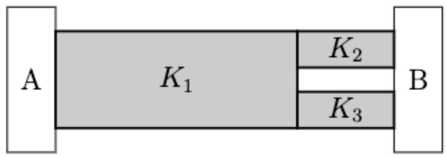
1 Problem 1

(8 points) System A is at 40°C and system B is at 0°C. The two systems are connected by a sequence of rods with conductances $K_1 = 100\text{W/K}$, $K_2 = 125\text{W/K}$ and $K_3 = 175\text{W/K}$, as shown below.



Calculate the rate of heat flow through each rod and the temperature in the middle where K_1 is connected to the parallel combination of K_2 and K_3.

1.1 Solution

The best move here would be to calculate the equivalent heat conductence of the entire system.

$$K_{23} = K_2 + K_3 \tag{1}$$

$$K_{\text{eq}} = \left(\frac{1}{K_1} + \frac{1}{K_2 + K_3}\right)^{-1} \tag{2}$$

This can be multiplied by the change in temperature to get the heat flow along the entire system.

$$\Delta T = 40\text{K} \tag{3}$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = K \cdot \Delta T = \left(\frac{1}{K_1} + \frac{1}{K_2 + K_3}\right)^{-1} * \Delta T \tag{4}$$

$$= \left(\frac{1}{100} + \frac{1}{125 + 175}\right)^{-1} * (40) = 3000\,\text{W} \tag{5}$$

This would equivalently be the rate at which the heat would flow through rod K_1, as well as the combination of rods two and three. It can also be first used to find the temperature at where K_1 is connected.

$$\Delta T = \frac{dQ/dt}{K_1} = \frac{3000}{100} = 30\,\text{K} \tag{6}$$

$$T_{\text{middle}} = 40°\text{C} - 30\,\text{K} = 10\,°\text{C} \tag{7}$$

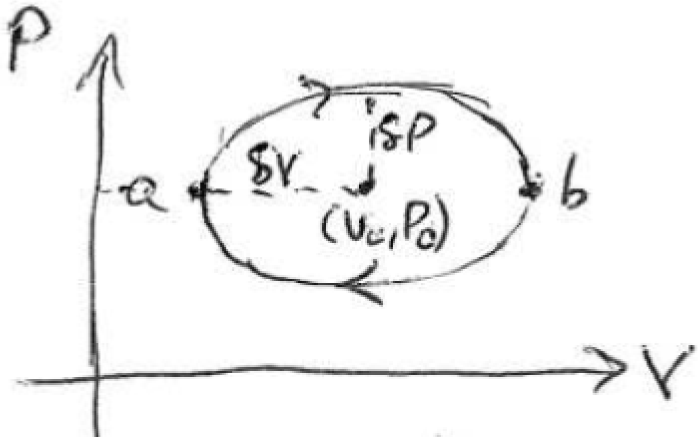
This equivalently makes ΔT between the middle and system B equal to $-10\,\mathrm{K}$. This in turn can be used to find the rate of heat flow along both K_2 and K_3.

$$\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)_2 = K_2 * \Delta T = 125 * (10) = 1250\,\mathrm{W} \tag{8}$$

$$\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)_3 = K_3 * \Delta T = 175 * (10) = 1750\,\mathrm{W} \tag{9}$$

2 Problem 2

2. (2 points) A particular system executes a cyclical process whose P-V graph is a clockwise-directed ellipse centered at (V_0, P_0) and radii δV and δP, respectively. Calculate the net work done and the net heat flow into/out of this system during one cycle.



2.1 Solution

We know that the process is cyclical, so the net change in internal energy will be zero. This, due to the first law of thermodynamics, means that the heat inserted is equal to the negative of the work done on the system, so we only have to calculate one of the two. To do this, we can set up an equation for the work done on the system.

$$W = -\int_{t_i}^{t_f} P(t)\,\frac{\mathrm{d}V(t)}{\mathrm{d}t}\,dt = -\int_a^b P(V)\,dV - \int_b^a P(V)\,dV \tag{10}$$

There is an equation for the ellipse.

$$\frac{(V(t)-V_0)^2}{\delta V^2} + \frac{(P(t)-P_0)^2}{\delta P^2} = 1 \tag{11}$$

We can solve this for $P(t)$ to get $P(V)$.

$$\frac{(P(t)-P_0)^2}{\delta P^2} = 1 - \frac{(V(t)-V_0)^2}{\delta V^2} \tag{12}$$

$$\frac{P(t)-P_0}{\delta P} = \pm\sqrt{1 - \frac{(V(t)-V_0)^2}{\delta V^2}} \tag{13}$$

$$P(t)-P_0 = \pm\delta P\sqrt{1 - \frac{(V(t)-V_0)^2}{\delta V^2}} \tag{14}$$

$$P(t) = P_0 \pm \delta P\sqrt{1 - \frac{(V(t)-V_0)^2}{\delta V^2}} \tag{15}$$

$$P(V) = P_0 \pm \delta P\sqrt{1 - \frac{(V-V_0)^2}{\delta V^2}} \tag{16}$$

This can be integrated. Looking above at the equation for work, the left integral can be given the positive square root, while the right integral can be given the negative square root.

$$W = -\int_a^b P(V)\,dV - \int_b^a P(V)\,dV \tag{17}$$

$$= -\int_a^b P_0 + \delta P\sqrt{1 - \frac{(V-V_0)^2}{\delta V^2}}\,dV - \int_b^a P_0 - \delta P\sqrt{1 - \frac{(V-V_0)^2}{\delta V^2}}\,dV \tag{18}$$

$$= \int_b^a P_0 + \delta P\sqrt{1 - \frac{(V-V_0)^2}{\delta V^2}}\,dV - \int_b^a P_0 - \delta P\sqrt{1 - \frac{(V-V_0)^2}{\delta V^2}}\,dV \tag{19}$$

$$= \int_b^a P_0 + \delta P\sqrt{1 - \frac{(V-V_0)^2}{\delta V^2}} - P_0 + \delta P\sqrt{1 - \frac{(V-V_0)^2}{\delta V^2}}\,dV \tag{20}$$

$$= \int_b^a \delta P\sqrt{1 - \frac{(V-V_0)^2}{\delta V^2}} + \delta P\sqrt{1 - \frac{(V-V_0)^2}{\delta V^2}}\,dV \tag{21}$$

$$= \int_b^a 2*\delta P\sqrt{1 - \frac{(V-V_0)^2}{\delta V^2}}\,dV = 2*\delta P\int_b^a \sqrt{1 - \frac{(V-V_0)^2}{\delta V^2}}\,dV \tag{22}$$

We can recall in this instance that $\frac{\mathrm{d}}{\mathrm{d}V}(V) = 1$, which we can insert into this.

$$W = 2*\delta P\int_b^a \frac{\mathrm{d}V}{\mathrm{d}V} * \sqrt{1 - \frac{(V-V_0)^2}{\delta V^2}}\,dV \tag{23}$$

Integration by parts is applicable here, with the following values in u and v.

$$\int u \frac{\mathrm{d}v}{\mathrm{d}x}\, dx = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x}\, dx \tag{24}$$

$$u = \sqrt{1 - \frac{\left(V - V_0\right)^2}{\delta V^2}} \qquad du = \frac{\sqrt{2(V - V_0)}}{2 * \delta V \sqrt{1 - \frac{(V - V_0)^2}{\delta V^2}}} \tag{25}$$

$$v = V \qquad dv = 1 \tag{26}$$

$$W = 2 * \delta P \int_b^a \sqrt{1 - \frac{\left(V - V_0\right)^2}{\delta V^2}}\, dV \tag{27}$$

$$= 2\,\delta P \left(V \sqrt{1 - \frac{\left(V - V_0\right)^2}{\delta V^2}} - \int_b^a V \frac{\sqrt{2(V - V_0)}}{2 * \delta V \sqrt{1 - \frac{(V - V_0)^2}{\delta V^2}}}\, dV \right) \tag{28}$$

We can focus on the integral in this equation.

$$\int_b^a V \frac{\sqrt{2(V - V_0)}}{2 * \delta V \sqrt{1 - \frac{(V - V_0)^2}{\delta V^2}}}\, dV \tag{29}$$

$$\frac{\sqrt{2}}{2 * \delta V} * \int_b^a V \frac{\sqrt{V - V_0}}{\sqrt{1 - \frac{(V - V_0)^2}{\delta V^2}}}\, dV \tag{30}$$

$$\tag{31}$$