1. Find the second order Taylor expansion about the point $(0, 0)$ of the function

$$f(x, y) = e^{xy}$$

We begin by computing the matrix of partial derivatives of f.

$$Df(x, y) = (e^{xy}y, e^{xy}x)$$

From this we compute the Hessian matrix

$$Hf(x, y) = \begin{pmatrix} e^{xy}y^2 & e^{xy} + e^{xy}xy \\ e^{xy} + e^{xy}xy & e^{xy}x^2 \end{pmatrix}$$

Then we evaluate at the point $(0, 0)$ and find

$$Df(0, 0) = (0, 0)$$
$$Dg(0, 0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Now we put these together to compute the degree 2 Taylor polynomial

$$T_2(x, y) = f(0, 0) + Df(0, 0) \cdot (x, y) + \frac{1}{2}(x, y)Hf(0, 0)\begin{pmatrix} x \\ y \end{pmatrix}$$

Expanding out the linear algebra we obtain

$$T_2(x, y) = 1 + xy$$

Which is the degree 2 Taylor polynomial about $(0, 0)$ of the function $f(x, y) = e^{xy}$

2. Find and classify all critical points of the function

$$f(x, y) = x^3 + x^2y + y^2 + xy + x + 1$$

As in the last problem we will begin by computing both the matrix of partial derivatives and the Hessian matrix.

$$Df(x, y) = (3x^2 + 2xy + y, x^2 + 2y + x)$$
$$Hf(x, y) = \begin{pmatrix} 6x + 2y & 2x + 1 \\ 2x + 1 & 2 \end{pmatrix}$$

Now to find the critical values we compute where $Df(x, y) = (0, 0)$

We easily find the following points $(0, 0)$, $(1, -1)$, and $(1/2, -3/8)$. Now using our criteria on each of these points we find that the points $(0, 0)$ and $(1, -1)$ are both saddle points, while $(1/2, -3/8)$ is a local minimum.

3. Compute the matrix of partial derivatives of $f \circ g$ at the point $(0,0)$ where

$$f(x,y) = (x^2 + y^2, x - y)$$
$$g(x,y) = (e^x - 3, 2y + 1)$$

This is a chain rule problem. We begin by computing Df and Dg

$$Df = \begin{pmatrix} 2x & 2y \\ 1 & -1 \end{pmatrix}$$
$$Dg = \begin{pmatrix} e^x & 0 \\ 0 & 2 \end{pmatrix}$$

We then recall where we must evaluate the matrix of partial derivatives. As we are computing $f \circ g$ we evaluate

$$g(0,0) = (-2, 1)$$
$$Dg(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

We then compute $Df(g(0,0))$

$$Df(-2,1) = \begin{pmatrix} -4 & 1 \\ 1 & -1 \end{pmatrix}$$

Now the chain rule tells us that $D(f \circ g) = Df(g(0,0)) \cdot Dg(0,0)$ which we can easily compute

$$\begin{pmatrix} -4 & 2 \\ 1 & -2 \end{pmatrix}$$

4. Using Green's Theorem, compute the area of the ellipse

$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$

To use green's theorem we will replace the double integral with a line integral over the function $\mathbf{F} = (-y/2, x/2)$, so that we are finding area. We then must parameterize the boundary of the ellipse, which we can easily do by modifying the equations of circles.

$$c(t) = (5\cos(t), 2\sin(t)) \qquad \text{for } 0 \le t \le 2\pi$$

We then can setup and compute the line integral

$$\frac{1}{2}\int_0^{2\pi}(-2\sin(t),5\cos(t))\cdot(-5\sin(t),2\cos(t))dt = \frac{1}{2}\int_0^{2\pi}10\sin^2(t)+10\cos^2(t)dt$$
$$= \frac{1}{2}\int_0^{2\pi}10dt$$
$$= 10\pi$$

5. Using Stokes's Theorem, compute the value of the line integral

$$\oint_C \mathbf{F} d\mathbf{S}$$

Where $\mathbf{F}(x,y,z) = (\tan(x^2+x, y-2yz, \cos(z^4))$ and C is the boundary of the region $z^2 = x^2+y^2$ above $z=0$ and below $z=1$ (with upward facing normal vector).

We will first parameterize the surface S as

$$s(r,\theta) = (2r\cos(\theta), 3r\sin(\theta), r) \quad \text{for } 0 \le r \le 1 \text{ and } 0 \le \theta \le 2\pi$$

We will also take a second and compute a normal vector to the surface, anticipating its use later.

$$\operatorname{Curl}\mathbf{F} = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{j} \\ 2\cos(\theta) & 3\sin(\theta) & 1 \\ -2u\sin(\theta) & 3r\cos(\theta) & 0 \end{pmatrix} = (-3r\cos(\theta), -2u\sin(\theta), 6r)$$

Using Stokes's theorem we can convert the integral over the boundary to a integral over the entire surface by changing the integrand into the curl. Lets compute Curl $\mathbf{F}$

$$\operatorname{Curl}\mathbf{F} = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \tan(x^2+x & y-2yz & \cos(z^4) \end{pmatrix} = (y^2, 0, 0)$$

Then we can easily setup the surface integral over Curl $\mathbf{F}$

$$\int_0^{2\pi}\int_0^1 (9r^2\sin^2(\theta),0,0)\cdot(-3r\cos(\theta),-2u\sin(\theta),6r)drd\theta = \int_0^{2\pi}\int_0^1(-27r^3\sin^2(\theta)\cos(\theta)drd\theta$$
$$= -27\int_0^{2\pi}\sin^2(\theta)\cos(\theta)\frac{r^4}{4}\bigg|_0^1 d\theta$$
$$= -27\int_0^{2\pi}\sin^2(\theta)\cos(\theta)d\theta$$
$$= 0$$

Which we is the value of the line integral.

6. Use the Divergence Theorem to compute the value of the flux integral

$$\iint_S \mathbf{F} d\mathbf{S}$$

Where $\mathbf{F}(x, y, z) = (y^3 + 3x, xz + y, z + x^4 \cos(x^2 y))$ and S is the boundary of the region bounded by $x^2 + y^2 = 1$, $x \geq 0$, $y \geq 0$ and $0 \leq z \leq 1$

Using the divergence theorem we will compute this rather difficult integral into a (hopefully) simpler triple integral over the divergence of $\mathbf{F}$. To begin let us compute $\div\mathbf{F}$

$$\div(\mathbf{F}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(y^3 + 3x, xz + y, z + x^4 \cos(x^2 y)\right) = 5$$

We then can setup the triple integral as

$$\iiint_W 5dV$$

Looking at the region we want to compute in cylindrical coordinates as

$$\int_0^1 \int_0^{\frac{\pi}{2}} \int_0^1 5r dr d\theta dz = \frac{5\pi}{4}$$

Which is the value of the surface integral, by the Divergence theorem.