A Document Detailing the Vector Calculus Derivation of Magnetic Field Perpendicular to a Ring of Current

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This is a document meant for the sole purpose of deriving the formula for the total magnetic field a distance from the center of and on the axis perpendicular to a ring of current. We will be doing this in cartesian coordinates.

By the Biot-Savart law, we have an equation for the magnetic field from a level of current.

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \cdot \frac{d\vec{s} \times \hat{r}}{r^2} \tag{1}$$

Suppose our ring had a radius R, that our ring lay on the xy-plane, and that our point P were located at cartesian coordinates $(0, 0, z)$. This is a very idealized situation, but this is Physics, where we can afford to be idealized.

$d\vec{s}$ is an infinitesimally small arc length of the circle. It owuld roughly line on the vector tangent to the circle at each given, which would be the line of velocity of a point traveling around the circle.

$$d\vec{s} = \frac{d}{d\theta}\begin{pmatrix} R * \cos(\theta) \\ R * \sin(\theta) \\ 0 \end{pmatrix} d\theta = \begin{pmatrix} R\sin(\theta)d\theta \\ -R\cos(\theta)d\theta \\ 0 \end{pmatrix} \tag{2}$$

We know that $\hat{r} = \frac{\vec{r}}{|r|}$, and we can assemble formulae for each of these. We should bear in mind that $\vec{r}$ must be the vector from $d\vec{s}$ to the point P.

$$\vec{r} = \begin{pmatrix} -R\cos(\theta) \\ -R\sin(\theta) \\ r \end{pmatrix} \tag{3}$$

$$|r| = \sqrt{(R\cos(\theta))^2 + (R\sin(\theta))^2 + z^2} = \sqrt{R^2 + z^2} \tag{4}$$

$$\hat{r} = \frac{1}{\sqrt{R^2 + z^2}}\begin{pmatrix} -R\cos(\theta) \\ -R\sin(\theta) \\ r \end{pmatrix} \tag{5}$$

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Since we can draw out $\frac{1}{\sqrt{R^2+z^2}}$, we can calculate $d\vec{s} \times \vec{r}$ and apply $\frac{1}{\sqrt{R^2+z^2}}$ afterwards.

$$d\vec{s} \times \vec{r} = \begin{pmatrix} R\sin(\theta)d\theta \\ -R\cos(\theta)d\theta \\ 0 \end{pmatrix} \times \begin{pmatrix} -R\cos(\theta) \\ -R\sin(\theta) \\ r \end{pmatrix} \tag{6}$$

$$= \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ R\sin(\theta)d\theta & -R\cos(\theta)d\theta & 0 \\ -R\cos(\theta) & -R\sin(\theta) & r \end{bmatrix} \tag{7}$$

$$= \begin{vmatrix} -R\cos(\theta)d\theta & 0 \\ -R\sin(\theta) & r \end{vmatrix} \hat{i} + \begin{vmatrix} 0 & R\sin(\theta)d\theta \\ r & -R\cos(\theta) \end{vmatrix} \hat{j} + \begin{vmatrix} R\sin(\theta)d\theta & -R\cos(\theta)d\theta \\ -R\cos(\theta) & -R\sin(\theta) \end{vmatrix} \hat{k} \tag{8}$$

$$= \begin{pmatrix} Rz\cos(\theta)d\theta \\ Rz\sin(\theta)d\theta \\ R^2\sin^2(\theta)d\theta + R^2\cos^2(\theta)d\theta \end{pmatrix} = \begin{pmatrix} Rz\cos(\theta) \\ Rz\sin(\theta) \\ R^2 \end{pmatrix} d\theta \tag{9}$$

This is something we can plug nto the Biot-Savart law to find our answer.

$$\vec{B} = \int d\vec{B} = \int_0^{2\pi} \frac{\mu_0 I}{4\pi} \cdot \frac{d\vec{s} \times \hat{r}}{r^2} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{d\vec{s} \times \hat{r}}{r^2} \tag{10}$$

$$= \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{1}{(R^2+z^2)^{3/2}} \begin{pmatrix} Rz\cos(\theta) \\ Rz\sin(\theta) \\ R^2 \end{pmatrix} d\theta \tag{11}$$

$$= \frac{\mu_0 I}{4\pi (R^2+z^2)^{3/2}} \int_0^{2\pi} \begin{pmatrix} Rz\cos(\theta) \\ Rz\sin(\theta) \\ R^2 \end{pmatrix} d\theta \tag{12}$$

$$= \frac{\mu_0 I}{4\pi (R^2+z^2)^{3/2}} \begin{pmatrix} Rz\sin(\theta) \\ -Rz\cos(\theta) \\ R^2\theta \end{pmatrix}_0^{2\pi} \tag{13}$$

$$= \frac{\mu_0 I}{4\pi (R^2+z^2)^{3/2}} \begin{pmatrix} Rz(\sin(2\pi) - \sin(0)) \\ -Rz(\cos(2\pi) - \cos(0)) \\ R^2 * 2\pi \end{pmatrix} \tag{14}$$

$$= \frac{\mu_0 I}{4\pi (R^2+z^2)^{3/2}} \begin{pmatrix} 0 \\ 0 \\ R^2 * 2\pi \end{pmatrix} \tag{15}$$

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We can complete this in unit-vector notation rather than matrix vector notation, since we are only dealing with one dimension and unit vector.

$$\vec{B} = \frac{\mu_0 I R^2 * 2\pi\hat{k}}{4\pi\left(R^2 + z^2\right)^{3/2}} = \frac{\mu_0 I R^2}{2\left(R^2 + z^2\right)^{3/2}}\hat{k} \tag{16}$$

This is indeed the answer we were looking for. This document has fulfilled its sole purpose.