California Polytechnic University San Luis Obispo

Mathematical Finance

2024 Senior Portfolio

Daniel Chamberlin

I Introduction to Financial Derivatives

1) Forward Contract Payoff Find the payoff from a long position in a forward contract on one unit of an asset. Find the payoff from a short position in a forward contract on one unit of an asset.

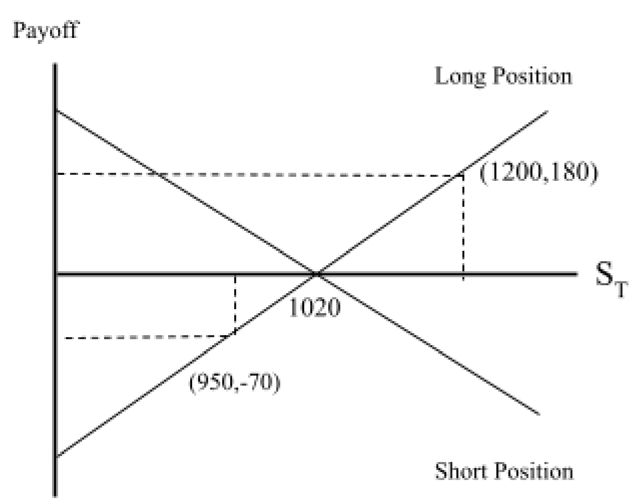
For delivery price K and spot price S_T, in the long position the holder will make a payoff of S_T - K. In the short position, the holder will make a payoff of K - S_T.

2) Forward Contract on Stock Index Suppose that the S&R 500 index has a current price of $1000, and the 6-month forward price is $1020. What happens if the index price is $950 in 6 months? $1200 in 6 months? Construct payoff diagrams for the long and short positions on this contract. What would be an advantage of using the forward contract to buy the index in 6 months, as opposed to buying it outright at time t = 0?

Case 1, index price of $950: Payoff to the long position: 950 - 1020 = -$70
Payoff to the short position: 1020 - 950 = $70

Case 2, index price of $1200: Payoff to the long position: 1200 - 1020 = $180
Payoff to the short position: 1020 - 1200 = -$180

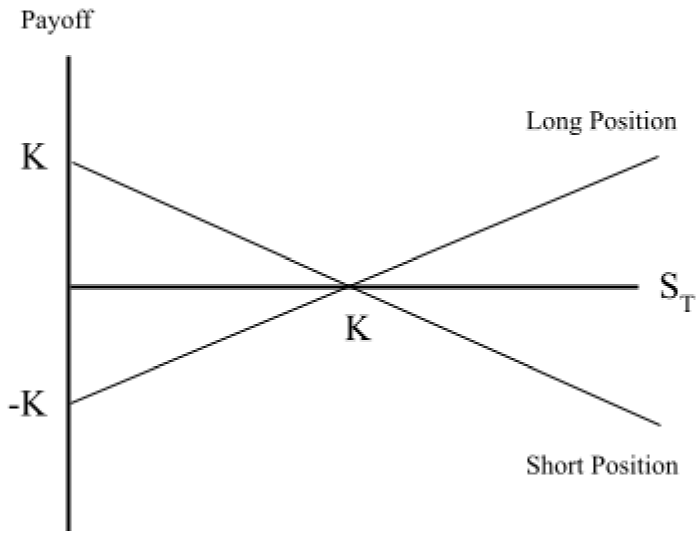
The advantage of using a forward contract as opposed to buying it outright is in the outright case you need to buy all of the shares up front. On the other hand with a forward contract, you are able to invest your money while you wait, thus accumulating money during the six month wait.



3) Payoff Diagrams for Forward Contract Construct payoff diagrams for the long and short positions in a forward contract with delivery price K and spot price S_T .

Payoff function to the long position: $f(S_T) = S_T - K$

Payoff function to the short position: $f(S_T) = K - S_T$



4) Forward Contract on Foreign Exchange Forward contracts on foreign exchange are very popular. Most large banks employ both spot and forward foreign-exchange traders. The table below shows quotes for the exchange rate between the British pound (GBP) and the U.S. dollar (USD) in May 2020. The quote is for the number of USD per GBP.

	Bid	Ask
Spot	1.2217	1.2220
1-month forward	1.2218	1.2222
3-month forward	1.2220	1.2225
6-month forward	1.2224	1.2230

Suppose that the treasurer of a U.S. corporation knows that it will pay 1 million GBP in 6 months, and wants to hedge against exchange rate changes. Suppose that the bank agrees to a

6-month forward contract to purchase 1 million GBP in 6 months. What happens if the spot exchange rate is 1.3000 in 6 months? What if the spot exchange rate is 1.2000 in 6 months?

Since the 6 month forward price is 1.2230, let K be 1.2230. Then since we are looking at the side who is doing the buying, this tells us we are in the long position. Thus from Exercise 2 we know the payoff will be equal to S_T - K. Now let us evaluate:

Spot exchange of 1.3000 => (1.3000 - 1.2230) * 1 million = 77,000 USD
Spot exchange of 1.2000 => (1.2000 - 1.2230) * 1 million = -23,000 USD

Thus, if the spot exchange is greater than K, the forward contract for the long position makes money and if less than K, the long position loses money.

5) An investor enters into a short forward contract to sell 100,000 British pounds for U.S. dollars at an exchange rate of 1.3000 USD per pound. How much does the investor gain or lose if the exchange rate at the end of the contract is (a) 1.2900 and (b) 1.3200?

a) 1.2900 => (1.3000 - 1.2900) * 100 thousand = $1000
b) 1.3200 => (1.3000 - 1.3200) * 100 thousand = -$2000

6) A trader enters into a short forward contract on 100 million yen. The forward exchange rate is $0.0090 per yen. How much does the trader gain or lose if the exchange rate at the end of the contract is (a) $0.0084 per yen and (b) $0.0101 per yen?

Let K = $0.0090. Then since we are in the short position the payoff will follow K - S_T

a) Let S_T = $0.0084. Payoff = (0.0090 - 0.0084) * 100 million = $60,000 gain
b) Let S_T = $0.0101. Payoff = (0.0090 - 0.0101) * 100 million = -$110,000 loss

II Introduction to Options

7) A Call Option Consider the following European call option. Let T denote the date exactly 10 days from now. At time T, the holder of the option may purchase one share of XYZ stock for \$250. To gain an understanding of how call options work and what might be reasonable for the price of this option, we will consider two possible situations that might occur on the expiry time T. Let S_T denote the price of one share of XYZ stock at time T. What happens if S_T = \$270? What happens if S_T = \$230?

a) \$270 => Will exercise option as $S_T > K$, thus payoff = 270 - 250 = \$20
b) \$230 => Will not exercise option as $S_T < K$, thus payoff = \$0

8) Suppose that the XYZ share in example 7 only takes the values \$230 or \$270 with equal probability. Find the expected payoff at time T on the call option. This expected value is a useful approximation for what a reasonable amount to pay for the call option would be.

If we have a probability of ½ for the two conditions then to find the expected payoff we need to multiply individual payoffs by their probabilities and add the values. Thus, the expected payoff, E = ½ (20) + ½ (0) = \$10

9) Let c denote the expected payoff that you obtained in Exercise 8. Although option pricing is, in general, more complicated, suppose that the holder of the option did pay \$c for this option. (a) What is his net profit or loss if S_T = \$270? Express the net profit or loss in this case as a percentage of the initial cost of the option. (b) What is his net profit or loss if S_T = \$230? Express the net profit or loss in this case as a percentage of the initial cost of the option.

a) \$270 => profit = payoff - initial cost = 20 - 10 = \$10. Then in terms of percentage of initial cost the profit is 100% of the original investment.
b) \$230 => profit = 0 - 10 = -\$10. Then the loss is 100% of the original investment.

10) Suppose that, instead, the investor purchased the share for \$250 instead of purchasing the option. Express his net profit or loss in each case (i.e. S_T = \$230 or S_T = \$270) as a percentage of the initial cost of purchasing the share. Compare with the results of the previous exercise.

a) \$270 => profit = 270 - 250 = \$20. Then the loss is 8% of the original investment.
b) \$230 => profit = 230 - 250 = -\$20. Then the loss is 8% of the original investment.

Then we notice that by investing in an option we are able to get a greater return on our investment.

11) A Put Option Consider an investor who buys a European put option to sell 100 shares of stock XYZ with a strike price of \$70. Suppose that the current stock price is \$65. What happens if S_T = \$55?

Since $S_T < K$ in this put option, the long position will exercise the option. Thus the long will make a payoff of (70 -55) * 100 = \$1,500.

12) It is important to observe that sometimes an investor chooses to exercise an option even though he may make a loss overall. For example, suppose that an investor buys a European call option with a strike price of \$100 per share to buy 100 shares of XYZ stock, and that the current stock price is \$98 per share. The price of the option to purchase these 100 shares is \$500. Suppose that the price of the stock is \$102 per share at expiry. Explain why it is preferable for the investor to exercise the option in this case, even though he makes a loss overall.

Let us look at the profits of this example if a) the option is exercised b) the option is not exercised.

a) (102 - 100) *100 - 500 = 200 - 500 = -\$300
b) 0 - 500 = -\$500

Even though there is a loss, we can see by the profits calculated above that it would still be a smaller loss if we exercise the option. This is because we still get the benefit from the expiry price being higher than the delivery price.

13) Payoffs from Positions in European Call Options Find each of the following.

(a) The payoff to the holder of a long position in a European call option.

$\max\{S_T - K, 0\}$

(b) The payoff to the holder of a short position in a European call option.

$-\max\{S_T - K, 0\} = \min\{K - S_T, 0\}$

(c) The payoff to the holder of a long position in a European put option.

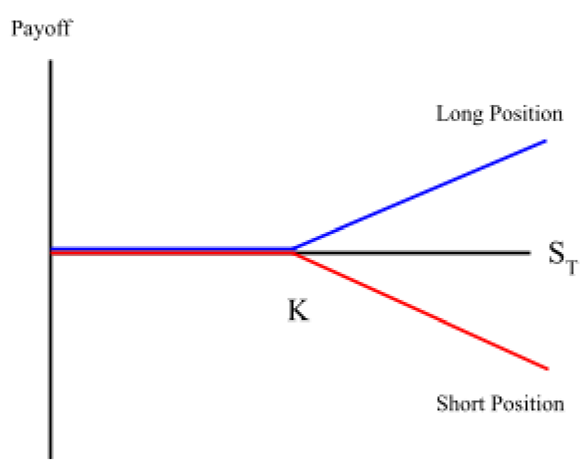
$\max\{K - S_T, 0\}$

(d) The payoff to the holder of a short position in a European put option.

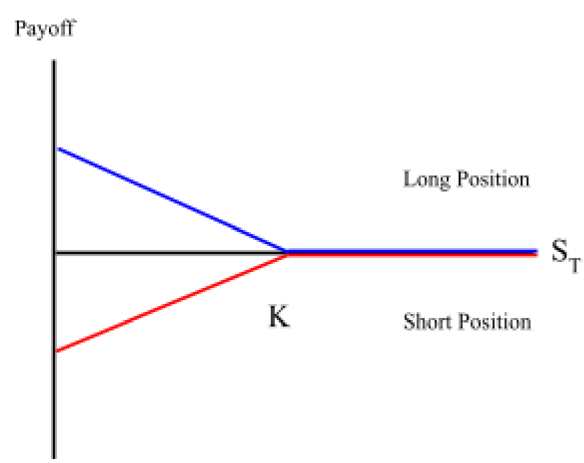
$-\max\{K - S_T, 0\}= \min\{S_T - K, 0\}$

14) Payoff Diagrams for Positions in European Call Options Construct payoff diagrams for each of the four positions above (long call, short call, long pout, short put).

European Call Option

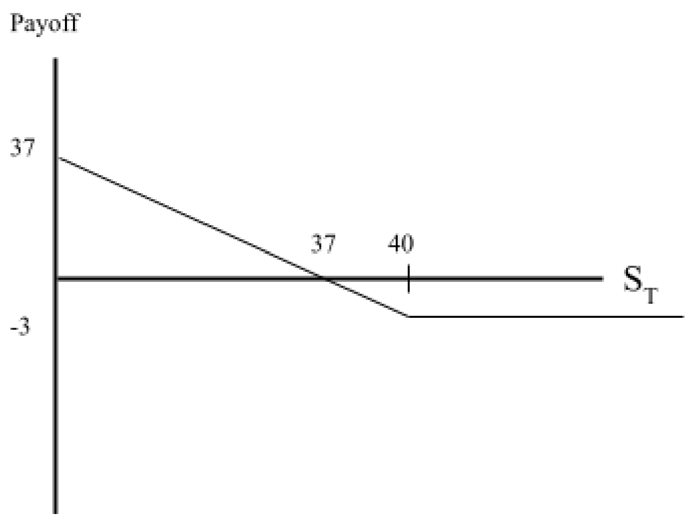


European Put Option



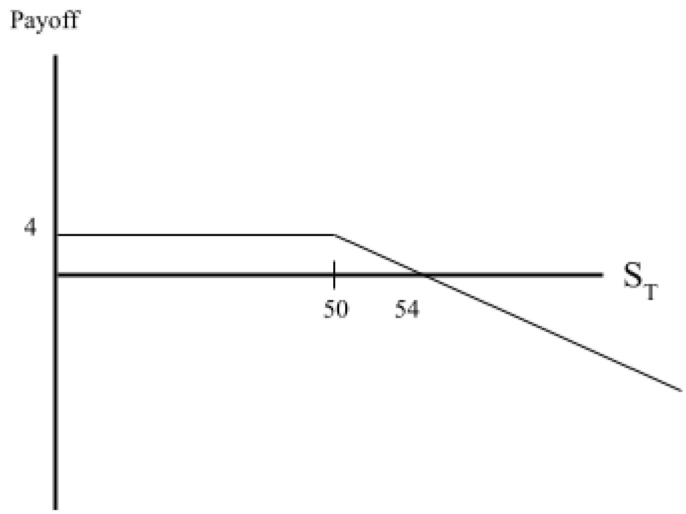
15) An investor buys a European put on a share for \$3. The stock price is \$42, and the strike price is \$40. Under what circumstances does the investor make a profit? Under what circumstances will the option be exercised? Draw a profit diagram illustrating the variation of the investor's profit (not the payoff) with the stock price at the maturity of the option.

From exercise 13 we know the payoff to the long position for a put option is max$\{K - S_K, 0\}$. Thus from this we know that the investor will make a profit when the payoff is greater than the initial cost, \$3. This will happen when the expiry cost is less than 40 - 3 = \$37. We can see this in the graph below.



16) An investor sells a European call on a share for \$4. The stock price is \$47, and the strike price is \$50. Under what circumstances does the investor make a profit? Under what circumstances will the option be exercised? Draw a diagram showing the variation of the investor's profit with the stock price at the maturity of the option.

From exercise 13 we know the payoff to the short position for a call option is min$\{K - S_K, 0\}$. So in order for the investor to make a profit he needs the long position not to exercise, this will happen when S_T is less than 50 + 4 = \$54.



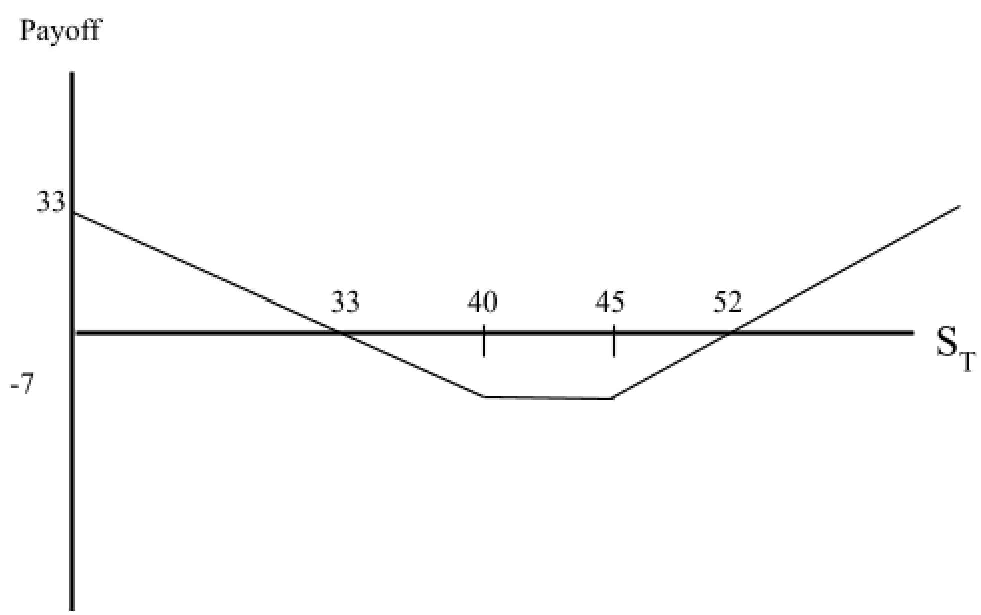
17) An investor sells a European call option with strike price K and maturity T, and buys a put with the same strike price and maturity. Describe the investor's position–describe all of the possible situations at maturity, explain which (if any) of the options the investor should exercise at maturity, and find the investor's payoff in each case.

Payoffs	$S_T > K$	$S_T \leq K$
Short Position ECO	$K - S_T$	0
Long Position EPO	0	$K - S_T$
Total	$K - S_T$	$K - S_T$

Thus all together the payoff = $K - S_T$ for each situation; note this is the same as the short position in a forward contract.

18) A trader buys a call option with a strike price of \$45 and a put option with a strike price of \$40. Both options have the same maturity. The call costs \$3 and the put costs \$4. Draw a diagram showing the variation of the trader's profit with the asset price. Note: this type of trading strategy is known as a strangle.

	Long Position ECO	Long Position EPO	Payoff	Profit
$S_T < 40$	0	$40 - S_T$	$40 - S_T$	$33 - S_T$
$40 < S_T < 45$	0	0	0	-7
$S_T > 45$	$S_T - 45$	0	$S_T - 45$	$S_T - 52$



Thus, will gain a profit if the stock price is less than \$33 or greater than \$52.

19) Explain why an American option is always worth at least as much as a European option on the same asset with the same strike price and exercise date.

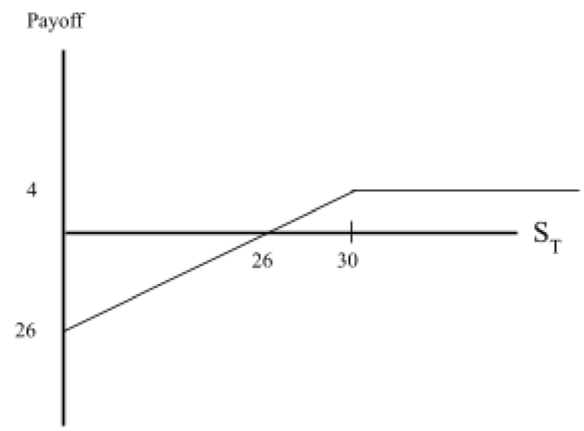
Holding the asset, strike price, and exercise date the same, an American option will always be worth at least as much as a European option since it can be exercised at any time, allowing for more flexibility. In other words, the European and American at the exercise date will be worth the same, so the price will be at least as much. Then since you are able to exercise early, allowing for the possibility of a higher payoff, a higher cost may be required.

20) Complete the following table to summarize the effect on the price of a stock option of increasing one variable while keeping all others fixed. Write a + to indicate that an increase in the variable causes the option price to increase, and write a − to indicate that an increase in the variable causes the option price to decrease. Write a ? if the relationship is uncertain.

Variable	European Call Option	European Put Option	American Call Option	American Put Option
Current Stock Price, S_0	+	-	+	-
Strike Price, K	-	+	-	+
Time to Expiration, T	?	?	+	+
Volatility, σ	+/?	+/?	+	+
Risk-free Interest Rate, r	?	?	?	?

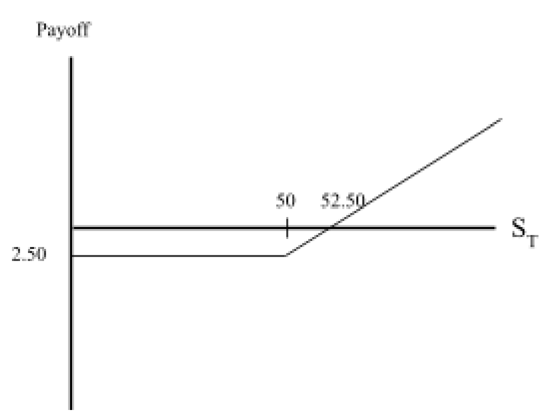
21) A trader writes a December put option with a strike price of $30. The price of the option is $4. Under what circumstances does the trader make a profit?

Since trader is in the short position on a put option, will make a profit when $S_T > 30 - 4 = \$26$



22) Suppose that a March call option to buy a share for $50 costs $2.50 and is held until March. Under what circumstances will the holder of the option make a profit? Under what circumstances will the option be exercised? Draw a diagram illustrating how the profit from a long position in the option depends on the stock price at maturity of the option.

Since the investor is in a long position of a call option, will make a profit if $S_T > 50 + 2.50 =$ $52.50. And the option will be exercised if $S_T > K = \$50$.



23) It is May and a trader writes a September call option with a strike price of $20. The stock price is $18 and the option price is $2. Describe the trader's cash flows if the option is held until September and the stock price is $25 at that time.

Since the trader is using a call option, then the option will be exercised if the strike price is greater than $20. In this case the strike price is $25, and since the trader is in the short position they will need to sell the stock at $20. Thus will have an inflow of $2 for the purchase of the option, but an outflow of 25 - 20 = -$5. Thus the total cash flow is -$3.

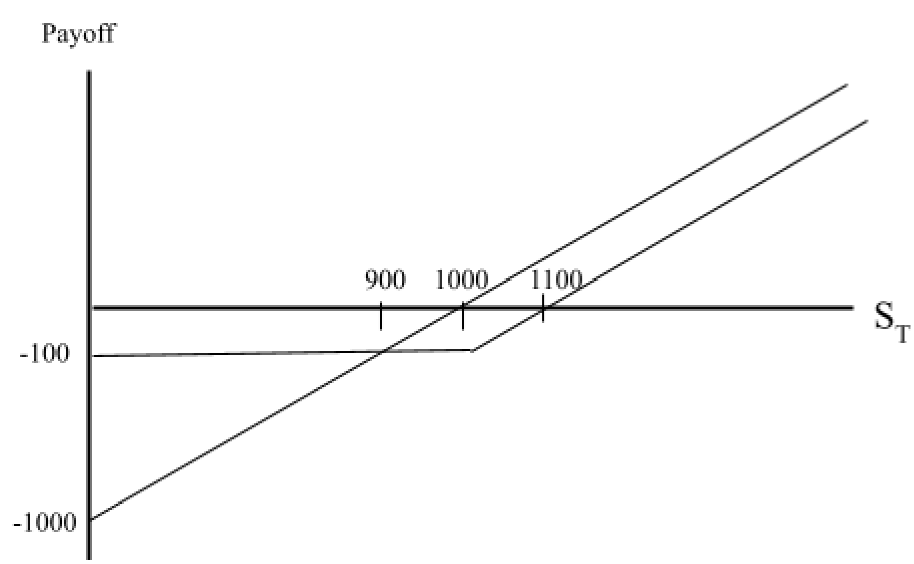
24) Trader A enters into a forward contract to buy an asset for $1,000 in one year. Trader B buys a call option to buy the asset for $1,000 in one year. The cost of the option is $100. What is the difference between the positions of the traders? Show the profit as a function of the price of the asset in one year for the two traders.

The first difference between the two contracts is the cost, in the forward contract there is no upfront cost where in the call option there is a $100 initial cost. The second difference is that in the forward contract you are required to buy the asset at the end of the year whereas in the call option the investor may choose to not exercise.

Forward Contract: Profit = S_T - K => S_T - 1000

Call Option: Profit = {-100, if $S_T < 1000$

S_T - 1100, if $S_T > 1000$}



Thus, the call option allows for the trader to lose less of his money as a result of the spot price falling below $900.

The tables below illustrate the bid and ask quotes for some of the call and put options trading on Apple (ticker symbol: AAPL), on May 21, 2020. The quotes are taken from the CBOE website. The Apple stock price at the time of the quotes was bid 316.23, ask 316.50. The bid–ask spread for an option (as a percent of the price) is usually much greater than that for the underlying stock and depends on the volume of trading.

Strike price	*June 2020*		*September 2020*		*December 2020*	
($)	*Bid*	*Ask*	*Bid*	*Ask*	*Bid*	*Ask*
290	3.00	3.30	12.70	13.65	20.05	21.30
300	4.80	5.20	15.85	16.85	23.60	24.90
310	7.15	7.85	19.75	20.50	28.00	28.95
320	11.25	12.05	24.05	24.80	32.45	33.35
330	17.10	17.85	28.75	29.85	37.45	38.40
340	24.40	25.45	34.45	35.65	42.95	44.05

Call option prices on AAPL, May 21, 2020

Strike price	*June 2020*		*September 2020*		*December 2020*	
($)	*Bid*	*Ask*	*Bid*	*Ask*	*Bid*	*Ask*
290	29.80	30.85	39.35	40.40	46.20	47.60
300	21.55	22.40	32.50	33.90	40.00	41.15
310	14.35	15.30	26.35	27.25	34.25	35.65
320	8.65	9.00	20.45	21.70	28.65	29.75
330	4.20	5.00	15.85	16.25	23.90	24.75
340	1.90	2.12	11.35	12.00	19.50	20.30

Put option prices on AAPL, May 21, 2020

25) A trader is considering two alternatives: buy 100 shares of the stock and buy 100 September call options. For each alternative, find each of the following: (a) the upfront cost, (b) the total profit if the stock price in September is \$400, and (c) the total loss if the stock price in September is \$300.

a) 100 shares: cost = 100 * 316.50 = \$31, 650
100 call options: cost = 100 * 21.70 = \$2170

b) 100 shares: profit = S_T - P = 100 * (400 - 316.50) = \$8,350
100 call options: profit = S_T - K - cost = 100 * (400 - 320 - 21.70) = \$5,830

c) 100 shares: profit = S_T - P = 100 * (300 - 316.50) = -\$1,650
100 call options: profit = S_T - K - cost = 100 * (300 - 320 - 21.70) = -\$2,170

26) On May 21, 2020, an investor owns 100 Apple shares. The investor is comparing two alternatives to limit risk. The first involves buying one December put option contract with a strike price of \$290. The second involves instructing a broker to sell the 100 shares as soon as Apple's price reaches \$290. Discuss the advantages and disadvantages of the two strategies.

Some advantages of the put option is that you are able to choose to exercise or not. Whereas in the stop-loss investment the investor is forced to sell their shares once it reaches a certain price point. This can cause the investor to either miss out on rising stock prices or can leave the investor stuck with no option to sell. By investing in the put option you are essentially buying insurance for if the price gets high; however, with the stop-loss investment there is a possibility of a higher payoff.

III Interest Rates and Time Value of Money

27) Suppose that a deposit of $150 attracts simple annual interest at a rate of 8%. Find the value of the deposit after 20 days.

Let us use the definition of simple interest, $V(t) = (1 + rt) * P$. In this example our initial deposit $P = \$150$, the annual rate $r = .08$, and time in years $t = 20/365$.
Thus, $V(20/365) = (1 + (20/365 * .08)) * 100 =$ <u>$150.66</u>

28) Find the principal to be deposited initially in an account attracting simple annual interest at a rate. of 8% if $1,000 is needed after three months.

In this example we need to solve for the initial investment P. So we need to solve the equation, $1000 = (1 + (1/4 * .08)) * P \Rightarrow P = 1000/1.02 \Rightarrow P =$ <u>$980.39</u>

29) Find and compare the future value after two years of a deposit of $100 attracting interest at an annual interest rate of 10% is compounded (a) annually and (b) monthly.

Let us use the definition of compounded interest, $V(t) = (1 + r/m)^{rm} * P$. Where m in this model represents the number of interest payments made per year.

a) $V(2) = (1 + .10/1)^{2} * 100 =$ <u>$121</u>

b) $V(2) = (1 + .10/12)^{24} * 100 =$ <u>$122.03</u>

30) Show that if m < k, then $(1 + r/m)^m < (1 + r/k)^k$

Suppose m > k. Suffices to show that $(1 + 1/x)^x$ is increasing for x > 0. Let $y = (1 + 1/x)^x$. Then taking the natural log, ln(y) = x * ln(1 + 1/x) or ln(y) = x * ln((x + 1)/x). Now taking the derivative we get 1/y * dy/dx = x * x/x + 1 - $1/x^2$ + ln((x + 1)/x). Isolating the derivative and plugging in our value for y we then arrive at dy/dx =$(1 + 1/x)^x$ * (ln((x+1)/x) - 1/(x+1)). Then since x > 0 suffices to show that ln((x+1)/x) - 1/(x+1) > 0, or ln((x+1)/x) >1/(x+1). By the use of power series for natural log, ln(1/(1 - z)) = Σ 2^n/n. Manipulating ln((x+1)/x) to follow the form for the infinite series we can represent as 1/(1 - (1 - x/(x+1))).
Now z = 1 - x/(x+1).
So, equal to Σ $1/n(1/(x + 1))^n$ = 1/(1 + x) + 1/2(1/$(1 + x)^2$ + … > 1/(1 + x). Thus increasing.

31) Continuous Compounding I In the case of continuously compounded interest, interest is added continuously to the principal. If V(t) is the amount in the bank at time t and if r is the constant interest rate, then we obtain the following differential equation for V as a function of t: dV/dt = r * V.

Given dV/dt = r·V and V(0) = p, find V(t). We will use separation of variables to solve: Start with dV/V = r * dt. Integrating both sides we then get ln|V| = rt + C. Next to isolate V, exponentiate each side by e: V = e^{rt} + C = Ce^{rt}. Now using our initial condition V(0) = P, V(0) = Ce^0 = C = P. Thus we are left with our equation for continuous compounding interest V(t) = Pe^{rt} .

32) Continuous Compounding II Continuously compounded interest can also be viewed as periodically compounded interest in which we take the limit as m (the number of interest payments made per year) goes to infinity, i.e $V(t)= \lim_{m\to\infty} (1 + r/m)^{tm} * P$

a) Show that $e = \lim_{x\to\infty} (1 + 1/x)^{x}$

First take the natural log of both sides leaving us with, $\ln(L) = \lim_{x\to\infty} x * \ln(1+ 1/x)$. This we can rewrite as $\ln(L) = \lim_{x\to\infty} \ln(1+ 1/x)/(1/x)$. Now we can use L'hopital's rule and take the derivative of the numerator and denominator, thus our equation can be represented as, $\ln(L) = \lim_{x\to\infty} (-1/ x^2)/(1 + 1/x) (-1/ x^2) = \lim_{x\to\infty} 1/(1 + 1/x) = 1$. Thus $\ln(L) = 1$, then exponentiating both sides by e leaves us with $L = e$ as desired.

b) Use the result above and $V(t) = \lim_{m\to\infty} (1 + r/m)^{tm} * P$ to obtain a closed- form expression for V(t). You should, of course, obtain the same expression that you obtained by solving the differential equation in Continuous Compounding I.

First manipulate the given equation such that, $V(t) = \lim_{m\to\infty} (1 + r/m)^{tr(m/r)} * P$. Then we let $x = m/r$, we can make this substitution since x goes to infinity as m goes to infinity. Therefore, now we can write the limit as, $\lim_{m\to\infty} (1 + 1/x)^{trx} * P$. Then, we can break it up and use the result found in part (a) such that the limit is $\lim_{m\to\infty} ((1 + 1/x)^{x})^{tr} * P$, which then simplifies to our result in exercise 31, $V(t) = Pe^{rt}$.

IV Properties of Options

33) Upper bound on c Show that the stock price is an upper bound on the option price: $c \leq S_0$.

We will prove by contradiction. Suppose $c > S_0$. Consider the following portfolio, an investor sells a european call option for \$c and buys a stock for S_0. At $t = 0$, the investor has an initial cash flow of $c - S_0 > 0$ by assumption. Then let us observe the profits made at expiry time,

Case 1: $S_T \leq K$.
Then the holder of the call option will not exercise the option, thus the investor makes a profit of $c - S_0 > 0$.

Case 2: $S_T > K$.
Now the holder will exercise the option and will buy the stock from the investor for \$K. Thus the investor makes the profit of $c - S_0 + K > 0$.

Therefore in both cases the investor makes a profit, thus violating the no-arbitrage principle.

34) Upper bound on p Show that the put option cannot be worth more than the present value of K today: $p \leq Ke^{-rt}$.

We will prove by contradiction. Suppose $p > Ke^{-rt}$ or equivalently, $Pe^{rt} - K > 0$. We can make riskless profit in the following way: sell a put option for \$P and invest \$P into a risk free interest market.

Case 1: $S_T \leq K$.
Then the holder will exercise so will need to pay K, thus the payoff is $Pe^{rT} - K$, which we know is positive by our assumption.

Case 2: $S_T > K$.
Then the holder will not exercise the option, thus the payoff is $Pe^{rT} > 0$.

Therefore since both cases yield a positive payoff, this represents an arbitrage opportunity, thus a contradiction.

35) Lower bound on c Show that $c \geq S0 - Ke^{-rt}$.

Consider the following two portfolios:

Portfolio A: one European call option and an amount of cash equal to Ke^{-rt}

Portfolio B: one share of the stock

At expiry:

The payoff to portfolio A: $\max\{S_T - K, 0\} + Ke^{-rt}(e^{rt}) = \max\{S_T - K, 0\} + K = \max\{S_T, K\}$

The payoff to portfolio B: S_T

Then since portfolio A can only be as big or greater than portfolio B we say that the value of Portfolio A is greater than or equal to at expiry. Then it must be the same at the time of their purchase ($t = 0$), by the no-arbitrage principle.

Value of portfolio A at $t = 0$: $c + Ke^{-rt}$

Value of portfolio B at $t = 0$: S_0

Therefore, $c + Ke^{-rt} \geq S_0$. Thus $c \geq S_0 - Ke^{-rt}$ as desired.

36) Lower bound on p Show that $p \geq Ke^{-rt} - S_0$.

Consider the following two portfolios:

Portfolio A: one European put option and one share in the stock

Portfolio B: an amount of cash equal to Ke^{-rt}

At expiry:

The payoff to portfolio A: $\max\{K - S_T , 0\} + S_T = \max\{K, S_T\}$

The payoff to portfolio B: $Ke^{-rt}(e^{rt}) = K$

As before we observe the relation to the values at expiry and use the no-arbitrage principle. Here, $A \geq B$. Now we must look at our values at $t = 0$.

Value of portfolio A at $t = 0$: $p + S_0$

Value of portfolio B at $t = 0$: Ke^{-rt}

Therefore, $p + S_0 \geq Ke^{-rt}$. Thus $p \geq Ke^{-rt} - S_0$ as desired.

37) What is a lower bound for the price of a 2-month European put option on a non-dividend-paying stock when the stock price is \$58, the strike price is \$65, and the risk- free interest rate is 5% per annum?

We know from exercise 36 that the lower bound on p, $p \geq Ke^{-rt} - S_0$. So let us simply plug in the values given, $p \geq 65e^{-(0.05)(1/6)} - 58 = \$6.46$. Thus the lower bound on p is \$6.46.

38) Put-Call Parity Show that $c + Ke^{-rt} = p + S_0$.

Consider the following two portfolios:

Portfolio A: one European call option and an amount of cash equal to Ke^{-rt}

Portfolio B: one European put option and one share of the stock

At expiry:

The payoff to portfolio A: $\max\{S_T - K, 0\} + K = \max\{S_T, K\}$

The payoff to portfolio B: $\max\{K - S_T, 0\} + S_T = \max\{K, S_T\}$

Here, A = B. And by the no-arbitrage principle:

Value of portfolio A at $t = 0$: $c + Ke^{-rt}$

Value of portfolio B at $t = 0$: $p + S_0$

Thus $c + Ke^{-rt} = p + S_0$ as desired.

39) The current price of a stock is S_0 = \$19 and the price of a 3-month European call option on the stock with a strike price of \$20 is \$1. The risk-free annual interest rate is 4%. What is the price of a 3-month European put option on the stock with strike price \$20?

In this exercise we can use put-call parity, $c + Ke^{-rt} = p + S_0$. Plugging in our values we get,

$$1 + 20e^{-(0.04)(.25)} = p + 19 \quad \Rightarrow \quad p = 1 + 19.8 - 19 \quad \Rightarrow \quad p = \$1.80.$$

Thus, the price of the European put option in this case is \$1.80.

40) The prices of European call and put options on a stock with an expiration date in 12 months and a strike price of \$120 are \$20 and \$5, respectively. The current stock price is \$130. What is the implied risk-free interest rate?

Using put-call parity again and plugging in our given values we can solve for r. We get,

$$20 + 120e^{-r(1)} = 5 + 130 \quad \Rightarrow \quad 120e^{-r} = 115 \quad \Rightarrow -r = \ln(115/120) \quad \Rightarrow \quad r = .0426.$$

Thus, the implied risk-free interest rate is 4.26%.

41) Suppose that the price of a stock is \$31, and that the price of a European call option on the stock with strike price \$30 is \$3, and that the price of a European put option on the stock with the same strike price and expiry is \$2.25. The risk-free interest rate is 10% per year, paid every quarter. Show that put-call parity does not hold, and construct an arbitrage opportunity. In particular, show that an arbitrager makes a risk-free profit by buying the call option and short-selling both the put and the stock.

Put-call parity tells us that $c + Ke^{-rt} = p + S_0$. So, let us check if the principle holds with our values: $3 + 30e^{(-0.1)(.25)} = 2.25 + 31$. Simplifying we reach, $e^{(-0.1)(.25)} = 30.25/30$. Taking the natural log of both sides we get, $(-0.1)(.25) = \ln(30.25/30)$. However, now we can observe that the left hand side must be negative and the right positive. Therefore it is impossible for this equality, this put-call parity does not hold.

Now to show that an arbitrage opportunity exists consider the following portfolio.
Buy one european call option, sell one european put option, and short the stock. So,

have $-c + p + S_0$, using the values given -3 + 2.25 + 31 = \$30.25. And after investing this money for three months, $(30.25)e^{(0.1)(.25)}$ = \$31.02 is the amount you have at expiry. Then,

Case 1: $S_T < K$.

Exercise the put option, need to buy stock for \$K. Therefore, profit = 31.02 - 30 = \$1.02

Case 2: $S_T \geq K$.

Exercise the call option, will buy the stock for \$K. Therefore, profit = 31.02 - 30 = \$1.02

Therefore in both cases, the arbitrageur has a guaranteed profit of \$1.02.

42) A 1-month European put option on a non-dividend-paying stock is currently selling for \$2.50. The stock price is \$47, the strike price is \$50, and the risk-free interest rate is 6% per annum. What opportunities are there for an arbitrageur?

First, note that the present value of the strike price, \$50, is equal to $50e^{-(0.06)(1/12)}$ = \$49.75. Then, the current stock price is \$47. Now since the options value \$2.50 < 49.75 - 47 = \$2.75, then the option is undervalued, so an arbitrage opportunity exists. Consider the following portfolio: buy the European put option for \$2.50, buy the stock for \$47, and borrow the total cost of \$49.50 and invest. (Note: at expiry this investment will gain \$.25) This gives the investor an initial cash flow of 49.50 - 47 - 2.50 = \$0. Now consider the following two cases:

Case 1: $S_T > 50$,

Then you will not exercise the option, and we can sell the stock in the market for at least \$50. So when we return our borrowed money, then we have the profit, 50 - 49.50 + .25 = \$.75.

Case 2: $S_T < 50$,

In this case you will exercise, allowing the investor to sell the stock at \$50, and as before will return borrowings to yield a profit of \$.75.

In both cases the profit is positive, thus a riskless profit possibility arises for an arbitrageur.

43) Explain why the arguments leading to put–call parity for European options cannot be used to give a similar result for American options.

The reason put–call parity does not hold for American options is because in an American option there are infinite times to exercise. Let us consider the following two portfolios:

Portfolio A: one American call option and an amount of cash equal to Ke^{-rt}
Portfolio B: one American put option and one share of the stock

Let us say that portfolio A decides to exercise at expiry, that is $t = T$. Then same as in exercise 38, payoff = $\max\{S_T, K\}$.
However since we are using American options, let us say portfolio B decides to exercise at some $t^* < T$. Then, payoff = $K - S_{t^*}$.

Therefore, because of the ability to exercise early, the two portfolios are not the same value at expiry. So, we are unable to use the no-arbitrage principle, thus put-call parity does not hold for American options.

44) Show that it is never optimal to exercise an American call option prior to expiry.

There are two main reasons for why it is not optimal to exercise early.
The first is that a call option provides a form of insurance. When a call option is held instead of a stock, you are in effect protecting yourself against the possibility of the stock's price falling. So the moment you buy the option, you lose the insurance provided by the option.
The second reason is the time value of money. From the point of view of the option holder, it is preferable to spend \$K later and keep your money invested in the risk-free interest market, then to spend \$K sooner.

45) Explain (with an example) why it can be optimal to exercise an American put option prior to expiry.

Suppose we have a stock where its value is close to zero. Then, knowing that the payoff for exercising a put option is $K - S_{t^*}$ for some time $t^* < T$, there is more to gain in exercising early as will be able to invest the $K into the risk-free interest market. This is optimal since the price of stocks tend to change based on the risk-free interest rate, thus the option is valued the same as exercising the option and investing, however with more risk.

46) Give an intuitive explanation for why the early exercise of an American put becomes more attractive as the risk-free rate increases and volatility decreases.

As the risk-free rate increases, people are more likely to pull out of the stock market and put into the fixed income market, reflecting an increase in the stock's prices.
As the volatility decreases, the probability of future payoff in options decreases as well. Thus it is more beneficial to exercise to minimize the downside risk.

47) Let C denote the value of an American call option to buy one share of a stock, and let P denote the value of an American put option to sell one share of a stock. Show that:
$S_0 - K \leq C - P \leq S_0 - Ke^{-rT}$.

We will solve this by proving two inequalities held by contradiction and combining.

First, suppose to the contrary that $S_0 - K > C - P$, and consider the following portfolio. Buy an American call option for $C and exercise and sell an American put option for $P. Then the initial cash flow is $S_0 - K - C + P$ and by our assumption this quantity is positive, thus violating the no-arbitrage principle as a riskless profit can be made.

Second, suppose to the contrary that $C - P > S_0 - Ke^{-rT}$, and consider the following portfolio. Sell an American call option for $C and exercise, buy an American put option for $P, and buy one share of the stock for S_0. In this case the initial cash flow is $C - P - S_0 + K$. Then since $t = 0$ at this cashflow$K = Ke^0 > Ke^{-rT}$, thus $C - P > S_0 - Ke^{-rT}$ also is true. Therefore by assumption, the initial inflow is positive, contradicting the no-arbitrage principle.

Thus putting the two together we get the desired result, $S_0 - K \leq C - P \leq S_0 - Ke^{-rT}$.

48) Different Strike Prices Suppose that $c(K_1)$, $c(K_2)$, and $c(K_3)$ are the prices of European call options with strike prices K_1, K_2, K_3, respectively, where $K_1 < K_2 < K_3$, and that $p(K_1)$, $p(K_2)$, and $p(K_3)$ are the prices of European put options with these strike prices. All options have the same maturity. Prove each of the following inequalities.

(a) $c(K_1) \geq c(K_2)$

We will prove by contradiction. Suppose $c(K_1) < c(K_2)$ and consider the following portfolio. Sell a call option with strike price K_2 and buy a call option with strike price K_1. So at $t = 0$, this portfolio guarantees an initial cash flow of $c(K_2) - c(K_1)$, which is positive by our assumption. Then at expiry we have the following three cases:

	Long call, $K = K_1$	Short call, $K = K_2$	Payoff
$S_T < K_1$	0	0	0
$K_1 < S_T < K_2$	$S_T - K_1$	0	$S_T - K_1$
$K_2 < S_T$	$S_T - K_1$	$K_2 - S_T$	$K_2 - K_1$

Then in all three cases the payoff is positive thus we have a contradiction to the no-arbitrage principle.

(b) $p(K_2) \geq p(K_1)$

By put-call parity, for all strike prices $K_i > 0$, $c(K_i) + K_ie^{-rT} = p(K_i) + S_0$. Therefore we can represent each call price by $c(K_i) = p(K_i) + S_0 - K_ie^{-rT}$. Now we have two new ways to represent our call prices, $c(K_1) = p(K_1) + S_0 - K_1e^{-rT}$ and $c(K_2) = p(K_2) + S_0 - K_2e^{-rT}$.

Subtracting the two we get the following equation:

$c(K_1) - c(K_2) = p(K_1) + S_0 - K_1e^{-rT} - [p(K_2) + S_0 - K_2e^{-rT}]$.

Simplifying we get, $c(K_1) - c(K_2) = p(K_1) - p(K_2) - K_1e^{-rT} + K_2e^{-rT}$. Then, since we know $K_2 > K_1$ then we know $-K_1e^{-rT} + K_2e^{-rT} = (K_2 - K_1)e^{-rT} > 0$, and from (a) $c(K_1) - c(K_2) \geq 0$. Thus, $0 \geq p(K_1) - p(K_2)$ or $p(K_2) \geq p(K_1)$ as desired.

(c) $c(K_1) - c(K_2) \leq K_2 - K_1$

We will prove by contradiction. Suppose $c(K_1) - c(K_2) > K_2 - K_1$ and consider the following portfolio. Sell a call option with strike price K_1, buy one call option with strike price K_2, and invest $K_2 - K_1$ into a risk-free interest rate. So at $t = 0$, this portfolio guarantees an initial cash flow of $c(K_1) - c(K_2)$, which is positive by our assumption as $K_2 > K_1$ so $K_2 - K_1 > 0$. Then at expiry we have the following three cases:

	Short call, $K = K_1$	Long call, $K = K_2$	Payoff
$S_T < K_1$	0	0	$(K_2 - K_1)e^{rT}$
$K_1 < S_T < K_2$	$K_1 - S_T$	0	$K_1 - S_T + (K_2 - K_1)e^{rT}$
$K_2 < S_T$	$K_1 - S_T$	$S_T - K_2$	$K_1 - K_2 + (K_2 - K_1)e^{rT}$

Then in all three cases the payoff is positive thus we have a contradiction to the no-arbitrage principle.

(d) $p(K_2) - p(K_1) \leq K_2 - K_1$

We will prove by contradiction. Suppose $p(K_2) - p(K_1) > K_2 - K_1$ and consider the following portfolio. Sell one put option with strike price K_2, buy one put option with strike price K_1, and invest $K_2 - K_1$ into a risk-free interest rate. So at $t = 0$, this portfolio guarantees an initial cash flow of $p(K_2) - p(K_1)$, which is positive by our assumption as $K_2 > K_1$ so $K_2 - K_1 > 0$. Then at expiry we have the following three cases:

	Long put, $K = K_2$	Short put, $K = K_1$	Payoff
$S_T < K_1$	$K_2 - S_T$	0	$(K_2 - K_1)e^{rT}$
$K_1 < S_T < K_2$	$K_2 - S_T$	$S_T - K_1$	$K_2 - K_1 + (K_2 - K_1)e^{rT}$
$K_2 < S_T$	0	$S_T - K_1$	$S_T - K_1 + (K_2 - K_1)e^{rT}$

Then in all three cases the payoff is positive thus we have a contradiction to the no-arbitrage principle.

49) Convexity Suppose that $c(K_1)$, $c(K_2)$, and $c(K_3)$ are the prices of European call options with strike prices K_1, K_2, K_3, respectively, where $K_1 < K_2 < K_3$. All options have the same maturity.

(a) Show that K_2 is a convex combination of K_1 and K_3, i.e. there exists a real number lambda such that $0 < \lambda < 1$ and $K_2 = \lambda K_1 + (1 - \lambda)K_3$.
Let K_i - K_j be the difference in strike prices for some option i and j. Then let $\lambda = K_3\text{-}K_2/K_3\text{-}K_1$. First let us check that $0 < \lambda < 1$. Since $K_2 > K_1$ then $K_3\text{-}K_2 < K_3\text{-}K_1$, thus $0 < \lambda < 1$ holds. Now, clearing the denominator we get, $\lambda(K_3\text{-}K_1) = K_3\text{-}K_2$. Isolating K_2 we get our desired result, $K_2 = \lambda K_1 + (1 - \lambda)K_3$. Thus we found a lambda $0 < \lambda < 1$ satisfying our assumption.

(b) Show that, with the value of λ found above, $c(K_2) \leq \lambda c(K_1) + (1 - \lambda)c(K_3)$.
We will prove by contradiction. Suppose $c(K_2) > \lambda c(K_1) + (1 - \lambda)c(K_3)$. Consider the following portfolio: buy λ European call options with strike price K_1, Buy λ-1 European call options with strike price K_3, and sell one European call option with strike price K_2. Therefore we have an initial cash flow of $-\lambda c(K_1) + c(K_2) - (1\text{-}\lambda)c(K_3) > 0$ by assumption. Then we have the following four cases:

	Long call, $K = K_1$	Short call, $K = K_2$	Long call, K - K_3	Payoff
$S_T < K_1$	0	0	0	0
$K_1 < S_T < K_2$	$\lambda(S_T - K_1)$	0	0	$\lambda(S_T - K_1)$
$K_2 < S_T < K_3$	$\lambda(S_T - K_1)$	$K_2 - S_T$	0	$\lambda(S_T - K_1) + K_2 - S_T$
$K_3 < S_T$	$\lambda(S_T - K_1)$	$K_2 - S_T$	$(1\text{-}\lambda)(S_T - K_3)$	$\lambda(S_T - K_1) + K_2 - S_T + (1\text{-}\lambda)(S_T - K_3)$

Then in all four cases the payoff is positive thus we have a contradiction to the no-arbitrage principle.

50) Suppose that $c(K_1)$, $c(K_2)$, and $c(K_3)$ are the prices of European call options with strike prices K_1, K_2, K_3, respectively, where $K_1 < K_2 < K_3$ and $K_3 - K_2 = K_2 - K_1$. All options have the same maturity. Show that $c(K_2) \leq (c(K_1) + c(K_3))/ 2$.

Using exercise 49 we know $c(K_2) \leq \lambda c(K_1) + (1 - \lambda)c(K_2)$. So using $\lambda = 1/2$, we get that $c(K_2) \leq 1/2c(K_1) + 1/2c(K_3)$ as desired.

51) State and prove the result corresponding to the exercise above for European put options.

Recall from exercise 50, $c(K_2) \leq (c(K_1) + c(K_3))/ 2$. Now using put-call parity we can represent each European call cost as $c(K_i) = p(K_i) + S_0 - K_ie^{-rT}$. Then we can plug the values into our result in exercise 50:

$p(K_2) + S_0 - K_2e^{-rT} = 1/2[p(K_1) + S_0 - K_1e^{-rT} + p(K_3) + S_0 - K_3e^{-rT}] =$
$1/2[2S_0 - (K_1 + K_3)e^{-rT} + p(K_1) + p(K_3)$. Then by our assumption $2K_2 = K_1 + K_3$, so we can simplify to reach the result that $p(K_2) \leq P(K_1 + K_3)$.

V Trading Strategies Involving Options

A **bull spread** consists of a long position on a European call option on a stock with strike price K_1 and a short position on a European call option on the same stock with strike price K_2, where $K_2 > K_1$. Both options have the same expiration date.

52) Consider a bull spread with strike prices as above. As usual, let ST denote the price of the stock at expiry of the options. Find the payoff from a bull spread. Is an investor who enters into a bull spread hoping that the stock price will increase or decrease?

The payoff from a bull spread is equal to the payoff to the long position on a European call option with strike price K_1 summed with the payoff from the short position on a European call option on the same stock with strike price K_2. Thus,

$$\text{Payoff} = \max\{S_T - K_1, 0\} + \min\{K_2 - S_T, 0\}$$

An investor in a bull spread would hope the stock price increases since $|S_T - K_1| > |K_2 - S_T|$.

53) An investor buys for \$3 a 3-month European call with a strike price of \$30 and sells for \$1 a 3-month European call with a strike price of \$35. Find the profit from this bull spread in each of the following cases:

The initial cash flow for this portfolio is -3 + 1 = -\$2

(a) $S_T = \$25$

Profit = max{25-30, 0} + min{35-25, 0} - 2 = 0 + 0 - 2 = -\$2

(b) $S_T = \$34$

Profit = max{34-30, 0} + min{35-34, 0} - 2 = 4 + 0 - 2 = \$2

(c) $S_T = \$40$

Profit = max{40-30, 0} + min{35-40, 0} - 2 = 10 - 5 - 2 = \$3

A **bear spread** consists of a long position on a European put option on a stock with strike price K_1 and a short position on a European put option on the same stock with strike price K_2, where $K_2 < K_1$. Both options have the same expiration date.

54) Consider a bear spread with strike prices as above. As usual, let ST denote the price of the stock at expiry of the options. Find the payoff from a bear spread. Is an investor who enters into a bear spread hoping that the stock price will increase or decrease?

The payout will follow the same intuition as exercise 52 but with put options. Thus,

$$\text{Payoff} = \max\{K_1 - S_T, 0\} + \min\{S_T - K_2, 0\}$$

An investor in a bull spread would hope the stock price decreases since $|K_1 - S_T| > |S_T - K_2|$.

55) An investor buys for \$3 a 3-month European put option with a strike price of \$35 and sells for \$1 a 3-month European put with a strike price of \$30. Find the profit from this bull spread in each of the following cases:

The initial cash flow for this portfolio is -3 + 1 = -\$2

(a) $S_T = \$25$

$$\text{Profit} = \max\{35-25, 0\} + \min\{25-30, 0\} - 2 = 10 - 5 - 2 = \$3$$

(b) $S_T = \$34$

$$\text{Profit} = \max\{35-34, 0\} + \min\{34-30, 0\} - 2 = 1 + 0 - 2 = -\$1$$

(c) $S_T = \$40$

$$\text{Profit} = \max\{35-40, 0\} + \min\{40-30, 0\} - 2 = 0 + 0 - 2 = -\$2$$

A **straddle** is a trading strategy that consists of a long position on a European call option and a long position on a European put option with the same strike price and expiry.

56) Find the payoff from a straddle. As usual, let K denote the strike price and let S_T denote the price of the stock at expiry.

The payoff from a straddle is equal to the payoff to the long position on a European call option with strike price K summed with the payoff from the long position on a European call option on the same stock with the same strike price. Thus,

$$\text{Payoff} = \max\{S_T - K, 0\} + \max\{K - S_T, 0\}$$

57) A call with a strike price of \$60 costs \$6. A put with the same strike price and expiration date costs \$4. Construct a table that shows the profit from a straddle. For what range of stock prices would the straddle lead to a loss?

The initial cash flow for this portfolio is -6 - 4 = -\$10.

S_T	Long call, K	Long put, K	Payoff
45	0	60 - 45	15
50	0	60 - 50	10
55	0	60 - 55	5
60	0	0	0
65	65 - 60	0	5
70	70 - 60	0	10
75	75 - 60	0	15

Here we will see a profit when the payoff is greater than \$10, so the range where the investor makes a profit is $S_T < \$50 \cup S_T > \$70$.

VI Binomial Tree Option Pricing

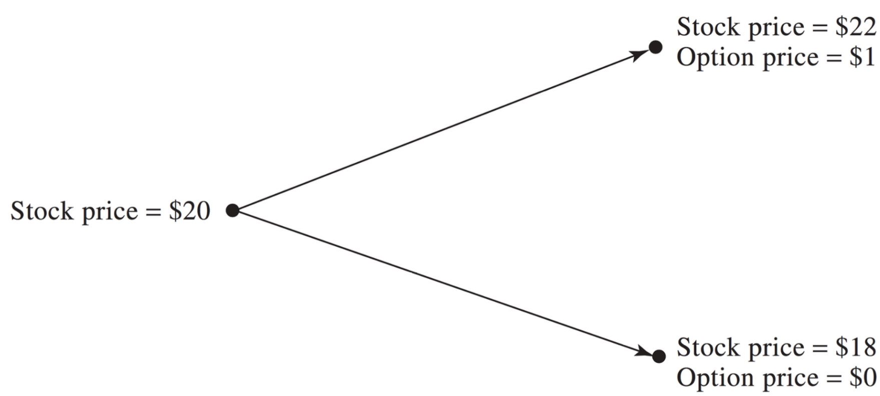
For Exercises 58 - 63 use the following information for a European call option:

S_0 = \$20,

S_T = \$22 or \$18,

K = \$21

58) Show that at time T , the value of the option is either \$1 or \$0, as illustrated in the one-step binomial tree below.



The value of the option is either \$1 or \$0 because if not an arbitrage opportunity would arise.

To price the option, we will construct a portfolio consisting of the stock and the option in such a way that there is no uncertainty about the value of the portfolio at time T. Then, we can argue that since the portfolio has no risk, the return it earns must equal the risk-free interest rate (otherwise, an arbitrage opportunity would exist).

Consider a portfolio consisting of a long position in Δ shares of the stock and a short position in one call option. (We will determine the value of Δ.)

59) Show that if $S_T = \$22$, then the total value of the portfolio at time T is $22\Delta - 1$.

Since $S_T > K$ the option will be exercised, so the value of the option will be $K - S_T = 21 - 22 = -\$1$ then each of the Δ stocks will be worth \$22 at T. Thus, the total value of the portfolio is $22\Delta - 1$.

60) Show that if $S_T = \$18$, then the total value of the portfolio at time T is 18Δ.

Here since $S_T < K$ the option will not be exercised, thus the option expires and the value of the portfolio is precisely the Δ shares valued at \$18. Thus, the total value of the portfolio at $t = T$ is 18Δ.

61) Find the value of Δ that makes this portfolio riskless, and conclude that, regardless of whether the stock price moves up or down, the value of the portfolio is always \$4.5 at time T.

The values will be riskless when both portfolios, for the two possibilities for S_T, are equal. Therefore will be riskless when $22\Delta - 1 = 18\Delta$. Thus, $\Delta = .25$. From this we can deduce that the value of each portfolio is $18(.25) = \$4.50$.

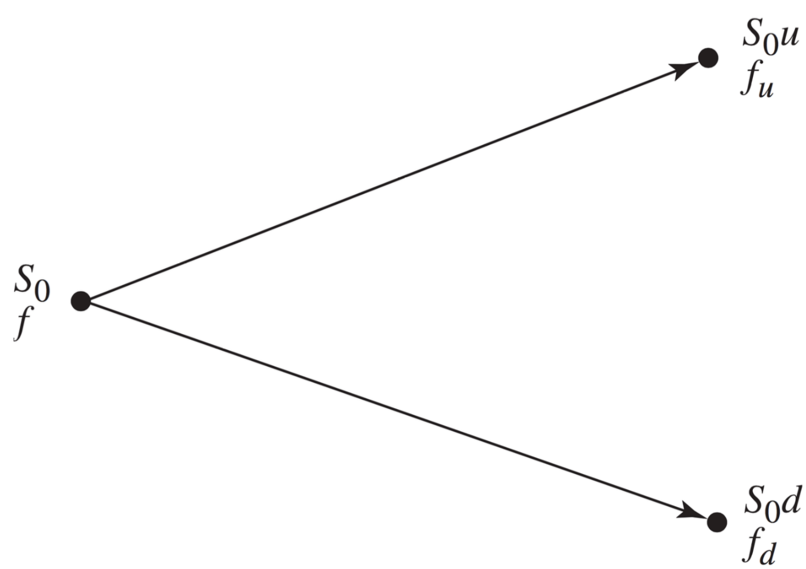
62) Suppose that the risk-free annual interest rate (compounded continuously) is 12% and that T = 3 months. Show that the value of the portfolio today is \$4.367.

To avoid the possibility for an arbitrage opportunity we need to account for the risk-free interest rate. Using $r = .12$ and $T = .25$, we see that $P = 4.50e^{-(0.12)(.25)} = \$4.367$. So if the portfolio is worth \$4.50 at expiry then because of the risk-free interest rate the value at $t = 0$ is \$4.367.

63) Show that, in the absence of arbitrage, the price of the call option today is \$0.633.

Let f = price of the call option at t = 0. Using our portfolio, at t = 0 the price of the stock is \$20, so the shares at t = 0 are worth 20Δ = \$5 and the price of the European call option is -f. Thus, 5 - f = 4.367. Simplifying we get that the price of the option is f = \$0.633.

A more general case now:



64) Recall that the portfolio is riskless if the portfolio is worth the same amount at time T in both cases (i.e. $S_T = S_0u$ or $S_T = S_0d$). Find the value of Δ that makes the portfolio riskless.

For an increase in the stock price, the value is $\Delta S_0u - f_u$. And for a decrease in the stock price, the value is $\Delta S_d - f_d$. Setting these equal we can then find a value Δ that makes this portfolio riskless: $\Delta = (f_u - f_d)/(S_0u - S_0d)$.

65) One-Step Binomial Tree Option Price Use the present value of the portfolio and the expression for Δ that you obtained in the previous exercise to show that $f = e^{-rT}[pf_u + (1-p)f_d]$, where $p = (e^{rT} - d) / (u - d)$.

Now that we know this Δ value, we know at expiry the portfolio has a value of $\Delta S_0 u - f_u$. Thus, by the no-arbitrage principle, the value of the portfolio at $t = 0$ is the discounted value of this when incorporating risk-free interest. So at $t= 0$, the portfolio is worth $(\Delta S_0 u - f_u)e^{-rT}$. Then, we also know at $t = 0$, the portfolio is also worth $\Delta S_0 - f$, so we can set the two equal to find the value for f.

$\Delta S_0 - f = (\Delta S_0 u - f_u)e^{-rT} \;\Rightarrow\; f = \Delta S_0 - (\Delta S_0 u - f_u)e^{-rT} \;\Rightarrow\; f = \Delta S_0(1 - ue^{-rT}) + f_u e^{-rT}$.

Next, let us plug in the Δ value we gpt in the previous exercise. So,

$f = ((f_u - f_d)/(S_0 u - S_0 d))S_0(1 - ue^{-rT}) + f_u e^{-rT}$

Now, let $p = (e^{rT} - d)/(u - d)$, making $1-p = (u - e^{rT})/(u - d)$. Simplifying the above equation, we get,

$f = e^{-rT}[(e^{rT} - d)/(u - d)f_u + f_d - (e^{rT} - d)/(u - d)f_d] \;\Rightarrow\; f = e^{-rT}[pf_u + (1- p)f_d]$, as desired.

66) Use this expression for f to show that f = \$.633 for the numerical example considered previously in this section.

Let us plug in the following values to verify result:

$r = .12,\ T = .25,\ S_0 = \$20,\ u =1.1,\ S_0 u = \$22,\ d = 0.9,\ S_0 d = \$18,\ f_u = \$1,\ f_d = \$0$.

Then, $p =(e^{(.12)(.25)} - 0.9) / (1.1 - 0.9) = .6523$.

So, $f = e^{-(.12)(.25)} [.6523(1) + 0] = .633$. Thus we get the result we were looking for.

67) A stock price is currently \$100. Over each of the next two 6-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a 1-year European call option with a strike price of \$100?

We will use the equation we will develop in exercise 71:

$$f = e^{-2r\Delta t}\,[p2f_{uu} + 2p(1-p)f_{ud} + (1-p)2f_{dd}] \text{ with } p = (e^{r\Delta t} - d)/(u - d)$$

So first, $p = (e^{(.08)(.5)} - 0.9)/(1.1 - 0.9) = .704$. Then, note that any value of a European call option will be $\max(S_T - K, 0)$, so since $f_{ud} = 99$ and $f_{dd} = 81$, both will be valued at zero at expiry. So, we will only consider $f_{uu} = 121$. Finally, $f = e^{-2(.08)(.5)}\,[(.704)(21)] = .923[10.397] = \$9.61$.

68) For the situation considered in the previous exercise, what is the value of a 1-year European put option with a strike price of \$100?

Now can use our equation from exercise 65:

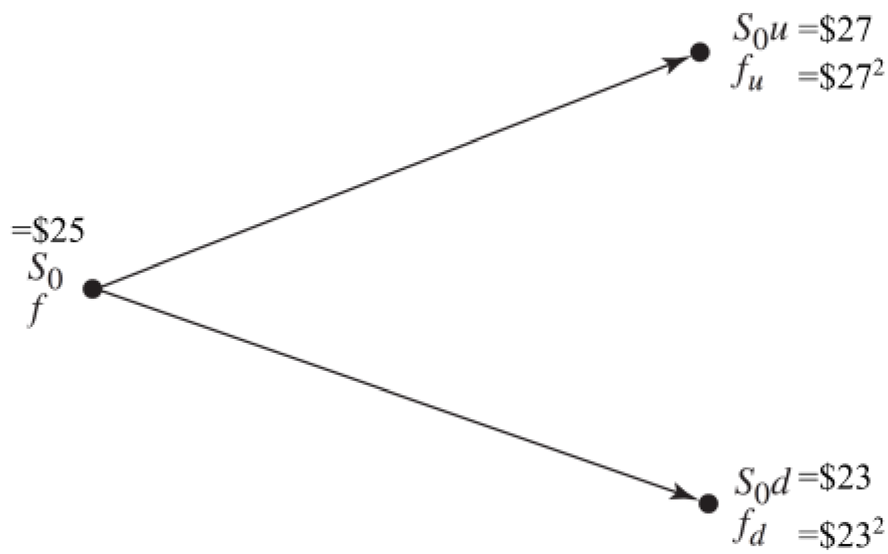
$$f = e^{-rT}\,[pf_u + (1-p)f_d], \text{ where } p = (e^{rT} - d) / (u - d).$$

Here, $p =$ $p = (e^{.08} - 0.9) / (1.1 - 0.9) = .916$. Again we will only need to consider the case where the stock price increases for the same reason. So, $f = e^{-(.08)}\,[(.916)(10)] = 1.04[9.16] = \$9.53$.

Now we have found the price of a call option with these parameters, so we must use put-call parity to find the price of a European put option, $p(K) = c(K) - S_0 + Ke^{-rT}$. Plugging in values we get, $P(K) = 9.53 - 100 + 100(e^{-(.08)}) = \$1.92$.

69) A stock price is currently \$25. It is known that at the end of 2 months, it will be either \$23 or \$27. The risk-free interest rate is 10% per annum with continuous compounding. Suppose S_T is the stock price at the end of 2 months. What is the value of a derivative that pays S_T^2 at this time?

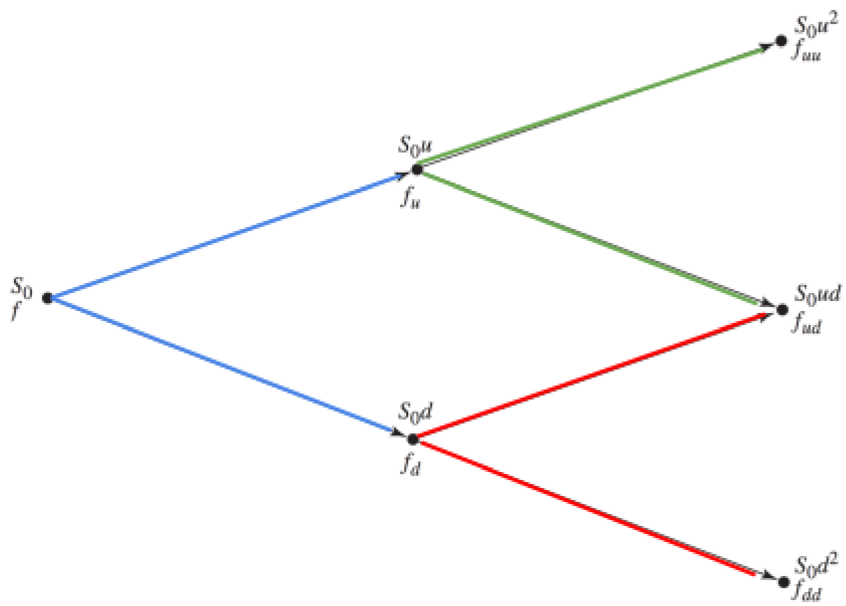
Consider the following binomial tree diagram:



Then, to get this result we need to have the payoffs equal at expiry, we will use delta hedging. Consider the portfolio where we long Δ shares of the stock and short the derivative. Then at expiry, $\Delta S_0u - f_u = \Delta S_0d - f_d \Rightarrow 27\Delta - 23\Delta = 27^2 - 23^2 \Rightarrow 4\Delta = 100$. Thus, $\Delta = 50$. So, we know the payoff at expiry is $(50)(27) - 27^2 = 621$. Then, the value at $t = 0$ is $\Delta S_0 - f$. Then by the no-arbitrage principle, $(\Delta S_0 - f)e^{rT} = 621$. Now, let us solve for f to find the price of the derivative, $f = \Delta S_0 - e^{-rT}(621)$. Thus, $f = (50)(25) - .983(621) = 1250 - 610.735 = \$639.30$.

70) Two-Step Binomial Tree Option Price Consider a stock whose current price is S0 and a European call option on the stock whose current price is f. Suppose that there are two-time steps, each of length Δt, before expiry, and that during each time step, the stock price either moves up to u times its initial value (u > 1) or down to d times its initial value (d < 1). This is illustrated in the following figure, using the same notation as in the previous figure for the payoff from the option at each stage. (For example, f_{uu} is the value of the option at time $2\Delta t = T$ if $ST = u^2S_0$, i.e.

if the stock increases in value at each time step.



Show that $f = e^{-2r\Delta t} [p2f_{uu} + 2p(1 - p)f_{ud} + (1 - p)2f_{dd}]$

To derive this formula we will need to think of this 2-step binomial tree in three sections as highlighted above in blue, green, and red. Since each step follows the same parameters they will all use the same p-value with $p = (e^{r\Delta t} - d) /(u - d)$. Then by our definition found in exercise 65, we know that the blue tree can be represented as $f = e^{-r\Delta t} [pf_u + (1-p)f_d]$, and this tree can be generalized to our other tree sections. Starting with the green tree, $f_u = e^{-r\Delta t} [pf_{uu} + (1-p)f_{ud}]$. Similarly, for the red tree section, $f_d = e^{-r\Delta t} [pf_{du} + (1-p)f_{dd}]$. Now, we have equations representing f_u and f_d so can plug into our blue section to get an expression to represent all steps of the tree:

$f = e^{-r\Delta t} [p[e^{-r\Delta t} [pf_{uu} + (1-p)f_{ud}]] + (1-p)[e^{-r\Delta t} [pf_{du} + (1-p)f_{dd}]]]$.

Simplifying we get our result, $f = e^{-2r\Delta t} [p2f_{uu} + 2p(1 - p)f_{ud} + (1 - p)2f_{dd}]$.

71) A stock price is currently \$50. Over each of the next two 3-month periods, it is expected to go up by 6% or down by 5%. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a 6-month European call option with a strike price of \$51?

Let us solve this 2-step binomial tree option price problem by the same steps we used to derive exercise 70. First, let us find our p value, $p = (e^{-(.05)(.25)} - 0.95)/(1.06 - 0.95) = .5689$. Now let us look at the expected payoffs for each case:

S_0u = \$53	S_0d = \$47.5
S_0uu = \$56.18	S_0ud = \$50.33
S_0dd = \$45.13	

Thus, since our strike price is \$51, we will only need to take into account f_u and f_{uu} for pricing. Let us find our price then for $f_u = e^{-r\Delta t}\,[pf_{uu} + (1-p)f_{ud}] = e^{-(.05)(.25)}\,[(.5689)(5.18) + 0] =$ \$2.19. Then we know f_d = \$0 as the strike price is greater than both f_{du} and f_{dd}. Thus we can plug our values into $f = e^{-r\Delta t}\,[pf_u + (1-p)f_d] = e^{-(.05)(.25)}[(.5689)(2.91) + 0] =$ \$1.635.

VII The Black-Scholes-Merton Model 1

72) Expected Value of a Binomial Distribution Let S_n be the number of successes in n Bernoulli trials with probability p of success on each trial. Show that $E[S_n] = np$.

We know that $S_n = X_1 + X_2 + \ldots + X_n$. And $E[X_i] = 1(p) + 0(1-p) = p$, for all $i = 1, \ldots, n$. Then because the expected value is a linear operator we can split up $E[S_n] = E[X_1] + E[X_2] + \ldots + E[X_n]$. Therefore, $E[S_n] = p + p + \ldots + p$, n times. Thus, $E[S_n] = np$ as desired.

73) Variance of a Discrete Random Variable Let X be a numerically-valued discrete random variable with expected value μ. Show that $V(X) = E[X_2] - \mu^2$.

We know by our definition of variance that $V(X) = E[(x-\mu)^2] = E[X^2 - 2X\mu + \mu^2]$. Then again since the expected value is a linear operator, $E[X^2 - 2X\mu + \mu^2] = E[X^2] - E[2X\mu] + E[\mu^2]$. Now since μ is a constant $E[\mu] = \mu$, thus we are left with $E[X^2] - 2\mu E[X] + \mu^2$. Then, by definition, $E[X] = \mu$. So, $V(X) = E[X^2] - 2\mu^2 + \mu^2 = E[X_2] - \mu_2$ as desired.

74) Variance of a Binomial Distribution Let S_n be the number of successes in n Bernoulli trials with probability p of success on each trial and $1 - p = q$. Show that $V[S_n] = \sigma^2 = npq$.

We know $V[X] = E[X^2] - (E[X])^2$, so $V[X] = E[S_n^2] - (E[S_n])^2$. Then from exercise 72, $E[S_n] = np$. Now we need to find $E[S_n^2]$.

$E[S_n^2] = E[(X_1 + X_2 + \ldots + X_n)^2] =$

$$= E[X_1^2 + X_2^2 + \ldots + X_n^2 + 2X_1X_2 + 2X_1X_3 + \ldots + 2X_{n-1}X_n]$$

$$= E[X_1^2] + E[X_2^2] + \ldots + E[X_n^2] + E[2X_1X_2] + E[2X_1X_3] + \ldots + E[2X_{n-1}X_n]$$

$$= np + n(n-1)p^2.$$

Thus, $V[X] = np + n(n-1)p^2 - (np)^2 = np + (n^2 - n)p^2 - n^2p^2 = np - np^2 = np(1-p) = npq$.

75) Show that for any x, $1 - N(x) = N(-x)$

We know that $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$. So, $1 - N(x) = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$. Which we can rewrite as $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} dt - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$. Then we notice that we can combine these two integrals to reach the expression, $\frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-t^2/2} dt$. Now we can solve the integral. First, let u = -t, then du = -dt. Substituting we get, $\frac{1}{\sqrt{2\pi}} \int_{-x}^{-\infty} e^{u^2/2} (- \, du)$. Then moving to the proper form, $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-x} e^{u^2/2} du = N(-x)$.

76) Price of a Put Option Show that the price of a European put option on a non-dividend paying stock with strike price K and expiry T is $p = Ke^{-rT} N(-d_2) - S_0N(-d_1)$.

Recall from put-call parity that $p = c + Ke^{-rT} - S_0$. Then, by our equation for the price of a call option found with the Black-Scholes-Merton formula, $c = S_0N(d_1) - Ke^{-rT}N(d_2)$. So,

$p = S_0N(d_1) - Ke^{-rT}N(d_2) + Ke^{-rT} - S_0$. Combining like terms then we get,
$p = S_0[N(d_1) - 1] + Ke^{-rT}[1 - N(d_2)]$. Then we know from exercise 75, $1 - N(x) = N(-x)$. So,
$p = Ke^{-rT} N(-d_2) - S_0N(-d_1)$, as desired.

77) A non-dividend paying stock has a current value S_0 = \$41. The volatility is $\sigma = 0.3$ and the risk-free interest rate is r = 8%. Find the price of a European call option on the stock with strike price K = \$40 and expiry T = 3 months.

Let us use the following equations:

$c = S_0N(d_1) - Ke^{-rT}N(d_2)$,

$d_1 = [\ln(S_0/K) + (r + \sigma/2)T]/\sigma\sqrt{T}$,

$d_2 = d_1 - \sigma\sqrt{T}$.

Then, plugging the numbers into an Excel program, we get

$d_1 = 0.3729$, $d_2 = 0.2229$, $N(d_1) = 0.6454$, $N(d_2) = 0.5882$, and $\underline{c = 3.399}$.

78) Find the price of a European put option on the stock in Exercise 77.

Now we can use the result from exercise 76, $p = Ke^{-rT}N(-d_2) - S_0N(-d_1)$. Now plugging into an Excel program, we get,

$d_1 = 0.3729$, $d_2 = 0.2229$, $N(-d_1) = 0.3557$ $N(-d_2) = 0..4129$, and $\underline{c = 1.607}$.

79) Find the binomial approximation for the call option in Exercise 77 with n = 1, n = 2, n = 10, n = 12, and n = 100. What happens to the approximation as n increases?

Need to use our equation for the cost of a call option using a binomial approximation,

$c = \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \max(S_0 u^j d^{n-j} - K, 0)$. Then for each n need to plug in the appropriate parameters, $u = e^{\sigma\sqrt{\frac{T}{n}}}$, $d = e^{-\sigma\sqrt{\frac{T}{n}}}$, and $p = \frac{e^{r(T/n)} - d}{u - d}$. This then results in the following for the given n values:

n	1	2	10	12	100
c	3.96	3.336	3.43	3.43	3.4

Then we can see that the more steps we have, the more accurate the cost of the call option gets to the exact price found in exercise 77.

80) A non-dividend paying stock has a current value S_0 = \$120. The volatility is $\sigma = 0.3$ and the risk-free interest rate is $r = 8\%$.

(a) Find the price of a European call option on the stock with strike price K = \$100 and expiry T = 1 year.

Let us use the following equations, $c = S_0N(d_1) - Ke^{-rT}N(d_2)$,

$d_1 = [\ln(S_0/K) + (r + \sigma/2)T]/\sigma\sqrt{T}$, and $d_2 = d_1 - \sigma\sqrt{T}$. Now we can plug in the given numbers to get $d_1 = 1.0244$ and $d_2 = .7244$. Now we can find that, $N(d_1) = .8461$ and $N(d_2) = .7642$. Thus, $c = 120(.8461) - 100e^{-(.08)}(.7642) = 30.987$.

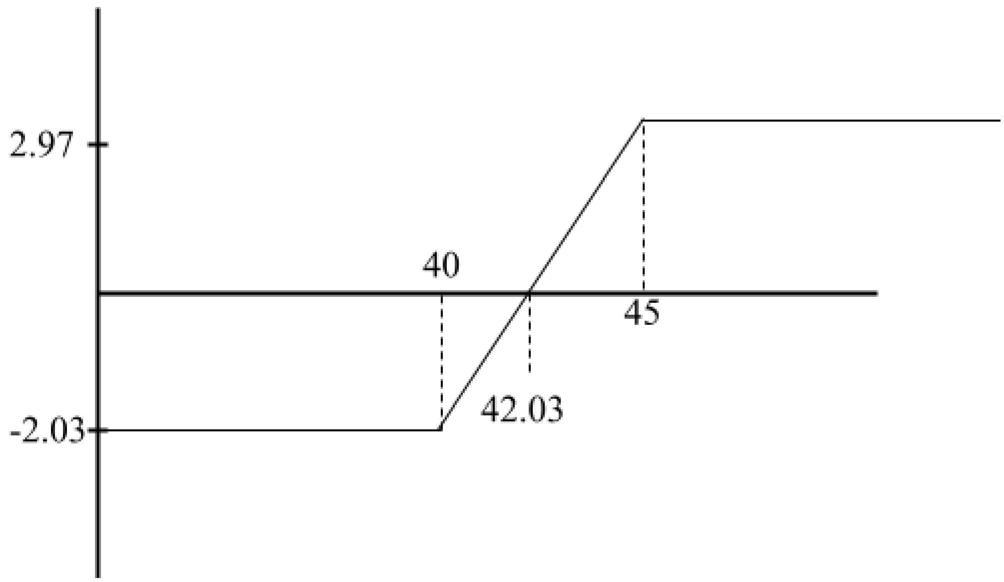
(b) Compute the price of the European call option for several values of large T. What happens to the price of the European call option as $T \to \infty$?

T	20	50	100	500
c	\$102.15	\$118.36	\$119.97	\$120

Mathematically, as $T \to \infty$, we notice that d_1 and d_2 both go to ∞. So, $N(d_1)$ and $N(d_2)$ will then go to 1. Now plugging into our equation we see $c = S_0(1) - Ke^{-rT}(1) = c = S_0 - Ke^{-rT}$. Finally as $T \to \infty$, $Ke^{-rT} \to 0$. Thus as $T \to \infty$ the price of the call option is the same as the initial stock value S_0.

81) Consider a bull spread where you buy a 40-strike call and sell a 45-strike call. Suppose $S_0 = \$40$, $\sigma = 0.30$, $r = 0.08$, and $T = 0.5$.Draw a graph with stock prices ranging from \$20 to \$60 depicting the profit on the bull spread after 1 day, 3 months, and 6 months.

Here we can see that the payoff = $(C_2 - C_1) + \max(S_T - K_1, 0) + \min(K_2 - S_T, 0)$. Using the Black-Scholes-Merton formula we get that the cost of the first option is \$4.16 and the second option costs \$2.13. Thus, payoff = $-2.03 + \max(S_T - 40, 0) + \min(45 - S_T, 0)$.



X Linear Programming, Optimization, and Operations Research

98) Graph the feasible region for the following example.

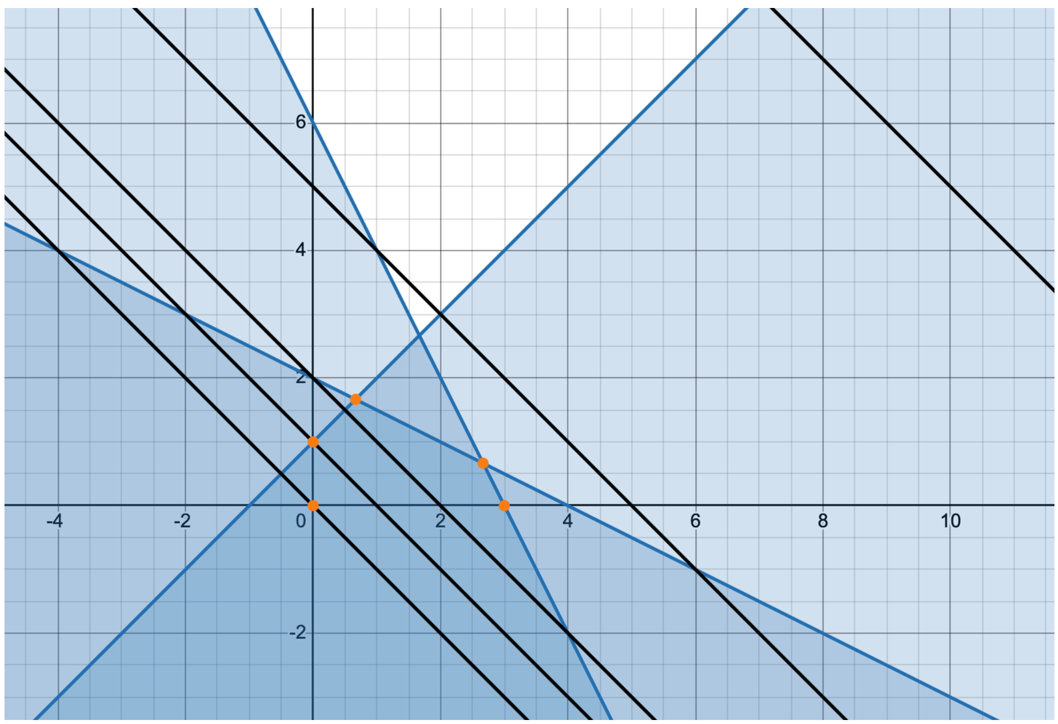
maximize $x_1 + x_2$ subject to the constraints

$x_1 + 2x_2 \leq 4$

$4x_1 + 2x_2 \leq 12$

$-x_1 + x_2 \leq 1$

where $x_1 \geq 0$, and $x_2 \geq 0$



99) Graph the lines $z = x_1 + x_2$ along with the feasible region for z = 0, 1, 2, 5, 15. Which of these represent possible values of the objective function?

Shown in black lines above.

100) Find all 5 of the corner points for the feasible region of exercise 98, and substitute each into the objective function. Verify that 10/3 is the maximum value of the objective function and that it occurs at $(x_1, x_2) = (8/3, 2/3)$.

The five points, highlighted in orange, are (0,0), (0,1), (3,0), (8/3,2/3), (2/3, 5/3).

Then to verify (8/3, 2/3) is the maximum we plug each into the objective function.

$0 + 0 = 0$

$0 + 1 = 1$

$3 + 0 = 3$

$8/3 + 2/3 = 10/3$

$2/3 + 5/3 = 7/3$

Then notice the greatest value is 10/3 corresponding to the point, (8/3,2/3) as desired.

101) Consider the following linear programming problem: Find x1 and x2 to

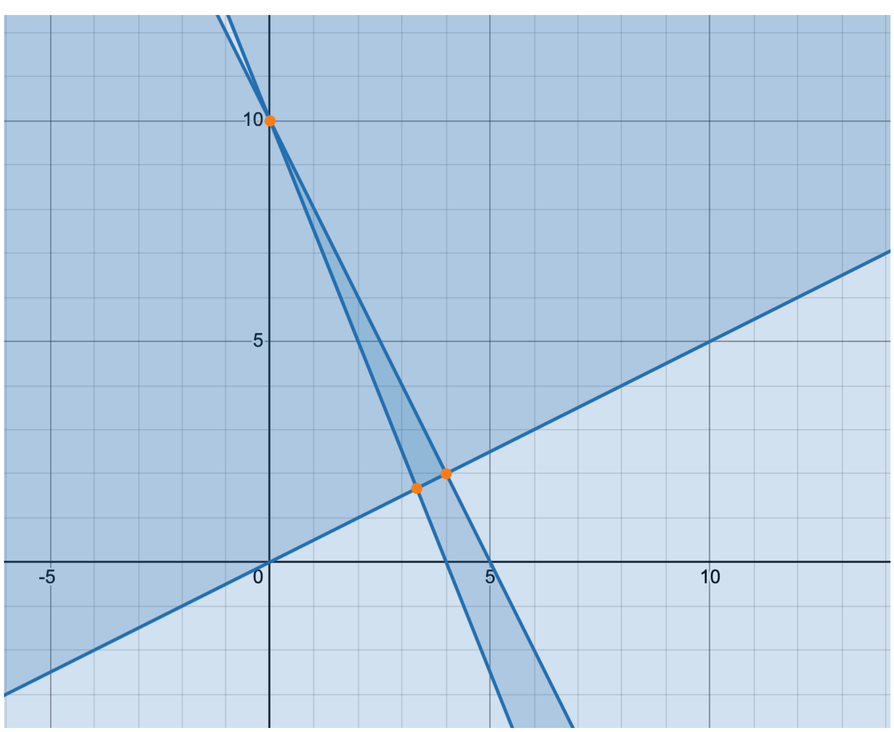
minimize $x_1 + 3x_2$ subject to the constraints:

$2x_1 + x_2 \leq 10$

$5x_1 + 2x_2 \geq 20$

$-x_1 + 2x_2 \geq 0$

where $x_1, x_2 \geq 0$



Graphing the feasible region and identifying the corner points, we can use the objective function to find the minimum.

$0 + 3(10) \quad = 30$

$4 + 3(2) \quad = 10$

$10/3 + 3(5/3) = 25/3$

Thus the minimum for this problem is 25/3 corresponding to the point (10/3, 5/3).

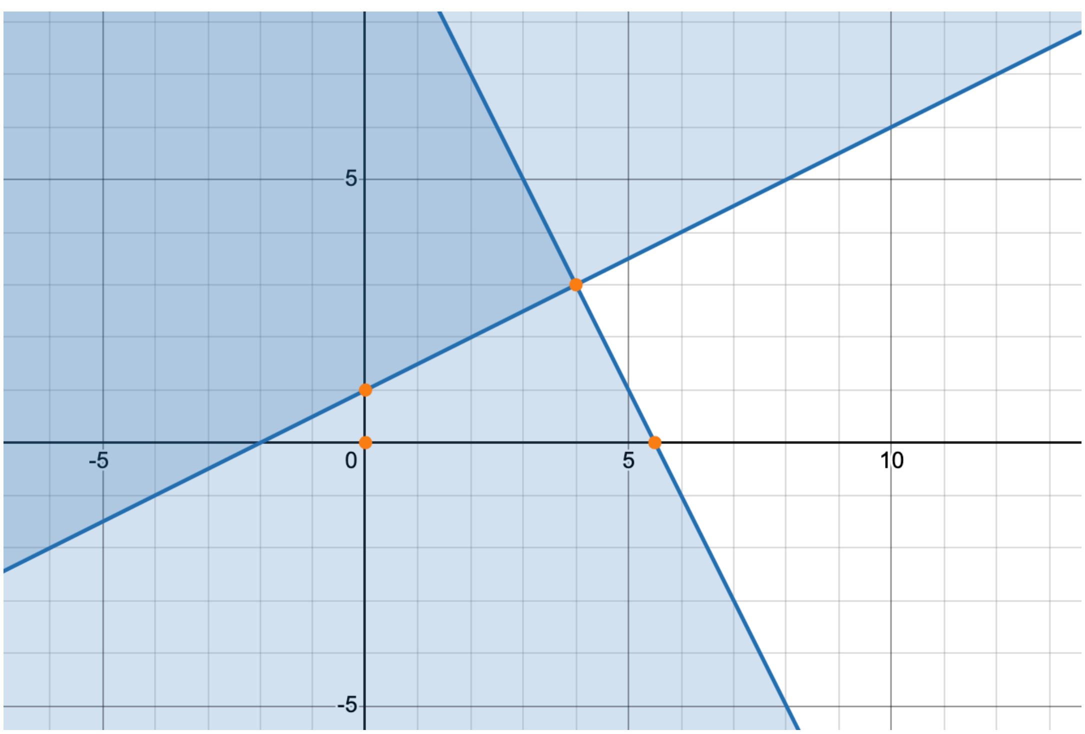
102) This linear programming problem was part of the Operations Research Examination of the Society of Actuaries. If $c_2 > 0$, determine the range of c_1/c_2 for which $(x_1,x_2) = (4,3)$ is an optimal solution of the problem

maximize $c_1x_1 + c_2x_2$ subject to the constraints

$2x_1 + x_2 \leq 11$

$-x_1 + 2x_2 \leq 2$

where $x_1,x_2 \geq 0$



Graphing the feasible region we find that the corner points are (0,0), (0,1), (5.5, 0), and (4,3). Now, using the objective function we can find the range that makes (4,3) an optimal combination.

Recall, $z = c_1x_1 + c_2x_2$

(0,0) -> $z = 0$
(0,1) -> $z = c_2$
(5.5,0) -> $z = 5.5c_1$
(4,3) -> $z = 4c_1 + 3c_2$

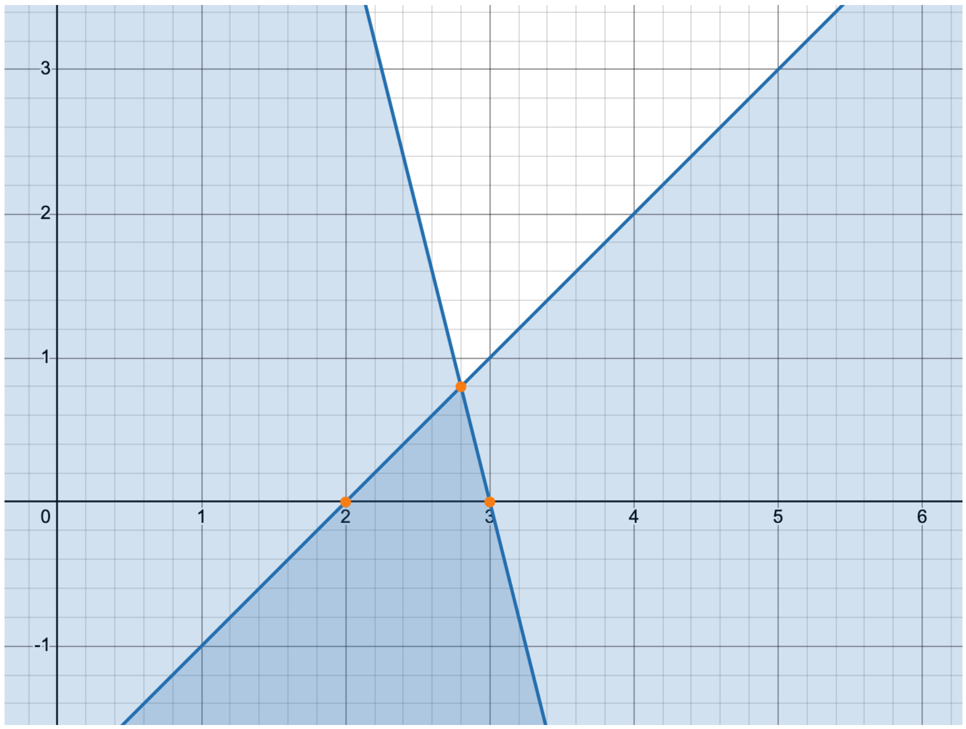
Then since we want (4,3) to be the maximum point this implies that $4c_1 + 3c_2 > 0$, $4c_1 + 3c_2 > c_2$, and $4c_1 + 3c_2 > 5.5c_1$. Now we can find the ratio c_1/c_2 for each inequality.

$4c_1 + 3c_2 > 0 \Rightarrow 4c_1 > 3c_2 \Rightarrow c_1/c_2 > 3/4$
$4c_1 + 3c_2 > c_2 \Rightarrow 4c_1 > -2c_2 \Rightarrow c_1/c_2 > -1/2$
$4c_1 + 3c_2 > 5.5c_1 \Rightarrow 3c_2 > 1.5c_1 \Rightarrow c_1/c_2 < 2$

Thus the range c_1/c_2 where (4,3) will satisfy all the inequalities and be the maximum point is $c_1/c_2 = (-1/2, 2)$.

103) Consider the following linear programming problem, where the value of c_1 has not yet been determined. Find x_1 and x_2 to determine the optimal solutions for the various possible values of c_1 (both positive and negative).

maximize $c_1x_1 + 2x_2$ subject to the constraints
$4x_1 + x_2 \leq 12$
$x_1 - x_2 \geq 2$
where $x_1, x_2 \geq 0$



Graphing the feasible region we find that the corner points are (2,0), (3,0), and (14/5, 4/5). Now, using the objective function we can find what corner points will be the maximum depending on certain values of c_1.

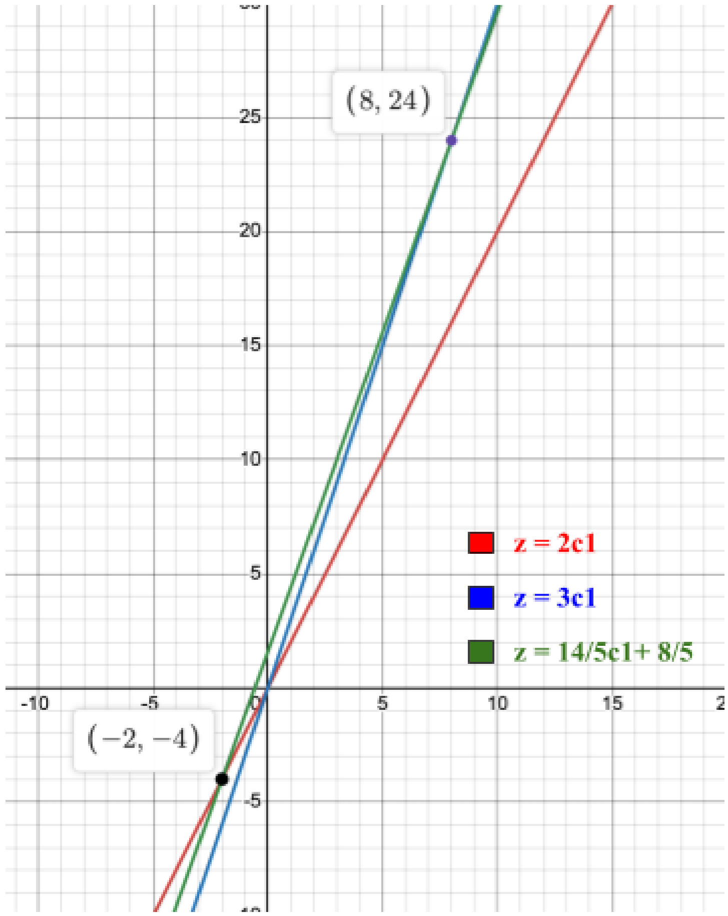
Recall, $z = c_1x_1 + 2x_2$

(2,0) -> $z = 2c_1$

(3,0) -> $z = 3c_1$

(14/5, 4/5) -> $z = 14/5c_1 + 8/5$

Now we can graph the three equations for z in terms of c_1 to find which is greater for different values.



Then looking at the graph above we can create ranges for c_1 where the different objective functions represent the maximum.

(2,0) for (-∞, -2), (3,0) for (-2,8), and (8, ∞) for (14/5, 4/5).

104) A manufacturing process requires that oil refineries manufacture at least 2 gallons of gasoline for every gallon of fuel oil. To meet the winter demand for fuel oil, at least 3 million gallons a day must be produced. The demand for gasoline is no more than 12 million gallons per day. It takes .25 hour to ship each million gallons of gasoline and 1 hour to ship each million gallons of fuel oil out of the warehouse. No more than 6.6 hours are available for shipping. If the refinery sells gasoline for $1.25 per gallon and fuel oil for $1 per gallon, how much of each should be produced to maximize revenue? Find the maximum revenue.

maximize $x_1 + 5/4x_2$ subject to the constraints:

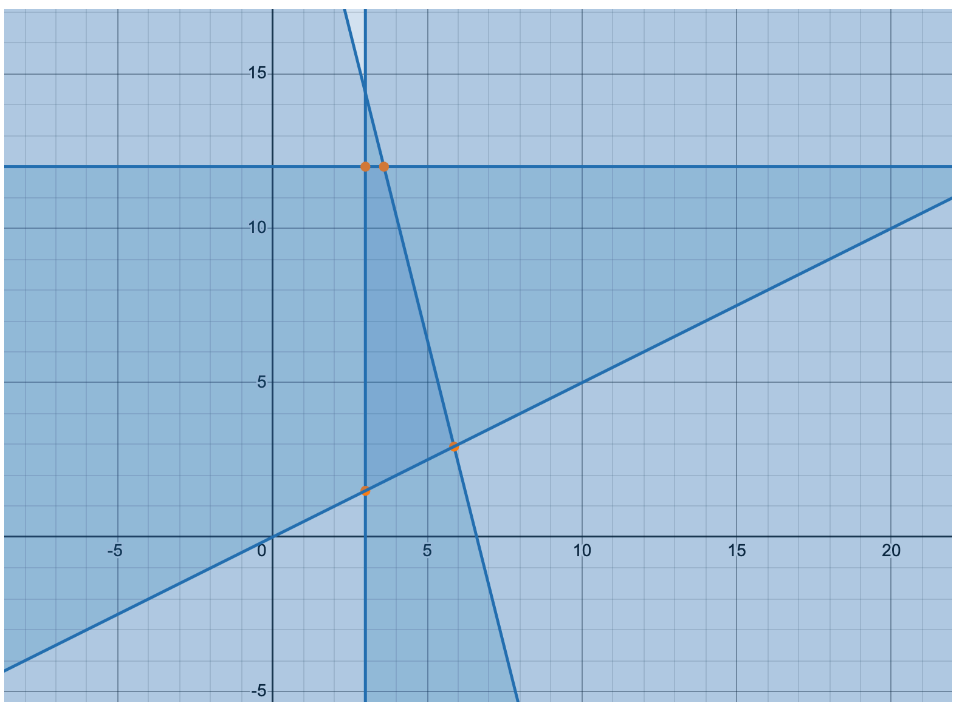
$x_1 + 1/4x_2 \geq 6.6$

$x_2 \leq 12$

$x_1 \geq 3$

$2x_2 \geq x_1$

where $x_1, x_2 \geq 0$



Graphing the feasible region we find that the corner points are (3,12), (3.6,12), (3, 1.5), and (5.87, 2.93). Now, using the objective function we can find the maximum

Recall $z = x_1 + 5/4x_2$

$3 + 5/4(12)$	$= 18$
$3.6 + 5/4(12)$	$= 18.6$
$3 + 5/4(1.5)$	$= 4.875$
$5.87 + 5/4(2.93)$	$= 9.532$

Thus to produce the maximum revenue, the refining company should produce 12 gallons of gasoline and 3.6 gallons of fuel oil.

105) Transform the following LP(Linear Program) to a standard maximum problem.

minimize $x_1 - 12x_2 - 2x_3$ subject to the constraints:

$5x_1 - x_2 - 2x_3 = 10$

$2x_1 + x_2 - 20x_3 \geq 30$

$x_2 \leq 0$

$1 \leq x_3 \leq 4$

First, we need to make a maximum problem by using the reciprocal.
maximize $-x_1 + 12x_2 + 2x_3$. Next, we need to break up our first constraint, the equality, into two less-than or equal-to inequalities. $5x_1 - x_2 - 2x_3 \leq 10$ and, $5x_1 - x_2 - 2x_3 \geq 10$. Then, we need to flip the inequality for our second inequality found above as well as our second constraint. $-5x_1 + x_2 + 2x_3 \leq -10$, and $-2x_1 - x_2 + 20x_3 \leq -30$. Now, we need to introduce a new variable to create a lower bound on x_2. Let $w_2 = -x_2$. Then, $w_2 \geq 0$. Finally, we need to get rid of the interval bound on our last constraint by introducing another variable.
Let $w_3 = x_3 - 1$. Then, $w_3 \geq 0$ and $w_3 \leq 3$. So all together we transformed the LP to

maximize $-x_1 + 12x_2 + 2x_3$ subject to the constraints:

$5x_1 - x_2 - 2x_3 \leq 10$ & $-5x_1 + x_2 + 2x_3 \leq -10$

$-2x_1 - x_2 + 20x_3 \leq 30$

$w_2 \geq 0$

$w_3 \geq 0$ & $w_3 \leq 3$

106) Rewrite the following LP by introducing slack variables so that each inequality constraint is replaced with an equality.

maximize $z = 2x_1 + 3x_2 + x_3$ subject to the constraints:

$x_1 + x_2 + 4x_3 \ \leq 100$

$x_1 + 2x_2 + x_3 \ \leq 150$

$3x_1 + 2x_2 + x_3 \leq 320$

where $x_1,x_2,x_3 \geq 0$

Gets changed to,

maximize $z = 2x_1 + 3x_2 + x_3$ subject to the constraints:

$x_1 + x_2 + 4x_3 + s_1 \ = 100$

$x_1 + 2x_2 + x_3 + s_2 \ = 150$

$3x_1 + 2x_2 + x_3 + s_3 = 320$

where $x_1,x_2,x_3 \geq 0$

Pivot

Pivot column

$$\begin{array}{rc} & \begin{array}{ccccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & & \end{array} \\ \begin{array}{r} \\ \text{Pivot row} \rightarrow \\ \\ \\ \end{array} & \left[\begin{array}{ccccccc|c} 1 & 1 & 4 & 1 & 0 & 0 & 0 & 100 \\ 1 & \boxed{2} & 1 & 0 & 1 & 0 & 0 & 150 \\ 3 & 2 & 1 & 0 & 0 & 1 & 0 & 320 \\ \hline -2 & -3 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]. \end{array}$$

107) In Exercise 106, the pivot column is x_2: Show that the quotients obtained in Step 2 of Algorithm 2 are 100, 75, and 160, and conclude that the pivot row is the second row. Thus, the pivot is the 2 indicated below:

Since -3 is the most negative indicator we look at its column to find the pivot. Then, following the algorithm, 100/1 = 100, 150/2 = 75, 320/2 = 160. Then since 75 is the least of the three this tells us the second row is the pivot row.

108) Use pivoting to obtain the following simplex tableau for Exercise 106.

$$\begin{array}{c}
\begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{array} \\
\left[\begin{array}{ccccccc|c}
\frac{1}{2} & 0 & \frac{7}{2} & 1 & -\frac{1}{2} & 0 & 0 & 25 \\
\frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 75 \\
2 & 0 & 0 & 0 & -1 & 1 & 0 & 170 \\
\hline
-\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{3}{2} & 0 & 1 & 225
\end{array}\right]
\end{array}.$$

$$\begin{array}{c}
\begin{array}{ccccccc} x1 & x2 & x3 & s1 & s2 & s3 & z \end{array} \\
\left[\begin{array}{ccccccc|c}
1 & 1 & 4 & 1 & 0 & 0 & 0 & 100 \\
1 & 2 & 1 & 0 & 1 & 0 & 0 & 150 \\
3 & 2 & 1 & 0 & 0 & 1 & 0 & 320 \\
\hline
-2 & -3 & -1 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}
\Rightarrow
\begin{array}{c}
\begin{array}{ccccccc} x1 & x2 & x3 & s1 & s2 & s3 & z \end{array} \\
\left[\begin{array}{ccccccc|c}
1/2 & 0 & 7/2 & 1 & -1/2 & 0 & 0 & 25 \\
1/2 & 1 & 1/2 & 0 & 1/2 & 0 & 0 & 75 \\
2 & 0 & 0 & 0 & -1 & 1 & 0 & 170 \\
\hline
-1/2 & 0 & 1/2 & 0 & 3/2 & 0 & 1 & 225
\end{array}\right]
\end{array}.$$

109) In the simplex tableau obtained above, select a new pivot and perform the pivoting.

$$\begin{array}{c}
\begin{array}{ccccccc} x1 & x2 & x3 & s1 & s2 & s3 & z \end{array} \\
\left[\begin{array}{ccccccc|c}
1/2 & 0 & 7/2 & 1 & -1/2 & 0 & 0 & 25 \\
1/2 & 1 & 1/2 & 0 & 1/2 & 0 & 0 & 75 \\
2 & 0 & 0 & 0 & -1 & 1 & 0 & 170 \\
\hline
-1/2 & 0 & 1/2 & 0 & 3/2 & 0 & 1 & 225
\end{array}\right]
\end{array}
\Rightarrow
\begin{array}{c}
\begin{array}{ccccccc} x1 & x2 & x3 & s1 & s2 & s3 & z \end{array} \\
\left[\begin{array}{ccccccc|c}
1 & 0 & 7 & 2 & -1 & 0 & 0 & 50 \\
0 & 1 & -3 & -1 & 1 & 0 & 0 & 50 \\
0 & 0 & -14 & -4 & 1 & 1 & 0 & 70 \\
\hline
0 & 0 & 4 & 1 & 1 & 0 & 1 & 250
\end{array}\right]
\end{array}.$$

110) Find the maximum value of $z = 2x_1 + 3x_2 + x_3$ in Exercise 106 and the values of x_1 , x_2 , x_3 that produce the maximum.

Looking at the bottom row we get the equation $4x_3 + s_1 + s_2 + z = 250$, so to maximize z we need $x_3, s_1, s_2 = 0$. Then knowing this we can solve for the remaining values by looking at the rows of the matrix.

$x_1 + 7x_3 + 2s_1 - s_2 = 50 \;\Rightarrow\; x_1 = 50$

$x_2 - 3x_3 - s_1 + s_2 = 50 \;\Rightarrow\; x_2 = 50$

Thus the maximum value for exercise 106 corresponds to the point (50, 50, 0).

111) An office manager needs to purchase new filing cabinets. She knows that Ace cabinets cost \$40 each, require 6 square feet of floor space, and hold 8 cubic feet of files. On the other hand, each Excello cabinet costs \$80, requires 8 square feet of floor space, and holds 12 cubic feet. The budget permits her to spend no more than \$560, while the office has room for no more than 72 square feet of cabinets. The manager desires the greatest storage capacity within the limitations imposed by funds and space. How many of each type of cabinet should she buy?

maximize $z = 8x_1 + 12x_2$ subject to the constraints:

$40x_1 + 80x_2 \le 560$

$6x_1 + 8x_2 \le 72$

where $x_1, x_2 \ge 0$

$$\begin{array}{c} \begin{array}{cccccc} \mathbf{x1} & \mathbf{x2} & \mathbf{s1} & \mathbf{s2} & \mathbf{z} & \end{array} \\ \left[\begin{array}{ccccc|c} 40 & 80 & 1 & 0 & 0 & 560 \\ 6 & 8 & 0 & 1 & 0 & 72 \\ \hline -8 & -12 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \Rightarrow \begin{array}{c} \begin{array}{cccccc} \mathbf{x1} & \mathbf{x2} & \mathbf{s1} & \mathbf{s2} & \mathbf{z} & \end{array} \\ \left[\begin{array}{ccccc|c} 1/2 & 1 & 1/80 & 0 & 0 & 7 \\ 2 & 0 & -1/10 & 1 & 0 & 16 \\ \hline -2 & 0 & 3/20 & 0 & 1 & 84 \end{array}\right] \end{array}.$$

$$\begin{array}{c} \begin{array}{cccccc} \mathbf{x1} & \mathbf{x2} & \mathbf{s1} & \mathbf{s2} & \mathbf{z} & \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 1 & 3/80 & -1/4 & 0 & 3 \\ 1 & 0 & -1/20 & 1/2 & 0 & 8 \\ \hline 0 & 0 & 1/20 & 1 & 1 & 100 \end{array}\right] \end{array}.$$

Now looking at the bottom row we get the equation $1/20s_1 + s_2 + z = 100$, thus $s_1, s_2 = 0$. Then, $x_2 + 3/80s_1 - 1/4s_2 = 3 \Rightarrow x_2 = 3$ and $x_1 + -1/20s_1 + 1/2s_2 = 8 \Rightarrow x_1 = 8$. So the office manager should buy 8 Ace cabinets and 3 Excello cabinets to maximize storage capacity.

112) Use the simplex method to solve the following LP.

maximize $z = x_1 + 8x_2 + 2x_3$ subject to the constraints:

$x_1 + x_2 + x_3 \leq 90$

$2x_1 + 5x_2 + x_3 \leq 120$

$x_1 + 3x_2 \leq 80$

where $x_1, x_2, x_3 \geq 0$

$$\begin{array}{c}\begin{array}{ccccccc} x1 & x2 & x3 & s1 & s2 & s3 & z \end{array}\\ \left[\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 90 \\ 2 & 5 & 1 & 0 & 1 & 0 & 0 & 120 \\ 1 & 3 & 0 & 0 & 0 & 1 & 0 & 80 \\ \hline -1 & -8 & -2 & 0 & 0 & 0 & 1 & 0 \end{array}\right]\end{array} \Longrightarrow \begin{array}{c}\begin{array}{ccccccc} x1 & x2 & x3 & s1 & s2 & s3 & z \end{array}\\ \left[\begin{array}{ccccccc|c} 3/5 & 0 & 4/5 & 1 & -1/5 & 0 & 0 & 66 \\ 2/5 & 1 & 1/5 & 0 & 1/5 & 0 & 0 & 24 \\ -1/5 & 0 & -3/5 & 0 & -3/5 & 1 & 0 & 8 \\ \hline 11/5 & 0 & -2/5 & 0 & 8/5 & 0 & 1 & 192 \end{array}\right]\end{array}.$$

$$\begin{array}{c}\begin{array}{ccccccc} x1 & x2 & x3 & s1 & s2 & s3 & z \end{array}\\ \left[\begin{array}{ccccccc|c} 3/4 & 0 & 1 & 5/4 & -1/4 & 0 & 0 & 82.5 \\ 1/4 & 1 & 0 & -1/4 & 1/4 & 0 & 0 & 7.5 \\ 1/4 & 0 & 0 & 3/4 & -3/4 & 1 & 0 & 57.5 \\ \hline 5/2 & 0 & 0 & 1/2 & 3/2 & 0 & 1 & 225 \end{array}\right]\end{array}.$$

Then, from the last row $5/2x_1 + 1/2s_1 + 3/2s_2 + z = 225$, thus $x_1\ s_1, s_2 = 0$. Now we can solve for x_2 and x_3.

$3/4x_1 + x_3 + 5/4s_1 - 1/4s_2 = 82.5 \Rightarrow x_3 = 82.5$ and $1/4x_1 + x_2 - 1/4s_1 + 1/4s_2 = 7.5 \Rightarrow x_2 = 7.5$. Thus the maximum values for this problem is 225 corresponding to the point (0, 7.5, 82.5).

113) Use the simplex method to solve the following LP.

maximize $z = 5x_1 + 4x_2 + x_3$ subject to the constraints:

$-2x_1 + x_2 + 2x_3 \leq 3$

$x_1 - x_2 + x_3 \leq 1$

where $x_1,x_2,x_3 \geq 0$

$$
\begin{array}{c}
\begin{array}{cccccccc} x1 & x2 & x3 & s1 & s2 & s3 & z & \end{array}\\
\left[\begin{array}{ccccccc|c} -2 & 1 & 2 & 1 & 0 & 0 & 0 & 3\\ 1 & -1 & 1 & 0 & 1 & 0 & 0 & 1\\ \hline -5 & -4 & -1 & 0 & 0 & 0 & 1 & 0\end{array}\right]
\end{array}
\Rightarrow
\begin{array}{c}
\begin{array}{cccccccc} x1 & x2 & x3 & s1 & s2 & s3 & z & \end{array}\\
\left[\begin{array}{ccccccc|c} 2 & -1 & 4 & 1 & 2 & 0 & 0 & 5\\ 1 & -1 & 1 & 0 & 1 & 0 & 0 & 1\\ \hline 0 & -9 & 4 & 0 & 5 & 0 & 1 & 5\end{array}\right]
\end{array}.
$$

Now since all pivots are negative, the problem is unbounded. Thus there is no maximum.

114) A farmer has 110 acres of available land he wishes to plant with a mixture of potatoes, corn, and cabbage. It costs him \$400 to produce an acre of potatoes, \$160 to produce an acre of corn, and \$280 to produce an acre of cabbage. He has a maximum of \$20,000 to spend. He makes a profit of \$120 per acre of potatoes, \$40 per acre of corn, and \$60 per acre of cabbage. (a) How many acres of each crop should he plant to maximize his profit? (b) If the farmer maximizes his profit, how much land will remain unplanted? What is the explanation for this?

maximize $z = 120x_1 + 40x_2 + 60x_3$ subject to the constraints:

$x_1 + x_2 + x_3 \leq 110$

$400x_1 + 160x_2 + 280x_3 \leq 20{,}000$

where $x_1,x_2,x_3 \geq 0$

$$
\begin{array}{c}
\begin{array}{ccccccc} x1 & x2 & x3 & s1 & s2 & z & \end{array}\\
\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & 110\\ 400 & 160 & 280 & 0 & 1 & 0 & 20k\\ \hline -120 & -40 & -60 & 0 & 0 & 1 & 0\end{array}\right]
\end{array}
\Rightarrow
\begin{array}{c}
\begin{array}{ccccccc} x1 & x2 & x3 & s1 & s2 & z & \end{array}\\
\left[\begin{array}{cccccc|c} 0 & 3/5 & 3/10 & 1 & -1/400 & 0 & 60\\ 1 & 2/5 & 7/10 & 0 & 1/400 & 0 & 50\\ \hline 0 & 8 & 24 & 0 & .3 & 1 & 6000\end{array}\right]
\end{array}.
$$

Thus by the bottom row, x_2, x_3 and s_2 are zero. Then by the second row,

$x_1 + 2/5x_2 + 7/10x_3 + 1/400s_2 = 50 \Rightarrow x_1 = 50$.

(a) To maximize his profit the farmer should on,y plant 50 acres of potatoes.

(b) If he uses the profit maximizing allocation he will have 60 acres of unused land. This can be explained by the slack variable being greater than zero. This tells us that in order to get more profit, by expanding our farming space we need to increase the budget.

115) A product can be made in three sizes, large, medium, and small, which yield a net unit profit of \$12, \$10, and \$9, respectively. The company has three centers where this product can be manufactured and these centers have a capacity of turning out 550, 750, and 275 units of the product per day, respectively, regardless of the size or combination of sizes involved. Manufacturing this product requires cooling water and each unit of large, medium, and small sizes produced require 21, 17, and 9 gallons of water, respectively. The centers 1, 2, and 3 have 10,000, 7000, and 4200 gallons of cooling water available per day, respectively. Market studies indicate that there is a market for 700, 900, and 340 units of the large, medium, and small sizes, respectively, per day. The problem is to determine how many units of each of the sizes should be produced at the various centers in order to maximize the profit. (a) Formulate and solve the linear programming model for this problem. How many units of each of the sizes should be produced at the various centers to maximize the profit?

For this problem we will have nine variables to keep track of the three sizes for the three centers. Below is the Excel file used to solve for all of the constraints.

DECISION VARIABLES

	Small	Medium	Large	Plant Totals
A	0	421.176471	128.823529	550
B	340	231.764706	0	571.764706
C	0	247.058824	0	247.058824
Size Totals	340	900	128.823529	

OBJECTIVE FUNCTION

	Small	Medium	Large
	340	900	128.823529
	12	10	9
Total Profit	14239.4118		

CONSTRAINTS

	A-S	A-M	A-L	B-S	B-M	B-L	C-S	C-M	C-L			
A Capacity	1	1	1							550	<=	550
A H20	9	17	21							9865.29412	<=	10000
B Capacity				1	1	1				571.764706	<=	750
B H20				9	17	21				7000	<=	7000
C Capacity							1	1	1	247.058824	<=	275
C H20							9	17	21	4200	<=	4200
Demand S	1			1			1			340	<=	340
Demand M		1			1			1		900	<=	900
Demand L			1			1			1	128.823529	<=	700

We see that the optimal combination of small, medium, and large at center A is (0, 421, 129), center B is (340, 231.76, 0), and center C is (0, 247.06, 0). Yielding a total profit of $14,239.41.

(b) Next, suppose that the following additional constraint is introduced: By company policy, the fraction (scheduled production)/(center's capacity) must be the same at all the centers. Using this additional information, formulate and solve a new linear programming model to maximize the profit. How has your result changed? Why do you think that this might be a policy that an actual manufacturing company might implement?

DECISION VARIABLES

	Small	Medium	Large	Plant Totals
A	0	419.294118	0	419.294118
B	340	231.764706	0	571.764706
C	0	209.647059	0	209.647059
Size Totals	340	860.705882	0	

OBJECTIVE FUNCTION

	Small	Medium	Large
	340	860.705882	0
	12	10	9
Total Profit	12687.0588		

CONSTRAINTS

	A-S	A-M	A-L	B-S	B-M	B-L	C-S	C-M	C-L			
A Capacity	1	1	1							419.294118	<=	550
A H20	9	17	21							7128	<=	10000
B Capacity				1	1	1				571.764706	<=	750
B H20				9	17	21				7000	<=	7000
C Capacity							1	1	1	209.647059	<=	275
C H20							9	17	21	3564	<=	4200
Demand S	1			1			1			340	<=	340
Demand M		1			1			1		860.705882	<=	900
Demand L			1			1			1	0	<=	700

Part B	A	B	C
	0.76235294	0.76235294	0.76235294

Adding in the constraint for we get that each center works at 76.2% capacity. This then changes the optimal production of shirts at each firm and brings the profit down to $12,687.06. Eventhough this statatgy decreases the profit of the company, they may want to use this policy to ensure all centers are perferming equally.

(c) Which, if any, of the water capacity constraints was binding? What happens to the solution and to the overall maximum profit if you increase the water capacity at each center by 1%? 5%? 10%? Vary each water capacity one at a time, while holding the others fixed, and write a report on your findings. Include graphs and tables to illustrate your results. Then do the same thing for the other constraints, and discuss your results in detail.

Only one of the centers has a binding water constraint, we can see as the water used for center B is the same as the capacity. Thus by increasing the water capacity for this center will increase the production of all the firms.

DECISION VARIABLES				
	Small	Medium	Large	Plant Totals
A	0	378.196078	84.2352941	462.431373
B	340	290.588235	0	630.588235
C	0	231.215686	0	231.215686
Size Totals	340	900	84.2352941	

OBJECTIVE FUNCTION			
	Small	Medium	Large
	340	900	84.2352941
	12	10	9
Total Profit	13838.1176		

CONSTRAINTS												
	A-S	A-M	A-L	B-S	B-M	B-L	C-S	C-M	C-L			
A Capacity	1	1	1							462.431373	<=	550
A H20	9	17	21							8198.27451	<=	10000
B Capacity				1	1	1				630.588235	<=	750
B H20				9	17	21				8000	<=	8000
C Capacity							1	1	1	231.215686	<=	275
C H20							9	17	21	3930.66667	<=	4200
Demand S	1			1			1			340	<=	340
Demand M		1			1			1		900	<=	900
Demand L			1			1			1	84.2352941	<=	700

Part B	A	B	C
	0.84078431	0.84078431	0.84078431

Here we see by increasing the water capacity in firm B from 7,000 to 8,000, all of the centers are now working at 84% capacity and the profit has now risen to $13,838.12.

Derivation of the Black-Scholes-Merton Equation for the Price of a European Call Option

1) Show that $c = e^{-rT} \sum_{j=0}^{n} \binom{n}{j} p^j(1-p)^{n-j} \max(S_0u^jd^{n-j} - K, 0)$.

We know for every time step in the binomial tree, the stock either goues up, u, or down, d. So at any step we have our original price $S_0 = S_0u$ or S_0d for $u > 1$ and $d < 1$. So in an n-step binomial tree at time $t = T$, if there has been j up movements then there are n-j doen movements and the value of the stock is $S_T = S_0u^jd^{n-j}$, for possible values $j = 0,1,...,n$. Thus the payoff at expiry for a ECO is $\max(S_T - K, 0) = \max(S_0u^jd^{n-j} - K, 0)$. Now, since the payoff of the ECO at $t = T$ is a discrete random variable, we can find the expected value equation to get $\sum_{j=0}^{n} \boldsymbol{\binom{n}{j} p^j(1-p)^{n-j}} \max(S_0u^jd^{n-j} - K, 0)$, where the bolded section represents the probability of exactly j upward stock movements. Then by the no-arbitrage principle, the initial price of a call option is the discounted value of the expected value at expiry. Thus,

$c = e^{-rT} \sum_{j=0}^{n} \binom{n}{j} p^j(1-p)^{n-j} \max(S_0u^jd^{n-j} - K, 0)$, as desired.

2) Show that the terms of the summation in the equation above are nonzero if and only if

$j > \alpha = \frac{n}{2} - \frac{\ln(So/K)}{2\sigma\sqrt{T/n}}$. Thus, $c = e^{-rT} \sum_{j>\alpha} p^j(1-p)^{n-j}(S_0u^jd^{n-j} - K) = e^{-rT}(S_0U_1 - KU_2)$.

In order to calculate we need to get rid of the max function in our summation, so to do this we need to find all of the j values where the terms in the sum are nonzero. So, we want j's such that $S_0u^jd^{n-j} > K$. Let us isolate j using a log function,

$\ln(S_0u^jd^{n-j}) > \ln(K) \Rightarrow \ln(S_0) + j\ln(u) + (n-j)\ln(d) > \ln(K)$. Now plugging in u and d we know that $\ln(u) = \sigma\sqrt{(T/n)}$ and $\ln(d) = - \sigma\sqrt{(T/n)}$. Now, we have
$\ln(S_0) + j(\sigma\sqrt{(T/n)}) + (n-j)(-\sigma\sqrt{(T/n)}) > \ln(K)$, and rearranging we get our desire result,
$j > \alpha = \frac{n}{2} - \frac{\ln(So/K)}{2\sigma\sqrt{T/n}}$. Now that we have a constant, α, to represent the non-zero j values we

can get rid of the max function and are left with $c = e^{-rT}\sum_{j>\alpha}^{n} p^j(1-p)^{n-j}(S_0u^jd^{n-j} - K)$. Now breaking up this sum into two parts, a $S_0u^jd^{n-j}$ part and a -K part, S_0 and K are constants so can pull out and will label the rest of the parts $U_1 = \sum_{j>\alpha}^{n} p^j(1-p)^{n-j}u^jd^{n-j}$ and $U_2 = \sum_{j>\alpha}^{n} p^j(1-p)^{n-j}$. This then leaves us with the the equation $c = e^{-rT}(S_0U_1 - KU_2)$.

3) Show the details in computing the following limit. $\lim_{n\to\infty} p(1-p) = \frac{1}{4}$.

We know $p = \frac{e^{\frac{rT}{n}} - d}{u-d}$, $u = e^{\sigma\sqrt{(T/n)}}$, and $d = e^{-\sigma\sqrt{(T/n)}}$. Let us use the first-order Taylor expansion for e^x, $1 + x$. Using this we can write $u = 1 + \sigma\sqrt{(T/n)}$ and $d = 1 - \sigma\sqrt{(T/n)}$. Then, $u - d = [1 + \sigma\sqrt{(T/n)}] - [1 - \sigma\sqrt{(T/n)}] = 2\sigma\sqrt{(T/n)}$. Next we can rewrite part of our p as well, $e^{r(T/n)} = 1 + rT/n$. With everything we have gathered now

$p = \frac{\frac{rT}{n} + \sigma\sqrt{\frac{T}{n}}}{2\sigma\sqrt{\frac{T}{n}}} = \frac{rT}{2\sigma\sqrt{T}\sqrt{n}} x \frac{1}{\sqrt{n}} + \frac{\sigma\sqrt{\frac{T}{n}}}{2\sigma\sqrt{\frac{T}{n}}} = \frac{rT}{2\sigma\sqrt{T}\sqrt{n}} x \frac{1}{\sqrt{n}} + \frac{1}{2}$. Then as n goes to infinity, p goes to ½. Therefore, $\lim_{n\to\infty} p(1-p) = \frac{1}{2}(\frac{1}{2}) = \frac{1}{4}$ as desired.

4) Use the Central Limit Theorem to show that $U_2 = N(d_2)$.

The Central Limit Theorem states that $\lim_{n\to\infty} S_n$ is equal to the normal distribution random variable with mean np and volatility $\sqrt{(np(1-p))}$. Equivalently we can say $\frac{Sn - np}{\sqrt{np(1-p)}}$ is a standard random variable, so $P(\frac{Sn-np}{\sqrt{np(1-p)}} \le x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\Pi}} e^{-\frac{1}{2}t^2} dt = N(x)$. That being said,

$U_2 = P(S_n > \alpha) = 1 - P(S_n \le \alpha) = 1 - P(\frac{Sn - np}{\sqrt{np(1-p)}} \le \frac{\alpha - np}{\sqrt{np(1-p)}}) = N(\frac{\alpha - np}{\sqrt{np(1-p)}})$. By exercise 75, $1 - N(X) = N(-X)$ and substituting $\alpha = \frac{n}{2} - \frac{ln(So/K)}{2\sigma\sqrt{T/n}}$.

$U_2 = N(\frac{ln(\frac{So}{k})}{2\sigma\sqrt{\frac{T}{n}}\sqrt{p(1-p)}} + \frac{\sqrt{n}(p - \frac{1}{2})}{\sqrt{p(1-p)}})$. Then plug in $p = \frac{e^{\frac{rT}{n}} - d}{u - d}$ and take the limit as n goes to infinity. Using our result from the previous exercise $\lim_{n \to \infty} p(1 - p) = \frac{1}{4}$, and that

$\lim_{n \to \infty} \sqrt{n(p - \frac{1}{2})} = \frac{(r - \frac{\sigma^2}{2})\sqrt{T}}{2\sigma}$ we get the following,

$U_2 = N(\frac{ln(\frac{So}{k}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}) = N(d_2)$ as desired.

6) Conclude that $c = S_0N(d_1) - Ke^{-rT}N(d_2)$.

Then from exercise 2 of this section $c = e^{-rT}(S_0U_1 - KU_2)$. From the pervious exercise we know $U_2 = N(d_2)$ and through a similar process can get that $U_1 = e^{rT}N(d_1)$. Thus we can conclude that $c = S_0N(d_1) - Ke^{-rT}N(d_2)$.

Estimating Volitility from Historical Data

1) Suppose that observations on a stock price (in dollars) at the end of each of 15 consecutive weeks are as follows: [30.2, 30, 31.1, 30.1, 30.2, 30.3, 30.6, 33, 32.9, 33, 33.5, 33.5, 33.7, 33.5, 33.2]. Estimate the stock price volatility. Assume that there are 52 trading weeks in one year.

```
#Volitility Q1
import numpy as np
import math


data = [30.2,30,31.1,30.1,30.2,30.3,30.6,33,32.9,33,33.5,33.5,33.7,33.5,33.2]

#find ui's
u = []
a = 0
for d in range(1,15):
    u.append(np.log(data[d]/data[a]))
    a += 1
print("u          :",u)

#find ubar
ubar = 0
for uu in u:
    ubar += uu
ubar = ubar/14
print("ubar    :",ubar)

#find standard deviation
s = 0
for ui in u:
    s += ((ui - ubar) ** 2)
s = (1/13) * s
s = math.sqrt(s)
print("s          :",s)

#find volitility
vol = s /(math.sqrt(14/52))
print("\nvolitility:",vol)

```

```
u         : [-0.006644542718668501, 0.03601043752303303, -0.03268264743035842, 0.003316752625
-0.003034903695154047, 0.003034903695154111, 0.015037877364540502, 0.0, 0.005952398527295352,
ubar      : 0.006764853681544219
s         : 0.024783834939625177

volitility: 0.04776460185442059
[Finished in 238ms]
```

With this data we were able to extrapolate an implied volatility of .0477

2) A call option on a non-divident paying stock has a market price of \$2.50. The stock price is\$15, the exercise price is \$13, the time to maturity is 3 months, and the risk-free interest rate is 5% per annum. What is the implied volatility?

```
#Volitility Q2
import numpy as np
import math
import scipy.stats as st

cost = 2.5
S0 = 15
K = 13
T = .25
r = .05
#imput guess for volitility as ß
ß = .4934

#with d values, use normal distribution to find and input N(d1) and N(d2)
d1 = (np.log(S0/K)+(r + (ß**2)/2)*T)/ß*math.sqrt(T)
d2 = d1 - ß*math.sqrt(T)
Nd1 = st.norm.cdf(d1)
Nd2 = st.norm.cdf(d2)

#finally see how close you are to the cost, $2.50
c = S0*Nd1 - K*(math.exp(-r*T))*Nd2
print("\n\nCost is,",c,",with volitility",ß,"\n\n\n\n\n")
```

```
Cost is, 2.5000460678707848 ,with volitility 0.4934
```

By guessing different volitilities we can get closer and closer to the actual volatility by comparing the cost, \$2.50, to what we get with the Black-Scholes-Merton model. We got that the volatility is about .4934.

3) Choose a stock, and estimate its historical volatility using the closing stock price for 21 consecutive days of trading. You may use the Historical Data section on Yahoo Finance to obtain this information (or any other source for historical data on prices of publicly dated stocks)

```python
#Volitility Q3
import numpy as np
import math

##TSLA 21 days##
data = [170.66,173.79,177.48,177.94,175.0,174.77,176.29,178.08,178.79,176.19,
176.75,179.24,173.74,180.11,186.6,174.95,177.46,174.84,173.99,177.55,171.89]

#find ui's
u = []
a = 0
for d in range(1,21):
    u.append(np.log(data[d]/data[a]))
    a += 1
print("u          :",u)

#find ubar
ubar = 0
for uu in u:
    ubar += uu
ubar = ubar/20
print("ubar    :",ubar)

#find standard deviation
s = 0
for ui in u:
    s += ((ui - ubar) ** 2)
s = (1/19) * s
s = math.sqrt(s)
print("s          :",s)

#find volitility
vol = s /(math.sqrt(20/252))
print("\nvolitility:",vol)
```

```
u         : [0.01817440066820363, 0.021010252734066374, 0.0025884883059126684, -0.0166604408
0.003979045236493854, -0.014648974561190933, 0.0031733465741444027, 0.013989385141260089, -0
0.01424501259349478, -0.014873961412077991, -0.004873443696488867, 0.02025443309047212, -0.0
ubar      : 0.00035907321158134864
s         : 0.024336623135423622

volitility: 0.08638644247223645
[Finished in 243ms]
```

For this question I used the last 21 days of Tesla's(TSLA) stock and use a similar approach to exercise 1 from this section. From the data we found that the historical volatility is .0863.

4) Consider a European call option on a stock with current price S_0= 100, strike price K= 50, r= 0.06, and T= 0.01. (a) Find the price of the option for values of σ between 0.05 and 1 (in increments of 0.05, i.e. σ = 0.05, 0.1, 0.15, . . .).

```python
#Volitility Q4
import numpy as np
import math
import scipy.stats as st
import warnings
warnings.filterwarnings("ignore", category=RuntimeWarning)

S0 = 100
K = 50
T = .01
r = .06

#increment the volitility by increments of .05
ß = 0
print("volitility   price")
while ß < 20:
    ß = ß/20
    d1 = (np.log(S0/K)+(r + (ß**2)/2)*T)/ß*math.sqrt(T)
    d2 = d1 - ß*math.sqrt(T)
    Nd1 = st.norm.cdf(d1)
    Nd2 = st.norm.cdf(d2)
    c = S0*Nd1 - K*(math.exp(-r*T))*Nd2
    print(ß,"    ",c)
    ß = ß*20
    ß += 1
```

```
volitility  price
0.0         50.02999100179973
0.05        45.933624238618265
0.1         37.98469230549555
0.15        34.19819070640842
0.2         32.18047755571636
0.25        30.966902998697567
0.3         30.176812974234906
0.35        29.6351995497923
0.4         29.251288701153126
0.45        28.973539329899207
0.5         28.770588170965855
0.55        28.622282868574665
0.6         28.515094773961394
0.65        28.43960914609013
0.7         28.389074201716497
0.75        28.358522892036273
0.8         28.34422028883221
0.85        28.343304096788575
0.9         28.353543964269985
0.95        28.373176223172404

 5.0        34.946363767052674
```

(b) Interpret the results that you obtained in (a) in financial terms.

In financial terms this tells us that as volatility gets bigger, the price of the option gets smaller. This makes sense because when volatility is high, the option is much riskier, thus people will only buy if they are cheaper than the more stable low volatility options.

(c) What happens if σ = 5?

When volatility is 5, the BSM model priced the option at $34.95.