

# Numerical Techniques 2024–2025

## 3. Space differencing

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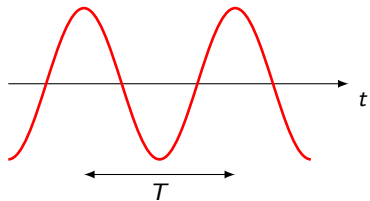
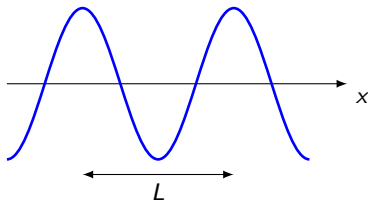
- Discretization and finite differences
- Consistency, convergence and stability
- Effects of time discretization on oscillation equation:
  - ▶ Damping or amplification
  - ▶ Acceleration or deceleration
  - ▶ Implicitness of 2TL schemes
  - ▶ Computational mode of 3TL schemes and filtering

The following function is oscillating in space and time:

$$\psi(x, t) = e^{i(kx - \omega t)}$$

where

- $\omega$  is the *frequency*:  $\omega = \frac{2\pi}{T}$  with  $T$  the period;
- $k$  is the *wavenumber*:  $k = \frac{2\pi}{L}$  with  $L$  the wavelength.



- If  $\omega = c k$ , then  $\psi(x, t)$  is a solution to the advection equation

$$\frac{\partial \psi}{\partial t} + c \frac{\partial \psi}{\partial x} = 0$$

- So there is a relation between the frequency  $\omega$  (or period  $T$ ), the wavenumber  $k$  (or wavelength  $L$ ) and the propagation speed  $c$ .
- We call this relation between  $\omega$  and  $k$  the *dispersion relation*.
- For the advection equation it is simply

$$\omega = ck$$

expressing that all waves propagate at the same speed.

- We define:

the phase speed  $\frac{\omega}{k}$       the group speed  $\frac{\partial \omega}{\partial k}$

- For the particular case of the 1D advection equation,  
group speed = phase speed =  $c$ .

- we will now determine the dispersion relation when discretizing  $x$  (not  $t$ !):  
 $\omega_d = f(k)$ .
- this allows to focus on effect of spatial discretization (cfr. Fourier decomposition to focus on temporal discretization)
- the comparison of the discrete dispersion relation with the exact dispersion relation will tell us
  - ▶ amplification: determined by imaginary part of  $\omega_d$
  - ▶ acceleration: determined by real part of  $\omega_d$

since

$$e^{i(kj\Delta x - \omega_d t)} = e^{\omega_i t} e^{i(kj\Delta x - \omega_r t)}$$

## Dispersion relation with the centered scheme

- Let's consider the centered finite difference scheme,

$$\frac{d\phi_j}{dt} + c \left( \frac{\phi_{j+1} - \phi_{j-1}}{2\Delta x} \right) = 0$$

- Suppose the solution is of the form,

$$\phi_j(t) = e^{i(kj\Delta x - \omega_{2c}t)}$$

- This will be the case if

$$-i\omega_{2c}\phi_j = -c \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x} \phi_j$$

$$\text{or } (e^{i\theta} = \cos \theta + i \sin \theta),$$

$$\omega_{2c} = c \frac{\sin k\Delta x}{\Delta x}$$

- $\omega_{2c}$  is real so there is no amplitude error.
- The phase speed is

$$\frac{\omega_{2c}}{k} = c \frac{\sin k\Delta x}{k\Delta x} \approx c \left[ 1 - \frac{1}{6}(k\Delta x)^2 \right]$$

This is a function of  $k$ , so the waves are *dispersive*.

- ▶ This scheme is second-order in  $k\Delta x$ .
- ▶ The phase speed is zero for  $k\Delta x = \pi$ , i.e. for the “ $2\Delta x$ ”-wave.

- The group speed is

$$\frac{\partial \omega_{2c}}{\partial k} = c \cos k\Delta x$$

The group velocity of the  $2\Delta x$  waves is  $-c$ !



- What about a higher-order discretization of  $\partial\psi/\partial x$ :

$$\frac{d\phi_j}{dt} + c \left[ \frac{4}{3} \frac{\phi_{j+1} - \phi_{j-1}}{2\Delta x} - \frac{1}{3} \frac{\phi_{j+2} - \phi_{j-2}}{4\Delta x} \right] = 0$$

- Similar calculations give the following discrete dispersion relation:

$$\omega_{4c} = \frac{c}{\Delta x} \left( \frac{4}{3} \sin k\Delta x - \frac{1}{6} \sin 2k\Delta x \right)$$

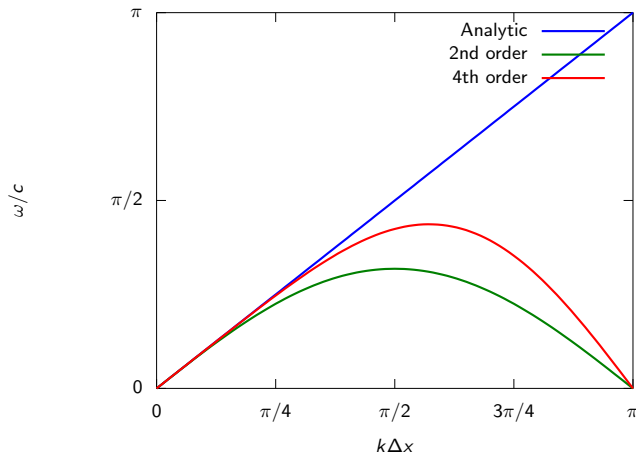
which is fourth-order accurate since the phase speed is

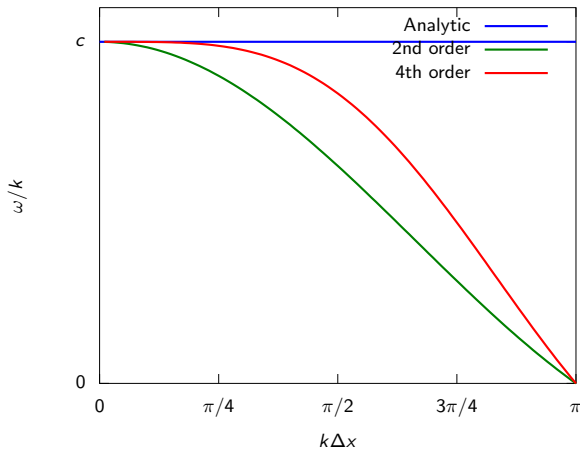
$$c = \frac{\omega_{4c}}{k} \approx c \left[ 1 - \frac{(k\Delta x)^4}{30} \right]$$

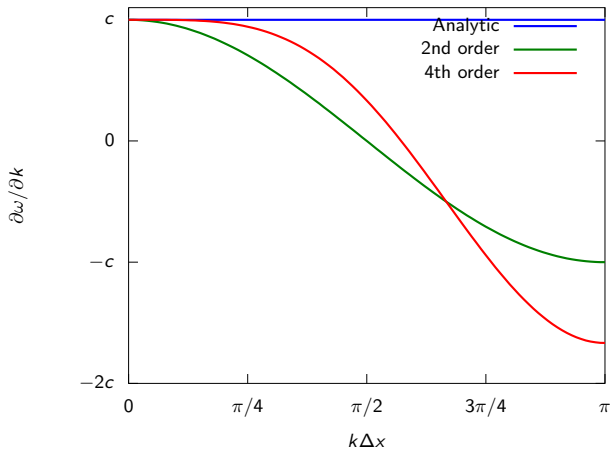
- The group velocity is given by

$$\frac{\partial \omega_{4c}}{\partial k} = c \left[ \frac{4}{3} \cos k \Delta x - \frac{1}{3} \cos 2k \Delta x \right]$$

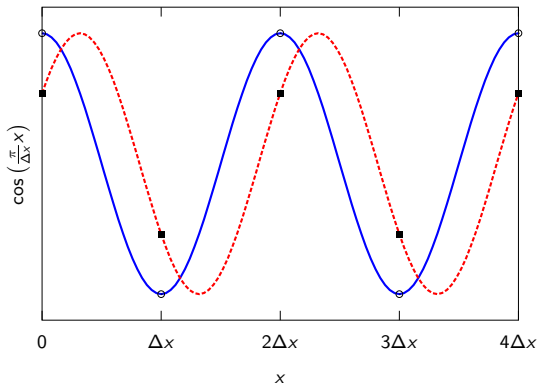
Remark: the  $2\Delta x$  wave has a group velocity of  $-\frac{5}{3}c$ , which is even worse than for the second-order scheme !



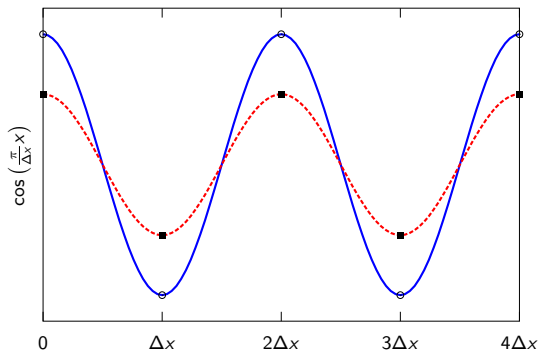




- There is a fundamental problem with representing the phase of the  $2\Delta x$  wave:

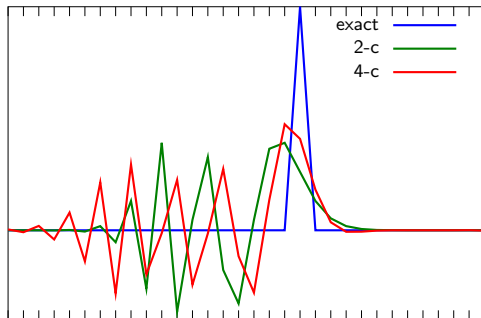


- There is a fundamental problem with representing the phase of the  $2\Delta x$  wave:



A phase shift looks like an amplitude change.<sup>x</sup>

- Advection of a sharp spike:





- Let us try the one-sided decentered derivatives

$$\frac{d\phi_j}{dt} + c \left( \frac{\phi_j - \phi_{j-1}}{\Delta x} \right) = 0$$

which is the decentered scheme that we have seen before (upstream scheme).

- The discrete dispersion relation becomes

$$\omega_{1s} = \frac{c}{i\Delta x} \left( 1 - e^{-ik\Delta x} \right) = \frac{c}{\Delta x} [\sin k\Delta x + i(\cos k\Delta x - 1)]$$

- Besides phase errors, there will be amplitude errors due to the imaginary part. The amplitude will grow or decay with a rate

$$\exp \left[ -\frac{c}{\Delta x} (1 - \cos k\Delta x) t \right]$$

In case  $c < 0$  this gives rise to an amplification, which is to be expected after what we saw in the first lesson.

- Another one-sided decentered scheme is

$$\frac{d\phi_j}{dt} + \frac{1}{6}c \frac{2\phi_{j+1} + 3\phi_j - 6\phi_{j-1} + \phi_{j-2}}{\Delta x} = 0$$

which is third-order.

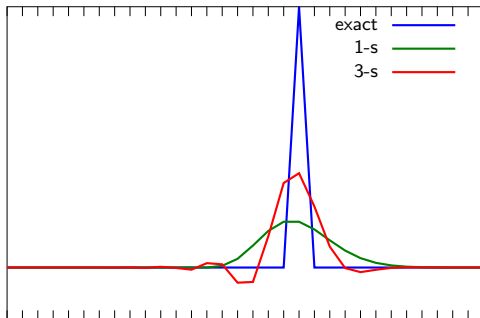
- The discrete dispersion relation becomes

$$\omega_{3s} = \frac{c}{\Delta x} \left[ \left( \frac{4}{3} \sin k\Delta x - \frac{1}{6} \sin 2k\Delta x \right) - \frac{i}{3} (1 - \cos k\Delta x)^2 \right]$$

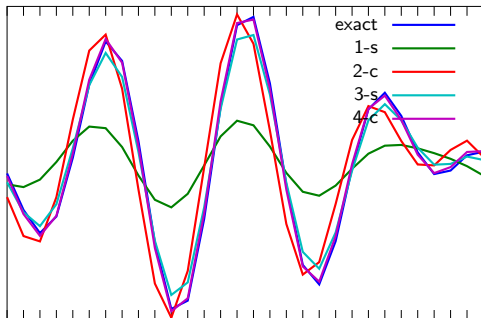
which has the same phase error as the fourth-order scheme, and an amplitude decay

$$\exp \left[ -\frac{c}{3\Delta x} (1 - \cos k\Delta x)^2 t \right]$$

- Advection of a sharp spike:



- Advection of a sum of two waves:



- The damping in the decentered schemes can be good or bad;
- In the spike test they damp the spurious trail which makes decentered schemes actually better than the centered ones.
- The third-order scheme somewhat better approximates the analytic solution than the first-order one
- However, the spike test is an extreme test! In the more realistic test with 2 modes it is not clear whether the third-order decentered scheme is better than the second-order centered one:
  - ▶ The amplitude is better for the second-order one
  - ▶ The third-order one does not make a phase error.

- A qualitative description of the error is obtained by examining the truncated terms; instead of solving the advection equation with zero RHS, we are in fact solving

$$\frac{\partial \phi}{\partial t} + c \frac{\partial \phi}{\partial x} = \epsilon_m \frac{\partial^m \phi}{\partial x^m} + \epsilon_{m+1} \frac{\partial^{m+1} \phi}{\partial x^{m+1}} + \dots$$

- an even order  $m$  will lead to a dissipative effect:

$$\frac{\partial \xi}{\partial t} = \frac{\partial^m \xi}{\partial x^m} \quad \Rightarrow \quad \xi(x, t) = C e^{ikx} e^{-\gamma t}$$

- an odd term will lead to a dispersive effect:

$$\frac{\partial \xi}{\partial t} = \frac{\partial^m \xi}{\partial x^m} \quad \Rightarrow \quad \xi(x, t) = C e^{i(kx - \omega t)}$$

- The centered derivatives have *no* numerical dissipation.  
(only odd terms in the truncation error)
- The first and second-order have the same leading-order dispersive terms, so they have the same dispersion! The same holds for the third- and fourth order schemes.
- In fact, the dispersive error in the odd-order schemes are dissipated away!
- So we are compensating one error by another! This is usually a bad idea.
- One way out for the even-order schemes is to introduce some (controlled) artificial dissipation.
- (We will see later that this is in fact also important in nonlinear systems).

- Until now we have been isolating the error coming from the *time differencing* and the *space differencing*
- The *fundamental* behaviour of the scheme can sometimes be deduced from the constituents.

For instance, a combination of a *forward difference scheme* with *centered derivatives* will be a combination of an amplifying time scheme with a neutral space scheme, yielding an amplifying scheme.

- But what will happen if we combine *forward differencing* with an *upstream one-sided space differencing*? Forward differencing will amplify, while the one-side space differencing may damp the mode. Further analysis is needed. . .
- What if we combine the leapfrog scheme with centered space differencing? Leapfrog is conditionally neutral; centered-space differencing is neutral. But leapfrog is accelerating, whereas centered differencing is decelerating?



- One can consider the solution to the time- and space-discretized equation of the form

$$\phi_j^n = e^{i(kj\Delta x - \omega n\Delta t)}$$

- Substituting this solution in the scheme and solving for  $\omega$  as a function of  $k$  gives the *discrete dispersion relation*.
- In fact, it summarizes the behaviour of a scheme completely: let  $\omega = \omega_r + i\omega_i$ , then
  - ▶  $\omega_i$  determines the amplification factor
  - ▶  $\omega_r$  determines the phase-speed error.

(leave the complex algebra to a computer or a mathematician. . . )

- We studied the influence of space discretization through the *dispersion relation*
- The dispersion relation tells how waves propagate; phase speed and group speed.
- All finite difference schemes have difficulties with the  $2\Delta x$  wave
- Decentered schemes also show damping, on top of dispersion
- Also combined time-space effects can be studied through the dispersion relation

- ① Determine the discrete dispersion relation for a 2nd-order centered leapfrog discretization of the advection equation:

$$\frac{\phi_j^{n+1} - \phi_j^{n-1}}{2\Delta t} + c \frac{\phi_{j+1}^n - \phi_{j-1}^n}{2\Delta x} = 0$$

You'll need  $\frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin \theta$ .

- ② Determine the discrete dispersion relation for the upstream scheme:

$$\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + c \frac{\phi_j^n - \phi_{j-1}^n}{\Delta x} = 0$$

You'll need  $\log(a + ib) = \frac{1}{2} \log(a^2 + b^2) + i \arctan(b, a)$ . This is pretty hard to calculate analytically, so use a computer to check the results...

Does this confirm what we saw in the first lesson?

$$\frac{\phi_j^{n+1} - \phi_j^{n-1}}{2\Delta t} + c \frac{\phi_{j+1}^n - \phi_{j-1}^n}{2\Delta x} = 0$$

$$\frac{\phi_j^{n+1} - \phi_j^{n-1}}{2\Delta t} + c \frac{\phi_{j+1}^n - \phi_{j-1}^n}{2\Delta x} = 0$$

Substituting a harmonic solution, discretized in space and time:  $\phi_j^n = e^{i(kj\Delta x - \omega n\Delta t)}$ , gives:

$$\frac{e^{i(kj\Delta x - \omega(n+1)\Delta t)} - e^{i(kj\Delta x - \omega(n-1)\Delta t)}}{2\Delta t} + c \frac{e^{i(k(j+1)\Delta x - \omega n\Delta t)} - e^{i(k(j-1)\Delta x - \omega n\Delta t)}}{2\Delta x} = 0$$

$$\frac{\phi_j^{n+1} - \phi_j^{n-1}}{2\Delta t} + c \frac{\phi_{j+1}^n - \phi_{j-1}^n}{2\Delta x} = 0$$

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Dividing both sides by  $e^{i(kj\Delta x - \omega n\Delta t)}$ , gives:

$$\frac{e^{-i\omega\Delta t} - e^{i\omega\Delta t}}{2\Delta t} + c \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x} = 0$$

$$\frac{\phi_j^{n+1} - \phi_j^{n-1}}{2\Delta t} + c \frac{\phi_{j+1}^n - \phi_{j-1}^n}{2\Delta x} = 0$$

Substituting a harmonic solution, discretized in space and time:  $\phi_j^n = e^{i(kj\Delta x - \omega n\Delta t)}$ , gives:

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$$\frac{e^{-i\omega\Delta t} - e^{i\omega\Delta t}}{2\Delta t} + c \frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x} = 0$$

Using  $e^{i\theta} - e^{-i\theta} = 2i \sin \theta$ , this is simplified to

$$\sin \omega\Delta t = \frac{c\Delta t}{\Delta x} \sin k\Delta x$$

The discrete dispersion relation is

$$\omega = \frac{1}{\Delta t} \arcsin \left( \frac{c\Delta t}{\Delta x} \sin k\Delta x \right)$$

So:

- for  $\left| \frac{c\Delta t}{\Delta x} \right| < 1$ ,  $\omega$  is real. The scheme is stable without amplitude error.
- for  $\left| \frac{c\Delta t}{\Delta x} \right| > 1$ ,  $\omega$  is complex. The scheme is unstable.
- for  $k\Delta x = \pi$ ,  $\omega = 0$ . So the scheme is decelerating.



$$\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + c \frac{\phi_j^n - \phi_{j-1}^n}{\Delta x} = 0$$

$$\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + c \frac{\phi_j^n - \phi_{j-1}^n}{\Delta x} = 0$$

Substituting a harmonic solution, discretized in space and time:  $\phi_j^n = e^{i(kj\Delta x - \omega n\Delta t)}$ , and dividing both sides by  $\phi_j^n$ , gives:

$$e^{-i\omega\Delta t} - 1 + \frac{c\Delta t}{\Delta x} (1 - e^{-ik\Delta x}) = 0$$

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The discrete dispersion relation becomes

$$\omega = \frac{i}{\Delta t} \log(1 - \mu(1 - \cos k\Delta x + i \sin k\Delta x))$$

with  $\mu$  the Courant number  $\mu = \frac{c\Delta t}{\Delta x}$ .

Stability is obtained if the imaginary part of  $\omega$  is smaller than zero, i.e. if

$$\log \left( (1 - \mu + \mu \cos k\Delta x)^2 + (\mu \sin k\Delta x)^2 \right) \leq 0$$

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Or

$$\mu(1 - \mu)(1 - \cos k\Delta x) \leq 0$$

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Or if

$$0 \leq \mu \leq 1$$