Homework #16

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1 Problem 1

Two disks are mounted (like a merry-go-round) on low-friction bearings on the same axle and can be brought together so that they couple and rotate as one unit. The first disk, with rotational inertia 3.30 $\mathrm{kg} * \mathrm{m}^2$ about its central axis, is set spinning counterclockwise at 450 rev/min. The second disk, with rotational inertia 6.60 $\mathrm{kg} * \mathrm{m}^2$ about its central axis, is set spinning counterclockwise at 900 rev/min. They then couple together. (a) What is their angular speed after coupling? If instead the second disk is set spinning clockwise at 900 rev/min, what are their (b) angular speed and (c) direction of rotation after they couple together?

1.1 Solution

1.1.1 Section (a)

We have a concept called conservation of angular momentum.

$$L_i = L_f \tag{1}$$

$$L_f = l_1 + l_2 = I_1\omega_1 + I_2\omega_2 \tag{2}$$

$$\omega_f = \frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2} = \frac{3.3 * 450 + 6.6 * 900}{3.3 + 6.6} \tag{3}$$

$$= \frac{1485 + 5940}{9.9} = \boxed{750\mathrm{rev/min}} \tag{4}$$

1.1.2 Section (b)

We just need to change a positive to a negative.

$$\omega_f = \frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2} = \frac{3.3 * 450 - 6.6 * 900}{3.3 + 6.6} \tag{5}$$

$$= \frac{1485 - 5940}{9.9} = \boxed{-450\mathrm{rev/min}} \tag{6}$$

1.1.3 Section (c)

Since the magnitude is negative and negative angular velocity corresponds to clockwise motion, the angular motion is $\boxed{\textit{clockwise}}$.

2 Problem 2

The Sun's mass is 2.0 $\times 10^{30}$ kg, its radius is 7.0 $\times 10^{5}$ km, and it has a rotational period of approximately 28 days. If the Sun should collapse into a white dwarf of radius 3.5 $\times 10^{3}$ km, what would its period be if no mass were ejected and a sphere of uniform density can model the Sun both before and after?

2.1 Solution

We can calculate the angular frequency of the sun by using the period formula $T = \frac{2\pi}{\omega}$.

$$T = \frac{2\pi}{\omega} \tag{7}$$

$$\omega = \frac{2\pi}{T} \tag{8}$$

Next, we can use the conservation of angular momentum and the formula for the inertia of the dwarf sun to find a formula for the final angular velocity and then final period.

$$L_f = L_i \tag{9}$$

$$I_f \omega_f = I_i \omega_i \tag{10}$$

$$I_f \frac{2\pi}{T_f} = I_i \frac{2\pi}{T_i} \tag{11}$$

$$\frac{I_f}{I_i} \cdot \frac{2\pi}{2\pi} = \frac{T_f}{T_i} \tag{12}$$

$$\frac{I_f}{I_i} * T_i = T_f \tag{13}$$

$$\frac{\frac{2}{5} M R_f^2}{\frac{2}{5} M R_i^2} * T_i = \frac{R_f^2}{R_i^2} * T_i = \frac{(3.5 \times 10^3)^2}{(7.0 \times 10^5)^2} * 28\text{days} = T_f \tag{14}$$

$$\frac{12.25 \times 10^6}{49.0 \times 10^{10}} * 28\text{days} = \frac{28\text{days}}{4 \times 10^4} = 7 \times 10^{-4}\text{days} = T_f \tag{15}$$

This means that the period is $\boxed{7 \times 10^{-4} \text{ days}}$.

3 Problem 3

The displacement from equilibrium of a particle is given by $x(t) = A\cos\left(\omega t - \frac{\pi}{3}\right)$. Which, if any, of the following are equivalent expressions:

$$a)\ x(t) = A\cos\left(\omega t + \frac{\pi}{3}\right) \tag{16}$$

$$b)\ x(t) = A\cos\left(\omega t + \frac{5\pi}{3}\right) \tag{17}$$

$$c)\ x(t) = A\cos\left(\omega t + \frac{\pi}{6}\right) \tag{18}$$

$$d)\ x(t) = A\cos\left(\omega t - \frac{5\pi}{6}\right) \tag{19}$$

3.1 Solution

We can see that the only change here is the part labeled ϕ in the format of simple harmonic motion. For an equivalent value, the value of the cosine must be the same at every point, which can only be true if $\phi = -\frac{\pi}{3} \mod 2\pi$.

	ϕ	$\phi \mod 2\pi$	Correct?
	$-\frac{\pi}{3}$	$\frac{5\pi}{3}$	Yes
a)	$\frac{\pi}{3}$	$\frac{\pi}{3}$	No
b)	$\frac{5\pi}{3}$	$\frac{5\pi}{3}$	Yes
c)	$\frac{\pi}{6}$	$\frac{\pi}{6}$	No
d)	$\frac{5\pi}{6}$	$\frac{5\pi}{6}$	No

4 Problem 4

In a block and spring system $m = 0.250\text{kg}$ and $k = 4.00\text{N/m}$. At $t = 0.150\text{s}$, the velocity is $v = -0.174\text{m/s}$ and the acceleration $a = +0.877\text{m/s}^2$. Write an expression for the displacement as a function of time, $x(t)$. (Hint, remember that the inverse tan function only returns the principal value, but there is a secondary value as well.)

4.1 Solution

We have some formulas for velocity and acceleration that we can use.

$$\begin{aligned}
\omega &= \sqrt{\frac{k}{m}} = \sqrt{\frac{4.0}{0.25}} = \sqrt{4^2} = 4\text{s}^{-1} && (20)\\
v(t) &= -\omega x_m \sin(\omega t + \phi) \quad \rightarrow v(0.15) = -0.174\text{m/s} = -4x_m \sin(0.6 + \phi) && (21)\\
a(t) &= -\omega^2 x_m \cos(\omega t + \phi) \rightarrow a(0.15) = \ 0.877\text{m/s}^2 = -16x_m \cos(0.6 + \phi) && (22)\\
\frac{a(0.15)}{v(0.15)} &= \frac{-16x_m \cos(0.6 + \phi)}{-4x_m \sin(0.6 + \phi)} = 4 * \frac{\cos(0.6 + \phi)}{\sin(0.6 + \phi)} && (23)\\
\tan(\phi) &= \frac{\sin(\phi)}{\cos(\phi)} = \frac{v(0)\sqrt{k}}{a(0)\sqrt{m}} && (24)\\
0.6 + \phi &= \arctan\left(4 * \frac{v(0)}{a(0)}\right) = \arctan\left(4 * \frac{-0.174}{0.877}\right) && (25)\\
&= \arctan\left(-\frac{0.696}{0.877}\right) = \begin{matrix} 3.812 \\ 6.954 \end{matrix} && (26)
\end{aligned}$$

One of these is in the second quadrant, the other is in the fourth quadrant. Knowing that ω is positive and trusting that x_m is positive, since the negative cosine is positive and the negative sine is negative, the cosine is negative and the sine is positive, so $0.6 + \phi$ is in the second quadrant. This means $0.6 + \phi = 3.812$ and $\phi = 3.212$. Last, we just neeed to find the value of x_m, which we will find using the value of $a(0)$.

$$a(0.15) = -16x_m \cos(0.6 + 3.212) \tag{27}$$

$$x_m = -\frac{a(0)}{16\cos(3.812)} = \frac{0.877}{0.7833} = 0.06998\text{m} \tag{28}$$

Lastly, we find the value of ω and use that to finalize the formula for $x(t)$.

$$\boxed{x(t) = 0.06998 * \cos(4t + 3.212)} \tag{29}$$

5 Problem 5

A 60.0 g block attached to a horizontal spring is held at 8.00 cm from its equilibrium position and released at $t = 0$. Its period is 0.900s. Find: (a) the displacement x at 1.20s; (b) the velocity when $x = -5.00$cm; (c) the acceleration when $x = -5.00$cm; (d) the total energy.

5.1 Solution

5.1.1 Section (a)

To find the position, we can use the simple harmonic motion formula. We can set $x_m = 8.0$cm. Next, we need to find ω. Since it starts from the fullest extension at $t = 0$, $\phi = 0$.

$$\omega = \frac{2\pi}{T} \tag{30}$$

$$x(t) = x_m \cos(\omega t + \phi) = x_m \cos\left(\frac{2\pi}{T} t + \phi\right) \tag{31}$$

$$= 8.0 * \cos\left(\frac{2\pi}{0.900} t\right) \tag{32}$$

$$x(1.2) = 8.0 * \cos\left(\frac{2\pi}{0.900} * 1.2\right) = 8.0 * \cos\left(\frac{24\pi}{9}\right) \tag{33}$$

$$= 8.0 * \cos\left(\frac{8\pi}{3}\right) = 8.0 * (-0.5) = -4.0\text{cm} \tag{34}$$

This means that the block is $\boxed{\text{4cm}}$ away from the equilibrium.

5.1.2 Section (b)

First, we find the time at whch $x = 5.00$cm.

$$-5\text{cm} = 8.0 * \cos\left(\frac{2\pi}{0.900} t\right) \tag{35}$$

$$\cos\left(\frac{2\pi}{0.900} t\right) = -\frac{5}{8} \tag{36}$$

By using the pythagorean theorem, we can find a value for $\sin\left(\frac{2\pi}{0.900}t\right)$.

$$\sin^2(\theta) = 1 - \cos^2(\theta) \tag{37}$$

$$\sin(\theta) = \sqrt{1 - \cos^2(\theta)} \tag{38}$$

$$\sin\left(\frac{2\pi}{0.900}t\right) = \sqrt{1 - \cos^2\left(\frac{2\pi}{0.900}t\right)} \tag{39}$$

$$\sin\left(\frac{2\pi}{0.900}t\right) = \sqrt{1 - \frac{5}{8}^2} = \frac{\sqrt{39}}{8} \tag{40}$$

The SHM velocity is the first derivatve of the SHM position.

$$x(t) = 8.0 * \cos\left(\frac{2\pi}{0.900}t\right) \tag{41}$$

$$\frac{dx(t)}{dt} = v(t) = -8.0 * \frac{2\pi}{0.900} * \sin\left(\frac{2\pi}{0.900}t\right) = -\frac{16\pi}{0.900} * \sin\left(\frac{2\pi}{0.900}t\right) \tag{42}$$

$$v(t_1) = -\frac{16\pi}{0.900} * \frac{\sqrt{39}}{8} = -\frac{20\pi\sqrt{39}}{9} = -43.598\text{cm/s} \tag{43}$$

This means that the velocity is $\boxed{43.598\text{m/s}}$.

5.1.3 Section (c)

From our in-class differential equations for SHM, we know that $\frac{d^2x(t)}{dt^2} = -\omega^2 x(t)$. We can work with this, recalling that $\omega = \frac{2\pi}{T}$.

$$\frac{d^2x(t)}{dt^2} = a(t) = -\omega^2 x(t) = \left(\frac{2\pi}{T}\right)^2 * x(t)$$

$$a = \frac{4\pi^2}{T^2} * x = \frac{4\pi^2}{0.9^2} * 5 = \boxed{243.694\text{cm/s}^2}$$

5.1.4 Section (d)

We can calculate this using the velocity where there is no potential energy (where x = 0). This can only be true where $cos(\theta) = 0$, since the equivalent of θ is the only variable without a set value (yet). With the pythagorean theorem, if $cos(\theta) = 0$, $\sin(\theta) = \pm 1$, with either one working, so we will be using -1.

$$v = -\frac{16\pi}{0.900} * \sin\left(\frac{2\pi}{0.900}t\right) = -\frac{16\pi}{0.900} * (-1) = \frac{16\pi}{0.900}\text{cm/s} \tag{44}$$

$$E_{total} = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2} * (60.0\text{g}) * \left(\frac{16\pi}{0.900}\text{cm/s}\right)^2 + 0 \tag{45}$$

$$= \boxed{93578 \text{ dyn} * \text{cm} = 9.3578 \times 10^{-3}\text{J}} \tag{46}$$

6 Problem 6

A wire has a torsional constant $\kappa = 2.00\mathrm{N} * \mathrm{m/rad}$. A solid disk of radius $R = 5.00\mathrm{cm}$ and mass $M = 100\mathrm{g}$ is suspended at its center as shown in the figure. What is the frequency of torsional oscillations?

6.1 Soluton

7 Problem 8

A uniform rod of mass M and length $L = 1.20\text{m}$ oscillates about a horizontal axis at one end. What is the length of the simple pendulum that would have the same period? The rotational inertia is $\frac{ML^2}{3}$.

7.1 Solution

What we have here is a physical pendulum, and we want to compare it to a simple pendulum. We can create an equality. Since we know that the two values are equal, we don't have separate kinds of the variable T.

$$T = 2\pi\sqrt{\frac{L_s}{g}} \tag{47}$$

$$T = 2\pi\sqrt{\frac{I}{mgh}} = 2\pi\sqrt{\frac{ML_p^2}{3Mgh}} \tag{48}$$

To be clear, the first eqution is for a simple equation, while the second is for a physical pendulum.

$$I = I_{com} + MR^2 = \frac{1}{3}ML^2 + \frac{1}{4}ML^2 = \frac{7}{12}ML^2 \tag{49}$$

$$2\pi\sqrt{\frac{L_s}{g}} = 2\pi\sqrt{\frac{7ML_p^2}{12Mgh}} \tag{50}$$

$$\sqrt{\frac{L_s}{g}} = \sqrt{\frac{7L_p^2}{12gh}} \tag{51}$$

$$\frac{L_s}{g} = \frac{7L_p^2}{12gh} \tag{52}$$

$$L_s = \frac{7L_p^2}{12h} \tag{53}$$

To conclude this, we can know that the center of mass (h) of the uniform rod pendulum is going to be $h = \frac{L_p}{2}$.

$$L_s = \frac{7L_p^2}{12h} = \frac{14L_p^2}{12h} = \frac{14}{12}L_p = \frac{14}{12} * 1.20\text{m} = \boxed{1.4\text{m}} \tag{54}$$