

# Numerical Techniques 2022–2023

## 1. Discretization and stability

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Postgraduate Studies in Weather and Climate Modeling

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- Discretization
- A prototype system: the 1D advection equation
- An example of a numerical scheme: the upstream scheme
- Accuracy and consistency
- Convergence
- Stability

- Meteorological fields (temperature, wind, pressure, humidity, ...) are continuous in time and space.
- On a computer you cannot represent a continuous field; you would need an infinity of points.
- So a field will be *discretized*, i.e. it is represented by the values in a set of discrete points:



replacing the continuous coordinate  $x$  by discrete points  $i\Delta x$  labeled by  $i$ . Note that  $i$  is variable, and  $\Delta x$  is constant!

- similarly in 2D:  $i\Delta x, j\Delta y$
- discretizing time:  $n\Delta t$

# Discretization of derivatives: finite differences

- Many physical laws are differential equations
- Some definition of the derivative in  $x$ :

$$\frac{df}{dx}(x_0) = \lim_{\delta x \rightarrow 0} \frac{f(x_0 + \delta x) - f(x_0)}{\delta x}$$

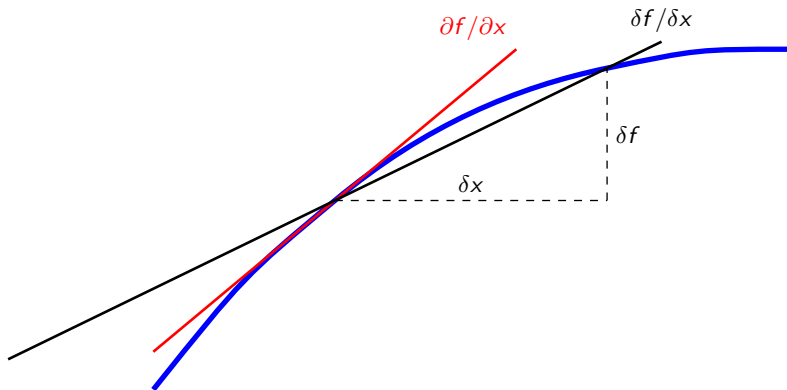
$$\frac{df}{dx}(x_0) = \lim_{\delta x \rightarrow 0} \frac{f(x_0) - f(x_0 - \delta x)}{\delta x}$$

$$\frac{df}{dx}(x_0) = \lim_{\delta x \rightarrow 0} \frac{f(x_0 + \delta x) - f(x_0 - \delta x)}{2\delta x}$$

- On our discrete grid we can't take the limit  $\delta x \rightarrow 0$ , so the derivative is computed with a finite value  $\delta x \rightarrow \Delta x$  (i.e. the grid distance).

# Discretization of derivatives: finite differences

- The limit is then better approximated by increasing the resolution



We will now consider the 1D advection equation with constant advection

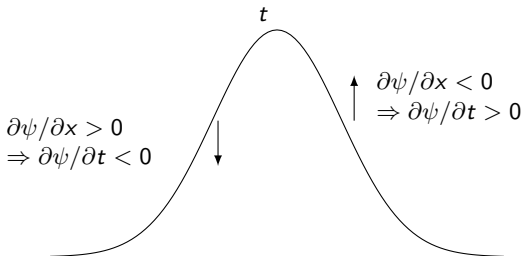
$$\frac{\partial \psi}{\partial t} + c \frac{\partial \psi}{\partial x} = 0$$

Why?

- hyperbolic systems can be decomposed into decoupled equations of this type
- most systems studied with atmospheric models are hyperbolic.

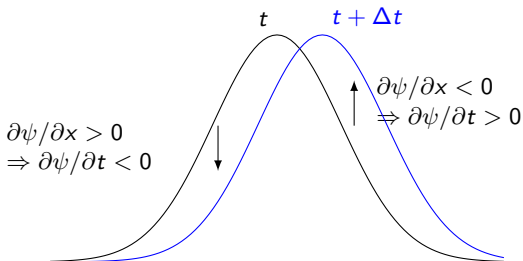
Physical interpretation of the advection equation:

- $\psi$  is any (meteorological) field: temperature, pressure, humidity, pollutant concentration, ...
- $c$  is the wind speed.
- time evolution: the field is *transported* with speed  $c$ :



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The exact solution of the advection equation

$$\frac{\partial \psi}{\partial t} + c \frac{\partial \psi}{\partial x} = 0$$

is given by

$$\psi(x, t) = \psi(x - ct, t = 0)$$

Let us call  $\phi_j^n$  the solution of the discretized system at time  $n\Delta t$  and at grid point  $j\Delta x$ .

We replace the derivatives by,

$$\begin{aligned}\frac{\partial \psi}{\partial t} &\rightarrow \frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} \\ \frac{\partial \psi}{\partial x} &\rightarrow \frac{\phi_j^n - \phi_{j-1}^n}{\Delta x}\end{aligned}$$

in the advection equation:

$$\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + c \frac{\phi_j^n - \phi_{j-1}^n}{\Delta x} = 0$$

This is called the *upstream scheme*.

This equation can be solved w.r.t.  $\phi_j^{n+1}$ :

$$\phi_j^{n+1} = (1 - \mu)\phi_j^n + \mu\phi_{j-1}^n$$

with  $\mu = c\Delta t/\Delta x$ .

So if the solution is known at time  $t = n\Delta t$ ,  
then it can be calculated at  $t = (n + 1)\Delta t$ .

- But we see that increasing the spatial resolution doesn't necessarily improve the solution !?

We say that the discretized system becomes *unstable*.

- Stability is a central concept in this course. This lesson will (try to) explain why.
- For this we need additional concepts: accuracy, consistency and convergence.

Using a Taylor expansion

$$f(x_0 + \Delta x) = f(x_0) + \Delta x \frac{df}{dx}(x_0) + \frac{\Delta x^2}{2} \frac{d^2 f}{dx^2}(x_0) + \frac{\Delta x^3}{6} \frac{d^3 f}{dx^3}(x_0) + \dots$$

the error in the approximation of the derivative is

$$\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} - \frac{df}{dx}(x_0) = \frac{\Delta x}{2} \frac{d^2 f}{dx^2}(x_0) + \frac{\Delta x^2}{6} \frac{d^3 f}{dx^3}(x_0) + \dots$$

The leading term is proportional to  $\Delta x$ , so this is called a first-order accurate method.

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Considering the centered finite-difference approximation

$$\frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x} - \frac{df}{dx}(x_0) = \frac{\Delta x^2}{6} \frac{d^3 f}{dx^3}(x_0) + \frac{\Delta x^4}{120} \frac{d^5 f}{dx^5}(x_0) + \dots$$

we see that it is second-order accurate.

By using more information (more gridpoints), it is possible to obtain a more accurate approximation for the derivative:

$$\begin{aligned} \frac{df}{dx}(x_0) = & \frac{4}{3} \left( \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x} \right) \\ & - \frac{1}{3} \left( \frac{f(x_0 + 2\Delta x) - f(x_0 - 2\Delta x)}{4\Delta x} \right) + O[\Delta x^4] \end{aligned}$$

A scheme is called *consistent* if the truncation error converges to zero when  $\Delta x \rightarrow 0$  and  $\Delta t \rightarrow 0$ .

For example, the truncation error for the upstream scheme is:

$$\frac{\psi_j^{n+1} - \psi_j^n}{\Delta t} + c \frac{\psi_j^n - \psi_{j-1}^n}{\Delta x} = \frac{\Delta t}{2} \frac{\partial^2 \psi}{\partial t^2} - c \frac{\Delta x}{2} \frac{\partial^2 \psi}{\partial x^2} + \dots$$

where  $\psi$  denotes the exact solution.



A finite-difference scheme is called *convergent* of order  $(p, q)$  if in the limit  $\Delta t, \Delta x \rightarrow 0$ , the numerical solution  $\phi_j^n$  satisfies

$$\|\psi(n\Delta t, j\Delta x) - \phi_j^n\| = O[\Delta t^p] + O[\Delta x^q]$$

Consistency tells you something about the equations, convergence tells you about the solution.

The upstream scheme example shows that these two are not identical: it is consistent but not convergent.

There's a problem with the usability of convergence: it relies on the exact solution, which is unknown in most cases.

The Lax equivalence theorem says that:

**If a finite-difference scheme is linear, stable, and consistent,  
then it is convergent**

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Importance: in practice we never know the true solution, but we  
can check if the scheme is *consistent* and *stable*.

But when is a scheme stable?

A sufficient condition for stability is that

$$\|\phi^n\| \leq \|\phi^0\| \quad \text{for all timesteps } n > 0$$

Again, we don't need to know the true solution for this!

- It is difficult to check the evolution of  $\|\phi^n\|$  in general.
- Von Neumann proposed to check stability by considering harmonic functions:

$$\phi(x) = \exp(ikx) = \cos(kx) + i \sin(kx)$$

**If a scheme is stable for all harmonic functions (i.e. all values of  $k$ ), then it is also stable for all (non-harmonic) functions.**

- Because harmonic functions are eigenfunctions of the differential operator, one can express their time evolution as a multiplication with an *amplification factor*  $A_k$ :

$$\phi^n = A_k \phi^{n-1} = (A_k)^n \phi^0$$

- So stability requires  $\|A_k\| \leq 1$ , for all  $k$ .

## Example: stability of the upstream scheme

Let us consider again the upstream scheme for the advection equation:

$$\phi_j^{n+1} = (1 - \mu)\phi_j^n + \mu\phi_{j-1}^n$$

with  $\mu = c\Delta t/\Delta x$ .

**What is the amplification factor?**

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**What is the amplification factor?**

First, assume that  $\phi_j^n = (A_k)^n e^{ikj\Delta x}$ , so

$$(A_k)^{n+1} e^{ikj\Delta x} = (A_k)^n (1 - \mu) e^{ikj\Delta x} + (A_k)^n \mu e^{ik(j-1)\Delta x}$$



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The amplification factor  $A_k$  then becomes

$$\begin{aligned} A_k &= 1 - \mu + \mu e^{-ik\Delta x} \\ &= (1 - \mu + \mu \cos k\Delta x) - i\mu \sin k\Delta x \end{aligned}$$

## Example: stability of the upstream scheme

**Is the amplification factor smaller than 1?**

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The modulus of the complex number  $A_k$  is given by

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This will be  $\leq 1$  for all values of  $k$  if

$$\mu(1 - \mu) \geq 0$$

i.e. if

$$0 \leq \frac{c\Delta t}{\Delta x} \leq 1$$

## Example: stability of the upstream scheme

- So the upstream scheme is *conditionally stable*, i.e. stability depends on the values of  $c$ ,  $\Delta x$  and  $\Delta t$ .
- In general,
  - ▶ higher spatial resolution is less stable
  - ▶ smaller timesteps is more stable
  - ▶ higher velocity is less stable
- $\mu = \frac{c\Delta t}{\Delta x}$  is called the Courant number.
- The condition  $\mu \leq 1$  is called the Courant-Fredrichs-Lewy (CFL) criterion.

Starting from a discretized equation:

- ① Check if the scheme is linear and consistent
- ② Fill in a harmonic solution  $\phi_j^n = (A_k)^n e^{i k j \Delta x}$  in the discretized equation
- ③ Calculate (the modulus of) the amplification factor  $A_k$
- ④ If  $|A_k| \leq 1$ , the scheme is stable
- ⑤ If the scheme is stable and consistent, it is convergent
- ⑥ If the scheme is convergent, the *error on the solution* can be reduced by increasing the resolution

- *Discretization* of a continuous function on a grid
- Approximation of derivatives with *finite differences*
- *Accuracy* and *Consistency* of a scheme (error on equations)
- *Convergence* of a scheme (error on solution)
- *Stability* of a scheme: Von Neumann analysis, Courant number and CFL criterion
- Applied to the upstream scheme to solve the 1D advection equation