Multivariable Calculus

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1 Single Variable Calculus

Theorem 1.1 (Fundamental Theorem of Calculus)**.**

$$\int_a^b f'(x)\,dx = f(b) - f(a) \tag{1}$$

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \tag{2}$$

Definition 1.1 (Length of Curve)**.**

$$L(a,b) = \int_a^b \sqrt{1 + (f'(x))^2}\,dx \tag{3}$$

Definition 1.2 (Area of Surface of Revolution)**.**

$$A(a,b) = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx \tag{4}$$

1.1 Integration/Derivation Techniques

Definition 1.3 (Chain Rule)**.**

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) \tag{5}$$

Definition 1.4 (Product Rule)**.**

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x) \tag{6}$$

Definition 1.5 (Quotient Rule)**.**

$$\frac{d}{dx}(\frac{f(x)}{f(x)}) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)} \tag{7}$$

Definition 1.6 (Integration by Parts)**.**

$$\int u\,dv = uv - \int v\,du \tag{8}$$

1.2 Trigonometry

Definition 1.7 (Trigonometric Identities).

$$\begin{cases} \sin^2\theta + \cos^2\theta = 1 \\ \sec^2\theta = \tan^2\theta + 1 \\ \csc^2\theta = \cot^2\theta + 1 \end{cases} \tag{9}$$

$$\begin{cases} \sin(2\theta) & = 2\sin\theta\cos\theta \\ \cos(2\theta) & = \cos^2\theta - \sin^2\theta \\ & = 2\cos^2\theta - 1 \\ & = 1 - 2\sin^2\theta \\ \tan(2\theta) & = \dfrac{2\tan\theta}{1-\tan^2\theta} \end{cases} \tag{10}$$

$$\begin{cases} \sin^2\theta & = \frac{1}{2}(1-\cos(2\theta)) \\ \cos^2\theta & = \frac{1}{2}(1+\cos(2\theta)) \\ \tan^2\theta & = \dfrac{1-\cos(2\theta)}{1+\cos(2\theta)} \end{cases} \tag{11}$$

$$\begin{cases} \sin\alpha\sin\beta & = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta)) \\ \cos\alpha\cos\beta & = \frac{1}{2}(\cos(\alpha-\beta) + \cos(\alpha+\beta)) \\ \sin\alpha\cos\beta & = \frac{1}{2}(\sin(\alpha+\beta) + \sin(\alpha-\beta)) \end{cases} \tag{12}$$

$$\begin{cases} \sin(\alpha\pm\beta) & = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\ \cos(\alpha\pm\beta) & = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \end{cases} \tag{13}$$

Definition 1.8 (Trigonometric Integration/Derivation).

$$\begin{cases} \frac{d}{dx}\sin x & = \cos x \\ \frac{d}{dx}\cos x & = -\sin x \\ \frac{d}{dx}\tan x & = \sec^2 x \\ \frac{d}{dx}\csc x & = -\cot x\csc x \\ \frac{d}{dx}\sec x & = \tan x\sec x \\ \frac{d}{dx}\cot x & = -\csc^2 x \end{cases} \tag{14}$$

$$\int \tan x\,dx = \ln|\sec x| + c \tag{15}$$

1.3 Conic Sections

Definition 1.9 (Circle).

$$(x-h)^2 + (y-k)^2 = r^2 \tag{16}$$

Definition 1.10 (Ellipse).

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \tag{17}$$

Definition 1.11 (Parabola).

$$\begin{cases} (x-h)^2 = 4p(y-k)^2 \\ (y-k)^2 = 4p(x-h)^2 \end{cases} \tag{18}$$

Definition 1.12 (Hyperbola).

$$\begin{cases} \dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1 \\ \dfrac{(y-k)^2}{b^2} - \dfrac{(x-h)^2}{a^2} = 1 \end{cases} \tag{19}$$

1.4 Parametrized Curve

Let $x = f(t), y = g(t)$ for $\alpha \leq t \leq \beta$.

Definition 1.13 (Derivation of Parametrized Curve).

$$\frac{dy}{dx} = \frac{\frac{d}{dt}y}{\frac{d}{dt}x} = \frac{g'(t)}{f'(t)} \tag{20}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{d}{dt}x} = \frac{g''(t)f'(t) - g'(t)f''(t)}{(f'(t))^3} \tag{21}$$

Definition 1.14 (Area under Parametrized Curve). Define positive curve as pointing right, or counterclockwise. Then

$$A = \pm \int_\alpha^\beta y\, dx = \pm \int_\alpha^\beta g(t)f'(t)\, dt \tag{22}$$

Definition 1.15 (Length of Parametrized Curve).

$$L = \int_\alpha^\beta \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_\alpha^\beta ds \tag{23}$$

Definition 1.16 (Area of Surface of Revolution for Parametrized Curve).

$$A = \int_\alpha^\beta 2\pi y(t)\, ds \tag{24}$$

1.5 Polar Coordinates

$$r = f(\theta), \quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad \Longrightarrow \begin{cases} x^2 + y^2 = r^2 \\ \tan\theta = \frac{y}{x} \end{cases}$$

Definition 1.17 (Area in Polar Coordinates)**.**

$$A = \frac{1}{2}\int_{\alpha}^{\beta} f^2(\theta)\, d\theta \tag{25}$$

Definition 1.18 (Length in Polar Coordinates)**.**

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + (r')^2}\, d\theta \tag{26}$$

2 Multivariable Calculus

2.1 Dot Product

Definition 2.1 (Dot Product)**.**

$$\vec{v} \cdot \vec{u} = \sum v_i u_i = \|\vec{v}\|\|\vec{u}\| \cos\theta \tag{27}$$