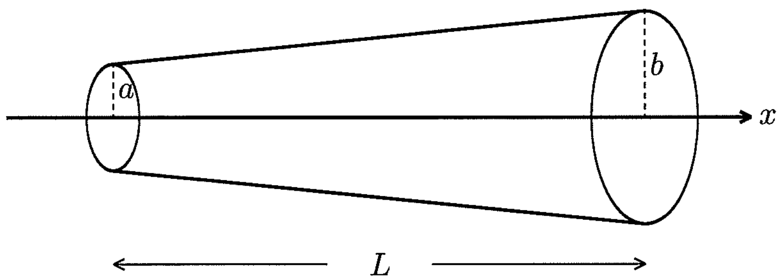
*. (10 points) The left face of a rod at $x = 0$ (shown below) is fixed at a temperature of $T(0)$ and the right face at $x = L$ is fixed at a temperature of $T(L)$. The radius varies from a to b uniformly as x varies from 0 to L. The material has a thermal conductivity k which is the same throughout the entire volume.



$\frac{dr}{dx} = \frac{b-a}{L}$

$r = a + \frac{b-a}{L} x$

a. (4 points) Show that the temperature profile is given by

$$T(x) - T(0) = -\frac{x}{k\pi a r}\frac{dQ}{dt}$$

where r is the radius at x. Hint: Recall that $dT/dx = (-1/k\pi r^2)dQ/dt$. This should be integrated over x, although the integral can be most easily done if you substitute $x \to r$.

$$\frac{dT}{dx} = -\frac{1}{kA}\frac{dQ}{dt} = -\frac{1}{k\pi r^2}\frac{dQ}{dt} \quad r(x) \to r$$

$$\Rightarrow T(x) - T(0) = \int_0^x -\frac{1}{k\pi r(x)^2}\frac{dQ}{dt}\,dx = \int_{r(0)\to a} -\frac{1}{k\pi r^2}\frac{dQ}{dt}\overbrace{\left[\frac{L}{b-a}dr\right]}^{dx}$$

$$= -\frac{1}{k\pi}\frac{L}{b-a}\frac{dQ}{dt}\int_a^r \frac{dr}{r^2} = -\frac{1}{k\pi}\frac{L}{b-a}\frac{dQ}{dt}\left(\frac{1}{a} - \frac{1}{r}\right)$$

$$= -\frac{1}{k\pi}\frac{L}{b-a}\frac{dQ}{dt}\frac{r-a}{ar} = -\frac{1}{k\pi a r}\frac{dQ}{dt}\boxed{L\frac{r-a}{b-a}} \leftarrow x$$

$$= -\frac{x}{k\pi a r}\frac{dQ}{dt} \quad \checkmark$$

b. (2 points) Determine the thermal conductance of the entire rod.

$$T(L) - T(0) = -\underbrace{\left(\frac{L}{k\pi a b}\right)}_{1/K}\frac{dQ}{dt} \Rightarrow \boxed{K = \frac{k\pi a b}{L}} \left(= \frac{k\pi R_{eff}^2}{L} \; ; \; R_{eff} = \sqrt{ab}\right)$$

c. (4 points) If $T(0) = 80°C$, $T(L) = 20°C$, and $b = 2a$, calculate $T(L/2)$.

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$$T(L/2) - T(0) = -\frac{L/2}{k\pi a \bar{r}}\frac{dQ}{dt} \quad \text{where } \bar{r} = r(L/2) = \tfrac{1}{2}(a+b)$$

$$\frac{T(L/2) - T(0)}{T(L) - T(0)} = \frac{L/2}{L}\cdot\frac{b}{\bar{r}} = \frac{1}{2}\frac{b}{\frac{1}{2}(a+b)} = \frac{b}{a+b} = \frac{2a}{a+2a} = \frac{2}{3}$$

$$\Rightarrow T(L/2) = T(0) + \frac{2}{3}(T(L) - T(0)) = 80°C + \frac{2}{3}(-60°C) = \boxed{40°C}$$