In contrast to this result, the next chapter will take up a property *of the circle which is not characteristic. We shall find that a whole* series of curves has this property, and we will see how the principle of the converse can lead to new concepts, in this case a remarkable class of curves.

25. Curves of Constant Breadth

1. A circle is defined as the curve all of whose points lie at a given distance from a fixed point, the center. The wheel is a direct practical application of this property of the circle. The hub of the wheel is held at a fixed height above the ground by the spokes of equal length, thus maintaining a smooth horizontal motion. In moving very heavy loads, the wheel and axle is sometimes not sufficiently strong. In this case one often resorts to the more primitive use of rollers. The load is merely rolled along over cylindrical rollers (Fig. 88) which are continually placed under the front. The load moves horizontally over these cylinders, whose cross sections are circles.

Obviously a wheel must be made in the form of a circle with the hub at the center, since any other form will produce an up-and-down motion. However, strange to say, it is not necessary that rollers have a circular cross section in order to perform their services properly. For rollers the *center* is no longer important. The property of the circle that allows it to be used for rollers is that every pair of parallel tangents is always at the same distance apart, no matter how the circle is turned. The circle has the same width

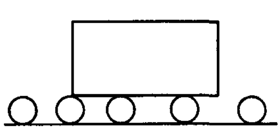

Fig. 88

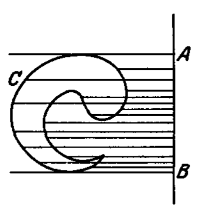

Fig. 89

in every direction; it is what is called a "curve of constant breadth". One might expect this property of the circle to be completely characteristic, as were the properties discussed in the last chapter.

But surprisingly, there are other curves with this same property. Indeed, there is a great multiplicity of curves of constant breadth which are not circles.

2. If we wish to determine the breadth of some curve C in a given direction, we can project each point of the curve perpendicularly on a line parallel to that direction (Fig. 89). The projection will fill up some segment AB of the line, and the length of this segment will be the breadth of the curve in the given direction.

The two lines of projection that are perpendicular to AB at A and B have at least one point in common with the curve C, while the entire curve lies on only one side of the line. We shall call a line with this property a 'supporting line' of the curve.

A closed curve has exactly two supporting lines in each direction. They can be found by the method of Fig. 89, or we can draw two parallel lines in the given direction, containing the curve between them, and then slide them together until they just touch the curve (Fig. 90).

A supporting line is not the same as a tangent line. In Fig. 91a the line t is a tangent at T, but is not a supporting line. In Fig. 91b s is a supporting line but is not tangent to the curve.

For a curve of constant breadth, the distance between every pair of supporting lines is a fixed amount b. If we draw two pairs of

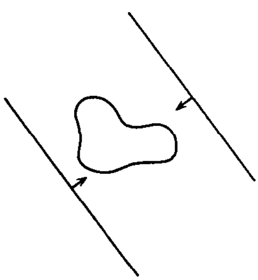

Fig. 90

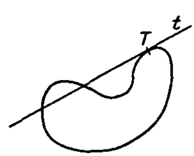

Fig. 91a

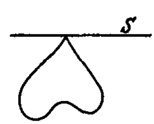

Fig. 91b

supporting lines to such a curve, the parallelogram which they form will be a rhombus (Fig. 92). If the pairs of supporting lines are perpendicular, the rhombus is a square of side b. Therefore all squares circumscribed about a curve of constant breadth are congruent. This can be nicely illustrated by cutting a piece of heavy cardboard into the shape of a figure of constant breadth and cutting a square hole in another piece of cardboard. If the square has sides equal to the breadth of the curve, then it will fit the curve no matter what the direction in which it is turned. The curved figure can be turned freely inside the square without ever having any room to spare. Both this and its converse are true: A curve of constant breadth can be rotated inside a square without any space to spare,

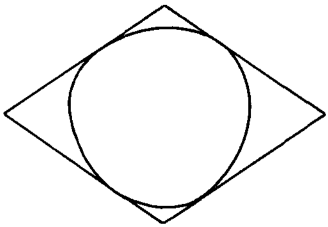

Fig. 92

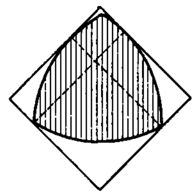

Fig. 93

and a curve that can be rotated inside a square is a curve of constant breadth.

3. The simplest curve of constant breadth that is not a circle is the curvilinear triangle pictured in Fig. 93. The three sides are equal arcs of circles, and the center of each arc is the opposite corner. The three arcs have equal radii which are equal to the constant breadth b of the curve. Of any two parallel supporting lines, one must touch at a corner and the other be tangent to the opposite side, or else both must touch at corners. In the first case, the distance between the supporting lines is clearly b. In the second case, each supporting line is tangent to the arc opposite the other corner, and their distance is again b.

This curvilinear triangle was first discovered, in the sense of a curve of constant breadth, by the technologist Reuleaux. He proved kinematically that this curve can be rotated inside a square without any space to spare. We have just seen that this property is characteristic of the curves of constant breadth.

4. The principle used to construct the Reuleaux triangle can be extended to figures with more sides. The essential idea is to draw a series of arcs of equal radii in such a way that the center of each arc is the opposite corner. We can start with any point B for the first corner and draw an arc with radius b and B as center. On this arc we choose two points A and C to be new corners. The arc of radius b with center C goes through B, since $BC = b$ by the previous construction. On this arc we choose another corner D. The arc of radius b, with D as center, goes through C. If we wish to end this process, we choose the corner E on this arc so that it is

also on an arc of radius b with center A. That is, E is the intersection of these two arcs. Finally, we join A and D by an arc with

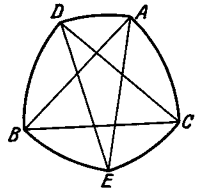

Fig. 94a

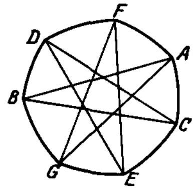

Fig. 94b

center E and obtain a curvilinear pentagon $ADBEC$ of constant breadth (Fig. 94a). Curvilinear polygons of more sides can be constructed in the same way by delaying the step at which the curve is closed. Fig. 94b is such a polygon of 7 sides. Since each corner is opposite a side which is an arc of radius b having the corner as its center, it immediately follows that this construction produces a curve of constant breadth b. For a later purpose we have joined each corner with the two ends of the opposite arc by means of radii. These radii form a self-intersecting polygon, all of whose sides are equal. The angle formed by each pair of radii through a corner is the central angle of the opposite arc.

All the curvilinear polygons constructed by this method will have an *odd* number of sides. To see this, we mark a corner and its opposite side. If we now pass around the curve starting at the marked corner, we will first pass a side, then a corner, and so on alternately until we pass a corner just before reaching the marked side. In all, we will have passed the same number, say n, of sides and corners in going from the marked corner to the marked side. Now if we start at the marked corner again and pass around the curve in the *opposite* direction, we will again pass n sides and n corners before reaching the marked side, since opposite each corner on the first path there is a side on the second path, and opposite each side there is a corner. Counting in the marked parts, there are then $2n + 1$ corners and the same number of sides.

5. The curves that we have constructed all have corners, that is, points where two sides meet at an angle. However, we can use these curves to obtain new curves of constant breadth that do not have any corners. Starting with one of our curves, we draw a curve parallel to it and outside it at a fixed distance d (Figs. 95a, b, c). This is easily done with the aid of the diagonal polygons

which we have drawn inside the curves. We merely replace each arc of the original curve by an arc having the same center, but

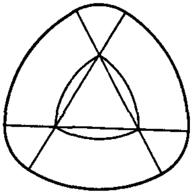

Fig. 95a

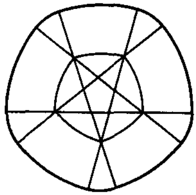

Fig. 95b

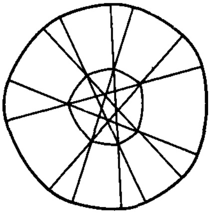

Fig. 95c

with the radius increased by *d*. The corners of the original figure are considered as arcs of radius 0, so they are replaced by arcs of radius *d*. The resulting figure is made up of an odd number of arcs of one radius and the same number of arcs of another radius. The arcs pair off, an arc of one radius being paired with one of the other radius having the same center (a corner of the original curve).

The same principle can be used to construct curvilinear polygons of constant breadth with arcs having radii of more than two different sizes. The opposite arcs are arranged so that they have the same center and so that their central angles form vertical angles (Fig. 96).

These methods allow us to construct an unlimited number of curves of constant breadth. However, these curves all have the

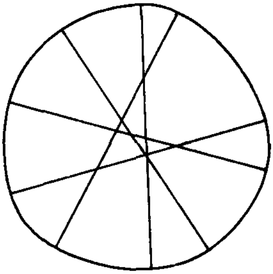

Fig. 96

special feature that they are formed of a number of circular arcs. In order to prevent a misunderstanding, we wish to emphasize that there are curves of constant breadth for which no part of the curve, no matter how small, is a circular arc.

6. Now that we have seen some examples of curves of constant breadth, we shall consider some of their general properties. In all of our examples the curves are convex curves, that is, curves which

have only two points in common with any line that cuts them. In order to simplify the discussion, we shall restrict our considerations to convex curves, and we shall always mean such curves even if we don't explicitly state that they are convex. As a matter of fact, this is really no restriction at all. It can be proved that every curve of constant breadth is convex. However, the proof would carry us too far afield, and we can avoid it by making the restriction.

Carefully defined, a convex curve is the boundary of a convex region. A convex region is characterized by the property that every two points of it can be joined by a straight line segment that is entirely in the region. Examples of convex regions are: a square, a circle, a triangle, an ellipse, and all the figures of constant breadth that we have mentioned. It is clear that a supporting line of a convex region will either have just one point or a whole segment in common with the boundary of the region. However, we shall prove the theorem:

Theorem I. *A curve of constant breadth has just one point in common with each of its supporting lines.*

Before proving this we make a simple observation:

Theorem II. *The distance between any two points on a curve of constant breadth b is at most equal to b.*

For if P and Q are two points on the curve (Fig. 97), then the two supporting lines perpendicular to the segment PQ must contain PQ between them. Therefore the distance between these lines is at least as large as the distance PQ. Since the distance between the supporting lines is b, the result is proved.

Turning to the proof of theorem I, we assume that it is false, that

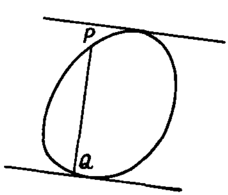

Fig. 97

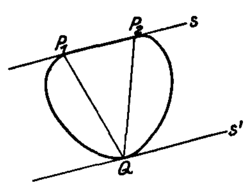

Fig. 98

two points P_1 and P_2 of the curve lie on the supporting line s (Fig. 98). We draw the supporting line s' parallel to s on the other side of the curve and let Q be a point of contact of s' and the curve. The distance between s and s' is again b.

The segments P_1Q and P_2Q cannot both be perpendicular to s, since the triangle P_1QP_2 cannot have two right angles. Consequently one of the segments is longer than b, but this contradicts theorem II. Therefore the assumption of two points of the curve on the supporting line is disproved and we have theorem I.

If we again use the fact that the perpendicular line joining two supporting lines has length b while any other joining line is longer, we immediately obtain the theorem:

Theorem III. *If a line joins the two points of contact of two parallel supporting lines of a curve of constant breadth, then it is perpendicular to the supporting lines.*

7. If we draw a circle of radius b about any point of the curve of constant breadth b as center, then, by II, the whole curve will be enclosed by the circle. We shall show that the curve cannot lie wholly in the interior of the circle, but that it must have at least one point on the circumference.

Let P be any point on the curve C of constant breadth b. With P as center we draw a circle K_1 (Fig. 99) which is large enough to enclose C but small enough to have a point Q of C on its circumference. The radius r of K_1 is at most equal to b, since the circle K

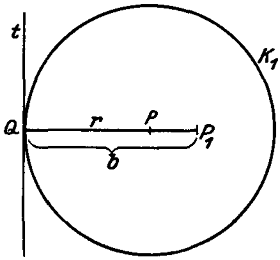

Fig. 99

of radius b and center P encloses C. Therefore K_1 is inside or at most identical with K.

The tangent t to the circle K_1 at Q goes through the point Q of C. Furthermore, C is enclosed by K_1 so it lies all on one side of t. Therefore t is a supporting line of C. The supporting line s, parallel to t on the other side of C, is at the distance b from t because C is of constant breadth b. According to theorem III, the point of contact P_1 of s is on the perpendicular to t through Q. If $r = b$, then P_1 falls on P; if $r < b$, then P lies between Q and P_1. But the latter is impossible. The three points Q, P, P_1 would belong to C and would lie on a line. Now a convex curve can be cut by a line in only two points. A line could have more than two points in common with a convex curve only if it were a supporting line. But we know, according to I, that a supporting line has only one point in common with a curve of constant breadth. Therefore P_1 must fall on P, and we have $r = b$.

In this proof P was an arbitrary point of C, and we have constructed the supporting line s of C through this point. Therefore we have also proved the result:

Theorem IV. *There is at least one supporting line through every point of a curve of constant breadth.*

A curve of constant breadth may have points at which there is more than one supporting line. Such points are called corners. In our earlier examples there were many curves of constant breadth having corners. At a corner, all the lines that lie in the angle formed by two supporting lines are clearly supporting lines themselves (Fig. 100). Therefore a convex curve has a whole *bundle* of supporting lines at each corner. Among these supporting lines there are two extreme ones that bound the bundle.

If P is an arbitrary point of C, then by theorem IV we can draw s, the (or a) supporting line of C, at this point. We draw the per-

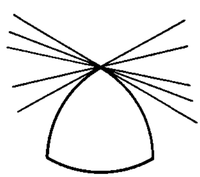

Fig. 100

pendicular to s through P. It will cut C in an opposite point Q, and PQ will have the length b. The circle of radius b with center Q will then enclose C and will have s as a tangent. This can be put in the form of the following theorem:

Theorem V. Through every point P of a curve of constant breadth, a circle of radius b can be drawn that encloses the curve and that is tangent, at P, to the supporting line of the curve, or to a predetermined supporting line if there are more than one.

8. The following theorem also relates to curves of constant breadth and circles:

Theorem VI. If a circle has three (or more) points in common with a curve of constant breadth b, then the length of the radius of the circle is at most b.

The Reuleaux triangle shows that such a circle may have a radius equal to b. If any one of the three arcs is extended to a full circle, this circle has radius b and has infinitely many points in common with the curve.

In order to prove theorem VI, we suppose that the circle k has the points P, Q, R in common with the curve C of constant breadth b. Of the three angles of the triangle PQR, there is at least one that is not exceeded by the other two, be it larger than both the others, equal to one and larger than the third, or, perhaps equal to both the others. We can suppose that this angle lies at P, and we shall call it α. Through P we now draw the (or a) supporting line of the curve of constant breadth. Then we draw the circle K of radius b, tangent to the supporting line at P and enclosing C. The points Q and R will lie inside or on the circumference of K. If both Q and R lie on the circumference, then K and k are identical, since there is only *one* circle that passes through three points P, Q, R. In this case there is nothing more to prove.

Otherwise we extend PQ and PR to their intersections, Q' and R', with K (Fig. 101). We now wish to prove that $Q'R'$ is longer than QR.

If Q and Q' happen to be the same point, then R and R' are different, since the case in which both Q and R lie on K has been settled. Here the triangles of Fig. 101 are related, as is shown in Fig. 102a. The angle $QRR' = \delta$ is an exterior angle of the triangle

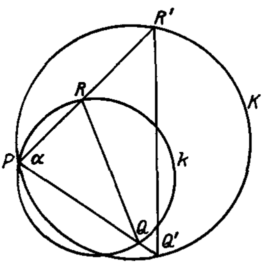

Fig. 101

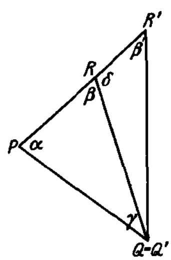

Fig. 102a

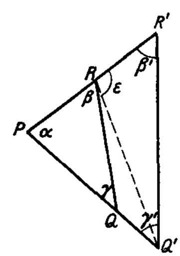

Fig. 102b

PQR. Now, by a theorem of elementary geometry, an exterior angle is greater then either of the two angles of the triangle which are not adjacent to it. In our case we have $\delta > \alpha$. Also, since β is an exterior angle of the triangle QRR', we have $\beta > \beta'$. Because we chose $\alpha \geqq \beta$ in the triangle PQR, we have $\delta > \alpha \geqq \beta > \beta'$, or $\delta > \beta'$. Therefore QR' is a side of the triangle QRR' which is opposite the angle δ, while QR is opposite the smaller angle β'. Then, by a well-known theorem of geometry, we have $QR' > QR$.

If Q differs from Q' and R differs from R', the triangles are related as in Fig. 102b. Because of the theorem concerning the sum of the angles in a triangle, we have $\beta + \gamma = 180° - \alpha = \beta' + \gamma'$. It is therefore impossible to have $\beta' > \beta$ and $\gamma' > \gamma$ at the same time. Let us suppose that it is the first of these inequalities which does not hold. We then have $\beta' \leqq \beta$. In the quadrilateral $QQ'R'R$ we draw the diagonal $Q'R$, the diagonal which does not divide the angle β'. (In the case $\gamma' \leqq \gamma$, we would draw QR'.) Designating the angle $Q'RR'$ by ε, we see that it is an exterior angle of the triangle $PQ'R$, and we have $\varepsilon > \alpha$. Furthermore, since we already have $\alpha \geqq \beta$ and $\beta \geqq \beta'$, we finally get $\varepsilon > \beta'$. Then, in the triangle $Q'R'R$, the side $Q'R'$ is opposite an angle greater than the angle opposite $Q'R$, and we have $Q'R' > Q'R$. Since our earlier argument can be used to show that we also have $Q'R > QR$, we finally obtain the desired result $Q'R' > QR$.

We have now obtained $Q'R' > QR$ for every case in which the circles k and K (Fig. 101) are different. Now α, the angle inscribed in the circle k and subtended by the chord QR, is also subtended in the circle K by the chord $Q'R'$. Therefore the chords QR and $Q'R'$ belong to the same central angle 2α in the circles k and K respectively. If we bring these central angles together, we obtain Fig. 103. Here we

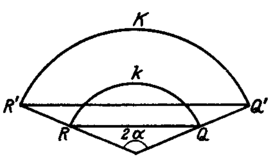

Fig. 103

recognize at once that the larger chord belongs to the larger circle, and therefore we see that the radius b of K is larger than the radius of k. This concludes the proof of theorem VI.

9. The simplest curve of constant breadth which is not a circle, the Reuleaux triangle, possesses corners. The following theorem shows that the Reuleaux triangle is outstanding among the curves of constant breadth because of its corners.

Theorem VII. A corner of a curve of constant breadth cannot be more pointed than 120°. *The only curve of constant breadth that has a corner of* 120° *is the Reuleaux triangle, which has three such corners.*

We measure the angle of a corner by means of the two extreme supporting lines of the bundle of supporting lines at the corner. If a corner Q has the angle ϑ, then the bundle of supporting lines occupies an angle of $180° - \vartheta$ (Fig. 104). The perpendiculars at Q to all these supporting lines form another bundle that occupies an angle of $180° - \vartheta$, the angle P_1QP_2. From theorem III we see that each of these perpendiculars crosses the curve at a point whose distance from Q is b .

Therefore the part of the curve opposite the corner Q is a circular arc of radius b and central angle $180° - \vartheta$. According to theorem II,

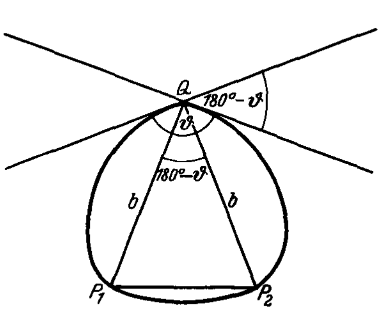

Fig. 104

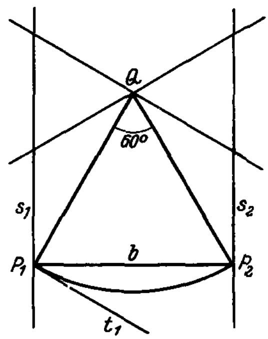

Fig. 105

the length of the chord P_1P_2 cannot exceed the width b. Then the isosceles triangle QP_1P_2 has legs of length b, and its base cannot exceed b in length. Therefore the angle P_1QP_2 is at most 60°. We have already seen that this angle P_1QP_2 is $180° - \vartheta$, so we have $180° - \vartheta \leqq 60°$, and hence $\vartheta \geqq 120°$. Since ϑ was the angle at an arbitrary corner, the first part of the theorem is proved.

Now if the corner angle ϑ is 120°, the angle P_1QP_2 is 60° and the isosceles triangle P_1QP_2 is equilateral (Fig. 105). Then P_1P_2 has the length b. Since this length is equal to the breadth of the curve, the two supporting lines perpendicular to P_1P_2 must pass through P_1 and P_2. From this we can see that P_1 and P_2 must also be corners of the curve. The part of the curve between P_1 and P_2 is an arc of a circle, as we have already seen. Not only is s_1 a supporting line at P_1, but so is the tangent t_1 to the circular arc at P_1. The inner angle between these two lines is easily seen to be 120°. Consequently s_1 and t_1 must be the extreme supporting lines of the bundle through P_1, since no corner can have an angle less than 120°. The corner at P_1 (and similarly at P_2) therefore has exactly the angle 120°. Then P_1 and P_2 have just the same properties as Q. Opposite each of them is a circular arc of radius b and central angle 60°. But this gives us exactly the Reuleaux triangle, and therefore the second part of theorem VII is proved.

10. In this chapter we first obtained some special curves of constant breadth by seeing how to construct them. Then we proved the general properties given by theorems I to VII. These properties hold for all convex curves of constant breadth, but they do not say anything about the existence of curves of constant breadth. We will now give a perfectly general construction that will yield every curve of constant breadth. This will give us a complete view of all possible curves of constant breadth. The property of our curves shown in theorem V is an especially important one. Curves of constant breadth are characterized by this property to the extent that one may arbitrarily choose one-half of such a curve between two opposite points, so long as it satisfies the conditions of theorem V. More accurately stated, we assert:

Theorem VIII. If a convex arc[1] *Γ has a chord of length b, if the entire arc lies between the two perpendiculars to the chord at its ends, and if it has the property of being enclosed by every circle of radius b tangent to a supporting line at its point of contact and lying on the same side of the line as the arc, then the curve can be extended to form a curve of constant breadth b.*

11. In proving this theorem it will be convenient to use the idea of *regions* of constant breadth. Since every region of constant breadth is bounded by a *curve* of constant breadth, we need only show the existence if a suitable region.

Before starting the proof we must make a remark about 'intersections' of regions. If a number of regions are given, then the part of the regions that is common to all of them is called their intersection. For example, the intersection of the two circles in Fig. 106 is the shaded region.

The intersection of an arbitrary set of convex regions is itself convex.

To prove this we must show that every two points of the intersection may be joined by a line segment that lies entirely within the intersection. But this is obvious. For if two points P and Q lie in the intersection, they lie in every region of the set. Then, because each region of the set is convex, the segment PQ is in all the regions. Since it is in all the regions, the segment PQ must also be in the intersection.

[1] That is, an arc which, together with its chord, bounds a convex region.

In this proof it is quite immaterial whether the set contains finitely or infinitely many convex regions.

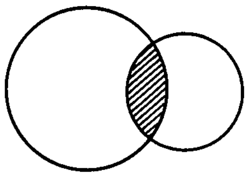

Fig. 106

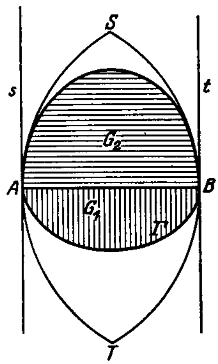

Fig. 107

12. We now suppose that the arc Γ (Fig. 107) has the properties required by theorem VIII: the chord AB has the length b. The arc, together with its chord, bounds the convex region G_1. The perpendiculars to AB at A and B are supporting lines of G_1. Every circle of radius b that is tangent to a supporting line of Γ at its point of contact encloses Γ.

To G_1 we add a new region ABS bounded by the chord AB, the circular arc AS with center B, and the circular arc BS with center A. We shall call this region G_2. The two convex regions G_1 and G_2 together form a convex region G which is bounded by Γ and the arcs AS and BS.

We now consider the totality of all circles of radius b whose centers lie on Γ. The region G and this infinite set of circles have a convex intersection D (the shaded part of the figure). We will now demonstrate that this region D is a region of constant breadth having the arc Γ as part of its boundary.

If Γ belongs to D it must certainly be on the boundary, since it is already on the boundary of G. But Γ is in all the circles of radius b with centers on Γ. To demonstrate that this is true, we need to show that no two points of Γ can be further apart than the distance b. Now, by assumption, Γ lies in both the circles of radius b with centers A and B, and therefore it lies in the curved figure $SATBS$. Since Γ lies on one side of AB, it must lie in the region G_2', which is the mirror image of G_2 in AB. The distance between any two points of G_2' is clearly at most b, so this must be true in particular for any two points on Γ. Therefore Γ belongs to D and it is a part of the boundary of D. Since D is convex, it must contain every chord joining two points of Γ. For this reason G_1 must be a part of D.

No two points of D can be further apart than the distance b. Since D is a part of G and G is made up of the two regions G_1 and G_2, we have three cases to consider: If both points are in G_1, then, as we mentioned before, they cannot be further apart than the distance b. If both points are in G_2 the same is true. Finally, if one point P_1 is in G_1 and the other point P_2 is in G_2, we can join P_1 with P_2 and extend the line until it cuts Γ, say at P. The three points will lie on this line in the order PP_1P_2. The circle of radius b with center P contains all of D, so it contains these three points. Consequently P_1 and P_2 lie on a radius of length b, and hence the distance between them cannot exceed b.

The result we have just obtained shows that the region D cannot have a breadth greater than b in any direction. We must now show that it has the breadth b in every direction. In the direction AB, the breadth b was prescribed by the theorem. We consider any other direction and draw the two supporting lines of D perpendicular to this direction. One of the two, say s_1, will have a point of contact Q on Γ. At Q we draw a perpendicular of length b to s_1 and call its end point M. Now M belongs to D. To prove this we must show that M is in G, as well as in every circle of radius b with center on Γ. The latter requires that we show that the distance of M from each point of Γ is at most b. This follows from the fact that the circle of radius b with center M is tangent to s_1 at Q. According to the assumptions, this circle must enclose the arc Γ, and this shows that the distance, from M to any point of Γ is not greater than b. Since in particular the distances AM and BM are not more than b, the point M must lie in the figure $SATBS$. Furthermore, since M lies on the opposite side of AB from Γ, it also lies in G_2. Therefore it lies in G as well as in all the circles. Consequently M lies in the intersection D.

Since QM is perpendicular to s_1 and s_2, and Q and M belong to D, the distance between the supporting lines s_1 and s_2 must be at least as large as $QM = b$. The distance cannot be greater than b, since this would mean that the distance between the two points of contact was greater than b, and we have seen that this is impossible for two points of D. Therefore the distance between s_1 and s_2 is exactly b; the region D has breadth b in every arbitrary direction.

This theorem shows that the arc Γ, satisfying certain requirements, can be extended to form a curve C of constant breadth b. It can easily be seen that there is only one way in which the extension can be made, and that the curve C is therefore uniquely determined.

13. In conclusion we mention, without proof, another remarkable property of these curves: All curves of constant breadth having the same breadth b have the same perimeter. This perimeter must obviously equal the circumference of a circle of diameter b. This fact can easily be verified for the examples constructed in §§ 4 and 5, making use of the similarity of circular arcs with the same central angle. However, the proof for general curves requires ideas and methods which are beyond the scope of this book. The proof can be started only after a very careful analysis of the concept of the length of a curve.

26. The Indispensability of the Compass for the Constructions of Elementary Geometry

1. The constructions of elementary geometry are all carried out with the aid of a straightedge and compass. In fact, a distinguishing property of elementary geometry is the fact that the only implements allowed are the compass and straightedge. But these two instruments are not entirely necessary. There are many constructions in which one or the other can be dispensed with. More than this, according to the investigations of Mascheroni and the recently found earlier work of Mohr, the straightedge can be dispensed with entirely. All constructions that are possible with a straightedge and compass can be made with a compass alone. Since a line cannot be drawn without a straightedge, in these investigations a line is considered as being constructed as soon as two of its points are found. On the other hand, Jacob Steiner has found that all the constructions of elementary geometry can be made using only a straightedge, provided only that *a fixed circle and its center* have been drawn beforehand. It is not difficult to prove that this fixed circle is indispensable. We shall prove this by showing that a *fixed circle whose center is unknown* is not sufficient to allow all the constructions to be carried out with a straitghtedge alone. Indeed, *two non-intersecting circles* with unknown centers will not suffice. However, it is known that two intersecting circles without their centers, or three non-intersecting circles, are sufficient to replace the Steiner circle with center.