Section1

因为大部分与高中内容重合, 所以仅记录一些高中不常用到的内容

Definition1.1 坐标表示方法

1.1.1 Position Vector

我们用从原点出发, 指向某点的向量表示该点的坐标

1.1.2 parametric equation

对于一个 $\mathbb{R}^2$上的点集 (x,y) , 我们可以有如下的表示方式:

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

该点集也可以表示为:

$$v(t) = (x,y) = (f(t), g(t))$$

1.1.3 Polar Coordinates

在 $\mathbb{R}^2$ 上, 我们可以用有序二元对 (r,θ) 来表示点 P 的坐标.
其中 r 等于 $||OP||$ 的大小, θ 等于 $\vec{OP}$ 与x坐标轴的夹角.

Proposition 极坐标与笛卡尔坐标的转换

对于点 $P=(x,y)$, 我们有:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

也即

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \tan\theta = \frac{y}{x} \end{cases}$$

1.1.4 Cylindrical Coordinates 圆柱坐标系

在 $\mathbb{R}^3$ 上, 我们可以用有序三元对 (r, θ, z) 来表示点 P 的坐标.
其中 (r, θ) 表示P在xy平面上的投影, z为P的z坐标

1.1.5 Spherical Coordinates 球坐标

(ρ, ψ, θ)

在 $\mathbb{R}^3$ 上, 我们可以用有序三元对 $(\cancel{r}, \psi, \theta)$ 来表示点 P 的坐标.
~~其中 (r, θ) 定义同极坐标, ψ为$\vec{OP}$的与z轴的夹角.~~

$$\begin{cases} \rho : \|\vec{OP}\| \\ \psi : 与z夹角 \\ \theta : 在xy上投影与x夹角 \end{cases}$$

Proposition 笛卡尔坐标系与球坐标系的转换

对于点 $P = (x, y, z)$, 我们有:

$$\begin{cases} x = \rho \sin\psi \cos\theta \\ y = \rho \sin\psi \sin\theta \\ z = \rho \cos\psi \end{cases}$$

$$\begin{cases} \rho = \sqrt{x^2 + y^2 + z^2} \\ \tan\psi = \frac{\sqrt{x^2+y^2}}{z} \\ \tan\theta = \frac{y}{x} \end{cases}$$

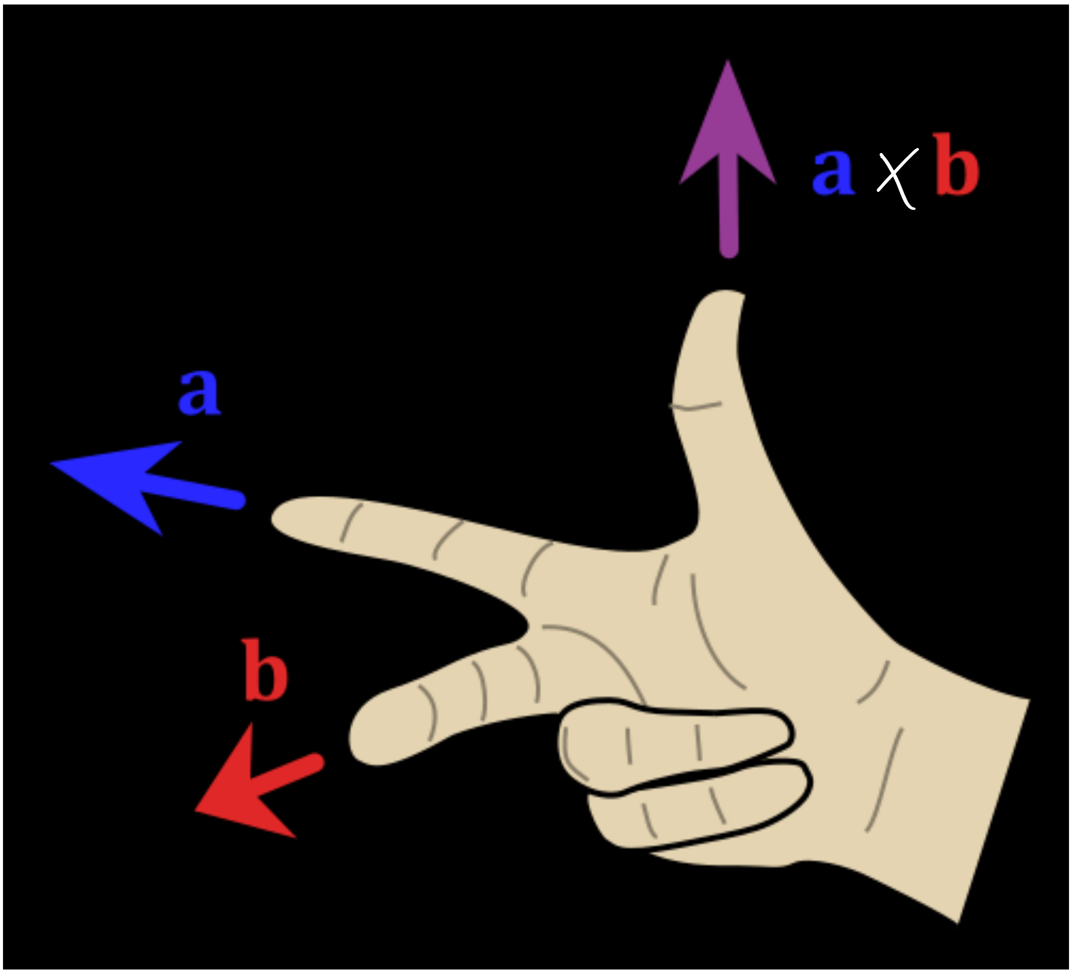
Definition1.2 Cross Product

我们定义两个 $\mathbb{R}^3$ 上的向量 u 与 v 的向量积(叉乘)满足:

- 是一个向量
- $u \times v \perp u, v$
- $\|u \times v\| = \|u\| \|v\| \sin\theta$, 其中 θ 是 u 与 v 的夹角

Proposition1.1 Right-Hand Rule

向量积的方向满足"右手法则":



Proposition1.2 三维向量的叉乘公式:

设 $u=(u_1,u_2,u_3), v=(v_1,v_2,v_3)$, 我们有:

$$u\times v=(u_2v_3-u_3v_2,u_3v_1-u_1v_3,u_1v_2-u_2v_1)$$
$$\begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

其中 &(i,j,k)$ 是空间向量的一组基底

Proposition1.3 三维向量叉乘的性质

1. (associativity) $(u+v)\times w=u\times w+v\times w$
2. (distributivity) $u\times(v+w)=u\times v+u\times w$
3. (anticommutativity) $u\times v=-v\times u$
4. $(cu)\times v=u\times(cv)=c(u\times v)$

Proposition1.4

三角形OAB的面积及可以表示为:

$$S_{\triangle OAB} = \frac{1}{2}\|u \times v\|$$

Proposition1.5 scalar triple product

在空间中, 三个不共面的位置向量: $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3), w = (w_1, w_2, w_3)$,其合成的平行六面体(parallelepiped)的体积可以表示为:

$$V = |(u \times v) \cdot w|$$

而 $(u \times v) \cdot w$ 也被称为 u, v, w 的标量三重积.

Definition1.3 Equations of Planes

我们称一个 $\mathbb{R}^3$ 上的平面可以被表示为:

$$\vec{u} + s\vec{v} + t\vec{w}$$

其中 s, t 为实参数

Definition1.4 normal vector

一个 $\mathbb{R}^3$ 上平面的 **法向量(normal)** 垂直于这个平面上的所有向量

Proposition1.6

我们设在 $\mathbb{R}^3$ 上, 点P的位置向量为(a, b, c) ,那么经过点P法向量为 $n = (A, B, C)$ 的平面的方程可以表示为:

$$A(x-a) + B(y-b) + C(z-c) = 0$$

Section2

Definition2.1 injective, surjective and bijective

对于函数$f : X \to Y$, 我们称

(1) injective

f 是 **单射的(injective)/one-to-one** 当 $\forall a \neq b \in X, f(a) \neq f(b)$ 时.

$a \neq b \Leftrightarrow f(a) \neq f(b)$

(2) surjective

f 是 **满射的(surjective)/onto** 当 $\forall y \in Y, \exists x \in X, s.t. f(x) = y$ 时.

$\exists f^{-1} : Y \to X$ (本身 f 就保证了 $\forall x \in X, \exists f(x) \in Y$)

(3) bijective

f 是 **双射的(bijective)/one-to-one correspondence** 当 f 既是单射又是满射时.

Proposition2.1

所有的双射函数都具有反函数.

Definition2.2 inverse function

对于函数 $f : X \to Y$, 我们称 $g : Y \to X$ 为其反函数, 当且仅当:

$$f \circ g = g \circ f = identity\ function$$

我们通常将 f 的反函数记作 f^{-1}

Definition2.3 vector-valued function

我们考虑一个 **向量值函数(vector-valued, 值为向量)** $f : X \to \mathbb{R}^m$, 其中 $X \subseteq \mathbb{R}^n$.
我们可以写出:

$$f(\vec{v}) = (f_1(\vec{v}), f_2(\vec{v}), ..., f_m(\vec{v}))$$

其中 $f_i : X \to \mathbb{R}$
我们称 $f_1, f_2, ..., f_m$ 为 f 的**分量函数(component functions)**. 这样我们就只关注那些 **标量值函数(scalar-valued)** 了.

Definition2.4 graph

我们称函数 $f : X \to \mathbb{R}$, 其中 $X \subseteq \mathbb{R}^2$ 的 **图(graph)** 为点集:

$$\{(x, y, f(x, y)) \in \mathbb{R}^3 | (x, y) \in X\}$$

称其在高度 c 的 **轮廓曲线(contour curve)** 为点集: 三维点集,

$$\{(x, y, f(x, y)) \in \mathbb{R}^3 | (x, y) \in X, f(x, y) = c\}$$

二维点集

称其在高度 c 的 **等高线(level curve)** 为点集:

$$\{(x, y) \in \mathbb{R}^2 | (x, y) \in X, f(x, y) = c\}$$

Definition2.5 quadric surface

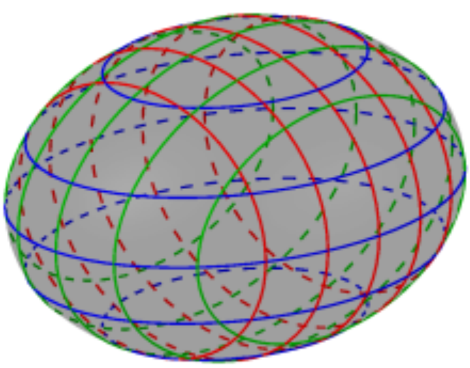
在 $\mathbb{R}^3$, 我们称一个 **二次曲面(quadric surface)** 通过如下公式定义(字典序):

$$Ax^2 + Bxy + Cxz + Dy^2 + Eyz + Fz^2 + Gx + Hy + Iz + J = 0$$

以下是常见的二次曲面

椭圆面(ellipsoid):

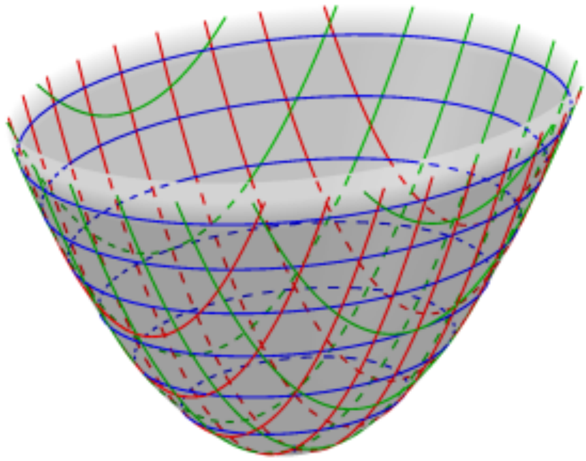
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



当 $a = b = c$ 时, 为一 **球面(sphere)**

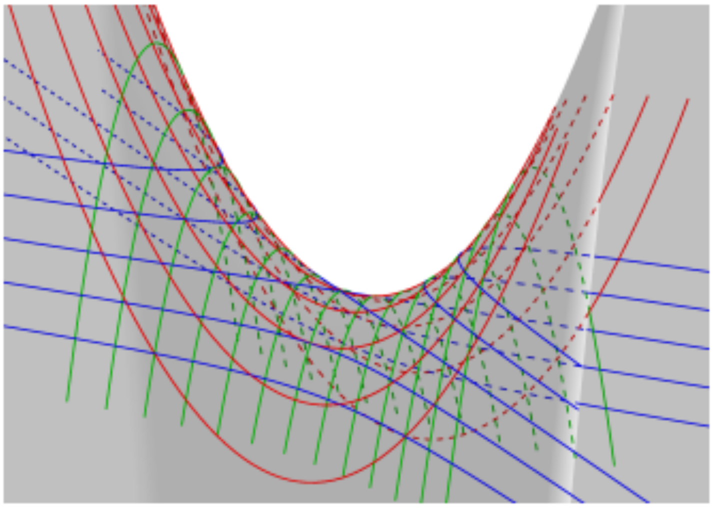
椭圆抛物面(elliptic paraboloid):

$$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$



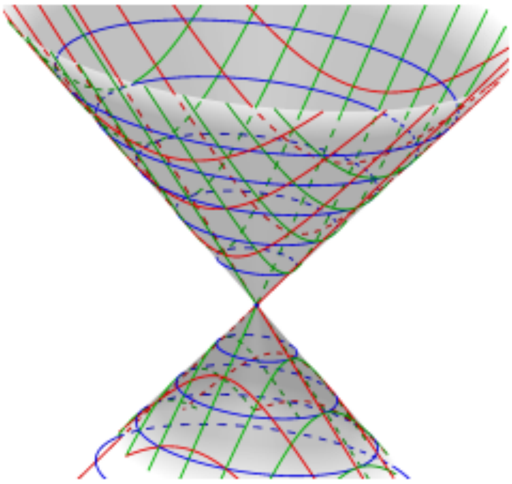
双曲抛物面(hyperbolic paraboloid)

$$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$$



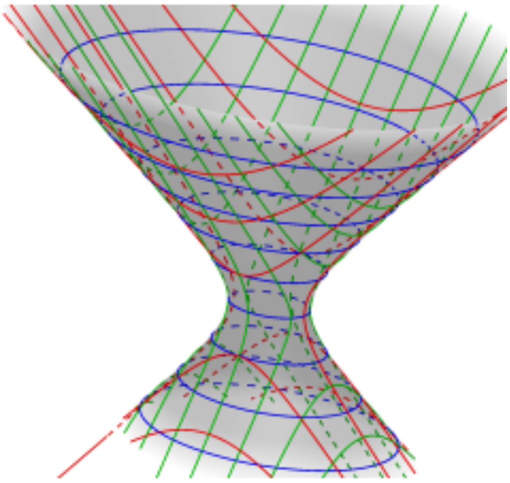
椭圆锥面(elliptic cone)

$$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$



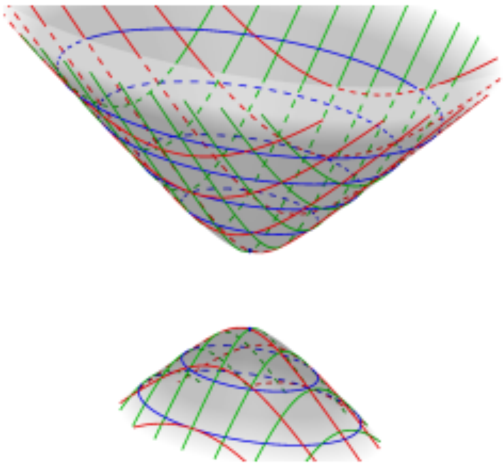
单叶双曲面(hyperboloid of one sheet)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$



双叶双曲面(hyperboloid of two sheet)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$$



Definition2.6

对于任意的 $a \in \mathbb{R}^n$ 与 $r > 0$, 我们称;
$\mathbb{R}^n$上以 a 为中心, r 为半径的 **开球(open ball)** 为集合:

$$B(a,r) = \{x \in \mathbb{R}^n | \|x-a\| < r\}$$

$\mathbb{R}^n$上以 a 为中心, r 为半径的 **闭球(close ball)** 为集合:

$$\bar{B}(a,r) = \{x \in \mathbb{R}^n | \|x-a\| \leq r\}$$

Definition2.7

对于 $X \subseteq \mathbb{R}^n$, 我们称:
X 的**内点(interior point)** 满足:
存在以 a 为中心的开球, 开球上所有点都属于 X
我们记 X 的内点组成的集合为 **内部(interior)**

X 的 **边界点(boundary point)** $a \in \mathbb{R}^n$ 满足:
对所有以 a 为中心的开球, 都同时存在属于 X 的点与不属于 X 的点.
我们记 X 的临界点组成的集合为 **分界线(boundary)**, 记作 ∂X

注意: X 的内点一定属于 X, 临界点不一定属于 X

Definition2.8

我们称集合 $X \subseteq \mathbb{R}^n$ 是 $\mathbb{R}^n$的 **开集(open set)** 若 X 中的所有点都是 X 的 **内点(interior point)**
我们称集合 $X \subseteq \mathbb{R}^n$ 是 $\mathbb{R}^n$的 **闭集(close set)** 若 $\mathbb{R}^n \setminus X$ 是一个 $\mathbb{R}^n$上的开集

Proposition2.2

对于 $X \subseteq \mathbb{R}^n$, X 是闭集当且仅当 $\partial X \subseteq X$

Definition2.9

对于 $X \subseteq \mathbb{R}^n$, 我们称:

$a \in \mathbb{R}^n$ 是 X 的 **极限点(limit point or accumulation point)**, 若 a 满足:

$$\forall \delta > 0, \exists x \neq a (x \in X \cap B(a, \delta))$$

在任意距离，a的开球都有X的点

$a \in X$ 是 X 的 **孤立点(isolated point)** , 若 a 不是极限点.

Proposition2.3

X 的极限点中不在 X 中的一定是 X 的临界点.

Definition2.10 Lim

对于 $f : X \to \mathbb{R}^m, X \subset \mathbb{R}^n$, a 是 X 的一个极限点. 我们称 f 在 a 处的**极限(limit)** 为 $L \in \mathbb{R}^m$, 满足:

$$\forall \epsilon > 0, \exists \delta > 0, s.t. (x \in X \wedge 0 < \|x - a\| < \delta \to \|f(x) - L\| < \epsilon)$$

我们记做:

$$\lim_{x \to a} f(x) = L$$

Proposition2.4

若极限存在, 则在该点处的极限是被 **唯一确定(uniquely determined)** 的

Proposition2.5 极限的性质

对于

$$f, g : X \to \mathbb{R}^m, x \subset \mathbb{R}^n$$
$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

我们有:

(1)

$$\lim_{x \to a}(f \pm g)(x) = L \pm M$$

(2)

$$\forall c \in \mathbb{R}, \lim_{x \to a}(cf)(x) = cL$$

(3)若 $m = 1$,

$$\lim_{x \to a}(fg)(x) = LM$$

(4)若 $m = 1, M \neq 0$,

$$\lim_{x \to a}(\frac{f}{g})(x) = \frac{L}{M}$$

Theorem2.1 夹逼定理/三明治定理 略

Proposition2.6

对于 $f : X \to \mathbb{R}^m, X \subset \mathbb{R}^n$, 且有 $f = (f_1, f_2, ..., f_m)$, a 是 X 的极限点, 那么我们有:

$$\lim_{x \to a} f(x) = (l_1, l_2, ..., l_m)$$

其中,

$$\lim_{x \to a} f_i(x) = l_i \ for \ i = 1, 2, ..., m$$

Definition2.11

对于 $f : X \to \mathbb{R}^m, X \subset \mathbb{R}^n$, 我们称 f 在 a 是 **连续的**, 若满足如下条件中任意一个:

$$a\text{是}X\text{中的一个孤立点} \tag{1}$$
$$\lim_{x\to a} f(X) = f(a) \tag{2}$$

否则, 则称 f 在 a 处不连续.

我们称 f 是 **连续函数** , 当它在 X 的每个点处都连续, 反之则称为 **不连续函数** .

Proposition2.7

若 f 与 g 在 a 处连续, 我们有:
(1) $f+g$ 在 a 处连续
(2) $\forall c \in \mathbb{R}, cf$ 在 a 处连续
(3) 若 $m=1$, fg 在 a 处连续
(4) 若 $m=1$且$g(a) \neq 0$, $\frac{f}{g}$ 在 a 处连续

Proposition2.8

对于 $f: X \to \mathbb{R}^m, X \subset \mathbb{R}^n$, 且有 $f=(f_1, f_2, ..., f_m)$.
f 在 a 处连续当且仅当 f_i 在 a 处连续

Proposition2.9

对于 $f: X \to \mathbb{R}^m, X \subset \mathbb{R}^n; g: Y \to \mathbb{R}^k, f(X) \subset T \subset \mathbb{R}^m$, a 是 X 的一个极限点. 若 $\lim_{x\to a} f(x) = L \in Y$ 且g在L处连续, 那么有:

$$\lim_{x\to a}(g\circ f)(x) = g(L)$$

作为推论, 若 f 与 g 是连续的, 那么 $f\circ g$ 也是连续的.

Definition2.12 partial derivative

对于 $\mathbb{R}^n$ 上的变量 X, 与 $f: X \to \mathbb{R}$, 我们称 f 在 $\vec{a}$ 点处 x_j 分量的偏导数(部分微分)为:

$$\frac{\partial f}{\partial x_j}(a) = \lim_{h\to 0}\frac{f(\vec{a}+h\vec{e_j}) - f(\vec{a})}{h}$$

也被记作 $f_{x_j}(a)$ or $D_{x_j}f(a)$

Definition2.13 Tangent Plane

对于 $f: X \to \mathbb{R}, X \subset \mathbb{R}^2$, (a,b) s是 X 的内点. 我们称 f 的graph S 在 $x = (a,b,f(a,b))$ 的 **切面** 是包含了 x 处 S 的所有切线的平面.

懒得打字了, 转战平板

Prop 2.10 $f: X \to R$, $X \subset R^2$, (a,b)是 X 的内点,

在 (a,b) 处偏导 f_x 与 f_y 存在. 若在点 $(a,b,f(a,b))$ 处切面存在, 则切面方程为:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

更一般的: 对于 $f: X \to R$, $X \subset R^n$, ...

则 $(a_1, a_2 \cdots a_n, f(a_1, a_2 \cdots a_n))$ 处切面方程为:

$$X_{n+1} = f(\vec{a}) + \sum_{k=1}^{n} f_{x_k}(\vec{a})(x_k - a_k)$$

Definition 2.14

$f: X \to R$, $X \subset R^n$ 是开集.

若 $f_{x_j}(\vec{a})$ 对每个存在, 我们定义:

$$h(\vec{x}) = f(\vec{a}) + \sum_{k=1}^{n} f_{x_k}(\vec{a})(x_k - a_k)$$

则我们称 f 在 $\vec{a}$ 处是 '可微的 differentiable' 当:

$$\lim_{\vec{x} \to \vec{a}} \frac{f(\vec{x}) - h(\vec{x})}{\|\vec{x} - \vec{a}\|} = 0$$

我们称 f 是可微的 当 f 在 X 上每个点都可微.

Definition 2.15

$f: X \to R$，$X \subset R^n$．a 是 X 的内点．若对 $j = 1, 2, \cdots n$，$f_{x_j}(a)$ 存在．

我们称 f 在 a 的"梯度 gradient"为：→ 指向函数增长最快的方向

$$\text{grad } f(a) = \nabla f(a) = (f_{x_1}(a), \cdots, f_{x_n}(a))$$

由此定义，我们发现："可微分"的定义可以被写为：

$$0 = \lim_{\vec{x} \to \vec{a}} \frac{f(\vec{x}) - h(\vec{x})}{\|\vec{x} - \vec{a}\|} = \lim_{\vec{x} \to \vec{a}} \frac{f(\vec{x}) - f(\vec{a}) - \nabla f(\vec{a})(\vec{x} - \vec{a})}{\|\vec{x} - \vec{a}\|}$$

可微 $\Leftrightarrow \lim_{h \to 0} \frac{R(\vec{h})}{\|\vec{h}\|} = 0$

Definition 2.16

$f: X \to R$，$X \subset R^n$，a 是 X 上内点．若对 $j = 1, 2, \cdots n$，$f_{x_j}(a)$ 存在．

我们称 f 在 a 的"导数/微商 derivative"为：

$$Df(a) = (f_{x_1}(a) \quad f_{x_2}(a) \quad \cdots \quad f_{x_n}(a))$$

注意：$\nabla f(a)$ 是一个列向量，$Df(a)$ 是一个行向量！

不难发现：$\nabla f(a) \bullet (x - a) = Df(a)(x - a)$

Theorem: 2.2

$f: X \to R$，$X \subset R^n$，a 是 X 上内点

若 f 在一个包含 a 的开集上有连续的偏导，那么 f 在 a 上"可微 differentiable"

Prop 2.10

$f: X \to R$，$X \subset R^n$，a 是 X 上内点．

若 f 在 a 可微，则 f 在 a 连续．

Prop 2.11

$f, g: X \to R^m$，$X \subset R^n$，a 是 X 上内点．f, g 在 a 上可微．

我们有：(1) $f + g$ 在 a 可微，$D(f+g)(a) = Df(a) + Dg(a)$

(2) cf 在 a 上可微，$D(cf)(a) = cDf(a)$　　$c \in R$

(3) 若 $m=1$，则 fg 在 a 上可微，$D(fg)(a)=Df(a)\cdot g(a)+Dg(a)\cdot f(a)$

(4) 若 $m=1$ 且 $g(a)\neq 0$，有 $\frac{f}{g}$ 在 a 可微，且有：

$$D\left(\frac{f}{g}\right)(a)=\frac{g(a)Df(a)-f(a)Dg(a)}{g(a)^2}$$

Definition 2.17

我们考虑更高阶的微分，对于 $f: X\to R$，$X\subset R^n$，我们称：

二阶导 (2nd-order partial derivative)

$$\frac{\partial^2 f}{\partial x_i\partial x_j}=f_{x_ix_j}=\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$$

更一般的：
$$\frac{\partial^k f}{\partial x_{i_k}\partial x_{i_{k-1}}\cdots\partial x_{i_1}}=f_{x_{i_1}x_{i_2}\cdots x_{i_k}}=\frac{\partial}{x_{i_k}}\left(\frac{\partial}{x_{i_{k-1}}}\left(\cdots\left(\frac{\partial f}{\partial x_{i_1}}\right)\cdots\right)\right)$$

Definition 2.18.

$f: X\to R$，$X\subset R^n$，X是开集.

我们称 f 是 C^k-class 函数 若：f 的阶数 $\leq k$ 的偏导存在且连续.

我们称 f 是光滑的，若 f 是 C^∞-class 函数.

对于向量值函数 $f: X\to R^m$，我们称 f 是 C^k iff 对任何分量均为 C^k

Theorem 2.4.

$f: X\to R$，是 C^k，$X\subset R^n$ 是开集，$k\in N^+$

若 $(j_1, j_2\cdots j_k)$ 是 $(i_1,\cdots i_k)$ 的一个排列，$1\leq i\leq n$，有：

$$f_{x_{i_1}}f_{x_{i_2}}\cdots f_{x_{i_k}}=f_{x_{j_1}}f_{x_{j_2}}\cdots f_{x_{j_k}}$$

Theorem 2.5. Chain rule

$f: Y\to R$ (R^n)，$Y\subset R^n$；$g: X\to R^n$ (R)，$X\subset R$，$g(X)\subset Y$，$g=(x_1,x_2,\cdots,x_n)$

令 t_0 是 X 的内点，$y_0=g(t_0)$ 是 Y 的内点.　　$f(\overset{\to R^n}{y})=f(g(\overset{\to R}{x}))$

若 g 在 t_0 处可微，f 在 y_0 处可微，则：

① $f\circ g: X\to R$ 在 t_0 处可微.

② $\frac{df}{dt}(t_0)=\sum_{k=1}^{n}\frac{\partial f}{\partial x_k}(y_0)\cdot\frac{dx_k}{dt}(t_0)$，note that

写实上应为 $\frac{d\, f\circ g}{dt}(t_0)$ 更严谨

⑤ $D(f\circ g)(t_0) = \frac{d f\circ g}{dt}(t_0) = \underset{\text{行}}{Df(y_0)}\,\underset{\text{列}}{Dg(t_0)}$

我们考虑极坐标下的性质：

Prop 2.12.

$f: X\to R$, $X\subset R^2$ 是开集

我们假定 X上每个点 都有 Cartesian 坐标 (x,y) 与极坐标 (r,θ)

我们有：

$g(r,\theta) = (x,y) = (r\cos\theta, r\sin\theta)$

$$\begin{cases} \dfrac{\partial f}{\partial r} = \dfrac{\partial f}{\partial \sqrt{x^2+y^2}} = \cos\theta\, f_x + \cos\theta\, f_y = Df(x,y)\,Dg_r(r,\theta) = (f_x \;\; f_y)\begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix} \\ \dfrac{\partial f}{\partial \theta} = -r\sin\theta\, f_x + r\cos\theta\, f_y \end{cases}$$

$=$

$$\begin{cases} \dfrac{\partial f}{\partial x} = \cos\theta\, f_r - \dfrac{\sin\theta}{r} f_\theta \\ \dfrac{\partial f}{\partial y} = \sin\theta\, f_r + \dfrac{\cos\theta}{r} f_\theta \end{cases}$$

($r=0$时，该点为$[0,\infty)$的端点.)

Definition 2.19

$f: X\to R$, $X\subset R^n$, a是X的内点.

若u是R^n上的一个单位向量，那么我们称 f 在u方向上的"方向导数 directional derivative"为：

$$D_u f(a) = \lim_{h\to 0}\frac{f(\vec a + h\vec u) - f(\vec a)}{h} = \frac{d f(a+tu)}{dt} \in R.$$

Prop 2.13.

$f: X\to R$, $X\subset R^n$, a是X的内点, f在a处可微.

对任意方向向量u, 我们有：

$$D_u f(a) = \nabla f(a)\cdot u \le \|\nabla f(a)\|$$

注意，反之不一定成立，不可微也可能存在某方向的方向导数.

Prop 2.14

$f: X\to R$, $X\subset R^n$, a是X的内点, f在a处可微, $\nabla f(a)\neq 0$

$D_u f(a)$ 最大当与 $\nabla f(a)$ 同向 $= \|\nabla f(a)\|$

最小当与 $\nabla f(a)$ 异向

补充：n维下的球坐标.

$$(x_1, \ldots x_n) \longrightarrow (\rho, \varphi_1, \varphi_2 \cdots \varphi_{n-2}, \phi)$$

$$\begin{cases} x_1 = \rho\cos\varphi_1 \\ x_2 = \rho\sin\varphi_1\cos\varphi_2 \\ \quad \cdots \\ x_{n-1} = \rho\sin\varphi_1 \cdots \sin\varphi_{n-2}\cos\phi \\ x_n = \rho\sin\varphi_1 \cdots \sin\varphi_{n-2}\sin\phi \end{cases}$$

e.g. ，四维球坐标：$(\rho, \alpha, \beta, \gamma)$

$$\begin{cases} x_1 = \rho\cos\alpha \\ x_2 = \rho\sin\alpha\cos\beta \\ x_3 = \rho\sin\alpha\sin\beta\cos\gamma \\ x_4 = \rho\sin\alpha\sin\beta\sin\gamma \end{cases}$$

Section 3.

Definition 3.1 一元 Taylor's Theorem.

$f: X \to \mathbb{R}$, $X \subset \mathbb{R}$ 是开集, f 至少 k 维可微,

$\forall a \in X$, k阶泰勒多项式为: (Taylor polynomial of degree k)

$$P_x = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= \sum_{i=0}^{k} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Taylor polynomial 表明:

$$f(x) = P_k(x) + R_k(x,a)$$

其中: $\lim_{x \to a} \frac{R_k(x,a)}{(x-a)^k} = 0$, $R_k(x,a)$ 被称作 Remainder term

有: $R_k(x,a) = \frac{f^{(k+1)}(\xi)}{(k+1)!}(x-a)^{k+1}$ 若 f $k+1$ 阶可微

Taylor's Theorem.

$f: X \to \mathbb{R}$ 可微, $X \subset \mathbb{R}^n$ 是开集

$\forall \vec{a} \in X$, 我们定义:

$$P_1(\vec{x}) = f(\vec{a}) + Df(\vec{a})(\vec{x} - \vec{a})$$

有:

皮亚诺余项 Peano Remainder

$$f(\vec{x}) = P_1(\vec{x}) + R_1(\vec{x}, \vec{a})$$

其中:

$$\lim_{\vec{x} \to \vec{a}} \frac{R_1(\vec{x}, \vec{a})}{\|\vec{x} - \vec{a}\|} = 0$$

Proof: 原式可化为 $\lim_{x \to a} \frac{f(x) - [f(a) + Df(a)(x-a)]}{\|x-a\|}$ → f 在 a 处可微的充要条件

Definition 3.2

$f: X \to R$, 可微, $X \subset R^n$ 是开集,

若 $a \in X$, $h \in R^n$, $a+h \in X$, 则

① f 从 a 到 $a+h$ 的增量 (Incremental change) 被定义为:

$$\Delta f = f(a+h) - f(a)$$

② f 在 a 处沿 h 变化的全微分 (Total Differential) 为:

$$df(a,h) = Df(a)h$$

Prop 3.1

由一阶泰勒定理知: 当 $\|h\|$ 很小时, $\Delta f \approx df$

($f(x) - df$ 比 $\|h\|$ 趋于 0 更快)

proof: $$0 = \lim_{h \to 0} \frac{f(a+h) - f(a) - Df(a)h}{\|h\|} = \lim_{h \to 0} \frac{\Delta f - df(a,h)}{\|h\|}$$

$\Rightarrow \Delta f \approx df(a,h)$ 当 $\|h\|$ 很小时

Theorem 3.2 Taylor's theorem.

$f: X \to R$, 是 C^k 类函数, $X \subset R^n$ 是开集

$\forall a \in R^n$, 我们有:

$$P_K(X) = f(a) + \sum_{j=1}^{k} \frac{1}{j!} \left(\sum_{i_1, i_2, \cdots, i_j = 1}^{n} (a) \prod (x_{i_1} - a_{i_1}) \right)$$

那么有:

$$f(x) = P_K(X) + R_K(x,a)$$

其中:

$K=1$, C^1, $f(x) = P_1(a) + \frac{1}{2!} f_x$

$$\lim_{x \to a} \frac{R_K(x,a)}{\|x-a\|^K} = 0$$

Prop 3.L $f: X \to \mathbb{R}$, 是 C^{k+1}, $X \subset \mathbb{R}^n$ 是开集.

$\forall a \in X$, 有:

拉格朗日余项 对所有的 $(x_2 - a_2)$

$$P_k(x) = f(a) + \sum_{j=1}^{k} \frac{1}{j!} \left(\sum_{i_1, i_2 \cdots i_j = 1}^{n} f_{x_{i_1} x_{i_2} \cdots x_{i_j}} (a) \prod (x_{i_1} - a_{i_1}) \right)$$

所有可能的 j 阶偏导

$\exists \xi$ 在 a 与 x 的连线上, s.t.

$$f(x) = P_k(x) + \frac{1}{(k+1)!} \sum_{i_1, i_2 \cdots i_{k+1} = 1} f_{x_{i_1} x_{i_2} \cdots x_{i_{k+1}}} (\xi) \prod (x_{i_1} - a_{i_1})$$

$$= \sum_{i=0}^{k} \frac{1}{i!} D^i f(a) : (x-a)^{\otimes i}$$

Frobenius inner product

$k=1$

Definition 3.2

$f: X \to \mathbb{R}$ 是 C^2, $X \subset \mathbb{R}^n$ 是开集, 则

f 的 Hessian 为:

$$Hf = \begin{pmatrix} f_{x_1 x_1} & f_{x_1 x_2} & \cdots & f_{x_1 x_n} \\ f_{x_2 x_1} & f_{x_2 x_2} & \cdots & f_{x_2 x_n} \\ \cdots & \cdots & \ddots & \cdots \\ f_{x_n x_1} & f_{x_n x_2} & \cdots & f_{x_n x_n} \end{pmatrix}$$

借此定义, 我们有:

$$P_2(x) = f(a) + Df(a)(x-a) + \frac{1}{2}(x-a)^T Hf(a)(x-a)$$

详见 P23

i.e. $\exists \xi$ between $\vec{x}$ and $\vec{a}$, i.e.

$$f(x) = P_1(x) + \frac{1}{2}(x-a)^T Hf(\xi)(x-a)$$

Example 3.2. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = e^x \sin(x+y)$ and consider $\mathbf{a} = \mathbf{0}$. Then

$$Hf = \begin{pmatrix} 2e^x \cos(x+y) & e^x(-\sin(x+y) + \cos(x+y)) \\ e^x(-\sin(x+y) + \cos(x+y)) & -e^x \sin(x+y) \end{pmatrix}$$

and

$$P_2(x,y) = x + y + x^2 + xy.$$

Example 3.3. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = \ln(x^2 + y^2 + 1)$ and consider $\mathbf{a} = \mathbf{0}$. For any (x,y) with $|x|, |y| \leqslant 0.1$, we have

$$f(x,y) = x^2 + y^2 + E(x,y),$$

where $|E(x,y)| \leqslant 0.00083$.

3.2

3.2 Extrema of Functions of Several Variables

全局最值

Definition 3.3. Let $f : X \to \mathbb{R}$ be a function where $X \subset \mathbb{R}^n$. We say that f has a *global minimum* (plural: *global minima*) at $\mathbf{a} \in X$ if $f(\mathbf{x}) \geqslant f(\mathbf{a})$ for all $\mathbf{x} \in X$.

Similarly, we say that f has a *global maximum* (plural: *global maxima*) at $\mathbf{a} \in X$ if $f(\mathbf{x}) \leqslant f(\mathbf{a})$ for all $\mathbf{x} \in X$.

局部极值

Definition 3.4. Let $f : X \to \mathbb{R}$ be a function where $X \subset \mathbb{R}^n$. We say that f has a *local minimum* (plural: *local minima*) at $\mathbf{a} \in X$ if there exists an open set U containing $\mathbf{a}$ such that $f(\mathbf{x}) \geqslant f(\mathbf{a})$ for all $\mathbf{x} \in (X \cap U)$.

Similarly, we say that f has a *local maximum* (plural: *local maxima*) at $\mathbf{a} \in X$ if there exists an open set U containing $\mathbf{a}$ such that $f(\mathbf{x}) \leqslant f(\mathbf{a})$ for all $\mathbf{x} \in (X \cap U)$.

If $\mathbf{a}$ is a point of minimum or maximum, then we say that it is a point of (local or global) *extremum* (plural: *extrema*).

Theorem 3.3. Let $f : X \to \mathbb{R}$ be a function where $X \subset \mathbb{R}^n$, and let $\mathbf{a}$ be an interior point of X. If f has a local extremum at $\mathbf{a}$ and $\nabla f(\mathbf{a})$ exists, then $\nabla f(\mathbf{a}) = \mathbf{0}$.

The converse of this theorem does not hold. This means $\nabla f(\mathbf{a}) = \mathbf{0}$ does not imply f has a local extremum at $\mathbf{a}$.

Definition 3.5. Let $f : X \to \mathbb{R}$ be a function where $X \subset \mathbb{R}^n$, and let $\mathbf{a}$ be an interior point of X. We say that $\mathbf{a}$ is a *critical point* of f if $\nabla f(\mathbf{a})$ does not exist or $\nabla f(\mathbf{a}) = \mathbf{0}$.

临界点

It follows that an interior local extremum point must be a critical point.

Definition 3.6. Let $f : X \to \mathbb{R}$ be a function where $X \subset \mathbb{R}^n$, and let $\mathbf{a}$ be an interior point of X. We say that $\mathbf{a}$ is a *saddle point* of f if $\nabla f(\mathbf{a}) = \mathbf{0}$ but $\mathbf{a}$ is not a point of local extremum.

鞍点

Example 3.4. Suppose there is a mountain whose surface is the graph of

$$f(x,y) = -x^2 - 4y^2 + 6x - 8y + 24$$

with $f(x,y) \geqslant 0$. The highest point of this mountain is $(3,-1,37)$.

Example 3.5. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x,y,z) = x^2 + 2y^2 - z^2 + 2yz + x + 3z - 5.$$

Then $\nabla f(x,y,z) = (2x+1, 4y+2z, -2z+2y+3)$. The only critical point of f is $\left(-\frac{1}{2}, -\frac{1}{2}, 1\right)$, which is a saddle point. Also, f does not have global extrema.

Definition 3.7. Let A be an $n \times n$ real symmetric matrix. We say that A is *positive definite* (*negative definite*) if $\mathbf{x}^T A\mathbf{x} > 0$ ($\mathbf{x}^T A\mathbf{x} < 0$, respectively) for all nonzero $\mathbf{x} \in \mathbb{R}^n$.

正定 负定

If A is not positive or negative definite, it is called an *indefinite matrix*.

未定

or all of it's eigenvalue is positive

Theorem 3.4. (Second partial derivative test) Let $f : X \to \mathbb{R}$ be a function of class C^2 where $X \subset \mathbb{R}^n$ is open. Let $\mathbf{a}$ be a critical point of f. Then the following hold.

(a) If $Hf(\mathbf{a})$ is positive definite, then f has a local minimum at $\mathbf{a}$.

(b) If $Hf(\mathbf{a})$ is negative definite, then f has a local maximum at $\mathbf{a}$.

(c) If $\det Hf(\mathbf{a}) \neq 0$ and $Hf(\mathbf{a})$ is indefinite, then $\mathbf{a}$ is a saddle point of f.

Definition 3.8. Let A be an $n \times n$ matrix. For $k = 1, 2, \ldots, n$, let A_k be the $k \times k$ submatrix of A in the top left corner. Then $d_k = \det A_k$ is called a *leading principal minor* of A.

第k顺序主子式

det(Hf(a)) { +: local extrema
-: saddle point
=0: 不确定，进一步分析

Proof of Theorem 3.4

考虑 $\vec{a}$ 为 f 的一个 cirtical point. 在 $\vec{a}$ 点我们有:

$$P_2(\vec{x}) = f(\vec{a}) + f'(\vec{a})(\vec{x}-\vec{a}) + f''(\vec{a})(\vec{x}-\vec{a})^{\otimes 2}$$

由于 $\vec{a}$ 的性质, $f'(\vec{a})=0$, 我们有:

$$\forall |\vec{h}| \to 0,\ \delta f \approx P_2(\vec{a}+\vec{h}) - f(\vec{a})$$

$$= f''(\vec{a}) : (h \otimes h)$$

$$= h^T Hf(a)\, h$$

$\therefore$ Hf 正定 $\longrightarrow \forall h,\ \delta f > 0 \to$ 局部极小值

反之亦然.

Proof of Sylvester's criterion

必要性:

$\nearrow$ A 的前 K 行与 K 列

引理: 若 A 正定, 则 A_K 正定

引理证明: $\forall x_K \in R^K$, 令 $x = \begin{bmatrix} \vec{x}^K \\ \vec{0} \end{bmatrix} \in R^n$.

有 $x_K^T A_K x_K = x^T A x > 0$

$\therefore$ A_K 正定. Q.E.D.

而 $d_K = |A_K|$ 为其所有特征值乘积

$\therefore A$ 正定 $\to \forall K,\ A_K$ 正定 $\to d_K > 0$

充分性: 不会

proof

×

Theorem 3.5. (Sylvester's criterion) Let A be an $n \times n$ real symmetric matrix. Let $d_1, d_2, \dots, d_n$ be the leading principal minors of A. Then the following hold.

(a) A is positive definite if and only if $d_k > 0$ for $k = 1, 2, \dots, n$

(b) A is negative definite if and only if $d_k < 0$ for odd k and $d_k > 0$ for even k

$- + - + - +$

$1\ 2\ 3\ 4\ 5\ 6$

Example 3.6. Define

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -3 & 1 & 2 \\ 1 & -1 & -1 \\ 2 & -1 & -4 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & -2 \end{pmatrix}.$$

Then A is positive definite, B is negative definite, and C is indefinite.

Example 3.7. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = x^3 + xy^2 + x^2 + y^2 + 3z^2.$$

Then $\nabla f(x, y, z) = (3x^2 + y^2 + 2x, 2xy + 2y, 6z)$. The only critical points of f are $(0, 0, 0)$ and $\left(-\frac{2}{3}, 0, 0\right)$. By the second partial derivative test, we find that f has a local minimum at $(0, 0, 0)$, while $\left(-\frac{2}{3}, 0, 0\right)$ is a saddle point.

有界的

Definition 3.9. Let X be a subset of $\mathbb{R}^n$. We say that X is *bounded* if there exists $M > 0$ such that $\|\mathbf{x}\| \leqslant M$ for all $\mathbf{x} \in X$.

2 **Proposition 2.1.** A subset X of $\mathbb{R}^n$ is closed if and only if it contains all its boundary points.

紧致的 definition 2.8 (in note)

Definition 3.10. Let X be a subset of $\mathbb{R}^n$. We say that X is *compact* if it is closed and bounded.

1 **Definition 2.12.** A subset X of $\mathbb{R}^n$ is called an *open set* in $\mathbb{R}^n$ if every point in X is an interior point of X.

2 **Definition 2.13.** A subset X of $\mathbb{R}^n$ is called a *closed set* in $\mathbb{R}^n$ if the complement $\mathbb{R}^n - X$ is an open set in $\mathbb{R}^n$.

Theorem 3.6. (Extreme value theorem) Let $f : X \to \mathbb{R}$ be a continuous function where $X \subset \mathbb{R}^n$ is compact. Then f attains its global maximum and global minimum.

Example 3.8. An experiment is carried out in a device of size 8×8. We use the Cartesian coordinates to represent each point by (x, y) where $-4 \leqslant x, y \leqslant 4$. Suppose the temperature at the point (x, y) is given by

$$x^2 - xy + y^2$$

degree Celsius. Then the point $(0, 0)$ has the lowest temperature, while the points $(4, -4)$ and $(-4, 4)$ have the highest temperature.

Example 3.9. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = e^{x^2+y^2}\left(x^2 + y^2 - \frac{8}{3}x + 2\right).$$

Then f has a global minimum at $(x, y) = (1, 0)$ and it has no global maximum.

3.3

3.3 Lagrange Multipliers

Example 3.10. Define $f : S \to \mathbb{R}$ by

$$f(x, y) = x^2 + y^2 - x + y$$

where $S = \{(x, y) \in \mathbb{R}^2 : x + y = 1\}$. Then the global minimum of f is attained at $(x, y) = (1, 0)$.

拉格朗日乘数法

Theorem 3.7. (Lagrange multiplier rule) Let $f, g : X \to \mathbb{R}$ be functions of class C^1 where $X \subset \mathbb{R}^n$ is open. Let $S = \{\mathbf{x} \in X : g(\mathbf{x}) = c\}$ be the level set of g at height c. If the restriction $f|_S$ of f on S has a local extremum at $\mathbf{a} \in S$ and $\nabla g(\mathbf{a}) \neq \mathbf{0}$, then there exists $\lambda \in \mathbb{R}$ such that

f_S 在 $\vec{a}$ 的极值 且 $\nabla g(a) \neq 0$ $\rightarrow$
$$\nabla f(\mathbf{a}) = \lambda \nabla g(\mathbf{a}).$$

The restriction $f|_S$ means the function $f|_S : S \to \mathbb{R}$ defined by $f|_S(\mathbf{x}) = f(\mathbf{x})$. This means we want to optimize $f(\mathbf{x})$ given that $g(\mathbf{x}) = c$.

Example 3.11. Define $f : S \to \mathbb{R}$ by

$\nabla f(x,y) = (3x^2, 3y^2)$ $\quad x = y$

$$f(x, y) = x^3 + y^3$$

$\nabla g(x,y) = (2x, 2y)$

where $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Then the global minimum of f is attained at $(x, y) = (0, -1), (-1, 0)$ and the global maximum is attained at $(x, y) = (0, 1), (1, 0)$.

Example 3.12. A box of volume 4 without lid is to be made. The cost is proportional to the surface area of the box. To minimize the cost, it is the same as minimizing the function

$$f(x, y, z) = xy + 2xz + 2yz$$

under the constraints $xyz = 4$ and $x, y, z > 0$. The global minimum is attained at $(x, y, z) = (2, 2, 1)$.

Theorem 3.8. (Lagrange multiplier rule) Let $f, g_1, g_2, \dots, g_k : X \to \mathbb{R}$ be functions of class C^1 where $X \subset \mathbb{R}^n$ is open and $k < n$. Let

$$S = \{\mathbf{x} \in X : g_j(\mathbf{x}) = c_j \text{ for } j = 1, 2, \dots, k\}$$

where $c_1, c_2, \dots, c_k$ are constants. If the restriction $f|_S$ of f on S has a local extremum at $\mathbf{a} \in S$ and $\{\nabla g_1(\mathbf{a}), \nabla g_2(\mathbf{a}), \dots, \nabla g_k(\mathbf{a})\}$ is linearly independent, then there exist $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{R}$ such that

$$\nabla f(\mathbf{a}) = \lambda_1 \nabla g_1(\mathbf{a}) + \lambda_2 \nabla g_2(\mathbf{a}) + \cdots + \lambda_k \nabla g_k(\mathbf{a}).$$

A set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is said to be *linearly independent* if the only solution to

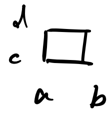
$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$$

where $a_1, a_2, \dots, a_k \in \mathbb{R}$ is $a_1 = a_2 = \cdots = a_k = 0$. In particular, when $k = 2$, this means each of $\mathbf{v}_1$ and $\mathbf{v}_2$ is not a multiple of the other.

Example 3.13. In the Euclidean space, there is a ring which is the intersection of the cylinder $x^2 + y^2 = 3$ and the plane $x + z = 1$. The charge density of the ring at the point (x, y, z) is yz coulombs per cubic unit. Then the charge density is maximized when $(x, y, z) = (-1, \sqrt{2}, 2)$, and is minimized when $(x, y, z) = (-1, -\sqrt{2}, 2)$.

4 Integration in Several Variables

4.1 Multiple Integration



Definition 4.1. Let $f : R \to \mathbb{R}$ be a function where $R = [a,b] \times [c,d]$ is a rectangle in $\mathbb{R}^2$. Consider a partition of R into the rectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ for $i, j = 1, 2, \ldots, n$ where

$$a = x_0 < x_1 < \cdots < x_n = b,$$
$$c = y_0 < y_1 < \cdots < y_n = d.$$

可看作 $[a,b] \times [c,d]$, $h=f$ 的体积.

For $i, j = 1, 2, \ldots, n$, let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_j = y_j - y_{j-1}$, and choose an arbitrary point $\mathbf{x}_{ij} \in R_{ij}$. Then the sum

$$\sum_{i=1}^{n} \sum_{j=1}^{n} f(\mathbf{x}_{ij}) \Delta x_i \Delta y_j$$

is called a *Riemann sum* of f on R.

Definition 4.2. Let $f : R \to \mathbb{R}$ be a function where $R = [a,b] \times [c,d]$ is a rectangle in $\mathbb{R}^2$. We say that f is *integrable* over R if the limit of the Riemann sum

$$\lim_{\Delta x, \Delta y \to 0} \sum_{i=1}^{n} \sum_{j=1}^{n} f(\mathbf{x}_{ij}) \Delta x_i \Delta y_j$$

exists, where the limit is taken over all partitions of R into rectangles such that $\Delta x = \max_{1 \leqslant i \leqslant n} \Delta x_i$ and $\Delta y = \max_{1 \leqslant j \leqslant n} \Delta y_j$ approach 0. In that case, we denote this limit by

$$\iint_R f \, dA$$

and call it the *double integral* of f over R.

These definitions can be easily extended to higher dimensional cases. For a *triple integral*, we can consider the integrability over a box $B = [a,b] \times [c,d] \times [q,r]$ and use the notation

$$\iiint_B f \, dV.$$

Proposition 4.1. Let $f : R \to \mathbb{R}$ be a function where $R = [a,b] \times [c,d]$ is a rectangle in $\mathbb{R}^2$. Suppose $f(\mathbf{x}) \geqslant 0$ for all $\mathbf{x} \in R$. Then the volume under the graph of f over R is given by

$$\iint_R f \, d\underbrace{A}_{\text{Area}},$$

if the integral exists.

二重积分的几何意义

一个集合 $E \subset \mathbb{R}^n$ 称为**零测集**（或勒贝格测度为零），如果对任意给定的 $\varepsilon > 0$，都存在**可数个开矩形**（或开球、开区间）$\{U_k\}_{k=1}^{\infty}$，使得：

Open Ball

参见 section 2

- $E \subset \bigcup_{k=1}^{\infty} U_k$（这些开集覆盖了 E）；
- 这些开集的总体积（即测度）小于 ε：

$$\sum_{k=1}^{\infty} \mathrm{vol}(U_k) < \varepsilon.$$

因为 ε 可以任意小（比如 0.001、0.000001……），这就说明 E "几乎没占空间"。

Proposition 4.2. Let $f : R \to \mathbb{R}$ be a bounded function where $R = [a, b] \times [c, d]$ is a rectangle in $\mathbb{R}^2$. Suppose the set of discontinuities of f on R has zero area. Then $\iint_R f\, dA$ exists.

不连续集是"零测集"

测度.

有界函数.

A function f is said to be *bounded* if there exists $M > 0$ such that $|f(\mathbf{x})| \leqslant M$ for all $\mathbf{x}$ in the domain of f. As a corollary of this proposition, (a continuous function defined on R must be integrable. In $\mathbb{R}^3$, the condition that the set of discontinuities has zero area is replaced by having zero volume.) 推论

Proposition 4.3. Let $f, g : R \to \mathbb{R}$ be integrable functions where $R = [a, b] \times [c, d]$ is a rectangle in $\mathbb{R}^2$. Then the following hold.

(a) $f \pm g$ is integrable over R and $\iint_R (f \pm g)\, dA = \iint_R f\, dA \pm \iint_R g\, dA$

(b) cf is integrable over R and $\iint_R (cf)\, dA = c \iint_R f\, dA$ for any $c \in \mathbb{R}$

(c) if $f(\mathbf{x}) \leqslant g(\mathbf{x})$ for all $\mathbf{x} \in R$, then $\iint_R f\, dA \leqslant \iint_R g\, dA$

(d) $|f|$ is integrable over R and $\left| \iint_R f\, dA \right| \leqslant \iint_R |f|\, dA$

Analogous results hold for higher dimensional cases.

Definition 4.3. Let $f : R \to \mathbb{R}$ be a function where $R = [a, b] \times [c, d]$ is a rectangle in $\mathbb{R}^2$. We define the *iterated integrals* by

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx$$

and

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy.$$

Theorem 4.1. (Fubini's theorem) Let $f : R \to \mathbb{R}$ be a bounded function where $R = [a, b] \times [c, d]$ is a rectangle in $\mathbb{R}^2$. Suppose the set of discontinuities of f on R has zero area, and every line parallel to the coordinate axes meets this set in at most finitely many points. Then

$$\iint_R f\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

Analogous definitions and results hold for higher dimensional cases.

Example 4.1. We have $\displaystyle\int_{-3}^{3}\int_{-1}^{1}\int_{0}^{2} (3y^2 + 2xy + 2xz)\, dz\, dy\, dx = 24$.

Example 4.2. The volume under the graph of $z = \sin x \cos y$ over the rectangle $[0, \pi] \times \left[0, \frac{\pi}{2}\right]$ is

$$\int_0^{\pi}\int_0^{\frac{\pi}{2}} \sin x \cos y \, dy\, dx = 2.$$

Definition 4.4. Let D be a subset of $\mathbb{R}^2$. We say that D is an *elementary region* if it is of one of the following types.

- (Type 1) $D = \{(x, y) : a \leqslant x \leqslant b,\ g(x) \leqslant y \leqslant h(x)\}$ where g and h are continuous
- (Type 2) $D = \{(x, y) : c \leqslant y \leqslant d,\ g(y) \leqslant x \leqslant h(y)\}$ where g and h are continuous
- (Type 3) D is of both type 1 and type 2

Definition 4.5. Let $f : D \to \mathbb{R}$ be a continuous function where D is an elementary region in $\mathbb{R}^2$. Let R be a rectangle containing D and define $\widetilde{f} : R \to \mathbb{R}$ by

$$\widetilde{f}(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D, \\ 0 & \text{if } (x, y) \notin D. \end{cases}$$

Then we define

$$\iint_D f\, dA = \iint_R \widetilde{f}\, dA.$$

Proposition 4.4. Let $f : D \to \mathbb{R}$ be a continuous function where D is an elementary region in $\mathbb{R}^2$. Suppose $f(\mathbf{x}) \geqslant 0$ for all $\mathbf{x} \in D$. Then the volume under the graph of f over D is given by

$$\iint_D f\, dA.$$

Proposition 4.5. Let D be an elementary region in $\mathbb{R}^2$. Then the area of D is

$$\iint_D 1\, dA.$$

The same definition and results apply if D is a more general region in $\mathbb{R}^2$, given that the integrals are well-defined.

Theorem 4.2. (Fubini's theorem) Let $f : D \to \mathbb{R}$ be a continuous function where D is an elementary region in $\mathbb{R}^2$. Then the following hold.

(a) If $D = \{(x, y) : a \leqslant x \leqslant b,\ g(x) \leqslant y \leqslant h(x)\}$ where g and h are continuous, then

$$\iint_D f\, dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y)\, dy\, dx.$$

(b) If $D = \{(x, y) : c \leqslant y \leqslant d,\ g(y) \leqslant x \leqslant h(y)\}$ where g and h are continuous, then

$$\iint_D f\, dA = \int_c^d \int_{g(y)}^{h(y)} f(x, y)\, dx\, dy.$$

Example 4.3. Let D be the triangle in $\mathbb{R}^2$ with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. Then $\displaystyle\iint_D x^2 y\, dA = \int_0^1 \int_0^{1-x} x^2 y\, dy\, dx = \frac{1}{60}$.

Example 4.4. Let D be the bounded region in $\mathbb{R}^2$ enclosed by the straight line $y = x$ and the parabola $y = -2x^2 - 2x + 2$. Then

$$\iint_D x\, dA = \int_{-2}^{\frac{1}{2}} \int_x^{-2x^2-2x+2} x\, dy\, dx = -\frac{125}{32}.$$

Example 4.5. Let $a, b, c > 0$ be real numbers. The volume of the tetrahedron in $\mathbb{R}^3$ with vertices $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ is

$$\int_0^a \int_0^{b(1-\frac{x}{a})} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy\, dx = \frac{1}{6}abc.$$

Example 4.6. We have $\displaystyle\int_0^2 \int_{\frac{y}{2}}^1 e^{x^2}\, dx\, dy = \int_0^1 \int_0^{2x} e^{x^2}\, dy\, dx = e - 1.$

Proposition 4.6. Let $f : D \to \mathbb{R}$ be a continuous function where $D \subset \mathbb{R}^2$ is bounded. If $D = D_1 \cup D_2$ where $D_1 \cap D_2$ has zero area, then

$$\iint_D f\, dA = \iint_{D_1} f\, dA + \iint_{D_2} f\, dA$$

if the integrals exist.

The definition of elementary regions can be extended to $\mathbb{R}^3$ and even $\mathbb{R}^n$. For example, a subset D of $\mathbb{R}^3$ satisfying

$$D = \{(x, y, z) : a \leqslant x \leqslant b,\ g(x) \leqslant y \leqslant h(x),\ \varphi(x, y) \leqslant z \leqslant \psi(x, y)\}$$

for some continuous functions g, h, φ and ψ is called an elementary region. In that case, for any continuous function $f : D \to \mathbb{R}$, we have

$$\iiint_D f\, dV = \int_a^b \int_{g(x)}^{h(x)} \int_{\varphi(x,y)}^{\psi(x,y)} f\, dz\, dy\, dx.$$

This gives the 4-dimensional volume under the graph of f over D. Also, the volume of D is

$$\iiint_D 1\, dV.$$

Example 4.7. The volume of a solid hemisphere of radius r is

$$\int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \int_0^{\sqrt{r^2-x^2-y^2}} 1\, dz\, dy\, dx = \frac{2}{3}\pi r^3.$$

Example 4.8. Let D be the bounded region in $\mathbb{R}^3$ enclosed by the elliptic paraboloid $y = x^2 + 4z^2$ and the plane $y = 4$. Then

$$\iiint_D \frac{1}{\sqrt{1-z^2}}\, dV = \int_{-1}^{1} \int_{-2\sqrt{1-z^2}}^{2\sqrt{1-z^2}} \int_{x^2+4z^2}^{4} \frac{1}{\sqrt{1-z^2}}\, dy\, dx\, dz = \frac{128}{9}.$$

4.2

4.2 Change of Variables

$$= \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1} = \cancel{\frac{1}{\det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}}}$$

Definition 4.6. Let $x, y : \mathbb{R}^2 \to \mathbb{R}$ be functions of class C^1 in the variables u and v. We define the *Jacobian* by

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} \overset{\frac{\partial x}{\partial u}}{x_u} & x_v \\ y_u & y_v \end{pmatrix} = x_u y_v - x_v y_u.$$

若 $x = x(u,v)$, $y = y(u,v)$
看作从[illegible] $(u,v) \to (x,y)$ 的映射,
则 $\frac{\partial(x,y)}{\partial(u,v)} = \det$(映射矩阵)

Theorem 4.3. Let $\mathbf{g}(u, v) = (x, y)$ be an injective function of class C^1 that maps from the uv-plane to the xy-plane. Let D and D' be elementary regions in the xy-plane and the uv-plane respectively such that $\mathbf{g}(D') = D$. For any integrable function $f : D \to \mathbb{R}$, we have

$$\iint_D f(x, y)\, dx\, dy = \iint_{D'} (f \circ \mathbf{g})(u, v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv.$$

对 $\int_{g(b)}^{g(a)} f((g(x)))\, d(g(x)) = \int_b^a f(g(x))g'(x)\,dx$ 的推广

In other words, if we want to make a change of variables from x, y to u, v, we need to change the region as well as multiplying the integrand by the Jacobian.

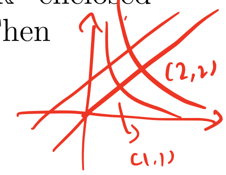
Example 4.9. Let D be the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,-1)$ and $(2,1)$. Then

$$\iint_D (x+y)^2(x-2y)^2\,dx\,dy = \iint_{D'} \frac{1}{3}u^2v^2\,du\,dv = \frac{27}{20},$$

let $u = (x+y)$, $v = (x-2y)$
$\Rightarrow x = \frac{2u+v}{3}$, $y = \frac{u-v}{3}$
$\Rightarrow$ Jacobian $\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{3}$

where D' is the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(3,0)$ and $(0,3)$.

Example 4.10. Let D be the bounded region in the first quadrant of $\mathbb{R}^2$ enclosed by the hyperbolas $xy = 1$, $xy = 4$ and the straight lines $y = x$, $y = x+2$. Then



$$\iint_D (x^2-y^2)e^{xy}\,dx\,dy = \iint_{D'} -ve^u\,du\,dv = 2(e-e^4),$$

$u = xy$, $v = y-x$
$\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1} = \frac{1}{\det\begin{pmatrix} y & x \\ -1 & 1 \end{pmatrix}} = \frac{1}{x+y}$

where D' is the rectangle $[1,4]\times[0,2]$.

$\iint_D (x+y)(x-y)e^{xy}\,dx\,dy = \iint_D (x+y)\cdot(-v)e^u \cdot \frac{1}{x+y}\,du\,dv = \sim$.

Example 4.11. Let D be the disc in $\mathbb{R}^2$ with centre $\mathbf{0}$ and radius a. Then

$$\iint_D e^{-\frac{x^2+y^2}{2}}\,dx\,dy = \iint_{D'} re^{-\frac{r^2}{2}}\,dr\,d\theta = 2\pi\left(1-e^{-\frac{a^2}{2}}\right),$$

$x = r\cos\theta$ $y = r\sin\theta$
$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r$

where D' is the rectangle $[0,a]\times[0,2\pi]$.

$\sim = \iint_{D'} re^{-\frac{r^2}{2}}\,dr\,d\theta$

In general, the conversion between the area elements of the Cartesian coordinates and polar coordinates is

以r为半径的圆

$$dA = dx\,dy = r\,dr\,d\theta.$$

One easily extends the change of variable formula for double integrals to multiple integrals. For example, in $\mathbb{R}^3$, the Jacobian is

三元 Jacobian
更多维同理

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$$

and the change of variable formula becomes

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{D'} (f\circ\mathbf{g})(u,v,w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|du\,dv\,dw.$$

The conversion between the volume elements of the Cartesian coordinates and cylindrical coordinates is

$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz,$$

while the corresponding formula for spherical coordinates is

$$dV = dx\,dy\,dz = \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

Example 4.12. The volume of a cone with radius a and height h is

$$\iiint_{\text{cone}} 1\,dV = \int_0^{2\pi}\int_0^a\int_0^{h(1-\frac{r}{a})} r\,dz\,dr\,d\theta = \frac{\pi}{3}a^2h.$$

Example 4.13. A lemon shape is the region that lies inside both of the spheres $x^2+y^2+(z-1)^2=5$ and $x^2+y^2+(z+1)^2=5$ in $\mathbb{R}^3$. The volume of this lemon shape is

$$\iiint_{\text{lemon}} 1\,dV = \int_0^{2\pi}\int_0^2\int_{1-\sqrt{5-r^2}}^{-1+\sqrt{5-r^2}} r\,dz\,dr\,d\theta = \frac{20\sqrt{5}-28}{3}\pi.$$

若 $f(x_1, x_2 \cdots x_n) = \prod f_i(x_i)$

积分区域 $D = [a_1, b_1] \times \cdots [a_n, b_n]$

则 $\overbrace{\int \cdots \int_D}^{n} = \prod \int_{a_i}^{b_i} f_i(x_i)\,dx_i$

5 Vector Calculus

5.1 Length of Curves

Definition 5.1. A *path* in $\mathbb{R}^n$ is a continuous function $\gamma : I \to \mathbb{R}^n$ where I is an interval in $\mathbb{R}$. The image set $\gamma(I)$ is called a *curve*.

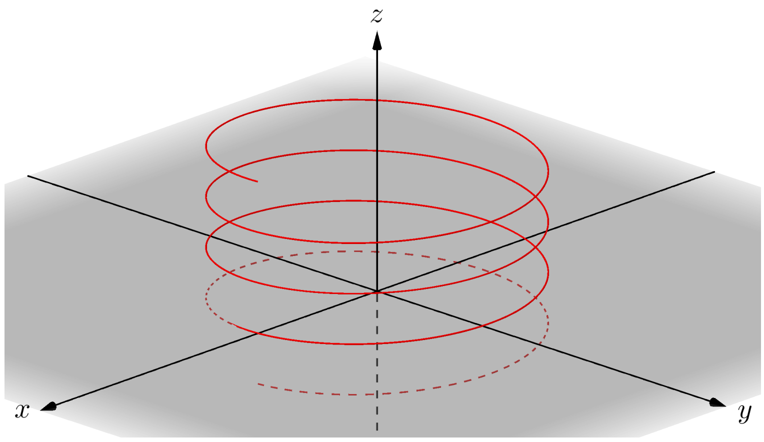
区间 图像集

If $I = [a, b]$ is a closed and bounded interval, we call $\gamma(a)$ and $\gamma(b)$ the *endpoints* of γ.

Example 5.1. Define $\gamma : \mathbb{R} \to \mathbb{R}^3$ by

$$\gamma(t) = (a \cos t, a \sin t, bt)$$

where a and b are some positive constants. The image of γ is called a *circular helix*.



Let $\gamma(t)$ be a path which is differentiable. We use $\gamma'(t)$ to denote the vector containing all derivatives of the component functions of $\gamma(t)$.

Definition 5.2. Let $\gamma : I \to \mathbb{R}^n$ be a differentiable path. Then $\mathbf{v}(t) = \gamma'(t)$ is called the *velocity vector* of the path, and its length $\|\mathbf{v}(t)\|$ is called the *speed* of the path.

It can be shown that $\mathbf{v}(t)$ is parallel to the tangent vector to $\gamma(t)$.

Example 5.2. Consider the path $\gamma : \mathbb{R} \to \mathbb{R}^3$ defined by

$$\gamma(t) = (a\cos t, a\sin t, bt).$$

Then we have

$$\mathbf{v}(t) = (-a\sin t, a\cos t, b)$$

and hence $\|\mathbf{v}(t)\| = \sqrt{a^2 + b^2}$.

Proposition 5.1. Let $\gamma : [a,b] \to \mathbb{R}^n$ be a differentiable path of class C^1, where $a, b \in \mathbb{R}$. Then the length of γ is given by

$$\int_a^b \|\gamma'(t)\|\, dt.$$

If a path is only piecewise C^1, which means it can be partitioned into a finite number of C^1 paths, then we can find its length by adding up the lengths of the pieces.

Example 5.3. The length of the path $\gamma : [0, 4\pi] \to \mathbb{R}^2$ defined by $\gamma(t) = (\cos t, \sin t)$ is

$$\int_0^{4\pi} \|(-\sin t, \cos t)\|\, dt = 4\pi.$$

5.2 Differential Operators

向量场 $R^n \to R^n$

Definition 5.3. A *vector field* on $\mathbb{R}^n$ is a function $\mathbf{F} : X \to \mathbb{R}^n$ where $X \subset \mathbb{R}^n$.

$x \longrightarrow F(x)$

We can visualize a vector field by drawing a vector $\mathbf{F}(\mathbf{x})$ at each point $\mathbf{x} \in X$ whose tail is at $\mathbf{x}$.

标量场 $R^n \to R$

Definition 5.4. A *scalar field* is a function $f : X \to \mathbb{R}$ where $X \subset \mathbb{R}^n$.

梯度场 保守向量场

Definition 5.5. A *gradient field* or a *conservative vector field* on $\mathbb{R}^n$ is a vector field $\mathbf{F} : X \to \mathbb{R}^n$ which is the gradient of some function $f : X \to \mathbb{R}$, where $X \subset \mathbb{R}^n$.

Example 5.4. Let c be a nonzero constant. The vector field $\mathbf{F} : \mathbb{R}^3 - \{\mathbf{0}\} \to \mathbb{R}^3$ defined by

$$\mathbf{F}(\mathbf{v}) = \frac{c}{\|\mathbf{v}\|^3}\mathbf{v}$$

is a gradient field. In fact, we have $\nabla f = \mathbf{F}$ where $f : \mathbb{R}^3 - \{\mathbf{0}\} \to \mathbb{R}$ is defined by

$F: X \to R^n,\ X \subset R^n$ 是一个 gradient field,

$$f(\mathbf{v}) = -\frac{c}{\|\mathbf{v}\|}.$$

① X 是单连通的开集

② 必要

①+② 充分

② $\forall\, i, j$, 有: $\frac{\partial f_i}{\partial x_i} = \frac{\partial F_j}{\partial x_j}$

or Nabel operator

Definition 5.6. The *del operator* in $\mathbb{R}^n$ is defined by

$$\nabla = \frac{\partial}{\partial x_1}\mathbf{e}_1 + \frac{\partial}{\partial x_2}\mathbf{e}_2 + \cdots + \frac{\partial}{\partial x_n}\mathbf{e}_n.$$

It maps a scalar field to a vector field.

For example, given any differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$, we have

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

This is how we define the gradient of a scalar-valued function.

Definition 5.7. Let $\mathbf{F} : X \to \mathbb{R}^n$ be a differentiable vector field where $X \subset \mathbb{R}^n$. The *divergence* of $\mathbf{F}$ is defined by

散度.

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n},$$

where $F_1, F_2, \ldots, F_n$ are the component functions of $\mathbf{F}$. The divergence maps a vector field to a scalar field.

- **散度的结果是一个标量场**：

  虽然输入的是一个向量场 F，但输出的 $\operatorname{div} F$ 在每一点上只是一个**实数**（标量），因此整体是一个**标量函数**（标量场）。

- **物理意义（直观理解）**：

  散度描述了向量场在某一点处的“**源强度**”或“**发散程度**”：

  - 如果 $\operatorname{div} F > 0$，表示该点像一个“源头”，流体从这里流出；
  - 如果 $\operatorname{div} F < 0$，表示该点像一个“汇点”，流体向这里汇聚；
  - 如果 $\operatorname{div} F = 0$，表示没有净流出或流入（例如不可压缩流体的流动）。

Example 5.5. Define $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\mathbf{F}(x, y, z) = (x^3, xyz, 2y + z + 1)$$

Then $\operatorname{div}\mathbf{F} = 3x^2 + xz + 1$.

Definition 5.8. Let $\mathbf{F} : X \to \mathbb{R}^3$ be a differentiable vector field where $X \subset \mathbb{R}^3$. The *curl* of $\mathbf{F}$ is defined by

旋度

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}, \quad = \det\begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}$$

where F_1, F_2, F_3 are the component functions of $\mathbf{F}$. The curl maps a vector field to a vector field.

Note that the curl only acts on vector fields in $\mathbb{R}^3$ (or $\mathbb{R}^2$ by regarding them as vector fields in $\mathbb{R}^3$).

重要性质

- **旋度的结果仍然是一个向量场**：输入是向量场 F，输出 $\operatorname{curl} F$ 也是一个三维向量场。
- **物理意义（直观理解）**：
  - 旋度描述了向量场在某一点附近的“**旋转程度**”或“**涡旋强度**”。
  - 如果把向量场看作流体的速度场，那么 $\operatorname{curl} F$ 的方向表示**旋转轴的方向**（右手定则），其大小表示**旋转的快慢**。
  - 若 $\operatorname{curl} F = \mathbf{0}$（零向量），称该向量场为**无旋场**（irrotational field）。

Example 5.6. Define $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\mathbf{F}(x, y, z) = (x^3, xyz, 2y + z + 1).$$

Then $\operatorname{curl}\mathbf{F}$ is the function mapping (x, y, z) to $(2 - xy, 0, yz)$.

Proposition 5.2. Let $f : X \to \mathbb{R}$ be a scalar field of class C^2 where $X \subset \mathbb{R}^3$. Then

$$\nabla \times (\nabla f) = \mathbf{0}.$$

$$= \nabla \times (f_x, f_y, f_z) = (f_{zy} - f_{yz}, \dots) = 0$$

Proposition 5.3. Let $\mathbf{F} : X \to \mathbb{R}^3$ be a vector field of class C^2 where $X \subset \mathbb{R}^3$. Then

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

dS 为面积微元，

ds 为弧长微元 (Arc length element)

则有：弧长的长度为 $\int_D ds$. ds 表示一段极小的长度.

$$ds = \begin{cases} \|r'(t)\| \, dt & \text{, 在参数方程中 . e.g. } r(t) = (x(t), y(t)) \\ \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx & \text{, 在显函数 } y = f(x) \text{ 中} \end{cases}$$

5.3 Line Integrals

Definition 5.9. Let $\gamma : [a, b] \to X$ be a path of class C^1 where $X \subset \mathbb{R}^n$. Let $f : X \to \mathbb{R}$ be a continuous function. The *scalar line integral* of f along γ is defined by

"第一类"曲线积分

$$\int_\gamma f \, ds = \int_a^b f(\gamma(t)) \, \|\gamma'(t)\| \, dt.$$

If γ is only piecewise C^1 or f is piecewise continuous, we can define the integral by adding up the contributions from different pieces.

Example 5.7. Define $\gamma : [0, 1] \to \mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$\gamma(t) = (2t, 3t, 6t) \quad \text{and} \quad f(x, y, z) = x^2 + yz.$$

Then

$$\int_\gamma f \, ds = \int_0^1 154t^2 \, dt = \frac{154}{3}.$$

Example 5.8. Consider a wire in $\mathbb{R}^2$ represented by the path $\gamma(t) = (t, |t|)$ with $-1 \leqslant t \leqslant 1$. Suppose the density function of the wire is $f(x, y) = x^2 y + 1$. Then the mass of the wire is

$$\int_\gamma f \, ds = \int_{-1}^0 \sqrt{2}(-t^3 + 1) \, dt + \int_0^1 \sqrt{2}(t^3 + 1) \, dt = \frac{5\sqrt{2}}{2}.$$

Definition 5.10. Let $\gamma : [a, b] \to X$ be a path of class C^1 where $X \subset \mathbb{R}^n$. Let $\mathbf{F} : X \to \mathbb{R}^n$ be a continuous function. The *vector line integral* of $\mathbf{F}$ along γ is defined by

> 也是 R^n 上

$$\int_\gamma \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\gamma(t)) \cdot \gamma'(t) \, dt.$$

$$= \int_a^b \sum F_i (r_i(t)) \, r'(t) \, dt$$

$$= \int_a^b \sum F_i(x) \, dx$$

If γ is only piecewise C^1 or $\mathbf{F}$ is piecewise continuous, we can define the integral by adding up the contributions from different pieces. If the component functions of $\mathbf{F}$ are $F_1, F_2, \ldots, F_n$ respectively, we can also write

$$\int_\gamma \mathbf{F} \cdot d\mathbf{s} = \int_\gamma \left(F_1\, dx_1 + F_2\, dx_2 + \cdots + F_n\, dx_n\right).$$

Example 5.9. Define $\gamma : [0,6] \to \mathbb{R}^3$ and $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\gamma(t) = (2t, t^2 + t, t + 1) \quad \text{and} \quad \mathbf{F}(x,y,z) = (y,z,x).$$

Then

$$\int_\gamma \mathbf{F} \cdot d\mathbf{s} = \int_0^6 (4t^2 + 7t + 1)\, dt = 420.$$

Example 5.10. Define $\gamma : [0,1] \to \mathbb{R}^2$ by $\gamma(t) = (t^2, 2t^4)$. Then

$$\int_\gamma (2xy)\, dx + (7xy^2)\, dy = \int_0^1 (8t^7 + 224t^{13})\, dt = 17.$$

参数化

Definition 5.11. Let C be a curve in $\mathbb{R}^n$. A *parametrization* of C is a path $\gamma : [a,b] \to \mathbb{R}^n$ of class C^1 such that ① the image of γ is C and ② γ is injective except possibly at finitely many points.

$\gamma([a.b]) = C$

几乎处处单射
i.e. 只有有限个交点

We can extend the definition by using a piecewise C^1 path. A path $\gamma : [a,b] \to \mathbb{R}^n$ is *closed* if $\gamma(a) = \gamma(b)$, and is *simple* if γ is injective, except possibly that $\gamma(a) = \gamma(b)$. A curve is closed or simple if the corresponding parametrization is closed or simple.

重参数化

Definition 5.12. Let $\gamma_1 : [a,b] \to \mathbb{R}^n$ and $\gamma_2 : [c,d] \to \mathbb{R}^n$ be paths of class C^1. We say that γ_2 is a *reparametrization* of γ_1 if there exists a bijective function $\phi : [c,d] \to [a,b]$ of class C^1 such that $\gamma_2 = \gamma_1 \circ \phi$ and the inverse $\phi^{-1} : [a,b] \to [c,d]$ is of class C^1.

保向的·

In addition, we say that γ_2 is *orientation-preserving* if $\phi(c) = a$ and $\phi(d) = b$; and say that γ_2 is *orientation-reversing* if $\phi(c) = b$ and $\phi(d) = a$.

反向的

反向

Given a path $\gamma : [a,b] \to \mathbb{R}^n$, we usually use $-\gamma$ to denote the path defined by $(-\gamma)(t) = \gamma(a + b - t)$ where $t \in [a,b]$. This gives a reparametrization of γ which is orientation-reversing.

Example 5.11. A parametrization of the unit circle in $\mathbb{R}^2$ with centre $(0,0)$ is given by $\gamma_1 : [0, 2\pi] \to \mathbb{R}^2$ where $\gamma_1(t) = (\cos t, \sin t)$.

A reparametrization of γ_1 is given by $\gamma_2 : [0,\pi] \to \mathbb{R}^2$ where $\gamma_2(t) = (\cos 2t, -\sin 2t)$. This reparametrization is orientation-reversing. The two paths are closed and simple.

Proposition 5.4. Let $\gamma_1 : [a,b] \to \mathbb{R}^n$ and $\gamma_2 : [c,d] \to \mathbb{R}^n$ be paths of class C^1 where γ_2 is a reparametrization of γ_1. Let $f : X \to \mathbb{R}$ be a continuous function where $X \subset \mathbb{R}^n$ contains the image of γ_1. Then

$$\int_{\gamma_1} f\,ds = \int_{\gamma_2} f\,ds.$$

$\gamma_2(t) = (\gamma_1 \circ \phi)(t)$

Proposition 5.5. Let $\gamma_1 : [a,b] \to \mathbb{R}^n$ and $\gamma_2 : [c,d] \to \mathbb{R}^n$ be paths of class C^1 where γ_2 is a reparametrization of γ_1. Let $\mathbf{F} : X \to \mathbb{R}^n$ be a continuous function where $X \subset \mathbb{R}^n$ contains the image of γ_1. Then the following hold.

(a) $\int_{\gamma_1} \mathbf{F} \cdot d\mathbf{s} = \int_{\gamma_2} \mathbf{F} \cdot d\mathbf{s}$ if the reparametrization is orientation-preserving

(b) $\int_{\gamma_1} \mathbf{F} \cdot d\mathbf{s} = -\int_{\gamma_2} \mathbf{F} \cdot d\mathbf{s}$ if the reparametrization is orientation-reversing

In view of these results, we can define the scalar line integral along a curve C by choosing a parametrization γ of C and then define

$$\int_C f\,ds = \int_\gamma f\,ds.$$

Similarly, if the orientation of C is given, then the vector line integral $\int_C \mathbf{F} \cdot d\mathbf{s}$ is well-defined. If C is a closed curve or can be decomposed as a finite number of closed curves, we sometimes use the notations

$$\oint_C f\,ds \quad \text{and} \quad \oint_C \mathbf{F} \cdot d\mathbf{s}$$

to mean $\int_C f\,ds$ and $\int_C \mathbf{F} \cdot d\mathbf{s}$ respectively.

Example 5.12. Let C be the segment in $\mathbb{R}^2$ joining $(0,0)$ and $(2,2)$. Consider two parametrizations $\gamma_1 : [0,2] \to \mathbb{R}^2$ and $\gamma_2 : [0,1] \to \mathbb{R}^2$ defined by

$$\gamma_1(t) = (t,t) \quad \text{and} \quad \gamma_2(t) = (2-2t, 2-2t).$$

Define $f : \mathbb{R}^2 \to \mathbb{R}$ and $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ by

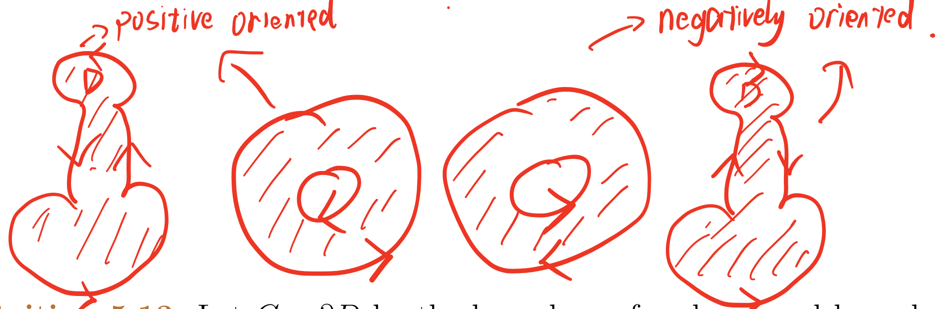
$$f(x,y) = x + 2y \quad \text{and} \quad \mathbf{F}(x,y) = (x+y, x-y).$$

Then

$$\int_{\gamma_1} f\,ds = \int_0^2 3\sqrt{2}t\,dt = 6\sqrt{2},$$
$$\int_{\gamma_2} f\,ds = \int_0^1 12\sqrt{2}(1-t)\,dt = 6\sqrt{2}$$

and

$$\int_{\gamma_1} \mathbf{F} \cdot d\mathbf{s} = \int_0^2 2t\,dt = 4,$$
$$\int_{\gamma_2} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 -8(1-t)\,dt = -4.$$



Definition 5.13. Let $C = \partial D$ be the boundary of a closed and bounded region D in $\mathbb{R}^2$. Suppose C is the union of finitely many simple closed curves. We say that C is *positively oriented* if D always lies on the left when one transverses C. Otherwise, we say that C is *negatively oriented*.

Theorem 5.1. (Green's theorem) Let $C = \partial D$ be the boundary of a closed and bounded region D in $\mathbb{R}^2$. Suppose C is the union of finitely many simple closed piecewise C^1 curves and C is positively oriented. Let $\mathbf{F} : X \to \mathbb{R}^2$ be a vector field of class C^1 where $X \subset \mathbb{R}^2$ contains D. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \oint_C F_1\,dx + F_2\,dy = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy$$

where F_1 and F_2 are the component functions of $\mathbf{F}$.

If we regard $\mathbf{F}$ as a vector field in $\mathbb{R}^3$ with k-component zero, then we can rewrite the result as

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\,dA.$$

Example 5.13. Consider the unit disc $D = \overline{B}(\mathbf{0}, 1)$ in $\mathbb{R}^2$ and orient $C = \partial D$ in the anticlockwise direction. Define $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{F}(x, y) = (x + y, 2x - y)$. On the one hand, we have

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} (-\sin^2 t - 2\sin t\cos t + 2\cos^2 t)\,dt = \pi.$$

On the other hand, we have

$$\iint_D \left(\frac{\partial(2x - y)}{\partial x} - \frac{\partial(x + y)}{\partial y} \right) dx\,dy = \iint_D 1\,dA = \pi.$$

Example 5.14. Consider the square $D = [0, 1] \times [0, 1]$ and orient $C = \partial D$ in the anticlockwise direction. Define $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{F}(x, y) = (x^2 + xy, x^2 y + 2y + 1)$. By Green's theorem, we have

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_D \left(\frac{\partial(x^2 y + 2y + 1)}{\partial x} - \frac{\partial(x^2 + xy)}{\partial y} \right) dx\,dy = \iint_D (2xy - x)\,dx\,dy = 0.$$

Theorem 5.2. (Divergence theorem in $\mathbb{R}^2$) Let $C = \partial D$ be the boundary of a closed and bounded region D in $\mathbb{R}^2$. Suppose C is the union of finitely many simple closed piecewise C^1 curves. Let $\mathbf{F} : X \to \mathbb{R}^2$ be a vector field of class C^1 where $X \subset \mathbb{R}^2$ contains D. If $\mathbf{n}$ is the outward unit normal vector to D, then

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \nabla \cdot \mathbf{F}\, dA.$$

Here, a normal vector to the region D means a vector which is perpendicular to the tangent vector to C.

Example 5.15. Consider the unit disc $D = \overline{B}(\mathbf{0}, 1)$ in $\mathbb{R}^2$ and let $C = \partial D$. Define $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ by $\mathbf{F}(x, y) = (x, y)$. On the one hand, we have

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_0^{2\pi} 1\, ds = 2\pi.$$

On the other hand, we have

$$\iint_D \nabla \cdot \mathbf{F}\, dA = \iint_D 2\, dA = 2\pi.$$

$\phi(s,t) = (x(s,t), y(s,t), z(s,t))$

- 所以，Φ 就像一个“模具”或“变形器”，把平面上的区域 D “弯曲”、“拉伸”或“卷起来”，嵌入到三维空间中，形成一个曲面。

补充说明：

- 如果 Φ 是**光滑且正则**（即偏导数 $\frac{\partial \Phi}{\partial s}$ 和 $\frac{\partial \Phi}{\partial t}$ 线性无关），那么这个映射在局部就像一个“没有折叠、没有撕裂”的光滑变形。
- 即使 Φ 只是连续且单射（如定义 5.14 所述），它也能定义一个“拓扑曲面”——可能不够光滑，但仍然是一个合理的曲面。

✅ 所以你的理解非常准确：**Φ 就是把平面上的一块区域“映射”或“展开”成三维空间中的一个曲面。**

5.4 Surface Integrals

Definition 5.14. Let $\Phi : D \to \mathbb{R}^3$ be a continuous function where $D \subset \mathbb{R}^2$ is open and connected, possibly together with some or all of its boundary points. If Φ is injective, except possibly along ∂D, then we say that Φ is a *parametrized surface*, and we say that $\Phi(D)$ is the *surface* parametrized by Φ.

仅边界上允许重合映射

A subset D is *connected* if every two points in D can be joined by a path in D.

Example 5.16. Define $\Phi : [0, \pi] \times [0, 2\pi] \to \mathbb{R}^3$ by

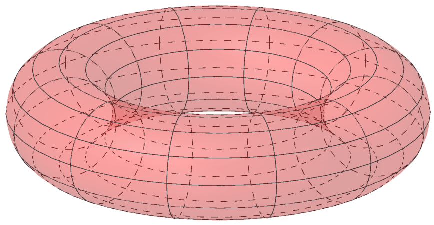
$$\Phi(s, t) = (\sin s \cos t, \sin s \sin t, \cos s).$$

Then Φ defines a surface in $\mathbb{R}^3$ which is a sphere.

Example 5.17. Define $\Phi : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3$ by

$$\Phi(s,t) = ((3 + \cos s)\cos t, (3 + \cos s)\sin t, \sin s).$$

Then Φ defines a surface in $\mathbb{R}^3$ which is called a *torus*.



Definition 5.15. Let $\Phi : D \to \mathbb{R}^3$ be a parametrized surface where $D \subset \mathbb{R}^2$. Let $(a,b) \in D$. The *s-coordinate curve* at $t = b$ is the image of the function $f : I_1 \to \mathbb{R}^3$ defined by

固定另一个参数

$$f(s) = \Phi(s,b)$$

where $I_1 \subset \mathbb{R}$ consists of all values of s such that $(s,b) \in D$.

Similarly, the *t-coordinate curve* at $s = a$ is the image of the function $g : I_2 \to \mathbb{R}^3$ defined by

$$g(t) = \Phi(a,t)$$

where $I_2 \subset \mathbb{R}$ consists of all values of t such that $(a,t) \in D$.

切向量

We use the notations $\mathbf{T}_s = \dfrac{\partial \Phi}{\partial s}(s,b)$ and $\mathbf{T}_t = \dfrac{\partial \Phi}{\partial t}(a,t)$ to denote the *tangent vectors* to the coordinate curves.

Definition 5.16. Let $\Phi : D \to \mathbb{R}^3$ be a parametrized surface of class C^1 where $D \subset \mathbb{R}^2$. Let $(a,b) \in D$. The *standard normal vector* (with respect to the parametrization Φ) is defined by

标准法向量

$$\mathbf{N}(a,b) = \mathbf{T}_s(a,b) \times \mathbf{T}_t(a,b).$$

We say that the surface $S = \Phi(D)$ is *smooth* at the point $\Phi(a,b)$ if $\mathbf{N}(a,b) \neq \mathbf{0}$. If S is smooth at every point, then it is said to be *smooth*.

分片

We can extend the definition of parametrized surfaces to piecewise parametrized surfaces. Loosely speaking, this simply means the union of a finite number of disjoint surfaces, possibly except the boundary. Similarly, we can define a piecewise smooth surface.

分片光滑.

- 法向量 $\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t$ 垂直于曲面在该点的**切平面**。
- 如果 $\mathbf{N} = \mathbf{0}$，说明两个切方向“塌陷”到同一直线上（甚至为零向量），曲面在该点可能有**尖点**、**褶皱或自交**，无法定义唯一的切平面——这样的点称为**奇点**（singular point）。
- 要求 $\mathbf{N} \neq \mathbf{0}$ 就是为了保证曲面在局部看起来像一张“光滑展开的纸”，没有撕裂或折叠。

“光滑”在这里的数学含义就是：参数化在每一点都能给出一个良好定义的、非零的法向量。

Example 5.18. Consider the parametrization $\Phi : D \to \mathbb{R}^3$ of the sphere defined by

$$\Phi(s,t) = (\sin s \cos t, \sin s \sin t, \cos s)$$

where $D = [0,\pi] \times [0, 2\pi]$. Then

$$\begin{aligned} \mathbf{T}_s(s,t) &= (\cos s \cos t, \cos s \sin t, -\sin s), \\ \mathbf{T}_t(s,t) &= (-\sin s \sin t, \sin s \cos t, 0), \\ \mathbf{N}(s,t) &= (\sin^2 s \cos t, \sin^2 s \sin t, \sin s \cos s). \end{aligned}$$

Proposition 5.6. Let $\Phi : D \to \mathbb{R}^3$ be a smooth parametrized surface of class C^1 where $D \subset \mathbb{R}^2$. Then the surface area of $S = \Phi(D)$ is

$$\iint_D \|\mathbf{N}(s,t)\| \, dA.$$

i.e. $dS = \|N(s,t)\| dA$

此外，命题还指出：

> 在大多数情况下，即使曲面在有限个点处不光滑（即法向量在这些点为零或未定义），上述公式仍然适用。

这是因为：

- 有限个点的“缺陷”在二维积分中**不影响整体积分值**（它们的测度为零），
- 只要曲面**几乎处处光滑**（即除了有限个孤立奇点外都光滑），表面积仍可由该二重积分正确计算。

直观理解：

- $\|\mathbf{N}(s,t)\|$ 实际上衡量了参数平面上一个无穷小矩形 $ds \times dt$ 被映射到三维空间后所“拉伸”成的平行四边形的面积。
- 对整个参数域 D 积分这个“局部面积放大因子”，就得到整个曲面的真实表面积。

Example 5.19. Consider the parametrization $\Phi : D \to \mathbb{R}^3$ of the sphere defined by

$$\Phi(s,t) = (\sin s \cos t, \sin s \sin t, \cos s)$$

where $D = [0,\pi] \times [0, 2\pi]$. Then the surface area of this sphere is

$$\int_0^{2\pi} \int_0^{\pi} \sin s \, ds \, dt = 4\pi.$$

Definition 5.17. Let $\Phi : D \to \mathbb{R}^3$ be a smooth parametrized surface where $D \subset \mathbb{R}^2$. Let $f : S \to \mathbb{R}$ be a continuous function where $S = \Phi(D)$. The *scalar surface integral* of f along Φ is defined by

$$\iint_\Phi f \, dS = \iint_D f(\Phi(s,t)) \, \|\mathbf{N}(s,t)\| \, ds \, dt.$$

If S is only piecewise smooth or f is piecewise continuous, we can define the integral by adding up the contributions from different pieces.

分片积分，相加.

Example 5.20. Define $\Phi : D \to \mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$\Phi(s,t) = ((3 + \cos s)\cos t, (3 + \cos s)\sin t, \sin s) \quad \text{and} \quad f(x,y,z) = z + 1$$

where $D = [0, 2\pi] \times [0, 2\pi]$. Then

$$\iint_\Phi f \, dS = \int_0^{2\pi} \int_0^{2\pi} (\sin s + 1)(3 + \cos s) \, ds \, dt = 12\pi^2.$$

Definition 5.18. Let $\Phi : D \to \mathbb{R}^3$ be a smooth parametrized surface where $D \subset \mathbb{R}^2$. Let $\mathbf{F} : S \to \mathbb{R}^3$ be a continuous function where $S = \Phi(D)$. The ~~*vector*~~ *surface integral* of $\mathbf{F}$ along Φ is defined by

$$\iint_{\Phi} \mathbf{F} \cdot d\mathbf{S} = \iint_{D} \mathbf{F}(\Phi(s,t)) \cdot \mathbf{N}(s,t)\, ds\, dt.$$

If S is only piecewise smooth or $\mathbf{F}$ is piecewise continuous, we can define the integral by adding up the contributions from different pieces.

Example 5.21. Define $\Phi : D \to \mathbb{R}^3$ and $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\Phi(s,t) = (s, t, 4 - s^2 - t^2) \quad \text{and} \quad \mathbf{F}(x,y,z) = (x, y, z - 2y)$$

where $D = \overline{B}(\mathbf{0}, 2)$ is a subset of $\mathbb{R}^2$. Then

$$\iint_{\Phi} \mathbf{F} \cdot d\mathbf{S} = \iint_{D} (s^2 + t^2 - 2t + 4)\, dA = 24\pi.$$

Definition 5.19. Let $\Phi_1 : D_1 \to \mathbb{R}^3$ and $\Phi_2 : D_2 \to \mathbb{R}^3$ be parametrized surfaces where $D_1, D_2 \subset \mathbb{R}^2$. We say that Φ_2 is a *reparametrization* of Φ_1 if there exists a bijective function $\phi : D_2 \to D_1$ such that $\Phi_2 = \Phi_1 \circ \phi$. If Φ_1 and Φ_2 are smooth and both ϕ and ϕ^{-1} are of class C^1, we say that Φ_2 is a *smooth reparametrization* of Φ_1.

Example 5.22. Define $\Phi_1 : \{(s,t) : 1 \leqslant s^2 + t^2 \leqslant 4\} \to \mathbb{R}^3$ by

$$\Phi_1(s,t) = (s, t, 4 - s^2 - t^2).$$

A smooth reparametrization of Φ_1 is given by $\Phi_2 : [1,2] \times [0, 2\pi) \to \mathbb{R}^3$ where

$$\Phi_2(s,t) = (s \cos t, s \sin t, 4 - s^2).$$

Definition 5.20. Let $\Phi_1 : D_1 \to \mathbb{R}^3$ and $\Phi_2 : D_2 \to \mathbb{R}^3$ be parametrized surfaces where $D_1, D_2 \subset \mathbb{R}^2$. Suppose Φ_2 is a smooth reparametrization of Φ_1 where $\Phi_2 = \Phi_1 \circ \phi$. We say that the reparametrization is *orientation-preserving* if the Jacobian of ϕ is always positive, and *orientation-reversing* if the Jacobian is always negative.

Proposition 5.7. Let $\Phi_1 : D_1 \to \mathbb{R}^3$ and $\Phi_2 : D_2 \to \mathbb{R}^3$ be smooth parametrized surfaces where $D_1, D_2 \subset \mathbb{R}^2$. Suppose Φ_2 is a smooth reparametrization of Φ_1. Let $f : X \to \mathbb{R}$ be a continuous function where $X \subset \mathbb{R}^3$ contains the image of Φ_1. Then

$$\iint_{\Phi_1} f\, dS = \iint_{\Phi_2} f\, dS.$$

Proposition 5.8. Let $\Phi_1 : D_1 \to \mathbb{R}^3$ and $\Phi_2 : D_2 \to \mathbb{R}^3$ be smooth parametrized surfaces where $D_1, D_2 \subset \mathbb{R}^2$. Suppose Φ_2 is a smooth reparametrization of Φ_1. Let $\mathbf{F} : X \to \mathbb{R}^3$ be a continuous function where $X \subset \mathbb{R}^3$ contains the image of Φ_1. Then the following hold.

(a) $\iint_{\Phi_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{\Phi_2} \mathbf{F} \cdot d\mathbf{S}$ if the reparametrization is orientation-preserving

(b) $\iint_{\Phi_1} \mathbf{F} \cdot d\mathbf{S} = -\iint_{\Phi_2} \mathbf{F} \cdot d\mathbf{S}$ if the reparametrization is orientation-reversing

In view of these results, we can define the scalar surface integral along a smooth surface S by choosing a parametrization Φ of S and then define

$$\iint_S f\, dS = \iint_\Phi f\, dS.$$

Similarly, if the orientation of S is given, then the vector surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ is well-defined. A surface is called a *closed surface* if it is compact and has no boundary. (In this context, the boundary ∂S of a surface S is different from Definition 2.11.) If S is a closed surface or can be decomposed as a finite number of closed surfaces, we sometimes use the notations

$$\oiint_S f\, dS \quad \text{and} \quad \oiint_S \mathbf{F} \cdot d\mathbf{S}$$

to mean $\iint_S f\, dS$ and $\iint_S \mathbf{F} \cdot d\mathbf{S}$ respectively.

Example 5.23. Let S be the whole surface of the cylinder with radius 3 and height 15, whose axis of symmetry is the z-axis and whose circular faces are located at $z = 0$ and $z = 15$ respectively. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x, y, z) = z$. Then

$$\iint_S f\, dS = \int_0^{2\pi} \int_0^{15} 3t\, dt\, ds + \int_0^3 \int_0^{2\pi} 15s\, dt\, ds = 810\pi.$$

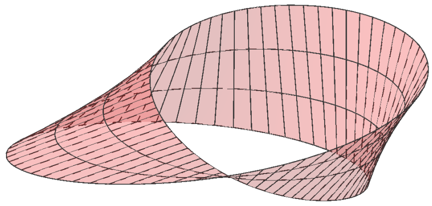
有内外/上下之

Definition 5.21. A smooth and connected surface S is *orientable* if we can define a single unit normal vector at each point of S so that the normal vectors vary continuously. Otherwise, S is *nonorientable*.

Example 5.24. Define $\Phi : [0, 2\pi] \times \left[-\frac{1}{2}, \frac{1}{2}\right] \to \mathbb{R}^3$ by

$$\Phi(s,t) = \left(\left(1 + t\cos\frac{s}{2}\right)\cos s, \left(1 + t\cos\frac{s}{2}\right)\sin s, t\sin\frac{s}{2}\right).$$

Then Φ defines a surface in $\mathbb{R}^3$ which is called a *Möbius strip*. This surface is nonorientable.



Definition 5.22. Let S be a bounded, piecewise smooth, orientable surface in $\mathbb{R}^3$ with a given orientation. Suppose the boundary ∂S of S is the union of finitely many simple closed piecewise C^1 curves. We say that ∂S is *oriented consistently* if the orientation of each closed curve in ∂S is chosen such that the right-hand rule is satisfied (which means if we use the fingers of the right hand to curl in the direction of the curve, then the thumb will point in the direction of the normal to S).

Theorem 5.3. (Stokes' theorem) Let S be a ① bounded, ② piecewise smooth, ③ orientable surface in $\mathbb{R}^3$ with a given orientation. Suppose the boundary ∂S of S is the union of finitely many simple closed piecewise C^1 curves which are oriented consistently with S. Let $\mathbf{F} : X \to \mathbb{R}^3$ be a vector field of class C^1, where $X \subset \mathbb{R}^3$ contains S. Then

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

Green's theorem is a special case of Stokes' theorem in $\mathbb{R}^2$. On the other hand, Stokes' theorem has a generalization in $\mathbb{R}^n$ using differential forms. The case $n = 1$ corresponds to the fundamental theorem of calculus. For this reason, Stokes' theorem and Green's theorem (together with Gauss's theorem) are sometimes known as the fundamental theorems of multivariable calculus.

Example 5.25. Consider the hemisphere $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1,\ z \geqslant 0\}$ in $\mathbb{R}^3$. Orient the surface using the outward normals. Define $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\mathbf{F}(x, y, z) = (x + y, -x + y, 0).$$

On the one hand, we have

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = -\int_0^{2\pi} \int_0^{\frac{\pi}{2}} 2 \sin s \cos s \, ds \, dt = -2\pi.$$

On the other hand, if ∂S is oriented consistently with S, then

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} -1 \, dt = -2\pi.$$

✅ **定理 5.3：Stokes 定理（斯托克斯定理）**

设：

- S 是 $\mathbb{R}^3$ 中一个**有界、分片光滑、可定向**的曲面，已给定定向；
- 其边界 ∂S **是有限条简单闭合、分段 C^1 曲线**，且定向与 S 一致；
- $\mathbf{F} : X \to \mathbb{R}^3$ 是 C^1 类向量场，X 包含 S。

则：

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$

> **物理意义**：旋度在曲面上的通量 = 向量场沿边界曲线的环量。

> **特例**：

- 在 $\mathbb{R}^2$ 中，Stokes 定理退化为 **Green 定理**。
- 在 $\mathbb{R}^1$ 中，对应**微积分基本定理**。
- 因此，Stokes 定理、Green 定理、Gauss 定理统称为**多元微积分的基本定理**。

✅ **定理 5.4：Gauss 定理（散度定理 / 高斯定理）**

设：

- $D \subset \mathbb{R}^3$ 是一个**有界立体区域**；
- 其边界 ∂D 是**有限个分片光滑、闭合、可定向曲面**的并集；
- 边界曲面的定向取为**指向 D 外部的法向量**（即"向外"）；
- $\mathbf{F} : X \to \mathbb{R}^3$ 是 C^1 类向量场，X 包含 D。

则：

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D (\nabla \cdot \mathbf{F})\, dV.$$

> **物理意义**：向量场从区域 D 流出的总通量 = D 内部源/汇的总量（散度积分）。

> 常用于电磁学（高斯电场定律）、流体力学等。

Example 5.26. Consider the surface S defined by $z = e^{-(x^2+y^2)}$ where $x^2 + y^2 \leqslant 1$. Orient the surface using normals pointing upwards. Define $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\mathbf{F}(x, y, z) = (e^{y+z} - 2y, xe^{y+z} + y, e^{x+y}).$$

Then

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = 2\pi.$$

Theorem 5.4. (Gauss's theorem/Divergence theorem) Let D be a bounded solid region in $\mathbb{R}^3$. Suppose the boundary ∂D of D is the union of finitely many piecewise smooth, closed orientable surfaces which are oriented by normals pointing away from D. Let $\mathbf{F} : X \to \mathbb{R}^3$ be a vector field of class C^1, where $X \subset \mathbb{R}^3$ contains D. Then

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F}\, dV.$$

Example 5.27. Consider the solid $D = \{(x, y, z) : 0 \leqslant z \leqslant 9 - x^2 - y^2\}$ in $\mathbb{R}^3$. Orient the surface using outward normals. Define $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\mathbf{F}(x, y, z) = (x, y, z).$$

On the one hand, we have

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \int_{-3}^{3} \int_{-\sqrt{9-t^2}}^{\sqrt{9-t^2}} (s^2 + t^2 + 9)\, ds\, dt + 0 = \frac{243}{2}\pi.$$

On the other hand, we have

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \iiint_D 3\, dV = \frac{243}{2}\pi.$$

Example 5.28. Consider the surface S defined by $z = (1 - x^2 - y^2)e^{1-x^2-3y^2}$ where $x^2 + y^2 \leqslant 1$. Orient the surface using normals pointing upwards. Define $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$\mathbf{F}(x, y, z) = (e^y \cos z, \sqrt{x^3 + 1} \sin z, x^2 + y^2 + 3).$$

Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{7}{2}\pi.$$

$\iint_R f \, dA$ double integrable

$\iiint_B f \, dV$

$\nabla \cdot F = \sum \frac{\partial F_i}{\partial x_i}$ $\quad \nabla \times F = \det\begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{pmatrix}$

$\int_r f \, ds \qquad dS = \|N(s,t)\| \, ds$

$= \int_r f(r(t)) \, r'(t) \, dt$

$\oint_C f \, ds \rightarrow$ C为闭合曲线

$F' = (F_1, F_2, 0)$

$\oint_C F \, ds = \oint_C F_1 dx + F_2 dy = \iint_D (\nabla \times F') \cdot (0,0,1) \, dA$

$\oint_C F \cdot n \, ds = \iint_D \nabla \cdot F \, dA = \iint \sum \frac{\partial F_i}{\partial x_i} \, dA$

$\iint_\Phi F \, dS = \iint_D F(\Phi) N(\Phi) \, dA$

$\iint_S \nabla \times F \, dS = \oint_{\partial S} F \cdot ds \qquad \oiint_{\partial D} F \, dS = \iiint_D \nabla \cdot F \, dV$