Homework #10

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Problem 1

A 1.95-kg particle is projected with an initial speed of 4.00 m/s along a surface for which $\mu_k = 0.600$. Find the distance it travels given that: (a) the surface is horizontal; (b) the particle moves up a 30° incline; (c) the particle moves down a 30° incline.

Solution

Work is defined by $W = \vec{F} \cdot \vec{d}$. In this instance, we can use the frictional force to expand this into the kinetic frictional work using g = 9.81m/s^2. The friction force will also be antiparallel to the motion in this instance until the block stops.

$$\begin{aligned} W_k &= \vec{f_k} \cdot \vec{d} = f_k * d * \cos(180°) = -f_k * d \\ &= -\mu_k * F_N * d = -\mu_k * m * g * d \end{aligned}$$

We next can calculate the work done by gravity.

$$W_g = \vec{F_g} \cdot \vec{d} = F_g * d * \sin(\theta) = m * g * d * \sin(\theta)$$

We use the formula for the kinetic energy.

$$\begin{aligned} K &= \frac{1}{2}mv^2 \\ \Delta K &= K_f - K_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \end{aligned}$$

We combine these three to get a formula for the distance here. We should place special attention on the formula $K_f = K_i + W_\Sigma$.

$$\begin{aligned} K_f &= K_i + W_\Sigma = K_i + W_g - W_k \\ \frac{1}{2}mv_f^2 &= \frac{1}{2}mv_i^2 + m * g * d * \sin(\theta) - \mu_k * m * g * d \\ \frac{1}{2}(v_f^2 - v_i^2) &= d * g * (\sin(\theta) - \mu_k) \\ d &= \frac{v_f^2 - v_i^2}{2 * g * (\sin(\theta) - \mu_k)} \end{aligned}$$

We will call this last formula "Formula 1", not named after the racecar competition.

Section (a)

We apply Formula 1.

$$d = \frac{v_f^2 - v_i^2}{2 * g * (\sin(\theta) - \mu_k)} = \frac{(0\text{m/s})^2 - (4\text{m/s})^2}{2 * (9.81\text{m/s}^2) * (\sin(0^\circ) - 0.600)}$$
$$= \frac{-16\text{m}^2/\text{s}^2}{19.62\text{m/s}^2 * (-0.600)} = \frac{16\text{m}^2/\text{s}^2}{11.772\text{m/s}^2} = \boxed{1.359\text{m}}$$

Section (b)

We apply Formula 1.

$$d = \frac{v_f^2 - v_i^2}{2 * g * (\sin(\theta) - \mu_k)} = \frac{(0\text{m/s})^2 - (4\text{m/s})^2}{2 * (9.81\text{m/s}^2) * (\sin(30^\circ) - 0.600)}$$
$$= \frac{-16\text{m}^2/\text{s}^2}{19.62\text{m/s}^2 * (0.500 - 0.600)} = \frac{16\text{m}^2/\text{s}^2}{1.962\text{m/s}^2} = \boxed{8.155\text{m}}$$

Section (c)

We apply Formula 1.

$$d = \frac{v_f^2 - v_i^2}{2 * g * (\sin(\theta) - \mu_k)} = \frac{(0\text{m/s})^2 - (4\text{m/s})^2}{2 * (9.81\text{m/s}^2) * (\sin(-30^\circ) - 0.600)}$$
$$= \frac{-16\text{m}^2/\text{s}^2}{19.62\text{m/s}^2 * (-0.500 - 0.600)} = \frac{16\text{m}^2/\text{s}^2}{21.582\text{m/s}^2} = \boxed{0.741\text{m}}$$

Problem 2

Problem 3

A 1.75-kg block starts from rest at an initial height of 40.0 cm and slides down a frictionless circular ramp, as shown in the figure below. It slides for 79.0 cm on the horizontal surface before coming to a stop. What is the coefficient of kinetic friction? Assume that only the horizontal stretch has friction, and the curved portion is frictionless.

Solution

First, we find the velocity when the block switches from the frictionless to the frictional ground by using the conservation of energy since all energy on the frictionless part is conservative.

$$K_f - U_f = K_i - U_i$$

$$\frac{1}{2}mv_f^2 - m*g*y_f = \frac{1}{2}mv_i^2 - m*g*y_i$$

The final y-position is zero, as is the initial velocity.

$$\frac{1}{2}mv_f^2 = -m*g*y_i$$

$$v_f^2 = -2*g*y_i$$

We keep in mind that in this instance, $g = -9.81\text{m/s}^2$, so we need not here worry about the velocity being imaginary.
From here, we can use a kinetic energy and work formula to find the final kinetic energy. We here use that $F_N = -F_g = -mg$.

$$\begin{aligned} K_f &= K_i + W_k \\ \frac{1}{2}mv_f^2 &= \frac{1}{2}mv_i^2 - F_N*\mu_k*d = \frac{1}{2}mv_i^2 + m*g*\mu_k*d \\ g*\mu_k*d &= \frac{1}{2}(v_f^2 - v_i^2) \\ \mu_k &= \frac{v_f^2 - v_i^2}{2*g*d} = \frac{(0\text{m/s}^2 + 2*g*y_i)}{2*g*d} = \frac{y_i}{d} = \frac{40\text{cm}}{79\text{cm}} = \boxed{0.506} \end{aligned}$$

Problem 4

The potential energy shared by two atoms separated by a distance r in a diatomic molecule is given by the Lennard-Jones function (U_0 and r_0 are constants):

$$U(r) = U_0 \left[\left(\frac{r_0}{r} \right)^{12} - 2 \left(\frac{r_0}{r} \right)^{6} \right]$$

(a) Where is U(r) = 0 ? (b) Show that the minimum potential energy is $-U_0$ and that it occurs at r_0. (c) Where is $F_r = 0$? (d) Sketch U(r).

Solution

Section (a)

This is simple algebra, with U(r) = 0.

$$U(r) = U_0 \left[\left(\frac{r_0}{r} \right)^{12} - 2 \left(\frac{r_0}{r} \right)^{6} \right]$$

$$0 = U_0 \left[\left(\frac{r_0}{r} \right)^{12} - 2 \left(\frac{r_0}{r} \right)^{6} \right] = \left(\frac{r_0}{r} \right)^{12} - 2 \left(\frac{r_0}{r} \right)^{6} = \left(\frac{r_0}{r} \right)^{6} - 2$$

$$2 = \left(\frac{r_0}{r} \right)^{6} \rightarrow r^6 = \frac{r_0^6}{2} \rightarrow \boxed{r = \frac{r_0}{\sqrt[6]{2}}}$$

Section (b)

We can first calculate $\frac{d}{dr}U(r)$ and $\frac{d^2}{dr^2}U(r)$.

$$\begin{aligned}
\frac{d}{dr}U(r) &= \frac{d}{dr}\left(U_0 \left[\left(\frac{r_0}{r} \right)^{12} - 2 \left(\frac{r_0}{r} \right)^{6} \right] \right) \\
&= U_0 \frac{d}{dr} \left(r_0^{12} r^{-12} - 2 r_0^6 r^{-6} \right) = U_0 \left(-12 * r_0^{12} r^{-13} + 12 * r_0^6 r^{-7} \right) \\
&= 12 * U_0 \left(r_0^6 r^{-7} - r_0^{12} r^{-13} \right) \\
\frac{d^2}{dr^2}U(r) &= \frac{d}{dr} \left(12 * U_0 \left(r_0^6 r^{-7} - r_0^{12} r^{-13} \right) \right) = 12 * U_0 \left(13 r_0^{12} r^{-14} - 7 r_0^6 r^{-8} \right)
\end{aligned}$$

Next, we can calculate $U(r_0)$, $U'(r_0)$, and $U''(r_0)$, which would prove the claim, assuming that $U_0 \geq 0$.

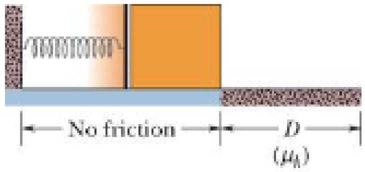
$$\begin{aligned}
U(r_0) &= U_0 \left[\left(\frac{r_0}{r_0} \right)^{12} - 2 \left(\frac{r_0}{r_0} \right)^{6} \right] = U_0 \left[1^{12} - 2 * 1^6 \right] = -U_0 \\
U'(r_0) &= 12 * U_0 \left(r_0^6 r_0^{-7} - r_0^{12} r_0^{-13} \right) = 12 * U_0 \left(r_0^{-1} - r_0^{-1} \right) = 0 \\
U''(r_0) &= 12 * U_0 \left(13 r_0^{12} r_0^{-14} - 7 r_0^6 r_0^{-8} \right) = 12 * U_0 \left(13 r_0^{-2} - 7 r_0^{-2} \right) \\
&= \frac{72 * U_0}{r_0^2} \geq 0
\end{aligned}$$

This concludes that $\boxed{\text{it is the minimum}}$.

Section (c)

The formula for force is $F(r) = -\frac{dU(r)}{dr}$.
We previously calculated that $\frac{dU(r)}{dr} = 12 * U_0 \left(r_0^6 r^{-7} - r_0^{12} r^{-13} \right)$ and that $U'(r) = 0$ at $r = r_0$.
This leads us to that $-U'(r) = -0 = 0$ at $r = r_0$, so $F(r) = -U'(r) = 0$ at $r = r_0$.

Problem 5



In the figure below, a 3.50 kg block is accelerated from rest by a compressed spring of spring constant 640 N/m. The block leaves the spring at the spring's relaxed length and then travels over a horizontal floor with a coefficient of kinetic friction $\mu_k = 0.25$. The frictional force stops the block in distance D = 7.80 m. What are (a) the increase in the thermal energy of the block-floor system, (b) the maximum kinetic energy of the block, and (c) the original compression distance of the spring?

Solution

Section (a)

We observe that the sole thermal energy would be frictional. We can apply the formula for that.

$$\Delta E_{th} = f_k d = F_N \mu_k d = mg\mu_k d = 3.50 * 9.81 * 0.25 * 7.80\text{J} = \boxed{66.95325\text{J}}$$

Section (b)

The fastest it would go would be at the intersection of the frictional and the frictionless surfaces. Since all the lost energy on the frictional surface is termal, we can use our answer in part (a) to answer this. The final kinetic energy is zero because it ends at a halt.

$$\begin{aligned}
\Delta K = -\Delta E_{th} &\rightarrow K_i - K_f = mg\mu_k d \\
\frac{1}{2}mv_i^2 = mg\mu_k d &\rightarrow v_i^2 = 2gd\mu_k \\
v_i = \sqrt{2gd\mu_k} &= \sqrt{2 * 9.81 * 7.80 * 0.25} = \boxed{6.1854\text{m/s}}
\end{aligned}$$

Section (c)

The compression distance can be computed using the kinetic energy and the conservation of energy.

$$U = K \rightarrow \frac{1}{2}kx^2 = \frac{1}{2}mv^2$$
$$x^2 = \frac{v^2}{k} = \frac{2gd\mu_k}{k}$$
$$x = \sqrt{\frac{2gd\mu_k}{k}} = \boxed{0.2445\text{m}}$$

Problem 6

The masses and positions of three particles in the xy plane are as follows: 2.05 kg at (-2.00, 3.00) m; 3.00 kg at (-3.00, 4.00) m, and; and 5.00 kg at (3.00, -1.00) m. What is the position of the CM?

Solution

We can set some values first.

$$
\begin{aligned}
m_1 &= 2.05\text{kg} & \vec{r}_1 &= \left(\begin{smallmatrix} -2.00\text{m} \\ 3.00\text{m} \end{smallmatrix}\right) \\
m_2 &= 3.00\text{kg} & \vec{r}_2 &= \left(\begin{smallmatrix} -3.00\text{m} \\ 4.00\text{m} \end{smallmatrix}\right) \\
m_3 &= 5.00\text{kg} & \vec{r}_3 &= \left(\begin{smallmatrix} 3.00\text{m} \\ -1.00\text{m} \end{smallmatrix}\right)
\end{aligned}
$$

We can then use the formula for center of mass.

$$
\begin{aligned}
\vec{r}_{com} &= \frac{\sum m * \vec{r}}{\sum m} = \frac{m_1 * \vec{r}_1 + m_2 * \vec{r}_2 + m_3 * \vec{r}_3}{m_1 + m_2 + m_3} \\
&= \frac{2.05\text{kg} * \left(\begin{smallmatrix} -2.00\text{m} \\ 3.00\text{m} \end{smallmatrix}\right) + 3.00\text{kg} * \left(\begin{smallmatrix} -3.00\text{m} \\ 4.00\text{m} \end{smallmatrix}\right) + 5.00\text{kg} * \left(\begin{smallmatrix} 3.00\text{m} \\ -1.00\text{m} \end{smallmatrix}\right)}{2.05\text{kg} + 3.00\text{kg} + 5.00\text{kg}} \\
&= \frac{\left(\begin{smallmatrix} -4.10\text{m} \\ 6.15\text{m} \end{smallmatrix}\right) + \left(\begin{smallmatrix} -9.00\text{m} \\ 12.00\text{m} \end{smallmatrix}\right) + \left(\begin{smallmatrix} 15.00\text{m} \\ -5.00\text{m} \end{smallmatrix}\right)}{10.05} = \frac{1}{10.05}\begin{pmatrix} 1.90\text{m} \\ 13.15\text{m} \end{pmatrix} = \boxed{\begin{pmatrix} 0.189\text{m} \\ 1.308\text{m} \end{pmatrix}}
\end{aligned}
$$

Problem 7

A block of mass m_1 = 2.00 kg has velocity $\vec{u}_1$ = 5.00 $\hat{\imath}$ - 3.00 $\hat{\jmath}$ + 4.00 $\hat{k}$ m/s and another block of mass m_2 = 6.00 kg has a velocity $\vec{u}_2$ = -3.00 $\hat{\imath}$ + 2.00 $\hat{\jmath}$ - 1.00 $\hat{k}$ m/s. (a) What is the velocity of the CM? (b) What is the total momentum of the system of two blocks?

Solution

For the sake of convenience, section(b) is done first, prior to section (a).

Section (b)

We add together the masses times the velocities.

$$\vec{p} = \Sigma m\vec{v} = m_1 * \vec{v}_1 + m_2 * \vec{v}_2 = 2.00\text{kg} * \begin{pmatrix} 5.00 \\ -3.00 \\ 4.00 \end{pmatrix} \text{m/s} + 6.00\text{kg} * \begin{pmatrix} -3.00 \\ 2.00 \\ -1.00 \end{pmatrix} \text{m/s}$$

$$= \begin{pmatrix} 10.00 \\ -6.00 \\ 8.00 \end{pmatrix} \text{kg} \cdot \text{m/s} + \begin{pmatrix} -18.00 \\ 12.00 \\ -6.00 \end{pmatrix} \text{kg} \cdot \text{m/s} = \boxed{\begin{pmatrix} -8.00 \\ 6.00 \\ 2.00 \end{pmatrix} \text{kg} \cdot \text{m/s}}$$

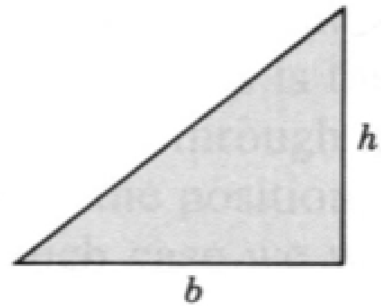
Section (a)

We divide the total momentum by the total mass.

$$\vec{v} = \frac{1}{m_1 + m_2} \begin{pmatrix} -8.00 \\ 6.00 \\ 2.00 \end{pmatrix} \text{kg} \cdot \text{m/s} = \frac{1}{8\text{kg}} \begin{pmatrix} -8.00 \\ 6.00 \\ 2.00 \end{pmatrix} \text{kg} \cdot \text{m/s} = \boxed{\begin{pmatrix} -1.00 \\ 0.75 \\ 0.25 \end{pmatrix} \text{kg} \cdot \text{m/s}}$$

Problem 8

Use integration to locate the CM of the triangular plate of base b and height h shown in the figure below. The plate has a uniform areal mass density σ (mass per unit area).



Solution

We know our equations for center of mass on the x and y axes. First, however, we calculate the formula for center of mass using area and mass density.

$$
\begin{aligned}
\frac{dm}{dx} &= x * \frac{h}{b} * \sigma \\
dm &= x * \frac{h}{b} * \sigma \ dx \\
x_{com} &= \frac{1}{M} \int x \ dm = \frac{1}{\sigma * \frac{1}{2} bh} \int_0^b x^2 * \frac{h}{b} * \sigma \ dx \\
&= \frac{2}{b^2} \int_0^b x^2 \ dx = \frac{2}{b^2} \left[\frac{x^3}{3} \right]_0^b = \frac{2 * b^3}{3 * b^2} = \frac{2}{3} b \\
\frac{dm}{dy} &= \left(b - y * \frac{b}{h} \right) * \sigma \\
dm &= \left(b - y * \frac{b}{h} \right) * \sigma \ dy \\
y_{com} &= \frac{1}{M} \int y \ dm = \frac{1}{\sigma * \frac{1}{2} bh} \int_0^h y * b * \left(1 - y * \frac{1}{h} \right) * \sigma \ dy \\
&= \frac{2}{h} \int_0^h y - \frac{y^2}{h} \ dx = \frac{2}{h} \left[\frac{y^2}{2} - \frac{y^3}{3h} \right]_0^h \\
&= \frac{2}{h} \left(\frac{h^2}{2} - \frac{h^2}{3} \right) = \frac{h}{3}
\end{aligned}
$$

This concludes that, relative to the bottom left corner at (0,0), the point of the center of mass is $\boxed{\left(\frac{2b}{3}, \frac{h}{3} \right)}$.