Chapter 4 Subspaces and Their Properties

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Subspaces of $\mathbb{R}^n$

Subspace:

A subset $W \subseteq \mathbb{R}^n$ is said to be a ***subspace*** of $\mathbb{R}^n$ if it satisfies the following:

- $\vec{0} \in W$
- If $\vec{x}, \vec{y} \in W$, then $\vec{x} + \vec{y} \in W$(i.e. W is closed under addition)
- If $\vec{x} \in W$and $c \in \mathbb{R}$, then $c\vec{x} \in W$(i.e. W is closed under scalar multiplication)

For each $m \times n$ matrix A, we have:

Row space

Notion: $Row\ A$

Subspace of $\mathbb{R}^n$

Span of the rows of A

Column space

Notion: $Col\ A$

Subspace of $\mathbb{R}^m$

Span of the columns of A

Null space

Notion: $Null\ A$

Subspace of $\mathbb{R}^n$

Solution set of $A\vec{x} = 0$

Basis:

Let V be a subspace of $\mathbb{R}^n$. A linearly independent generating set for V is called a basis for V.

Remarks:

- The plural for basis is ***bases***. (单词basis的复数形式是bases)
- A basis for V must be a subset of V.

Every basis for $\mathbb{R}^n$ consists of exactly n vectors.

How to find a basis for each of the row space, column space and the null space of a matrix A:

If R is the RREF of A

1. The set of non-zero rows of R will form a basis for $Row\ A$.
   i.e. $dim(Row\ A)$ is equal to the numbers of non-zero rows of R
2. The set of leading columns will form a basis for $Col\ A$.
3. The set of **special solution vectors corresponding to the free variables in R** will form a basis for $Null\ A$.

Reduction Theorem (约简定理) and Extension Theorem (扩展定理):

Let V be a non-zero subspace of $\mathbb{R}^n$. We have the following:

- (Reduction theorem) Every finite generating set of V contains a basis.

- (Extension theorem) Every linearly independent subset of V can be extended to a basis.

(By convention we say that only basis of the zero subspace is the empty set.)

Dimension

Any two bases for V contain the same number of vectors. This number is said to be the dimension of V and is denoted by $dim(V)$.

(By convention the zero subspace is defined to have dimension 0.)

If V and W are subspaces of $\mathbb{R}^n$ such that $V \subseteq W$, then $dim(V) \leq dim(W)$. Equality holds if and only if $V = W$.

Coordinate vector:

lemma:

Let $B = \{b_1, b_2 ... b_k\}$ be an ***ordered basis*** for a subspace V.
Then each $v \in V$ can be written as a unique linear combination of the vectors in B.

In the forms as:

$$v = c_1 b_1 + c_2 b_2 + ... + c_k b_k$$

Hence we define :

$$[v]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \cdots \\ c_k \end{bmatrix} = B^{-1} v$$

to be the ***coordinate vector of v relative to B*** (or ***B-coordinate vector of v***).

For a linear transformation T_A, we have:

$$[T(v)]_B = [T]_B [v]_B$$

$$[T]_B = [[T(b_1)]_B \ [T(b_2)]_B \ ... \ [T(b_k)]_B]$$

Similarity of matrices

Let A and B be square matrices. We say A is **similar** to B if

$$B = P^{-1}AP$$

for some invertible matrix P.

Thus in some sense, similar matrices can be seen as matrices representing the same linear transformation with respect to different bases.