cheatsheet

PART1: definition

1. Triangular matrix:

A matrix that only upper or lower triangular entry is non-zero.
  A matrix is **upper triangular** if the (i j)-entry is 0 whenever $i > j$.
  A matrix is **lower triangular** if the (i j)-entry is 0 whenever $i < j$.

2. Diagonal matrix:

A square matrix that both upper and lower triangular
(i.e. Only diagonal entries is non-zero)

3. Standard vectors

A vector in whose only i-th entry is 1 is denoted by e_i : $e_1, e_2, ..., e_n$ are collectively known as the **standard vectors** e_i
(also called **standard unit vectors** or **standard basis vectors**)

4. Identity Matrix I:

A matrix that only diagonal entries are 1, others are all 0.

5. Equicalent

The two systems are **equivalent** if they have the same solution set.

6. Augmented matrix:

We append RHS constants vector to coefficient matrix, written as the following form: $[A \mid b]$

7. EROs: elementary row operations

(I) **Exchange** two rows
(II) **Multiply** a row by a nonzero constant

(III) **Add** a multiple of a row to another row

8. REF row echelon matrix:

(1) Zero rows must be at the bottom of the matrix (if any)
(2) The **leading entry** (i.e. first non-zero entry, also called **pivot**) of non-zero row must be on the right of the leading entries in the rows above (i.e. the entries below a leading entry must be 0)
We call it is in **row echelon form (REF)**

9. RREF reduced row echelon matrix:

(3) The column that each leading entry in only has one non-zero entry (i.e. pivot itself)
(4) Every leading entry of non-zero rows are 1
We call it is in **reduced row echelon form (RREF)**, or **row canonical form**

Inconsistent:

If there is a linear system with no solution at all, it's **inconsistent.**
i.e. It's RREF has $[0\ 0\ ...\ 0 \mid 1]$

10. **Rank** and **Nullity**:

The **rank** of A (denoted by $rank(A)$) is the number of **pivots** in the RREF of A(also the REF of A)
The **nullity** of A (denoted by $nullity(A)$) is defined to be the number of **free variables** in the solutions of $Ax = 0$

11. **Span** and **Generation set**:

Let S be a finite non-empty set of vectors in $\mathbb{R}^n$, the **span** of S (denoted by $span(S)$) is defined to be the set of all linear combinations of the vectors in S
And S is called the **generation set** of $span\{s\}$;

12. Elementary matrix

An **elementary matrix** is a matrix obtained from the identity matrix by performing a **single ERO**. Specifically:

- **Type I Elementary Matrix**:
  - Corresponds to **swapping** two different rows.

    - e.g. swapping the first row with the second row in a 3x3 identity matrix.
- **Type II Elementary Matrix**:
    - Corresponds to **multiplying** a row by a non-zero scalar.
    - e.g. multiplying the second row of the identity matrix by a non-zero constant k.
- **Type III Elementary Matrix**:
    - Corresponds to **adding** a multiple of one row to another row.
    - e.g. in the identity matrix, adding k times the second row to the first row (where k is any constant).

13. LU Decomposition

LU decomposition of A is a factorization of the form $A = LU$ in which L is a unit lower triangular (square) matrix (i.e. all entries on the diagonal are 0) and U is upper triangular (not necessarily square).

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = LU$$
$$\text{其中：} L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

14. PLU Decomposition

When a matrix **hasn't LU decomposition**, we can find an **invertible** permutation matrix P, so that $P^(-1)A$ has LU decomposition.

$$P^{-1}A = LU$$
$$A = PLU$$

As addition, because permutation matrix is an **Orthogonal Matrix**, so we have:

$$P^{-1} = P^T$$

15. Matrix Transformation

Let A be an m×n matrix. The function:

$$T_a:\ R^n \to R^m$$
$$defined\ by\ T_a(x) = Ax$$

is said to be the matrix transformation induced by A (such A is called the standard matrix of T).

16. Linear Transformation

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be a linear transformation if

$$T(u+v) = T(u) + T(v)$$
$$and$$
$$T(cu) = cT(u)$$

for any $u, v \in \mathbb{R}^n$ and $c \in \mathbb{R}$,
i.e. T **preserves** addition and scalar multiplication.

17. Injectivity and surjectivity

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a **linear transformation** with standard matrix A. Then:
(a) T is **injective** if and only if $rankA = n$ (equivalently, T has null space{0}).
(b) T is **surjective** if and only if $rankA = m$.

18. Null space of T (aka null space of A, kernel of T)

The **preimage**(原像) of $\{0\}$
i.e. the set of all v such that $T(v) = 0$ is called the **null space** of T.

19. cofactor

Let A = $[a_{ij}]$ be an $n \times n$ matrix.
A_{ij} is a submatrix of A that obtained by deleting the $i-th$ row and $j-th$ column.
C_{ij} is (i, j)-cofactor, defined by

$$C_{ij} = (-1)^{i+j} \times det(A_{ij})$$

20. Determination

For an $n \times n$ matrix $A = [a_{ij}]$, we define:

$$detA = \sum_{i=1}^{n} a_{ij} \cdot C_{ij} \; or \; \sum_{j=1}^{n} a_{ij} \cdot C_{ij}$$
$$where \; i, j \in \mathbb{R}$$

Knowns as “The **cofactor expansion** along the **j-th column** or **i-th row**”

21. adjoint

The adjoint of A, denoted by $adj(A)$.
$adj(A) = [C_{ij}]$

22. Subspace

A subset $W \subseteq \mathbb{R}^n$ is said to be a ***subspace*** of $\mathbb{R}^n$ if it satisfies the following:

- $\vec{0} \in W$
- If $\vec{x}, \vec{y} \in W$, then $\vec{x} + \vec{y} \in W$(i.e. W is closed under addition)
- If $\vec{x} \in W$ and $c \in \mathbb{R}$, then $c\vec{x} \in W$(i.e. W is closed under scalar multiplication)

For each $m \times n$ matrix A, we have:

23. Row space

Notion: $Row\ A$

Subspace of $\mathbb{R}^n$

Span of the rows of A

24. Column space

Notion: $Col\ A$

Subspace of $\mathbb{R}^m$

Span of the columns of A

25. Null space

Notion: $Null\ A$

Subspace of $\mathbb{R}^n$

Solution set of $A\vec{x} = 0$

26. Basis:

Let V be a subspace of $\mathbb{R}^n$. A linearly independent generating set for V is called a basis for V.
Remarks:

- The plural for basis is ***bases***. (单词basis的复数形式是bases)
- A basis for V must be a subset of V.
- Every basis for $\mathbb{R}^n$ consists of exactly n vectors.

27. Reduction Theorem (约简定理) and Extension Theorem (扩展定理):

Let V be a non-zero subspace of $\mathbb{R}^n$. We have the following:

- (Reduction theorem) Every finite generating set of V contains a basis.
- (Extension theorem) Every linearly independent subset of V can be extended to a basis.
  (By convention we say that only basis of the zero subspace is the empty set.)

28. Dimension

Any two bases for V contain the **same number** of vectors. This number is said to be the **dimension of** V and is denoted by $dim(V)$.
(By convention the **zero subspace** is defined to have **dimension 0**.)

28. Coordinate vector:

lemma:

Let $B = \{b_1, b_2 ... b_k\}$ be an ***ordered basis*** for a subspace V.
Then each $v \in V$ can be written as a unique linear combination of the vectors in B.
In the forms as:

$$v = c_1 b_1 + c_2 b_2 + ... + c_k b_k$$

Hence we define :

$$[v]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \dots \\ c_k \end{bmatrix} = B^{-1} v$$

to be the ***coordinate vector of v relative to B*** (or ***B-coordinate vector of v***).

29. Similarity of matrices

Let A and B be square matrices. We say A is **similar** to B if

$$B = P^{-1}AP$$

for some invertible matrix P.
Thus in some sense, **similar matrices** can be seen as **matrices representing the same linear transformation with respect to different bases**.

30. eigencalue, eigenvactor and eigenspace

Let A be a square matrix. If $Ax = \lambda x$ for some non-zero vector x and scalar t
λ is called to be an **eigenvalue** of A ;
x is called an **eigenvector** of A corresponding to the eigenvalue ,or in short, a of A.
and we call $Null(A - tI)$ as *eigenspace* of A corresponding of t, or in shorter, the t-eigenspace of A

31. character equation and character polynomial

In the progess of finding the eigenvalue, we define **character equation** is:

$$det(A - \lambda I) = 0$$

and character polynomial is LHS:

$$det(A - \lambda I)$$

32. Algebraic multiplicity and geometric multiplicity

lemma:

A degree n polynomial (where $n > 1$) with coefficients in $\mathbb{C}$ has exactly n zeros(零点) in $\mathbb{C}$ (counting multiplicities，重根).
It thus follows that a $n \times n$ matrix A has exactly n eigenvalues in $\mathbb{C}$ (counting multiplicities)

- The ***algebraic multiplicity*** (or simply ***multiplicity***) of an eigenvalue is the number of times it appears as a zero of the character polynomial.
- The ***geometric multiplicity*** of an eigenvalue is the dimension of its corresponding eigenspace.

33. Diagonalization(对角化)

diagonalizable:

An $n \times n$ matrix A is **diagonalizable** IFF it has n **linear independent** eigenvectors.

- Eigenvectors of a matrix A that correspond to distinct eigenvalues are linearly independent.
- If a matrix A has an eigenvalue whose geometric multiplicity is less than the algebraic multiplicity, then A is not diagnosable.

Thus, A matrix A is diagonalizable if and only if for each of its eigenvalues, the algebraic and geometric multiplicities are equal.

Process

If A is a diagonalizable $n \times n$ matrix, it has n **eigenvalues**:$\{\lambda_1, \lambda_2, ...\lambda_n\}$ and n **corresponding linear independent eigenvectors**:$\{v_1, v_2, ...v_n\}$

$$(i.e. \forall i \in \mathbb{R} \ Av_i = \lambda_i v_i)$$

Then we have:

$$A = PDP^-1$$

where P is **eigenvector matrix** and D is **diagonal matrix of eigenvalues**

$$P = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}, \ D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

PART2: property

1. Given the system Ax = b, the following statements are equivalent.

(a) The system is consistent.

(b) The vector b is a linear combination of the columns of A.

(c) The reduced row echelon form of the augmented matrix of the system has no row of the form $[0\ 0\ \dots\ 0 \mid 1]$.

更一般地，在解决具有无限多解的线性系统时，我们可以将增广矩阵转换为简化行最简形式。设定那些对应于非主元列的变量为自由变量(free variables)，而那些对应于主元列的变量为基础变量(basic variables)。需要注意的是，简化行最简形式使得基础变量很容易用自由变量表示出来。

2. Let A be an $m \times n$ matrix. The following statements are equivalent.

(a) $Ax = b$ is consistent for every $\boldsymbol{b} \subseteq \mathbb{R}^m$.

(b) The span of the columns of A is $\mathbb{R}^m$.

(c) The RREF of A has no zero row.

(c') The RREF of $[A \mid b]$ has no row of the form $[0\ 0\ \dots\ 0 \mid 1]$ for every $\boldsymbol{b} \subseteq \mathbb{R}^m$

(d) rank(A) = m

3. Let A be an $m \times n$ matrix. The following statements are equivalent.

(a) The columns of A are linearly independent.

(b) $Ax = b$ has at most one solution for every $\boldsymbol{b} \subseteq \mathbb{R}^m$.

(c) $nullity(A) = 0$

(d) $rank(A) = n$

(e) The RREF of A is $[e_1\ e_2\ \dots\ e_n]$

(f) The system $Ax = 0$ only has the **trivial solution**.

4. Equivalent conditions about invertibility:

The following statements are equivalent for an n×n matrix
(1) A is invertible
(2) The RREF of A is I.
(3) The span of the columns of A is $\mathbb{R}^n$
(4) $rank(A) = n$.(i.e. $nullity(A) = 0$)
(5) $Ax = b$ is consistent for every $b \in \mathbb{R}^n$
(6) The columns of A are linearly independent.
(7) $Ax = 0$ only has the trivial solution.
(8) There exists a matrix B such that $BA = I$.
(9) There exists a matrix C such that $AC = I$.
(10) A is a product of elementary matrices.
(11) $det(A) \neq 0$

5. Common geometric transformation

(1) Reflection on x/y - axis

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ -y \end{bmatrix} \ or \ \begin{bmatrix} -x \\ y \end{bmatrix}$$

just multiply following matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ or \ \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

(2)Translation upward by 1 unit

Not exist a linear transformation fot it

(3)Enlargement about the origin by a factor of k

$$\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} kx \\ ky \end{bmatrix}, \ k \in \mathbb{R}$$

just multiply following matrix:

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

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6. A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is **linear** if and only if it is a **matrix transformation**.

7. Simplification of evaluating the determinant

- Find the tow/column that with more zero.
- The determinant of the triangular matrix, is equal to the product of the non-zero matrix.
  - $det\begin{pmatrix} A_{11} & * & \dots & * \\ 0 & A_{22} & \dots & * \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & A_{nn} \end{pmatrix} = \prod_{i=1}^{n} A_{ii}$
  - Generalized, if each A_{ii} is a block matrix, then we have:
    - $det\begin{pmatrix} A_{11} & * & \dots & * \\ 0 & A_{22} & \dots & * \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & A_{nn} \end{pmatrix} = \prod_{i=1}^{n} det(A_{ii})$
- ERO's effect on determinant:
  - Type 1 EROs—Exchange two rows
    - $det(E_1 A) = (-1) \times det(A)$
  - Type 2 EROs—Multiply one row by a constant k:
    - $det(E_2 A) = k \times det(A)$
  - Type 3 EROs—Add a row to another row:
    - $det(E_3 A) = det(A)$
  - Essentially, that is because:
    - $det(EA) = det(E) \times det(A)$

8. Properties of determinants

Let A be a square matrix. Then

- A is invertible if and only if $det(A) \neq 0$
- $det(AB) = det(A) \times det(B)$(if they have the same size)
- $det(A^T) = det(A)$
- $det(A^{-1}) = \frac{1}{det(A)}$
- Let $T: R^n \to R^n$ be an invertible linear transformation with standard matrix A. Then for any "**sufficiently nice region**" $S \in R^n$ (Usually refers to the region that can calculate the volumn), **the n-dimensional volume of T(S) is equal to** $|det(A)|$ **times the n-dimensional volume of S**.

9. Use determinant to solve the inverse matrix.

we have:
$A^{-1} = \frac{1}{det(A)} \times adj(A)$

10. Cramer's rule:

We have $Ax = b$
so $\vec{x} = A^{-1}\vec{b} = \frac{1}{det(A)} \times det(adj(A)) \times \vec{b}$
Let A_i denote the $i-th$ column of A
then we have:

$$x_i = \frac{det([\vec{A_1} \text{ ... } \vec{A_{i-1}} \ \vec{b} \ \vec{A_{i+1}} \text{ ... } \vec{A_n}])}{det(A)}$$

11. How to find a basis for each of the row space, column space and the null space of a matrix A:

If R is the RREF of A

1. The set of non-zero rows of R will form a basis for $Row\ A$.
   i.e. $dim(Row\ A)$ is equal to the numbers of non-zero rows of R
2. The set of leading columns will form a basis for $Col\ A$.
3. The set of **special solution vectors corresponding to the free variables in R** will form a basis for $Null\ A$.

12. If V and W are subspaces of $\mathbb{R}^n$ such that $V \subseteq W$, then $dim(V) \leq dim(W)$. Equality holds if and only if $V = W$.

13. For a linear transformation T_A, we have:

$$[T(v)]_B = [T]_B[v]_B$$

$$[T]_B = [[T(b_1)]_B \ [T(b_2)]_B \text{ ... } [T(b_k)]_B]$$

14. Finding eigenvalues

Let A be an $n \times n$ matrix.
The equation $det(A - tI) = 0$ is called the ***characteristic equation*** of A.
The LHS of the characteristic equation, is said to be the ***characteristic polynomial*** of A.
Eigenvalues of A are thus roots of its characteristic equation, or zeros of its characteristic polynomial.
It can be proved by induction on n that the characteristic polynomial of A is indeed a polynomial (with degree n).

P.S. Some authors prefer to use $det(tI - A)$ instead of $det(A - tI)$.

15. The algebraic multiplicity of an eigenvalue is always greater than or equal to its geometric multiplicity.

PART3: glossary

1. Triangular matrix: 三角矩阵
2. upper triangular matrix: 上三角矩阵
3. lower triangular matrix: 下三角矩阵
4. Diagonal matrix: 对角线矩阵
5. EROs: elementary row operations
6. REF: row echelon form
7. RREF: reduced row echelon form
8. Inconsistent: 不一致的
9. rank: 秩
10. nullity: 零度
11. Span: 张量空间
12. Generation set: 生成集
13. Elementary matrix: 初等矩阵
14. factorization: 因子分解
15. Orthogonal Matrix: 正交矩阵
16. Matrix Transformation: 矩阵变换
17. Linear Transformation: 线性变换
18. Injectivity: 单射性
19. Surjectivity: 满射性
20. Null space: 零空间
21. Kernel: 核

22. preimage: 原像
23. cofactor: 余子式
24. Determination: 特征值
25. cofactor expansion: 代数余子式展开
26. block matrix:分块矩阵
27. adjoint: 伴随矩阵
28. Subspace: 子空间
29. Row space: 行空间
30. Column space: 列空间
31. Basis: 基, 复数为bases
32. Reduction Theorem: 约简定理
33. Extension Theorem: 扩展定理
34. Dimension: 维度
35. zero subspace: 零子空间
36. Coordinate vector: 坐标向量
37.