Ay190 – Worksheet 10
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Solution of a simple BVP

We are solving

$$\frac{dy}{dx} = u(x),$$
$$\frac{du}{dx} = 12x - 4,$$
$$y(0) = 0,\ y(1) = 0.1.$$

using both Euler and RK2 integrators.

Implementing both for 10, 50, 200, and 1000 points, we find that all the Euler integrator only needs two iterations for the error to go below the threshold of 10^{-12}, even with the outlandish choices of z_0 and z_1, for all of the different numbers of points. The RK2 integrator only needs one! Below are the final values of z calculated by the integrators.

npoints	z_{final} (Euler)	z_{final} (RK2)
10	0.495061728395	0.124691357836
50	0.17996668055	0.100832985714
200	0.119999494962	0.100050503388
1000	0.103999995992	0.100002001971

The values of z seem to be converging to z = 0.1 for large numbers of points. It should do this because the actual solution to the BVP is

$$y = 2x^3 - 2x^2 + 0.1x,$$

which means $z = y'(0) = 0.1$.

To demonstrate convergence, we look at

$$Q = \frac{|y(x;h_2) - y(x)|}{|y(x;h_1) - y(x)|} = \left(\frac{h_2}{h_1}\right)^n$$

where $y(x)$ is the actual solution, and h_1 and h_2 are step sizes with $h_1 > h_2$. Choosing the step sizes $h_1 = 1/(10-1) = 1/9$ and $h_2 = 1/(1000-1) = 1/999$, we get that $Q = (1/111)^n \approx (0.009)^n$, where n is the order of convergence.

For the Euler Integrator, we get that $Q = 0.00912161248173$ (using the maximum differences between the calculated and actual solutions in the formula), which clearly indicates that the order of convergence is $n = 1$ as we expect.

For the RK2 integrator, we get the strange value $Q = 10.792093469$. To see why this is happening, we examine the plots of the two integrators with the actual solution.

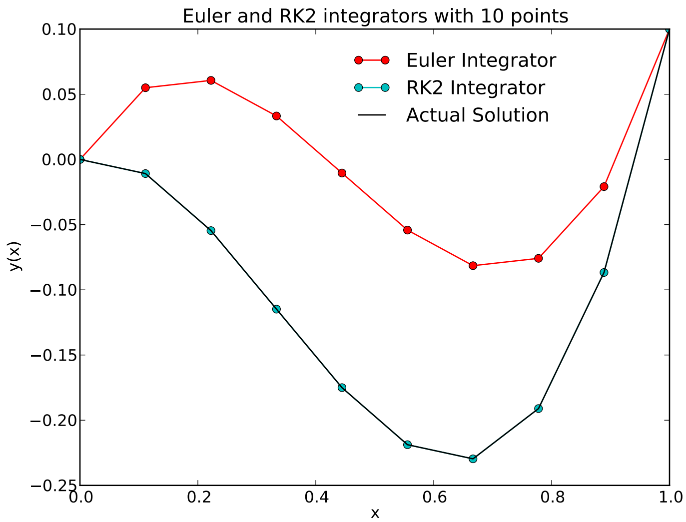

Figure 1: Euler and RK2 integrators with 10 points

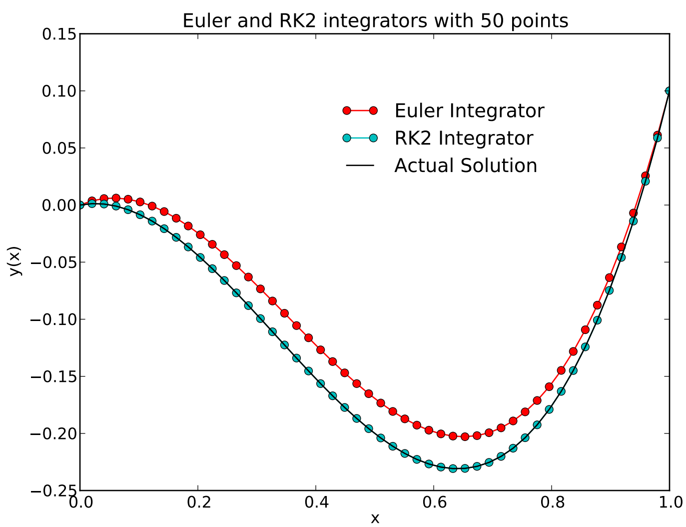

Figure 2: Euler and RK2 integrators with 50 points

We see that the RK2 integrator, even for the relatively small value of 10 points, seems to plot the actual solution exactly! Therefore, it makes sense that the Q factor is so weird because we are dividing by essentially 0. To explain why this happens, we note that the error in the RK2 integrator contains order h^3 terms and higher, which correspond to the third derivative of $u(x)$ in the equation

$$\frac{dy}{dx} = u(x).$$

That is, when we expand $u(x)$, the lowest order error term for the RK2 method is $\frac{h^3}{6}u'''(x_0)$. But, since $u'(x) = 12x - 4$, $u'''(x) = 0$ as do all higher derivatives. Therefore, all error terms are 0, which means we are actually calculating the exact answer no matter how many points we use.

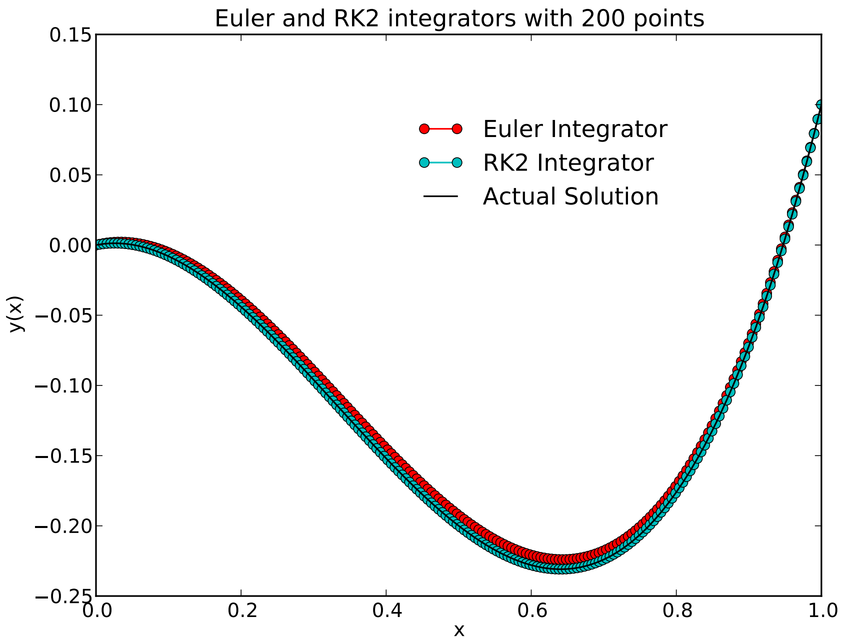

Figure 3: Euler and RK2 integrators with 200 points

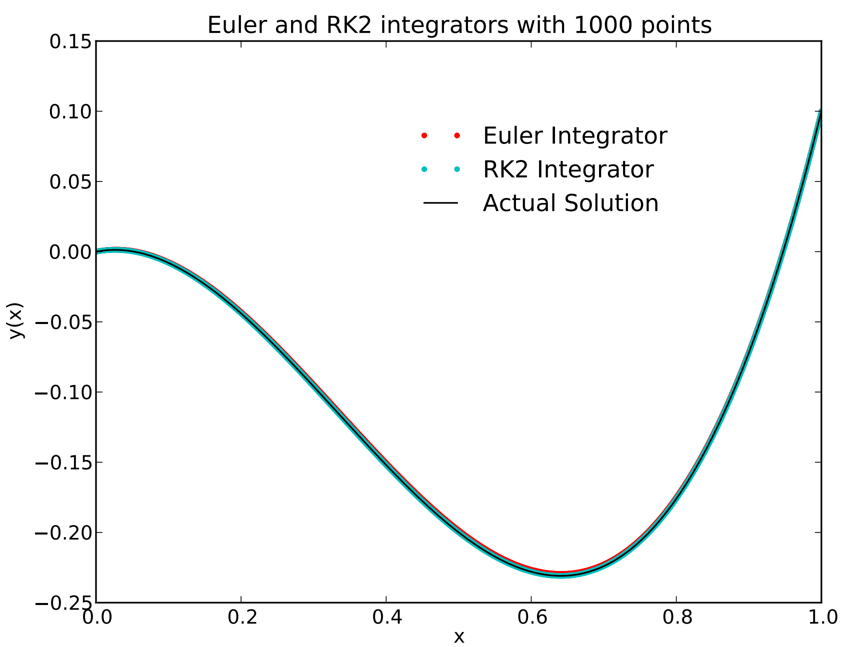

Figure 4: Euler and RK2 integrators with 1000 points