California Polytechnic University San Luis Obispo

Investigation on Lattice Geometry

2023 Senior Project

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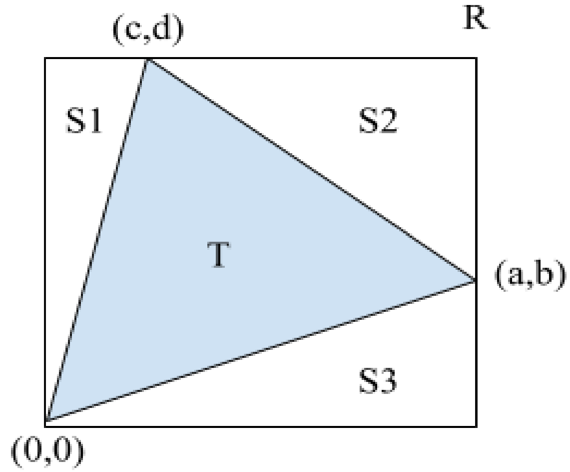
1) Is it possible to construct a regular lattice 3-gon (i.e. an equilateral and equiangular lattice triangle)? If so, provide an example. If not, prove it

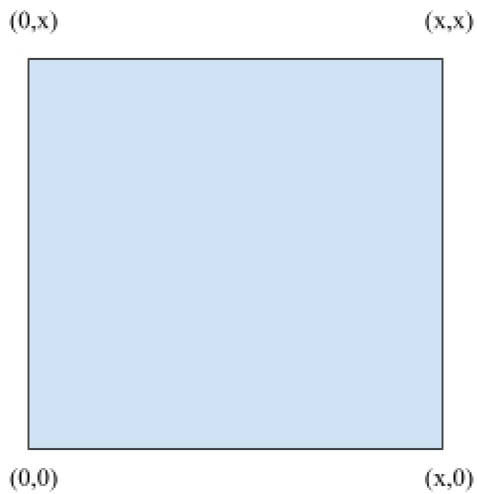
No this is not possible. Assume we have a regular lattice triangle T. Looking at the diagram below we can see that to find the area of T we can use the following equation. A(T) = A(R) - A(S1) - A(S2) - A(S3), where R is the a x d rectangle encompassing T. Decomposing each area, A(T) = ad - 1/2cd - 1/2(a-c)(d-b) -1/2ab. Since this is a sum of rationals, it implies A(T) must also be rational. Thus by assuming a lattice triangle, we end up with a rational area for T.



Now we will investigate the area of an equilateral triangle and compare with the area of our assumed lattice triangle above. Since equilateral, all side lengths will be length b. Then we can cut the triangle in half to make a right triangle so we can find the height, h, of the triangle. Then by pythagorean theorem, $(b/2)^2 + h^2 = s^2$. Thus $h =(b\sqrt{3})/2$. Now by area equation of a triangle, A(T) = $1/2(b)(h) = (b^2\sqrt{3})/2$. And since we know that $b = \sqrt{x^2 + y^2}$ for some x,y integer coordinates representing a vertex connecting to the origin, b^2 must be an integer; and consequently $b^2/2$ a rational. However, a rational multiplied by an irrational, $\sqrt{3}$, is irrational. So the final area for our equilateral triangle is irrational, a contradiction to the fact that a lattice triangle has a rational area as found in the first part of this problem. Thus it is impossible to construct a regular lattice 3-gon. ∎

2) Is it possible to construct a regular lattice 4-gon (i.e. a square)? If so, provide an example. If not, prove it.

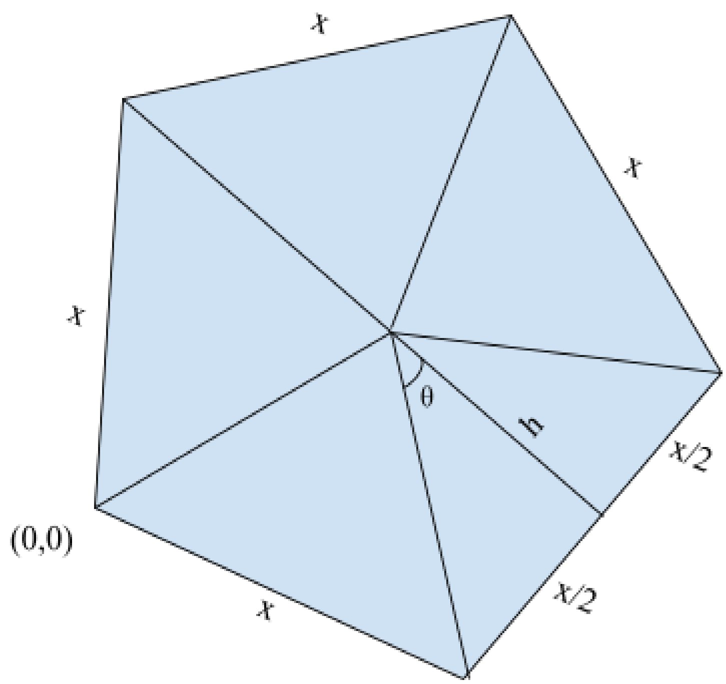
Yes it is possible. An example to show this would be to start at the origin and choose a length $x \in Z$. Since a square has all side lengths equal and all angles 90 degrees, we can ensure each vertex will land on a lattice point by starting at a lattice point, the origin, and moving around the square in a perpendicular direction x distance.



3) Is it possible to construct a regular lattice pentagon (5-gon)? If so, provide an example. If not, prove it.

We will show by contradiction that it is not possible to construct a regular lattice pentagon. So suppose by way of contradiction that there exists such a pentagon, P. We will obtain a contradiction by computing the area of P in 2 ways.
Without loss of generality let one of the vertices of P be the origin., and by definition, the others are lattice points. For our first way of computing A(P) we will enclose the pentagon in a bounding rectangle and subtract the triangular sections, $T_1,...,T_5$, that are not part of P. So

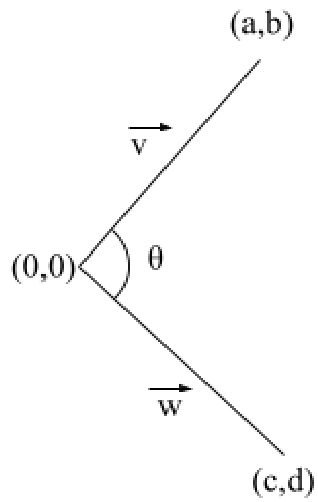
$A(P) = A(R) - A(T_1) - \ldots - A(T_5)$. Then since all vertices are lattice points we know A(R) is an integer and each $A(T_i)$ is rational by the formula for the area of a triangle. Thus A(P) is a rational number from this method.



Now, let x denote the side lengths of P. Let one of the adjacent vertices to the origin be (a,b). Then $x = \sqrt{a^2 + b^2}$, and let us split P into equilateral triangles T, where all are the same since regular polygon. Then $x^2 = a^2 + b^2$ which is an integer since a and b are integers. Now, we know $A(P) = 5A(T)$, and $A(T) = (1/2)bh$, so we need to find the height, h, of each triangle. We know that since P is regular each interior angle is $2\pi/5$ and we want to find the height so we can cut our equilateral in half to make a right triangle, with an angle, θ, $\pi/5$. Then to solve for h we can use $\tan(\pi/5) = (x/2)/h$. Solving for h we get the expression $h = x/(2\tan(\pi/5))$, thus $A(T) = (1/2)(x)(x/(2\tan(\pi/5)) = s^2/(4\tan(\pi/5))$, but $\tan(\pi/5)$ is irrational, which tells us A(P) is irrational when solving for area in this method, a contradiction to the fact that A(P) was rational in our first derivation of the area. Therefore, P is not a lattice polygon, thus it is not possible to construct a regular lattice pentagon. ■

4) Show that the cosine of each interior and exterior angle of any regular lattice polygon must be rational.

Let P be a regular lattice Polygon. Without loss of generality, let the vertex of P be located at the origin and let (a,b) and (c,d) be the two adjacent lattice point vertices to the origin. We know that to find $\cos\theta$ we can imagine the two lines leading to the adjacent points as two vectors $v = \langle a,b \rangle$ and $w = \langle c,d \rangle$. Then $\cos\theta = v \cdot w / \|v\| \|w\|$ by a property of vectors. Note that since P is a regular polygon $\|v\| = \|w\|$ so $\|v\| \|w\| = \|v\|^2$. Putting it all together, $\cos\theta = (ac + bd) / (a^2 + b^2)$; and since P is a lattice polygon a,b,c,d are all integers. Thus, $\cos\theta$, cosine of the interior angle, is a ratio of integers which means it is rational.



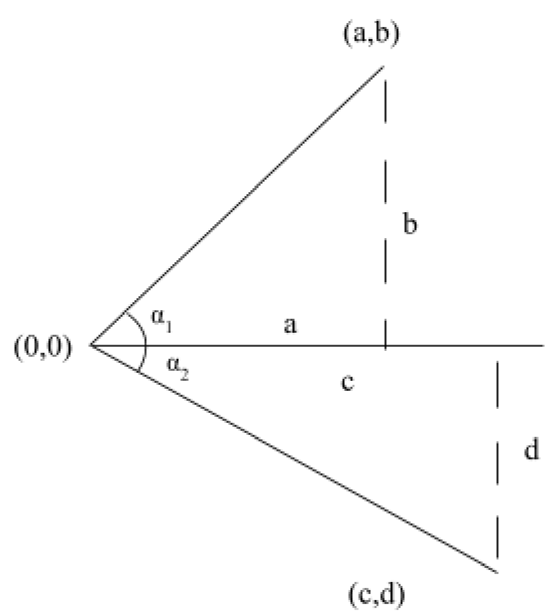
Then, to find the cosine of the exterior angle we know interior + exterior = π. Thus exterior = $\pi - \theta$ and by an identity of cos, $\cos(\pi - \theta) = -\cos\theta$. Therefore, since $\cos\theta$ is rational so is $-\cos\theta$. Thus the cosine of each interior and exterior angle of any regular lattice polygon must be rational. ▌

5) Use Exercise 4 to show that it is not possible to construct a regular lattice octagon (8-gon).

By Exercise 4 we know a regular lattice polygon must have a rational cosine of the exterior and interior angles. First, to find the total degree of a n-gon we can use the equation (n-2)180 degrees, and since we are dealing with a regular polygon, all angles will be the same so we are left with $\theta = (n-2)180/n$. So for our octagon we get that each interior angle will be 135 degrees, which tells us the exterior angles will be 45 degrees. From the unit circle we know that $\cos(\pi/4) = \sqrt{2}/2$ but $\sqrt{2}$ is irrational so we found that the exterior angle is not rational so we can conclude that it is impossible to construct a regular lattice octagon. ▌

6) Show that if α is a lattice angle and if the measure of α is not equal to $\pi/2$ or an odd integer multiple of $\pi/2$, then the tangent of α is a rational number.

Suppose α is a lattice angle. Then α is an angle with a vertex at a lattice point, and with sides that are lattice lines. Without loss of generality, let the vertex be the origin. Then we can split up α with a line such that we can make a right angle with both vertices at the ends of the lattice lines, as shown in the diagram below. Let us call the split up α, α_1 and α_2. Now since α is connected to two lattice lines, we know that a, b, c, and d are integers. Since they make up our right triangles, $\tan\alpha_1 = b/a$ and $\tan\alpha_2 = d/c$, which are ratios of integers, thus rational. Now we will use the angle formula: $\tan\alpha = (\tan\alpha_1 + \tan\alpha_2) / (1 - \tan\alpha_1\tan\alpha_2)$. Then since $\tan\alpha_1$ and $\tan\alpha_2$ are both rational, then $\tan\alpha$ is also rational. As long as α is not equal to $\pi/2$ or an odd integer multiple of $\pi/2$; since this would give us an undefined tan.▌

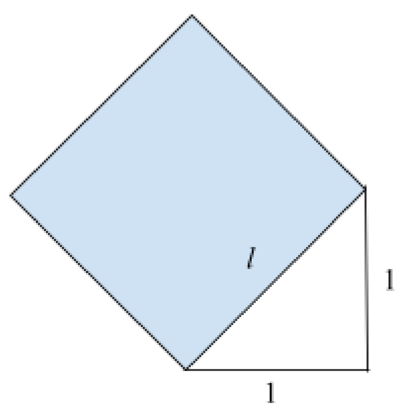


7) Make a conjecture about the positive integers n for which it is possible to construct a regular lattice n-gon. Although you do not need to provide a formal proof of your conjecture here, you should provide sufficient justification and reasoning (beyond simply citing the exercises that you have already solved above) to indicate why you believe your conjecture is valid.

In any regular lattice n-gon, we know each angle will measure 180(n-2)/n. By splitting one of the angles in half, we can always find a triangle inside the n-gon. In order for the n-gon to be on lattice points, we need the length and height of the triangle to be rational in order for it to be possible to have all lattice points. This means cos(180(n-2)/2n) and sin(180(n-2)/2n) must be rational. Therefore, we can say that for a regular lattice n-gonb to exist, tan(180(n-2)/2n) must be rational. ∎

8) Is it possible to construct a lattice square whose area is not a perfect square? If so, provide an example. If not, prove it.

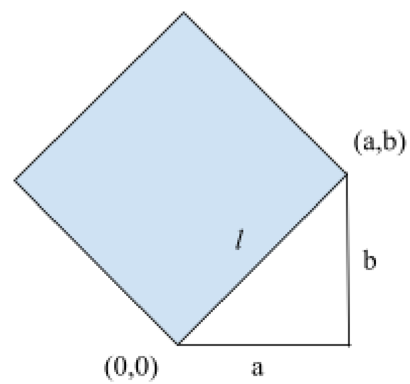
Yes it is possible. A perfect square is one where the square root of the number is an integer. Take the square with lattice points (1,0), (2,1), (1,2), and (0,1).



Then by evaluating the length of one side we can get the length, *l,* of all so take (1,0) and (2,1), then $l = \sqrt{(2-1)^2 + (1-0)^2} = \sqrt{1+1} = \sqrt{2}$. Then by the area formula of a square A = $\sqrt{2}^2$ = 2. But by definition 2 is not a perfect square, thus we found a lattice square whose area is not a perfect square. ∎

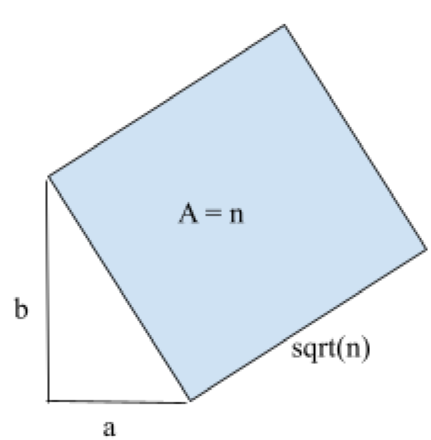
9) Is it possible to construct a lattice square whose area is not an integer? If so, provide an example. If not, prove it.

No it is not possible. Assume you have a lattice square. Without loss of generality, let one vertex be the origin. Then we can find the length of a side length, *l*, by the pythagorean theorem, here we will evaluate the length between the origin and the lattice point (a,b).



So $l = \sqrt{a^2 + b^2}$. Then by the area formula of a square A = $(\sqrt{a^2 + b^2})^2 = a^2 + b^2$. But we know a and b are integers since they are the x and y coordinates of a lattice point so $a^2 + b^2$ must also be an integer. Thus it is not possible to construct a lattice square whose area is not an integer. ∎

10) Show that there exists a lattice square with area n, where n is a positive integer, if and only if there exist non-negative integers a and b such that n = $a^2 + b^2$.



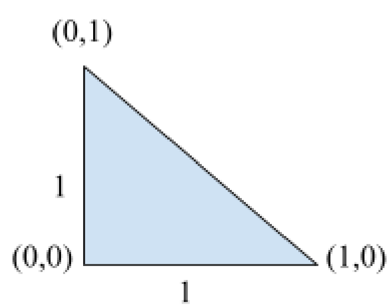
(=>) Suppose there exists a lattice square with area n, a positive integer. Then each side of the square must have length $\sqrt{n}$.

Without loss of generality, assume one of the vertices is at the origin, then we can represent one of the edge lengths as $\sqrt{a^2 + b^2}$. Now, $\sqrt{n} = \sqrt{a^2 + b^2}$ and $n = a^2 + b^2$.

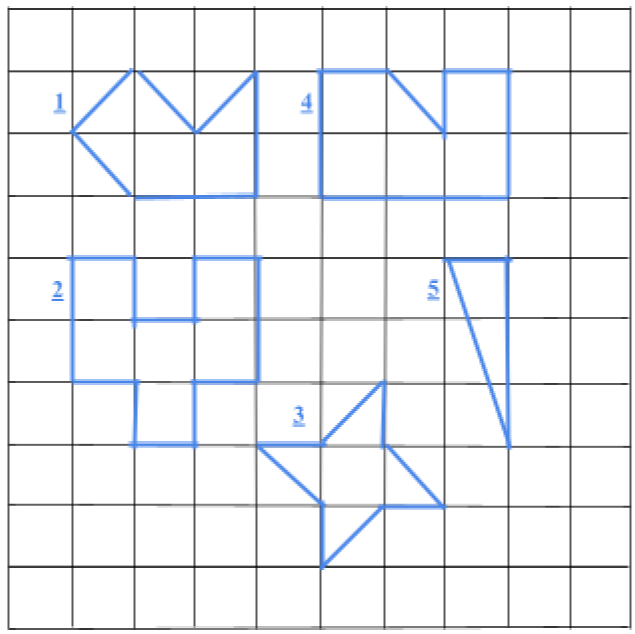
(<=) Suppose $n = a^2 + b^2$. Since a,b are integers, we can use them as points of our lattice square. Then we can see by taking the square root of $a^2 + b^2$ we are left with the distance formula for between vertex (a,b) and the origin. Without loss of generality, if we let both be vertices of our square we find the length of that edge is $\sqrt{a^2 + b^2}$, and by our supposition, can see $\sqrt{a^2 + b^2} = \sqrt{n}$. so we get an edge length of $\sqrt{n}$, which shows by the area formula that the area of the square is n, and by definition of a square the other two vertices can only differ by a in the x direction and b in the y direction, thus all points are lattice points. ▌

11) Is it possible to construct a lattice triangle whose area is not an integer? If so, provide an example. If not, prove it.

Yes, we can show by an example. The triangle with vertices (0,0), (1,0), and (0,1) has base and height = 1. Thus ½ (bh) = ½ (1)(1) = ½ , an element of the rational numbers and not an integer. ▌



12) Construct (at least) 5 lattice polygons with different areas. To keep this problem interesting, at least 3 of your polygons should be non-convex. Find the area of each polygon. What do you observe? Make a conjecture about the possible values of the area of a lattice polygon based on your computations in this problem. For example, do you think it's possible to achieve all integer areas? All rational areas? All real areas?

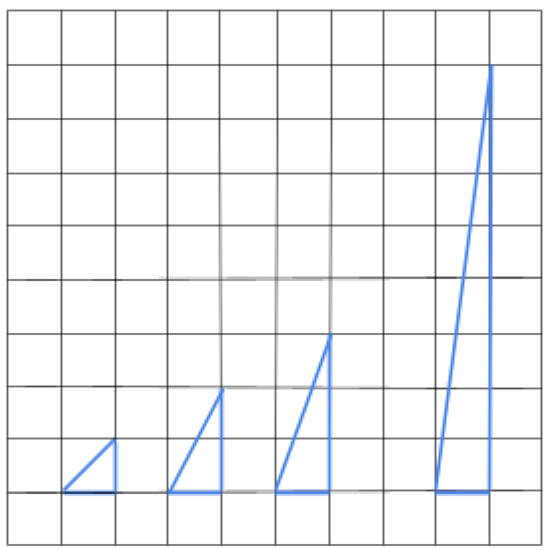


1. Area = 4 2. Area = 6 3. Area = 3 4. Area = 5.5 5. Area = 1.5

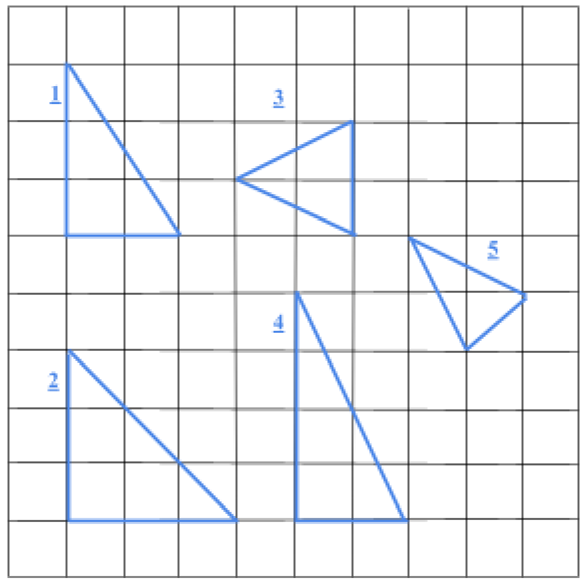
Here we notice that all of the polygons created are either an integer value or are a multiple of ½. A conjecture we can make is that all areas can be constructed by combinations of triangles, and since the area formula for a triangle is ½ (bh), we can see how we are getting multiples of ½. I think yes it is possible to achieve all integer areas, from previous conjecture. However I don't think we can create areas with all rational or real areas.

13) Let T be a lattice triangle with I(T) = 0. What are the possible values of B(T)? Prove your result.

The possible values of B(T) are infinite. We can show this with a visual (below).
Notice starting at our 1-by-1 triangle we have three boundary lattice points and there is no option for an interior lattice point. In the next triangle, a 2-by-1, we see that we add a new boundary lattice point by extending the height. Also note that since our base is only 1, the hypotenuse crosses height 1 at exactly halfway between two lattice points, this happens since it needs to have a slope of 2. Similarly for a 1-by-3 we add a boundary point, but keep zero interior points as our slope is now 3 and only can travel 1 unit in the x direction. Generalizing, for any triangle with base length 1, we can have the height be any integer and will never get an interior point. Thus B(T) = |h| + 2 for all integers h, thus infinitely many values of B(T) can be represented in a lattice triangle with I(T) = 0. ∎

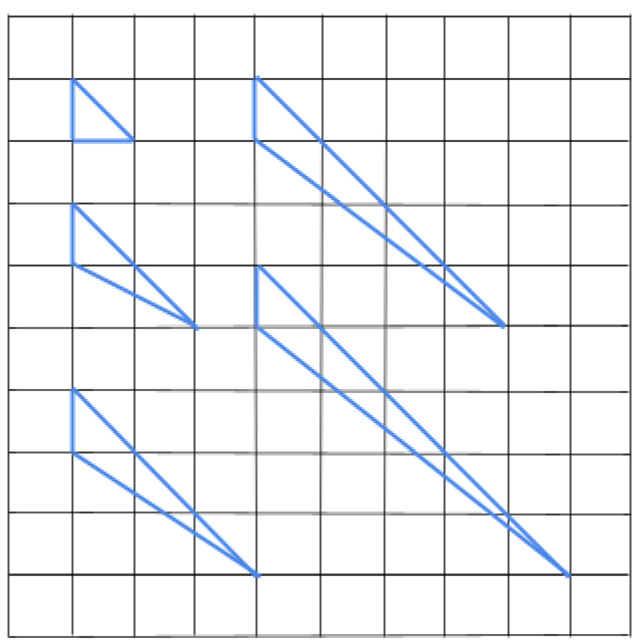


14) Make a conjecture about the possible values of B(T) (the number of lattice points on the boundary) for a lattice triangle T with I(T) = 1. Although you do not need to provide a formal proof of your conjecture here, you should provide sufficient justification and reasoning (beyond simply citing computational work) to indicate why you believe your conjecture is valid.



In the above diagram, we are looking at the 5 options for the number of boundary points for a lattice triangle with I(T) = 1. 1 has B(T) = 6, 2 has B(T) = 9, 3 has B(T) = 4, 4 has B(T) = 8, and 5 has B(T) = 3. Since we are constricting our triangles to only have 1 interior point, we can no longer make any other configuration yielding a new value of B(T) as any small change to these five triangles results in a second lattice point being created.

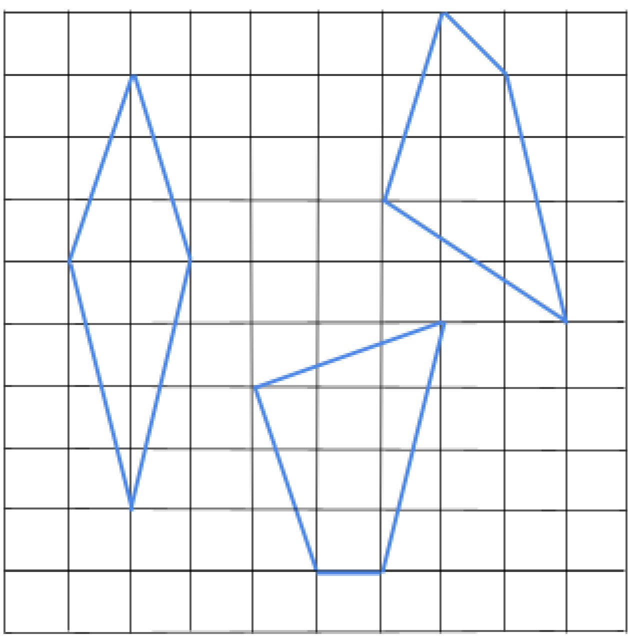
15) Construct (at least) 5 different non-congruent primitive lattice triangles, and find their area. Make a conjecture about the value of the area of a primitive lattice triangle.



In the diagram above we can see our five primitive lattice triangles, constructed by extending the hypotenuse and reforming the bottom line to match. The first triangle clearly has area 1/2. Moving on to the second triangle we can subtract from a surrounding rectangle to find A(T) = 4 - 1/2(2)(2) - 1/2(2)(1) = 4 - 2 - 1/2 = 1/2. Similarly for the rest of the triangles you get an area of 1/2. Thus a conjecture we can make is that the value of the area of a primitive lattice triangle is always 1/2.

16) Construct several (at least 5) different polygons that contain 4 boundary lattice points and 6 interior lattice points. Keep in mind that the polygons do not need to be convex! Find the area of each polygon. What do you observe? Make a conjecture based on your observations in this exercise.

Based on our constructions below, we can make the conjecture that each polygon with B(P) = 4 and I(P) = 6 will have an area of 7.



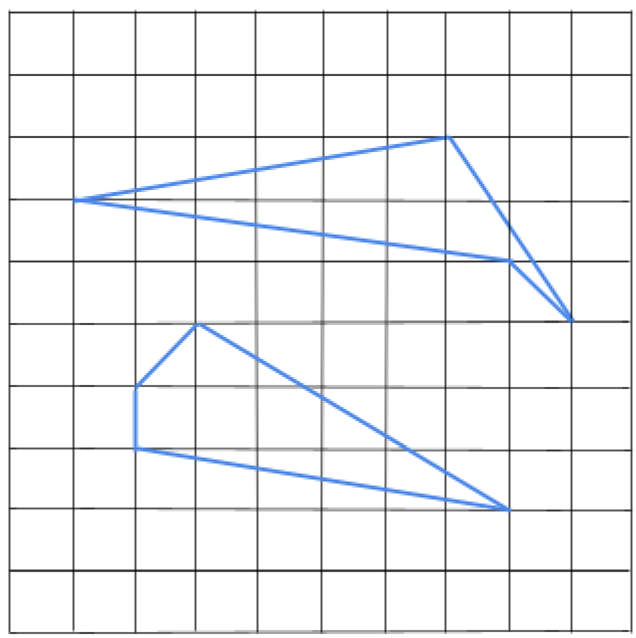
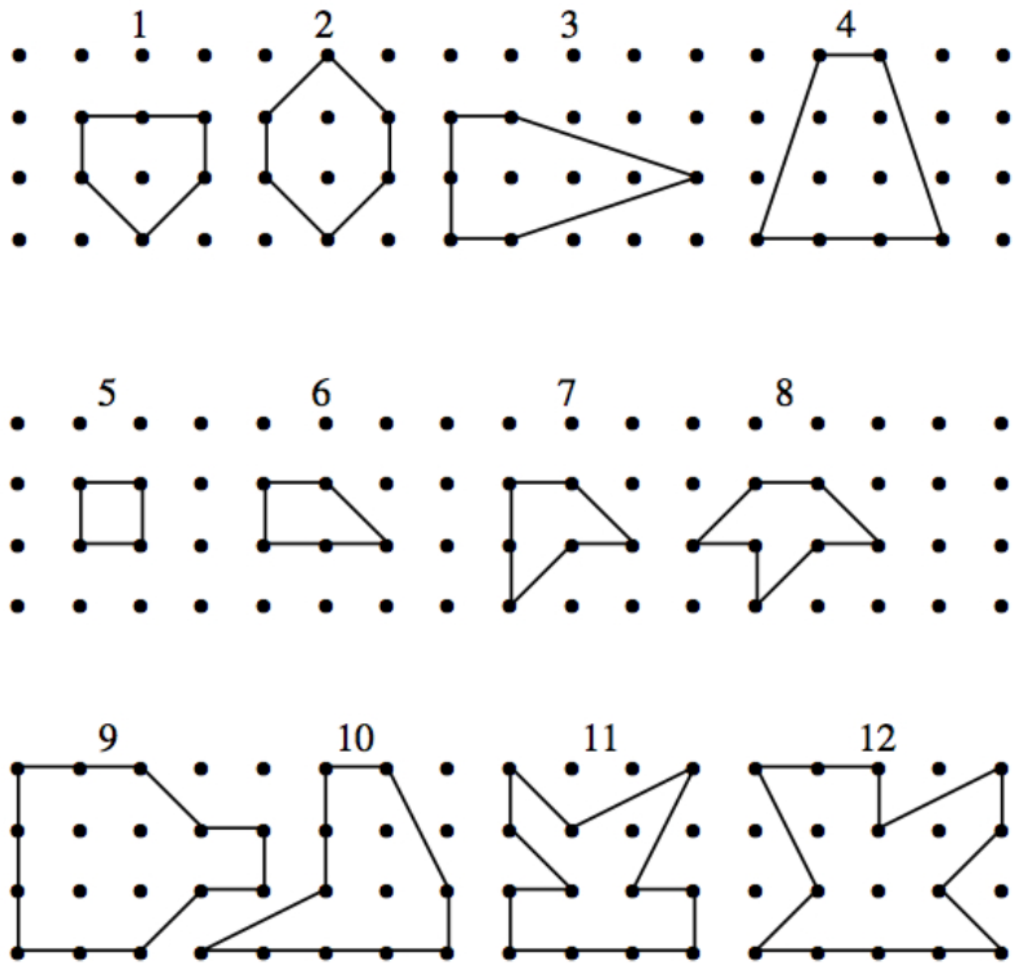


Figure 1)

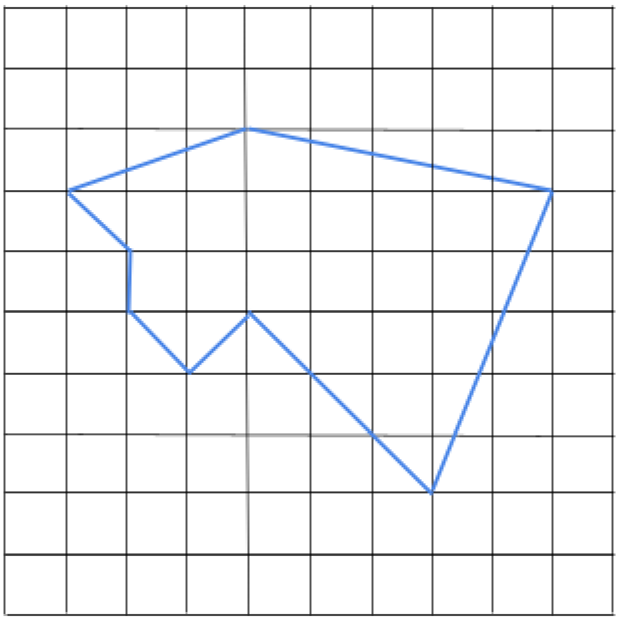


17) Find the area of each of the lattice polygons in Figure 1 (above). Make a table that contains the following information for each polygon: the area of the polygon, the number of lattice points inside the polygon (I), and the number of lattice points on the boundary of the polygon (B).

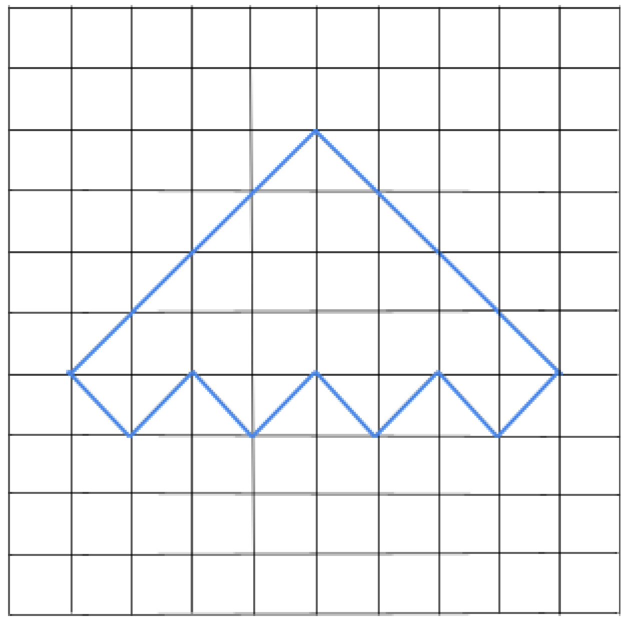
Polygon	Area	I	B
1	3	1	6

2	4	2	6
3	5	3	6
4	6	4	6
5	1	0	4
6	1.5	0	5
7	2	0	6
8	2.5	0	7
9	9	4	12
10	6	2	10
11	6	1	12
12	8.5	3	13
18.1	25	21	10
18.2	20	13	16
18.3	41.5	37	11
18.4	24	17	16
18.5	19.5	9	21
19	5.5	5	3

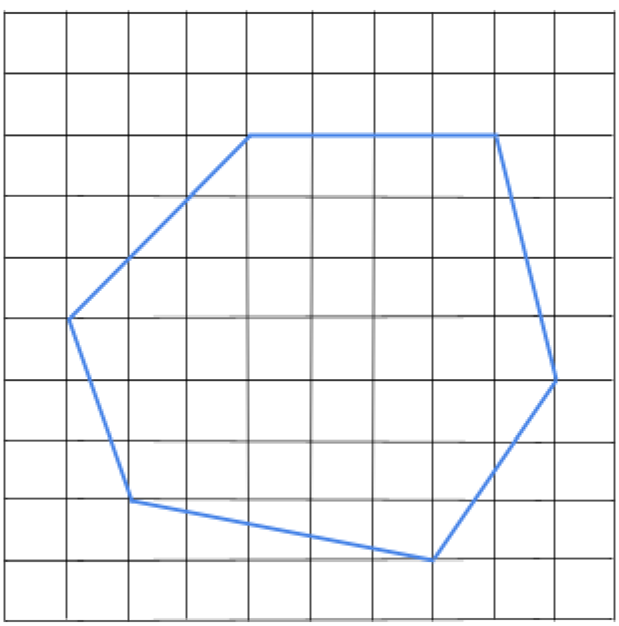
18) Construct 5 different lattice polygons. To keep this problem interesting, at least 3 of your polygons should be non-convex. All of your polygons should have at least 6 sides, and at least 10 boundary lattice points and at least 8 interior lattice points. For each of these 5 polygons, find the area, the number of lattice points inside the polygon, and the number of lattice points on the boundary of the polygon.



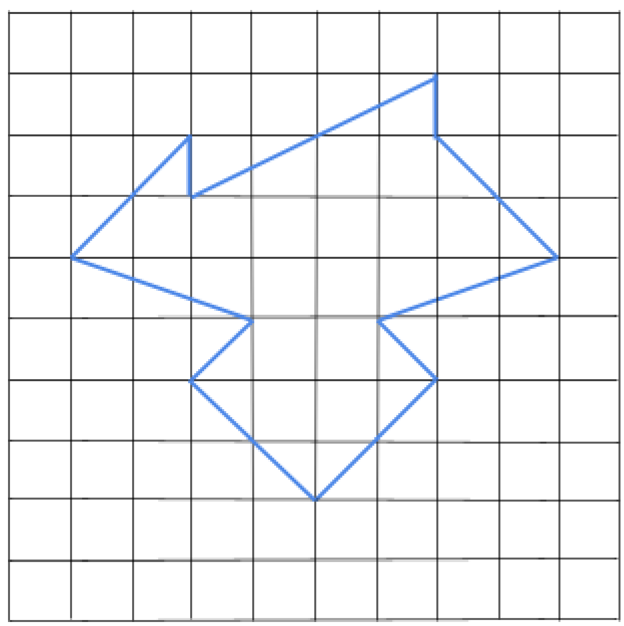
A(P) = 25, B(P) = 10, I(P) = 21



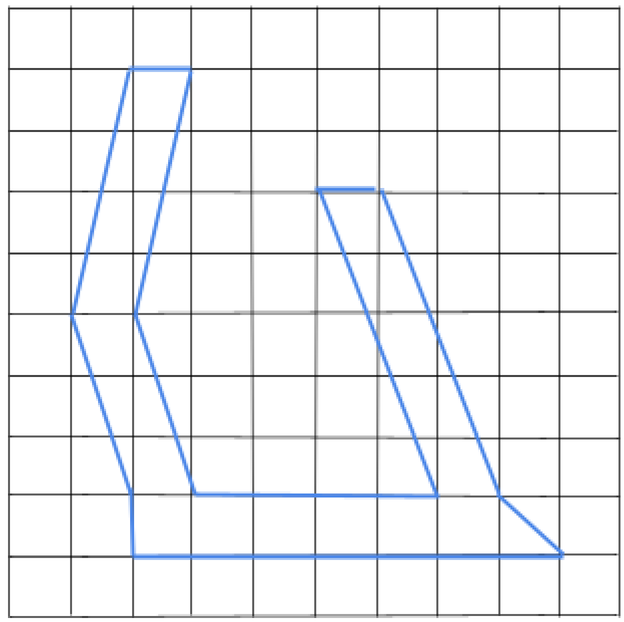
A(P) = 20, B(P) = 16, I(P) = 13



A(P) = 41.5, B(P) = 11, I(P) = 37

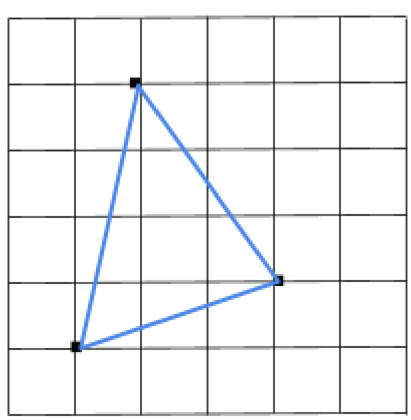


A(P) = 24, B(P) = 16, I(P) = 17



A(P) = 19.5, B(P) = 21, I(P) = 9

19) Let P be the triangle with vertices (0, 0), (3, 1), and (1, 4). Find the area of P , the number of lattice points inside the polygon, and the number of lattice points on the boundary of the polygon. Add this information to your table from Exercise 17



As seen from the diagram, this triangle has three 3 boundary points and 5 interior points. Then by some geometry, A(P) = 12 - 3/2 - 3 - 2 = 5½.

20) Exercise 20 Based on your work so far, make a conjecture about the area of a lattice pentagon in the case when B (the number of lattice points on the boundary) is even and in the case when B is odd.

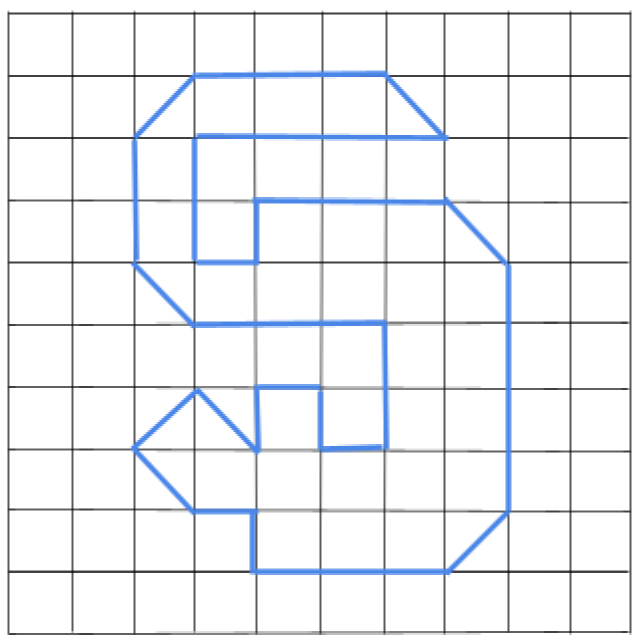
Based on our work so far we can make the conjecture that if the number of boundary lattice points is even then the area will be an integer. If odd, then the area will be rational; more specifically a multiple of 1/2.

21) Based on your work so far, conjecture a formula that relates the area of a lattice polygon to the number of lattice points inside the polygon and the number of lattice points on the boundary of the polygon. Explain how you obtained your conjecture, and why you think it makes sense.

Let P be a lattice polygon. From Exercise 20, we noticed that when B(P) is even, A(P) is an integer value and when B(P) is odd, A(P) is a multiple of 1/2. Knowing there may be a relationship between boundary points and area, we should first look at a case where the interior points have no impact. Take the 1X1 lattice square, having B(P) = 4 and I(P) = 0. We know the area of this polygon is 1 and we know that B(P) has some relationship to multiples of 1/2. Thus, we can make our first guess at a formula. A(P) = 1/2(4) = 2, but we know A(P) = 1 so we need to subtract one, leaving us with <u>A(P) = 1/2(B(P)) -1</u>. Now let us check with another case where I(P) = 0. Take Polygon #7 from Exercise 17. This polygon has B(P) = 6, I(P) = 0, A(P) = 2. Using our formula, A(P) = 1/2(6) - 1 = 3 -1 = 2; thus, so far the formula is working. Now, let us investigate the relationship between area and interior points when the number of boundary points are held constant. Take Polygon #1 from Exercise 17, this has B(P) = 6, I(P) = 1, A(P) = 3. Start with our equation of area with boundary points, A(P) = 1/2(6) -1 = 2. Now we need to make a relation with the interior points to get the last unit of area. Since we have one interior point for our one unit of area, let us assume it is a one-to-one relationship, leaving us with a new formula <u>A(P) = 1/2(B(P)) -1 + I(P)</u>. We can now check if it is indeed a one-to-one relationship by checking Polygon #2 from Exercise 17, which has the same parameters as before but one more interior point. By same investigation we find A(P) = 1/2(6) -1 = 2; however, now the area of the polygon is 4. This means there are 2 unaccounted units of area to two unaccounted interior points, thus the relationship must be one-to-one. Thus, we can make a conjecture that the area formula for lattice polygons is A(P) = 1/2(B(P)) + I(P) - 1.

22) Construct your own lattice polygon, and verify that the formula that you conjectured in Exercise 21 is satisfied. Your polygon should be at least somewhat interesting–for example, make it non-convex, and have at least 10 boundary lattice points and at least 8 interior lattice points.

This Polygon has 38 boundary points and 13 interior lattice points. We can then count the area easily in this picture as we have mostly a combination of 1-by-1 squares and 1-by-1 triangles. We get 26 squares, 8 1-by-1 triangles, and 1 4-by-1 triangle. Therefore A(P) = 26 + ½(8) = 30. Now checking with our equation we get A(P) = ½(42) + 10 - 1 = 30**.** Thus our equation is verified.



23) Assuming that your conjecture in Exercise 21 is valid, make a conjecture about the possible values of the area of a lattice polygon. Explain how you obtained your conjecture, and why you think it makes sense (including a proof or partial proof if you have ideas). Although you do not need to provide a formal proof here, you should provide sufficient justification and reasoning to indicate why you believe your conjecture is valid.

Assuming our conjecture in 21 is valid, we have the following formula, A(P) = 1/2(B(P)) + I(P) - 1. If this is true then we are able to say that for any lattice polygon, the area must be rational. On top of this, that rational number must also be a multiple of 1/2.

24) Does every lattice line segment have rational length? If so, prove it. If not, provide an example of a lattice line segment with non-rational length.

No, consider for example the line (0,0), (1,1). Both vertices fall on lattice points so this is a lattice line, however by the distance formula $\sqrt{1^2 + 1^2} = \sqrt{2}$ which is irrational. ▮

25) Make and prove a conjecture about the square of the length of any lattice line segment.

The square of the length of any lattice line is a positive integer. Consider the lattice line segment (a,b), (c,d). Then a, b, c, and d are all integers. By the distance formula the length of a line segment is, $l = \sqrt{(a - c)^2 + (b - d)^2}$. Then $l^2 = (a - c)^2 + (b - d)^2$ but since all components of this expression are integers then l^2 must be an integer and the squared terms make it positive. Thus the square of the length of any lattice line is a positive integer. ▮

26) Let L be a line with rational slope in the plane. Show that if there is a lattice point on L, then the y-intercept of L is rational.

Let the slope of line L be a rational number a/b. Then the equation for the line is y = (a/b)x + c, for some y-intercept c. Let us call the lattice point on L (m,n) with m and n integers. Now can plug in our lattice point to get n = (a/b)m + c. Solving for c we get c = n - (a/b)m. From this we know n is an integer and (a/b)m is rational since an integer times a rational is rational. Therefore our y-intercept is the difference between an integer and a rational, thus c is a rational number. ∎

27) Let L be a line with rational slope in the plane. Show that if there is one lattice point on L, then there are infinitely many lattice points on L.

Let the slope of line L be a rational number a/b. Then we can write the equation of the line as y = (a/b)x + c. From the statement, we know there must be some lattice point on L, call it (m,n). Plugging this point into our equation gives us (1) n = (a/b)m + c. Now we need to show there is an infinite number of lattice points on this line. Let S be the set of points in the form (m + kb, n + ka) for some integer k. Notice that all points in this set are lattice points as m, n, a, b, and k are all integers and the sum and multiplication of integers stay integers. Now we need to show that these points do in fact lie on line L. To do this we need to show that plugging in a random x value gives us our desired y value. Take x = m + kb. Then, y = (a/b)(m+kb) + c. Distributing we get y = (a/b)m + ka + c. Then from (1) we know that n = (a/b)m + c, so our equation is now y = n + ka which is our desired result. Thus a line with rational slope will have infinitely many lattice points on L if you know there exists one. ∎

28) Let p = (m,n) be a lattice point in the plane with gcd(m,n) = 1. Show that there are no lattice points strictly between the origin 0 = (0,0) and p on the line segment 0p.

Suppose, by way of contradiction, that there exists a lattice point (a,b) strictly between 0 and p on the line segment L, such that a and b are integers. The slope of L is then (n-0)/(m-0) = n/m and goes through the origin, thus the equation of L is y = (n/m)x. Then since (a,b) is on L; b = (n/m)a , so mb = na. Now, let us define divisibility of integers. S divides T, S|T, if there exists an integer k such that Sk = T. Then we have a theorem that states, if a|bc and gcd(a,c) = 1, then a|b. Additionally by a special case of the above theorem, if p is prime and p|ab, then p|a or p|b. Putting it all together, since b is an integer and m and n are relatively prime, gcd(m,n) = 1, m|na implies that m|a. But this is a contradiction since if m|a that would mean there exists an integer k such that mk = a. However, m and a are positive by our construction so this would imply a ≥ m. But we supposed in our statement that the point (a,b) was strictly between 0 and p. Thus there are no lattice points strictly between the origin 0 and p on the line segment 0p. ▮

29) Show that if p = (m,n) is a visible point on the lattice line L that goes through the origin (0,0), then any lattice point on L is of the form (tm, tn) for some integer t.

By definition of a visible point, there is no lattice point between 0 and p on 0p. Then, similar to exercise 28, the slope of the line L is n/m and goes through the origin, so the equation of the line 0p is y = (n/m)x. Now let (a,b) be a lattice point on the line L such that a and b are integers. Then by the slope of the line we know a must be a multiple of m, so let a = tm for some integer t. Then using our equation for L, b = (n/m)tm. Simplifying we get that b = nt. Thus we have shown that our arbitrary lattice point (a,b) must be in the form (tm,tn) for some integer t. ▮

30) Let m and n be nonnegative integers. Show that there are exactly gcd(m, n) − 1 lattice points on the line segment between the origin and the point (m, n), not including the endpoints.

Let gcd(m,n) = d > 1; note that if d = 1 then we have the case of Exercise 28. Then we can say gcd(m/d, n/d) = 1. Then let L be the line containing the points (0,0) and (m,n). By Exercise 28, there are no lattice points between the points (0,0) and (m/d, n/d) since gcd = 1. Then by Exercise 29, every lattice point is of the form (tm/d, tn/d) for some integer t. Thus the lattice points on L between (0,0) and (m,n) are {(m/d, n/d), (2m/d, 2n/d), (3m/d, 3n/d),... …, ((d-1)m/d, (d-1)n/d)}. Where Exercise 28 tells us that (m/d, n/d) must be the first lattice point in between, and Exercise 29 says there is no possible lattice point between (tm/d, tn/d) and ((t+1)m/d, (t+1)n/d) for some t = {1,2,3,...,(d-1)}. Thus we have d-1, or gcd(m, n) - 1 lattice points between the origin and (m, n) not including the endpoints. ▮

31) Let P be a lattice n−gon with vertices $p_1 = (a_1,b_1)$, $p_2 = (a_2,b_2),\ldots,p_n = (a_n,b_n)$. Let $d_i = \gcd(a_i+1 - a_i, b_i+1 - b_i)$ for $i = 1, 2, \ldots, n-1$ and let $d_n = \gcd(a_1 - a_n, b_1 - b_n)$. Show that the number of lattice points on the boundary of P is given by n $B(P) = \sum_{i=1}^{n} d_i$.

From Exercise 30, using a mapping back to the origin, d_i will give us the number of lattice points from p_i to p_{i+1} not including p_i. Therefore, d_i finds the number of boundary points for the ith side of a polygon. Thus, the total number of boundary lattice points of P is given by $B(P) = \sum_{i=1}^{n} d_i$. We know this mapping preserves lattice structure since adding or subtracting from a lattice point by an integer in the x or y direction will preserve a lattice point as an integer plus an integer in an integer, thus we can safely do this mapping to the origin.

32) Express $v = \begin{bmatrix} 8 \\ -12 \end{bmatrix}$ as a Z-linear combination of 2 vectors in Z .

$2\begin{bmatrix} 5 \\ -6 \end{bmatrix} + (-2)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ -12 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ -12 \end{bmatrix}$. Then note that this is a Z-linear combination as all coefficients 2, -2 are integers as well as all entries in our two vectors 5, -6, 1, 0.

33) Are the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ linearly independent or dependent?

We want to solve the set of equations $\left\{ \begin{matrix} c1 + 3c2 = 0 \\ 2c1 + 4c2 = 0 \end{matrix} \right\}$ and show that the only solution is when c1 = c2 = 0. By the first equation we have c1 = -3c2, now can plug into the second equation to get 2(-3c2) + 4c2 = 0. Simplifying we get -2c2 = 0 which forces c2 = 0. Then plugging this into equation 1 we get c1 = 0. So we found the only solution to the system of equations is the trivial one, thus linearly independent.

34) Are the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \end{bmatrix}$ linearly independent or dependent?

Since, (-3)$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -6 \end{bmatrix}$, we found a scalar c such that $cv_1 = v_2$, telling us these vectors are linearly dependent.

35) $\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}\}$ (a) Is the set a basis for R^2? Prove your result.

(b) Is the set a Z-basis for Z^2? Prove your result.

To show if a set is a basis we need to show if the vectors are linearly independent and span the space. We have already shown this set of vectors is linearly independent so now we need to show that (a) they span R^2 and (b) they span Z^2. To do this we need to solve the system of equations $\left\{ \begin{matrix} c1 \quad + 3c2 = x \\ 2c1 + 4c2 \ = y \end{matrix} \right\}$ for some vector < x,y > in the space. If we can find ways to express c1 and c2 in terms of x and y then we have shown the vectors span the whole space. To start we can divide the second equation by 2 to get, $\left\{ \begin{matrix} c1 \quad + 3c2 = x \\ c1 + 2c2 \ = (1/2)\, y \end{matrix} \right\}$. Now we can subtract equation 2 from equation 1 to get x-y/2 = 3c2 - 2c2, so c2 = x - y/2. Finally, we can plug this into equation 1 to get c1 = (3/2)y - 2x. So for any vector < x,y > in R^2 there exists a solution c1, c2 such that the system of equations is solved, thus spanning R^2. However for Z^2 notice in our equations for c1 and c2 we have fractions. So there exists < x,y > vectors in Z^2 that cannot be formed as our c1, and c2 values could be rational. Thus the set of vectors is a basis for R^2 but not for Z^2. ∎

36) Use Definition 22 to show that the matrix A=$\begin{bmatrix} 1 & 4 \\ 3 & 11 \end{bmatrix}$ is invertible over R.

Definition 22 states that a matrix is invertible if there exists another matrix B such that AB = I. Let B = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then AB = $\begin{bmatrix} 1 & 3 \\ 4 & 11 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + 3c & b + 3d \\ 4a + 11c & 4b + 11d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Now solving for each entry we get the set, {**a + 3c = 1** , <u>b + 3d = 0</u>, **4a + 11c = 0**, <u>4b + 11d = 1</u>}. By equation 1, we have a = 1 - 3c. Plugging into equation 3 we then get 4(1-3c) + 11c = 0. Rearranging we get that c = 4, which tells us a = -11. Then for the other two equations, we get from equation 2, b = -3d. Plugging into equation 4 we get 4(-3d) + 11d = 1, so d = -1, which tells us b = 3. Thus we found a matrix B = $\begin{bmatrix} -11 & 3 \\ 4 & -1 \end{bmatrix}$, such that AB = I. ▮

37) Use Definition 22 to show that the matrix A= $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is invertible over R.

Let B = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then AB = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Now solving for each entry we get the set, {**a + 2c = 1** , <u>b + 2d = 0</u>, **3a + 4c = 0**, <u>3b + 4d = 1</u>}. By equation 1, we have a = 1 - 2c. Plugging into equation 3 we then get 3(1-2c) + 4c = 0. Rearranging we get that c = 3/2, which tells us a = -2. Then for the other two equations, we get from equation 2, b = -2d. Plugging into equation 4 we get 3(-2d) + 4d = 1, so d = -1/2, which tells us b = 1. Thus we found a matrix B = $\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$, such that AB = I. ▮

38) Use Definition 22 to show that the matrix A= $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is *not* invertible over R.

Let B = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then AB = $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 2a+4c & 2b+4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Now solving for each entry we get the set, {**a + 2c = 1** , <u>b + 2d = 0</u>, **2a + 4c = 0**, <u>2b + 4d = 1</u>}. By equation 1, we have a = 1 - 2c. Plugging into equation 3 we then get 2(1-2c) + 4c = 0. But then 2 - 4c + 4c = 0, and simplifying we get 2 = 0. Thus there is no solution to B, and thus *not* invertible. ▮

39) (a) Construct (at least) 3 different (non-identity) matrices with real entries that are invertible over R. Show that each of your matrices is invertible over R using Definition 22. Then find the determinant of each of your matrices.

Let A = $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, B = $\begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}$, and C = $\begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$. We will show these three matrices are invertible over R. and compute their determinants.

Let D = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then AD = $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a+c & 2b+d \\ a+3c & b+3d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$**.** Now solving for each entry we get the set, {**2a + c = 1** , 2<u>b + d = 0</u>, **a + 3c = 0**, <u>b + 3d = 1</u>}. By equation 1, we have a = (1 - c)/2. Plugging into equation 3 we then get (1-c)/2 + 3c = 0. Rearranging we get that c = -1/5, which tells us a = 3/5. Then for the other two equations, we get from equation 2, b = -(1/2d). Plugging into equation 4 we get (-1/2d) + 3d = 1, so d = 2/5, which tells us b = -1/2. Thus we found a matrix BD= $\begin{bmatrix} 3/5 & -1/5 \\ -1/5 & 2/5 \end{bmatrix}$, such that AD = I. ∎

Let D = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then BD = $\begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4a & 4b \\ a+5c & b+5d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$**.** Now solving for each entry we get the set, {**4a = 1** , <u>4b = 0</u>, **a + 5c = 0**, <u>b + 5d = 1</u>}. By equation 1, we have a = 1/4. Plugging into equation 3 we then get 1/4 + 5c = 0. Rearranging we get that c = -1/20. Then for the other two equations, we get from equation 2, b = 0. Plugging into equation 4 we get 5d = 1, so d = 1/5, Thus we found a matrix D = $\begin{bmatrix} 1/4 & 0 \\ -1/20 & 1/5 \end{bmatrix}$, such that BD = I. ∎

Let D = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then CD = $\begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a+2c & -b+2d \\ 3a+c & 3b+d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$**.** Now solving for each entry we get the set, {**-a + 2c = 1** , -<u>b + 2d = 0</u>, **3a + c = 0**, <u>3b + d = 1</u>}. By equation 1, we have a = 2c - 1. Plugging into equation 3 we then get 3(2c-1) + c = 0. Rearranging we get that c = 3/7, which tells us a = -1/7. Then for the other two equations, we get from equation 2, b = 2d. Plugging into equation 4 we get 3(2d) + d = 1, so d = 1/7, which tells us b = 2/7. Thus we found a matrix D = $\begin{bmatrix} -1/7 & 3/7 \\ 2/7 & 1/7 \end{bmatrix}$, such that CD = I. ∎

Now looking at the determinants of these matrices we get det(A) = 5, det(B) = 20 , and det(C) = -7.

(b) Construct (at least) 3 different matrices with real entries that are not invertible over R. Show that each of your matrices is not invertible over R using Definition 22. Then find the determinant of each of your matrices.

Let A = $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, B = $\begin{bmatrix} 0 & 3 \\ 0 & 6 \end{bmatrix}$, and C = $\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix}$. We will show these three matrices are invertible over R. and compute their determinants.

Let D = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then AD = $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Now solving for each entry we get the set, {**a + c = 1** , <u>b + d = 0</u>, **a + c = 0**, <u>b + d = 1</u>}. By equation 1, we have a = 1 - c. Plugging into equation 3 we then get (1-c) + c = 0. But then 1 - c + c = 0, and simplifying we get 1 = 0. Thus there is no solution to B, and thus *not* invertible.

Let D = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then BD = $\begin{bmatrix} 0 & 3 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3c & 3d \\ 6c & 6d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Now solving for each entry we get the set, {**3c = 1** , <u>3d = 0</u>, **6c = 0**, <u>6d = 1</u>}. By equation 1, we have c = 1/3. Plugging into equation 3 we then get 6(1/3) = 0. But then 2 = 0. Thus there is no solution to B, and thus *not* invertible.

Let D = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then CD = $\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -3a+2c & -3b+2d \\ 3a-2c & 3b-2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Now solving for each entry we get the set, {**-3a + 2c = 1** , <u>-3b + 2d = 0</u>, **3a -2c = 0**, <u>3b -2d = 1</u>}. By equation 1, we have a = 2/3c - 1/3. Plugging into equation 3 we then get 3(2/3c - 1/3) -2c = 0. But then 2c - 1 - 2c = 0, and simplifying we get -1 = 0. Thus there is no solution to B, and thus *not* invertible.

Now looking at the determinants of these matrices we get det(A) = 0, det(B) = 0 , and det(C) = 0.

(c) Performing additional computations if necessary, make a conjecture about the determinant of a matrix with real entries that is invertible over R.

Based on our examples in parts a and b, we can make the conjecture that if the determinant of a matrix is non-zero, then the matrix is invertible over R.

40) Consider The 2×2 matrix A= $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ .Show that if ad - bc ≠ 0, then A is invertible and A^{-1} = 1/(ad−bc) $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Show that If ad - bc = 0, then is not invertible.

Suppose det(A) ≠ 0. Let B = $\begin{bmatrix} b1 & b1 \\ b3 & b4 \end{bmatrix}$. By definition, a matrix A is said to be invertible if there exists a matrix B such that AB = I. So let us investigate what each b_i entry will have to be in B.

AB = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} b1 & b2 \\ b3 & b4 \end{bmatrix} = \begin{bmatrix} ab1 + bb3 & ab2 + bb4 \\ cb1 + db3 & cb2 + db4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, Now solving for each entry we get the set, {**ab1 + bb3 = 1** , <u>ab2 + bb4 = 0</u>, **cb1 + db3 = 0**, <u>cb2 + db4 = 1</u>} Solving for b3 from equation 1 we get b3 = (1-ab1)/b, plugging into equation 3 we then get, cb1 + d((1-ab1)/b) = 0. Simplifying to solve for b1 we end up with b1 = d/(ab-bc). From this we can derive the other b_i's, we get b2 = -b/(ab-bc), b3 = -c/(ab-bc), and b4 = a/(ab-bc). Then since ab-bc ≠ 0 these values exist so we get our desired result A^{-1} = 1/(ad−bc) $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Now suppose det(A) = 0. Then looking at our derivations from our proof above, the b_i's all have a form where an element in A is divided by the determinant, but if the determinant is zero this is impossible, thus if the det(A) = 0, A cannot be invertible. ∎

41) Use Definition 24 to show that the matrix A= $\begin{bmatrix} 1 & 3 \\ 4 & 11 \end{bmatrix}$ is invertible over Z.

We know that A is invertible over R since det(A) = 11 - 12 = -1 ≠ 0. We also know that there exists a matrix A^{-1} = 1/(ad−bc) $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, such that AA^{-1} = I. So it suffices to show that all entries in A^{-1} are integers. Since all entries in A are integers and we are only dividing by -1, then the negative of the entries are still integers. Thus, there exists a matrix $A^{-1} = \begin{bmatrix} -11 & 4 \\ 3 & -1 \end{bmatrix}$ such that $AA^{-1} = I$.

42) Use Definition 24 to show that the matrix is invertible over Z. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is not invertible over Z.

Alternative to exercise 41, we will show by our formula that the entries in A^{-1} are not integers. Notice det(A) = 4 - 6 = -2. Then we can see immediately that the third entry in A^{-1} would be -(3/-2) = -3/2 which is not an integer. Thus A is not invertible over Z.

43) Construct (at least) 3 different (non-identity) matrices with integer entries that are invertible over Z.

Let $A = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} -5 & 2 \\ -7 & 3 \end{bmatrix}$. We will show these three matrices are invertible over Z. and compute their determinants.

(a) Show that each of your matrices is invertible over Z.

Det(A) = 2 - 3 = -1 so we can be sure all entries will remain integers after invertible matrix formula so, $A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix}$.

Det(B) = 2 - 1 = 1 so we can be sure all entries will remain integers after invertible matrix formula so, $A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.

Det(C) = -15 +14 = -1 so we can be sure all entries will remain integers after invertible matrix formula so, $A^{-1} = \begin{bmatrix} -3 & 2 \\ 7 & -3 \end{bmatrix}$.

(b) Find the determinant of each of your matrices. What do you observe?

From calculations above we see that all determinants are either 1 or -1.

(c) Performing additional computations if necessary, make a conjecture about the determinant of a matrix with integer entries that is invertible over Z.

A conjecture we can make is that a matrix A is only invertible over Z if the determinant of A is 1 or -1.

44) Let A be a 2×2 matrix with entries in Z. Show that A is invertible over Z if and only if det(A) = ±1. Note that you only need to prove this result here for 2 × 2 matrices, but the same result holds for more general n × n matrices with entries in Z.

(=>) Suppose A is a 2x2 matrix that is Z invertible. Since invertible we know that the determinant of A is nonzero. By exercise 40, A^{-1} = 1/(ad - bc)$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and need each entry to be an integer. By definition of divisibility, there exists x1, x2, x3, and x4 such that (ad - bc)x1 = d, (ad - bc)x2 = b, (ad - bc)x3 = c, and (ad - bc)x4 = a. Then from these equations, (ad-bc) = (ad - bc)x4(ad - bc)x1 - (ad - bc)x2(ad - bc)x3 = $(ad - bc)^2$ (x4x1 - x2x3). Now can divide by the determinant on both sides to reach 1 = (ad - bc)(x4x1 - x2x3). Essentially, we have 1 = xy so we have three cases: x = 1 implying y = 1, x = -1 implying y = -1, or x = a implying y = 1/a. Then we only have two options since the third makes it so y is not an integer. Thus our only solutions are 1 and -1.

(<=) Suppose the determinant of A is 1 or -1, then we have two cases.
If det(A) = 1, let B = $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, then AB = $\begin{bmatrix} ad-bc & -ab+ab \\ -cd+cd & ad-bc \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
if det(A) = -1, let B = $\begin{bmatrix} -d & b \\ c & -a \end{bmatrix}$, then AB = $\begin{bmatrix} -ad+bc & ab-ab \\ -cd+cd & -ad+bc \end{bmatrix} = \begin{bmatrix} -(ad-bc) & 0 \\ 0 & -(ad-bc) \end{bmatrix} =$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus in both cases we can find matrices B such that AB = I. ∎

45) Which real numbers have a multiplicative inverse in R? Prove your statement.

Every real number has a multiplicative inverse in R except for 0. For any a in the reals, 1/a is also a real number and a(1/a) = 1. Then zero does not since there is no number a such that 0(a) = 1.∎

46) Which integers have a multiplicative inverse that is also an integer? Prove your statement.

The only integers that have a multiplicative inverse are 1 and -1. We know from inverses of R that if you have a number a, its multiplicative inverse is 1/a. However 1/a is not an integer for any a not equal to 1 or -1. So the only integers with an integer multiplicative inverse are 1 and -1, having inverses of themselves. ∎

47) Construct 3 different examples of bases {v = ⟨v1, v2⟩ , w = ⟨w1, w2⟩} of R^2.

$\begin{bmatrix} 5 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ 2) $\begin{bmatrix} 8 \\ 11 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \end{bmatrix}$ 3) $\begin{bmatrix} -9 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

(a) Show that each of your examples is actually a basis of R^2.

To show each is a basis of R^2 need to show two things; linear independence and span. Let us show in detail how we do this for basis 1.

Linear Independence:

Solve, $c1\begin{bmatrix} 5 \\ 4 \end{bmatrix} + c2\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. get 5c1 = 0 and 4c1 - c2 = 0. From the first equation c1 =0, and plugging into equation two we get c2 = 0. Thus we found the only linear combination of these vectors to get the zero vector are c1 = c2 = 0, thus is linearly independent.

Span:

Solve, c1$\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ + c2$\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ = $\begin{bmatrix} x \\ y \end{bmatrix}$. get 5c1 = x and 4c1 - c2 = y. So from equation 1 we get c1 = x/5. Plugging into equation 2 we get 4(x/5) - c2 = y, so c2 = (4/5x) - y. Thus, since we found c1 and c2 in terms of an arbitrary x and y, we can say these vectors span R^2.

Therefore these vectors form a basis for R^2.

Following similar steps for basis 2 and 3 we get that both are linearly independent and span with (c1,c2) being ((-81y+6x)/136, (8y-11x)/68) for basis 2 and ((2y+171x/1863), (y-18x)/414) for basis 3.

(b) For each basis, compute the determinant of the matrix $\begin{bmatrix} v1 & w1 \\ v2 & w2 \end{bmatrix}$. What do you observe?

Det(#1) = -5 -0 = -5, Det(#2) = 24 - (-44) = 68, Det(#3) = -54 + 8 = -46

And I notice that all of the determinants are non-zero.

(c) Doing more computations if necessary, state and prove a conjecture about the determinant of a matrix whose columns form a basis for R^2.

Claim: For a 2x2 matrix A, the column forms a basis for R^2 if and only if the determinant of A is not equal to zero.

(=>) Assume the column vectors form a basis for R^2. Then by definition, the vectors are linearly independent and span R^2. From this we know through Gaussian elimination that we can transform our matrix to a diagonal matrix without affecting the determinant. So for any arbitrary matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the corresponding diagonal matrix D = $\begin{bmatrix} a & b \\ 0 & d-(c/a)b \end{bmatrix}$. Now we need to look at two cases; a = 0 and a $\neq$ 0. If a = 0 then det(A) = -bc. Then we are looking for the determinant to be nonzero, but notice the only way for the determinant to be zero would be if b or c equals zero. However this would result in a zero row or column vector which will cause the vectors to be linear dependent. So det(A) $\neq$ 0. If a $\neq$ 0, then we can look at the diagonal entries of D to find our determinant, so det(A) = det(D) = a(d-(c/a)b). Then we know $a \neq 0$ and since the vectors form a basis we know that $d-(c/a)b \neq 0$ or else would be linear dependent by zero row. Thus the determinant of A cannot be zero.

(<=) In this direction we need to show the vectors are linearly independent and span R^2. To start we will assume the vectors are not linearly independent and show this is impossible. By linear dependence there exists a non-trivial solution c such that $Ac = 0$. So we get the system of equations $\{ac1 + bc2 = 0 , cc1 + dc2 = 0\}$. Since $c \neq 0$ we know at least one of c1 or c2 is not equal to zero. Without loss of generality, let $c2 \neq 0$. Then by the first equation $c1 = (-b/a)c2$. Substituting into the second equation, $c(-b/a)c2 + dc2 = 0$ so $c2(-(bc/a+d) = 0$. Then since $c2 \neq 0$, $-(bc/a+d) = 0$. Rearranging we get $ad-bc = 0$ or $\det(A) = 0$. Thus if we assume the vectors are linearly dependent, then the determinant must equal zero, which tells us if $\det(A) \neq 0$, then the vectors are linearly independent.

Now we need to prove that the vectors span R^2. So we need to show that for any x, there exists c such that $Ac = x$. So we get the system of equations $\{ac1 + bc2 = x1 , cc1 + dc2 = x2\}$. From equation 1, $c1 = (x-bc2)/a$ and from equation 2 we get that $c1 = (x2-dc2)/c$. Setting them equal to each other we get $cx - cbc2 = ax2 - adc2$. Simplifying we $c2 = (ax2-cx)/(ad-bc)$ and since $\det(A) \neq 0$ we are able to do this. Then subbing c2 into equation 1 we get $c1 = (x-b(ax2-cx/ad-bc))/a$. Thus we found constants c1 and c2 for any arbitrary <x1,x2>, thus spanning R^2. Therefore the column forms a basis for R^2 if and only if the determinant of A is nonzero. ▮

48) Construct 3 different examples of Z-bases $\{v = \langle v1, v2\rangle , w = \langle w1, w2\rangle\}$ of Z^2.

1) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 11 \end{bmatrix}$ 2) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ 3) $\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(a) Show that each of your examples is actually a Z−basis of Z^2.

Need to show vectors are linearly independent and span Z^2.Let us show in detail how we do this for basis 1.

Linear independence follows from being linearly independent in R. So since the determinant of basis 1 as a matrix A = $\begin{bmatrix} 1 & 4 \\ 3 & 11 \end{bmatrix}$ is det(A) = 11-12 = -1 $\neq$ 0. So this is a basis in R^2 thus is linearly independent.

Now we will show span:

Solve, c1$\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ + c2$\begin{bmatrix} 4 \\ 11 \end{bmatrix}$ = $\begin{bmatrix} x \\ y \end{bmatrix}$, with c1,c2 integers. Then we get the system of equations, {c1+4c2 = x, 3c1+11c2 = y}. From equation 1 we get c1 = x- 4c2, and plugging into equation 2 we have 3(x- 4c2) + 11c2 = y. Simplifying we get c2 = 3x-y and then solving for c1 we get c1 = -11x+4y. Then since x,y, c1,and c2 are integers, these span Z^2.

Therefore these vectors form a basis for Z^2.

Following similar steps for basis 2 and 3 we get that both are linearly independent and span with (c1, c2) being ((-5x+2y), (3x-y)) for basis 2 and ((x-2y), (-x+y)) for basis 3.

(b) For each basis, compute the determinant of the matrix $\begin{bmatrix} v1 & w1 \\ v2 & w2 \end{bmatrix}$ What do you observe?

Det(#1) = 11 - 12 = -1, Det(#2) = 5 - 6 = -1, Det(#3) = 3 -2 = 1

And I notice that all of the determinants are either 1 or -1.

(c) Doing more computations if necessary, state and prove a conjecture about the determinant of a matrix whose columns form a Z-basis for Z^2.

Claim: For a 2x2 matrix A, the column forms a basis for Z^2 if and only if the determinant of A is equal to 1 or -1.

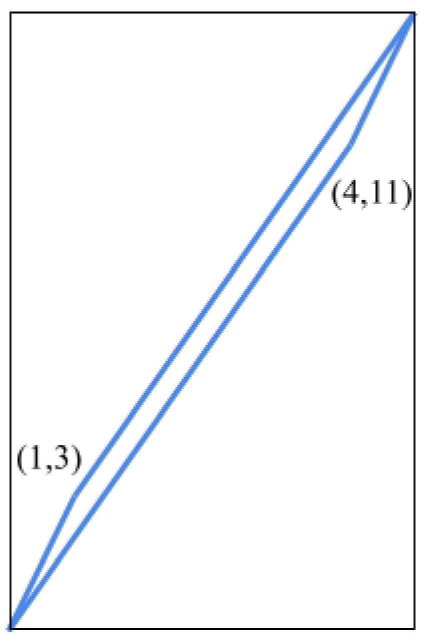
Let v1, v2 be vectors in Z^2 such that they form a Z-basis for Z^2. By Exercise 44, if we can show A = [v1,v2] is invertible over Z, then we can conclude the det(A) = 1 or -1. Let v1 = $\begin{bmatrix} a \\ b \end{bmatrix}$ and v2 = $\begin{bmatrix} c \\ d \end{bmatrix}$. Since v1 and v2 are in Z^2, a,b,c, and d are integers. So A = $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$

and we want to show there exists a matrix B such that AB = I. Since v1 and v2 form a Z-basis for Z^2, all $\begin{bmatrix} x \\ y \end{bmatrix}$ in Z^2 can be expressed as a Z-linear combination. In particular, since $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are elements of Z^2, there exists integers c1,c2,c3, and c4 such that c1v1+c2c2= $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

And c3v1 + c4v2 = $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus, $\begin{bmatrix} c1 & c3 \\ c2 & c4 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then since c1,c2,c3, and c4 are integers we found a matrix in Z^2 such that AB = I, so by exercise 44, the determinant of A is 1 or -1 since we have a Z invertible matrix. ▮

49) Sketch the parallelogram spanned by each of the Z−bases for Z^2 that you constructed in Exercise 48.

1)



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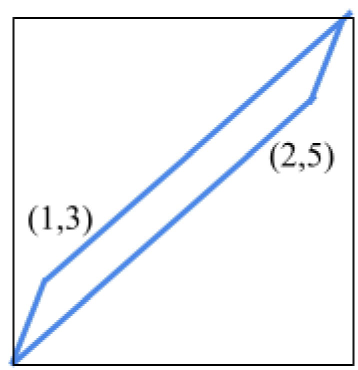

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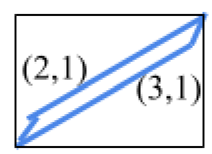


(a) Find the area of the parallelogram spanned by each of the Z−bases for Z^2 that you constructed in Exercise 48.

A(P) = (a+c)(b+d) - 2ad -ab -cd

1) A(P) = (5)(14) - 2(1)(11) - (1)(3) - (4)(11) = 70 - 22 - 3 - 44 = 1

2) A(P) = (3)(8) - 2(1)(5) - (1)(3) - (2)(5) = 24 - 10 - 3 - 10 = 1

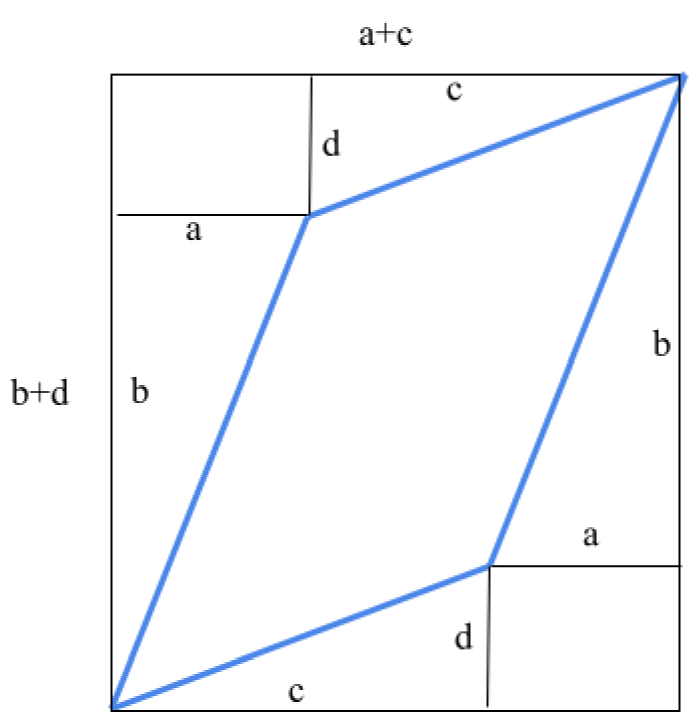
3) A(P) = (5)(2) - 2(3)(1) - (2)(1) - (3)(1) = 10 - 6 - 3 = 1

(b) State and prove a conjecture about the area of a lattice parallelogram P (v, w), where v and w form a basis of Z^2.

Claim: The area of the parallelogram P(v,w) formed by a Z-basis of Z^2 is equal to 1.

For two basis vectors v = $\begin{bmatrix} a \\ b \end{bmatrix}$ and w = $\begin{bmatrix} c \\ d \end{bmatrix}$. We can find the area of P(v,w) by encasing P(v,w) in a rectangle and subtracting the areas that are not part of P(v,w), like how we did in part a. Below we can see a way to break up that rectangle which leaves us with the following formula.

A(P(v,w)) = (a+c)(b+d) - 2ad -ab -cd = ab +ad +cb +cd -2ad - ab - cd = cd-ad = -(ad-cd) = -det(A). But we know from exercise 48 that the determinant of A is 1 or -1 and we can't have a negative area so that tells us the area of P(v,w) must be 1. ∎

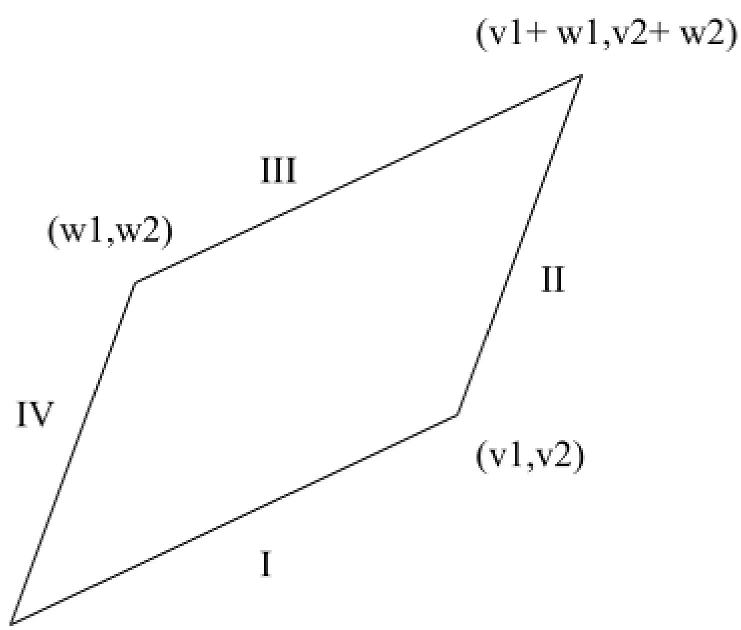


50) Sketch the parallelogram spanned by each of the Z−bases for Z2 that you constructed in Exercise 48.

(a) How many lattice points are on the sides of the parallelogram? How many lattice points are in the interior?

Based on the sketches there are 4 boundary lattice points and 0 interior lattice points.

(b) Doing more computations if necessary, make and prove a conjecture about the number of lattice points on the sides and in the interior of a lattice parallelogram P(v,w), where v and w form a Z−basis for Z^2.



Claim:

All P(v,w), where v and w form a Z−basis for Z^2 will have exactly 4 boundary lattice points and 0 interior lattice points.

To show B(P(v,w)) = 4 we need to show the only lattice points on the sides of P(v,w) are the vertices. For side I, by exercise 30, there are gcd(v1-0, v2-0)-1 = gcd(v1, v2) -1 lattice points excluding the endpoints, and similarly for side IV there are gcd(w1, w2) -1 lattice points.. By exercise 49 determinant is v1w2 - v2w1 = 1, so by Bezue's lemma gcd(v1,v2) = 1 and gcd(w1,w2) = 1 as well, so that means there are no lattice points on the line segments I or IV. Now I need to show there are no lattice points on II and III. We know that on line segment II, we get this form for the gcd; gcd(v1+w1 - v1, v2+w2 - v2) = 1. On segment III we get; gcd(v1+w1 - w1, v2+w2 - w2) = 1. Then, following from exercise 30, we know that there are gcd(v1,v2) - 1 lattice points on II and gcd(w1, w2) -1 on III, but we know the gcd of these are 1. Therefore for each line segment I,II,III, and IV, there are no lattice points besides the endpoints, thus the number of boundary lattice points is 4.

Now need to show there are no Interior lattice points in P(v,w). Suppose (x,y) is an element of Z^2 in or on P(v,w). Since v and w form a Z-basis, there exists integers c1 and c2 such that c1v + c2w = (x,y). On the other hand, (x,y) is in or on P(v,w). Thus, $0 \leq$ c1, c2 ≤ 1. So since c1 and c2 are integers, c1 = c2 = 0 or 1. If c1 = c2 = 0, (x,y) = (0,0). If c1 = 1, c2 = 0, (x,y) = v. If c1 = 0, c2 = 1, (x,y) = w. And if c1 = c2 =1, (x,y) = v + w. So only lattice points on P(v,w) are the vertices, so I(P) = 0.

Thus, All P(v,w), where v and w form a Z−basis for Z^2 will have exactly 4 boundary lattice points and 0 interior lattice points. ▮

51) Suppose that the parallelogram P(v,w), where v, w $\in Z^2$, is a primitive lattice parallelogram. Show that v and w form a Z−basis for Z^2.

First let's note that since v and w form a parallelogram, they must be linearly independent or else would create a line. Then since v and w are linearly independent over R, and hence Z, and there are two vectors for a two dimensional space, v and w also must span R^2, so they form a basis in R^2. Now let u = <u1, u2> be a vector in Z^2. We want to show there exists integers c1 and c2 such that u = c1v + c2w. Since we know v and w form a basis for $R^{2.}$, there exists c1, c2 in R such that u = c1v + c2w. So what is left to show is that c1 and c2 are integers.Let us write $c1 = \lfloor c1 \rfloor + \alpha 1$ and $c2 = \lfloor c2 \rfloor + \alpha 2$, for $0 \leq \alpha 1, \alpha 2 < 1$, where $\lfloor c1 \rfloor$ and $\lfloor c2 \rfloor$ represents the floor of c1 and c2 respectfully. Then to show c1 and c2 are integers we need to show $\alpha 1 = \alpha 2 = 0$. Let $u' = \alpha 1 v + \alpha 2 w$ and observe that $u' = u - (\lfloor c1 \rfloor v + \lfloor c2 \rfloor w)$, since $w = c1v + c2w = (\lfloor c1 \rfloor + \alpha 1)v + (\lfloor c2 \rfloor + \alpha 2)w$. Then we know u is an integer and $(\lfloor c1 \rfloor v + \lfloor c2 \rfloor w)$ is as well since $\lfloor c1 \rfloor$ and $\lfloor c2 \rfloor$ are integers, so u' must also be an integer since it is equal to the difference of integers. Now recall our definition for P(v,w). It says P(v,w) can be represented as $\{av+bw, 0 \leq a,b \leq 1\}$. So u' is an element of P(v,w). Thus u' is an element of Z^2 in or on P(v,w), but the only lattice points on P(v,w) are the vertices. This means that u' = 0, v, w, or v+w. Let us check the different cases: u' = 0 implies $\alpha 1 = \alpha 2 = 0$ by linear independence. Then $c1 = \lfloor c1 \rfloor$ and $c2 = \lfloor c2 \rfloor$ as we desired. Now let's look at the other cases. If u' = v then, $\alpha 1 v + \alpha 2 w = v$, but this means that $\alpha 1 = 1$, a contradiction since we defined $\alpha 1$ as $0 \leq \alpha 1 < 1$. Similarly with the other cases, they force either $\alpha 1$ or $\alpha 2$ to be 1. Thus the only possibility is that $\alpha 1 = \alpha 2 = 0$. So v and w form a Z−basis for Z^2. ▮

52) Let T be a primitive lattice triangle. If v and w are vectors corresponding to adjacent sides of T, show that v and w form a Z−basis of Z^2.

Let T be a primitive lattice triangle with side v and w. Since T is primitive, v and w are elements of Z^2 and B(T) = 3 and I(T) = O. Now let us show how v and w form a basis. Let P(v,w) be the parallelogram formed by v and w. Since T is primitive, so is P(v,w). By exercise 51, v and w form a Z-basis for Z^2. ▮

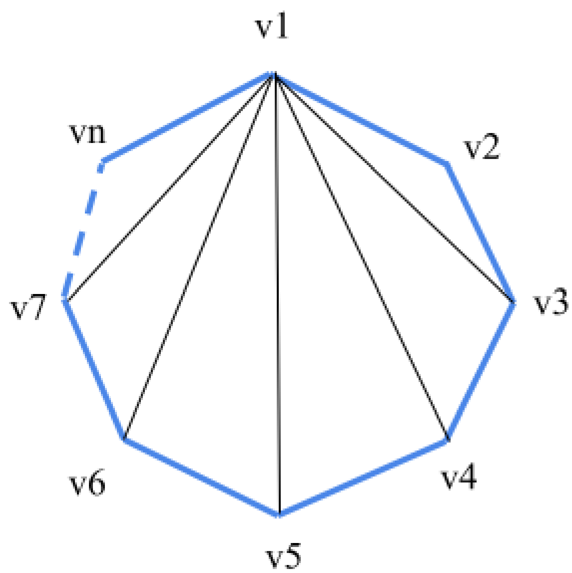
53) Prove that the area of a primitive lattice triangle is equal to 1/2.

Let T be a primitive lattice triangle. By exercise 52, v and w form a Z-basis for Z^2. Then by exercise 49, A(P(v,w)) = 1, then since our primitive triangle is half of thai parallelogram, we get A(T) = 1/2 for all primitive lattice triangles. ▮

54) Prove Theorem: Every n-gon can be dissected into n−2 triangles by means of non intersecting diagonals. The vertices of the triangles in this dissection by diagonals are vertices of the original polygon.

Let P be a polygon with n vertices. Then we have two cases; P convex or P not convex.
If P is convex, then we can construct P as follows. Let v1, v2,…, vn be the vertices of P in

the clockwise direction. Then since convex we can construct the diagonals v1v3, v1v4,..., v1vn-1. These diagonals dissect P into n-2 triangles where each triangle is non-intersecting and the vertices of the triangles are the vertices of P by construction. Thus if convex we can dissect P into n-2 triangles.



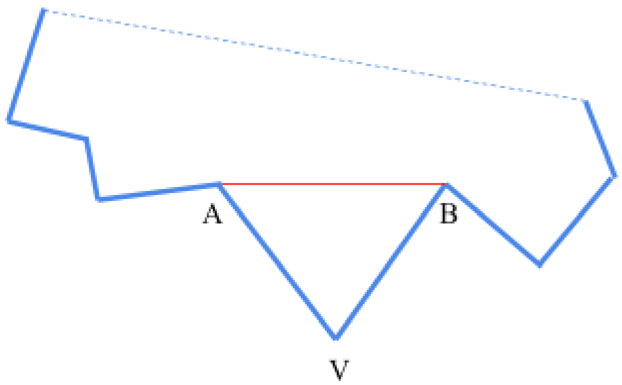
Now suppose P is non-convex. We will prove this result through induction on the number of vertices. We will first prove a lemma that will help us later in our inductive step.

Lemma: P must have some diagonal.

Let P be an n-gon with vertices (x1,y1), (x2,y2),…, (xn,yn) such that each (xi,yi) are lattice points. Then there exists k, 1 ≤ k ≤ n, such that yk ≤ yi for all i = 1, 2,..., n. In other words, there exists a vertex with a minimal y-coordinate; let V denote this vertexand let A and B denote the two vertices adjacent to V. Then we are faced with three possibilities.

1) There are no vertices of P in or on the triangle AVB
2) There are vertices of P inside the triangle AVB
3) There are triangles of P in the line segment AB but not in the triangle AVB\

1)



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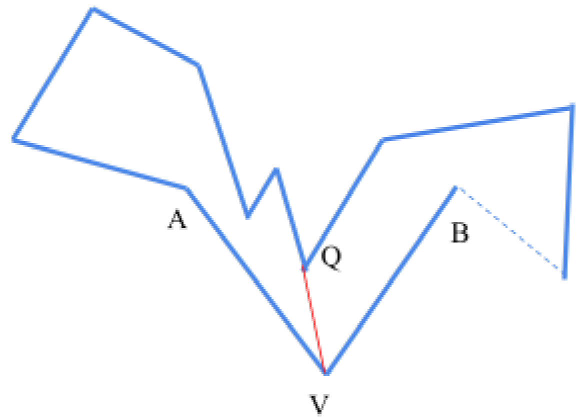

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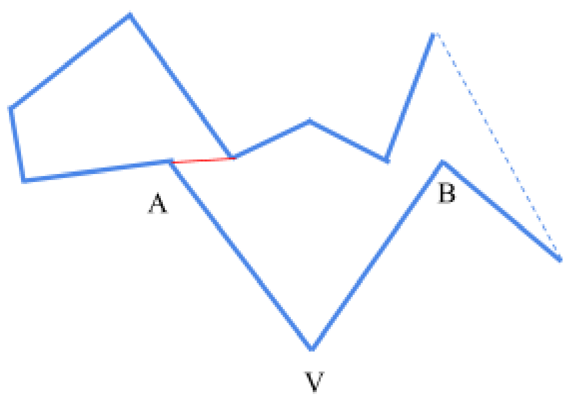


Looking at our diagrams above we can see where we need to create our diagonals. In case 1 we have no interruption on the line segment AB so our diagonal should be the line AB. In case 2 we do have interruption so we need to choose the vertex in AVB with the minimal y-coordinate, ca it Q. Then since A and B are adjacent there cannot be disturbance between this lowest point and V so we can make a diagonal VQ. Finally in case 3, we have disturbance but only on the line, so we can draw from A to the first disturbance on AB. Thus we have shown that any n-gon P must have a diagonal. ∎

Now let us prove the statement by induction.

Base Case:

Let n = 3. Then P is a triangle. So we have n-2 = 3-2 = 1 triangle. Thus P can be dissected into n-2 triangles, with no intersecting diagonals and vertices are the three of P.

Inductive Hypothesis:

Assume any P with k < n vertices can be dissected into k-2 triangles by means of non intersecting diagonals and the vertices of the triangles in this dissection by diagonals are vertices of the original polygon.

Inductive Step:

Let P be a polygon with n vertices. By lemma we know that there must exist a diagonal in P, so we can dissect P into the two parts of the polygon P1 and P2. Let n1 be the number of vertices in P1 and n2 be the number of vertices in P2. Then n1 + n2 = n + 2, since both P1 and P2 include two of the same vertices. Now we know n1 and n2 are less than n, so we can invoke our inductive hypothesis. P1 can be dissected into n1 - 2 triangles and P2 into n2 - 2 triangles. Then these two dissections of P1 and P2 collectively form a dissection of P, so the total number of triangles of P is (n1 -2) + (n2 -2) = n1 + n2 - 4 = n + 2 -4 = n - 2. Thus we have shown a non-convex polygon P can be dissected into n-2 triangles.

Thus any n-gon can be dissected into n-2 triangles by means of non intersecting diagonals, where the vertices of the triangles in this dissection by diagonals are vertices of the original polygon. ▮

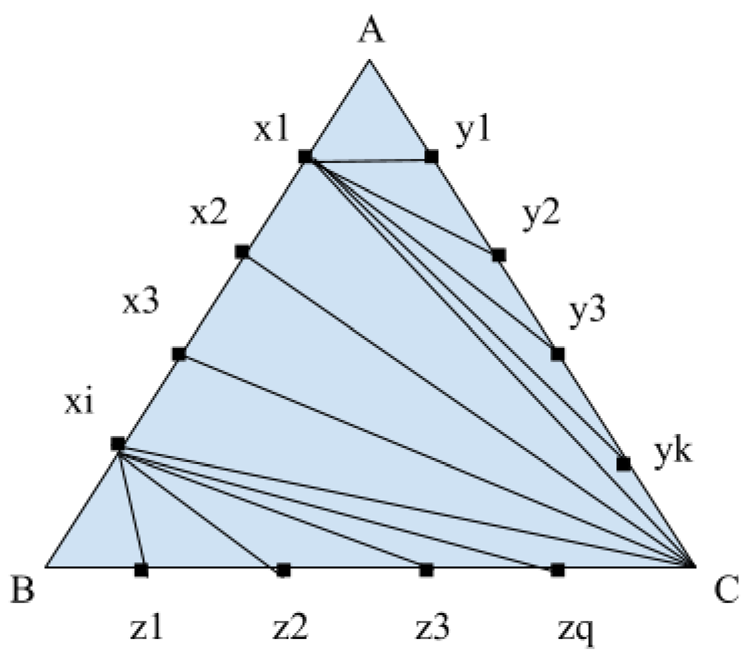
55) Prove Lemma: Every lattice triangle can be dissected into primitive lattice triangles.

We will prove this lemma by induction on the number of interior lattice points.

Base Case:

Let ABC be a triangle with zero interior lattice points. Then if ABC has three boundary points, ABC is a lattice triangle by definition and we are done. So without loss of generality, let us assume there exists at least 1 boundary point on the line segment AB; denote them x1, x2,..., xi. Then we can dissect ABC with line segments Cx1, Cx2,..., Cxi. This creates i+1 lattice triangles. Now we have only let AB have boundary lattice points and notice there is still the possibility of boundary lattice points on AC or BC. Without loss of generality, assume there exist boundary lattice points on AC denoted y1, y2,..., yk. Then by a similar method we can dissect this section into x1y1, x1y2,..., x1yk. This then leaves the final case that BC has boundary lattice points z1, z2,..., zq. Then we can dissect this section by the line segments xiz1, xiz2,...,xizq. Now note that each triangle formed is a primitive lattice triangle as we know there are no interior lattice points in ABC and each triangle has exactly three boundary points. Thus we have shown that any lattice triangle with zero interior lattice points can be dissected into primitive lattice triangles.

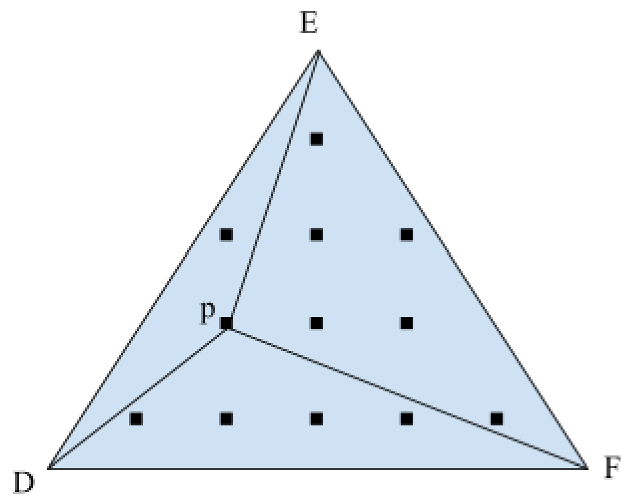
Inductive Hypothesis:



Assume that any lattice triangle with $k > 0$ interior lattice points can be dissected into primitive lattice triangles.

Inductive Step:

Let DEF be a lattice triangle with k+1 interior points. Without loss of generality, let p be some interior point of DEF. Then dissect DEF such that we form the line segments Dp, Ep, and Fp; this forms three lattice triangles. Then since p was originally an interior lattice point and now DEF is dissected into three lattice triangles using p as a boundary point, we can ensure each of the formed triangles have less than k interior points. Therefore by our inductive hypothesis each triangle can be dissected into primitive lattice triangles. Thus all of DEF can be dissected into primitive lattice triangles.

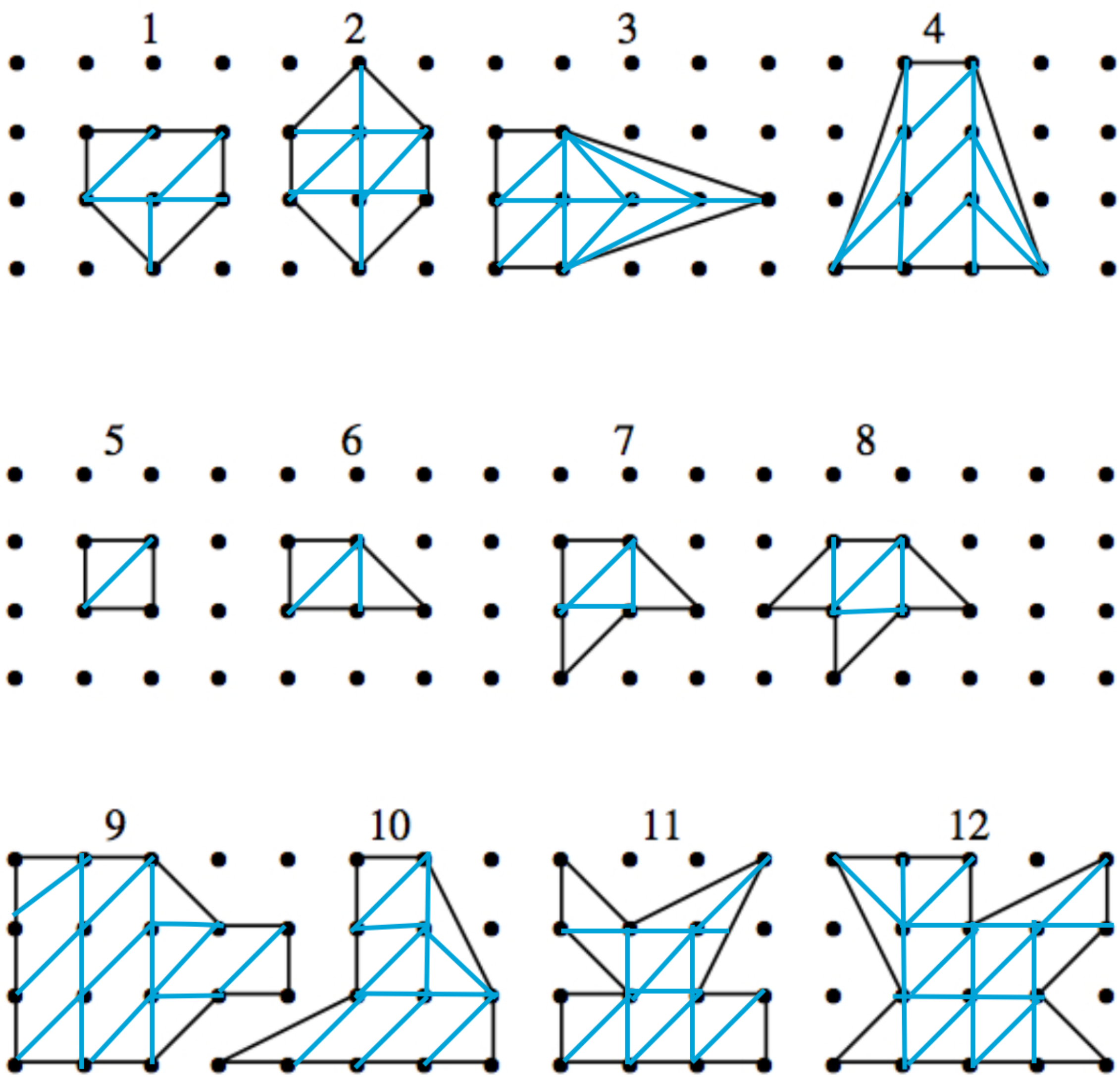


Thus by mathematical induction we have shown any lattice triangle can be dissected into primitive lattice triangles. ▮

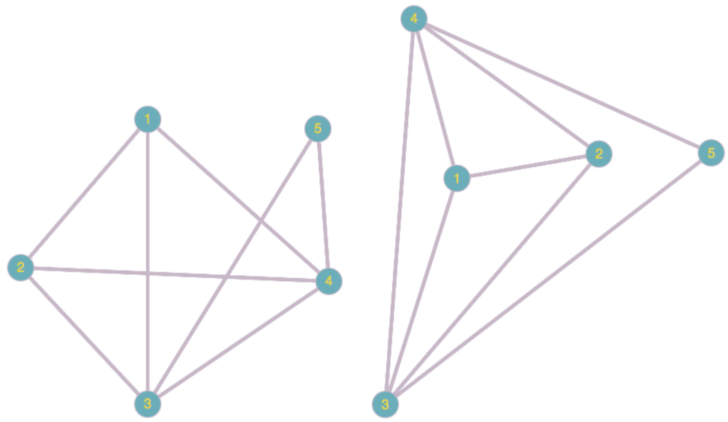
56) Prove Theorem. Every lattice polygon can be dissected into primitive lattice triangles.

Let P be a lattice polygon with n vertices. By exercise 54, P can be dissected into n-2 triangles where we know they are lattice since they are formed by the vertices of a lattice polygon P. Then by exercise 55, we can dissect any lattice triangle into primitive lattice triangles. Thus every lattice polygon can be dissected into primitive lattice triangles. ▮

57) Dissect the twelve lattice polygons into primitive lattice triangles.

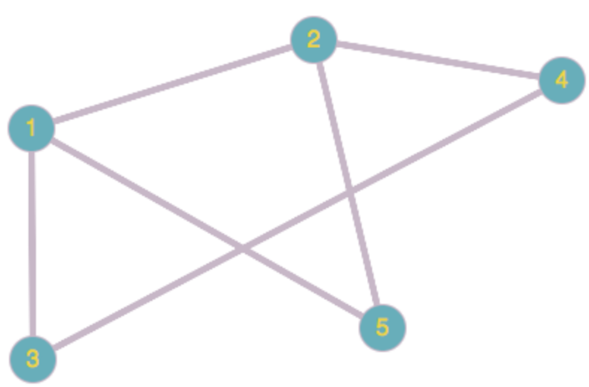


58) Show that the two drawings actually represent the same graph.

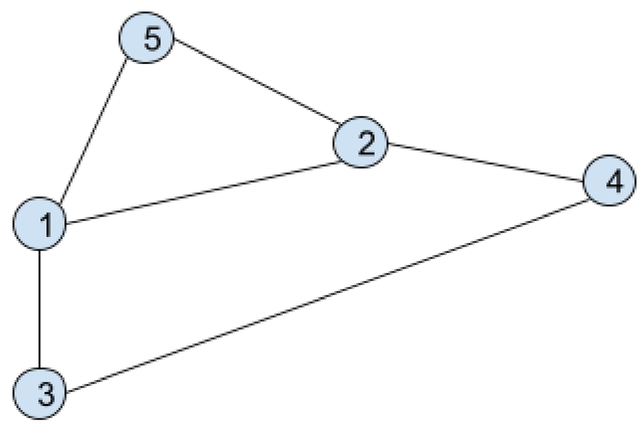


We can show these are the same graph by looking at their vertex and edge sets. For the left representation the vertex set V = {1, 2, 3, 4, 5} and the edge set E = {(1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (3,5), (4,5)}. Then looking at the right representation we see that V and E for this graph are the same at the first. Thus by definition of a graph they are the same.

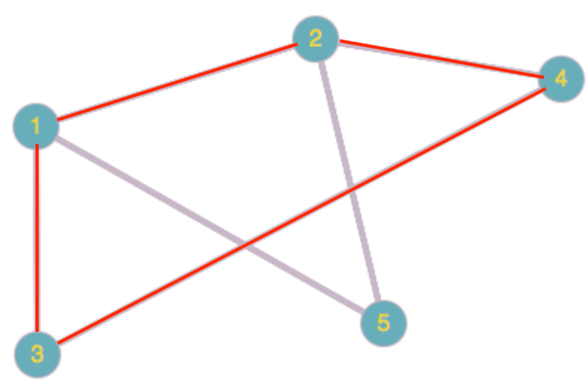
59) Show that the graph shown below is a planar graph.



A graph is planar if it can be drawn such that no two edges cross. Therefore we can see from our drawing that if vertex 5 was moved above vertices 1 and 2 then it would not cross any edge and thus is planar. Note there are many ways to show this graph is planar, but here is what the representation we talked about would look like.



60) Show that the graph illustrated contains a circuit.



A circuit in a graph is a path that starts and ends at the same vertex. So we can find the circuit on this graph as {1, 2, 4, 3, 1} as highlighted in red.

61) Let v denote the number of vertices and let e denote the number of edges of G. Prove that if G is a tree, then e = v − 1.

We will prove by induction on the number of vertices.

Base Case:

Let G be a single vertex. Then G is connected and has no cycles, so G is a tree. Then e = 0 and v = 1. Thus 0 = 1 - 1 = 0 so e = v - 1 holds for our base case.

Inductive Hypothesis:

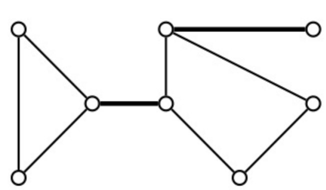
If G is a tree with v vertices and e edges, then e = v - 1 for all G such that 1 ≤ v ≤ n.

Inductive Step:

Let G be a graph with n vertices and e edges. We will remove one edge from G. This splits G into two graphs; call them G1 and G2 with n1 and n2 vertices in each respectively. Then G1 and G2 are locally connected and contain no cycles. We have n = n1 + n2 thus n1, n2 < n, so by hypothesis G1 has n1 - 1 edges and G2 has n2 - 1 edges. Now that we know the number of edges for the two subgraphs so we can add the two together and connect them by the edge we took out. This leaves us with e = n1 - 1 + n2 - 1 + 1 = (n1 + n2) - 1 = n - 1 = v - 1.

Thus by mathematical induction we have shown that if G is a tree then e = v - 1. ∎

62) For each of the graphs below, find the number of vertices v, edges e, and regions f. Then compute v − e + f.



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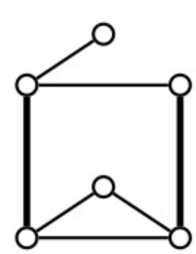

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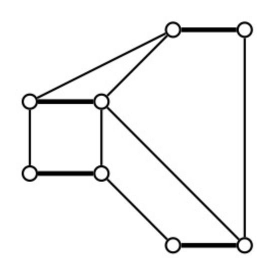

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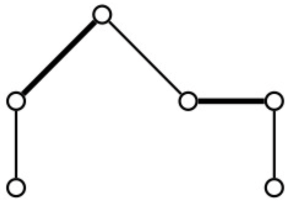

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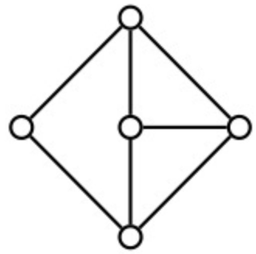

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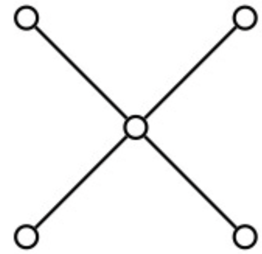

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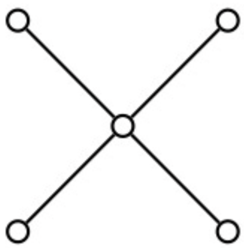

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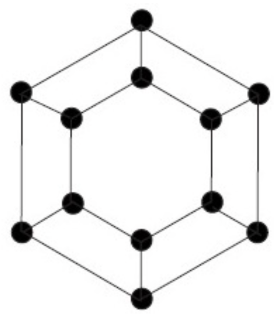

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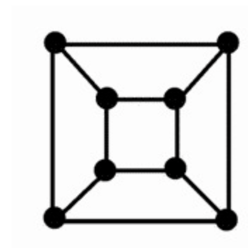

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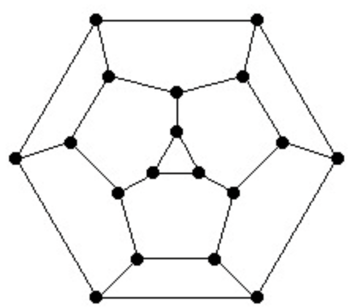


10.

Graph	Vertices	Edges	Regions	V - e + f
1	8	9	3	2
2	6	7	3	2
3	8	11	5	2
4	6	5	1	2
5	5	7	4	2

6	5	4	1	2
7	6	8	4	2
8	12	18	8	2
9	8	12	6	2
10	18	27	11	2

63) Prove Euler's Formula. If G is a connected, planar graph with v vertices, e edges, and f regions, then: $v - e + f = 2$.

We will prove by induction on the number of edges.

Base Case:

Let G be a single vertex. Then G is planar since no crossing edges, as there are none. So, $v = 1$, $e = 0$, and $f = 1$. So $1 - 0 + 1 = 2$, thus $v - e + f = 2$ holds for our base case.

Inductive Hypothesis:

Let $v - e + f = 2$ for all planar graphs with k edges.

Inductive Step:

Now we will show that the addition of a new edge will keep the same property, giving us the $k + 1$ case. We are faced with two cases. For our first case we say the addition of an edge creates a circuit. In this case, along with the addition of an edge we also add a new region by the creation of a circuit. So we get the equation,

$v - (e + 1) + (f + 2) = v - e + f + (1 -1) = v - e + f = 2$, as we were hoping.

For our second case we say the addition of an edge does not cause a circuit to be made. Thus in this case, along with the addition of an edge we also need to add a new vertex. So we get the equation $(v + 1) - (e + 1) + f = v - e + f + (1 -1) = v - e + f = 2$, again as we hoped.

Thus by mathematical induction we have shown that if G is connected and planar with v vertices, e edges, and f regions, then $v - e + f = 2$. ▮

64) For each of the polygons in Figure 3, use your dissection into primitive lattice triangles from Exercise 57 to construct the graph G. Let ei denote the number of edges of G inside the polygon P and let eb denote the number of edges of G that are on the boundary of the original polygon P . Finally, compute the quantity $2v - eb - 1$. Complete the following table. What do you observe?

Polygon	Area of P	f = # of regions of G	2v - eb - 1
1	3	7	7
2	4	9	9
3	5	11	11
4	6	13	13
5	1	3	3
6	1.5	4	4
7	2	5	5
8	2.5	6	6
9	9	19	19
10	6	13	13
11	6	13	13
12	8.5	18	18

I notice through the table that the number of regions is the same as the expression $2v - eb - 1$

65) Use Exercise 53 to state and prove an equation that relates the area of P to f .

$A(P) = 1/2(f - 1)$

We know that there are f - 1 primitive lattice triangles in the dissection of P. Then by exercise 53, we found that the area of a primitive lattice triangle is 1/2. So if there are f - 1 primitive lattice triangles with area 1/2, then $A(P) = 1/2(f - 1)$. ▮

66) The next step in the proof of Pick's Theorem is to compute e and relate it to f. Let ei denote the number of edges of G inside the polygon P and let eb denote the number of edges of G that are on the boundary of the original polygon P. Show that $f = 2v - eb - 1$.

Consider a polygon P. Then by exercise 56, we can dissect any polygon into primitive lattice triangles. In this dissected, planar version of P we get f - 1 primitive lattice triangles inside P. Then since each triangle has three edges, we get the equation $E = 3(f - 1)$. However, E represents all the edges of P where interior edges are double counted. So we can also write the equation $E = 2ei + eb$. Manipulating this we can come to the equation $E = 2(ei + eb) - eb$ and since the total number of edges is equal to the sum of the interior and boundary edges, $ei + eb = e$. Therefore we get $E = 2e - eb$. Now we have two expressions that represent E, so we can set them equal to each other and solve for f to try and match our desired form. To start, $3(f - 1) = 2e - eb$. Then distributing the three and isolating f we find the equation $f = 2(e - f) - eb + 3$. Next by Euler's Formula, from exercise 63, $v - e + f = 2$ or rewritten, $e - f = v - 2$. So plugging into our equation we get $f = 2(v - 2) - eb + 3$. Distributing out, $f = 2v - eb - 4 + 3$. Now simplifying, $f = 2v - eb - 1$. Thus we have proven that for a lattice polygon P, dissected into primitive lattice triangles, the number of regions $f = 2v - eb - 1$. ▮

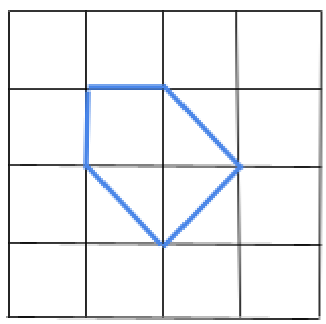
67) Prove Pick's Theorem: $A(P) = 1/2B(P) + I(P) - 1$.

Let P be a lattice polygon and let $G = (V, E)$ be the graph obtained by the dissection on P into primitive lattice triangles. Then G is planar by construction. Let $|E| = e$, $|V| = v$, and f be the number of regions in G. By exercise 65, $A(P) = 1/2(f - 1)$. Let ei denote the number of interior edges and eb the number of boundary edges. Then $e = ei + eb$ and by exercise 66,

$f = 2v - eb - 1$. Then we know $V = I(P) + B(P)$ since we are constructing our primitive lattice triangles with the interior and boundary lattice points of P. Then eb alone represents B(P), as each boundary edge connects to one of the boundary lattice points. Then we can plug into the equation, thus $f = 2(I(P) + B(P)) - B(P) - 1$. Now we have f in terms of interior and boundary points of P, so we can plug into our area formula, $A(P) = 1/2(f - 1)$, to get the equation $A(P) = 1/2(2(I(P) + B(P)) - B(P) - 1 - 1) = 1/2(2I(P) + B(P) -2) = I(P) + 1/2B(P) - 1$, as desired. Thus we have proven Pick's Theorem through graph theory. ▌

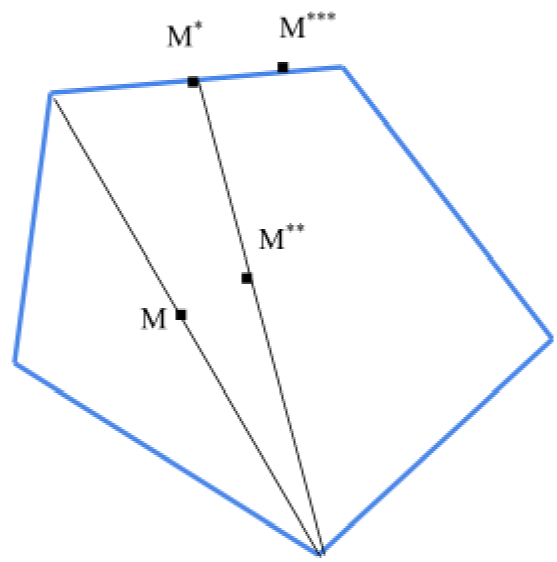
82) Show that if P is a convex lattice pentagon, then the area of P must be greater than or equal to 5/2. Is this bound strict? In other words, is it possible to construct a convex lattice pentagon with area equal to 5/2?

To start let us show there does exist a lattice pentagon with area 5/2.



We can see above that this pentagon has 5 boundary points and 1 interior point, thus by Pick's Theorem, $A(P) = 5/2$. Also note that by Pick's Theorem, we can reach an area of 5/2 by having 7 boundary points and zero interior points, but no other combination. Now we will show that these are the smallest possible cases. First note that in any lattice pentagon there must be at least 5 boundary points to represent the 5 vertices of the pentagon. Additionally, any lattice point can only be in one of the four forms: (even, even), (even, odd), (odd, even), and (odd, odd). Therefore by the pigeonhole principle, there must be at least 2 vertices of P that have the same pairing. Let $M = (a_i + a_j / 2 , b_i + b_j / 2)$ be the midpoint between any two vertices, a and b, of P. Then we know that M is a lattice point if and only if both $a_i + a_j$ and $b_i + b_j$ are even. Then this only happens if a and b are the same pairing of even or odd integers, and by pigeonhole principle there exists an a and b with this property. Now we have two cases, the vertices are adjacent or nonadjacent.

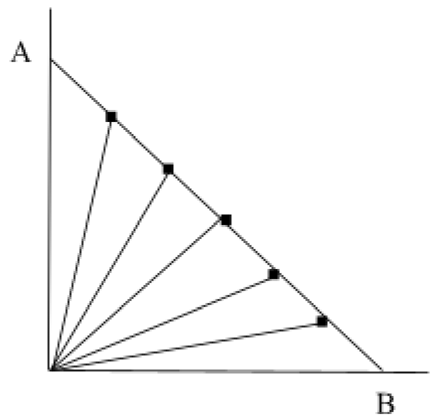
If the vertices are nonadjacent, then there exists a lattice point M within P, thus $I(P) \geq 1$. Then since all other pairings of vertices have different forms of even and odd integers, there are no other boundary points. Thus, if there are 5 boundary points there must exist 1 interior point, creating a lower bound of 5/2 in this case.



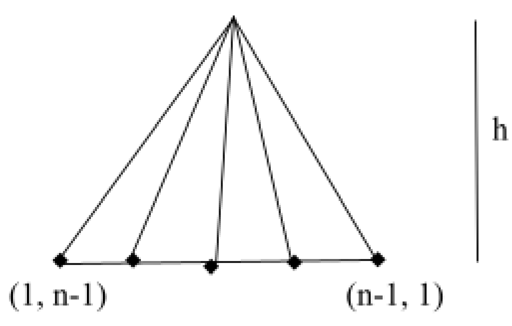
Now if the vertices are adjacent, then there exists a lattice point M^* on the boundary of P, thus $B(P) \geq 6$. But now our new point must have the same lattice point form as another vertex again by pigeonhole principle. If this point is nonadjacent to M^* that would imply there exists a lattice point M^{**} within P thus $I(P) \geq 1$. However, by Pick's Theorem $A(P) = 6/2 + 1 - 1 = 3$. Thus we disregard this case as it is greater than a lower bound we have already found. In the other case, if the vertices are adjacent, then there exists a lattice point M^{***} on the boundary of P, thus $B(P) \geq 7$. So again by Pick's Theorem $A(P) = 7/2 + 0 - 1 = 5/2$, creating a lower bound.

Therefore there is no way to construct a lattice pentagon with an area less than 5/2. ∎

83) Let A denote the point (n, 0) and let B denote the point (0, n). There are n − 1 lattice points, each of the form (i, n−i), for i=1,2,3,...,n−1,between A and B. Connect each one of them with the origin O(0, 0). The lines divide △OAB into n small triangles. It is clear that the 2 triangles next to the axes (i.e. the triangle adjacent to the x-axis and the triangle adjacent to the y-axis) contain no lattice points in their interior. Prove that if n is prime, then each of the remaining triangles contains exactly the same number of interior lattice points. Find an expression (in terms of n) for the number of interior lattice points in each of these triangles.



Let (k, n-k), k = 1, 2, 3, ..., n-1 represent the lattice points on the line segment AB. By exercise 30, there are gcd(k, n-k) - 1 lattice points on any line from the origin to a lattice point. Let gcd = d. Then d divides k and n-k, thus there exist integers z1 and z2 such that dz1 = k and dz2 = n - k. By substitution, dz2 = n - dz2 = d(z1 + z2) = n. Then since z1 and z2 are integers and d divides n, with n being prime, we are left with two possibilities; d =1 or d = n. Right away we can see d cannot equal n since k > 0, so n can't divide a smaller number n - k. Therefore d = 1, so the number of lattice points on each line from the origin to each lattice point is zero. Now we want to show the area of each triangle to show they have the same number of interior points.



By our construction, we know that each base will have a length of $\sqrt{2}$. By our diagram we can see that each triangle also has the same height of h, thus they all have the same area. Then since the areas and the number of boundary points are all the same, by Pick's Theorem the number of interior points must also be the same. Thus all the triangles have the same number of interior points. ∎

Now we will make an expression for the number of interior points on each triangle. We will do this by finding the total boundary points and area of the whole triangle and breaking it down to smaller cases. First notice that on the line segment A0 and B0 the number of lattice points are gcd(0, n) -1 = n-1. Then on the segment AB there are also n-1 lattice points, by construction. Therefore the total number of boundary points for our triangle is $B(T) = 3(n -1) + 3$, where we add three to account for our three vertices that were not included. Manipulating this we get that $B(T) = 3n$. Then by the definition of the area of a triangle, $A(T) = 1/2(b)(h)$ but base and height are the same with length n. So $A(T) = 1/2(n^2)$. Now using Pick's Theorem, $n^2/2 = 3n/2 + I(T) -1$, so $I(T) = n^2/2 - 3n/2 + 1$. Then we can rewrite this as $2I(T) = n^2 -3n + 1$, and by factorization $2I(T) = (n - 2)(n - 1)$. Therefore, $I(T) = (n - 1)/2 (n - 2)$. Then note that we have n-2 total interior triangles so we can conclude that each triangle has $(n-1)/2$ interior points.

84) Let n be an integer greater than or equal to 3. Prove that there is a set of n points in the plane such that the distance between any 2 points is irrational and each set of three points determines a non-degenerate triangle with rational area.

Let $C = \{(k, k^2) \mid k \geq 0, k \in Z\}$, and note $k \geq 0$ so that we don't have rational distance across the y-axis. Now we need to show any two points in out set S are irrational, call them $A = (a, a^2)$ and $B = (b, b^2)$. Then using the distance formula $d = \sqrt{((b - a)^2 + (b^2 - a^2))}$. Expanding, we get $d = \sqrt{((b - a)^2 + (b - a)^2 + (b + a)^2)}$. Therefore $d = |(b - a)|\sqrt{(1 + (b + a)^2)}$. Then the sum inside of the square root is one away from a perfect square, but by the nature of the growth of perfect squares there are no two perfect squares that are 1 apart. Thus the square root function produces an irrational number causing the distance to be irrational. Then since all points in C are lattice points, the area will be rational by Pick's Theorem. Thus there is a set of n points such that the distance between any 2 points is irrational and the area is rational. ▮

85) Show that if T is a lattice triangle with I(T) = 1, then B(T) = 3, 4, 6, 8, or 9.

Using Pick's Theorem, A(T) = B(T)/2 for the case of I(T) = 1. Without loss of generality, let the vertices of T be (0, 0), (a1, a2), and (b1, b2). Let a = gcd(a1 - 0, a2 - 0), b = gcd(b1 - 0, b2 - 0), and c = gcd(b1 - a1, b2 - a2) representing the gcd values on each of the three line segments of the triangle T. So by exercise 31, B(T) = a + b + c.

Lemma 1: If T is a lattice triangle with I(T) = 1, then ab, ac, and bc are divisors of B(T).

We will compute the area in two different ways. The first is by Pick's theorem, which we have already shown is A(T) = B(T)/2. Then using geometry we have that A(T) = 1/2|a1b2 - a2b1|. Manipulating we get A(T) = (ab)/2|a1b2/ab - a2b1/ab|, and this gives us A(T) = (abk)/2 for some integer k. Thus, B(T)/2 = (abk)/2; so ab divides B(T). Then by a similar argument we can show ac and bc also divide B(T).

Lemma 2: If T is a lattice triangle with I(T) = 1, then B(T) divides 6, 8, or 9.

By exercise 30, B(T) = a + b + c. Without loss of generality, let us assume that a ≥ b ≥ c. Then, a ≥ B(T)/3. But from lemma 1, a divides B(T), so a = B(T)/2 or a = B(T)/3. We can now list the possible values of a, b, and c.

(a, b, c) = (B(T)/2, B(T)/3, B(T)/6) or (B(T)/2, B(T)/4, B(T)/4) or (B(T)/3, B(T)/3, B(T)/3). Then since ab divides B(T), the integers $(B(T))^2/6$, $(B(T))^2/8$, and $(B(T))^2/9$ divide B(T). Thus, B(T) divides 6, 8, and 9.

Thus we can conclude that B(T) = 3, 4, 6, 8, or 9. ▮

86) Find Farey Sequence for F6 and F7.

F6 = {0/1, 1/6, 1/5, 1/4, 1/3, 2/5, 1/2, 3/5, 2/3, 3/4, 4/5, 5/6, 1/1}

F7 = {0/1, 1/7, 1/6, 1/5, 1/4, 2/7, 1/3, 2/5, 3/7, 1/2, 4/7, 3/5, 2/3, 5/7, 3/4, 4/5, 5/6, 6/7 1/1}

87) Properties of Farey Sequences. Prove each of the following statements. (a) Fn contains Fk for all $k \leq n$. (b) Let |Fn| denote the number of fractions in Fn. For $n > 1$, |Fn| is odd.

a) Consider Fk such that $k \leq n$. Then by definition, Fk is a sequence of fractions of the form a/b $0 \leq a \leq b \leq k$ with: ① gcd(a, b) = 1, ② $0 \leq a/b \leq 1$, and ③ $b \leq k$. Then since $k \leq n$, it follows that $0 \leq a \leq b \leq n$ and still follows the three conditions but with the third being that $b \leq n$, so Fn contains Fk.

b) Let a/b ∈ Fn, then gcd(a, b) = 1, $0 \leq a/b \leq 1$, and $b \leq n$. Let $n > 1$. We can form pairs a/b and 1-a/b = (b-a)/b. Suppose gcd(b-a, b) = c, then c divides b-a and c divides b, so there exist integers x and y such that cx = b - a and cy = b. Using substitution, cx = cy - a. So by a rearrangement a = c(y - x). Therefore c divides a. Since c divides a and c divides b the c must equal 1 since gcd(a, b) = 1. Then gcd(b - a, b) = 1. It follows since $0 \leq a/b \leq 1$ then $0 \leq (b-a)/b \leq 1$. So since $b \leq n$ all conditions of a Farey sequence are met, thus (b - a)/b ∈ Fn. So we have shown that for every rational a/b in the Farey sequence there is a matching element 1 - a/b. However looking at the case a/b = 1/2, (b - a)/b = 1/2. So, 1/2 is its own pair which lets us say there are an odd number of elements in a Farey sequence. Now since 1/2 is a part of F2, by part a 1/2 is a part of every Farey sequence with $n > 1$. Thus all Farey sequences have an odd cardinality. ∎

88) Show that |Fn| = |Fn−1| + φ(n), where φ(n) denotes the number of positive integers less than or equal to n that are relatively prime to n.

We know from exercise 87 part a that Fn−1 is contained in Fn, so the only elements that are not in Fn are the ones unique to Fn. By definition of the Farey sequence, an element has to be in the form a/b where a and b are relatively prime. Then since all of the cases where elements are simplified to a smaller denominator are covered, the new elements will be the pairs where gcd(a,n) = 1 for a = 0,1,....,n. Then this is the exact definition of $\varphi(n)$.

Thus $|Fn| = |Fn-1| + \varphi(n)$. ▮

89) The Mediant Property. Unfortunately, addition of fractions is not as easy as we would like it to be. For example, (a) Looking at the Farey sequences F4 and F5, how does 1/4 relate to 1/5 and 1/3?

They are all consecutive in the sequence and 1+1/3+5 = 2/8 = 1/4

(b) Can you find other Farey sequences in which you observe this phenomena? In particular, choose a Farey sequence Fn and choose 3 consecutive terms of Fn, say p1/q1, p2/q2, p3/q3. Compute (p1 + p3) /(q1 + q3) What do you observe?

Yes looking at F7 we can take the three consecutive terms 1/4, 2/7, 1/3. Notice then that 1+1/4+3 = 2/7 which is the median of the three.

90) (a) The fractions 2/5 and 3/7 are adjacent terms of the Farey sequence F7. Compute $5 \cdot 3 - 2 \cdot 7$.

15 - 14 = 1.

(b) Choose two other adjacent terms p1/q1 and p2/q2 of F7 and compute p2q1 − p1q2.

3/4 and 4/5, then (4)(4) - (3)(5) = 16 - 15 = 1.

(c) Choose two adjacent terms p1/q1 and p2/q2 of F5 and compute p2q1 − p1q2.

1/2 and 3/5, then (3)(2) - (1)(5) = 6-5 = 1

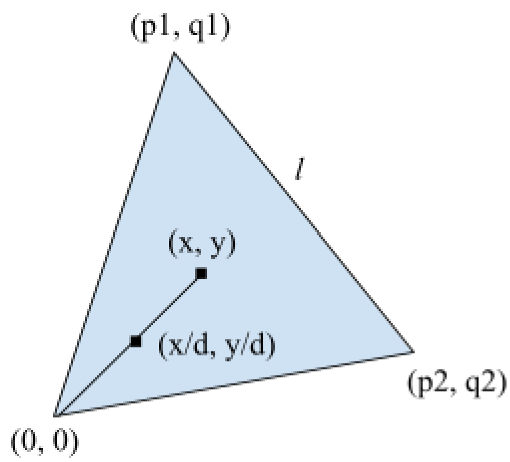
(d) Suppose that p1/q1 and p2/q2 are two successive terms of a Farey sequence Fn. Make a conjecture about the value of p2q1 − p1q2. We will use Pick's Theorem to prove this conjecture!

If $p1/q1$ and $p2/q2$ are two consecutive terms, then $p2q1 - p1q2 = 1$.

91) Suppose that $p1/q1$ and $p2/q2$ are two successive terms of Fn. In this problem, we will use Pick's Theorem to prove that $p2q1 - p1q2 = 1$. Let T be the triangle with vertices (0, 0), (p1, q1), and (p2, q2).

(a) Show that T has no lattice points in its interior, i.e. $I(T) = 0$.

Suppose $p1/q1$ and $p2/q2$ are adjacent in Fn. Then the following are true: $p1/q1 < p2/q2$, $\gcd(p1, q1) = 1$, $\gcd(p2, q2) = 1$, $0 \leq p1/q1 < p2/q2 \leq 1$, and $q1, q2 \leq n$. Then let T be a triangle with the vertices (0, 0), (p1, q1), and (p2, q2). Then since we know the slopes of the points, $q1/p1 > q2/p2$ we can accurately draw our triangle.



Now suppose there exists an interior lattice point (x, y). Then, looking at slopes, $q2/p2 < y/x < q2/p2$. By the reciprocal, we get $p1/q1 < x/y < p2/q2$. Now if x/y is an element of the Farey sequence then we have found a contradiction as we already assumed $p1/q1$ and $p2/q2$ are adjacent. By property 1, $0 \leq x/y \leq 1$ because $x/y < p2/q2 < 1$ and $0 \leq p1/q1 < x/y$. By property 2, $y \leq n$ because $y < q1 \leq n$. Then all that remains is property 3. If $\gcd(x, y) = 1$ then x/y is an element of Fn and $p1/q1 < x/y < p2/q2$ a contradiction. Else if $\gcd(x, y) = d \neq 1$, then the point $(x/d, y/d)$ is still in the interior and $\gcd(x/d, y/d) = 1$. Therefore, $(x/d)/(y/d)$ is an element of Fn and is in between $p1/q1$ and $p2/q2$, a contradiction. Thus There are no interior lattice points in T. ▮

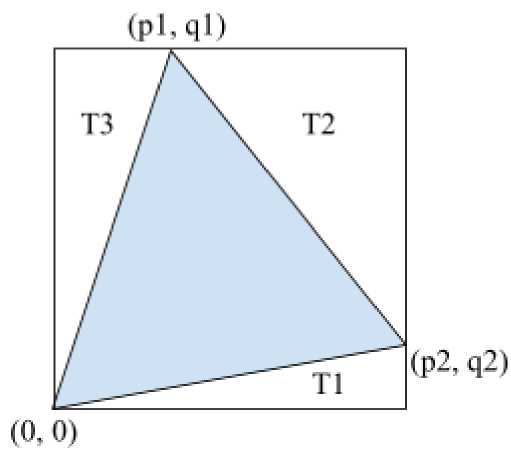
(b) Show that the only boundary points of T are the vertices of the triangle, i.e. $B(T) = 3$.

Since $\gcd(p1, q1) = \gcd(p2, q2) = 1$ and the points extend off of the origin, by exercise 30 there are no boundary lattice points on the line segments from the origin to these points. Then we must show there are no boundary points on the line segment from (p1, q1) to (p2, q2), call it *l*. Suppose there is a boundary lattice point (j, k) on *l*. Then $q2/p2 < k/j < q1/p1$, and reciprocating $p1/q1 < j/k < p2/q2$; a contradiction if j/k is an element of Fn. Similar to part a's explanation, properties 1 and 2 come from the same explanation. Suffices to show that if $\gcd(j, k) = 1$ we have a contradiction. But we have already shown there are no interior lattice points, so (j, k) must be a visible point. Therefore by definition $\gcd(j, k) = 1$, a contradiction since p1/q1 and p2/q2 are consecutive. Thus the only boundary lattice points of T are the vertices, so $B(T) = 3$. ▮

(c) Conclude, using Pick's Theorem, that $A(T) = 1/2$.

$A(T) = B(T)/2 + I(T) - 1$. So, $A(T) = 3/2 + 0 - 1 = 1/2$.

(d) Use geometry to show that $A(T) = 1/2\,(p2q1 - p1q2)$.



Enclosing T in a bounding rectangle lets us use the formula $A(T) = bh - T1 - T2 - T3$.
Then, $bh = (p2)(q1)$, $T1 = 1/2(p1)(q1)$, $T2 = 1/2(p1 - p2)(q1 - q2)$, $T3 = 1/2(p2)(q2)$.
Putting it all together $A(T) = (p2)(q1) - 1/2(p1)(q1) - 1/2(p1 - p2)(q1 - q2) - 1/2(p2)(q2)$.
So, $A(T) = 1/2(p2q1 - p1q2)$.

(e) Conclude that $p2q1 - p1q2 = 1$.

Now we have two equations for A(T). So, $1/2 = 1/2(p2q1 - p1q2)$. SImplifying we get the desired result $p2-q1 - p1q2 = 1$.

92) Prove that if $0 < a/b < c/d < 1$, then $a/b < a+c/b+d < c/d$.

First note that a, b, c, and d are all positive and greater than zero or else our original statement $0 < a/b < c/d < 1$ couldn't be true. We will show the result in two steps. First we will show $a/b < a+c/b+d$. Start with $a/b < c/d$ and cross-multiply to get $ad > bc$. Next add ab to both sides, $ad + ab < bc + ab$, and notice we can factor a term on both sides leaving us with, $a(b + d) < b(a + c)$. Then by rearranging $ab < a+c/b+d$ as desired. Now for the other side of the inequality $a+c/b+d < c/d$. Start with $a/b < c/d$ giving us $ad < bc$ again. Now we will add dc to both sides to get $ad + dc < bc + dc$. Notice again we can factor to get $d(a + c) < c(b + d)$. Rearranging we get the result $a+c/b+d < c/d$. Putting the two together we get the result $a/b < a+c/b+d < c/d$. ∎

93) Prove that if a/b and c/d are adjacent in some Fn , then $\gcd(a + c, b + d) = 1$.We thus have the following algorithm for computing Fn using Fn−1:
1) Copy Fn−1 in order.
2) Insert the mediant fraction a+c between a and c if $b+d \leq n$. (If $b+d > n$, the mediant a+c will appear in a later sequence).

Suppose a/b and c/d are adjacent in Fn. Then we know $a/b < c/d$. By exercise 91, $bc - ad = 1$. Then by Bezout's lemma, if there exist integers x and y such that $x(a + c) + y(b + d) = 1$ then the $\mathrm{gdc}(a+c, b+d) = 1$. So, let $x = b$ and $y = -a$. By Bezout let us look at $b(a+c) - a(b+d)$. Distributing we get the expression $ab + bc - ab - ad$. Simplifying we can get rid of the ab terms leaving us with $bc - ad$. But then by exercise 91, $bc - ad = 1$; so we found an x and y such that $x(a + c) + y(b + d) = 1$. Thus by Bezout's lemma, we get $\gcd(a+c, b+d) = 1$. ∎

94) Use Algorithm to compute F8 using F7.

Given F7 = {0/1, 1/7, 1/6, 1/5, 1/4, 2/7, 1/3, 2/5, 3/7, 1/2, 4/7, 3/5, 2/3, 5/7, 3/4, 4/5, 5/6, 6/7, 1/1}. We will go through the first couple of terms of F8. First take the first two elements of F7, 0/1 and 1/7. Then add the numerators and denominators, we get 0+1/1+7 = 1/8. Then since $8 \le n = 8$ we will include this in F8. Now look at the second and third terms of F7, 1/7 and 1/6. By the same technique we get 1+1/6+7 = 2/13, but $13 > n = 8$ so this term is not included. Continuing in this fashion we get new terms of 1/8, 3/8, 5/8, and 7/8. Thus, F8 = {0/1, 1/8, 1/7, 1/6, 1/5, 1/4, 2/7, 1/3, 3/8, 2/5, 3/7, 1/2, 4/7, 3/5, 5/8, 2/3, 5/7, 3/4, 4/5, 5/6, 6/7, 7/8, 1/1}.

95) Without listing out all of the fractions in F100, find the fraction a/b immediately before and the fraction c/d immediately after 61/79 in F100.

By exercise 92, 0 < a/b < a+c/b+d < c/d if 0 < a/b < c/d < 1. So we have that a+c = 61 and b+d = 79. Then we can use the division algorithm to deconstruct 79 to smaller factors.
79 = (61)(1) + 18, 61 = (18)(3) + 7, 18 = (7)(2) + 4, 7 = (4)(1) + 3, 4 = (3)(1) + 1. Now we have a chain of equations that lead to 1. So we can reconstruct these equations to get a form for Bezout's lemma 1 = x(b+d) + y(a+c) to conclude gcd(a+c, b+d) = 1 showing these are the correct boundary elements in F100. Start with 1 = 4 - 3 and we will continue by substituting equivalent expressions into the equation.

$1 = 4 - 3 \;\Rightarrow\; 1 = 4 - (7 - 4) \;\Rightarrow\; 1 = 2(4) - 7 \;\Rightarrow\; 1 = 2(18 - (7)(2)) - 7 \;\Rightarrow$

$1 = (2)(18) - (5)(7) \;\Rightarrow\; 1 = (2)(18) - (5)(61 - (18)(3)) \;\Rightarrow\; (17)(18) - (5)(61) \;\Rightarrow$

$1 = (17)(79 - 61) - (5)(61) \;\Rightarrow\; 1 = (17)(79) - (22)(61)$. Then since 61/79 and c/d are consecutive, by Bezout's lemma 61d + 79c = 1. Therefore we have found a c and d that satisfy this. Thus the fraction after 61/79 = 17/22. Then we know a+c = 61 and b+d = 79 so we can find that a = 44 and b = 57. Therefore the consecutive sequence we find in F100 is 44/57, 61/79, 17/22.