Numerical Techniques 2024–2025

5. Beyond the 1D advection equation

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- 2D problems
- Diffusion
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  - Burger's equation
  - (Barotropic vorticity equation)
  - Fibrillation

- **Systems of equations**
- 2D problems
- Diffusion
- Nonlinearity

- We consider here systems with multiple dependent variables plus interaction between these variables (e.g. wind-pressure).

- Example: the linearized 1D shallow water equations (SWE) with unknowns u and h:

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0$$
$$\frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} + H\frac{\partial u}{\partial x} = 0$$

with U and H constants.

These equations are used very often to study numerical schemes.

Systems of equations

- In matrix notation:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{C}\frac{\partial \mathbf{v}}{\partial x} = \mathbf{0}$$

where $\mathbf{v}$ is a vector containing the unknown fields

$$\mathbf{v} = \begin{pmatrix} u \\ h \end{pmatrix}$$

and $\mathbf{C}$ is a 2×2 matrix:

$$\mathbf{C} = \begin{pmatrix} U & g \\ H & U \end{pmatrix}$$

- Considering a harmonic shape for the solution,

$$\mathbf{v}_k = \begin{pmatrix} u_k \\ h_k \end{pmatrix} e^{ikx},$$

the time evolution can be written as

$$\mathbf{v}_k^{n+1} = \mathbf{A}_k \mathbf{v}_k^n$$

with $\mathbf{A}_k$ the *amplification matrix*.

- A definition of stability (analogous to oscillation equation) is then:

$$\|\mathbf{v}_k^n\| = \|\mathbf{A}_k^n \mathbf{v}_k^0\| \leq \|\mathbf{v}_k^0\|$$

- Given the property that $\|\mathbf{A}_k^n\| \leq \|\mathbf{A}_k\|^n$, a sufficient condition for stability is

$$\|\mathbf{A}_k\| \leq 1$$

- This is a sufficient condition, but not a necessary one (see *Durran* for an example).
- Different matrix norms can be defined; the most common is the maximal eigenvalue:

$$\|\mathbf{A}\| = \max |\mathrm{eig}(\mathbf{A})|$$

Systems of equations: dispersion relation

- Like for the advection equation, a more powerful tool is the discrete dispersion relation.
- The dispersion relation is retrieved by assuming discretized harmonic waves (in space *and* time) for u and h

$$
\begin{aligned}
u_j^n &= \hat{u} e^{i(kj\Delta x - \omega n \Delta t)} \\
h_j^n &= \hat{h} e^{i(kj\Delta x - \omega n \Delta t)}
\end{aligned}
$$

- For example, considering leapfrog time discretization and centered spatial differences, the SWE become

$$
\begin{aligned}
\left(-\frac{\sin \omega \Delta t}{\Delta t} + U \frac{\sin k \Delta x}{\Delta x}\right) \hat{u} + g \frac{\sin k \Delta x}{\Delta x} \hat{h} = 0 \\
H \frac{\sin k \Delta x}{\Delta x} \hat{u} + \left(-\frac{\sin \omega \Delta t}{\Delta t} + U \frac{\sin k \Delta x}{\Delta x}\right) \hat{h} = 0
\end{aligned}
$$

This homogeneous system only has a nontrivial (i.e. nonzero) solution if its determinant is zero.

- The determinant is zero if

$$\left(-\frac{\sin \omega\Delta t}{\Delta t} + U\frac{\sin k\Delta x}{\Delta x}\right)^2 - gh\left(\frac{\sin k\Delta x}{\Delta x}\right)^2 = 0$$

or, with $c = \sqrt{gH}$

$$\sin \omega\Delta t = \frac{(U \pm c)\Delta t}{\Delta x}\sin k\Delta x$$

- This is the *dispersion relation* of the SWE for this discretization. It should be compared with the exact dispersion relation for the SWE:

$$\omega = (U \pm c)k$$

- The $\pm$ symbol denotes that there are *two* wave solutions (left-travelling and right-travelling), each with their own speed. In the atmosphere, even more wave-types exist: Rossby-wave, inertia-gravity waves, sound waves.

- Note that the stability condition can be derived from the dispersion analysis, by requiring that ω is real:

$$\left|\frac{(U \pm c)\Delta t}{\Delta x}\right| \leq 1$$

- It is not strictly necessary to define all dependent variables in the same gridpoints.

- E.g. for the linearized 1D shallow water equations you can define u and h at different gridpoints:

  Not staggered:

  $$u_{j-1},\ h_{j-1} \qquad u_j,\ h_j \qquad u_{j+1},\ h_{j+1}$$

  Staggered:

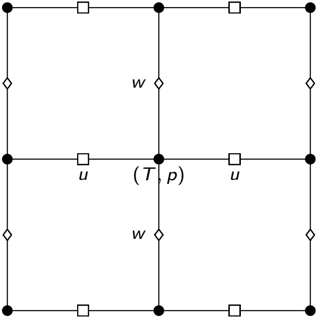
  $$u_{j-1} \qquad h_{j-\frac{1}{2}} \qquad u_j \qquad h_{j+\frac{1}{2}} \qquad u_{j+1}$$

- Advantage: spatial derivatives are calculated over a smaller grid distance (hence more accurate):

$$\left.\frac{\partial h}{\partial x}\right|_j \approx \frac{h_{j+\frac{1}{2}} - h_{j-\frac{1}{2}}}{\Delta x}$$

Systems of equations: Staggering

- Staggering solves the problem of negative groupspeeds of the shortest waves
- In more spatial dimensions, the *Arakawa C grid* is quite popular. Velocities are offset w.r.t. other variables (pressure, temperature, ...)



- Interested? – go for the student's project!

- Systems of equations
- **2D problems**
- Diffusion
- Nonlinearity

More than two independent variables

- Example: the 2D advection equation

$$\frac{\partial\psi}{\partial t} + U\frac{\partial\psi}{\partial x} + V\frac{\partial\psi}{\partial y} = 0$$

- Stability analysis of a scheme by inserting the wave solution

$$\phi^j_{m,n} = e^{i(km\Delta x + \ell n\Delta y - \omega j\Delta t)}$$

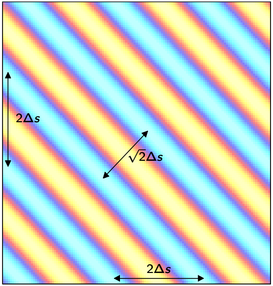
For the space-centered leapfrog scheme and $\Delta x = \Delta y = \Delta s$, this yields:

$$(|U| + |V|)\frac{\Delta t}{\Delta s} < 1$$

or, if C is the maximum speed in the domain,

$$C\frac{\Delta t}{\Delta s} < \frac{1}{\sqrt{2}}$$

- Reason for this more restrictive stability condition:



So a wave with wavelength $2\Delta s$ in the x-direction and $2\Delta s$ in the y-direction has a wavelength $\sqrt{2}\Delta s$ along the first bisector.

- Systems of equations
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Diffusion

- Diffusion is modeled with the following equation:

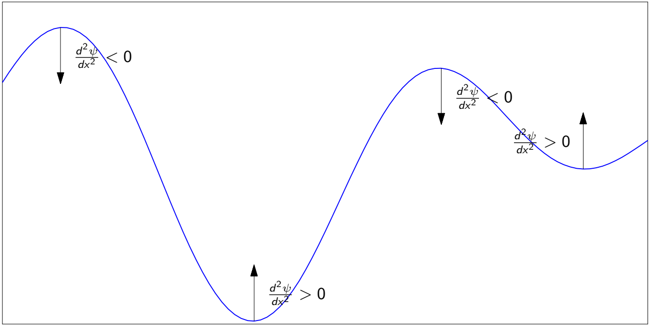
$$\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial x} K \frac{\partial \psi}{\partial x}$$

  with $K > 0$.

- Note that this is not a hyperbolic system (i.e. no wave solutions).

- In NWP models such terms arise in the physics parameterisation (see *Physical Meteorology: Surface and Turbulence*).

- Diffusion flattens the peaks in a signal:



- So it should be inherently stable...

- Let us discretize with a forward-time, centered-space scheme:

$$\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} = K\delta_x^2\phi_j^n$$

where $\delta_x^2\phi_j = \frac{\phi_{j+1} - 2\,\phi_j + \phi_{j-1}}{\Delta x^2}$

- Von Neuman stability analysis shows that the amplification factor is

$$A_k = 1 - 2\nu\,(1 - \cos k\Delta x)$$

with

$$\nu = \frac{K\Delta t}{\Delta x^2}$$

The scheme is stable provided $|A_k|^2 \leq 1$, i.e. if

$$0 < \nu \leq \tfrac{1}{2}$$

- Note that due to the form of $\nu = \frac{K\Delta t}{\Delta x^2}$ this scheme does not stay stable when $\Delta t, \Delta x \to 0$ unless Δt decreases much more rapidly than Δx!

- This makes this scheme very inefficient at high resolutions!

- With trapezium time differencing

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = \frac{K}{2}\left(\delta_x^2 \phi_j^{n+1} + \delta_x^2 \phi_j^n\right)$$

the amplification factor becomes

$$A_k = \frac{1 - \nu\,(1 - \cos k\Delta x)}{1 + \nu\,(1 - \cos k\Delta x)}$$

So if $K > 0$ then $|A_k| \leq 1$ for all Δt.

- However, one now has to solve a tridiagonal system (in 1D):

$$\begin{pmatrix} \ddots & & & & \\ \cdots & 1+\nu & -\nu/2 & 0 & \cdots \\ \cdots & -\nu/2 & 1+\nu & -\nu/2 & \cdots \\ \cdots & 0 & -\nu/2 & 1+\nu & \cdots \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ \phi_{j-1} \\ \phi_j \\ \phi_{j+1} \\ \vdots \end{pmatrix}^{n+1} = \ldots$$

- So the question becomes, does the increase in the time step Δt outweigh the increase in computing cost for the solver?

- Systems of equations
- 2D problems
- Diffusion
- **Nonlinearity**

Nonlinearity

We will consider the following examples of nonlinear systems:

- Variable advection speed and aliasing
- Burger's equation and shock development
- (The barotropic vorticity equation)
- Fibrillation due to nonlinear diffusion

- We consider the 1D advection equation with variable wind $c(x)$,

$$\frac{\partial \psi}{\partial t} + c(x)\frac{\partial \psi}{\partial x} = 0$$

- Let us consider a discretization on N gridpoints on a domain $[0, 2\pi)$. Suppose that the wind $c(x)$ and the initial state $\psi(t = 0)$ are composed of waves with wavenumbers 0, $N/4$ and $N/2$ (i.e. wavelengths ∞, $4\Delta x$ and $2\Delta x$):

$$c(x_j) = c_0 + (c_r + ic_i)e^{i\pi j/2} + (c_r - ic_i)e^{-i\pi j/2} + c_n e^{i\pi j}$$
$$\phi(x_j) = a_0 + (a_r + ia_i)e^{i\pi j/2} + (a_r - ia_i)e^{-i\pi j/2} + a_n e^{i\pi j}$$

- We may expect aliasing, e.g.

$$e^{i\pi j/2}e^{i\pi j} = e^{i3\pi j/2} = e^{-i\pi j/2}$$

- One can show that with centered spatial differences, the exact time evolution of the spectral coefficients a_0, a_r, a_i, a_n is given by:

$$\frac{da_0}{dt} = 2\frac{a_i c_r - a_r c_i}{\Delta x} \qquad \frac{da_n}{dt} = 2\frac{a_i c_r + a_r c_i}{\Delta x}$$

$$\frac{da_r}{dt} = a_i \frac{c_n + c_0}{\Delta x} \qquad \frac{da_i}{dt} = a_r \frac{c_n - c_0}{\Delta x}$$

- Eliminating a_i, we find for the time evolution of a_r:

$$\frac{d^2 a_r}{dt^2} = \frac{c_n^2 - c_0^2}{\Delta x^2} a_r$$

which has an exponential solution!

- So with this discretization, we get an instability if $c_n > c_0$, regardless of the time discretization!.
- This growth is unphysical since the solution is bounded by the maximum value of ψ.

- Another example of nonlinear instability is Burger's equation:

$$\frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi}{\partial x} = 0$$

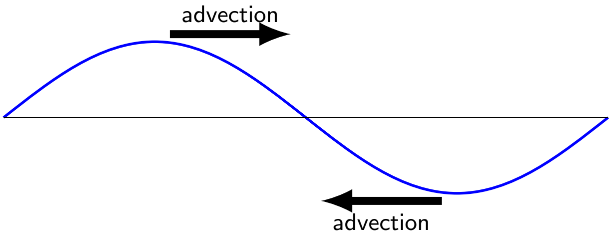
- If the initial condition is $\psi(x, t = 0) = f(x)$, then the solution can be written implicitly as

$$\psi(x, t) = f(x - \psi(x, t)t)$$

So ψ is constant along the so-called *characteristic curves* in the (x, t) plane:

$$x - \psi(x, t)t = cst.$$

- We can expect some problems: consider a sine-like initial condition:



- Eventually, a shock will develop.

- One can show that the ℓ_2-norm of the exact solution over a periodic domain $[0, 1]$ is conserved (so the shock will remain limited in the exact solution):

$$\frac{\partial}{\partial t}\int_0^1 \psi(x,t)^2 dx = 0$$

- However, when discretizing Burger's equation as

$$\frac{d\phi_j}{dt} + \phi_j\left(\frac{\phi_{j+1} - \phi_{j-1}}{2\Delta x}\right) = 0$$

the ℓ_2-norm is not conserved:

$$\frac{d}{dt}\sum_j \phi_j^2 = \sum_j \phi_j\phi_{j+1}\left(\frac{\phi_{j+1} - \phi_j}{\Delta x}\right)$$

- We can try a different shape ('flux-shape') of Burger's equation:

$$\frac{\partial \psi}{\partial t} + \frac{1}{2}\frac{\partial \psi^2}{\partial x} = 0$$

which is discretized as:

$$\frac{d\phi_j}{dt} + \tfrac{1}{2}\frac{\phi_{j+1}^2 - \phi_{j-1}^2}{2\Delta x} = 0$$

- This leads to

$$\frac{d}{dt}\sum_j \phi_j^2 = -\tfrac{1}{2}\sum_j \phi_j \phi_{j+1}\frac{\phi_{j+1} - \phi_j}{\Delta x} \neq 0$$

so it doesn't conserve the norm either.

- It's possible to make a combination of these two alternatives:

$$\frac{d\phi_j}{dt} + \tfrac{2}{3}\frac{\phi_{j+1}^2 - \phi_{j-1}^2}{2\Delta x} + \tfrac{1}{3}\frac{\phi_{j+1} - \phi_{j-1}}{2\Delta x}\phi_j = 0$$

- The contributions of both schemes to the norm evolution will cancel out, so this scheme conserves the norm

$$\frac{d}{dt}\sum_j \phi_j^2 = 0$$

and doesn't blow up.

- In reality there is always some diffusion

$$\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi}{\partial x} = \nu\frac{\partial^2\psi}{\partial x^2}$$

- Then the true solution never develops a shock. However, if discretized with the advective form, a scheme can develop a shock for sufficiently small values of ν.
- This is another reason for introducing some dosis of (artificial) numerical diffusion.

- The barotropic vorticity equation (see *Dynamic Meteorology*) is written as:

$$\frac{\partial \zeta}{\partial t} + \mathbf{u} \cdot \nabla \zeta = 0$$

with $\mathbf{u}$ the geostrophic wind

$$\mathbf{u} = \mathbf{k} \times \nabla \psi$$

and the vorticity

$$\zeta = \mathbf{k} \cdot \nabla \times \mathbf{u} = \nabla^2 \psi$$

- Although this a nonlinear equation, one can show that there is no *net* energy transfer between scales (the average wavenumber is conserved).
- Here too, a clever discretization satisfies this conservation constraint, and allows to suppress the nonlinear instability.

– see student's projects.

Fibrillation

- Nonlinear instabilities also have an effect in the diffusion equation.
- This can be studied in its most easy form in the damping equations,

$$\frac{\partial \psi}{\partial t} = -\left(K\psi^{P}\right)\psi + S$$

  with K a constant representing the amount of damping, and P the degree of nonlinearity. S is an external forcing.

- This equation has been studied by Kalnay and Kanamitsu (1988) (Mon. Wea. Rev., 116, 1945-1958) and is considered as the reference test for diffusion schemes in atmospheric models.

Fibrillation

- The equation is discretized as

$$\frac{\phi_{n+1} - \phi_n}{\Delta t} = -K\phi_n^P \left[\gamma\phi_{n+1} + (1-\gamma)\phi_n\right] + S_n$$

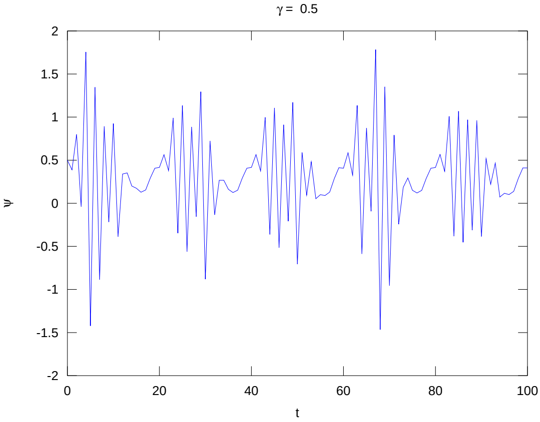
  with $\Delta t = 1$, and the forcing is set to

$$S_n = 1 + \sin\left(\frac{2\pi n}{20}\right)$$

  (think of this as a radiative forcing).
- The parameter γ determines the degree of implicitness ($\gamma = 0.5$ is trapezium-like).

Fibrillation

With $K = 10$ and $P = 2$ and with $\gamma = 1/2$, we get following results:



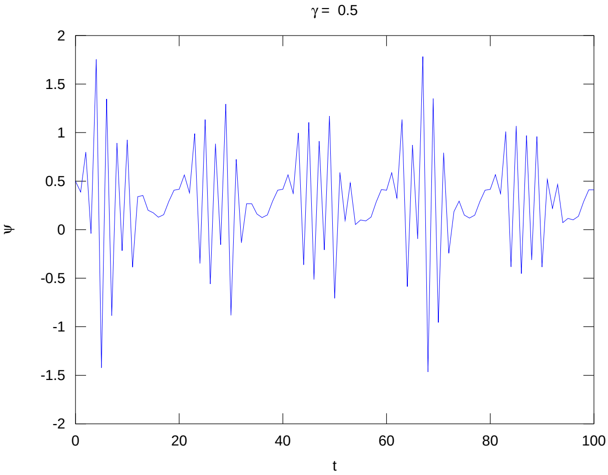
Similar behavior is also encountered in a 3D NWP model.

Fibrillation

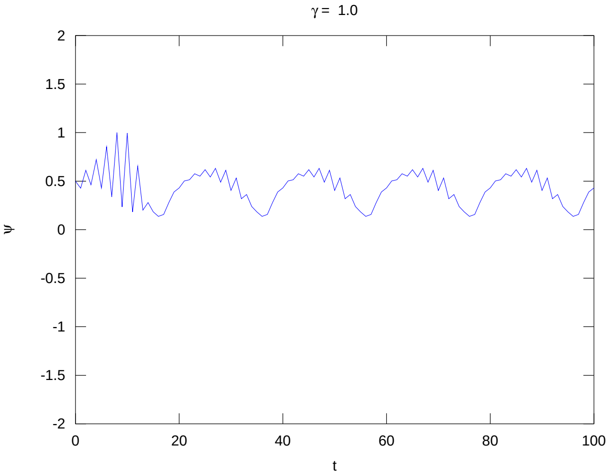
- We see nonsensical oscillations in time.
- These are not unstabilities in the sense that the model blows up.
- They can not be understood by linear stability analysis.
- Contrary to Burger's equations, we do not know the exact solution, so we can not analyse it on paper.
- However, they should be eliminated: invent schemes and choose the most convenient one (see *Kalnay and Kanamitsu*).

  A popular solution is *over-implicitness* (i.e. $\gamma > 1$).

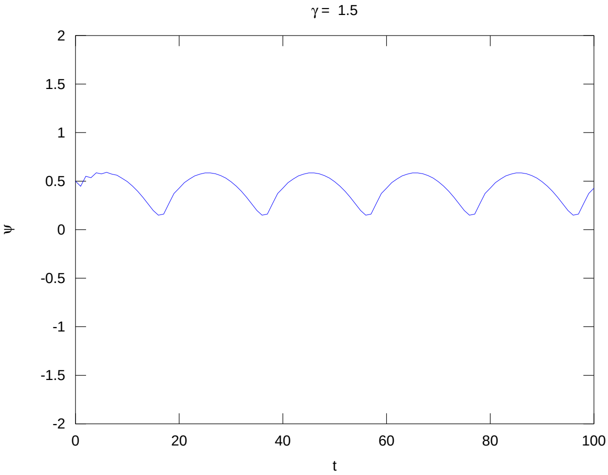
Fibrillation



Fibrillation



Fibrillation



Some overview remarks

- Try to see links with other lessons and other courses:
  - Stability and dispersion for systems of equations is (almost) the same as for scalar equations, with matrices replacing scalars.
  - Aliasing is closely related to nonlinearity
  - Barotropic vorticity equation $\Leftarrow$ *Dynamic Meteorology*
  - Diffusion equation $\Leftarrow$ *Physical Meteorology*

- Methodology is more important than algebra! Think in terms of waves...

- This course is about *surprising* results due to numerical discretization.
  - stability in 2D is more stringent than in 1D
  - even for the diffusion equation, unstable behaviour may occur
  - nonlinearity may lead to instability, even with supposedly stable schemes.
  - ...

- If something is unclear, you should pick the corresponding student's project!