2 - \left(\frac{R}{r}\right)^5 - \frac{3}{2}\left(\frac{R}{r}\right)^2 \right]. \end{aligned}$$

At the sphere surface the first term, which is $(1/r)\frac{\partial u_r}{\partial \theta}$, vanishes, and we obtain

$$\tau_{r\theta}(r = R) = -\frac{3}{2}\frac{\mu u_0}{R} \sin\theta.$$

This looks very similar to the pressure, with a sin instead of a cos, which means the stress is maximal at the equator, which seems reasonable. Notice the sign: by definition this stress is the stress on a surface of fixed r (the surface of the sphere) in the θ direction. Both coordinates are signed. If we pick an angle between 0 and π we get a negative stress. This means the upper part of the sphere will be subject to a force in the negative θ direction. Intuitively, since the velocity field is slower close to the sphere than far away, the liquid tries to rotate the sphere so that the velocities smooth out (such is the role of viscosity).

The sphere does not rotate as a whole, because the force at angle θ has a counterpart in the lower hemisphere at angle $-\theta$ — however, these stresses share a common horizontal component that does not cancel and produce a net drag. (This symmetry is actually fixed by our assumption of axisymmetric flow.) See Exercise **??** about stresses for some velocity fields.

It would seem pressure and stress contribute equally, since their expressions are so similar, but the contribution to the drag force is higher — twice higher, to be precise. This is because the sphere contributes more from the equator than from the poles. Indeed,

$$D_\tau = \int_S \tau_{r\theta}(-\sin\theta)\, dA,$$

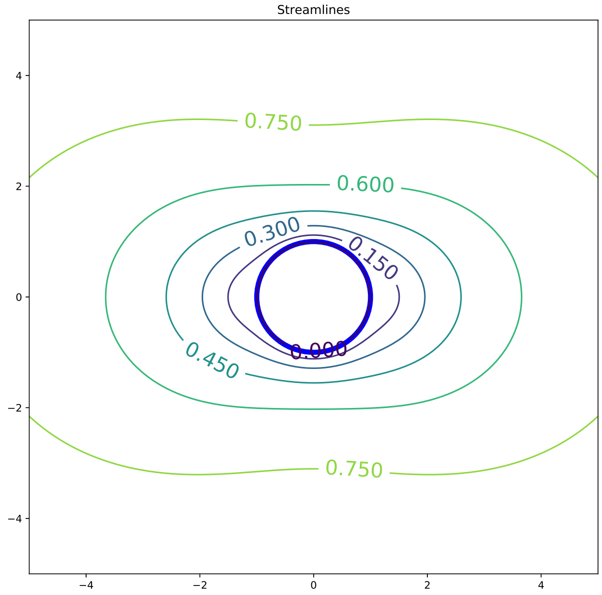

Figure 12.4:

with a minus sign on the $\sin\theta$ from projecting from the tangential unit vector $\hat{\mathbf{e}}_\theta$ to the horizontal $\hat{\mathbf{e}}_z$. We integrate in spherical coordinates, with $dA = 2\pi R^2 \sin\theta d\theta$. Then

$$D_\tau = -2\pi R^2 \int_0^\pi d\theta \tau_{r\theta} \sin^2\theta = 3\pi\mu u_0 R \int_0^\pi d\theta \sin^3\theta.$$

The primitive of $\sin^3\theta$ is, through $\cos^2\theta = 1 - \sin^2\theta$, equal to $(1/3)\cos^3\theta - \cos\theta$. The definite integral is then $4/3$, and

$$D_\tau = 4\pi\mu u_0 R.$$

The final drag force is the celebrated Stokes drag law:

$$D = 6\pi\mu u_0 R,$$

where $1/3$ of the drag is due to pressure and $2/3$ to shear stress.

Despite the shear having a contribution that is twice that two pressure, notice that by writing

$$\begin{aligned} -p(r=R) + p_0 &= \bar{p}\cos\theta, \\ -\tau_{r\theta}(r=R) &= \bar{p}\sin\theta, \end{aligned}$$

where $\bar{p} := -(3/2)\frac{\mu u_0}{R}$, it is apparent that the normal force, due to pressure, and the tangential one, due to shear stress, result from a decomposition of a force upon the sphere surface:

$$\mathbf{f} := \frac{3}{2}\frac{\mu u_0}{R}\hat{\mathbf{e}}_z.$$

It is remarkable that this force is equal for every point on the sphere.

In engineering, it is customary to cast forces upon objects to dimensionless quantities, called "force coefficients". Of course, the correct combination with the dimension of force is $\mu R u_0$, from which a drag coefficient would be

$$C_{\text{visc}} = \frac{D}{\mu R u_0} = 6\pi.$$

A typical experiment would be to check whether this value is indeed obtained for a variety of fluids and physical conditions, at low Reynolds number. The numerical value of 6π is also dependent on the body being a sphere, hence another line would be to measure and compute this coefficient for other bluff bodies.

Nevertheless, the weight of the tradition favors defining these coefficients with respect to the typical force due to inertial effects (even if these effects are neglected in the creeping limit.) The force is given by the typical pressure in this regime, $\rho u_0^2/2$, times the projected area of the sphere, πR^2 . Then,

$$C_{\text{inertial}} = \frac{2D}{\rho u_0^2 \pi R^2} = \frac{12\mu}{\rho u_0 R} = 24\frac{1}{\text{Re}},$$

where the Reynolds number is, as usual, defined in term of the diameter $2R$:

$$\mathrm{Re} = \frac{\rho u_0 (2R)}{\mu}$$

Then, in White's words ([12], p. 171), a Reynolds number effect is introduced when none exists. In the graphs of drag coefficient as a function of Reynolds number a vertical assymptote is reached at low values. This is misleading: the drag approaches a constant value, independent of the Reynolds number. Of course, any quantity that is divided by a increasingly small number will turn become very high, even if there is no dependence.

12.3 Exercises

1. Repeat the calculation of Stokes flow around a sphere using cylindrical coordinates. Use the fact that the identity of Eq. 6.25 leads to

   $$\nabla^2 \mathbf{A} = \frac{1}{\rho} \mathcal{L}[\psi] \, \hat{\mathbf{e}}_\phi ,$$

   where the operator $\mathcal{L}$ is defined as

   $$\mathcal{L} := \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} .$$

2. Calculate the stress of Eq. 12.18 for these purely polar fields ($u_r = 0$): $u_\theta = u_0 r^2 / R^2$, $u_\theta = u_0 r / R$, $u_\theta = u_0$, $u_\theta = u_0 R / r$, $u_\theta = u_0 R^2 / r^2$, and discuss the results for the stress on a sphere of radius R.

12.4 Exploring Stokes paradox*

Stokes paradox is very soon encountered. Our task is to solve the bi-Laplace equation, Eq. 12.17, for a 2D case, in which the vector potential **A** only has a z component, equal to the stream function ψ. Then,

$$\nabla^4 \psi = 0,$$

from which $u = \nabla \times \psi$. Finally, the pressure field is obtained from Stokes equation, 12.5: $\nabla p = \mu \nabla^2 \mathbf{u}$.

In polar coordinates,

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} . \qquad (12.19)$$

`{eq:Lapl_in_polar}`

Trying a function $\psi = r^\alpha \sin\theta$, we find that

$$\nabla^2 r^\alpha \sin\theta = r^{\alpha-2} \sin\theta \left(\alpha^2 - 1 \right) .$$

This means the only two possibilities are $\alpha = \pm 1$. Indeed, the potential solution for the flow past a cylinder was built from these two solutions (see e.g. Eq 6.6).

The bi-Laplacian is then

$$\nabla^4 r^\alpha \sin\theta = r^{\alpha-4} \sin\theta \left(\alpha^2 - 1\right) \left((\alpha-2)^2 - 1\right).$$

The "new" possible exponents are now $\alpha = 3$ and $\alpha = 1$. The first one is of no use, since it leads to the wrong behavior away from the cylinder. The second one is not new at all! We are therefore stuck with the same functional set as for potential flow. This is, in a nutshell, the paradox.

This means that the only sensible solution is the potential one, which of course features free-slip boundary conditions on the surface of the cylinder. The resulting pressure field will be constant, since $\nabla p = \mu \nabla^2 \mathbf{u}$, but these two functions have null Laplacian. This field would, however, have a non-zero shear stress on the cylinder, as we will discuss below.

The traditional way of dealing with this situation is to reject Stokes equation 12.5 as an adecuate model of the situation. The perturbation in the velocity field is so great that any Reynolds number is bound to be not so small after that — in the sense that the relevant length in Re=$\rho L u_0/\mu$ is surely not the radius of the cylinder, but a much larger value.

We present here an exploration of another way past this issue, which could be interesting, but is – we caution the reader – fraught with tricks.

Let us consider the next term, $\psi = r^\alpha \sin(2\theta)$. Then

$$\nabla^2 r^\alpha \sin(2\theta) = r^{\alpha-2} \sin(2\theta) \left(\alpha^2 - 4\right)$$
$$\nabla^4 r^\alpha \sin(2\theta) = r^{\alpha-4} \sin(2\theta) \left((\alpha-2)^2 - 4\right).$$

The solutions are then: $\alpha = \pm 2$ for the Laplace equation, and $\alpha = 0$ and 4 for the bi-Laplacian. Again, the 2 and 4 exponents are to be rejected, but we may combine the 0 and -2 ones (remembering the latter is a solution of Laplace equation.)

Let us write

$$\psi = R u_0 \left[\frac{r}{R} - \frac{1}{r}\right] \sin\theta - R w_0 \left[1 - \left(\frac{R}{r}\right)^2\right] \sin(2\theta),$$

where w_0 is some unknown velocity.

The second term is chosen this way because the radial velocity component is then

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = u_0 \left[1 - \left(\frac{R}{r}\right)^2\right] \cos\theta - 2 w_0 \left[\frac{R}{r} - \left(\frac{R}{r}\right)^3\right] \cos(2\theta),$$

which vanishes at the cylinder surface, $r = R$.

The tangential velocity is

$$u_\theta = -\frac{\partial\psi}{\partial r} = -u_0 \left[1 + \left(\frac{R}{r}\right)^2\right] \sin\theta + 2 w_0 \left(\frac{R}{r}\right)^3 \sin(2\theta).$$

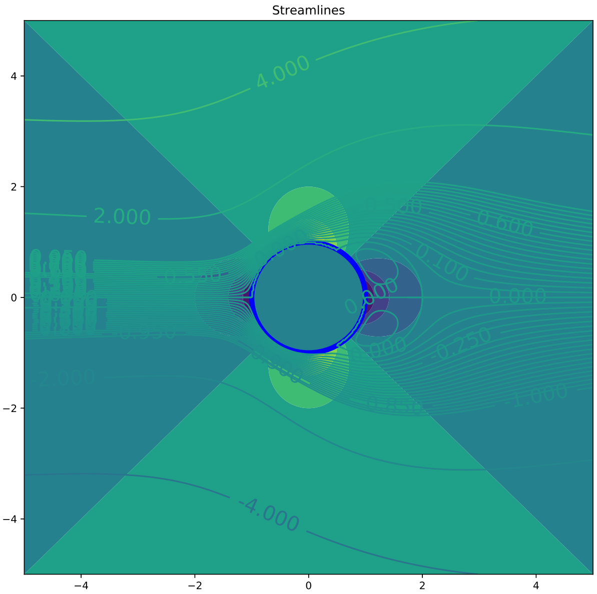

Figure 12.5:

At the surface, then

$$u_\theta(R) = -2u_0 \sin\theta + 2w_0 \sin(2\theta).$$

The value of w_0 therefore determines at which points of the cylinder is the velocity null (these are actually lines, in the z direction). Common sense dictates these points occur at angles between 0 and $\pi/2$. This means w_0 should be positive, and comparable to u_0. The simplest choice will be taken here, $w_0 = u_0$, resulting in angles $\theta = \pm\pi/3$, at which the velocity is zero. The resulting streamlines are plotted in 12.5. The figure is very pleasing, which is our original motivation in exploring this matter. The streamlines as we move with the cylinder, Fig. 12.6 are equally pleasing.

The resulting pressure field only has a non-trivial contribution from the $\alpha = 0$ term, since all other three are harmonic (i.e. solutions of Laplace equation.)

The vector Laplacian is given by

$$(\nabla^2\mathbf{u})_r = \nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}$$
$$(\nabla^2\mathbf{u})_\theta = \nabla^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta},$$

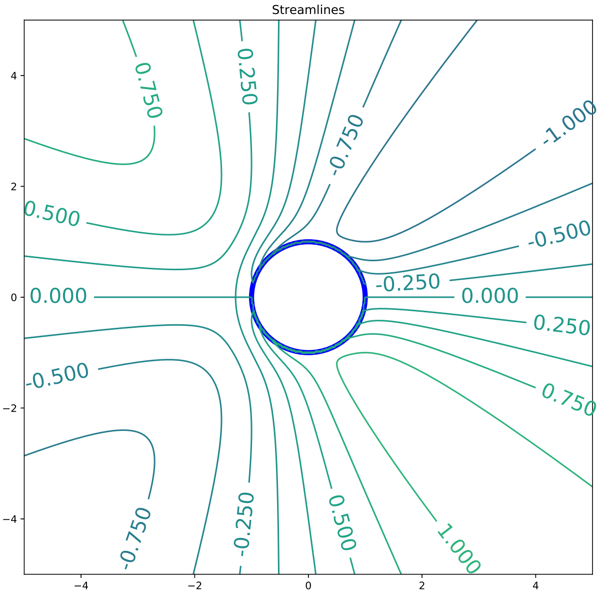
Streamlines
0.750
0.250
-0.750
-1.000
0.500
-0.500
-0.250
0.000
0.000
0.250
-0.500
0.750
-0.750
-0.250
0.500
1.000
4
2
0
−2
−4
−4
−2
0
2
4

Figure 12.6:

with the scalar ∇^2 expression as before in Eq. 12.19. We obtain

$$(\nabla^2 \mathbf{u})_r = \frac{8Ru_0}{r^3}\cos(2\theta)$$
$$(\nabla^2 \mathbf{u})_\theta = \frac{8Ru_0}{r^3}\sin(2\theta).$$

The pressure field is obtained from $\nabla p = \mu \nabla^2 \mathbf{u}$, which in polar coordinates reads

$$(\nabla p)_r = \frac{\partial p}{\partial r} = \mu (\nabla^2 \mathbf{u})_r$$
$$(\nabla p)_\theta = \frac{1}{r}\frac{\partial p}{\partial \theta} = \mu (\nabla^2 \mathbf{u})_\theta,$$

whose solution is

$$p = -\frac{4\mu u_0 R}{r^2}\cos(2\theta) = -\frac{4\mu u_0}{R}\left(\frac{R}{r}\right)^2 \cos(2\theta).$$

In the last expression we make it clear that the pressure value is set by $\mu u_0/R$, and therefore the total pressure force upon the cylinder is $D_p \sim pLR \sim \mu u_0 L$. The drag force would then be independent of the radius, as predicted in Eq. 12.14. This is all consistent, if somewhat surprising, but nevertheless moot: this pressure field, with its $\cos(2\theta)$, has a d'Alembert's paradox! Indeed, by the same arguments as in the classical potential solution, the net pressure force is null. This is obvious in Fig. 12.5, where the pressure field shows a left-right symmetry. Mathematically,

$$\frac{D_p}{L} = R\int_0^{2\pi} d\theta p\cos\theta = 4\mu u_0 \int_0^{2\pi} d\theta \cos\theta\cos(2\theta) = 0. \tag{12.20}$$

{eq:pressure_drag_on_cyl_creep

Also, the sign of the pressure is troublesome: the low-pressure areas are at the fore and aft of the cylinder, contrary to what may be expected (and what is found for the potential solution, Eq. 6.9.)

There is now another source of drag: shear stress on the surface. The expression for the relevant part of the stress tensor is

$$\tau_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right)$$

(i.e. it looks exactly as in spherical coordinates, Eq. 12.18).

After some computations, at the surface we obtain

$$\tau_{r\theta}(R) = \frac{4\mu u_0}{R}\left(\sin\theta - 2\sin(2\theta)\right).$$

The surprise is, then, that the term due to the potential solution, $\sin\theta$, will result in a net drag force, but the new one will not. Indeed, this force will be

$$\frac{D_\tau}{L} = R\int_0^{2\pi} d\theta (\tau_{r\theta})(-\sin\theta) = -4\mu u_0 \int_0^{2\pi} d\theta \sin^2\theta = -4\pi\mu u_0. \tag{12.21}$$

{eq:viscous_drag_on_cyl_creepi

Notice there is a minus sign due to projection on the x axis from the tangential direction, as in the shear stress on the sphere. The result is, in a sense, worse than the vanishing pressure force: the drag acts in the upstream direction! Notice that the $\sin\theta$ term, which is the only one contributing, comes from the potential solution. In this solution there are two zone, at the upper and lower points of the cylinder, at which the velocity is quite higher than u_0. This means that, in the upper part of the cylinder, the shear force will tend to rotate the cylinder counter-clockwise, in the positive direction, to equalize velocities. The same will happen in the lower part, with a vanishing rotation but a non-vanishing drag in the "wrong" direction.

If we nevertheless define a drag coefficient as for the sphere (with an exposed area now given by $2RL$), this would be

$$C_{\text{inertial}} = \frac{2|D|}{\rho u_0^2(2RL)} = \frac{4\pi\mu u_0}{\rho u_0^2 RL} = 8\pi\frac{1}{\text{Re}},$$

with a Reynolds number defined in term of the diameter $2R$:

$$\text{Re} = \frac{\rho u_0(2R)}{\mu}.$$

Notice that we have only employed two terms of what may be an infinite expansion in terms $\sin(n\theta)$. In this way, by means of Fourier analysis, we would be able to produce any tangencial velocity we wished on the surface of the cylinder. The question is, of course, which is the "right" distribution, since our main target, a null velocity, is the one that is the one that is ruled out. For instance, we could make the velocity null at the trailing edge of the cylinder only.

Even if we did this, the potential would not be free of the d'Alambert's paradox: all terms would ultimately be of the form $\cos(n\theta)$, and the resulting drag would be zero for the same reason as for $n = 2$ in Eq. 12.20.

Moreover, the shear stress would also not change! Indeed, the new terms in the stress tensor would be of the form $\sin(n\theta)$ — their contribution to the drag force would be null: in Eq. 12.21 it is clear that only $n = 1$ may contribute to this force. Since the only velocity contributing is the potential solution, whose value is fixed by the up-stream velocity, it cannot change.

Chapter 13

The viscous boundary layer

13.1 Stagnation flow

In order to provide a glimse into the difficulties that are encountered as soon as one ventures into slightly more complicated problems, let us work out a solution of Navier-Stokes equations for a simple stagnation situation.

The potential flow solution for a stady-state incompressible 2D stagnation flow pattern close to a flat wall is given by the stream function

$$\psi = Bxy,$$

from which,

$$\left\{ \begin{array}{l} u_x = \frac{\partial \psi}{\partial y} = Bx \\ u_y = -\frac{\partial \psi}{\partial x} = -By. \end{array} \right.$$

The parameter B has units of inverse time, and is given in practical situations by u_0/L, where u_0 is a relevant upstream velocity, and L a relevant size.

Recalling what we learned about potential flow and its relationship with complex analysis, since $\psi = (B/2)\Im(z^2)$, it is easy to guess the correct potential: $\phi = \Re(z^2) = (B/2)(x^2 + y^2)$.

This flow pattern looks roughly correct, see Fig. 13.1 left, where the streamlines are shown, along with the pressure. The latter follows from Bernoulli principle:

$$p = -\frac{1}{2}\left[u_x^2 + u_y^2\right] = -\frac{1}{2}\left[(Bx)^2 + (-By)^2\right]. \qquad (13.1)$$ {eq:p_stag_pot}

But of course, at the wall ($y = 0$) the fluid flows freely along the wall. This means the boundary conditions are of the "slip" kind, instead of the more realistic "no-slip" kind.

In order to find a correct flow, let us us the ansatz, originally investigated by Hiemenz in 1911[1].

$$\psi = Bxf(y),$$

---

[1] Hiemenz was a student of Prandtl, the founder of the theory of viscous boundary layers.

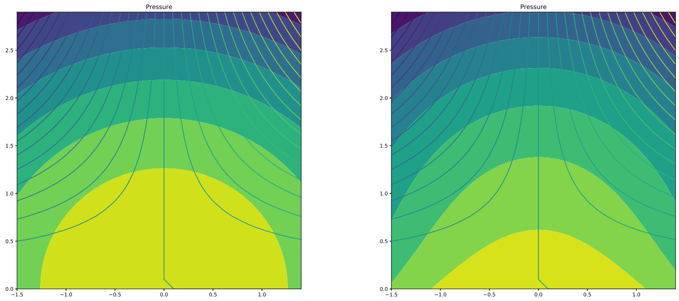

Figure 13.1:

where f is a function of y only. This is basically a separation of variables, also guessing a linear dependence on x.

The velocity is now,

$$\begin{cases} u_x = Bxf' \\ u_y = -Bf. \end{cases}$$

The correct no-slip condition then implies $f(0) = f'(0) = 0$. We will also require $f'(\infty) \to 1$ in order to recover our previous, potential flow, solution.

Now, the steady 2D Navier-Stokes equations read

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = -\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right) \tag{13.2}$$

$$u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} = -\frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} \right). \tag{13.3}$$

(Here, p is not the true pressure, but the "kinematic pressure", p/ρ. For convenience, a ρ factor is assimilated into it, in the same way that $\nu = \mu/\rho$.)

The x equation is then reduced to

$$(Bxf')(Bf') + (-Bf)Bxf'' = -\frac{\partial p}{\partial x} + \nu Bxf''',$$

which may be written as

$$Bff'' - B(f')^2 + \nu f''' = \frac{1}{Bx}\frac{\partial p}{\partial x}.$$

Now, the left part of the equation is a function of y only. This means the pressure can only have this form:

$$p(x,y) = Cx^2 + h(y),$$

with a constant C and a function of h to be determined later. Moreover, as y gets large we want to recover the potential solution. In this limit,

$$f \to a + y \qquad f' \to 1 \qquad f'' \to 0 \qquad f''' \to 0,$$

so the equation in this limit is

$$-B \to \frac{2Cx}{Bx},$$

which means $C = -B/2$, to be used later when solving for the pressure.

We must then solve

$$B\left(ff'' + 1 - (f')^2\right) + \nu f''' = 0.$$

Now, a lot may be learned from the shape of an equation without even solving it. Let us look for a similarity transformation of the form

$$f(y) = bg(ay),$$

so that equation for f may be cast as an equation for g with no parameters. With this prescription,

$$f' = bag' \qquad f'' = ba^2 g'' \qquad f''' = ba^3 g''',$$

So then the equation reads

$$B\left(b^2a^2 gg'' + 1 - b^2a^2(g')^2\right) + \nu ba^3 g''' = 0.$$

Because of the "1" in the parenthesis, it is clear $b = 1/a$, so then,

$$B\left(gg'' + 1 - (g')^2\right) + \nu a^2 g''' = 0.$$

Therefore, if $a = \sqrt{B/\nu}$,

$$gg'' + 1 - (g')^2 + g''' = 0, \tag{13.4}$$ {eq:Z_ode}

with no parameters, as we wanted. This means that, whichever solution we find, our f is going to be given by

$$f(y) = \sqrt{\frac{\nu}{B}} g\left(\sqrt{\frac{B}{\nu}} y\right) = \sqrt{\frac{\nu}{B}} g\left(\frac{y}{\sqrt{\nu/B}}\right) = \ell g\left(\frac{y}{\ell}\right).$$

Clearly, $\ell = \sqrt{\nu/B}$ sets the scale of variation of the flow away from its potential solution.

Notice that the velocities will be

$$\begin{cases} u_x = Bxg'(y/\ell) = B\ell \frac{x}{\ell} g'(y/\ell) \\ u_y = -B\ell g(y/\ell), \end{cases}$$

so that the velocity scale is set by $B\ell = \sqrt{B\nu}$.

Our task is then to integrate the non-linear ODE 13.4, subject to these boundary conditions:

$$g(0) = 0 \qquad g'(0) = 0 \qquad g''(x \to \infty) \to 0.$$

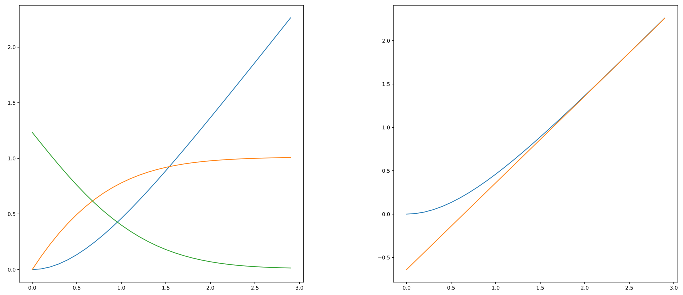

Figure 13.2:

If we had a condition on $g''(0)$, the problem would be a straight-forward exercise in integration. However, we have instead a condition on the other side of the integration domain, which makes this problem somewhat harder. The technique should then be a "shooting method", in which $g''(0)$ is adjusted until a vanishing small of g'' far away is found. The procedure may be made systematic, but we can also fiddle a bit with the parameters in order to find a reasoable approach. It is quite easy to arrive at $g''(0) \approx 1.234$, as in the jupyter python notebook at Supplementary Material.

In Figure 13.2 left, the function g and its first and second derivatives are plotted. With these, it is easy to plot the resulting streamlines, Figure 13.1 right. In the latter, the potential streamlines are shown in the left. It is apparent how the flow is "moved upwards" due to viscous effects near the wall. This displacement is readily quanfified by the asympotic behavour of g, as shown in Figure 13.2 left. A reasonable approximation is

$$g(y) \approx -0.64 + y.$$

This provides an estimate of the boundary layer thickness as given by

$$\delta = 0.64\ell = 0.64\sqrt{\nu/B},$$

which then increases as the root square of ν, and decreases as, basically, the root square of the velocity far away from the wall (through B).

To get some numbers, if air approaches a 10 cm diameter cylinder at $u_0 = 10\,\mathrm{m\,s^{-1}}$, then $B = 4u_0/D = 400\,\mathrm{s^{-1}}$ (there is a factor of 4 involved, see [12] §7.3). With $\nu \approx 1.5 \times 10^{-5}\,\mathrm{m^2\,s^{-1}}$, we get $\delta = 0.12\,\mathrm{mm}$, a very thin layer. The thickness of the layer can be defined in other ways, for example as the distance at which $f'(y) \approx 0.99$ (so that the value of $u_x = Bxf'$ is 99% its potential solution value, Bx.) This yields $\approx 2.4\ell$, which is about 3.7 times larger than δ. Still, only approximately 0.44 mm, beyond which the effect of the wall is quite neglibible.

This fact is a blessing, since it restores our faith in potential-based solutions: many flows look potential flows when viewed at some distance from obstacles (indeed, away from any region that may generate vorticity and involve shear, such as jets, wakes . . .). It also solves d'Alembert's paradox, since the velocity field can still comply with the no-slip boundary condition and exert drag forces on obstacles (in general, they will be due both to pressure and to wall shear stress). It is also a curse mathematically: the details of a proper matching between it the layer and the flow around it must be worked out carefully. Also computationally, the fact that the boundary layer is so thin compared with other dimensions of the problem leads to prohibitively large simulation meshes, or the necessity to refine the meshes close to surfaces. The latter fact favours methods such as the finite element method, or the finite volume method, in which refinement is easier, over other ones like finite differences.

The pressure was partly known already, $p = -(B^2/2)x^2 + h(y)$. The other Navier-Stokes equation, which we have still not used, reads

$$BfBf' = -\frac{\partial p}{\partial y} - \nu B f''.$$

This means

$$h' = -B^2 f f' - \nu B f'',$$

which may be easily integrated :

$$h = -B^2 \frac{1}{2} f^2 - \nu B f'.$$

So, finally:

$$p = -\frac{1}{2}\left[(Bx)^2 + (Bf)^2\right] - \nu B f' = -\frac{1}{2}\left[(u_x/f')^2 + u_y^2\right] - \nu B f',$$

where the last equality shows that the pressure is not so different from the potential solution, Eq. 13.1. Notice it is not u_X which features, but rather u_x/f', which does tend to the u_x away from the wall. An additional, viscous term appears, which makes the pressure have a constant offset $-\nu B$ with respect to its potential counterpart, as we move away from the wall.

Pressures are also included in Fig. 13.1. To make a more quantitative comparison, isobars are plotted in Fig. 13.3. It is interesting that viscosity causes pressure to be "pushed" against the wall, flattening the isobars (and, interestingly, making them not normal to the wall). While, far away from the wall, they approach the same circular shape, but with a constant offset. In these figures, pressures are plotted in their reduced form. It is easy to check that the pressure scale is given by $B\nu$ (units of velocity squared — remember this is the dynamic pressure).

Another interesting feature of the flow is its vorticity, which is readily computed from

$$\omega_z = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} = 0 - Bxf''(y) = -Bx^* g''(y^*).$$

Hence the wall is seen to induce vorticity close to it, with a sign change on both sides on the $x = 0$ symmetry plane. The shear stress is given by a similar expression,

$$\tau_{xy} = \mu\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) = \mu Bxf''(y) = \mu Bx^* g''(y^*).$$

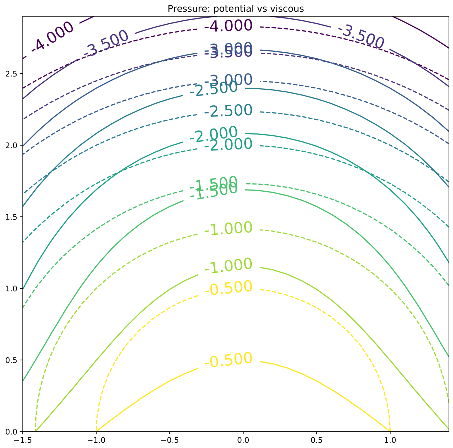

Figure 13.3:

In particular, the wall stress is

$$\tau_\mathrm{w} := \tau_{xy}(y=0) = \mu B x^* g''(0) \approx 1.234 \mu B x^*$$

The (horizontal) skin friction coefficient may be defined, for this problem, as

$$C_\mathrm{f} := \frac{2\tau_\mathrm{w}}{\rho (Bx)^2}.$$

The denominator features the horizontal velocity away from the wall, Bx. Then,

$$C_\mathrm{f} := \frac{2\sqrt{B\nu} g''(0)}{Bx} =: \frac{2g''(0)}{\sqrt{\mathrm{Re}_x}},$$

where the local Reynolds number is defined as

$$\mathrm{Re}_x = \frac{(Bx)x}{\nu}.$$

I.e. a Reynolds number where the typical velocity is the horizontal velocity far from the wall, Bx, and the distance is that to the impact point, x. A dependence of the friction coefficient with the inverse square root of a local Reynolds number is a common feature of laminar boundary layers.

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