Discrete Math Question Set 2

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1. Determine whether each of these functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

   a) $f(x) = 2x + 1$

   To prove this function is bijective, we need to prove it is injective and surjective.
   A function is injective if f(a) = c and f(b) = c and a = b. Therefore, we get
   $c = 2a + 1$ & $c = 2b + 1$

$$\begin{aligned} &2a + 1 = 2b + 1 \\ &2a = 2b \\ &a = b \end{aligned} \tag{1}$$

   Since $f(a) = f(b)$, and we proved that a = b, this function is injective. Now we must prove this function is surjunctive.

$$\begin{aligned} &f(x) = y \\ &2x + 1 = y \\ &x = \frac{y-1}{2} \\ &y = 2x + 1 \\ &y = 2(\frac{y-1}{2}) + 1 \\ &y = y \end{aligned} \tag{2}$$

   Because $f(x) = y$, this function is surjective. From our previous comclusions, it is also injective, and therefore also Bijective.

   b) $f(x) = x^2 + 1$

   Use same logic as before.

$$\begin{aligned} &a^2 + 1 = b^2 + 1 \\ &a^2 = b^2 \\ &\pm a = \pm b \end{aligned} \tag{3}$$

   Therefore, this function is injective.
   For this function to be surjective, we need to prove f(x) = y.

$$
\begin{aligned}
f(x) &= y \\
x^2 + 1 &= y \\
x^2 &= y - 2 \\
x &= \sqrt{y-2} \\
f(x) &= f(\sqrt{y-2}) \\
f(\sqrt{y-2}) &= \sqrt{y-2}^2 + 1 \\
f(\sqrt{y-2}) &= y - 2 + 1 \\
f(\sqrt{y-2}) &= y + 1
\end{aligned} \tag{4}
$$

Since $f(x) \neq y$, this function is not surjective and therefore not bijective.

c) $f(x) = \frac{x^2+1}{x^2+2}$

Use the same logic for proving injective-ness.

$$
\begin{aligned}
&\frac{a^2+1}{a^2+2} = c \\
&\frac{b^2+1}{b^2+2} = c \\
&\frac{a^2+1}{a^2+2} = \frac{b^2+1}{b^2+2} \\
&(a^2+1)(b^2+2) = (a^2+2)(b^2+1) \\
&a^2b^2 + 2a^2 + b^2 + 2 = a^2b^2 + a^2 + 2b^2 + 2 \\
&a^2 = b^2 \\
&\pm a = \pm b
\end{aligned} \tag{5}
$$

This function is therefore not injective. For example. if we make $x = \pm 1$, we get the same output of $\frac{1}{2}$. This already makes this function not bijective.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \lfloor x \rfloor$, the greatest integer in x. Find $f^-1(B)$ for the following subset B of $\mathbb{R}$. $B = [0, 2)$

$$
f^{-1}([0,2)) = \{x \in \mathbb{R} | \lfloor x \rfloor = 0\} \cup \{xx \in \mathbb{R} | \lfloor x \rfloor = 1\} \tag{6}
$$

The floor function does not have an inverse. The floor function is also defined to take in only integers. I.E. $f(\mathbb{R}) = \mathbb{Z}$. Therefore, the only values which I can find the inverse of are 0 and 1 (2 is not included in the set B). The "inverse" of this function is any x whose floor is either 0 or 1 and therefore is what makes answer a disjoint set. No floor(x) is equal to 2 different numbers.

3. Does the formula $f(x) = \frac{1}{x^2-2}$ define a function $f : \mathbb{R} \to \mathbb{R}$? What about $f : \mathbb{Z} \to \mathbb{R}$?

   a) $f : \mathbb{R} \to \mathbb{R}$ The function will be undefined if the denominator is equal to 0.

$$\begin{aligned} x^2 - 2 &= 0 \\ x^2 &= 2 \\ x &= \pm\sqrt{2} \end{aligned} \tag{7}$$

   At $\pm\sqrt{2}$, the function is not well defined as there is a hole there.

   b) $f : \mathbb{R} \to \mathbb{Z}$ Since the only number that makes the function undefined, $\pm\sqrt{2}$, is a real number, it is not part of the domain, which is $\mathbb{Z}$, and therefore makes the function well defined.

4. For each of the following functions $f : \mathbb{Z} \to \mathbb{Z}$, determine whether the function is one-to-one and whether it is onto. If the function is not onto, determine the range of $f(\mathbb{Z})$.

   a) $f(x) = -x+5$ To prove if this function is one to one, we can check to see if two different inputs a and b give us the same output.

$$\begin{aligned} -a + 5 &= c \\ -b + 5 &= c \\ -a + 5 &= -b + 5 \\ -a &= -b \\ a &= b \end{aligned} \tag{8}$$

   Since we determined that a is equal to b, this function is one-to-one.
   To determine if a function is onto, we need to find its inverse (if it has one) and plug it back to the equation. It will be onto if $f(x) = y$.
   Take any $y \in \mathbb{Z}$. Also $x \in \mathbb{Z}$.

$$\begin{aligned} f(x) &= y \\ -x + 5 &= y \\ (x &= 5 - y) \in \mathbb{Z} \\ f(5-y) &= -(5-y) + 5 \\ f(5-y) &= -5 + y + 5 \\ f(5-y) &= y \end{aligned} \tag{9}$$

   This function is onto as well and its domain is $\mathbb{R}$. We made $f(x) = y$ with $y \in \mathbb{R}$ and proved the first equality.

   b) $f(x) = x^2$ Once again, we can use the same principles from before.

$$\begin{aligned} a^2 &= c \\ b^2 &= c \\ a^2 &= b^2 \\ \pm a &= \pm b \end{aligned} \tag{10}$$

   This function is not one-to-one because of the plus minus. We can also logically see that if $x = \pm 1$ both output 1.

This fucntion is also not onto because it does not map $f : \mathbb{R} \to \mathbb{R}$. There is no integer such that $x^2 = -1$. The domain therefore for this function is $\mathbb{Z}^+$ because the input domain was $\mathbb{Z}$ and we determined that the function has no input (within the domain) whose output is negative.