Homework #15

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1 Problem 1

In unit-vector notation, what is the torque about the origin on a particle located at coordinates (0, -4.0 m, 3.0 m) if that torque is due to (a) force $\vec{F_1}$ with components $F_{1x} = 2.0$ N, $F_{1y} = F_{1z} = 0$, and (b) force $\vec{F_2}$ with components $F_{2x} = 0$, $F_{2y} = 2.0$ N, $F_{2z} = 4.0$ N?

1.1 Solution

In this instance, we can use the cross product for both parts.

1.1.1 Part (a)

$$\begin{aligned}
\tau = \vec{r} \times \vec{F} &= \begin{pmatrix} 0 \\ -4.0 \\ 3.0 \end{pmatrix} \times \begin{pmatrix} 2.0 \\ 0 \\ 0 \end{pmatrix} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -4.0 & 3.0 \\ 2.0 & 0 & 0 \end{vmatrix} && (1) \\
&= \begin{vmatrix} -4.0 & 3.0 \\ 0 & 0 \end{vmatrix} \hat{i} - \begin{vmatrix} 0.0 & 3.0 \\ 2.0 & 0 \end{vmatrix} \hat{j} + \begin{vmatrix} 0.0 & -4.0 \\ 2.0 & 0 \end{vmatrix} \hat{k} && (2) \\
&= 0\hat{i} + 6.0\hat{j} + 8.0\hat{k} = \boxed{\begin{pmatrix} 0 \\ 6.0 \\ 8.0 \end{pmatrix} \mathrm{N} * \mathrm{m}} && (3)
\end{aligned}$$

1.1.2 Part (b)

$$\begin{aligned}
\tau = \vec{r} \times \vec{F} &= \begin{pmatrix} 0 \\ -4.0 \\ 3.0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2.0 \\ 4.0 \end{pmatrix} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -4.0 & 3.0 \\ 0 & 2.0 & 4.0 \end{vmatrix} && (4) \\
&= \begin{vmatrix} -4.0 & 3.0 \\ 2.0 & 4.0 \end{vmatrix} \hat{i} - \begin{vmatrix} 0.0 & 3.0 \\ 0 & 4.0 \end{vmatrix} \hat{j} + \begin{vmatrix} 0.0 & -4.0 \\ 0 & 2.0 \end{vmatrix} \hat{k} && (5) \\
&= -22.0\hat{i} + 0\hat{j} + 0\hat{k} = \boxed{\begin{pmatrix} -22.0 \\ 0 \\ 0 \end{pmatrix} \mathrm{N} * \mathrm{m}} && (6)
\end{aligned}$$

2 Problem 2

At one instant, force $\vec{F} = 4.0\hat{j}$ N acts on a 0.25 kg object that has position vector $r = (2.0\hat{i} - 2.0\hat{k})$ m and velocity vector $v = (-5.0\vec{i} + 5.0\vec{k})$ m/s. About the origin and in unit-vector notation, what are (a) the object's angular momentum and (b) the torque acting on the object?

2.1 Solution

2.1.1 Section (a)

The angular momentum is equivalent to $\vec{\ell} = \vec{r} \times \vec{p}$.

$$\vec{\ell} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \times 0.25 \begin{pmatrix} -5 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} -\frac{5}{4} \\ 0 \\ \frac{5}{4} \end{pmatrix} \tag{7}$$

$$\vec{\ell} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -2 \\ -\frac{5}{4} & 0 & \frac{5}{4} \end{vmatrix} = \begin{vmatrix} 0 & -2 \\ 0 & \frac{5}{4} \end{vmatrix} \hat{i} - \begin{vmatrix} 2 & -2 \\ -\frac{5}{4} & \frac{5}{4} \end{vmatrix} \hat{j} + \begin{vmatrix} 2 & 0 \\ -\frac{5}{4} & 0 \end{vmatrix} \hat{k} \tag{8}$$

$$\vec{\ell} = 0\hat{i} + 0\hat{j} + 0\hat{k} = \boxed{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{kg} * \text{m/s}} \tag{9}$$

This means that the object is not moving around the origin at all, and is instead moving towards the origin.

2.1.2 Section (b)

We can use the torque-force-moment-arm equation.

$$\tau = \vec{F} \times \vec{r} = \begin{pmatrix} 0 \\ 4.0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2.0 \\ 0 \\ -2.0 \end{pmatrix} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 4.0 & 0 \\ 2.0 & 0 & -2.0 \end{vmatrix} \tag{10}$$

$$\tau = \begin{vmatrix} 4.0 & 0 \\ 0 & -2.0 \end{vmatrix} \hat{i} - \begin{vmatrix} 0 & 0 \\ 2.0 & -2.0 \end{vmatrix} \hat{j} + \begin{vmatrix} 0 & 4.0 \\ 2.0 & 0 \end{vmatrix} \hat{k} \tag{11}$$

$$\tau = -8.0\hat{i} - 0\hat{j} - 8.0\hat{k} = \boxed{\begin{pmatrix} -8.0 \\ 0 \\ -8.0 \end{pmatrix} \text{N} * \text{m}} \tag{12}$$

3 Problem 3

At the instant the displacement of a 2.00 kg object relative to the origin is $d = (2.00\text{m})\hat{\imath} + (4.00\text{m})\hat{\jmath} - (3.00\text{m})\hat{k}$, its velocity is $v = -(6.00\text{m/s})\hat{\imath} + (3.00\text{m/s})\hat{\jmath} + (3.00\text{m/s})\hat{k}$ and it is subject to a force $F = (6.00\text{N})\hat{\imath} - (8.00\text{N})\hat{\jmath} + (4.00\text{N})\hat{k}$. Find (a) the acceleration of the object, (b) the angular momentum of the object about the origin, (c) the torque about the origin acting on the object, and (d) the angle between the velocity of the object and the force acting on the object (See HW 3, problem 4).

3.1 Solution

3.1.1 Section (a)

We can use Newton's second law to find this.

$$\vec{a}_{com} = \frac{\vec{F}_{com}}{m} = \frac{(6.00\text{N})\hat{\imath} - (8.00\text{N})\hat{\jmath} + (4.00\text{N})\hat{k}}{2.00\text{kg}} = \boxed{\begin{pmatrix} 3.00 \\ -4.00 \\ 2.00 \end{pmatrix} \text{m/s}^2} \tag{13}$$

3.1.2 Section (b)

The angular momentum is equivalent to $\vec{\ell} = \vec{r} \times \vec{p}$.

$$\vec{\ell} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} \tag{14}$$

$$\vec{\ell} = \begin{pmatrix} 2.00 \\ 4.00 \\ -3.00 \end{pmatrix} \times 2.00 \begin{pmatrix} -6.00 \\ 3.00 \\ 3.00 \end{pmatrix} = \begin{pmatrix} 2.00 \\ 4.00 \\ -3.00 \end{pmatrix} \times \begin{pmatrix} -12.00 \\ 6.00 \\ 6.00 \end{pmatrix} \tag{15}$$

$$\vec{\ell} = \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2.00 & 4.00 & -3.00 \\ -12.00 & 6.00 & 6.00 \end{vmatrix} \tag{16}$$

$$\vec{\ell} = \begin{vmatrix} 4.00 & -3.00 \\ 6.00 & 6.00 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 2.00 & -3.00 \\ -12.00 & 6.00 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 2.00 & 4.00 \\ -12.00 & 6.00 \end{vmatrix} \hat{k} \tag{17}$$

$$\vec{\ell} = 42.00\hat{\imath} + 24.00\hat{\jmath} + 60.00\hat{k} = \boxed{\begin{pmatrix} 42 \\ 24 \\ 60 \end{pmatrix} \text{kg} * \text{m/s}} \tag{18}$$

3.1.3 Section (c)

We can use the torque-force-moment-arm equation.

$$\tau = \vec{r} \times \vec{F} = \begin{pmatrix} 2.00 \\ 4.00 \\ -3.00 \end{pmatrix} \times \begin{pmatrix} 6.00 \\ -8.00 \\ 4.00 \end{pmatrix} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2.00 & 4.00 & -3.00 \\ 6.00 & -8.00 & 4.00 \end{vmatrix} \tag{19}$$

$$\tau = \begin{vmatrix} 4.00 & -3.00 \\ -8.00 & 4.00 \end{vmatrix} \hat{i} - \begin{vmatrix} 2.00 & -3.00 \\ 6.00 & 4.00 \end{vmatrix} \hat{j} + \begin{vmatrix} 2.00 & 4.00 \\ 6.00 & -8.00 \end{vmatrix} \hat{k} \tag{20}$$

$$\tau = -8.00\hat{i} + -26.00\hat{j} + -40.00\hat{k} = \boxed{\begin{pmatrix} -8.00 \\ -26.00 \\ -40.00 \end{pmatrix} \mathrm{N} * \mathrm{m}} \tag{21}$$

3.1.4 Section (d)

We have two vector equations that we can use for finding an angle, one for the cross product $\left|\vec{a} \times \vec{b}\right| = ab\sin(\phi)$, and one for the dot product $\vec{a} \cdot \vec{b} = ab\cos(\phi)$. Since cosine is a continuous and inversible function between 0 and π, we will be using the dot product.

$$\vec{a} \cdot \vec{b} = ab\cos(\phi) \tag{22}$$

$$\cos(\phi) = \frac{\vec{a} \cdot \vec{b}}{ab} = \frac{\vec{F} \cdot \vec{v}}{Fv} \tag{23}$$

$$F = \left|\vec{F}\right| = \sqrt{6.0^2 + (-8.0)^2 + 4.0^2} \tag{24}$$

$$= \sqrt{36 + 64 + 16} = \sqrt{116} = 2\sqrt{29}\mathrm{N} \tag{25}$$

$$v = |\vec{v}| = \sqrt{(-6.0)^2 + 3.0^2 + 3.0^2} \tag{26}$$

$$= \sqrt{36 + 9 + 9} = \sqrt{54} = 3\sqrt{5}\mathrm{m/s} \tag{27}$$

$$\vec{F} \cdot \vec{v} = 6.0 * (-6.0) + (-8.0) * 3.0 + 4.0 * 3.0 \tag{28}$$

$$= -36 - 24 + 12 = -48\mathrm{N} * \mathrm{m/s} \tag{29}$$

$$\cos(\phi) = \frac{\vec{F} \cdot \vec{v}}{Fv} = -\frac{48\mathrm{N} * \mathrm{m/s}}{15\sqrt{29 * 5}\mathrm{N} * \mathrm{m/s}} \tag{30}$$

$$\phi = \arccos\left(-\frac{48}{15\sqrt{29 * 5}}\right) \approx \boxed{105.4^\circ} \tag{31}$$

4 Problem 4

The angular momentum of a flywheel having a rotational inertia of 0.140 kg $*$ m^2 about its central axis decreases from 3.00 to 0.800 kg $*$ m^2/s in 1.50 s. (a) What is the magnitude of the average torque acting on the flywheel about its central axis during this period? (b) Assuming a constant angular acceleration, through what angle does the flywheel turn? (c) How much work is done on the wheel? (d) What is the average power of the flywheel?

4.1 Solution

4.1.1 Section (a)

We can find the average torque with the inertia and the average acceleration.

$$\tau_{net} = \frac{d\ell}{dt} = \frac{0.800 - 3.00\text{kg} * \text{m}^2/\text{s}}{1.50\text{s}} = -\frac{2.20}{1.50}\text{N} * \text{m} = \boxed{-1.467\text{N} * \text{m}} \quad (32)$$

4.1.2 Section (b)

Assuming the listed angular momentum is the total angular momentum of the flywheel, we have the formula for the angular velocity and a formula for the angular acceleration from the torque. We can then apply those to a kinematic equation.

$$L = I\omega \qquad \tau_{net} = I\alpha \quad (33)$$

$$\omega^2 = \omega_0^2 + 2\alpha\Delta\theta \quad (34)$$

$$\omega_f = \frac{L_f}{I} \ \&\& \ \omega_i = \frac{L_f}{I} \ \&\& \ \alpha = \frac{\tau_{net}}{I} \quad (35)$$

$$\Delta\theta = \frac{\omega_f^2 - \omega_i^2}{2\alpha} = \frac{\frac{L_f}{I}^2 - \frac{L_i}{I}^2}{2\frac{\tau_{net}}{I}} = \frac{L_f^2 - L_i^2}{2\tau_{net}I} = \frac{0.800^2 - 3.00^2}{2 * (-\frac{22}{15}) * 0.140} \quad (36)$$

$$= \frac{900 - 64}{\frac{22}{15} * 28} = \frac{836 * 15}{22 * 28} = \frac{6270}{308} \approx \boxed{20.357} \quad (37)$$

Note: this is the radial measurement, not the angular measurement.

4.1.3 Section (c)