Chapter 5 Eigenvalues, Eigenvectors and Diagonalization

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Eigenvalues and eigenvectors

Definition:

Let A be a square matrix. If $Ax = \lambda x$ for some non-zero vector x and scalar t, is said to be an **eigenvalue** of A ;

x is called an **eigenvector** of A corresponding to the eigenvalue ,or in short, a of A.

Finding eigenvalues

Let A be an $n \times n$ matrix.

The equation $det(A - tI) = 0$ is called the ***characteristic equation*** of A.

The LHS of the characteristic equation, is said to be the ***characteristic polynomia****l* of A.

Eigenvalues of A are thus roots of its characteristic equation, or zeros of its characteristic polynomial.

It can be proved by induction on n that the characteristic polynomial of A is indeed a polynomial (with degree n).

P.S. Some authors prefer to use $det(tI - A)$ instead of $det(A - tI)$.

Then we call $Null(A-tI)$ as ***eigenspace*** of A corresponding of t, or in shorter, the t-eigenspace of A

Algebraic multiplicity and geometric multiplicity

A degree n polynomial (where $n > 1$) with coefficients in $\mathbb{C}$ has exactly n zeros(零点) in $\mathbb{C}$ (counting multiplicities，重根).

It thus follows that a $n \times n$ matrix A has exactly n eigenvalues in $\mathbb{C}$ (counting multiplicities)

The ***algebraic multiplicity*** (or simply ***multiplicity***) of an eigenvalue is the number of times it appears as a zero of the character polynomial.

The ***geometric multiplicity*** of an eigenvalue is the dimension of its corresponding eigenspace.

It can be shown that

The algebraic multiplicity of an eigenvalue is always greater than or equal to its geometric multiplicity.

Diagonalization(对角化)

diagonalizable:

An $n \times n$ matrix A is **diagonalizable** IFF it has n **linear independent** eigenvectors.

- Eigenvectors of a matrix A that correspond to distinct eigenvalues are linearly independent.
- If a matrix A has an eigenvalue whose geometric multiplicity is less than the algebraic multiplicity, then A is not diagnosable.

Thus, A matrix A is diagonalizable if and only if for each of its eigenvalues, the algebraic and geometric multiplicities are equal.

Process

If A is a diagonalizable $n \times n$ matrix, it has n **eigenvalues**:$\{\lambda_1, \lambda_2, ...\lambda_n\}$ and n **corresponding linear independent eigenvectors**:$\{v_1, v_2, ...v_n\}$

$$(i.e. \forall i \in \mathbb{R}\ Av_i = \lambda_i v_i)$$

Then we have:

$$A = PDP^{-}1$$

where P is **eigenvector matrix** and D is **diagonal matrix of eigenvalues**

$$P = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}, \ D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$