Ay190 – Worksheet 12
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Solving the Poisson Equation

We want to solve:

$$\nabla^2\Phi = 4\pi G\rho$$

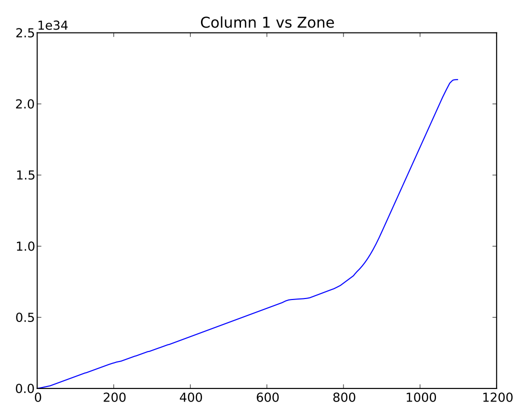
Before we do so, we examine the data to determine what columns contain radius and density.

1

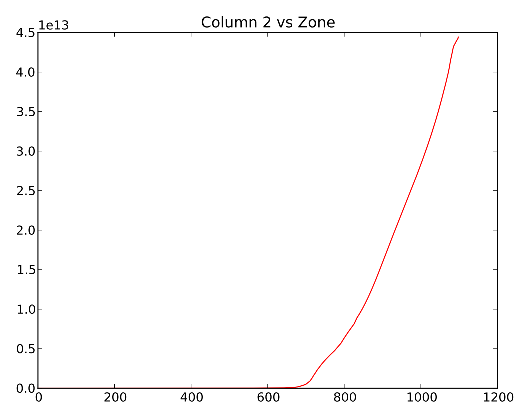
Clearly column 0 is just the zone, and there are 7 other columns. Graphs of all of these columns vs the zone number are shown in figures 1 through 3 (column 7 is all 0s, so it's just a horizontal line). The radius must increase with the zone, so it has to be either column 1 or 2. Noting that the radius of the sun is on the order of $7 * 10^{10}$ cm, the radius here must be column 2 (in cm), which goes up to 10^{13}, as opposed to column 1, which goes up to 10^{34}. In fact, noting that the *mass* of the sun is of order 10^{33} g, and mass must also increase with the zone, we realize that column 1 is the mass (in g).

Figure 2 shows a parameters that decreases with zone, so one is probably temperature and the other density. Unfortunately, the orders of magnitude of those two plots are very close, so we can't decide based on that. However, we know that density cannot increase at all for an increasing radius (because of the linearly increasing mass from zones 0 to 600), whereas temperature can, which means column 3 is temperature (in K) and column 4 is the density (in $\mathrm{g\,cm^{-3}}$).

Columns 5, 6, and 7 turn out to be infall velocity (it's negative for much of the star, which indicates a collapse is occuring), electron fraction (values from about 0.4 to 0.8), and angular velocity (0 for nonspinning star).

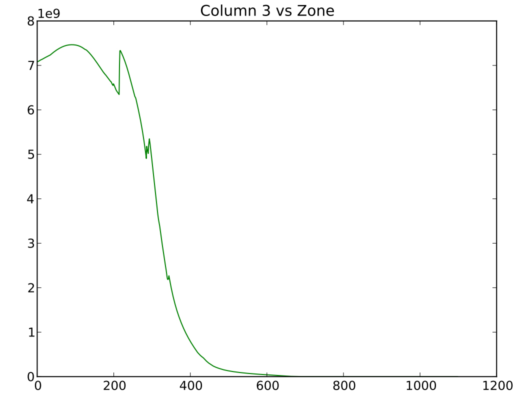


Column 1 (mass) vs Zone

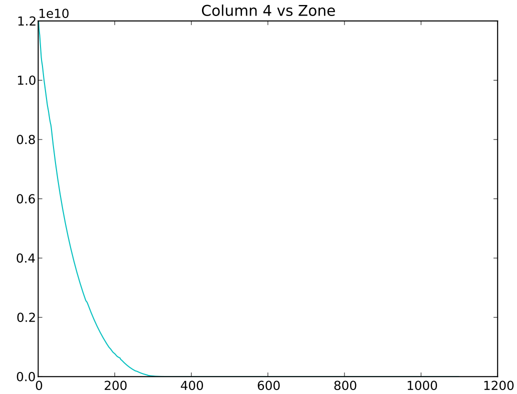


Column 2 (radius) vs Zone

Figure 1

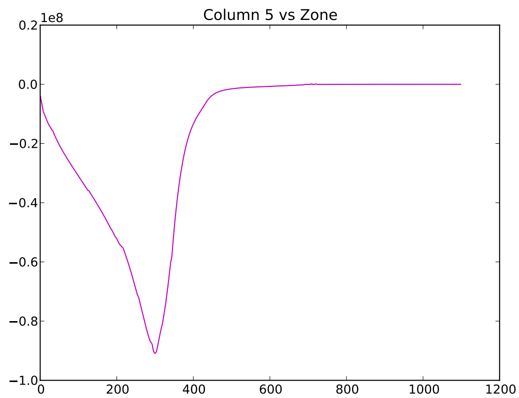


Column 3 (temperature) vs Zone

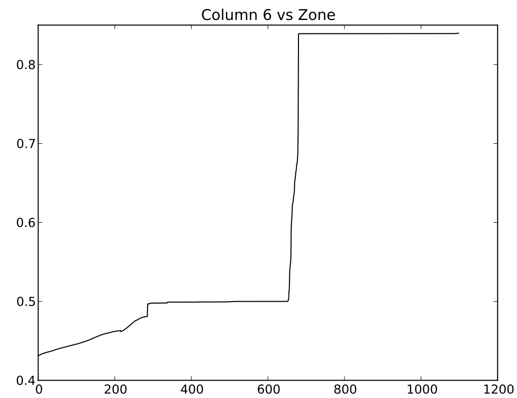


Column 4 (density) vs Zone

Figure 2



Column 5 (infall velocity) vs Zone



Column 6 (electron fraction) vs Zone

Figure 3

Plotting the log of density as a function of the log of radius, restricting radius $< 10^9$ cm, we can see the density profile of the interior star.

2

We choose linear interpolation. First, we edit the radius and density arrays using the lines

```
rad = np.concatenate(([0.], rad[rad <= 10.**9]))
rho = np.concatenate((rho[:1], rho[:(rad.size -1)]))
```

This cuts off the radius at 10^9 cm and the associated value of the density. It also adds a radius = 0 point (assumed to have the same density as the first value in the density array). We then interpolate using the numerical derivative array:

```
rhoprime = (rho[1:] - rho[:-1]) / (rad[1:] - rad[:-1])
```

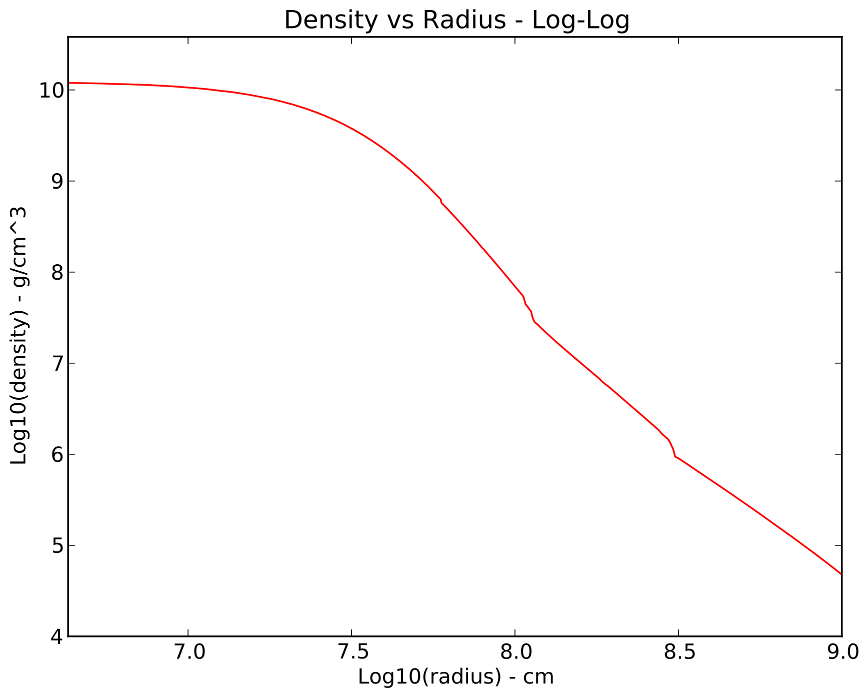


Figure 4: Plot of $\log \rho$ vs $\log r$

3

Now, we go back to the Poisson equation. Spherical symmetry simplifies it to the second order equation

$$\frac{d^2\Phi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr} = 4\pi G\rho(r)$$

which we can turn into a system of two first order ODE's:

$$\frac{d\Phi}{dr} = z$$
$$\frac{dz}{dr} = 4\pi G\rho - \frac{2}{r}z$$

In implementing the forward Euler method, we use the boundary conditions

$$\frac{d\Phi(r=0)}{dr} = 0$$
$$\frac{dz(r=0)}{dr} = 4\pi G\rho_c - \text{ constant?}$$

as well as the temporary condition

$$\Phi(r=0) = 0$$

with the understanding that we can add a constant to our answer at the end to satisfy the real condition

$$\Phi(r = R_{outer}) = -\frac{GM(r = R_{outer})}{R_{outer}}$$

[Note: the "-constant?" is there because I'm not sure how to properly handle the $-2z/r$ term, which is equal to $2*0/0$ at $r=0$. It assumed it is a constant since that's what it is for the constant density case. Further, I tried many different values for that constant of many orders of magnitude, and none seemed to significantly affect my answer. So, I chose the simplest route and made it 0.]

Now, we test the code assuming that the density is constant, so

$$\Phi(r) = \frac{2}{3}\pi G\rho(r^2 - 3R_{outer}^2)$$

Since

$$\Phi(0) = -2\pi G\rho R_{outer}^2,$$

we must add that number to our array of the potential so that the central value is correct. Then, we can check to see that the value at R_{outer} converges to the expected value:

$$\Phi(R_{outer}) = -\frac{4}{3}\pi G\rho R_{outer}^2.$$

We use two step sizes h_1 and h_2 (by choosing 3000 points and 6000 points respectively) so that

$$Q = \frac{|\Phi(h_2) - \Phi(R_{outer})|}{|\Phi(h_1) - \Phi(R_{outer})|} = \left(\frac{1}{2}\right)^n.$$

Computing this, we get that:

$$Q = \frac{8.33333333327 * 10^{-5}\Phi(R_{outer})}{0.000166666666666\Phi(R_{outer})} = \frac{1}{2}$$

, so the forward Euler method is first order convergent as it should be.

So, now that we know that the code works, we go back to our nonuniform density.

We get the mass at the outer radius by choosing the appropriate values from the mass array (at the same location as the max radius):

```
tot_mass = mass[rad.size-1]
```

Then, we correct the phi array with the line

```
phiarray += -ggrav*tot_mass/max(newrad) - phiarray[-1]
```

so that the outer boundary, represented by "phiarray[-1]", matches the boundary condition. Finally, we plot the gravitational potential of the star as a function of radius.

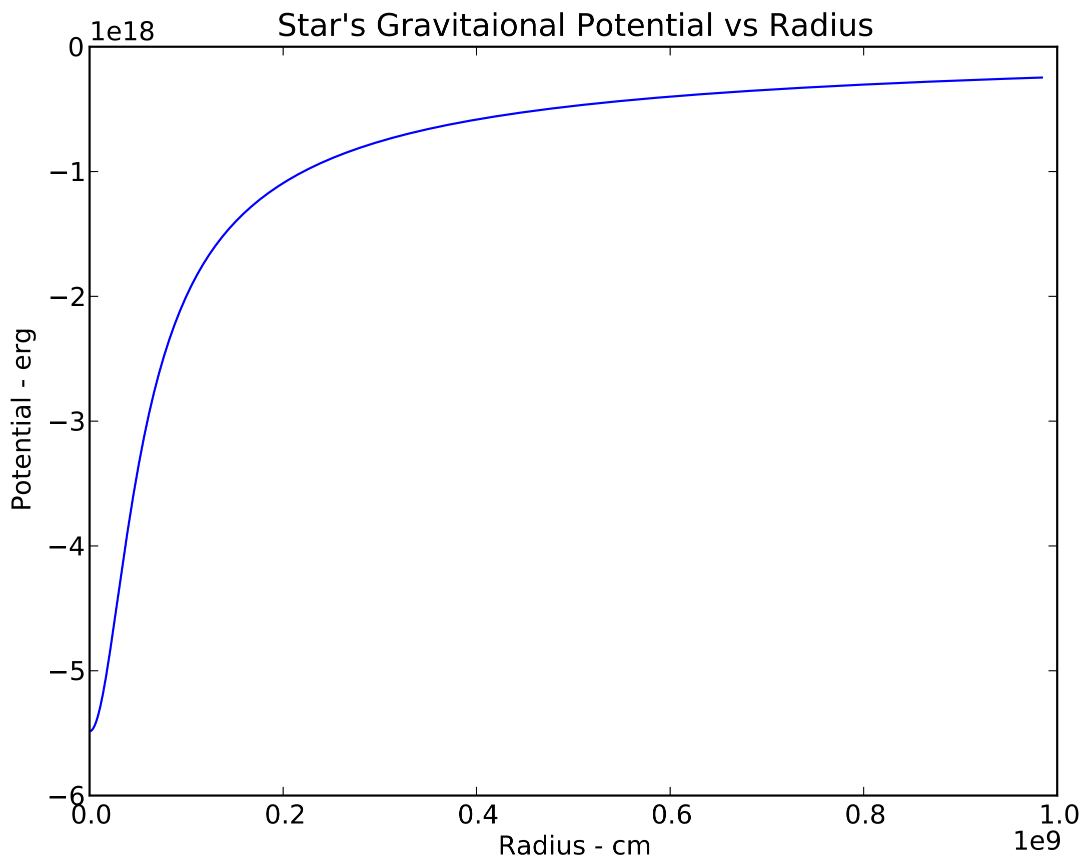


Figure 5: Gravitational Potential vs Radius