Limitations of Some Common Lateral Boundary Schemes used in Regional NWP Models

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ABSTRACT

A brief critical assessment is presented of several lateral boundary schemes currently employed in regional weather prediction models. Simple flow models are used to determine the nature and cause of the primary shortcomings of each of the considered schemes. An awareness of these deficiencies can prove helpful in the implementation and further refinement of these schemes, and also in the interpretation of the resulting prediction errors.

1. Introduction

The treatment of the lateral boundaries is an intrinsic and distinctive problem associated with the formulation of regional weather prediction models. It is a problem that has bedevilled modelers from the earliest days of numerical weather prediction. [In this respect note also the apposite account given by Platzman (1979) of the apocryphal boundary problems encountered during the ENIAC computations of 1950.] The troublesome nature of this problem is not altogether surprising since fundamental and esoteric questions of existence, uniqueness and well-posedness of fluid flows arise in the study of the correct specification of lateral boundary conditions. Indeed the spectre of rabid ill-posedness has often been attached to "open-boundary" regional NWP models. Nevertheless predictions with significant measures of skill continue to be made with a rapidly increasing number of such models, and invariably these models employ one of a range of pragmatic lateral boundary treatments. It has been concluded that—'over-specified boundary conditions, for example relaxation toward the large scale values in a boundary zone, is satisfactory for practical applications' (WMO Rep., 1978, 1979).

In this study attention is confined to these pragmatic formulations and we consider four specific schemes, viz. boundary zones of diffusive damping, of tendency modification and of flow relaxation and also the pseudo-radiation type of boundary scheme. In the next section an analysis is undertaken of the behaviour of the schemes in a simple but meteorologically significant flow system. Consideration of some more complicated flow systems is given in the subsequent section. Three aspects given particular attention in our examination are:

1) the manner in which each scheme attempts to circumvent the overspecification problem,
2) the properties of the differential equations that govern the behaviour of the flow system, and
3) the properties of certain conventional finite-difference representations of the system.

The first aspect relates to the criteria for uniqueness and well-posedness of flow solutions in an open domain. In principle, only the subset of dynamic quantities that correspond to the transfer of information into the domain should be specified at the boundary (Charney, 1962; Davies, 1973; Oliger and Sundström, 1978). In this context the four schemes under consideration fall into two categories. The boundary-zone schemes modify the flow system to allow an increase in the number of variables that can be specified at the boundaries (Davies, 1976). In the pseudo-radiation type of boundary formulation an attempt is made to apply a "local" form of a radiation condition. This latter approach is invariably an approximate one for all but the simplest flow systems.

The objective of the other two aspects mentioned before is to examine the degree to which solutions in the interior region are adversely affected by the boundary formulations. In particular the possibility is explored that inherent adverse computational effects can occur even if the continuous system is well posed and its solutions exhibit the required properties.

2. Study of the boundary schemes in a simple system

The behaviour of the various boundary schemes is examined for a particularly simple flow system. Consider small amplitude, hydrostatic perturbations in the x, z or x, σ plane of an inviscid and non-rotating atmosphere about an isothermal basic state in uniform flow ($\bar{U}$). The linear equations for the horizontal structure of each vertical eigenfunction of this system can be written (see Appendix for the derivation) in the shallow-water form

$$\left.\begin{aligned} u'_t + \bar{U}u'_x &= -gh'_x \\ h'_t + \bar{U}h'_x &= -Hu'_x \end{aligned}\right\} . \tag{1a}$$

Here H refers to a separation constant whose value depends upon the particular eigenfunction, whilst u' is the perturbation horizontal velocity field and h' is related to the perturbation component of the geopotential or surface pressure field.

The set (1a) can be recast in the compact characteristic form,

$$u_t + cu_x = 0, \tag{1b}$$

where $u_{1,2} = (H^{1/2}u' \pm g^{1/2}h')$, and $c_{1,2} = \bar{U} \pm (gH)^{1/2}$. If the flow system occupies a limited horizontal domain $0 \leqslant x \leqslant L$ then (1b) implies that $u_{1,2}$ should be specified at a boundary only if $c_{1,2}$ is directed into the domain at that boundary. Thus, if $c > 0$ then (1b) is to be solved subject to the initial conditions

$$u(x, 0) = g(x) \quad \text{for} \quad 0 \leqslant x \leqslant L \tag{2}$$

and boundary condition

$$u(0, t) = f_1(t) \quad \text{for} \quad t \geqslant 0. \tag{3}$$

We now consider each boundary scheme in turn.

a. Diffusive damping scheme

A straightforward approach to alleviate the noise problem generated in the vicinity of the lateral boundaries due to overspecification or inappropriate boundary data is to introduce a marginal zone of large diffusion for the prognostic variables in the vicinity of the lateral boundaries (e.g., Benwell *et al.*, 1971; Burridge, 1975; Mesinger, 1977).

For this boundary scheme we consider the replacement of the prototype meteorological system of (1) by the following two schemes:

$$\text{Scheme A:} \quad \left.\begin{aligned} u'_t + \bar{U}u'_x &= -gh'_x + (\nu u'_x)_x \\ h'_t + \bar{U}h'_x &= -Hu'_x \end{aligned}\right\} , \tag{4a}$$

$$\text{Scheme B:} \quad u_t + cu_x = (\nu u_x)_x. \tag{4b}$$

In each case the diffusion coefficient, $\nu = \nu(x)$, is assigned appreciable values only within two marginal zones of width l ($\ll L$) located at $x = 0, L$. Furthermore, an additional boundary condition, $u'(L, t)$ or $u(L, t) = f_2(t)$ is also enforced. This second boundary condition would overspecify (1) but is legitimate for the revised systems given by (4a, b). Scheme A merely corresponds to the addition of a viscous diffusion term. For simplicity in considering this scheme we shall assume $\bar{U} \equiv 0$. A similar system was studied recently by Israeli and Orszag (1981). For Scheme B diffusion terms are added for *all* prognostic variables, i.e. in terms of the original primitive equations (outlined in the Appendix) this involves diffusion terms for momentum, potential temperature and surface pressure.

Sharp shear zones and thermal gradients induced by the diffusion zone might themselves be the seat of physical or computational instabilities. However, there are more basic shortcomings to this approach. Consider the effect of forcing a time-periodic disturbance of frequency ω at one end of a boundary zone of constant ν. In such a zone the solutions of (4a, b) take the respective forms

$$u = Ie^{-m_i x'}e^{i(m_r x' - \omega t)} + re^{m_i x'}e^{-i(m_r x' + \omega t)}, \tag{5a}$$

$$u = Ie^{-(n_i - \mathrm{Re})x'}e^{i(n_r x' - \omega t)} + re^{(n_i + \mathrm{Re})x'}e^{-i(n_r x' + \omega t)}, \tag{5b}$$

where I and r are amplitude constants, $x' = x/l$, $\mathrm{Re} = \tfrac{1}{2}(cl/\nu)$, and

$$(m_r^2, m_i^2) = 2\,\mathrm{Re}^2(1 + c^2/\omega\nu)^{-1} \times \{\mp 1 + [1 + (\omega\nu/c^2)^2]^{1/2}\}, \tag{6a}$$

$$(n_r^2, n_i^2) = \tfrac{1}{2}\,\mathrm{Re}^2\{\mp 1 + [1 + (\omega l^2/\nu)^2(1/\mathrm{Re})^4]^{1/2}\}. \tag{6b}$$

The I terms in (5a, b) represent a wave traveling in the positive x-direction and decreasing in amplitude in the direction of travel. Again the r terms represent a wave traveling with the same speed but in the opposite direction and it also experiences a decrease in amplitude in its direction of travel.

Two criteria that the boundary scheme should satisfy are that

1) Within the inflow zone (i.e., $0 < x < l$) the incoming wave I is transmitted without appreciable change of phase or amplitude, and

2) At an outflow zone (i.e., $L - l < x < L$) the reflected wave r does not enter the inner domain ($x < L - l$) with appreciable amplitude.

From (5a) and (6a) we see that for Scheme A these criteria require respectively that

$$(\omega l/m_r) \rightarrow c, \quad m_i \ll 1 \quad \text{and} \quad m_i \gg 1. \tag{7a}$$

In view of these incompatible requirements upon m_i we infer directly, in harmony with Israeli and Orszag's conclusion, that this boundary scheme involving only the addition of a momentum diffusion term is unsuitable. For Scheme B the requirements are that

$$(\omega l/n_r) \rightarrow c, \ (n_i - \mathrm{Re}) \ll 1 \ \text{and} \ (n_i + \mathrm{Re}) \gg 1. \tag{7b}$$

From (6b) we deduce that for $n_i \sim \mathrm{Re}$ then

$$n_i^2 - \mathrm{Re}^2 \approx \tfrac{1}{4}(\omega l^2/\nu)^2\,\mathrm{Re}^{-2}$$

and hence

$$(n_i - \mathrm{Re}) \approx \tfrac{1}{8}(\omega l^2/\nu)^2\,\mathrm{Re}^{-3}.$$

The first two requirements of (7b) are thus formally met if

$$(\omega l^2/\nu) \ll 2\sqrt{2}\,\mathrm{Re}^{3/2}.$$

This inequality can be rewritten as

$$2\pi(l/\mathcal{L}) \ll \sqrt{2}\,\mathrm{Re}^{1/2}, \qquad (8)$$

where $\mathcal{L} = 2\pi(c/\omega)$ is the wavelength of a wave solution of (1b) of frequency ω. Thus it follows that, in principle, the criteria (7b) are all met for the continuous differential system (4b) provided Re is sufficiently large.

In practice the numerical representation of (4b) will introduce constraints upon the width of the boundary zone l and the Reynolds number Re. Moreover the difference equation representation of (4b) should also satisfy the above criteria. Here we examine the properties of a leapfrog finite-difference representation of (4b) with a Dufort–Frankel formulation of the diffusion term, i.e.,

$$u_j^{n+1} = u_j^{n-1} - \alpha(u_{j+1}^n - u_{j-1}^n) + \mu(u_{j+1}^n + u_{j-1}^n - u_j^{n+1} - u_j^{n-1}), \qquad (9)$$

with $x = j(\Delta x)$, $t = n(\Delta t)$ for $j = \cdots -2, -1, 0, 1, 2, \ldots$, $n = 0, 1, 2, \cdots$ and $\alpha = c(\Delta t)/(\Delta x)$, $\mu = 2\nu(\Delta t)/(\Delta x)^2$. Consider solutions of (9) that oscillate with a frequency ω, i.e.

$$u_j^n = \xi_j e^{i\omega(n\Delta t)}.$$

The coefficients ξ_j satisfy the equation

$$(1 - \nu^*)\xi_{j+1} + 2[\nu^* \cos(\omega\Delta t) + \alpha^{-1}\sin(\omega\Delta t)]\xi_j - (1 + \nu^*)\xi_{j-1} = 0, \qquad (10)$$

where $\nu^* = \mu/\alpha = 2\nu/c(\Delta x)$, and the difference analogue of (5b) takes the form,

$$u_j^n = (\lambda_1)^j e^{-i(\theta x - \omega t)} + (-1)^j(\lambda_2)^j e^{i(\theta x + \omega t)}, \qquad (11)$$

with x and t being defined only at discrete increments apart.

Now using (11) we consider in turn the extent to which Scheme B satisfies the two criteria. First, to avoid deleterious damping of incoming waves (criterion 1), λ_1 must tend to unity from below. Some limiting forms of λ_1 will suffice to illustrate the shortcomings of the scheme. For an incoming wave of good temporal resolution ($\omega\Delta t \to 1$), then it can be shown that

$$\lambda_1 \approx [(1 - \omega^{*2})^{1/2} - \nu^*]/(1 - \nu^*) \qquad (12)$$

for either $\nu^* \ll 1$ or with $\nu^* \neq 1$ and $\omega^* = (\omega\Delta t/\alpha) \ll 1$. It follows that $\lambda_1 \approx [1 + (\tfrac{1}{2}\omega^{*2})/\nu^*]$ for $\omega^* \ll 1$, $\nu^* \gg 1$. Thus there is an undesirable computational effect of spatial growth for $\nu^* > 1$. Again

$$\lambda_1 \approx (1 - \omega^{*2})^{1/2} \quad \text{for} \quad \nu^* \ll 1.$$

For there to be less than, say, 5% damping at an inflow boundary zone $l = s\Delta x$, we require

$$(1 - \omega^{*2})^{s/2} \geq 0.95,$$

i.e., $\omega^* \leq (1 - 0.9^{1/s})^{1/2}$, and, since $\omega^* = 2\pi(l/\mathcal{L}) \times (1/s)$, this condition can be written as

$$2\pi(l/\mathcal{L}) \leq s(1 - 0.9^{1/s})^{1/2}. \qquad (13)$$

This finite-difference analogue of inequality (8) is far more stringent. With $s = 2, 3, 4, 5$ the rhs of (13) takes the values 0.44, 0.55, 0.64, 0.7.

Thus, in relation to criterion (1) we conclude from the inequality (13) that the diffusive boundary scheme B is effective only if the wavelength ($\mathcal{L}$) of the basic system is considerably longer than $6l$, where l is the width of the diffusion zone. If this inequality is not satisfied, then the incoming flow field will be degraded by the zone and can suffer a significant amplitude reduction and minor phase modification. This shortcoming will curtail the time-span for which the regional NWP model output is useful. In particular the transmission of comparatively smaller scale synoptic disturbances into a limited-area forecast model could be adversely affected.

We now turn to assess the behaviour of (9) in relation to the damping of the reflected wave (i.e. criterion 2) we consider the following system (portrayed in Fig. 1). Let $\nu \equiv 0$ for $j \leq 0$, and assume a wave of unit amplitude impinging on a boundary zone $j = 1, s$ with an imposed boundary condition $u_s^n \equiv 0$. The smallness of the amplitude of the reflected wave in the zone $j \leq 0$ will be a measure of the effectiveness of the diffusion zone. It can be shown that, for $s = 2, 3, 4$ and in the limit $\omega^* \ll 1$, this amplitude is

$$|r_{s=2}| \approx |(1 - \nu^*)/(1 + 3\nu^*)|,$$

$$|r_{s=3}| \approx |(1 - \nu^*)^2/(1 + 2\nu^* + 5\nu^{*2})|,$$

$$|r_{s=4}| \approx |(1 - \nu^*)^3/(1 + 5\nu^* + 3\nu^{*2} + 7\nu^{*3})|.$$

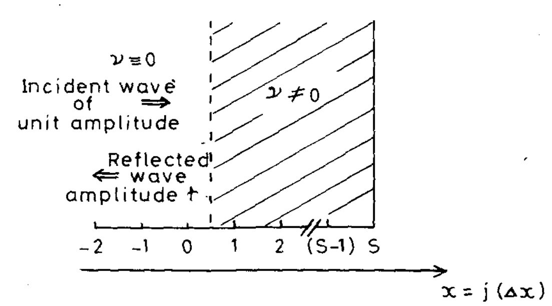

FIG. 1. Schematic depiction of the numerical flow system under consideration. A boundary zone is located in domain $j = 1, s$. In domain $j \leq 0$ an incident wave of unit amplitude impinges upon boundary zone from $-\infty$, and a reflected wave also exists.

The actual values are displayed in Fig. 2. These results, together with the requirement that $\nu^* < 1$ from (12), indicate that the commonly used finite-difference scheme (9) is an inappropriate boundary zone scheme. Other schemes must be judged on their own merit, but this example serves to illustrate the potential shortcomings of a diffusion boundary zone.

b. Tendency modification scheme

In this scheme the tendencies of the model prediction variables are modified in the marginal zone. The tendencies are assigned a weighted average of the externally specified fields and the internally determined fields such that the weighting associated with the external field varies from one at the boundary to zero at the inner extremity of the marginal zone (Kessel and Winninghoff, 1972; Perkey and Kreitzberg, 1976; Fritsch and Chappell, 1980; Maddox *et al.*, 1981). In addition to the tendency modification the variables in the marginal zone are also subjected to a scale-selective spatial filtering procedure.

With this scheme the system represented by (16) is replaced by the equation,

$$u_t + cu_x = -\gamma(u - \tilde{u})_t. \tag{14}$$

Here the $\tilde{u} = \tilde{u}(x, t)$ field is prescribed externally and, if consistent, is itself a solution of (1). It follows that the equation for the 'error,' $u' = u - \tilde{u}$, takes the form

$$u'_t + c^*(x)u'_x = 0, \tag{15}$$

where $c^* = c/(1 + \gamma)$, with γ varying from zero in the interior to infinity at the boundary. The error field is advected along at the modified speed c^* which reduces to zero at the boundary. Thus the problem of overspecification is again nominally overcome, but the 'error energy' (u'^2) accumulates in the boundary zone. An incident wave of a given frequency encountering a decrease in c^* across the boundary zone will undergo a concomitant decrease in its local wavelength. Hence the use of the spatial filter. However, the filter has to be applied to the u-field (not the u'-field), and thus it makes this scheme susceptible to the same shortcoming as that outlined earlier for the previous boundary diffusion-zone scheme.

A numerical shortcoming can be readily illustrated with a simple example. Let us adopt a leapfrog finite-difference representation of (15) (with the prime suffix omitted)

$$u_j^{n+1} = u_j^{n-1} - \alpha(u_{j+1}^n - u_{j-1}^n), \tag{16}$$

with, as before, $x = j(\Delta x)$, $t = n(\Delta t)$, $\alpha = c^*\Delta t/\Delta x$, and $j = \cdots -2, -1, 0, 1, 2, \ldots$. Assume that c^* has a step function discontinuity such that

$$\begin{aligned} \alpha &= \alpha_1 \quad \text{in} \quad L_1\ (j \leqslant -1), \\ &= \alpha_2\ (\alpha_2 < \alpha_1) \quad \text{in} \quad L_2\ (j \geqslant 0). \end{aligned}$$

Now consider the time-periodic situation of a wave of unit amplitude and of frequency ω impinging from $-\infty$ on the phase speed interface between L_1 and L_2. This interface corresponds to a change in the 'refractive index' of the computational system and it can act to trigger a reflected wave. Thus we assume the existence of a reflected computational wave of amplitude r in the semi-infinite domain L_1, and a transmitted wave of amplitude T in the other semi-infinite domain L_2.

From (16) we deduce that

$$\text{in } L_1\text{: } u_j = 1e^{-i(k_1x-\omega t)} + r(-1)^j e^{i(k_1x+\omega t)}, \tag{17}$$

$$\text{in } L_2\text{: } u_j = Te^{-i(k_2x-\omega t)}, \tag{18}$$

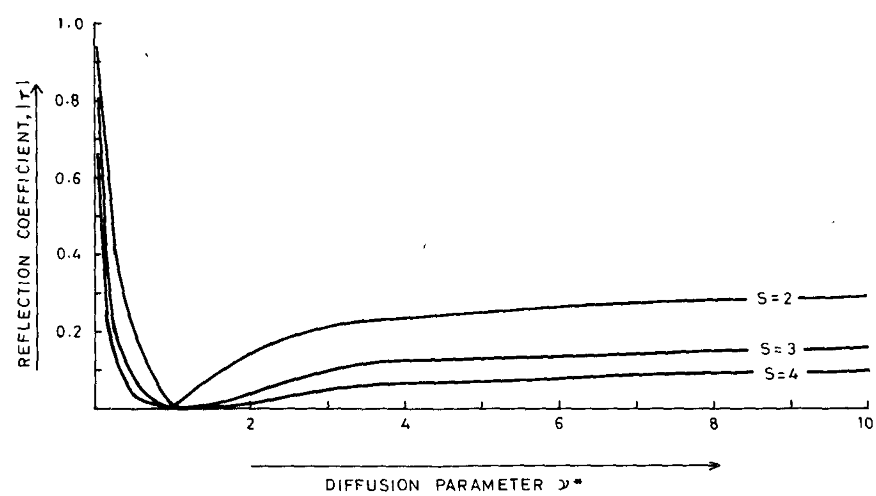

FIG. 2. A plot of the reflection $|r|$ as a function of ν^* $(=2\nu/c\Delta x)$ in the limit of ω^* $(=\omega\Delta x/c)$ $\ll 1$ for diffusion zones of different widths with boundaries located at $s = 2, 3, 4$.

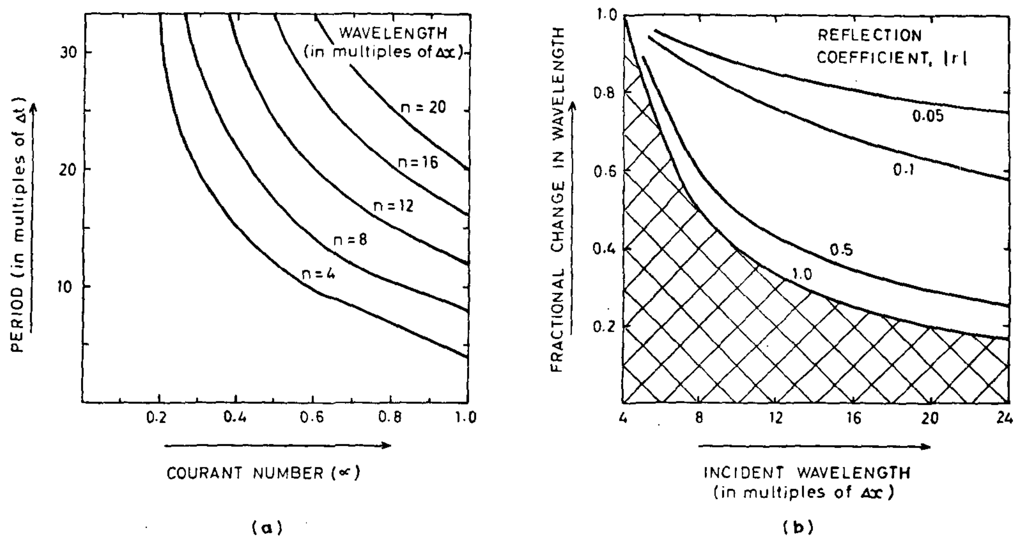

FIG. 3. The wavelength (in multiples of Δx) of the wave in L_1 or L_2 is displayed in (a) as a function of the Courant number (α) and the period of the wave. In (b) the amplitude of the reflected wave $|r|$ is shown as a function of the incident wavelength in domain L_1 and the fractional change in wavelength (η_2/η_1) on transmission to L_2. The cross-hatched area is the region of parameter space where there is total reflection of the incident wave.

where

$$k_s(\Delta x) = \theta_s = \sin^{-1}[\alpha_s^{-1} \sin\omega(\Delta t)], \quad s = 1, 2. \tag{19}$$

Continuity of the u field at the two points $j = -1$, 0 gives, using (17, 18), the compatibility relationships

$$\left.\begin{aligned} 1 + r &= T \\ e^{i\theta_1} - re^{-i\theta_1} &= Te^{i\theta_2} \end{aligned}\right\}.$$

From these relationships it follows that

$$r = + \frac{e^{i\theta_1} - e^{i\theta_2}}{e^{i\theta_2} + e^{-i\theta_1}},$$

and hence we deduce that

$$|r| = \frac{|\sin\theta_1 - \sin\theta_2|}{1 + \cos(\theta_1 + \theta_2)}. \tag{20}$$

The response of the system depends crucially upon the values of the two parameters $\delta_{s=1,2} = \alpha_s^{-1} \sin\omega(\Delta t)$. Two limiting cases reveal this dependency. First, at low frequencies ($\omega\Delta t \rightarrow 0$) and moderate Courant number ($\alpha_s \leqslant 1$), then $\delta_s \ll 1$ and the limiting form of (20) becomes

$$|r| = \frac{1}{2}\,\omega(\Delta t)\,\frac{\alpha_1 - \alpha_2}{\alpha_1\alpha_2}.$$

Second, when $\delta_2 > 1$ there is total reflection. This reflection arises because in this circumstance a transmitted propagating physical mode cannot occur in L_2.

The reflective properties of the interface for the full range of δ_s can be inferred from Figs. (3a, b). For a given change in α between the two domains the change in wavelength of the incident wave on transmission from L_1 to L_2 can be deduced from Fig. 3a. With a knowledge of this fractional change in wavelength (η_2/η_1) the amplitude of the reflected wave can then be estimated by inspection of Fig. 3b.

This example demonstrates that a boundary zone of tendency modification can induce appreciable numerical reflection. In practice only a gradual spatial variation of the weighting factors have been imposed across the marginal zone, but this places constraints upon the width of such a zone. A rationale for this gradual variation will be examined in a slightly different context in the next sub-section. It is worth noting that the saw-toothed, $(-1)^j$, amplitude variation of the reflected wave [see (17)] makes this particular wave readily amenable to spatial filtering. However, note also that the direct application of this scheme to the gravity wave system (1a) would complicate this convenient circumstance since then a realistic reflected gravity wave of the same wavelength as the incident wave could also be triggered by the change in the refractive index of the medium.

c. *Flow relaxation scheme*

In this scheme the prognostic variables are subjected to a forcing in the marginal zone that constrains them to relax towards the externally specified field on a time scale that again varies with distance from the lateral boundary (Davies, 1976; Kallberg and Gibson, 1977a,b; Lepas *et al.*, 1977; Gauntlett *et al.*, 1978; Ninomiya and Tatsumi, 1980; Leslie *et al.*, 1981).

For the linear advection case (1b) the original modified equation for this scheme is

$$u_t + cu_x = -K(u - \tilde{u}). \tag{21a}$$

However, Tatsumi (1980), following a suggestion of Hovermale, extended the scheme to include a diffusion relaxation term, i.e.

$$u_t + cu_x = -K(u - \tilde{u}) + [\nu(u - \tilde{u})_x]_x. \tag{21b}$$

We first consider (21a). Now it is the relaxation coefficient, $K = K(x)$, that is non-zero only in the boundary zones, and $\tilde{u}$ is again the externally specified field. In this case the error equation takes the form

$$u'_t + cu'_x = -Ku'. \tag{22}$$

Thus as the error field u' is advected into the boundary zone its amplitude is reduced due to the relaxation damping at a net rate determined by the values of c and K. Note also that at inflow only the departures of the field away from the specified values are subject to the relaxation effect. Hence again in this scheme the effect of overspecification is mitigated but now without inducing a deleterious effect in the inflow zone.

However, this scheme is also not immune to numerical shortcomings. Consider the following finite-difference representations of (22),

$$u_j^{n+1} = u_j^{n-1} - \alpha(u_{j+1}^n - u_{j-1}^n) - K(2\cdot\Delta t)F, \tag{23}$$

with F assigned one of the following forms (i) u_j^{n-1}, (ii) $\tfrac{1}{2}(u_j^{n+1} + u_j^{n-1})$, or (iii) u_j^{n+1}. For the comparatively large values of $(K\cdot 2\Delta t)$ that may be required in NWP models the scheme (i) experiences a well-established instability problem, whilst the damping properties of scheme (ii) can become inadequate (Davies and Turner, 1978). The implicit scheme (iii) is stable and produces suitable damping, but it is salutary to note that the leading order truncation term, i.e. $(K\cdot 2\Delta t)u_t$, is reminiscent of the effect of a phase speed change.

To examine the possible numerical shortcomings of scheme (iii) we consider a similar problem to that studied in Section 2b, viz. the propagation of a wave from $-\infty$ toward a step change in K from zero to a constant value at a location between grid points $j = (0, 1)$. In effect this problem isolates one feature that is basic to the relaxation scheme at the interface between the interior and the marginal zone.

We now have that,

$$\text{in } L_1\text{: } u_j = 1e^{-i(k_1x-\omega t)} + r(-1)^j e^{i(k_1x+\omega t)}, \tag{24}$$

and

$$\text{in } L_2\text{: } u_j = Te^{-\mu x}e^{-i(k_2x-\omega t)}. \tag{25}$$

The wavenumber k_1 is defined as before, but now $e^{-\mu\Delta x}$ and $e^{ik_2\Delta x}$ are derived from the complex roots of the following quadratic equation,

$$\phi^2 + 2(a + ib)\phi - 1 = 0, \tag{26}$$

with $a = K^* \cos(\omega\Delta t)$, $b = (K^* + 1/\alpha)\sin(\omega\Delta t)$, and $K^* = K(\Delta t)/\alpha = K(\Delta x)/c$.

Continuity of the u-field at the two points $j = (-1, 0)$ leads to the relationship

$$r = \frac{(\Gamma e^{i\theta_2} - e^{i\theta_1})}{(\Gamma e^{i\theta_2} + e^{-i\theta_1})},$$

and hence

$$|r| = \frac{[(1 - \Gamma^2)^2 + 4(1 - \Gamma^2)\gamma\sin\theta_1 + 4\gamma^2]^{1/2}}{[1 + \Gamma^2 + 2\Gamma\cos(\theta_1 + \theta_2)]}, \tag{27}$$

where $\Gamma = e^{\mu\Delta x}$ and $\gamma = (\Gamma\sin\theta_2 - \sin\theta_1)$. In the limiting case of waves whose temporal and spatial dependency in L is well represented by the finite-difference representation (i.e., $\omega\Delta t \to 0$, $k_1\Delta x \to 0$) the reflection coefficient reduces to

$$|r| = \frac{1}{2}K^* \frac{|\Lambda|}{(1 + K^*\Lambda)^2},$$

where $\Lambda = K^* - (1 + K^{*2})^{1/2}$. Thus $|r| \to \tfrac{1}{2}K^*$ as $K^* \to 0$ and $|r| \to 1$ as $K^* \to \infty$. On stipulating a maximum amplitude $r_{\max}$ ($\ll 1$) for the reflected wave, then we must require that

$$K^* \leq 2r_{\max}. \tag{28}$$

A similar result is obtained in the same limit for the other two representations of the relaxation term F of (23). Thus in this case again the boundary-zone interface can induce a significant numerical reflection of impinging waves. This result indicates that to effectively damp an outgoing wave utilizing a boundary zone with a small constant value for the relaxation coefficient K, the zone must be excessively wide.

A possible alternative is to allow the relaxation coefficient K to vary gradually across the marginal zone. The spatial variation of K to be such as to allow the incoming wave to penetrate the zone without appreciable loss of amplitude and then to be damped. Too sharp a variation in K will induce reflection, whereas insufficiently high values of K will not damp the outgoing waves adequately and in effect produce a reflection off the boundary itself.

To illustrate these effects and to outline a strategy for specifying the appropriate form for the variation of K we consider the following system: the finite-difference system (23 iii) with $K \equiv 0$ for $j \leq 0$ and $K = K(j)$ within the boundary zone $j = 1, s - 1$. The boundary condition $u_s^n \equiv 0$ is applied at $j = s$. As before we examine the response to a wave of unit amplitude and given frequency impinging from $-\infty$ upon the relaxation boundary zone.

The reflection coefficient r for waves that are temporally well resolved in $j \leq 0$ is given to a good approximation by the relation,

$$r = \frac{1 - \mu}{1 + \mu}, \tag{29}$$

with

$$\mu = 2K_1^* + (2K_2^* + (2K_3^* + \cdots + (2K_{s-1}^*)^{-1} \cdots)^{-1})^{-1}, \quad (30)$$

where K_j^* refers to the value of K^* at the point j of the boundary zone. It is preferable that the boundary zone be as narrow as possible and thus we restrict our attention to boundary zones with $s \leqslant 5$. Now $K^* = K(\Delta x)/c$ with $c = \bar{U} \pm (gH)^{1/2}$, and from (1b) we note that H assumes a different value for the different vertical eigensolutions. Therefore, it is necessary to examine the dependency of the reflection coefficient upon K^*. A lower bound, $K_{\min}^*$, is set by the mode with the largest phase velocity, i.e. $c_{\max} = |U + (gH)^{1/2}|_{\max}$. Also a finite upper bound for the K^* values of interest must also exist in practice since very slowly moving waves will not penetrate sufficiently into the boundary zone during the period of integration to produce reflection. An estimate of the cut-off velocity $c_{\min}$ is that $c_{\min} \geqslant 2s(\Delta x)/\Delta t$. Thus there is a finite band of K^* values that are of interest with $(K_{\max}^*/K_{\min}^*) \approx (c_{\max}/c_{\min})$.

In Fig. 4 the reflection coefficient $|r|$ is shown as a function of K^* for boundary zones of constant K values and of varying widths, i.e. $s = 2, 3, 4, 5$. The effects of insufficient damping on the one hand and computational reflection on the other are evident in the respective limits of $K^* \ll 1$ and $K^* \gg 1$. A feature of particular interest is the occurrence of zero reflection at certain discrete K^* values.

The effect of allowing a spatial variation of K^* for $s = 3$ is shown in Fig. 5, where r is displayed as a function of $\tilde{K}^*$ $(=K_2^*)$ for a range of values of $\alpha = K_1^*/K_2^*$. In the range $0 < \alpha < 1/4$ there are now two discrete values of K^* that induce no reflection. This result and the form of the relations (29) and (30) show that the added degree(s) of freedom gained by permitting a spatial variation in K^* allows up to $(s - 1)$ points of zero reflection for a boundary zone terminating at $j = s$.

It is this feature that forms the basis for our strategy to specify an appropriate form for the spatial variation of K, viz. to suitably locate the points of zero reflection within the K^* band of interest so as to obtain comparatively small values of reflection across the entire width of the band. To illustrate this procedure we assume that $c_{\max}$ and $c_{\min}$ are such that $K_{\min}^* = K_{\max}^*/100$, and then on the basis of the results displayed in Figs. 4 and 5 we choose $K_{\min}^* \sim 1/2$. With $s = 4$ we relate the points of zero reflection $K^* = (a, \mu a, \gamma a)$ by the relations $\mu = 6.5$ and $\gamma = 65$. Some algebraic manipulation shows the zero points to be located at (0.577, 3.75, 37.5), and thus $K_1^* = 0.0119K_3^*$ and $K_2^* = 0.128K_3^*$. The comparable values for the $s = 5$ case for zero points $(a, \mu a, \gamma a, \epsilon a)$ with $\mu = 4$, $\gamma = 16$ and $\epsilon = 65$ gives the zero point locations (0.633, 2.53, 10.1, 41.1) with $K_1^* = 0.00919K_4^*$, $K_2^* = 0.0491K_4^*$, $K_3^* = 0.208K_4^*$. The resulting reflection curves are given in Fig. 6. [Note the change in scale of the ordinate.] There is a marked improvement in extending the boundary zone from $s = 4$ to $s = 5$. In the latter case the maximum reflection for the band $(0.47 < K^* < 47)$ is less than 8% and the reflection coefficient is less than 0.05 over a substantial fraction of that K^* band.

This example illustrates both the potential and the limitations of the boundary relaxation scheme. However it should be recalled that this analysis applies only to one particular finite-difference scheme and is subject to the limit $\omega^* \ll 1$. The adoption of a semi-implicit scheme or the addition of a tendency modification term would have the beneficial effect of decreasing the width of the K^* band, and hence modify the form for the preferred spatial variation of K. Israeli and Orszag (1981) indicated some advantages

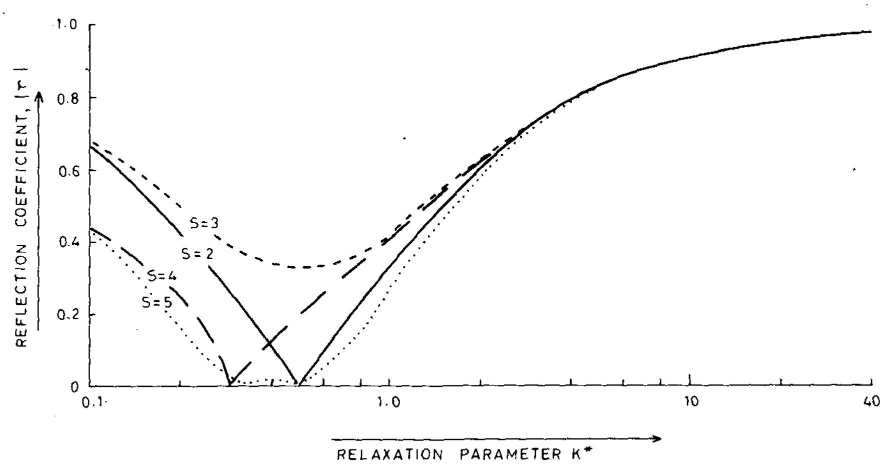

FIG. 4. Reflection $|r|$ shown (for $\omega^* \ll 1$) as a function of $K^* = K(\Delta x)/c$ for relaxation zones of constant K^* values but of different widths. Plots shown for boundaries located at $s = 2, 3, 4, 5$ (i.e. K^* non-zero, respectively, at 1, 2, 3, 4 points in the interior).

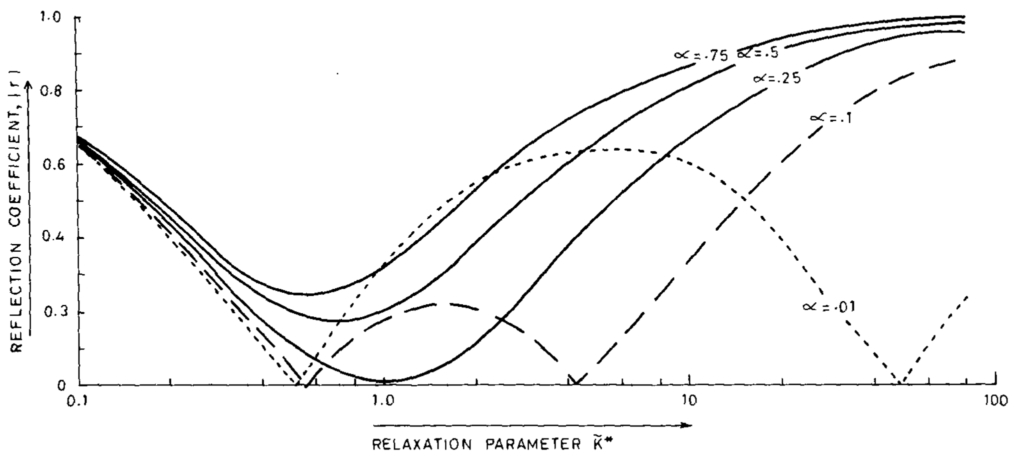

FIG. 5. Reflection $|r|$ shown (for $\omega^* \ll 1$) as a function of $\hat{K}^*$ ($=K_2^*$) for a relaxation zone with boundary located at $s = 3$ for a range of values of $\alpha = K_1^*/K_2^*$. Here K_j^* refers to the value of K^* at the grid location j.

of combining a damping and tendency modification boundary formulation for a system of differential equations, but it must be remembered that there are inherent computational effects associated with their introduction in a difference scheme. Again Tatsumi (1980) showed that the scheme of (21b) is helpful. In this case the additional diffusive relaxation term does not damp the incoming meteorological information, but as indicated in Section 2a care must be exercised in the choice of a finite-difference form for this additional term.

d. The pseudo-radiation boundary scheme

This scheme differs from those already considered in that there is no direct modification of the prognostic variables in the marginal zones, but only a direct specification or calculation of the variables at the lateral boundary itself. For the system (1a, b) their characteristic form shows that the dynamical quantities $u_{1,2}$ should be specified only at boundaries of inflow, i.e. where $c_{1,2}$ is directed into the domain. Thus the correct application of this criterion requires the prior decomposition of the original variables into their vertical eigenfunctions. In practice an alternative approach has usually been adopted based on the *generally unverifiable* assumption that all prognostic variables of the set of primitive equations satisfy individually a relationship similar to (1b) at the boundary. If at a given instant a boundary pseudo-advection velocity, say $c^*_{x=L}$, is directed into the domain then the associated prognostic boundary variable is specified externally, whereas the same variable is evaluated internally using an equation of the form of (1b) if the flow velocity $c^*_{x=L}$ is out of the limited domain. The required pseudo-advection velocity is either specified externally by invoking some *a priori* knowledge of the flow system, or evaluated by sampling the system in the vicinity of the boundary. In the latter case some finite-difference approximation of the relationship

$$c^*_{x=L} = -(u_t/u_x)_{x=L-\Delta x} \tag{31}$$

is usually used. Variants of this scheme for hydrostatic and non-hydrostatic systems have been employed by Williamson and Browning (1974), Pearson (1974), Orlanski (1976), Miyakoda and Rosati (1977), Klemp and Lilly (1978), Klemp and Wilhelmson (1978), Clark (1979), Camerlengo and O'Brien (1980), Miller and Thorpe (1981), Ross and Orlanski (1982).

A detailed study of the reflection and transmission properties of this approach for the system (1a) has already been given by Klemp and Lilly (1978). Overspecification at the boundary, specification of the correct number of variables but with inaccurate val-

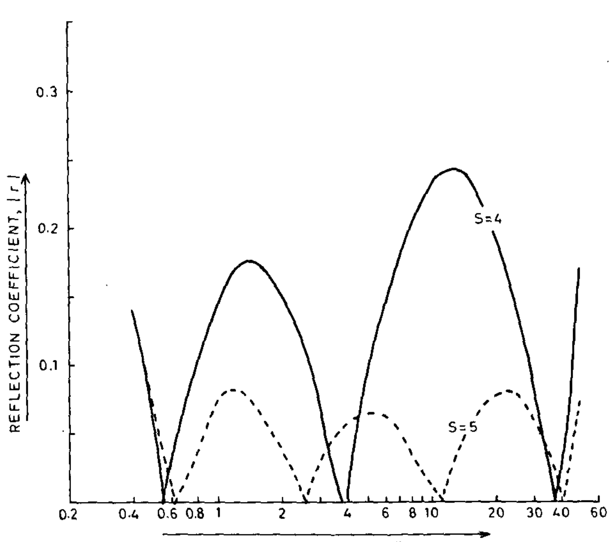

FIG. 6. A plot of the reflection $|r|$ for $\omega^* \ll 1$ as a function of K_s^* for relaxation zones with boundaries at $s = 4$ and 5, and with 'tuned' specified spatial variation for $K^*_{j=1,s-1}$ (see text for details).

ues, and errors in the estimate of $c^*_{x=L}$, all contribute to partial reflection of outward propagating waves. The first two sources of error arise if a vertical decomposition of the fields is not undertaken. The schemes of Pearson (1974), Klemp and Lilly (1978), Klemp and Wilhelmson (1978) seek to use *a priori* knowledge of the system to investigate this effect. Better estimates of $c^*_{x=L}$ can be derived with higher order finite-difference schemes. Miller and Thorpe (1981) present a hierarchy of such schemes.

3. Application of schemes to more complicated flow systems

In this section we consider briefly some aspects of the relaxation scheme and the pseudo-radiation scheme when the basic system (1a, b)—that represents perturbations in the (x, σ) plane of a uniform flow of a stratified non-rotating atmosphere—is complicated by the inclusion of shear of the basic flow, rotation, and three dimensional, diabatic and non-linear effects.

a. Consideration of flow relaxation scheme

It has been shown (Davies, 1976) that the relaxation scheme adequately handles shear in the basic flow field. On the inclusion of rotation it is appropriate to extend the scheme to include the relaxation of the potential vorticity in the boundary zone (Davies, 1976). However most model formulations using the relaxation scheme operate with apparent success without this additional term. The extension of the flow system to include other effects does not introduce additional conceptual problems. Tatsumi (1980) shows that the addition of a ∇_h^2 diffusion relaxation term is helpful. It may act to compensate for the absence of a potential vorticity relaxation term.

For the non-linear barotropic vorticity equation Bennett and Kloeden (1978) pointed out that solutions of the continuous differential equations appropriate to limited area models can generate spurious discontinuities in the fields of the prognostic variables at boundary locations where the flow field is tangential to the boundary. In practice sharp gradients in the finite-difference values of the fields would occur at such points. The marginal zones of the relaxation boundary scheme would provide a region for the internal flow field to adjust to the possible incompatibility between the externally specified fields and internally evaluated fields and hence render the pragmatic boundary-zone schemes less sensitive to this particular problem.

The foregoing consideration suggests that the possible ill-posedness of limited area models using the relaxation boundary schemes may not be a crucial issue. This inference is supported by the comparatively successful routine use of such models as numerical weather prediction tools.

b. Consideration of pseudo-radiation scheme

A method for extending the pseudo-radiation boundary scheme to more complicated systems is not readily apparent. The rationale for the approach adopted derives from a dictum attributable to Sommerfeld. He noted that if wave-energy is radiating away from localized sources then in the far-field the specification of the phase relationship between the flow variables should ensure an outward directed flux of energy (see for instance Sommerfeld, 1949, pp. 188). This requirement has long been recognized in the problem of the upper boundary condition of meteorological wave and tidal problems (Wilkes, 1949; Eliassen and Palm, 1961).

The rigorous application of this requirement to only slightly more complex lateral boundary flow problems (Bennett, 1976; Beland and Warn, 1975) indicates that this pristine approach demands an inordinate computer storage of boundary data. Engquist and Majda (1977) proposed an attractive method for obtaining *approximate* 'local' boundary conditions that circumvents this storage requirement. For a two-dimensional wave equation their method generates a sequence of higher order boundary formulations. The pseudo-radiation scheme (31) is the lowest order accurate scheme in this sequence. In this case the reflection is proportional to the angle of departure of the wave propagation away from normal incidence.

It is of interest to note that the inclusion of vertical shear in the simple flow system considered in the previous section can mitigate the problem emphasized by Oliger and Sundström (1978) of overspecification. This follows from the following considerations. For such a system the natural modes of the system are split into two types. One half possess a horizontal phase (and group) velocity greater than the maximum basic flow velocity, and for the other half the corresponding velocities will be less than the minimum velocity of the basic flow. This result, coupled with the expectation of low basic wind speeds near the ground, indicates that a good estimate can be made of the nature and number of variables that need to be specified at the lateral boundaries of such a simple system.

In line with Sommerfeld's reasoning the validity, and hence the usefulness, of a 'radiation-type' boundary scheme must be in doubt if there are energy sources (e.g., diabatic heating, frictional effects) or phase perturbing effects (e.g., orography) in the vicinity of the boundary. Lilly (1981) has shown that the pseudo-radiation scheme can nevertheless handle non-propagating, temporally growing perturbations of a simple flow system. Here we indicate the kind of error that can ensue when using the pseudo-radiation scheme in an inappropriate flow situation.

Consider the following slight generalization of (1b) to include a constant forcing term $(-F)$, i.e.

$$u_t + cu_x = -F. \quad (32)$$

On rewriting (25) in the compact form,

$$(u + Ft)_t + c(u + Ft)_x = 0, \quad (33)$$

we deduce that an appropriate formalism at the boundaries is to specify $(u + Ft)$ at $x = 0$ and to evaluate $(u + Ft)$ at $x = L$. However, we shall now show that this is not necessarily the outcome obtained when using the pseudo-radiation boundary scheme.

The general solution of (33) is of the form,

$$(u + Ft) = f(X) \quad \text{where} \quad X = x - ct.$$

It follows that at the boundary the value of $c^*_{x=L}$ of (31) is given by

$$c^*_{x=L} = c + (F/f_X)_{x=L}.$$

In the usual method of implementing the pseudo-radiation scheme, the value of u at the $x = L$ boundary will be either specified or evaluated depending upon whether $c \gtrless (F/f_X)$. Overspecification will result if the lower inequality prevails, and with a numerical scheme such as (11) this will result in at least transitory reflection. Again numerical problems can ensue if the upper inequality prevails with $c^*_{x=L}$ such that $(c^*\Delta t/\Delta x) > 1$. This computational problem is clearly divorced from the true physical problem and hence is another potential shortcoming of this approach.

4. Further remarks

In this study we have sought to draw attention to the underlying problems associated with various pragmatic lateral boundary schemes. These problems and their effects need to be borne in mind in the implementation and further refinement of the schemes and in the interpretation of the resulting model predictions.

APPENDIX

Equations for Wave Perturbations in a σ Coordinate System.

Consider small, hydrostatic, perturbations of an inviscid non-rotating atmosphere about an isothermal basic state in uniform flow $\bar{U}$. The linear equations governing the behaviour of the system in an (x, σ) coordinate formulation are

$$u''_t + \bar{U}u''_x = -(\phi'' + \bar{\mu}\pi'')_x,$$

$$\phi''_\sigma = -K\pi''/\bar{p} - (\bar{\pi}/\bar{p})(\theta''/\bar{\theta}),$$

$$\pi''_t + \bar{U}\pi''_x + \pi\dot{\sigma}''_\sigma = 0,$$

$$\theta''_t + \bar{U}\theta''_x + \dot{\sigma}''(\theta_\sigma) = 0,$$

where the overbar and double prime denote respectively basic state and perturbation variables, and $\bar{\mu} = R\bar{T}/\bar{\pi}$. The other notation is conventional for a 'σ'-system. The last three equations can be combined to the form,

$$\left(\frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial x}\right)[\sigma^2\phi''_\sigma] - KR\bar{T}\frac{\partial u}{\partial x} = 0.$$

A separation of the horizontal and vertical dependency can be achieved on writing

$$u'' = u'(x, y, t)R(\sigma),$$

$$(\phi'' + \bar{\mu}\pi'') = gh'(x, y, t)R(\sigma).$$

The vertical structure equation then takes the form

$$[\sigma^2 R_\sigma]_\sigma = -\lambda^2 R,$$

and the horizontal structure is given by the shallow water system,

$$u'_t + \bar{U}u'_x = -gh_x,$$

$$h'_t + \bar{U}h'_x = -Hu_x.$$

Here H is the separation constant, $KR\bar{T}/(g\lambda^2)$, and takes on different values for the different vertical eigensolutions.

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