Homework #16

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1 Problem 1

Two disks are mounted (like a merry-go-round) on low-friction bearings on the same axle and can be brought together so that they couple and rotate as one unit. The first disk, with rotational inertia 3.30 $\text{kg} * \text{m}^2$ about its central axis, is set spinning counterclockwise at 450 rev/min. The second disk, with rotational inertia 6.60 $\text{kg} * \text{m}^2$ about its central axis, is set spinning counterclockwise at 900 rev/min. They then couple together. (a) What is their angular speed after coupling? If instead the second disk is set spinning clockwise at 900 rev/min, what are their (b) angular speed and (c) direction of rotation after they couple together?

1.1 Solution

1.1.1 Section (a)

We have a concept called conservation of angular momentum.

$$L_i = L_f \tag{1}$$
$$L_f = l_1 + l_2 = I_1\omega_1 + I_2\omega_2 \tag{2}$$
$$\omega_f = \frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2} = \frac{3.3 * 450 + 6.6 * 900}{3.3 + 6.6} \tag{3}$$
$$= \frac{1485 + 5940}{9.9} = \boxed{750\text{rev/min}} \tag{4}$$

1.1.2 Section (b)

We just need to change a positive to a negative.

$$\omega_f = \frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2} = \frac{3.3 * 450 - 6.6 * 900}{3.3 + 6.6} \tag{5}$$
$$= \frac{1485 - 5940}{9.9} = -450\text{rev/min} \tag{6}$$
$$|\omega_f| = \boxed{450\text{rev/min}} \tag{7}$$

1.1.3 Section (c)

Since the value is negative and negative angular velocity corresponds to clockwise motion, the angular motion is $\boxed{\textit{clockwise}}$.

2 Problem 2

The Sun's mass is 2.0 $\times 10^{30}$ kg, its radius is 7.0 $\times 10^{5}$ km, and it has a rotational period of approximately 28 days. If the Sun should collapse into a white dwarf of radius 3.5 $\times 10^{3}$ km, what would its period be if no mass were ejected and a sphere of uniform density can model the Sun both before and after?

2.1 Solution

We can calculate the angular frequency of the sun by using the period formula $T = \frac{2\pi}{\omega}$.

$$T = \frac{2\pi}{\omega} \tag{8}$$
$$\omega = \frac{2\pi}{T} \tag{9}$$

Next, we can use the conservation of angular momentum and the formula for the inertia of the dwarf sun to find a formula for the final angular velocity and then final period.

$$L_f = L_i \tag{10}$$
$$I_f \omega_f = I_i \omega_i \tag{11}$$
$$I_f \frac{2\pi}{T_f} = I_i \frac{2\pi}{T_i} \tag{12}$$
$$\frac{I_f}{I_i} \cdot \frac{2\pi}{2\pi} = \frac{T_f}{T_i} \tag{13}$$
$$\frac{I_f}{I_i} * T_i = T_f \tag{14}$$
$$\frac{\frac{2}{5}MR_f^2}{\frac{2}{5}MR_i^2} * T_i = \frac{R_f^2}{R_i^2} * T_i = \frac{(3.5 \times 10^3)^2}{(7.0 \times 10^5)^2} * 28\text{days} = T_f \tag{15}$$
$$\frac{12.25 \times 10^6}{49.0 \times 10^{10}} * 28\text{days} = \frac{28\text{days}}{4 \times 10^4} = 7 \times 10^{-4}\text{days} = T_f \tag{16}$$

This means that the period is $\boxed{7 \times 10^{-4} \text{ days}}$.

3 Problem 3

The displacement from equilibrium of a particle is given by $x(t) = A\cos\left(\omega t - \frac{\pi}{3}\right)$. Which, if any, of the following are equivalent expressions:

$$a)\ x(t) = A\cos\left(\omega t + \frac{\pi}{3}\right) \tag{17}$$

$$b)\ x(t) = A\cos\left(\omega t + \frac{5\pi}{3}\right) \tag{18}$$

$$c)\ x(t) = A\cos\left(\omega t + \frac{\pi}{6}\right) \tag{19}$$

$$d)\ x(t) = A\cos\left(\omega t - \frac{5\pi}{6}\right) \tag{20}$$

3.1 Solution

We can see that the only change here is the part labeled ϕ in the format of simple harmonic motion. For an equivalent value, the value of the cosine must be the same at every point, which can only be true if $\phi = -\frac{\pi}{3} \mod 2\pi$.

	ϕ	$\phi \mod 2\pi$	Correct?
	$-\frac{\pi}{3}$	$\frac{5\pi}{3}$	Yes
a)	$\frac{\pi}{3}$	$\frac{\pi}{3}$	No
b)	$\frac{5\pi}{3}$	$\frac{5\pi}{3}$	Yes
c)	$\frac{\pi}{6}$	$\frac{\pi}{6}$	No
d)	$\frac{5\pi}{6}$	$\frac{5\pi}{6}$	No

4 Problem 4

In a block and spring system $m = 0.250\text{kg}$ and $k = 4.00\text{N/m}$. At $t = 0.150\text{s}$, the velocity is $v = -0.174\text{m/s}$ and the acceleration $a = +0.877\text{m/s}^2$. Write an expression for the displacement as a function of time, $x(t)$. (Hint, remember that the inverse tan function only returns the principal value, but there is a secondary value as well.)

4.1 Solution

We have some formulas for velocity and acceleration that we can use.

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{4.0}{0.25}} = \sqrt{4^2} = 4\text{s}^{-1} \tag{21}$$

$$v(t) = -\omega x_m \sin(\omega t + \phi) \quad \rightarrow v(0.15) = -0.174\text{m/s} = -4x_m \sin(0.6 + \phi) \tag{22}$$

$$a(t) = -\omega^2 x_m \cos(\omega t + \phi) \rightarrow a(0.15) = \ 0.877\text{m/s}^2 = -16x_m \cos(0.6 + \phi) \tag{23}$$

$$\frac{a(0.15)}{v(0.15)} = \frac{-16x_m \cos(0.6 + \phi)}{-4x_m \sin(0.6 + \phi)} = 4 * \frac{\cos(0.6 + \phi)}{\sin(0.6 + \phi)} \tag{24}$$

$$\tan(\phi) = \frac{\sin(\phi)}{\cos(\phi)} = \frac{v(0)\sqrt{k}}{a(0)\sqrt{m}} \tag{25}$$

$$0.6 + \phi = \arctan\left(4 * \frac{v(0)}{a(0)}\right) = \arctan\left(4 * \frac{-0.174}{0.877}\right) \tag{26}$$

$$= \arctan\left(-\frac{0.696}{0.877}\right) = \begin{matrix} 2.471 \\ 5.612 \end{matrix} \tag{27}$$

One of these is in the second quadrant, the other is in the fourth quadrant. Knowing that ω is positive and trusting that x_m is positive, since the negative cosine is positive and the negative sine is negative, the cosine is negative and the sine is positive, so $0.6 + \phi$ is in the second quadrant. This means $0.6 + \phi = 2.471$ and $\phi = 1.871$. Last, we just neeed to find the value of x_m, which we will find using the value of $a(0)$.

$$a(0.15) = -16x_m \cos(0.6 + 1.871) \tag{28}$$

$$x_m = -\frac{a(0)}{16\cos(2.471)} = \frac{0.877}{0.7833} = 0.06998\text{m} \tag{29}$$

Lastly, we find the value of ω and use that to finalize the formula for $x(t)$.

$$\boxed{x(t) = 0.06998 * \cos(4t + 1.871)} \tag{30}$$

5 Problem 5

A 60.0 g block attached to a horizontal spring is held at 8.00 cm from its equilibrium position and released at $t = 0$. Its period is 0.900s. Find: (a) the displacement x at 1.20s; (b) the velocity when $x = -5.00$cm; (c) the acceleration when $x = -5.00$cm; (d) the total energy.

5.1 Solution

5.1.1 Section (a)

To find the position, we can use the simple harmonic motion formula. We can set $x_m = 8.0$cm. Next, we need to find ω. Since it starts from the fullest extension at $t = 0$, $\phi = 0$.

$$\omega = \frac{2\pi}{T} \tag{31}$$

$$x(t) = x_m \cos(\omega t + \phi) = x_m \cos\left(\frac{2\pi}{T} t + \phi\right) \tag{32}$$

$$= 8.0 * \cos\left(\frac{2\pi}{0.900} t\right) \tag{33}$$

$$x(1.2) = 8.0 * \cos\left(\frac{2\pi}{0.900} * 1.2\right) = 8.0 * \cos\left(\frac{24\pi}{9}\right) \tag{34}$$

$$= 8.0 * \cos\left(\frac{8\pi}{3}\right) = 8.0 * (-0.5) = -4.0\text{cm} \tag{35}$$

This means that the block is $\boxed{4\text{cm}}$ away from the equilibrium.

5.1.2 Section (b)

First, we find the time at whch $x = 5.00$cm.

$$-5\text{cm} = 8.0 * \cos\left(\frac{2\pi}{0.900} t\right) \tag{36}$$

$$\cos\left(\frac{2\pi}{0.900} t\right) = -\frac{5}{8} \tag{37}$$

By using the pythagorean theorem, we can find a value for $\sin\left(\frac{2\pi}{0.900}t\right)$.

$$\sin^2(\theta) = 1 - \cos^2(\theta) \tag{38}$$

$$\sin(\theta) = \sqrt{1 - \cos^2(\theta)} \tag{39}$$

$$\sin\left(\frac{2\pi}{0.900}t\right) = \sqrt{1 - \cos^2\left(\frac{2\pi}{0.900}t\right)} \tag{40}$$

$$\sin\left(\frac{2\pi}{0.900}t\right) = \sqrt{1 - \frac{5}{8}^2} = \frac{\sqrt{39}}{8} \tag{41}$$

The SHM velocity is the first derivatve of the SHM position.

$$x(t) = 8.0 * \cos\left(\frac{2\pi}{0.900}t\right) \tag{42}$$

$$\frac{dx(t)}{dt} = v(t) = -8.0 * \frac{2\pi}{0.900} * \sin\left(\frac{2\pi}{0.900}t\right) = -\frac{16\pi}{0.900} * \sin\left(\frac{2\pi}{0.900}t\right) \tag{43}$$

$$v(t_1) = -\frac{16\pi}{0.900} * \frac{\sqrt{39}}{8} = -\frac{20\pi\sqrt{39}}{9} = -43.598\text{cm/s} \tag{44}$$

This means that the velocity is $\boxed{43.598\text{m/s}}$.

5.1.3 Section (c)

From our in-class differential equations for SHM, we know that $\frac{d^2x(t)}{dt^2} = -\omega^2 x(t)$. We can work with this, recalling that $\omega = \frac{2\pi}{T}$.

$$\frac{d^2x(t)}{dt^2} = a(t) = -\omega^2 x(t) = \left(\frac{2\pi}{T}\right)^2 * x(t)$$

$$a = \frac{4\pi^2}{T^2} * x = \frac{4\pi^2}{0.9^2} * 5 = \boxed{243.694\text{cm/s}^2}$$

5.1.4 Section (d)

We can calculate this using the velocity where there is no potential energy (where x = 0). This can only be true where $cos(\theta) = 0$, since the equivalent of θ is the only variable without a set value (yet). With the pythagorean theorem, if $cos(\theta) = 0$, $\sin(\theta) = \pm 1$, with either one working, so we will be using -1.

$$v = -\frac{16\pi}{0.900} * \sin\left(\frac{2\pi}{0.900}t\right) = -\frac{16\pi}{0.900} * (-1) = \frac{16\pi}{0.900}\text{cm/s} \tag{45}$$

$$E_{total} = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2} * (60.0\text{g}) * \left(\frac{16\pi}{0.900}\text{cm/s}\right)^2 + 0 \tag{46}$$

$$= \boxed{93578 \text{ dyn} * \text{cm} = 9.3578 \times 10^{-3}\text{J}} \tag{47}$$

6 Problem 6

A wire has a torsional constant $\kappa = 2.00\text{N} * \text{m/rad}$. A solid disk of radius $R = 5.00\text{cm}$ and mass $M = 100\text{g}$ is suspended at its center as shown in the figure. What is the frequency of torsional oscillations?

6.1 Solution

First, I want to convert the whole thing to the SI unit system, so the radius is 0.05m and the mass is 0.1kg. The frequency is equal to the reciprocal of the period, and we have a formula for the period.

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{I}{\kappa}} \tag{48}$$

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{\kappa}{I}} \tag{49}$$

$$= \frac{1}{2\pi}\sqrt{\frac{2\kappa}{MR^2}} = \frac{1}{2\pi}\sqrt{\frac{2 * 2}{0.1 * 0.05^2}} \tag{50}$$

$$= \frac{1}{2\pi}\sqrt{\frac{4}{2.5 \times 10^{-4}}} = \frac{1}{\pi\sqrt{2.5 \times 10^{-4}}} \tag{51}$$

$$= \boxed{20.13\text{Hz}} \tag{52}$$

7 Problem 7

The total energy of a block and spring system is 0.200J. The mass of the block is 120g and the spring constant is 40N/m. Find: (a) the amplitude; (b) the maximum speed; (c) the displacement from equilibrium when the speed is 1.30m/s; (d) the maximum acceleration.

7.1 Solution

7.1.1 Section (a)

When there is no velocity (and the sine value is zero), the cosine value is one. All the energy would also be spring potential energy.

$$x = x_m \cos(\omega t + \phi) = x_m \tag{53}$$

$$0.200 = \frac{1}{2}kx^2 \tag{54}$$

$$0.400 = 40 * x^2 \tag{55}$$

$$0.01 = x^2 \tag{56}$$

$$x = \boxed{x_m = 0.1\text{m}} = \sqrt{0.01} \tag{57}$$

7.1.2 Section (b)

The maximum speed will be achieved when all the energy is kinetic.

$$0.200 = \frac{1}{2}mv^2 \tag{58}$$

$$v^2 = \frac{0.400}{0.12} \tag{59}$$

$$v = \sqrt{\frac{10}{3}} = \frac{\sqrt{30}}{3} = \boxed{1.8257\text{m/s}} \tag{60}$$

7.1.3 Section (c)

We can plug in values into the energy and isolate the postional value.

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \tag{61}$$

$$kx^2 = 2E - mv^2 \tag{62}$$

$$x = \sqrt{\frac{2E - mv^2}{k}} = \sqrt{\frac{2 * 0.200 - 0.12 * 1.3^2}{40}} = \sqrt{\frac{0.4 - 0.2028}{40}} \tag{63}$$

$$= \sqrt{\frac{0.1972}{40}} = \sqrt{4.93 \times 10^{-3}} = \boxed{0.07021\text{m}} \tag{64}$$

7.1.4 Section (d)

We can use our differential equation to find this, given our maximum position and an unchanging ω.

$$\frac{d^2x(t)}{dt^2} = a(t) = -\omega^2 x(t) = -\frac{k}{m} * x(t) \tag{65}$$

$$|a| = \frac{k}{m} * x = \frac{40}{0.12} * 0.1 = \boxed{\frac{100}{3}\text{m/s}^2 = 33.33\text{m/s}^2} \tag{66}$$

8 Problem 8

A uniform rod of mass M and length $L = 1.20$m oscillates about a horizontal axis at one end. What is the length of the simple pendulum that would have the same period? The rotational inertia is $\frac{ML^2}{3}$.

8.1 Solution

What we have here is a physical pendulum, and we want to compare it to a simple pendulum. We can create an equality. Since we know that the two values are equal, we don't have separate kinds of the variable T.

$$T = 2\pi\sqrt{\frac{L_s}{g}} \tag{67}$$

$$T = 2\pi\sqrt{\frac{I}{mgh}} = 2\pi\sqrt{\frac{ML_p^2}{3Mgh}} \tag{68}$$

To be clear, the first eqution is for a simple equation, while the second is for a physical pendulum.

$$I = \frac{1}{3}ML^2 \tag{69}$$

$$2\pi\sqrt{\frac{L_s}{g}} = 2\pi\sqrt{\frac{ML_p^2}{3Mgh}} \tag{70}$$

$$\sqrt{\frac{L_s}{g}} = \sqrt{\frac{L_p^2}{3gh}} \tag{71}$$

$$\frac{L_s}{g} = \frac{L_p^2}{3gh} \tag{72}$$

$$L_s = \frac{L_p^2}{3h} \tag{73}$$

To conclude this, we can know that the center of mass (h) of the uniform rod pendulum is going to be $h = \frac{L_p}{2}$.

$$L_s = \frac{L_p^2}{3h} = \frac{2L_p^2}{3h} = \frac{2}{3}L_p = \frac{2}{3} * 1.20\text{m} = \boxed{0.8\text{m}} \tag{74}$$