Chapter 3 Determinants

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Definition of Determinant:

Let A = $[a_{ij}]$ be an $n \times n$ matrix.

A_{ij} is a submatrix of A that obtained by deleting the $i-th$ row and $j-th$ column.

C_{ij} is (i, j)-cofactor, defined by

$$C_{ij} = (-1)^{i+j} \times det(A_{ij})$$

So we have:

$$det(A) = \sum_{i=1}^{n} a_{mi} \times C_{mi} \quad or \quad \sum_{i=1}^{n} a_{im} \times C_{im}$$
$$m \in \{1, 2, 3...n\}$$

Knowns as “The cofactor expansion along the $m-th$ row/column”

Simplification of evaluating the determinant

- Find the tow/column that with more zero.
- The determinant of the triangular matrix, is equal to the product of the non-zero matrix.
  - $det \begin{pmatrix} A_{11} & * & \dots & * \\ 0 & A_{22} & \dots & * \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & A_{nn} \end{pmatrix} = \prod_{i=1}^{n} A_{ii}$
  - Generalized, if each A_{ii} is a block matrix, then we have:
    - $det \begin{pmatrix} A_{11} & * & \dots & * \\ 0 & A_{22} & \dots & * \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & A_{nn} \end{pmatrix} = \prod_{i=1}^{n} det(A_{ii})$
- ERO’s effect on determinant:
  - Type 1 EROs—Exchange two rows
    - $det(E_1 A) = (-1) \times det(A)$

- Type 2 EROs—Multiply one row by a constant k:
    - $det(E_2A) = k \times det(A)$
- Type 3 EROs—Add a row to another row:
    - $det(E_3A) = det(A)$
- Essentially, that is because:
    - $det(EA) = det(E) \times det(A)$

Properties of determinants

Let A be a square matrix. Then

- A is invertible if and only if $det(A) \neq 0$
- $det(AB) = det(A) \times det(B)$(if they have the same size)
- $det(A^T) = det(A)$
- $det(A^{-1}) = \frac{1}{det(A)}$
- Let T : $R^n \rightarrow R^n$ be an invertible linear transformation with standard matrix A. Then for any "sufficiently nice region" S in R^n(Usually refers to the region that can calculate the volumn), the n-dimensional volume of T(S) is equal to $|det(A)|$ times the n-dimensional volume of S.

Use determinant to solve the inverse matrix.

The adjoint of A, denoted by $adj(A)$.

$adj(A) = [C_{ij}]$

then we have:

$A^{-1} = \frac{1}{det(A)} \times adj(A)$

- proof

$$\begin{aligned}
& det(A) \times det\left(\frac{1}{det(A)} \times adj(A)\right) \\
& = det\left(A\ adj(A)\right) \times \frac{1}{det(A)} \\
& = det(\left[\sum_{k=1}^{n} a_{ik} \times C_{kj}\right]) \times \frac{1}{det(A)} \\
& = det(A) \times \frac{1}{det(A)} \\
& = 1
\end{aligned}$$

考虑 A 和它的伴随矩阵 $\operatorname{adj}(A)$ 的乘积：

$A \cdot \operatorname{adj}(A) = [\sum_{k=1}^{n} a_{ik} C_{kj}]$

这里，$[A \cdot \operatorname{adj}(A)]_{ij}$ 表示乘积矩阵的 i, j 元素。根据伴随矩阵的定义，当 $i = j$ 时，这个表达式实际上就是沿着矩阵 A 的第 i 行展开的行列式，因此：

$[A \cdot \operatorname{adj}(A)]_{ii} = \det(A)$

这意味着在对角线上，$A \cdot \operatorname{adj}(A)$ 的每个元素等于 $\det(A)$。

而对于 $i \neq j$ 的情况，上述求和实际上是将 A 中第 i 行替换为第 j 行后的行列式，这导致了两行相同，从而使得行列式的值为0。因此，

$[A \cdot \operatorname{adj}(A)]_{ij} = 0, \quad i \neq j$

综上所述，我们得到：

$A \cdot \operatorname{adj}(A) = \det(A) I$

Use determinant to solve linear equation

Cramer’s rule:

We have Ax=b

so $\vec{x} = A^{-1}\vec{b} = \frac{1}{det(A)} \times det(adj(A)) \times \vec{b}$

Let A_i denote the $i-th$ column of A

then we have:

$$x_i = \frac{det([\vec{A_1} \ ... \ \vec{A_{i-1}} \ \vec{b} \ \vec{A_{i+1}} \ ... \ \vec{A_n}])}{det(A)}$$