

# Efficient Algorithms and Intractable Problems

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## 1 Complexity Analysis

### 1.1 Partial Sums

$$S_k = \sum_{n=1}^k a_n = \frac{k}{2}(a_1 + a_k) \quad (\text{Arithmetic Series})$$

$$S_k = \sum_{n=1}^k a_1(r)^n = a_1 \left( \frac{1 - r^k}{1 - r} \right) \quad (\text{Geometric Series})$$

### 1.2 Asymptotic Relations

$$f = O(g) \approx f(n) \leq c \cdot g(n)$$

$$f = o(g) \approx f(n) < c \cdot g(n)$$

$$f = \Omega(g) \approx f(n) \geq c \cdot g(n)$$

$$f = \omega(g) \approx f(n) > c \cdot g(n)$$

$$f = \Theta(g) \approx f(n) = c \cdot g(n)$$

*Remark.*  $O(i^n \mid i > 1) > O(n^j) > O(\log^k n)$

**Theorem 1.1** (Master Theorem). *If  $T(n) = aT([n/b]) + O(n^d)$  for some constants  $a > 0$ ,  $b > 1$ , and  $d \geq 0$ , then*

$$T(n) = \begin{cases} \Theta(n^d) & a < b^d \\ \Theta(n^d \log n) & a = b^d \\ \Theta(n^{\log_b a}) & a > b^d \end{cases}$$

## 2 Polynomial Interpolation

Given a degree  $n$  polynomial  $A(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ , the relationship between its values and coefficients can be represented by

$$\begin{bmatrix} A(x_0) \\ A(x_1) \\ \vdots \\ A(x_{n-1}) \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ & & \vdots & & \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} \quad (\text{evaluation})$$

where the matrix  $M$  is a *Vandermonde* matrix.

### 2.1 Fast Fourier Transform (FFT)

**Definition 2.1** (Discrete Fourier Transform Matrix). For polynomials of degree  $< n$  ( $n$  is even; polynomials can be 0-padded), the Discrete Fourier Transform can be represented by the matrix

$$M_n(\omega) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ & & \vdots & & \\ 1 & \omega^j & \omega^{2j} & \cdots & \omega^{(n-1)j} \\ & & \vdots & & \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{bmatrix}$$

where  $\omega = e^{2\pi i/n}$  is the  $n$ th root of unity, and  $M_n(\omega)$  is an unitary matrix whose columns forms the *Fourier Basis*.

*Remark.*  $M_n^{-1}(\omega) = \frac{1}{n} \overline{M_n(\omega)} = \frac{1}{n} M_n(\omega^{-1})$

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**Algorithm 1:** Fast Fourier transform

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**Input:** A coefficient vector,  $\vec{a} = \langle a_0, \dots, a_{n-1} \rangle$  and the  $n$ th root of unity,  $\omega$ .

**Output:**  $M_n(\omega)\vec{a}$

```

1 Function FFT( $\vec{a}, \omega$ ):
2   if  $\omega = 1$  then
3     return  $\vec{a}$ 
4   else
5     //  $A(x) = A_{\text{even}}(x^2) + xA_{\text{odd}}(x^2)$ 
6      $\langle A_e(0), \dots, A_e(n/2 - 1) \rangle \leftarrow \text{FFT}(\langle a_0, a_2, \dots, a_{n-2} \rangle, \omega^2)$ 
7      $\langle A_o(0), \dots, A_o(n/2 - 1) \rangle \leftarrow \text{FFT}(\langle a_1, a_3, \dots, a_{n-1} \rangle, \omega^2)$ 
8     for  $j := 0$  to  $n/2 - 1$  do
9        $A(j) \leftarrow A_e(j) + \omega^j A_o(j)$ 
10       $A(j + n/2) \leftarrow A_e(j) - \omega^j A_o(j)$ 
11   return  $\langle A(0), \dots, A(n-1) \rangle$ 

```

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*Remark.* If  $A$  is evaluated at points  $\pm\omega_0, \dots, \pm\omega_{n/2-1}$ , then  $A_e(x^2)$  and  $A_o(x^2)$  will only need to evaluate half the amount of points due to squaring.

$$T(n) = 2T(n/2) + O(n) = O(n \log n)$$

## 2.2 Applications of FFT

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**Algorithm 2:** Fast Polynomial Multiplication

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**Input:** Coefficient vectors,  $a$  and  $b$ , and the  $n$ th root of unity,  $\omega$ .

**Output:** The coefficient vector of  $A(x)B(x)$

```

1  $\hat{\vec{a}} \leftarrow M_n(\omega)\vec{a}$  (FFT)
2  $\hat{\vec{b}} \leftarrow M_n(\omega)\vec{b}$ 
3 for  $i = 0$  to  $n - 1$  do
4    $\hat{c}_i \leftarrow \hat{a}_i \hat{b}_i$ 
5 return  $\frac{1}{n} M_n(\omega^{-1}) \hat{\vec{c}}$  (inverse matrix)

```

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**Definition 2.2** (Cross-Correlation).  $\text{corr}(\vec{x}, \vec{y})[k] = \sum x_i y_{i-k}$ , which measures similarity.

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**Algorithm 3:** Cross-Correlation
 

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**Input:** Two signal vectors,  $\vec{x}$  and  $\vec{y}$ .

**Output:**  $\text{corr}(\vec{x}, \vec{y})$

- 1  $X(t) \leftarrow x_{m-1} + x_{m-2}t + \cdots + x_0t^{m-1}$
  - 2  $Y(t) \leftarrow y_0 + y_1t + \cdots + y_{n-1}t^{n-1}$
  - 3  $Q(t) \leftarrow X(t)Y(t)$     (Fast Polynomial Multiplication)
  - 4 **return**  $\vec{q}$
-

### 3 Graphs

**Definition 3.1** (Graph). A graph is a pair  $G = (V, E)$ , typically represented by an adjacency matrix or an adjacency list.

Table 1: Graph representations.

|                  | Space               | Connectivity               | getNeighbors( $u$ )        | DFS Runtime         |
|------------------|---------------------|----------------------------|----------------------------|---------------------|
| Adjacency Matrix | $\Theta( V ^2)$     | $O(1)$                     | $\Theta( V )$              | $\Theta( V ^2)$     |
| Adjacency List   | $\Theta( V  +  E )$ | $\Theta(\text{degree}(u))$ | $\Theta(\text{degree}(u))$ | $\Theta( V  +  E )$ |

#### 3.1 Depth-First Search

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**Algorithm 4:** Depth-first search

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**Input:**  $V, E$  of directed graph  $G$ .

1 **Function** DFS( $V, E$ ):

2      $n \leftarrow |V|$

3      $clk \leftarrow 1$

4      $visited \leftarrow \text{boolean}[n]$

5      $preorder, postorder = \text{int}[n]$

6     **for**  $v \in V$  **do**

7         **if**  $\neg visited[v]$  **then**

8             EXPLORE( $v$ )

9 **Function** EXPLORE( $v$ ):

10      $visited[v] \leftarrow \text{True}$

11      $preorder[v] \leftarrow clk++$

12     **for**  $(v, w) \in E$  **do**

13         **if**  $\neg visited[w]$  **then**

14             EXPLORE( $w$ )

15      $postorder[v] \leftarrow clk++$

/\* Preorder-postorder intervals are either nested or disjoint. \*/

/\*  $postorder[u] \leq postorder[v]$  iff  $(u, v)$  is a back edge. \*/

/\*  $G$  contains a cycle iff it contains a back edge. \*/

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### 3.1.1 Applications of DFS

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**Algorithm 5:** Topological sort.

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**Input:** A directed cyclic graph  $G$ .

**Output:** An ordered list of  $V$  such that  $u_i$  comes before  $v_i$  for all  $(u_i, v_i) \in E$  (i.e., ordered by decreasing dependency).

1  $post \leftarrow$  DFS-visited vertexes ordered by postorder visits

2 **return**  $reverse(post)$

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**Definition 3.2** (Strongly Connected Component). A SCC is a maximal partition of a directed graph in which every vertex is reachable from every other vertex.

$u$  is in sink SCC of graph  $G \Leftrightarrow u$  is in source SCC of reverse graph  $G$   
 $\Leftrightarrow u$  is in source SCC if highest postorder number.

### 3.2 Single-Source Shortest Path

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**Algorithm 6:** Single-Source Shortest Path

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**Input:** A directed graph  $G$  and a start vertex  $S$ .

**Output:** Two arrays  $prev[|V|]$  (shortest-path predecessor) and  $dist[|V|]$  (shortest-path distance).

1 **Function**  $BFS(G, S)$ :

$\lfloor$  /\* Must have uniform edge weights.  $O(|V| + |E|)$  runtime. \*/

2 **Function**  $Dijkstra(G, S)$ :

$\lfloor$  /\* Must have positive edge weights.  $O(|V| \log |V| + |E|)$  runtime if  
         implemented using Fibonacci heap. \*/

3 **Function**  $Bellman-Ford(G, S)$ :

$\lfloor$  /\* Can have arbitrary edge weights. \*/

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### 3.3 Minimum Spanning Tree

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**Algorithm 7:** Minimum spanning tree.

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**Input:** A graph  $G$  and a starting vertex  $v$ .

**Output:** The minimum spanning tree  $T$  of  $G$ .

*/\* Use the cut property. \*/*

1 **Function** Prim( $G, v$ ):

*/\* Sequentially adds the closest neighbor of the running set.  
      $O(|E| + \log |V|)$  runtime if implemented using Fibonacci heap. \*/*

2 **Function** Kruskal( $G, v$ ):

*/\* Sequentially adds the shortest edge that does not create a  
     cycle.  $O(|E| \log |V|)$  runtime if implemented using Union-Find  
     data structure. \*/*

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## 4 Greedy Algorithm

**Definition 4.1** (Greedy Algorithm). A greedy algorithm is one that builds the solution iteratively using a sequence of local choices.

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### Algorithm 8: Example greedy algorithms.

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**1 Function SCHEDULING:**

```

    /* Find the maximum set of jobs that can be completed within time
       by iteratively select the next job to have the smallest end
       time without conflicting existing schedule.  $O(n \log n)$  runtime
       if sorting the collection of jobs first. */

```

**2 Function HUFFMAN:**

```

    /* Find a prefix tree for prefix-free Huffman coding by
       iteratively combine the two least frequent elements of the
       alphabet and retrieve the order of the prefix tree accordingly.
        $O(n \log n)$  runtime if implemented with min-heap. */

```

**Input:** A set of partitions  $S = \{S_1, \dots, S_m\}$  that covers the universe  $\{1, \dots, n\}$ .

**Output:** The indices of the smallest sub-collection of  $S$  that covers the universe.

**3 Function SET-COVER:**

```

    /* Greedy search yields sub-optimal but competitive solution to
       the set-cover problem. If the optimal solution uses  $k$  sets,
       then the greedy solution uses at most  $k \ln n$  sets. */
4    $A \leftarrow \{1, \dots, n\}$ 
5    $B \leftarrow \emptyset$ 
6   while  $|A| > 0$  do
7       let  $i \in [m] \setminus B$  be s.t.  $|A \cap S_i|$  is maximum
8        $A \leftarrow A \setminus S_i$ 
9        $B \leftarrow B \cup i$ 
10  return  $B$ 

```

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## 5 Union-Find

**Definition 5.1** (Amortized Analysis). Suppose a data structure supports  $k$  operations. Then the amortized cost of each operation is  $t_i$  if for any sequence of operations with  $N_i$  of  $O_i$  operations, the total time is at most  $\sum_{i=1}^k t_i N_i$ .

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**Algorithm 9:** Union-find (disjoint forest implementation).

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```

/*  $O((m+n) \log^* n)$  runtime. */
1  $parent[1, \dots, n]$ 
2  $rank[1, \dots, n]$  // rank is defined as the height if no path compression
   occur.
3 Function MAKE-SET( $x$ ):
4    $parent[x] \leftarrow x$ 
5    $rank[x] \leftarrow 0$ 
6 Function FIND( $x$ ):
7   if  $x = parent[x]$  then
8     return  $x$ 
9    $parent[x] \leftarrow FIND(parent[x])$  // path compression
10  return  $parent[x]$ 
11 Function UNION( $x, y$ ):
12    $x \leftarrow FIND(x)$ 
13    $y \leftarrow FIND(y)$ 
14   if  $x = y$  then
15     return // no work needed
16   if  $rank[x] > rank[y]$  then
17     swap  $x$  and  $y$ 
18    $parent[x] \leftarrow y$  if  $rank[x] = rank[y]$  then
19      $rank[y] \leftarrow rank[y] + 1$ 

```

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*Remark.* Union-Find Invariants:

- a tree rooted at  $x$  has  $\geq 2^{r[x]}$  items;
- $(\forall x)$ , if  $x$  is not a root,  $r[p[x]] > r[x]$ ;
- the number of items of exactly rank  $k$  is  $\leq \frac{n}{2^k}$ .

## 6 Dynamic Programming

**Definition 6.1** (Top-Down DP/Memoization). Recursion + look-up table

**Definition 6.2** (Bottom-Up DP). Fill up the look-up table iteratively instead of recursively

*Remark.* Bottom-up DP sometimes have better memory

## 7 Linear Programming

**Definition 7.1** (Linear Programming). Linear programming (LP) describes a broad class of optimization tasks in which both the constraints and the optimization criterion are linear functions. The optimum of a linear program is achieved at a vertex of the convex feasible region.

*Remark.* A linear program does not have an optimum *iff* its feasible region is infeasible and/or unbounded.

**Definition 7.2** (Simplex Method). A standard greedy algorithm for solving LP by hill-climbing on vertices of the feasible region.

*Remark.* Solves real-life LP in polynomial time.

### 7.1 LP Conversion

1. (Maximization  $\leftrightarrow$  minimization) multiply the coefficients of the objective function by  $-1$
2. (Inequality  $\rightarrow$  equality)  $ax \leq b \rightarrow ax + s = b \mid s \geq 0$
3. (Equality  $\rightarrow$  Inequality)  $ax = b \rightarrow ax \leq b \wedge ax \geq b$
4. (Signed  $\leftrightarrow$  unsigned)  $x \leftrightarrow x^+ - x^- \mid x^+, x^- \geq 0$

### 7.2 Duality

**Theorem 7.1** (Duality theorem). *If a linear program has a bounded optimum, then so does its dual, and the two optimum values coincide (strong duality; weak duality states that primal opt.  $\leq$  dual opt.).*

Primal LP:

$$\begin{aligned} \max c^T x \\ Ax \leq b \\ b \geq 0 \end{aligned}$$

Dual LP:

$$\begin{aligned} \min y^T b \\ y^T A \geq c^T \\ y \geq 0 \end{aligned}$$

*Remark.* Dual/Primal unbounded  $\implies$  Primal/Dual unfeasible.

### 7.3 Network Flow

**Definition 7.3** (Flow). Given a directed graph  $G = (V, E)$  with capacities  $c_e > 0$  on all edges. The flow  $f$  from source  $s$  to sink  $t$  satisfies the constraints:

1. For all  $e \in E$ ,

$$0 \leq f_e \leq c_e$$

2. For all  $e \in E \setminus \{s, t\}$ ,

$$\sum_{(w,u) \in E} f_{wu} = \sum_{(u,z) \in E} f_{uz} \quad (\text{conservation of flow})$$

*Remark.* By the conservation principle,

$$\text{size}(f) = \sum_{(s,u) \in E} f_{su}$$

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#### Algorithm 10: Ford-Fulkerson

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```

// Simplex algorithm for solving max-flow problem. Pseudopolynomial
// time complexity;  $O(|E|F)$ 
Input:  $G = (V, E), s, t$ 
Output:  $f$ 
//  $G_f :=$  residual graph (also contains back-edges)
1 while  $\exists$  an augmenting path in  $G_f$  do
2   Find an arbitrary augmenting path  $P$  from  $s$  to  $t$ 
3   Augment flow  $f$  along  $P$ 
4   Update  $G_f$ 

```

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**Theorem 7.2** (Max-flow Min-cut Theorem). *The size of the maximum flow in a network equals the capacity of the smallest  $(s, t)$ -cut  $(L, R)$  (total capacity of the edges crossing the cut)*

*Remark.*  $L$  contains all reachable vertices from  $s$  in the final residual  $G^f$  and  $R$  contains all remaining vertices.

### 7.4 Zero-Sum Game

**Theorem 7.3** (Min-Max Theorem). *For zero-sum games, there exists an equilibrium such that*

$$\max_x \min_y \sum_{i,j} G_{i,j} x_i y_j = \min_y \max_x \sum_{i,j} G_{i,j} x_i y_j$$

*Remark.* To convert game strategies to LP:

$$\bullet \max_x \min_y \sum_{i,j} G_{i,j} x_i y_j \implies$$

$$\begin{aligned} & \max z \\ \forall y, & \sum_{i,j} G_{i,j} x_i y_j \geq z \\ & \sum_i x_i = 1 \\ & \forall i, x_i \geq 0 \end{aligned}$$

$$\bullet \min_y \max_x \sum_{i,j} G_{i,j} x_i y_j \implies$$

$$\begin{aligned} & \min w \\ \forall x, & \sum_{i,j} G_{i,j} x_i y_j \leq w \\ & \sum_j y_j = 1 \\ & \forall j, y_j \geq 0 \end{aligned}$$

It is apparent that the two LPs are dual; therefore, the equilibrium can be found in polynomial time via LP.

## 8 Multiplicative Weight Updates