# Contents

# Chapter 1

## 1.1   Exercise 1.1

Explain the error in the following "proof" that $2 = 1$.
Let $x = y$. Then

$$x^2 = xy$$
$$x^2 + y^2 = xy - y^2$$
$$(x + y)(x - y) = y(x - y)$$
$$x + y = y$$
$$2y = y$$
$$2 = 1$$

### Solution

Between $(x + y)(x - y) = y(x - y)$ and $x + y = y$, we divide by $x - y$, which from the given, is equal to zero. Hence, the step involves dividing by zero, itself undefined and not allowed to be performed.

## 1.2   Exercise 1.2

Which of the following statements are true?  Give a short explanation for each of your answers?
(a) For every $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that $m > n$.
(b) For every $m \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that $m > n$.
(c) There is an $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $m \geq n$.
(d) There is an $n \in \mathbb{N}$ such that for every $m \in \mathbb{N}$, $m \geq n$.
(e) There is an $n \in \mathbb{R}$ such that for every $m \in \mathbb{R}$, $m \geq n$.
(f) For every pair $x < y$ of integers, there is an integer $z$ such that $x < z < y$.
(g) For every pair $x < y$ of real numbers, there is a real number $z$ such that $x < z < y$.

### 1.2.1   Solution (a) True

*Proof.* We begin with our given, which is that $n \in \mathbb{N}$. Natural numbers have a quality that they are ordered such that successive numbers are a distance of 1 apart. Natural numbers are also unlimited in the direction of positive infinity. This means that for every $n \in \mathbb{N}$, there is a greater natural number. In other words, for every $n \in \mathbb{N}$, there exists an $m \in \mathbb{N}$ such that $m > n$.  □

### 1.2.2   Solution (b) False

To prove this false, we need but find a counterexample. For the counterexample, we look at the number 1. $1 \in \mathbb{N}$. However, there is no $n \in \mathbb{N}$ such that $1 > n$. As such, it is false.

### 1.2.3   Solution (c) False

There is not maximum number in the natural numbers. For this to be true, it would have to contradict part (a), which has been proven. As such it is false.

### 1.2.4   Solution (d) True

The contrapositive of this is there is no $n \in \mathbb{N}$ such that for every $m \in \mathbb{N}$, $m < n$. Since there are infinitely many $\mathbb{R}$ with no upward bound, there will be no number $n \in \mathbb{N}$ such that $m < n$ for every $m \in \mathbb{N}$. This makes it true.

### 1.2.5 Solution (e) True

The contrapositive of this is there is no $n \in \mathbb{R}$ such that for every $m \in \mathbb{R}$, $m < n$. Since there are infinitely many $\mathbb{R}$ with no upward bound, there will be no number $n \in \mathbb{R}$ such that $m < n$ for every $m \in \mathbb{R}$. This makes it true.

### 1.2.6 Solution (f) False

If $x$ and $y$ are adjacent, like 0 and 1, then the number $z$ such that $x < z < y$ would have to be of distance less than 1 from the two adjacent numbers. This contradicts the principle of integers, which says that all integers are of distance 1 from their adjacent numbers. The statement is false.

### 1.2.7 Solution (g) True

There are considered to be infinitely many real numbers between two other real numbers. This means that for any $x, y \in \mathbb{R}$, there does exist $z \in \mathbb{R}$ such that $x < z < y$.

## 1.3 Exercise 1.3

If $A$ and $B$ are two boxes (possibly with things inside), describe the following in terms of boxes:

$$(a) A \setminus B \qquad\qquad (b) \mathcal{P}(A) \qquad\qquad (c) |A|$$

### 1.3.1 Solution (a)

$A \setminus B$ means the set of objects in A not in B. $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$. In terms of boxes, this would mean the box with everything that is in A but not in B.

### 1.3.2 Solution (b)

$\mathcal{P}(A)$ or the *power set* of A means the set of all subsets of A. $\mathcal{P}(A) = \{X : X \subseteq B\}$. In terms of boxes, this would be the box containing every box containing a combination of only different objects contained in $A$.

### 1.3.3 Solution (c)

$|A|$ or the cardinatity of A means the count of all elements in $A$. In terms of boxes, this would be the number of objects contained in the box $A$.

## 1.4 Exercise 1.4

If $A_1, A_2, A_3, \ldots, A_n$ are all boxes (possibly with things inside), describe the following in terms of boxes:

$$(a)\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n \qquad (b)\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

### 1.4.1 Solution (a)

A box containing all the objects in each of the boxes $A_1, A_2, A_3, \ldots, A_n$.

### 1.4.2 Solution (b)

A box containing each object that is in every one of the boxes $A_1, A_2, A_3, \ldots, A_n$.

## 1.5   Exercise 1.5

Prove that each of the following holds for any sets $A$ and $B$.
(a) $A \cup B = A$ if and only if $B \subseteq A$.
(b) $A \cap B = A$ if and only if $A \subseteq B$.
(c) $A \setminus B = A$ if and only if $A \cap B = \emptyset$.
(d) $A \setminus B = \emptyset$ if and only if $A \subseteq B$.

### 1.5.1   Solution (a)

We have to prove $A \cup B = A$ if and only if $B \subseteq A$. To prove this, we must prove that $A \cup B = A \implies B \subseteq A$ and $B \subseteq A \implies A \cup B = A$. We begin with the first.

*Proof.* We know that $A \cup B = A$. This means that for any $b \in B$ or $b \in A$, $b \in A$. Taking the first half of this, we get $b \in B \implies b \in A$. This can be rewritten as $B \subseteq A$. QED $\qquad\square$

Now we prove the second.

*Proof.* We know that $B \subseteq A$. This can be written that $b \in A$ for every $b \in B$, or $\frac{b \in B}{b \in A}$.
From identity, we also know that $b \in A$ for every $b \in A$, or $\frac{b \in A}{b \in A}$.
Putting these together, we get $b \in A$ for every $b \in A$ or $b \in B$, rewriten as $\frac{(b \in A) \vee (b \in B)}{b \in A}$.
Rewriting this in sets, we get $\frac{\{b : b \in A \text{ or } b \in B\}}{\{b : b \in A\}}$.
Putting this in implies and sets notation, we finally end up with $A \cup B = A$. QED $\qquad\square$

Putting these together, we get the if and only if statement we were looking for.

### 1.5.2   Solution (b)

*Proof.* (1) It is given that $A \subseteq B$. From this, we know from definitional that for any $a \in A$, $a \in B$.
$$a \in A \implies a \in B$$
.
(2) In what I call "The 30s conjecture" (iff a person is in their thirties then

they are in their thirties and are in their thirties), we can say that if and only if $a \in A$ and $a \in A$, then $a \in A$.

$$A = \{a : a \in A, a \in A\}$$

(3) From step (1), we can put in the implication into part (2) that if and only if $a \in A$ and $a \in B$, then $a \in A$.

$$A = \{a : a \in A, a \in B\}$$

(4) We can rewrite that as $A \cap B = A$. QED $\qquad \square$

### 1.5.3 Solution (c)

Part 1: $A \cap B = \emptyset \implies A \setminus B = A$

*Proof.* If $A \cap B = \emptyset$, that means that $a \notin B$ for any $a \in A$ and $a \notin A$ for any $a \in B$. It also means that $A$ and $B$ are disjoint.

$$a \in B \implies a \notin A \tag{1.1}$$
$$a \in A \implies a \notin B \tag{1.2}$$

We can also define the set $A \setminus B$.

$$A \setminus B = \{x : x \in A, x \notin B\} \tag{1.3}$$

We can set the falsehood to its equivalent.

$$A \setminus B = \left\{x : x \in A, \overline{x \in B}\right\} \tag{1.4}$$

The equivalent takes in (1.2).

$$A \setminus B = \left\{x : x \in A, \overline{x \notin A}\right\} \tag{1.5}$$

We can turn the double negative into a positive.

$$A \setminus B = \{x : x \in A, x \in A\} \tag{1.6}$$

We can then combine the two and compare then to the identity.

$$A \setminus B = \{x : x \in A\} \tag{1.7}$$
$$A = \{x : x \in A\} \tag{1.8}$$
$$A \setminus B = A \tag{1.9}$$

QED $\qquad \square$

Part 2: $A \setminus B = A \implies A \cap B = \emptyset$

*Proof.* Given: $A \setminus B = A$.

$$x \in A \wedge x \notin B \leftrightarrow x \in A \tag{1.10}$$

We define the set $A \cap B$.

$$A \cap B = \{x : x \in A, x \in B\} \tag{1.11}$$

We can substtute in the equivalence (1.10) into (1.11).

$$A \cap B = \{x : x \in A \wedge x \notin B, x \in B\} \tag{1.12}$$
$$A \cap B = \{x : x \in A, x \notin B, x \in B\} \tag{1.13}$$

This then gives us a contradiction. An element cannot both be in and not in set $B$. As such, we can equate the equivalence with an empty set.

$$A \cap B = \emptyset \tag{1.14}$$

QED $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

## 1.5.4   Solution (d)

Prove $A \setminus B = \emptyset$ if and only if $A \subseteq B$.

*Proof.* We can define the relationship $A \subseteq B$ and the set $A \setminus B$.

$$x \in A \implies x \in B \tag{1.15}$$
$$A \setminus B = \{x : x \in A, x \notin B\} \tag{1.16}$$

We can substitute (1.15) into (1.16).

$$A \setminus B = \{x : x \in B, x \notin B\} \tag{1.17}$$

This generates a contradiction, which we can use to turn the set into a null set.

$$A \setminus B = \emptyset \tag{1.18}$$

QED $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

## 1.6 Exercise 1.6

Suppose f: $X \to Y$ and $A \subseteq X$ and $B \subseteq Y$.
(a) Prove that $f(f^{-1}(B)) \subseteq B$.
(b) Give an example where $f(f^{-1}(B)) \neq B$.
(c) Prove that $A \subseteq f^{-1}(f(A))$.
(d) Give an example where $A \neq f^{-1}(f(A))$.

### 1.6.1 Solution (a)

*Proof.* We can set a reflexive truth involving $B$.

$$B = B \tag{1.19}$$

Since $f : X \to Y$ and $B \subseteq Y$, we can say that $B$ is part of the codomain of $f$. We can as such apply the inverse function of $f$ to both sides of the reflexive property.

$$f^{-1}(B) = f^{-1}(B) \tag{1.20}$$

Lastly, we can apply the function $f$ to both sides, which may cancel out on the right side. Since there may be some values in $b \in B$ for which $f^{-1}(B)$ is undefined or does not exist, we can no longer set it as an equality. However, se can set it as a subset equality.

$$f(f^{-1}(B)) \subseteq B \tag{1.21}$$

QED $\square$

### 1.6.2 Solution (b)

$$f(x) = \frac{1}{x} \tag{1.22}$$

### 1.6.3   Solution (c)

*Proof.* We will first define the function $f$ and the function $f^{-1} \circ f$.

$$f(A) = \{f(x)|x \in A\} \tag{1.23}$$
$$f^{-1}(f(A)) = \{f^{-1}(x)|x \in f(A)\} \tag{1.24}$$

From this, we ca assume the existence of a generic element $x$ that is an element of $A$.

$$x \in A \tag{1.25}$$

Since $x \in A$, we can then apply the function $f$ to both sides.

$$f(x) \in f(A) \tag{1.26}$$

We can then apply the definition of the image to this.

$$x \in f^{-1}(f(A)) \tag{1.27}$$

Since $x$ is a generic element of $A$, we have shown that every element of $A$ is in $f^{-1}(f(A))$.

$$A \subseteq f^{-1}(f(A)) \tag{1.28}$$

QED $\square$

### 1.6.4   Section (d)

$$f(x) = \sqrt{x}$$

## 1.7   Exercise 1.7

Suppose that $f\colon X \to Y$ and $g\colon Y \to X$ are functions and that the composite $g \circ f$ is the identity function id: $X \to X$. (The identity function sends every element to itself: $\mathrm{id}(x) = x$.) Show that $f$ must be a one-to-one function and that $g$ must be an onto function.

### 1.7.1   Solution

First, we have to prove that $f$ must be a one-to-one function.

*Proof.* First, we know that $g \circ f(x) = g(f(x)) = x$. Suppose that we have two values $x, y \in X$ and that they have the same value when the function $f$ is applied to them.

$$f(x) = f(y) \tag{1.29}$$

We can use the function $g$ on both sides of this.

$$g(f(x)) = g(f(y)) \tag{1.30}$$

Since $g \circ f = \mathrm{id}$, we replace this.

$$\mathrm{id}(x) = \mathrm{id}(y) \tag{1.31}$$

This is equivalent to our ending point.

$$x = y \tag{1.32}$$

Putting it all together, we get an implication.

$$f(x) = f(y) \implies x = y \tag{1.33}$$

Q.E.D. □

Now, we have to prove that $g$ is an *onto* function. As a reminder, a function $f : A \to B$ is onto (or surjective) if for every $b \in B$, there exists some $a \in A$ such that $f(a) = b$.

*Proof by Contradiction.* For contradiction, suppose that $\exists x \in X$ such that for $\forall y \in Y$, $g(y) \neq x$. The given information we have is that $g \circ f$ is the

identity function id. By its very nature, id is always bijective. From that, we can establish that it applies to our value $x$, as does its equivalent.

$$\exists x \in X : x = \text{id}(x) \tag{1.34}$$

$$\exists x \in X : x = g(f(x)) \tag{1.35}$$

We can apply this to the given inequality.

$$g(y) \neq x \tag{1.36}$$

$$g(y) \neq g(f(x)) \tag{1.37}$$

Since $f : X \to Y$ and $x \in X$, we can establish that $\exists y \in Y : y = f(x)$.

$$\exists y \in Y : f(x) = y \tag{1.38}$$

We can take the final contradiction assumption, creating new values appropriately and apply substitution to it. Since the value of $y$ can be any value in $Y$, that value could be $f(x)$, which is established to exist.

$$\exists x \in X : g(y) \neq x \tag{1.39}$$

$$\exists x \in X : g(f(x)) \neq x \tag{1.40}$$

$$\exists x \in X : \text{id}(x) \neq x \tag{1.41}$$

This means that $x$ is not equal to its identity function. This contradicts the sheer fact of $\text{id}(x) = x$. This shows that $\text{id}(x) = x$ can only be true if our initial assumption is false. This proves our point by contradiction.
Q.E.D. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

## 1.8 Exercise 1.8

The following are special cases of De Morgan's laws.
(a) Prove that $(A \cap B)^c = A^c \cup B^c$.
(b) Prove that $(A \cup B)^c = A^c \cap B^c$.

### 1.8.1 Solution (a)

Prove that $(A \cap B)^c = A^c \cup B^c$.

## 1.9  Exercise 1.37

(a) Determine
$$\{1, 3, 5\} \cdot \{-3, 0, 1\}$$

(b) Give an example of sets $A$ and $B$ where $\sup(A \cdot B) \neq \sup(A) \cdot \sup(B)$.