# Chapter 7 Orthogonality

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Class: MATH2101

## Norm (or length) of vector:

$$||v|| = \sqrt{v_1^2 + v_2^2 + ... + v_n^2}$$

## Distance between $u$ and $v$:

$$d(u,\ v) = ||u - v||$$

## Orthogonal (or perpendicular):

We say that $u$ and $v$ are orthogonal (or perpendicular) if $u \cdot v = 0$

(i.e. $u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$=0)

## Prop:

1. $u \cdot u = ||u||^2$
2. $u \cdot u \geq 0$ , with equality if and only if $u = 0$
3. $u \cdot v = v \cdot u$
4. $u \cdot (v + w) = u \cdot v + u \cdot w$
5. $(cu) \cdot v = u \cdot (cv) = c(u \cdot v)$
6. $||cu|| = c||u||$

## Unit vector and normalizing:

For any non-zero vector v, consider

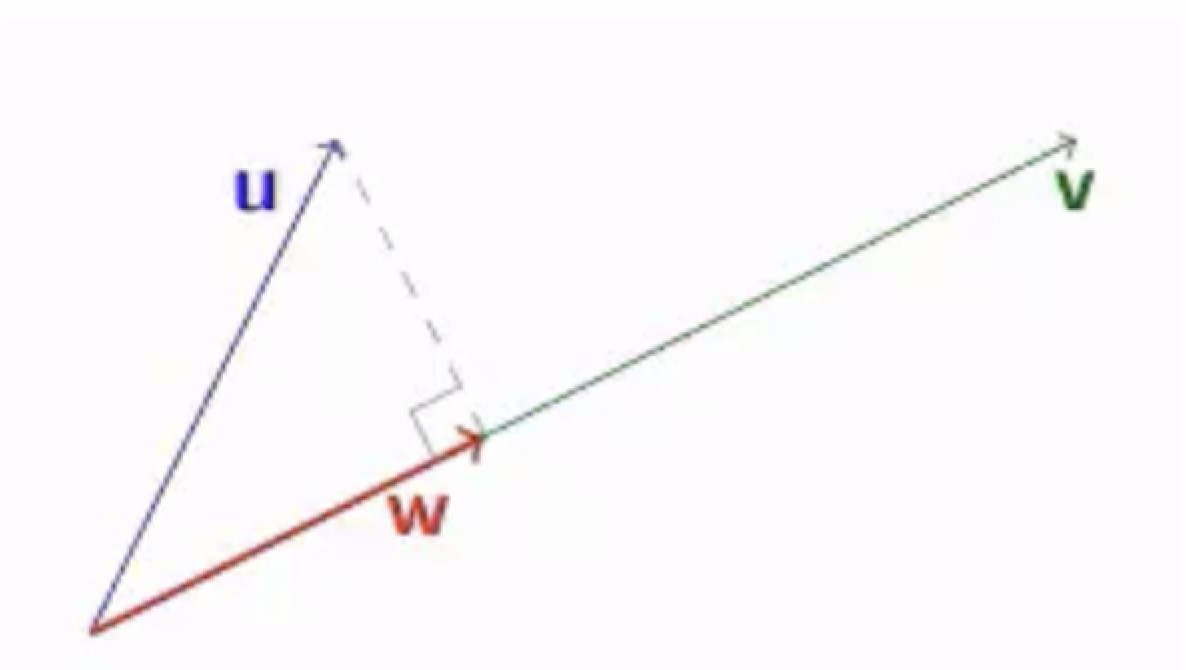
$$u = \frac{1}{||v||} v$$

Then $||u|| = 1$ and is called a **unit vector**. This process is known as **normalising** the vector $v$, producing a unit vector in the same direction as $v$.

## Orthogonal projection:

In general, the orthogonal projection of $u$ on a non-zero vector $v$ is given by:

$$w = \frac{u \cdot v}{||v||^2} v$$




## Pythagoras' Theorem

For two vectors:

Two vectors $u$ and $v$ in $\mathbb{R}^n$ are **orthogonal IFF:**

$$||u||^2 + ||v||^2 = ||u + v||^2$$

Generally:

For $m$ vectors $v_1,\ v_2 \ldots\ v_m$ in $\mathbb{R}^n$, they are pairwise orthogonal IFF:

$$||v_1||^2 + ||v_2||^2 + \ldots + ||v_m||^2 = ||v_1 + v_2 + \ldots + v_m||^2$$

## Cauchy-Schwarz inequality

For real numbers $u_1, u_2, ..., u_n$ and $v_1, v_2, ..., v_n$, we have:

$$(u_1v_1 + u_2v_2 + ... + u_nv_n)^2 \leq (u_1^2 + u_2^2 + ... + u_n^2)^2(v_1^2 + v_2^2 + ... + v_n^2)^2$$

Equality holds IFF $\forall i \neq j, u_iv_j = u_jv_i$

For vector, we have:

For any $u, v \in \mathbb{R}^n$, we have $|u \cdot v| \leq ||u|| \cdot ||v||$

Equality holds IFF $u \parallel v$

## Triangle inequality

For any $v_1, v_2, ..., v_k \in \mathbb{R}^n$, we have:

$$||v_1|| + ||v_2|| + ... + ||v_k|| \geq ||v_1 + v_2 + ... + v_k||$$

# Orthogonal set

## Definition:

Let $S$ be a subset of $\mathbb{R}^n$.

(a) $S$ is said to be an **orthogonal set** if any two vectors in $S$ are orthogonal.

(b) Furthermore, if every vector in $S$ has unit length (i.e. norm 1), then $S$ is said to be an orthonormal set.

Clearly, we can get an orthonormal set by normalizing each vector of an orthogonal set.

## Orthogonal basis

In general, suppose $B = v_1, v_2, ..., v_k$ is an orthogonal basis for a subspace $V$ of $\mathbb{R}^n$. Then for any $v \in V$, we have:

$$v = \sum \frac{v \cdot v_i}{||v_i||^2} v_i$$

Furthermore, if the basis B is orthonormal, the above expression can be simplified to:

$$v = \sum (v \cdot v_i) v_i$$

Prop: Every orthogonal set of non-zero vectors is linearly independent.

## *Gram Schmidt* process

Suppose $\{u_1, u_2, ...u_k\}$ is a basis for a subspace $W$ of $\mathbb{R}^n$. The ***Gram Schmidt*** process turns this basis into an orthogonal basis $\{v_1, v_2, ...v_n\}$ by:

$$v_1 = u_1$$
$$v_i = u_i - \sum_{k=0}^{i-1} \frac{u_i \cdot v_k}{||v_k||} v_k \; for \; 2 \leq i \leq k$$

# Orthogonal Complement of S:

## Definition:

Let $S$ be a subset of $\mathbb{R}^n$.
The ***Orthogonal complement***of $S$, denoted by $S^\perp$ is the set of vectors in $\mathbb{R}^n$ that are orthogonal to every vector in $S$.
i.e.

$$S^\perp = \{v \in \mathbb{R}^n : \forall u \in S,\ v \cdot u = 0\}$$

## prop:

1. Let $S$ be a subset of $\mathbb{R}^n$, then $S^\perp$is a subspace of $\mathbb{R}^n$.
2. Let $S$ be a finite subset of $\mathbb{R}^n$. Then $S^\perp = (Span\ S)^\perp$
3. Let $A$ be a matrix.Then$(Row\ A)^\perp = Null\ A$. (Here we identify the row vectors in $Row\ A as column vectors in the natural way.)

# Orthogonal Decomposition Theorem

## Definition

Let $W$ be $a$ subspace of $\mathbb{R}^n$.
Then every vector $u$ in $\mathbb{R}^n$ can be written in the form $u = w + z$ where $w \in W$ and $z \in W^\perp$ in a unique way.

## prop:

1. $dim\ W + dim\ W^\perp = n$
2. $B \cup B'$is a basis for $\mathbb{R}^n$, where $B$ is a basis for $W$ and $B'$ is a basis for $W^\perp$.

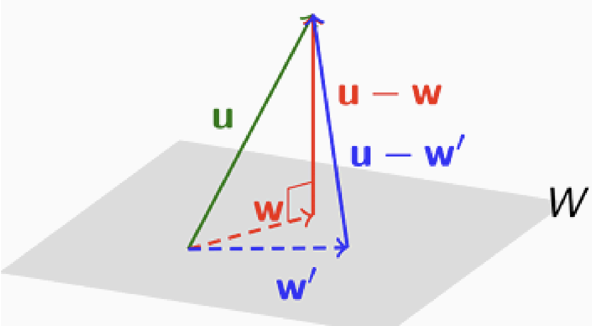
## Orthogonal Projection

Let $W$ be a subspace of $\mathbb{R}^n$. The ***orthogonal projection*** function
$U_W : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation. The standard matrix $P_W$ of $U_W$ is given by:
$P_W = C(C^TC)^{-1}C^T$, where $C$ is a matrix whose columns form a basis for $W$.
And we have:
Let $C$ be a matrix whose columns are linearly independent. Then $C^TC$ is invertible.

### prop:

$U_W(u)$ is the vector in $W$ that is closest to $u$.

Proof:




Show: $\|\mathbf{u}-\mathbf{w}\| < \|\mathbf{u}-\mathbf{w}'\|$ ,if $\mathbf{w} \neq \mathbf{w}'$.

$$\begin{aligned}\|\mathbf{u}-\mathbf{w}'\|^2 &= \|\mathbf{u}-\mathbf{w}\|^2 + \|\mathbf{w}-\mathbf{w}'\|^2 \\ &> \|\mathbf{u}-\mathbf{w}\|^2\end{aligned}$$

# Least Squares Fitting:

There are $n$ points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ on the plane.We use square deviation $E$ to describe the fitting level of the straight line $y = a_0 + a_1 x$:

$$\begin{aligned} E &= \sum_{i=1}^{n} [y_i - (a_0 + a_1 x_i)]^2 \\ &= \left\| \begin{bmatrix} y_1 - (a_0 + a_1 x_1) \\ y_2 - (a_0 + a_1 x_2) \\ \vdots \\ y_n - (a_0 + a_1 x_n) \end{bmatrix} \right\|^2 \\ &= ||\mathbf{y} - (a_0 \mathbf{1} + a_1 \mathbf{x})||^2 \end{aligned}$$

Hence we want to look for the vector in $Span\{1, x\}$ that is closest to $\mathbf{y}$,
naturally, we consider the orthofonal projection.

> 根据线性代数理论, $y$ 在子空间$Span\{1, x\}$上的正交投影是最小化误差$||\mathbf{y} - \mathbf{p}||$的唯一解

More generally, finding the 'Least Squares Fitting' is equivalent to finding the 'best approximation' of $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ . & . \\ . & . \\ . & . \\ 1 & x_n \end{bmatrix}, \mathbf{b} = \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_n \end{bmatrix}, \mathbf{x} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

It's equivalent to the 'best approximate solution' $\mathbf{z}$ so that $A\mathbf{z} = \mathbf{b}'$ is as close to $\mathbf{b}$ as possible. This amounts to solving the equation $A\mathbf{z} = \mathbf{b}'$ where \mathbf{b}^'$is the orthogonal projection of$ \mathbf{b}$on$Col\ A$. There are two cases:

1. There are infinitely solutions of $A\mathbf{x} = \mathbf{b}$
   The form of the solution can be

$$\mathbf{x} = \mathbf{x_0} + \mathbf{z}$$

   where $x_0$ is a **particular solution** of the linear equation
   $z$ is the **general solution** of the linear equation($Az = \mathbf{0}$, i.e. $z \in Null(A)$)
   We want to find the least norm solution, which means the $\mathbf{z}$ is clost to $\mathbf{0}$
2. There are no solution of $A\mathbf{x} = \mathbf{b}$
   So there also be infinitely many best approximate solutions to $A\mathbf{x} = \mathbf{b}'$