Generalized Multiplicative Extended Kalman Filter for Aided Attitude and Heading Reference System

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In this paper, we propose a "Generalized Multiplicative Extended Kalman Filter" (GMEKF) to estimate the position and velocity vectors and the orientation of a flying rigid body, using measurements from low-cost Earth-fixed position and velocity, inertial and magnetic sensors. Thanks to well-chosen state and output errors, the gains and covariance equations converge to constant values on a much bigger set of trajectories than equilibrium points as it is the case for the standard Multiplicative Extended Kalman Filter (MEKF). We recover thus the fundamental properties of the Kalman filter in the linear case, especially the convergence and optimality properties, for a large set of trajectories, and it should result in a better convergence of the estimation. We illustrate the good performance and the nice properties of the GMEKF on simulation and on experimental comparisons with a commercial system.

Nomenclature

ω	Instantaneous angular velocity vector, rad/s
ω_b	Constant vector bias on gyroscopes, rad/s
ω_m	Instantaneous angular velocity vector measured by the gyroscopes, rad/s
A	Constant gravity vector in North-East-Down coordinates, $A = ge_3$, m/s^2
a	Specific acceleration vector (all the non-gravitational forces divided by the body mass), m/s^2
a_m	Specific acceleration vector measured by the accelerometers, m/s^2
a_s	Positif constant scaling factor on accelerometers
B	Earth magnetic field in NED coordinates, T
e_i	Unit vectors pointing respectively North, East, Down, i=1,2,3
h_b	Constant scalar bias on altitude measurement provided by the barometric sensor, m
q	Unit quaternion representing the orientation of the body-fixed frame with respect to the Earth-fixed frame
V,X	Velocity and position vectors of the center of mass with respect to the Earth-fixed frame, m/s,m

I.   Introduction

Aircraft, especially Unmanned Aerial Vehicles (UAV), commonly need to know their orientation, velocity and position to be operated, whether manually or with computer assistance. When cost or weight is an issue, using very accurate inertial sensors for "true" (i.e. based on the Schuler effect due to a non-flat rotating Earth) inertial navigation is excluded. Instead, low-cost systems –sometimes called aided Attitude and Heading Reference Systems (aided AHRS)– rely on light and cheap "strapdown" gyroscopes, accelerometers and magnetometers "aided" by position and velocity sensors (provided for example in body-fixed coordinates by an air-data or Doppler radar system, or in Earth-fixed coordinates by a GPS engine). The various measurements are then "merged" according to the motion equations of the aircraft assuming a flat non-rotating Earth, with some filtering algorithm. Although many filtering techniques have been proposed and studied to design observers for aided AHRS (e.g. invariant nonlinear observers,[1–3]

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Sigma-point Kalman filtering,[4,5] adaptive filtering[6,7]), the Extended Kalman Filter (EKF) is still the most widely used approach to design such estimators. To design an EKF, the system is seen as a stochastic differential equation,

$$\dot{x} = f(x,u) + Mw \tag{1}$$
$$y = h(x,u) + Nv, \tag{2}$$

where x, u, y belong to an open subset of $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$; w, v are independent white gaussian noises of size n and p, and M, N are square matrices. The input u and output y are known signals, and the state x must be estimated. An estimation $\hat{x}(t)$ of $x(t)$ is then computed by the EKF

$$\dot{\hat{x}} = f(\hat{x},u) + K \cdot \left(y - h(\hat{x},u) \right) \tag{3}$$
$$\dot{P} = AP + PA^T + MM^T - PC^T(NN^T)^{-1}CP, \tag{4}$$

with $K = PC^T(NN^T)^{-1}$, $A = \partial_1 f(\hat{x},u)$ and $C = \partial_1 h(\hat{x},u)$ (∂_i means the partial derivative with respect to the ith argument). The rationale is to compute the gain K as in a linear Kalman filter since the estimation error $\Delta x = \hat{x} - x$ satisfies up to higher order terms the linear equation

$$\Delta\dot{x} = (A - KC)\Delta x - Mw + KNv. \tag{5}$$

Of course the convergence of the EKF is not guaranteed in general as in the linear case, see e.g. Ref. 8 for some (local) convergence results.

Another drawback of this "linear" approach is that it is often not adapted to the specific context of aerospace engineering. Indeed in this field, it is customary to use a unit quaternion q to represent the attitude of the aircraft, instead of Euler angles, since it provides a global parametrization of the body orientation, and is well-suited for calculations and computer simulations. In this case, the "linear" approach of the EKF does not respect the geometry of the quaternion space: the standard linear correction term does not preserve the norm of the quaternion, whereas the standard linear error $\hat{q} - q$ does not really make sense for quaternions. Therefore ad hoc modifications of the EKF are used, in particular the so-called Multiplicative EKF (MEKF), introduced in Refs. 9–11, see also Refs. 12, 13. The idea is to respect the geometry of the quaternion space, by using for the quaternion estimation a multiplicative correction term which preserves the unit norm, and by computing the error equation with the error $q^{-1} * \hat{q}$ (or equivalently $\hat{q}^{-1} * q = (q^{-1} * \hat{q})^{-1}$). However, the MEKF leads to matrices A and C that depend on the trajectory. We recover then the fundamental properties of the Kalman filter in the linear case (especially the convergence properties) only at the equilibrium points of the system: convergence issues may happen in many situations, even if the UAV follows a "smooth" trajectory.

In this paper we propose a modification of the MEKF, so-called "Generalized Multiplicative EKF" (GMEF), to overcome the limitations (regarding the convergence properties) of the MEKF. The GMEKF comes directly from the method developed in Ref. 14 to construct an Invariant Extended Kalman Filter (IEKF) for nonlinear systems possessing symmetries (built on the ideas developed in Refs. 15–17). The main benefit of the IEKF is that the matrices A and C are constant on a much bigger set of trajectories (so-called "permanent trajectories"[16]) than equilibrium points as it is the case for the EKF (and for the MEKF). Near such trajectories, we are back to the "true", i.e. linear, Kalman filter where convergence is guaranteed. Informally, this means the IEKF should in general converge at least around any slowly-varying permanent trajectory, rather than just around any slowly-varying equilibrium point for the EKF. Moreover, the filter proposed in this paper supersedes the examples developed in Ref. 14 (i.e. Right and Left IEKF).

The paper is organized as follows. In Section II, we first describe the considered system. Then, we design in Section III a "standard" MEKF and we compare its main features with the GMEKF designed in Section IV. Finally, we illustrate the good performance and the nice properties of the GMEKF on simulation (in Section V) and on experimental comparisons with a commercial system (in Section VI).

II. Physical equations and measurements

A. Motion equations

The motion of a flying rigid body (assuming the Earth is flat and defines an inertial frame) is described by

$$\dot{q} = \frac{1}{2}q * \omega$$
$$\dot{V} = A + q * a * q^{-1}$$
$$\dot{X} = V.$$

The first equation describes the kinematics of the body, the second is Newton's force law. For more details about this section and about the quaternions, see any good textbook on aircraft modeling, for instance Ref. 18, and appendix for useful formulas used in this paper.

B.   Measurements

We use four triaxial sensors, yielding twelve scalar measurements: 3 gyros measure $\omega_m = \omega + \omega_b$, where ω_b is a constant vector bias; 3 accelerometers measure $a_m = a_s a$, where $a_s > 0$ is a constant scaling factor; 3 magnetometers measure $y_B = q^{-1} * B * q$; the position and velocity vectors X and V are provided by the navigation solutions y_X and y_V of a GPS engine (the GPS velocity is obtained from the carrier phase and/or Doppler shift data, and not by differentiating the GPS position, hence is of rather good quality). A barometric sensor provides also a measure of the altitude $y_h = \langle X, e_3 \rangle - h_b$, where h_b is a constant scalar bias. There is some freedom in the modeling of the sensors imperfections, see Ref. 3 for a discussion. All these measurements are of course also corrupted by noise.

It is reasonable to assume each scalar sensor is corrupted by an additive gaussian white noise with identical variance for each of the three scalar sensors constituting a triaxial sensor, and all the noises mutually independent (this is technologically motivated for the accelerometers, gyros and magnetic sensors, though much more questionable for the GPS engine). Hence we can see each triaxial sensor as corrupted by a "coordinate-free vector noise" whose coordinates are gaussian in the body frame as well as the Earth frame (or any other smooth time-varying frame). Indeed, the mean and the auto-correlation time of such a noise is not affected by a (smoothly) time-varying rotation.

C.   The considered system

To design our observers we therefore consider the system

$$\dot{q} = \frac{1}{2} q * (\omega_m - \omega_b) \tag{6}$$

$$\dot{V} = A + \frac{1}{a_s} q * a_m * q^{-1} \tag{7}$$

$$\dot{X} = V \tag{8}$$

$$\dot{\omega}_b = 0 \tag{9}$$

$$\dot{a}_s = 0 \tag{10}$$

$$\dot{h}_b = 0, \tag{11}$$

where ω_m and a_m are seen as known inputs, together with the output

$$\begin{pmatrix} y_V \\ y_X \\ y_h \\ y_B \end{pmatrix} = \begin{pmatrix} V \\ X \\ \langle X, e_3 \rangle - h_b \\ q^{-1} * B * q \end{pmatrix}. \tag{12}$$

This system is observable provided $B \times (q * a_m * q^{-1}) \neq 0$ since all the state variables can be recovered from the known quantities $\omega_m, a_m, y_V, y_X, y_h, y_B$ and their derivatives: see Ref. 3 for details.

III.   Standard Multiplicative Extended Kalman Filter

We start with the design of a Multiplicative EKF in the spirit of Refs. 9–11, see also Refs. 12, 13.

A. Problem setting

We consider the noise enters the system as

$$\dot{q} = \frac{1}{2}q*(\omega_m - \omega_b) + q*M_q w_q \tag{13}$$

$$\dot{V} = A + \frac{1}{a_s}q*a_m*q^{-1} + q*M_V w_V*q^{-1} \tag{14}$$

$$\dot{X} = V + M_X w_X \tag{15}$$

$$\dot{\omega}_b = M_\omega w_\omega \tag{16}$$

$$\dot{a}_s = M_a w_a \tag{17}$$

$$\dot{h}_b = M_h w_h, \tag{18}$$

and the output as

$$\begin{pmatrix} y_V \\ y_X \\ y_h \\ y_B \end{pmatrix} = \begin{pmatrix} V + N_V v_V \\ X + N_X v_X \\ \langle X, e_3 \rangle - h_b + N_h v_h \\ q^{-1}*B*q + N_B v_B \end{pmatrix}, \tag{19}$$

with $M_q, M_V, M_X, M_\omega, N_V, N_X, N_B$ diagonal matrices, and M_a, M_h, N_h scalars. The driving and observation noises are thus consistent with a scalar additive noise on each individual sensor.

B. MEKF equations

The MEKF then takes the form

$$\dot{\hat{q}} = \frac{1}{2}\hat{q}*(\omega_m - \hat{\omega}_b) + \hat{q}*K_q E \tag{20}$$

$$\dot{\hat{V}} = A + \frac{1}{\hat{a}_s}\hat{q}*a_m*\hat{q}^{-1} + K_V E \tag{21}$$

$$\dot{\hat{X}} = \hat{V} + K_X E \tag{22}$$

$$\dot{\hat{\omega}}_b = K_\omega E \tag{23}$$

$$\dot{\hat{a}}_s = K_a E \tag{24}$$

$$\dot{\hat{h}}_b = K_h E, \tag{25}$$

where the output error is given by

$$E = \begin{pmatrix} \hat{y}_V - y_V \\ \hat{y}_X - y_X \\ \hat{y}_h - y_h \\ \hat{y}_B - y_B \end{pmatrix} = \begin{pmatrix} \hat{V} - V - N_V v_V \\ \hat{X} - X - N_X v_X \\ \langle \hat{X} - X, e_3 \rangle - \hat{h}_b + h_b - N_h v_h \\ \hat{q}^{-1}*B*\hat{q} - q^{-1}*B*q - N_B v_B \end{pmatrix}.$$

But for Eq. (20), the MEKF has the form of a standard EKF.

We consider the state error

$$\begin{pmatrix} \mu \\ \nu \\ \chi \\ \beta \\ \alpha \\ \lambda \end{pmatrix} = \begin{pmatrix} q^{-1}*\hat{q} \\ \hat{V} - V \\ \hat{X} - X \\ \hat{\omega}_b - \omega_b \\ \hat{a}_s - a_s \\ \hat{h}_b - h_b \end{pmatrix}.$$

A tedious but simple computation yields the error system

$$\dot{\mu} = -\frac{1}{2}\beta * \mu + \mu \times \hat{I}_\omega - M_q w_q * \mu + \mu * K_q E$$

$$\dot{v} = \hat{I}_a - \frac{1}{\hat{a}_s - \alpha}\hat{q}\mu^{-1} * a_m * \mu * \hat{q}^{-1} - \mu * \hat{q} * M_V w_V * \hat{q}^{-1} * \mu^{-1} + K_V E$$

$$\dot{\chi} = v - M_X w_X + K_X E$$

$$\dot{\beta} = -M_\omega w_\omega + K_\omega E$$

$$\dot{\alpha} = -\alpha M_a w_a + \alpha K_a E$$

$$\dot{\lambda} = -M_h w_h + K_h E,$$

where the output error is rewritten as

$$E = \begin{pmatrix} v - N_V v_V \\ \chi - N_X v_X \\ \langle \chi, e_3 \rangle - \lambda - N_h v_h \\ \hat{I}_B - \mu * \hat{I}_B * \mu^{-1} - N_B v_B \end{pmatrix},$$

and $\hat{I}_\omega = \omega_m - \hat{\omega}_b$, $\hat{I}_a = \frac{1}{\hat{a}_s}\hat{q} * a_m * \hat{q}^{-1}$ and $\hat{I}_B = \hat{q}^{-1} * B * \hat{q}$.

We next linearize this error system around $(\overline{\mu}, \overline{v}, \overline{\chi}, \overline{\beta}, \overline{\alpha}, \overline{\lambda}) = (1, 0, 0, 0, 0, 0)$, i.e. the estimated state equals the actual state, drop all the quadratic terms in noise and infinitesimal state error (see Ref. 14 for mathematical justification), and eventually find

$$\begin{pmatrix} \delta\dot{\mu} \\ \delta\dot{v} \\ \delta\dot{\chi} \\ \delta\dot{\beta} \\ \delta\dot{\alpha} \\ \delta\dot{\lambda} \end{pmatrix} = (A - KC)\begin{pmatrix} \delta\mu \\ \delta v \\ \delta\chi \\ \delta\beta \\ \delta\alpha \\ \delta\lambda \end{pmatrix} - M\begin{pmatrix} w_q \\ w_V \\ w_X \\ w_\omega \\ w_a \\ w_h \end{pmatrix} + KN\begin{pmatrix} v_V \\ v_X \\ v_h \\ v_B \end{pmatrix}.$$

This linearized error system has the desired form (5) with

$$A = \begin{pmatrix} -\hat{I}_{\omega\times} & 0_{33} & 0_{33} & -\frac{1}{2}I_3 & 0_{31} & 0_{31} \\ -2\hat{I}_{a\times}R(\hat{q}) & 0_{33} & 0_{33} & 0_{33} & -\hat{I}_a & 0_{31} \\ 0_{33} & I_3 & 0_{33} & 0_{33} & 0_{31} & 0_{31} \\ 0_{33} & 0_{33} & 0_{33} & 0_{33} & 0_{31} & 0_{31} \\ 0_{13} & 0_{13} & 0_{13} & 0_{13} & 0_{11} & 0_{11} \\ 0_{13} & 0_{13} & 0_{13} & 0_{13} & 0_{11} & 0_{11} \end{pmatrix}$$

$$C = \begin{pmatrix} 0_{33} & I_3 & 0_{33} & 0_{33} & 0_{31} & 0_{31} \\ 0_{33} & 0_{33} & I_3 & 0_{33} & 0_{31} & 0_{31} \\ 0_{13} & 0_{13} & e_3^T & 0_{13} & 0_{11} & -I_1 \\ 2\hat{I}_{B\times} & 0_{33} & 0_{33} & 0_{33} & 0_{31} & 0_{31} \end{pmatrix}$$

$$M = Diag(M_q, R(\hat{q})M_V, M_X, M_\omega, M_a, M_h)$$

$$N = Diag(N_V, N_X, N_h, N_B)$$

$$K = -(K_q, K_V, K_X, K_\omega, K_a, K_h)^T,$$

where the matrices $I_\times$ and $R(q)$ are defined by $I_\times u := I \times u$ and $R(q)u := q * u * q^{-1}$ for all for $u \in \mathbb{R}^3$. Similarly to the standard EKF, the matrix gain K is now computed as $K = PC^T(NN^T)^{-1}$ with P defined by (4).

C. Features of the MEKF

SOUND GEOMETRIC STRUCTURE FOR THE QUATERNION ESTIMATION EQUATION by construction, Eq. (20) preserves the unit norm of the estimated quaternion, since $K_q E$ is a vector of $\mathbb{R}^3$ (see Appendix).

POSSIBLE CONVERGENCE ISSUES IN MANY SITUATIONS indeed, the matrices A and C used for computing the gain matrix K are constant only in level flight, i.e. $\hat{I}_\omega \simeq 0$ and $\hat{I}_a \simeq -A$, because of the trajectory-dependent terms $R(\hat{q})$ in A and C.

IV.   Generalized Multiplicative Extended Kalman Filter

Following the method proposed in Ref. 14, we now design a Generalized MEKF.

A.   Problem setting

We consider the noise enters the system as

$$\dot{q} = \frac{1}{2}q * (\omega_m - \omega_b) + M_q w_q * q \tag{26}$$

$$\dot{V} = A + \frac{1}{a_s}q * a_m * q^{-1} + M_V w_V \tag{27}$$

$$\dot{X} = V + M_X w_X \tag{28}$$

$$\dot{\omega}_b = q^{-1} * M_\omega w_\omega * q \tag{29}$$

$$\dot{a}_s = a_s M_a w_a \tag{30}$$

$$\dot{h}_b = M_h w_h, \tag{31}$$

and the output as

$$\begin{pmatrix} y_V \\ y_X \\ y_h \\ y_B \end{pmatrix} = \begin{pmatrix} V + N_V v_V \\ X + N_X v_X \\ \langle X, e_3 \rangle - h_b + N_h v_h \\ q^{-1} * (B + N_B v_B) * q \end{pmatrix}, \tag{32}$$

with $M_q, M_V, M_X, M_\omega, N_V, N_X, N_B$ diagonal matrices, and M_a, M_h, N_h scalars. Notice the noise on y_B is different from the noise in (19): it is additive in body axes rather than in Earth axes; the same frame change appears for driving noise in Eqs. 26,27,29, see section II for a tentative physical justification.

B.   GMEKF equations

The GMEKF then takes the form

$$\dot{\hat{q}} = \frac{1}{2}\hat{q} * (\omega_m - \hat{\omega}_b) + K_q E * \hat{q} \tag{33}$$

$$\dot{\hat{V}} = A + \frac{1}{\hat{a}_s}\hat{q} * a_m * \hat{q}^{-1} + K_V E \tag{34}$$

$$\dot{\hat{X}} = \hat{V} + K_X E \tag{35}$$

$$\dot{\hat{\omega}}_b = \hat{q}^{-1} * K_\omega E * \hat{q} \tag{36}$$

$$\dot{\hat{a}}_s = \hat{a}_s K_a E \tag{37}$$

$$\dot{\hat{h}}_b = K_h E, \tag{38}$$

where the output error is given by

$$E = \begin{pmatrix} \hat{y}_V - y_V \\ \hat{y}_X - y_X \\ \hat{y}_h - y_h \\ B - \hat{q} * y_B * \hat{q}^{-1} \end{pmatrix} = \begin{pmatrix} \hat{V} - V - N_V v_V \\ \hat{X} - X - N_X v_X \\ \langle \hat{X} - X, e_3 \rangle - \hat{h}_b + h_b - N_h v_h \\ B - \hat{q} * q^{-1} * (B + N_B v_B) * q * \hat{q}^{-1} \end{pmatrix}.$$

We consider the state error

$$
\begin{pmatrix} \mu \\ v \\ \chi \\ \beta \\ \alpha \\ \lambda \end{pmatrix} = \begin{pmatrix} \hat{q} * q^{-1} \\ \hat{V} - V \\ \hat{X} - X \\ q * (\hat{\omega}_b - \omega_b) * q^{-1} \\ \frac{\hat{a}_s}{a_s} \\ \hat{h}_b - h_b \end{pmatrix}.
$$

The error system is

$$
\begin{aligned}
\dot{\mu} &= -\frac{1}{2}\mu * \beta - \mu * M_q w_q + K_q E \mu \\
\dot{v} &= \hat{I}_a - \alpha\mu^{-1} * \hat{I}_a * \mu - M_V w_V + K_V E \\
\dot{\chi} &= v - M_X w_X + K_X E \\
\dot{\beta} &= (\mu^{-1} * \hat{I}_\omega * \mu) \times \beta + M_q w_q \times \beta - M_\omega w_\omega + \mu^{-1} * K_\omega E * \mu \\
\dot{\alpha} &= -\alpha M_a w_a + \alpha K_a E \\
\dot{\lambda} &= -M_h w_h + K_h E,
\end{aligned}
$$

where the output error is rewritten as

$$
E = \begin{pmatrix} v - N_V v_V \\ \chi - N_X v_X \\ \langle \chi, e_3 \rangle - \lambda - N_h v_h \\ B - \mu * (B + N_B v_B) * \mu^{-1} \end{pmatrix},
$$

and $\hat{I}_\omega = \hat{q} * (\omega_m - \hat{\omega}_b) * \hat{q}^{-1}$, $\hat{I}_a = \frac{1}{\hat{a}_s}\hat{q} * a_m * \hat{q}^{-1}$.

We next linearize this error system around $(\overline{\mu}, \overline{v}, \overline{\chi}, \overline{\beta}, \overline{\alpha}, \overline{\lambda}) = (1,0,0,0,0,0)$, drop all the quadratic terms in noise and infinitesimal state error, and eventually find

$$
\begin{pmatrix} \delta\dot{\mu} \\ \delta\dot{v} \\ \delta\dot{\chi} \\ \delta\dot{\beta} \\ \delta\dot{\alpha} \\ \delta\dot{\lambda} \end{pmatrix} = (A - KC) \begin{pmatrix} \delta\mu \\ \delta v \\ \delta\chi \\ \delta\beta \\ \delta\alpha \\ \delta\lambda \end{pmatrix} - M \begin{pmatrix} w_q \\ w_V \\ w_X \\ w_\omega \\ w_a \\ w_h \end{pmatrix} + KN \begin{pmatrix} v_V \\ v_X \\ v_h \\ v_B \end{pmatrix},
$$

which has the desired form (5) with

$$
A = \begin{pmatrix}
0_{33} & 0_{33} & 0_{33} & -\frac{1}{2}I_3 & 0_{31} & 0_{31} \\
-2\hat{I}_{a\times} & 0_{33} & 0_{33} & 0_{33} & -\hat{I}_a & 0_{31} \\
0_{33} & I_3 & 0_{33} & 0_{33} & 0_{31} & 0_{31} \\
0_{33} & 0_{33} & 0_{33} & \hat{I}_{\omega\times} & 0_{31} & 0_{31} \\
0_{13} & 0_{13} & 0_{13} & 0_{13} & 0_{11} & 0_{11} \\
0_{13} & 0_{13} & 0_{13} & 0_{13} & 0_{11} & 0_{11}
\end{pmatrix}
$$

$$
C = \begin{pmatrix}
0_{33} & I_3 & 0_{33} & 0_{33} & 0_{31} & 0_{31} \\
0_{33} & 0_{33} & I_3 & 0_{33} & 0_{31} & 0_{31} \\
0_{13} & 0_{13} & e_3^T & 0_{13} & 0_{11} & -I_1 \\
2B_\times & 0_{33} & 0_{33} & 0_{33} & 0_{31} & 0_{31}
\end{pmatrix}
$$

$$
M = Diag(M_q, M_V, M_X, M_\omega, M_a, M_h)
$$

$$
N = Diag(N_V, N_X, N_h, N_B)
$$

$$
K = -(K_q, \ K_V, \ K_X, \ K_\omega, \ K_a, \ K_h)^T.
$$

As usual the matrix gain K is computed as $K = PC^T(NN^T)^{-1}$ with P defined by (4).

C. Features of the GMEKF

SOUND GEOMETRIC STRUCTURE FOR THE QUATERNION ESTIMATION EQUATION   by construction, Eq. (33) preserves the unit norm of the estimated quaternion, since $K_q E$ is a vector of $\mathbb{R}^3$ (see Appendix).

LARGER EXPECTED DOMAIN OF CONVERGENCE   the matrices A and C used for computing the gain matrix K are constant not only in level flight but also on every trajectory defined by constant $\hat{I}_\omega, \hat{I}_a$.

SYMMETRY-PRESERVING STRUCTURE   since the GMEKF is a specific Invariant EKF,[14] rotations, translations and scaling in the appropriated frames leave the error system unchanged, unlike the MEKF: it is meaningful from an engineering point of view, and it should result to a physically more "sensible" convergence of the estimations.

V.   Simulation results

We first illustrate the behavior of the proposed filters on simulations. The noises w_i, v_i are independent normally distributed random 3-dimensional vectors with mean 0 and variance 1. The tuning of the MEKF and GMEKF is made via the choice of covariance matrices $M_q = 0.1 I_3$, $M_V = 0.01 I_3$, $M_X = 0.1 I_3$, $M_\omega = 0.001 I_3$, $M_a = 0.01$, $M_h = 0.01$, $N_V = 0.1 I_3$, $N_X = 0.1 I_3$, $N_h = 0.1$, $N_B = 0.1 I_3$. The (scaled) Earth magnetic field is taken as $B = (1\ 0\ 1)^T$ (roughly the value in France).

To enforce $\|\hat{q}\| = 1$ despite numerical round off, we systematically add the term $k(1 - \|\hat{q}\|^2)\hat{q}$ in the estimated quaternion equation (otherwise the norm would slowly drift), which is a standard trick in numerical integration with quaternions. For instance for the GMEKF, we take

$$\dot{\hat{q}} = \frac{1}{2}\hat{q} * (\omega_m - \hat{\omega}_b) + K_q E * \hat{q} + k(1 - \|\hat{q}\|^2)\hat{q}.$$

We have used $k = 1$ (this value is not critical).

The system follows a trajectory $\mathcal{T}_1$, which is quite representative of a small UAV flight. The MEKF and GMEKF are initialized with the same values. Both filters give correct estimations after the initial transient, see Fig. 1.

We now show the GMEKF gain matrix K becomes as expected constant on the trajectory $\mathcal{T}_1$, while the MEKF gain does not, see Fig. 2. This is remarkable since $\mathcal{T}_1$ is far from being an equilibrium point.

Finally we illustrate the invariance property of the GMEKF: both EKF are initialized with three different initial conditions having the same initial state error. The MEKF behavior does depend on the initial conditions, while the GMEKF behavior does not, see Fig. 3 (only the norm of the velocity error $e_v = \|v\|$ is displayed, but similar results are obtained for other state errors).

VI.   Experimental results

We now compare the behavior of our observer with the commercial INS-GPS device MIDG II from Microbotics Inc. For each experiment we first save the raw measurements from the MIDG II gyros, accelerometers and magnetic sensors (at a 50Hz refresh rate), the position and velocity provided by the navigation solutions of its GPS engine (at a 5 Hz refresh rate) and the raw measurements from a barometer module Intersema MS5534B (at a 12.5 Hz refresh rate). A microcontroller on a development kit communicate with these devices and send the measurements to a computer via the serial port (see Fig. 4). On Matlab Simulink we feed offline the observer with these data and then compare the estimations of the observer to the estimations given by the MIDG II (computed according to the user manual by some Kalman filter). In order to have similar behaviors and considering the units of the raw measurements provided by the MIDG II, we have chosen $M_q = 1e - 3 I_3$, $M_V = 5e - 3 I_3$, $M_X = 0.01 I_3$, $M_\omega = 1e - 7 I_3$, $M_a = 2e - 6$, $M_h = 2e - 4$, $N_V = 5e - 3 I_3$, $N_X = 0.1 I_3$, $N_h = 0.1$, $N_B = 1e - 3 I_3$, and $k = 1$.

We wait a few minutes until the biases reach constant values, then move the system in all directions. The GMEKF and the MIDG II give similar results (see Fig. 5). On Fig. 5(d) it is in fact $h_b + 168m$ that is represented to plot a_s and h_b on the same axis. We can also notice on Fig. 5(b) that the estimation of V_Z given by our observer seems to be closer to the real value than the estimation provided by the MIDG II: we know that we let the system motionless at $t = 42s$, which is coherent with our estimated V_Z. In Fig. 6, we also plot the evolution of some coefficients of the gain matrix $K(t)$ (the same as in Section V). As expected, the gain matrix remains (almost) constant during the experiment, even if the trajectory followed by the system is far from the equilibrium point.
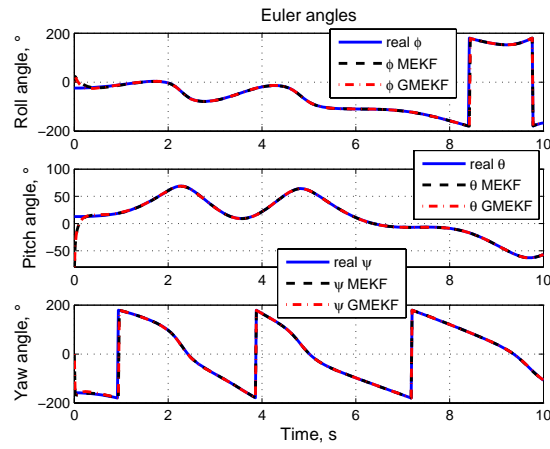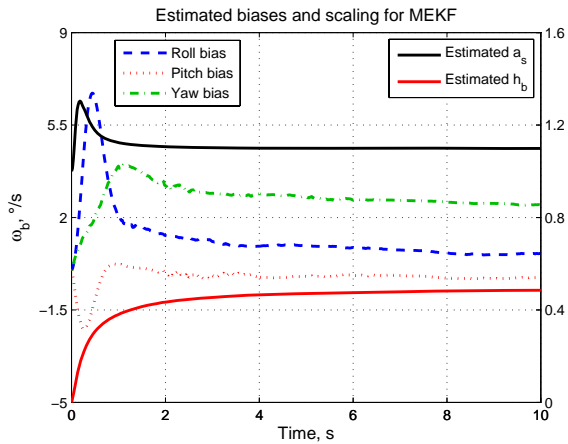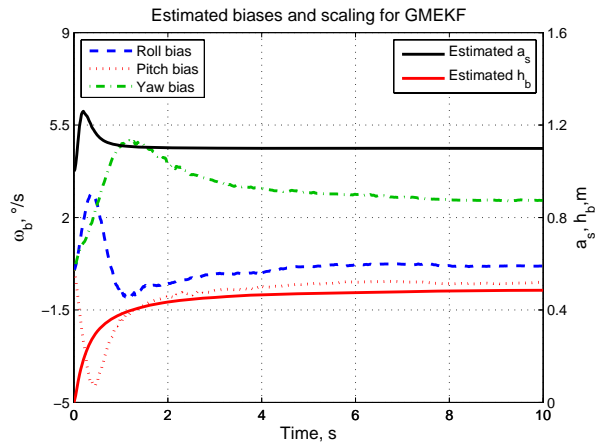
(a) Estimated Euler angles
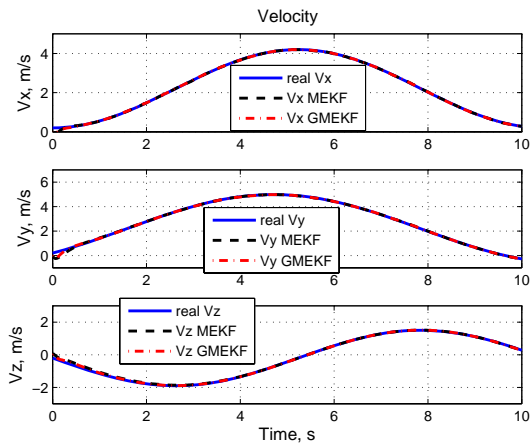
(b) Estimated velocity
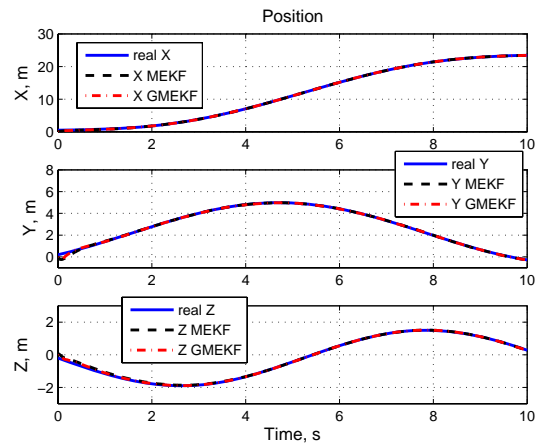
(c) Estimated position

(d) Estimated biases and scaling (MEKF)

(e) Estimated biases and scaling (GMEKF)
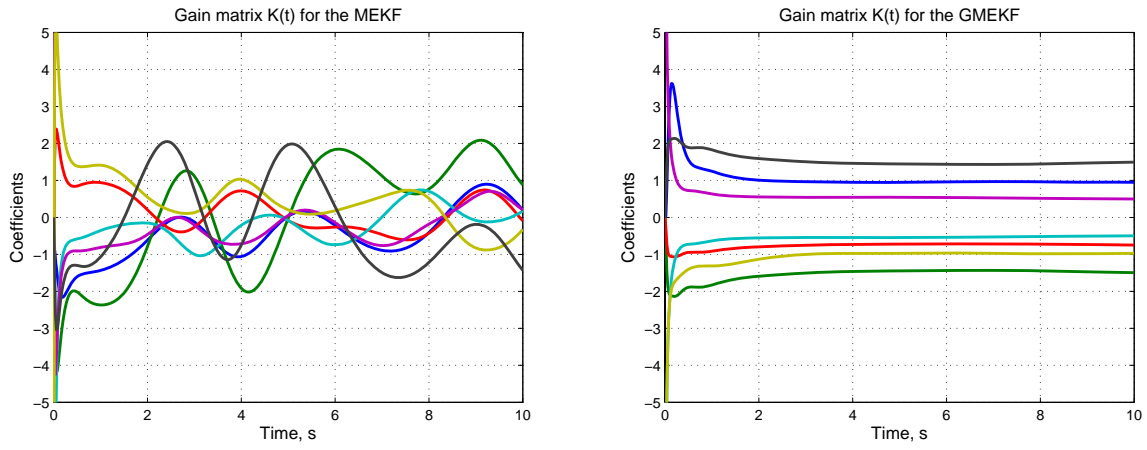
Figure 1. Simulation: Comparison MEKF / GMEKF

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Figure 2. Simulation: Evolution of gain matrix K (MEKF left, GMEKF right)
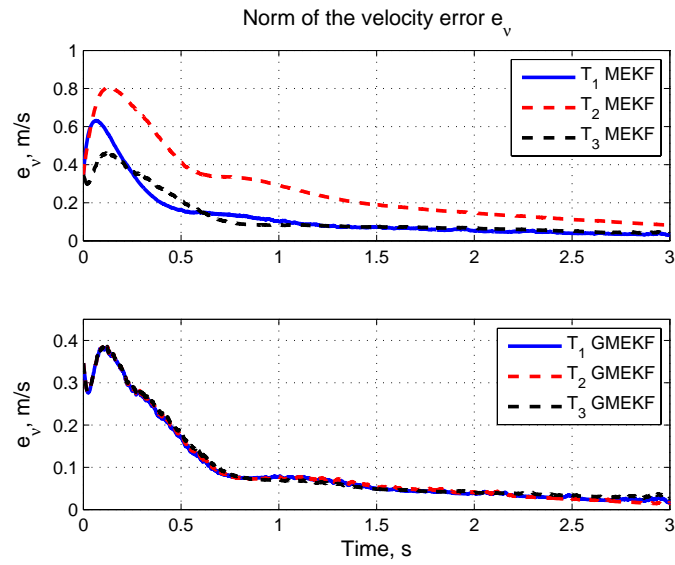
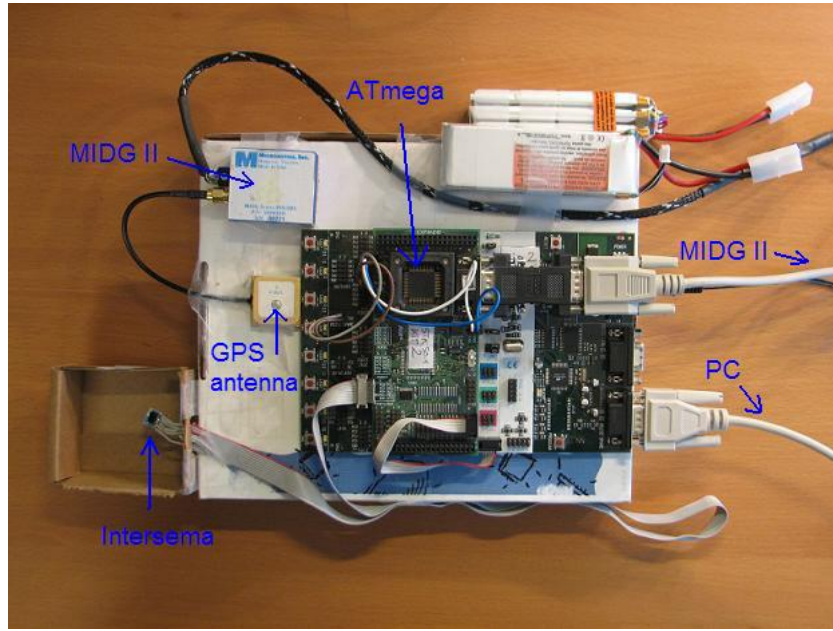**Figure 3. Simulation: Evolution of $e_v = \|v\|$ for three initial conditions**

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Figure 4. Experimental protocol

Appendix: Quaternions

Thanks to their four coordinates, quaternions provide a global parametrization of the orientation of a rigid body (whereas a parametrization with three Euler angles necessarily has singularities). Indeed, to any quaternion q with unit norm is associated a rotation matrix $R_q \in SO(3)$ by

$$q^{-1} * \vec{p} * q = R_q \cdot \vec{p} \quad \text{for all } \vec{p} \in \mathbb{R}^3.$$

A quaternion p can be thought of as a scalar $p_0 \in \mathbb{R}$ together with a vector $\vec{p} \in \mathbb{R}^3$,

$$p = \begin{pmatrix} p_0 \\ \vec{p} \end{pmatrix}.$$

The (non commutative) quaternion product $*$ then reads

$$p * q \triangleq \begin{pmatrix} p_0 q_0 - \vec{p} \cdot \vec{q} \\ p_0 \vec{q} + q_0 \vec{p} + \vec{p} \times \vec{q} \end{pmatrix}.$$

The unit element is $e \triangleq \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix}$, and $(p * q)^{-1} = q^{-1} * p^{-1}$.
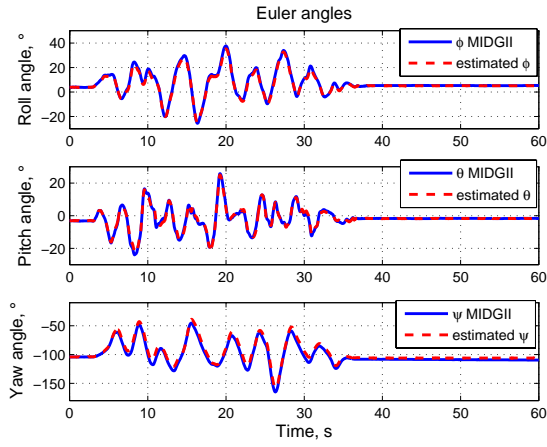
Any scalar $p_0 \in \mathbb{R}$ can be seen as the quaternion $\begin{pmatrix} p_0 \\ \vec{0} \end{pmatrix}$, and any vector $\vec{p} \in \mathbb{R}^3$ can be seen as the quaternion $\begin{pmatrix} 0 \\ \vec{p} \end{pmatrix}$.
We systematically use these identifications in the paper, which greatly simplifies the notations.
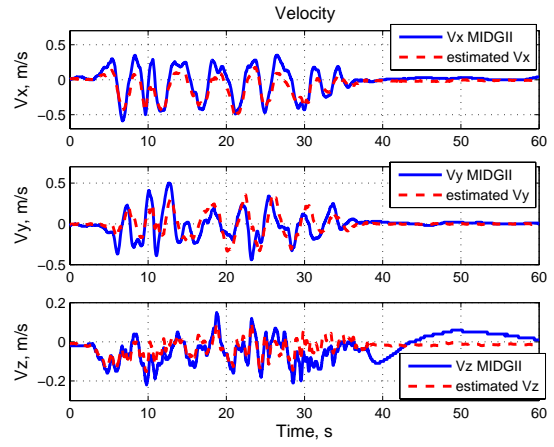
We have the useful formulas

$$p \times q \triangleq \vec{p} \times \vec{q} = \frac{1}{2}(p * q - q * p)$$

$$(\vec{p} \cdot \vec{q})\vec{r} = -\frac{1}{2}(p * q + q * p) * r.$$

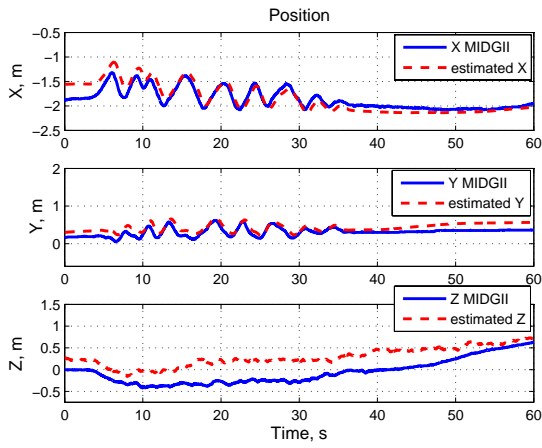If q depends on time, then $\dot{q}^{-1} = -q^{-1} * \dot{q} * q^{-1}$.
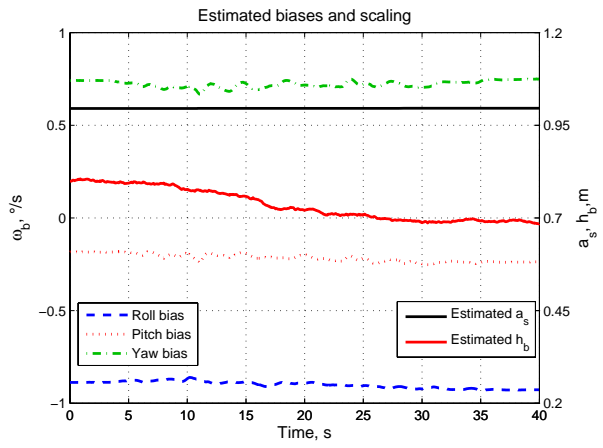
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(a) Estimated Euler angles

(b) Estimated velocity

(c) Estimated position

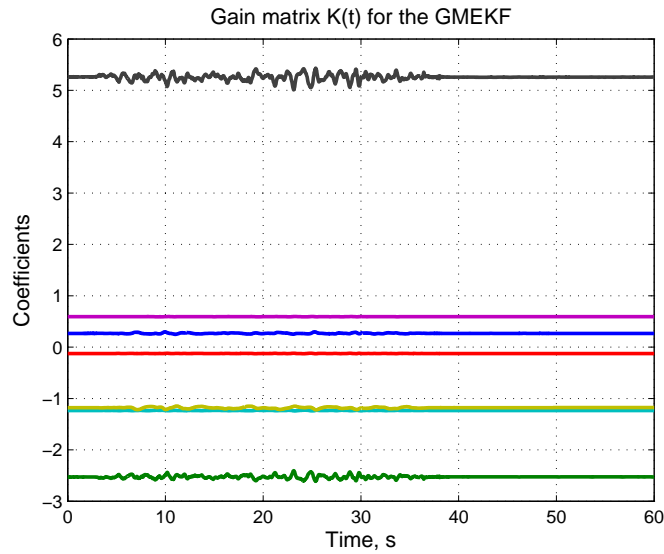(d) Estimated biases and scaling

Figure 5. Experiment : Dynamic behavior

Figure 6. Experiment: Evolution of gain matrix K

Finally, consider the differential equation $\dot{q} = q * u + v * q$ where u, v are vectors $\in \mathbb{R}^3$. Let q^T be defined by $\begin{pmatrix} q_0 \\ -\vec{q} \end{pmatrix}$. Then $q * q^T = \|q\|^2$. Therefore,

$$\widetilde{q * q^T} = q * (u + u^T) * q^T + \|q\|^2 (v + v^T) = 0$$

since u, v are vectors. Hence the norm of q is constant.

Acknowledgements

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