Homework #4

PHYS 4D: Modern Physics

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1 Questions

1.1 Question 4

Denoting the wave function of a particle by $\psi(x)$, write down an expression for the probability that the particle will be found between a and b.

1.1.1 Solution

Use an integral and the relationship between $\psi(x)$ and probability. $\psi^\star(x)$ denotes the complex conjugate of $\psi(x)$.

$$1 = \int_{-\infty}^{\infty} \psi^\star(x)\psi(x)\,dx \tag{1}$$

$$P(a < x < b) = \int_a^b \psi^\star(x)\psi(x)\,dx \tag{2}$$

1.2 Question 5

Denoting the wave function of a particle by $\psi(x)$, write down an equation for the average value of x.

1.2.1 Solution

This is done by integral over all values of x. Every value of x should be multiplied by its probability, which is given by the wave function of the particle. This is where we get (2.21).

$$\boxed{\langle x \rangle = \int_{-\infty}^{\infty} x \, |\psi(x)|^2 \, dx} \tag{2.21}$$

1.3 Question 6

Suppose that a particle, which is confined to move in one-dimension between 0 and L, is described by the wave function, $\psi(x) = Ax(L-x)$. What condition could be imposed upon the wave function $\psi(x)$ to determine the constant A?

1.3.1 Solution

I'm thinking something to do with integrating. Knowing the probability density curve, we can use something similar to (2.21).

$$\int_{-\infty}^{\infty} \psi^\star(x)\psi(x)\, dx = \int_{-\infty}^{\infty} A^2(Lx - x^2)^2 \, dx = 1 \tag{3}$$

This can be integrated after we pull out the A^2. We can also change the bounds to 0 and L because the particle is bound to that area.

$$1 = A^2 \int_0^L (Lx - x^2)^2 \, dx = A^2 \int_0^L L^2x^2 - 2Lx^3 + x^4 \, dx \tag{4}$$

$$= A^2 \left[\frac{L^2x^3}{3} - \frac{2Lx^4}{4} + \frac{x^5}{5}\right]_0^L = A^2\left(\frac{L^5}{3} - \frac{L^5}{2} + \frac{L^5}{5}\right) \tag{4b}$$

$$= A^2\left(\frac{10L^5 - 15L^5 + 6L^5}{30}\right) = A^2\left(\frac{L^5}{30}\right) \tag{4c}$$

This can be solved for A.

$$A^2 = \frac{30}{L^5} \tag{5}$$

$$\boxed{A = \sqrt{\frac{30}{L^5}}} \tag{6}$$

That being said, we could also just measure $\psi\left(\frac{L}{2}\right)$ and divide it by $\frac{L^2}{4}$. I guess it just depends on what equipment we have.

1.4 Question 7

Suppose that a perfectly elastic ball were bouncing back and forth between two rigid walls with no gravity. Which of the variables, p, $|p|$, E, would have a constant value?

1.4.1 Solution

In a perfectly elestic collision, no magnitude of momentum is lost, so such will be the case involving the magnitude of the momentum. However, since momentum is a vector and the ball has limitations of where it can go, the momentum itself will not have a constant value. The energy, however, will have a constant value, especially since this is a closed system.

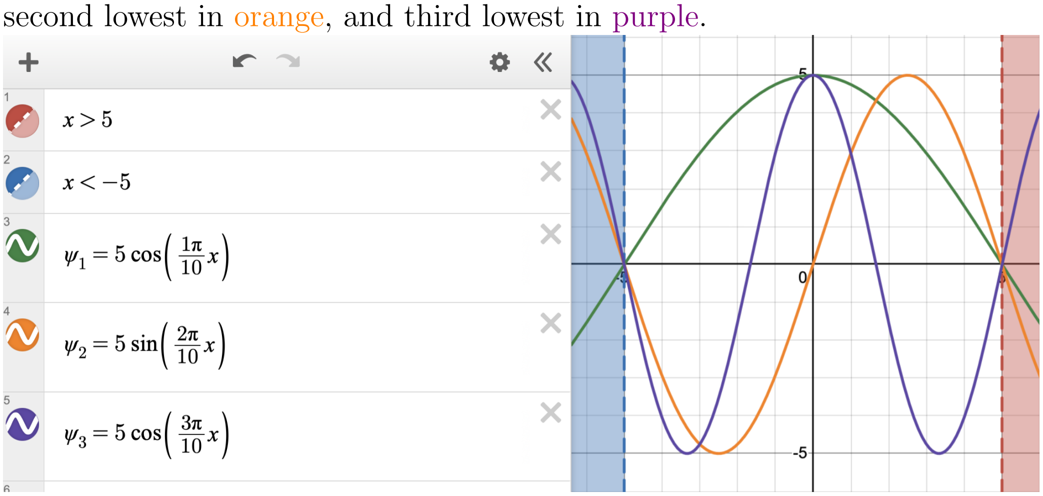
Variable	p	$\|p\|$	E
Const?	No	Yes	Yes

1.5 Question 8

Sketch the form of the wave functions corresponding to the three lowest energy levels of a particle confined to an infinite potential well.

1.5.1 Solution

I used Desmos to make the below image. The lowest energy level is in green, second lowest in orange, and third lowest in purple.



1.6 Question 10

What is the value of the kinetic energy of a particle at the classical turning points of an oscillator?

1.6.1 Solution

Assuming that a turning point is a point where the tangent line is horizontal (i.e. a local maximum/minimum), the oscillator would be unmoving and the kinetic energy woud be $\boxed{0}$.

1.7 Question 12

Suppose that a harmonic oscillator made a transition from the $n = 3$ to the $n = 2$ state. What would be the energy of the emitted photon?

1.7.1 Solution

I will be answering this in terms of the angular frequency.

$$\begin{aligned} E_{\text{photon}} &= -\Delta E = E_3 - E_2 = \hbar\omega * 3.5 - \hbar\omega * 2.5 \quad (7) \\ &= \hbar\omega(3.5 - 2.5) = \boxed{\hbar\omega} \quad (8) \end{aligned}$$

1.8 Question 13

Describe in qualitative terms the form of the wave functions of the harmonic oscillator between the classical turning points?

1.8.1 Solution

Between the turning points, it follows a sinusoidal wave.

1.9 Question 14

How does the form of the wave function of the harmonic oscillator change as x increases beyond the classical turning point.

1.9.1 Solution

The potential energy would be greater than the total energy, so it would taper off quickly.

1.10 Question 18

Describe the wave functions obtained by multiplying the stationary wave Ae^{ikx} by the function $e^{-i\omega t}$.

1.10.1 Solution

This is a traveling wave that moves over time.

2 Problem 3

An electron in a 10 nm-wide infinite well makes a transition from the $n = 3$ to the $n = 2$ state emitting a photon. Calculate (a) the energy of the photon and (b) the wavelength of the light.

2.1 Solution (a)

The equation of the energy in a well is given in equation (2.17).

$$E = \frac{n^2 h^2}{8mL^2} \tag{2.17}$$

This can be used to calculate the change in energy.

$$\begin{aligned}
\Delta E = E_2 - E_3 &= \frac{2^2 h^2}{8mL^2} - \frac{3^2 h^2}{8mL^2} = (2^2 - 3^2)\frac{h^2}{8mL^2} && (9)\\
&= -5\frac{h^2}{8mL^2} = -\frac{5 * (6.626 \times 10^{-34})^2}{8(9.109 \times 10^{-31})(10 \times 10^{-9})^2} && (10)\\
&= -3.01 \times 10^{-21}\ \mathrm{J} && (11)
\end{aligned}$$

The energy of the photon would be the negative of this.

$$E_{\text{photon}} = -\Delta E = \boxed{3.01 \times 10^{-21}\ \mathrm{J}} \tag{12}$$

2.2 Solution (b)

Turn energy to wavelength.

$$\lambda = \frac{hc}{E} = \frac{1.986 \times 10^{-25}\ \mathrm{J \cdot m}}{3.01 \times 10^{-21}\ \mathrm{J}} = \boxed{65.9\ \mu\mathrm{m}} \tag{13}$$

3 Problem 4

Show by direct substitution that the wave function (14) satisfies Eq. (2.32) for the harmonic oscillator. Calculate the corresponding energy.

$$\psi(x) = Ae^{-\frac{m\omega x^2}{2\hbar}} \tag{14}$$

$$-\frac{\partial^2\psi}{\partial x^2} + \left(\frac{m\omega^2 x^2}{\hbar^2}\right)\psi = \left(\frac{2mE}{\hbar^2}\right)\psi \tag{2.32}$$

3.1 Solution

Before active substitution, take the second derivative of the wave equation with respect to x.

$$\frac{\partial\psi}{\partial x} = -A\frac{m\omega x}{\hbar}e^{-\frac{m\omega x^2}{2\hbar}} \tag{15}$$

$$\frac{\partial^2\psi}{\partial x^2} = -A\frac{m\omega}{\hbar}e^{-\frac{m\omega x^2}{2\hbar}} + A\frac{m^2\omega^2 x^2}{\hbar^2}e^{-\frac{m\omega x^2}{2\hbar}} = \frac{m^2\omega^2 x^2}{\hbar^2}\psi - \frac{m\omega}{\hbar}\psi \tag{16}$$

This actually does not work with Equation (2.32), because of its term $\frac{m\omega^2 x^2}{\hbar^2}\psi$, which does not appear in (16). It doesn't even work with Equation (2.31), which is supposed to form (2.32) when divided by $\frac{\hbar^2}{2m}$. Equation (2.31) divided by $\frac{\hbar^2}{2m}$ actually is equal to what I'm calling (2.32ν).

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}m\omega x^2\psi = E\psi \tag{2.31}$$

$$-\frac{\partial^2\psi}{\partial x^2} + \left(\frac{m^2\omega^2 x^2}{\hbar^2}\right)\psi = \left(\frac{2mE}{\hbar^2}\right)\psi \tag{2.32ν}$$

Since (2.32ν) works with (16), I will assume a typo in the textbook and move forward using (2.32ν) instead. First, I will substitute in for $\frac{\partial^2\psi}{\partial x^2}$. First thing's first, cancel out all ϕ and cancel out $\pm\frac{m^2\omega^2 x^2}{\hbar^2}$.

$$\frac{m\omega}{\hbar}\cancel{\psi} - \cancel{\frac{m^2\omega^2 x^2}{\hbar^2}}\cancel{\psi} + \cancel{\left(\frac{m^2\omega^2 x^2}{\hbar^2}\right)}\cancel{\psi} = \left(\frac{2mE}{\hbar^2}\right)\cancel{\psi} \tag{17}$$

$$\frac{m\omega}{\hbar} = \frac{2mE}{\hbar^2} \tag{18}$$

This brings us to Equation (18). We can cancel out some more values and solve for E.

$$\frac{\cancel{m}\omega}{\cancel{\hbar}} = \frac{2\cancel{m}E}{\hbar^{\cancel{2}}} \tag{19}$$

$$\boxed{E = \frac{\hbar\omega}{2}} \tag{20}$$

4 Problem 5

Determine the constant A in the preceding problem by requiring that the wave function be normalized. Hint: For an arbitrary value of the constant a, the integral that arises in doing this problem may be evaluated using equation (21).

$$\int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \tag{21}$$

4.1 Solution

A Gaussian, I see. Recall the value of ψ.

$$\psi(x) = Ae^{-\frac{m\omega x^2}{2\hbar}} \tag{14}$$

The thing to remember about ψ is its relationship to the probability density function.

$$1 = \int_{-\infty}^{\infty} \psi^\star(x)\psi(x)\,dx \tag{1}$$

Our value of ψ is its own complex conjugate, so we can plug in values of ψ.

$$1 = \int_{-\infty}^{\infty} \left(Ae^{-\frac{m\omega x^2}{2\hbar}}\right)^2 dx = A^2 \int_{-\infty}^{\infty} e^{-\frac{m\omega x^2}{\hbar}}\,dx \tag{22}$$

With $a = \frac{m\omega}{\hbar}$, we can use (21).

$$1 = A^2 * \frac{1}{2}\sqrt{\frac{\pi}{\frac{m\omega}{\hbar}}} = A^2\sqrt{\frac{\pi\hbar}{4m\omega}} \tag{23}$$

$$A^2 = \sqrt{\frac{4m\omega}{\pi\hbar}} \rightarrow \boxed{A = \sqrt[4]{\frac{4m\omega}{\pi\hbar}}} \tag{24}$$

5 Problem 6

A particle is described by the below wave function where A and a are constants.

$$\psi(x) = \begin{cases} Ax\, e^{-ax}, & x > 0 \\ 0, & x \le 0 \end{cases} \tag{25}$$

(a) Sketch the wave function.

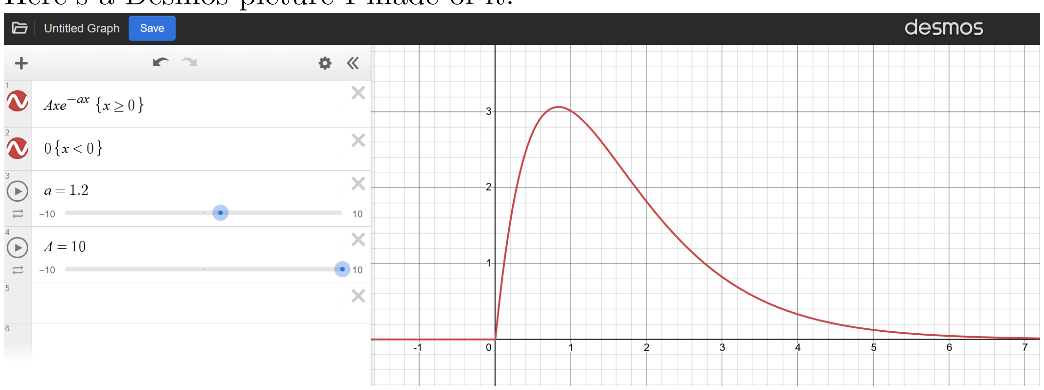
(b) Use the normalization condition to determine the constant A.

(c) Find the most probable position of the particle.

(d) Calculate the average value of the position of the particle.

5.1 Solution (a)

Here's a Desmos picture I made of it.



5.2 Solution (b)

I'm going to normalize it over the span of $[0, \infty)$. Since it's equal to zero at all ponts, we don't need to consider the span of $(-\infty, 0)$. Here, ψ is its own complex conjugate.

$$1 = \int_{-\infty}^{\infty} \psi^{\star}(x)\psi(x)\,dx \tag{1}$$

$$1 = \int_{0}^{\infty} \psi(x)^2\,dx = \int_{0}^{\infty} A^2 x^2\, e^{-2ax}\,dx = A^2 \int_{0}^{\infty} x^2\, e^{-2ax}\,dx \tag{26}$$

$$\begin{bmatrix} u = x^2 & du = 2x \\ v = -\frac{1}{2a}e^{-2ax} & dv = e^{-2ax} \end{bmatrix} \tag{27}$$

$$\frac{1}{A^2} = -\frac{x^2 e^{-2ax}}{2a} - \int_{0}^{\infty} -\frac{2x}{2a} e^{-2ax}\,dx = -\frac{x^2 e^{-2ax}}{2a} + \int_{0}^{\infty} \frac{x}{a} e^{-2ax}\,dx \tag{28}$$

$$\begin{bmatrix} u = x & du = 1 \\ v = -\frac{1}{2a^2}e^{-2ax} & dv = \frac{1}{a}e^{-2ax} \end{bmatrix} \tag{29}$$

$$= -\frac{x^2 e^{-2ax}}{2a} - \frac{x}{2a^2} e^{-2ax} + \int_{0}^{\infty} \frac{1}{2a^2} e^{-2ax}\,dx \tag{30}$$

$$= \left[-\frac{x^2 e^{-2ax}}{2a} - \frac{x e^{-2ax}}{2a^2} - \frac{e^{-2ax}}{4a^3} \right]_{0}^{\infty} \tag{31}$$

This is getting a bit big, so I'm just going to outsource this to Halliday and Resnick's Physics textbook, which has a citable integral for this in its Appendix E (10th Edition).

$$\int_{0}^{\infty} x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}} \tag{E.15}$$

We can use this for $n = 2$ and $a = 2a$.

$$\frac{1}{A^2} = \int_{0}^{\infty} x^2\, e^{-2ax}\,dx = \frac{2!}{8a^3} \tag{32}$$

$$\boxed{A = \sqrt{4a^3}} \tag{33}$$

5.3 Solution (c)

The most probable value is the maximum point of the graph, which we can fnd ourselves by first taking the derivative of ψ.

$$\frac{\mathrm{d}\psi}{\mathrm{d}x} = A\left(e^{-ax} - axe^{-ax}\right) = Ae^{-ax}\left(1 - ax\right) \tag{34}$$

The only point where this would be equal to zero is when $\boxed{x = \frac{1}{a}}$.

5.4 Solution (d)

Recall Equation (2.21) from Question 5.

$$\langle x \rangle = \int_{-\infty}^{\infty} x \left|\psi(x)\right|^2 \, dx \tag{2.21}$$

This is adaptable to our value of ψ. We use equation (E.15) here again, like how we used it in part (b). We ignore the range of $(-\infty, 0)$ for the same reason as in part (b).

$$\langle x \rangle = \int_{0}^{\infty} x^3 A^2 e^{-2ax} \, dx = A^2 \frac{3!}{16a^4} = \boxed{\frac{3A^2}{8a^4}} \tag{35}$$

6 Problem 7

(a) For a particle moving in the potential well shown in Figure 1, write down the Schrödinger equations for the region where $0 \leq x \leq L$ and the region where $x \geq L$. (b) Give the general form of the solution in the two regions. (c) Assuming that the potential is infinite at $x = 0$, impose boundary conditions that are natural for this problem and derive an equation that can be used to find the energy levels for the bound states.

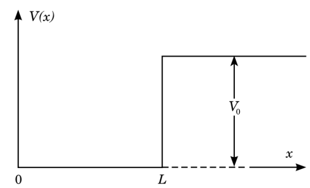


Figure 1: Potential Well

6.1 Solution (a)

Recall Schrödinger's time-independent equation.

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\,\psi = E\,\psi \tag{71}$$

This can be applied for both cases of the value of $V(x)$.

6.1.1 $0 \leq x \leq L$

Within this boundary, $V = 0$. We can plug it into (71).

$$\boxed{-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\,\psi} \tag{36}$$

6.1.2 $x \geq L$

Do the same for $V = V_0$.

$$\boxed{-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V_0\,\psi = E\,\psi} \tag{37}$$

6.2 Solution (b)

This would also be separated into areas. I will be finding solutions using exponentials.

6.2.1 $0 \leq x \leq L$

Start with the equation we found in part (a).

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = E\,\psi \tag{38}$$

Solve for $\dfrac{\partial^2\psi}{\partial x^2}$.

$$\frac{\partial^2\psi}{\partial x^2} = -\frac{2mE}{\hbar^2}\,\psi = \left(i\frac{\sqrt{2mE}}{\hbar}\right)^2\,\psi \tag{39}$$

This can be placed into an exponential.

$$\boxed{\psi = Ae^{i\frac{\sqrt{2mE}}{\hbar}x}} \tag{40}$$

6.2.2 $x \geq L$

Start with the equation from part (a).

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V_0\,\psi = E\,\psi \tag{41}$$

Collect all ψ and solve for $\dfrac{\partial^2\psi}{\partial x^2}$.

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = (E - V_0)\,\psi \tag{42}$$

$$\frac{\partial^2\psi}{\partial x^2} = -\frac{2m(E-V_0)}{\hbar^2}\,\psi = \frac{\sqrt{2m(V_0-E)}}{\hbar}\,\psi \tag{43}$$

This can be placed into an exponential.

$$\boxed{\psi = Ae^{\frac{\sqrt{2m(V_0-E)}}{\hbar}x}} \tag{44}$$

Note that this could be a complex exponential, dependant on whether $V_0 - E$ is positive or negative. It is equivalent to (40) when $V_0 = 0$.

6.3 Solution (c)

If the potential is infinite at $x = 0$, that means there is no chance in hell that the particle will be found there. We can put this as an inital condition.

$$\psi(0) = 0 \tag{45}$$

We can find a specific solution to ψ with the initial condition of (45).

$$\psi(0) = 0 = Ae^{\frac{\sqrt{2m(V_0-E)}}{\hbar}*0} + c \tag{46}$$

$$c = -Ae^0 = -A \tag{47}$$

$$\boxed{\psi(x) = Ae^{\frac{\sqrt{2m(V_0-E)}}{\hbar}x} - A} \tag{48}$$

There appears to be no indication of what $\dfrac{\partial\psi}{\partial x}$ is at $x = 0$.

7 Problem 8

Show that the wave function of a traveling wave (2.41) satisfies the time-dependent Schrödinger equation (2.47).

$$\psi(x,t) = Ae^{ikx} \cdot e^{-i\omega t} \tag{2.41}$$

$$\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right]\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{\partial t} \tag{2.47}$$

7.1 Solution

Show. Take the first derivative of (2.41) with respect to t, and the second derivative with respect to x.

$$\frac{\partial\psi}{\partial t} = -i\omega Ae^{ikx} \cdot e^{-i\omega t} = -i\omega\psi \tag{49}$$

$$\frac{\partial^2\psi}{\partial x^2} = -k^2 Ae^{ikx} \cdot e^{-i\omega t} = -k^2\psi \tag{50}$$

We can plug these into (2.47).

$$\frac{-\hbar^2}{2m}(-k^2)\psi + V(x,t)\psi = i\hbar(-i\omega)\psi \tag{51}$$

$$\frac{\hbar^2k^2}{2m} + V(x,t) = \hbar\omega \tag{52}$$

Recall a handful of equations involving both values in (52) and 2π.

$$\hbar = \frac{h}{2\pi} \tag{53}$$

$$k = \frac{2\pi}{\lambda} \tag{54}$$

$$\omega = 2\pi f \tag{55}$$

We can plug all of these into (52).

$$\frac{\left(\frac{h}{2\pi}\right)^2\left(\frac{2\pi}{\lambda}\right)^2}{2m} + V(x,t) = \frac{h}{2\pi} * 2\pi f \tag{56}$$

$$\frac{\left(\frac{h}{\cancel{2\pi}}\right)^2\left(\frac{\cancel{2\pi}}{\lambda}\right)^2}{2m} + V(x,t) = \frac{h}{\cancel{2\pi}} * \cancel{2\pi} f$$

$$\frac{h^2}{2m\lambda^2} + V(x,t) = hf \tag{57}$$

To proceed, we recall Planck's relation and De Broglie's relation.

$$E = hf \tag{58}$$

$$\lambda = \frac{h}{p} \rightarrow p = \frac{h}{\lambda} \tag{59}$$

These can be plugged into (57).

$$\frac{p^2}{2m} + V(x,t) = E \tag{60}$$

$$K + V(x,t) = E \tag{61}$$

Unless there's a new kind of energy that is not potential or kinetic, this we know is true. TENA

8 Problem 9

Show how the wave function of the even states of a particle in an infinite well extending from $x = -L/2$ to $x = L/2$ evolve in time.

8.1 Solution

There exist three sectons we can use for the potential energy of the particle.

$$V(x) = \begin{cases} \infty & x > \frac{L}{2} \\ 0 & -\frac{L}{2} \leq x \leq \frac{L}{2} \\ \infty & x < -\frac{L}{2} \end{cases} \tag{62}$$

The central prong of this is what interests us. It gives us a version of the Schrödinger time-dependant Equation where $V = 0$.

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = i\hbar\frac{\partial\psi}{\partial t} \tag{63}$$