Contents

Chapter 1

1.1 Exercise 1.1

Explain the error in the following "proof" that $2 = 1$.
Let $x = y$. Then

$$\begin{aligned} x^2 &= xy \\ x^2 + y^2 &= xy - y^2 \\ (x+y)(x-y) &= y(x-y) \\ x + y &= y \\ 2y &= y \\ 2 &= 1 \end{aligned}$$

Solution

Between $(x+y)(x-y) = y(x-y)$ and $x + y = y$, we divide by $x - y$, which from the given, is equal to zero. Hence, the step involves dividing by zero, itself undefined and not allowed to be performed.

1.2 Exercise 1.2

Which of the following statements are true? Give a short explanation for each of your answers?
(a) For every $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that $m > n$.
(b) For every $m \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that $m > n$.
(c) There is an $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$, $m \geq n$.
(d) There is an $n \in \mathbb{N}$ such that for every $m \in \mathbb{N}$, $m \geq n$.
(e) There is an $n \in \mathbb{R}$ such that for every $m \in \mathbb{R}$, $m \geq n$.
(f) For every pair $x < y$ of integers, there is an integer z such that $x < z < y$.
(g) For every pair $x < y$ of real numbers, there is a real number z such that $x < z < y$.

1.2.1 Solution (a) True

Proof. We begin with our given, which is that $n \in \mathbb{N}$. Natural numbers have a quality that they are ordered such that successive numbers are a distance of 1 apart. Natural numbers are also unlimited in the direction of positive infinity. This means that for every $n \in \mathbb{N}$, there is a greater natural number. In other words, for every $n \in \mathbb{N}$, there exists an $m \in \mathbb{N}$ such that $m > n$. $\square$

1.2.2 Solution (b) False

To prove this false, we need but find a counterexample. For the counterexample, we look at the number 1. $1 \in \mathbb{N}$. However, there is no $n \in \mathbb{N}$ such that $1 > n$. As such, it is false.

1.2.3 Solution (c) False

There is not maximum number in the natural numbers. For this to be true, it would have to contradict part (a), which has been proven. As such it is false.

1.2.4 Solution (d) True

The contrapositive of this is there is no $n \in \mathbb{N}$ such that for every $m \in \mathbb{N}$, $m < n$. Since there are infinitely many $\mathbb{R}$ with no upward bound, there will be no number $n \in \mathbb{N}$ such that $m < n$ for every $m \in \mathbb{N}$. This makes it true.

1.2.5 Solution (e) True

The contrapositive of this is there is no $n \in \mathbb{R}$ such that for every $m \in \mathbb{R}$, $m < n$. Since there are infinitely many $\mathbb{R}$ with no upward bound, there will be no number $n \in \mathbb{R}$ such that $m < n$ for every $m \in \mathbb{R}$. This makes it true.

1.2.6 Solution (f) False

If x and y are adjacent, like 0 and 1, then the number z such that $x < z < y$ would have to be of distance less than 1 from the two adjacent numbers. This contradicts the principle of integers, which says that all integers are of distance 1 from their adjacent numbers. The statement is false.

1.2.7 Solution (g) True

There are considered to be infinitely many real numbers between two other real numbers. This means that for any $x, y \in \mathbb{R}$, there does exist $z \in \mathbb{R}$ such that $x < z < y$.

1.3 Exercise 1.3

If A and B are two boxes (possibly with things inside), describe the following in terms of boxes:

$$(a) A \setminus B \qquad\qquad (b) \mathcal{P}(A) \qquad\qquad (c)\ |A|$$

1.3.1 Solution (a)

$A \setminus B$ means the set of objects in A not in B. $A \setminus B = \{x : x \in A \text{and} x \notin B\}$. In terms of boxes, this would mean the box with everything that is in A but not in B.

1.3.2 Solution (b)

$\mathcal{P}(A)$ or the *power set* of A means the set of all subsets of A. $\mathcal{P}(A) = \{X : X \subseteq B\}$. In terms of boxes, this would be the box containing every box containing a combination of only different objects contained in A.

1.3.3 Solution (c)

$|A|$ or the cardinatity of A means the count of all elements in A. In terms of boxes, this would be the number of objects contained in the box A.

1.4 Exercise 1.4

If $A_1, A_2, A_3, \ldots, A_n$ are all boxes (possibly with things inside), describe the following in terms of boxes:

$$(a)\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n \qquad (b)\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

1.4.1 Solution (a)

A box containing all the objects in each of the boxes $A_1, A_2, A_3, \ldots, A_n$.

1.4.2 Solution (b)

A box containing each object that is in every one of the boxes $A_1, A_2, A_3, \ldots, A_n$.

1.5 Exercise 1.5

Prove that each of the following holds for any sets A and B.
(a) $A \cup B = A$ if and only if $B \subseteq A$.
(b) $A \cap B = A$ if and only if $A \subseteq B$.
(c) $A \setminus B = A$ if and only if $A \cap B = \emptyset$.
(d) $A \setminus B = \emptyset$ if and only if $A \subseteq B$.

1.5.1 Solution (a)

We have to prove $A \cup B = A$ if and only if $B \subseteq A$. To prove this, we must prove that $A \cup B = A \implies B \subseteq A$ and $B \subseteq A \implies A \cup B = A$. We begin with the first.

Proof. We know that $A \cup B = A$. This means that for any $b \in B$ or $b \in A$, $b \in A$. Taking the first half of this, we get $b \in B \implies b \in A$. This can be rewritten as $B \subseteq A$. QED $\square$

Now we prove the second.

Proof. We know that $B \subseteq A$. This can be written that $b \in A$ for every $b \in B$, or $\frac{b \in B}{b \in A}$.
From identity, we also know that $b \in A$ for every $b \in A$, or $\frac{b \in A}{b \in A}$.
Putting these together, we get $b \in A$ for every $b \in A$ or $b \in B$, rewriten as $\frac{(b \in A) \vee (b \in B)}{b \in A}$.
Rewriting this in sets, we get $\frac{\{b : b \in A \text{ or } b \in B\}}{\{b : b \in A\}}$.
Putting this in implies and sets notation, we finally end up with $A \cup B = A$.
QED $\square$

Putting these together, we get the if and only if statement we were looking for.

1.5.2 Solution (b)

Part 1: $A \subseteq B \implies A \cap B = A$

Proof. (1) It is given that $A \subseteq B$. From this, we know from definitional that for any $a \in A$, $a \in B$.

$$a \in A \implies a \in B$$

.

(2) In what I call "The 30s conjecture" (iff a person is in their thirties then they are in their thirties and are in their thirties), we can say that if and only if $a \in A$ and $a \in A$, then $a \in A$.

$$A = \{a : a \in A, a \in A\}$$

(3) From step (1), we can put in the implication into part (2) that if and only if $a \in A$ and $a \in B$, then $a \in A$.

$$A = \{a : a \in A, a \in B\}$$

(4) We can rewrite that as $A \cap B = A$. QED □

Part 2: $A \cap B = A \implies A \subseteq B$

Proof. Suppose that $A \cap B = A$. We can establish definitions for this.

$$A \cap B = \{x : x \in A \text{ and } x \in B\} \tag{1.1}$$

From the transistive property, we can continue.

$$A = \{x : x \in A \text{ and } x \in B\} \tag{1.2}$$

We can then establish the reflexive definition for the set A and apply that to the above equation with the transistive property.

$$A = \{x : x \in A\} \tag{1.3}$$

$$\{x : x \in A\} = \{x : x \in A \text{ and } x \in B\} \tag{1.4}$$

We can use this to define a generic element x.

$$x \in A \Leftrightarrow x \in A \text{ and } x \in B \tag{1.5}$$

$$x \in A \implies x \in A \text{ and } x \in B \tag{1.6}$$

This can be separated into two implications.

$$x \in A \implies x \in A \tag{1.7}$$

$$x \in A \implies x \in B \tag{1.8}$$

We can turn the second point into $A \subseteq B$.

Q.E.D. □

1.5.3 Solution (c)

Part 1: $A \cap B = \emptyset \implies A \setminus B = A$

Proof. If $A \cap B = \emptyset$, that means that $a \notin B$ for any $a \in A$ and $a \notin A$ for any $a \in B$. It also means that A and B are disjoint.

$$a \in B \implies a \notin A \tag{1.9}$$
$$a \in A \implies a \notin B \tag{1.10}$$

We can also define the set $A \setminus B$.

$$A \setminus B = \{x : x \in A, x \notin B\} \tag{1.11}$$

We can set the falsehood to its equivalent.

$$A \setminus B = \left\{x : x \in A, \overline{x \in B}\right\} \tag{1.12}$$

The equivalent takes in (1.2).

$$A \setminus B = \left\{x : x \in A, \overline{x \notin A}\right\} \tag{1.13}$$

We can turn the double negative into a positive.

$$A \setminus B = \{x : x \in A, x \in A\} \tag{1.14}$$

We can then combine the two and compare then to the identity.

$$A \setminus B = \{x : x \in A\} \tag{1.15}$$
$$A = \{x : x \in A\} \tag{1.16}$$
$$A \setminus B = A \tag{1.17}$$

QED □

Part 2: $A \setminus B = A \implies A \cap B = \emptyset$

Proof. Given: $A \setminus B = A$.

$$x \in A \land x \notin B \leftrightarrow x \in A \tag{1.18}$$

We define the set $A \cap B$.

$$A \cap B = \{x : x \in A, x \in B\} \tag{1.19}$$

We can substtute in the equivalence (1.10) into (1.11).

$$A \cap B = \{x : x \in A \land x \notin B, x \in B\} \tag{1.20}$$
$$A \cap B = \{x : x \in A, x \notin B, x \in B\} \tag{1.21}$$

This then gives us a contradiction. An element cannot both be in and not in set B. As such, we can equate the equivalence with an empty set.

$$A \cap B = \emptyset \tag{1.22}$$

QED $\square$

1.5.4 Solution (d)

Part 1: $A \subseteq B \implies A \setminus B = \emptyset$.

Proof. We can define the relationship $A \subseteq B$ and the set $A \setminus B$.

$$x \in A \implies x \in B \tag{1.23}$$
$$A \setminus B = \{x : x \in A, x \notin B\} \tag{1.24}$$

We can substitute (1.15) into (1.16).

$$A \setminus B = \{x : x \in B, x \notin B\} \tag{1.25}$$

This generates a contradiction, which we can use to turn the set into a null set.

$$A \setminus B = \emptyset \tag{1.26}$$

QED $\square$

Part 2: $A \setminus B = \emptyset \implies A \subseteq B$.

Proof by Contradiction. For the sake of contradiction, we assume that $A \nsubseteq B$. We also have as given, that $A \setminus B = \emptyset$.

$$A \setminus B = \emptyset \tag{1.27}$$
$$A \setminus B = \{x : x \in A, x \notin B\} \tag{1.28}$$

We can infer from this the existence of an element x.

$$x \in A \tag{1.29}$$
$$x \notin B \tag{1.30}$$

Based on the definition, it would follow that x would be an element of $A \setminus B$. We can apply a transistive property to this.

$$x \in A \setminus B \tag{1.31}$$

$$x \in \emptyset \tag{1.32}$$

Since $\emptyset$ has no elements, it contradicts its definition to say that x is an element of $\emptyset$. As such, our assumtion cannot be true. This valudates that $A \setminus B = \emptyset \implies A \subseteq B$.
Q.E.D. $\square$

1.6 Exercise 1.6

Suppose f: $X \to Y$ and $A \subseteq X$ and $B \subseteq Y$.
(a) Prove that $f(f^{-1}(B)) \subseteq B$.
(b) Give an example where $f(f^{-1}(B)) \neq B$.
(c) Prove that $A \subseteq f^{-1}(f(A))$.
(d) Give an example where $A \neq f^{-1}(f(A))$.

1.6.1 Solution (a)

Proof. We can set a reflexive truth involving B.

$$B = B \tag{1.33}$$

Since $f : X \to Y$ and $B \subseteq Y$, we can say that B is part of the codomain of f. We can as such apply the inverse function of f to both sides of the reflexive property.

$$f^{-1}(B) = f^{-1}(B) \tag{1.34}$$

Lastly, we can apply the function f to both sides, which may cancel out on the right side. Since there may be some values in $b \in B$ for which $f^{-1}(B)$ is undefined or does not exist, we can no longer set it as an equality. However, se can set it as a subset equality.

$$f(f^{-1}(B)) \subseteq B \tag{1.35}$$

QED □

1.6.2 Solution (b)

$$f(x) = \frac{1}{x} \tag{1.36}$$

1.6.3 Solution (c)

Proof. We will first define the function f and the function $f^{-1} \circ f$.

$$f(A) = \{f(x)|x \in A\} \tag{1.37}$$
$$f^{-1}(f(A)) = \left\{f^{-1}(x)|x \in f(A)\right\} \tag{1.38}$$

From this, we ca assume the existence of a generic element x that is an element of A.

$$x \in A \tag{1.39}$$

Since $x \in A$, we can then apply the function f to both sides.

$$f(x) \in f(A) \tag{1.40}$$

We can then apply the definition of the image to this.

$$x \in f^{-1}(f(A)) \tag{1.41}$$

Since x is a generic element of A, we have shown that every element of A is in $f^{-1}(f(A))$.

$$A \subseteq f^{-1}(f(A)) \tag{1.42}$$

QED $\square$

1.6.4 Section (d)

$$f(x) = \sqrt{x}$$

1.7 Exercise 1.7

Suppose that $f\colon X \to Y$ and $g\colon Y \to X$ are functions and that the composite $g \circ f$ is the identity function id: $X \to X$. (The identity function sends every element to itself: $\mathrm{id}(x) = x$.) Show that f must be a one-to-one function and that g must be an onto function.

1.7.1 Solution

First, we have to prove that f must be a one-to-one function.

Proof. First, we know that $g \circ f(x) = g(f(x)) = x$. Suppose that we have two values $x, y \in X$ and that they have the same value when the function f is applied to them.

$$f(x) = f(y) \tag{1.43}$$

We can use the function g on both sides of this.

$$g(f(x)) = g(f(y)) \tag{1.44}$$

Since $g \circ f = \mathrm{id}$, we replace this.

$$\mathrm{id}(x) = \mathrm{id}(y) \tag{1.45}$$

This is equivalent to our ending point.

$$x = y \tag{1.46}$$

Putting it all together, we get an implication.

$$f(x) = f(y) \implies x = y \tag{1.47}$$

Q.E.D. □

Now, we have to prove that g is an *onto* function. As a reminder, a function $f : A \to B$ is onto (or surjective) if for every $b \in B$, there exists some $a \in A$ such that $f(a) = b$.

Proof by Contradiction. For contradiction, suppose that $\exists x \in X$ such that for $\forall y \in Y$, $g(y) \neq x$. The given information we have is that $g \circ f$ is the identity function id. By its very nature, id is always bijective. From that, we can establish that it applies to our value x, as does its equivalent.

$$\exists x \in X : x = \mathrm{id}(x) \tag{1.48}$$

$$\exists x \in X : x = g(f(x)) \tag{1.49}$$

We can apply this to the given inequality.

$$g(y) \neq x \tag{1.50}$$

$$g(y) \neq g(f(x)) \tag{1.51}$$

Since $f : X \to Y$ and $x \in X$, we can establish that $\exists y \in Y : y = f(x)$.

$$\exists y \in Y : f(x) = y \tag{1.52}$$

We can take the final contradiction assumption, creating new values appropriately and apply substitution to it. Since the value of y can be any value in Y, that value could be $f(x)$, which is established to exist.

$$\exists x \in X : g(y) \neq x \tag{1.53}$$

$$\exists x \in X : g(f(x)) \neq x \tag{1.54}$$

$$\exists x \in X : \mathrm{id}(x) \neq x \tag{1.55}$$

This means that x is not equal to its identity function. This contradicts the sheer fact of $\mathrm{id}(x) = x$. This shows that $\mathrm{id}(x) = x$ can only be true if our initial assumption is false. This proves our point by contradiction.
Q.E.D. $\square$

1.8 Exercise 1.8

The following are special cases of De Morgan's laws.
(a) Prove that $(A \cap B)^c = A^c \cup B^c$.
(b) Prove that $(A \cup B)^c = A^c \cap B^c$.

1.8.1 Solution (a)

Prove that $(A \cap B)^c = A^c \cup B^c$.

Proof by Truth Table. We will have all the following conditions apply for whether a given element x is part of it.

A	B	$A \cap B$	$(A \cap B)^c$	A	B	A^c	B^c	$A^c \cup B^c$
T	T	T	F	T	T	F	F	F
T	F	F	T	T	F	F	T	T
F	T	F	T	F	T	T	F	T
F	F	F	T	F	F	T	T	T

Since the columns of $(A \cap B)^c$ and $A^c \cup B^c$ are the same for the same values of A and B in their respective columns, we can conclude that $(A \cap B)^c = A^c \cup B^c$.
Q.E.D. □

1.8.2 Solution (b)

Prove that $(A \cup B)^c = A^c \cap B^c$.

Proof by Truth Table. We will have all the following conditions apply for whether a given element x is part of it.

A	B	$A \cup B$	$(A \cup B)^c$	A	B	A^c	B^c	$A^c \cap B^c$
T	T	T	F	T	T	F	F	F
T	F	T	F	T	F	F	T	F
F	T	T	F	F	T	T	F	F
F	F	F	T	F	F	T	T	T

Since the columns of $(A \cup B)^c$ and $A^c \cap B^c$ are the same for the same values of A and B in their respective columns, we can conclude that $(A \cup B)^c = A^c \cap B^c$.
Q.E.D. □

1.9 Exercise 1.9

(a) Prove that $\sqrt{3}$ is irrational.
(b) What goes wrong when you try to adapt your argument from part (a) to show that $\sqrt{4}$ is irrational (which is absurd)?
(c) In part (a) you proved that $\sqrt{3}$ to be irrational, and essentially the same proof shows that $\sqrt{5}$ is irrational. By considering their product or otherwise, prove that $\sqrt{3}+\sqrt{5}$ and $\sqrt{3}-\sqrt{5}$ are either both rational or both irrational. Deduce that they must both be irrational.

1.9.1 Solution (a)

Proof that $\sqrt{3}$ is irrational.

Proof by Contradiction. Suppose $\sqrt{3}$ is rational. This would mean that for some relatively prime $p, q \in \mathbb{Z}$, $\sqrt{3} = \frac{p}{q}$. Square both sides and solve for p^2.

$$\sqrt{3} = \frac{p}{q} \tag{1.56}$$

$$3 = \frac{p^2}{q^2} \tag{1.57}$$

$$p^2 = 3q^2 \tag{1.58}$$

By Euclid's Lemma, we know that if 3 divides p^2, then 3 divides p. As such, we can substitute in $p = 3j$ for some $j \in \mathbb{Z}$. We can continue with prior steps.

$$(3j)^2 = 3q^2 \tag{1.59}$$

$$3q^2 = 9j^2 \tag{1.60}$$

$$q^2 = 3j^2 \tag{1.61}$$

This tells us that 3 divides q^2 and by Euclid's Lemma that 3 divides q. This means that 3 divides both p and q. This means that p and q are not relatively prime, which contradicts our assumption. Therefore, $\sqrt{3}$ has to be irrational.
Q.E.D. $\square$

1.9.2 Solution (b)

Proof.

$$\sqrt{4} = \frac{p}{q} \tag{1.62}$$

$$4 = \frac{p^2}{q^2} \tag{1.63}$$

$$p^2 = 4q^2 \tag{1.64}$$

Now, the issue we have here is that Euclid's lemma only works for prime numbers. Since 4 is not prime, we can only say that if 4 divides p^2, then $\sqrt{4} = 2$ divides p. Here is what happens if we try to substitute in $p = 2j$ for sone $j \in \mathbb{Z}$.

$$(2j)^2 = 4q^2 \tag{1.65}$$

$$4j^2 = 4q^2 \tag{1.66}$$

$$j^2 = q^2 \tag{1.67}$$

We can substitute this into our other equations. However, we would be going in circles and would be unable to arrive at a contradiction. This proves that we cannot prove that $\sqrt{4}$ is irrational the way we did for $\sqrt{3}$. □

1.9.3 Solution (c)

Proof. We first consider the product of the two.

$$(\sqrt{3} + \sqrt{5})(\sqrt{3} - \sqrt{5}) = 3 - 5 = -2 \tag{1.68}$$

□

1.10 Problem 1.10

Prove that the multiplicative identity in a field is unique.

1.10.1 Solution

1.11 Problem 1.11

Given an ordered field $\mathbb{F}$, recall that we defined the positive elements to be a nonempty subset $P \subseteq \mathbb{F}$ that satisfies both of the following conditions:

(i) If $a, b \in P$, then $a + b \in P$ and $a \cdot b \in P$;

(ii) If $a \in F$ and $a \neq 0$, then either $a \in P$ or $-a \in P$, but not both.

(a) Give an example of some $P_1 \subseteq R$ that satisfies (i) but not (ii).

(b) Give an example of some $P_2 \subseteq R$ that satisfies (ii) but not (i).

1.11.1 Solution (a)

1.11.2 Solution (b)

1.12 Exercise 1.12

Assume that $\mathbb{F}$ is an ordered field and $a, b, c, d \in \mathbb{F}$ with $a < b$ and $c < d$.
(a) Show that $a + c < b + d$.
(b) Prove that it is not necessarily true that $ac < bd$.
Note whenever you use an axiom.

1.12.1 Solution (a)

1.12.2 Solution (b)

1.13 Exercise 1.13

Let a, b and ε be elements of an ordered field.
(a) Show that if $a < b + \varepsilon$ for every $\varepsilon > 0$, then $a \leq b$.
(b) Use part (a) to show that if $|a - b| < \varepsilon$ for all $\varepsilon > 0$, then $a = b$.
Note whenever you use an axiom.

1.13.1 Solution (a)

1.13.2 Solution (b)

1.14 Exercise 1.37

(a) Determine

$$\{1, 3, 5\} \cdot \{-3, 0, 1\}$$

(b) Give an example of sets A and B where $\sup(A \cdot B) \neq \sup(A) \cdot \sup(B)$.