Documenting Historical Mathematical Models

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1 Project Overview and Goals

This semester, our group analyzed a selection of the University of Illinois' historical collection of approximately 400 mathematical models, dating back to the late 1800s and early 1900s. Our work included examining wooden models and pieces from Schilling's Series 40, focusing on identifying models that lack existing documentation and providing mathematical descriptions for each. Additionally, we developed visual materials for a digital display in the university's digital library collection. As a result of the work, a selection of the models was showcased in an exhibit at the Library's Friends event on Friday, November 15, from 12:00 to 2:00 PM.

2 History

The University of Illinois owns one of the largest collections of historical mathematical models worldwide, comprising approximately 400 pieces from the late 19th and early 20th centuries. These include models purchased from Germany as well as those created by the faculty and students of the Department of Mathematics [3].

Figure 1: Part of the math models previously stored at Altgeld.

The renowned Mathematical Model Collection at the University of Illinois Urbana-Champaign, stored at the Altgeld Hall, is widely considered the largest of its type worldwide. Its origins trace back to Edgar Townsend, who began his role as an Assistant Professor of Mathematics during the 1893 Chicago World Columbian Exposition. There, he was inspired by Felix Klein's presentation of German mathematical models. After completing his doctoral studies under David Hilbert in Göttingen, Germany – a leading center for mathematical model production – Townsend returned to the University of Illinois. Dissatisfied with the existing models, he commissioned Arnold Emch in 1911 to create custom pieces unique to the collection.

Today, more than 380 of these models remain on display, illustrating complex geometric forms and topological properties across various mathematical disciplines. These include mechanisms and linkages for conics and trochoids, as well as demonstrations of problems in movement geometry [2].

Thirteen digitized model catalogs and related descriptive works are available in the digital library, which provide context and explanations of the Altgeld mathematical models collection. Most of these documents are in German and include illustrations and mathematical formulae that explain the phenomena described by each model. These resources are hosted on the UI Histories website. Several catalog covers are displayed, featuring different titles and decorative graphics, showcasing various mathematical models and formulae [3].

3 Wooden Models

3.1 Introduction

While the more advanced models were displayed in cases at Altgeld Hall, the Mathematics Library also housed thirty-eight wooden models in the stacks. These models are designed to help visualize a variety of geometric surfaces, illustrating different shapes and showcasing unique geometric features.

The collection includes **quadric surfaces** and **conic sections**, which are surfaces described by quadratic equations with three variables. Examples of quadric surfaces are ellipsoids and hyperboloids (both one-sheeted and two-sheeted), represented by models numbered 9, 10, 14, 21, 22, 23, 24, 25, 32, 34, and 35. **Prisms**, shown in models 1, 3, 4, 5, 16, 26, 28, and 30, are polyhedra with two parallel, congruent polygonal bases connected by rectangular lateral faces. **Cones**, represented in models 7, 15, 19, and 33, are three-dimensional shapes with a circular or elliptical base that narrows to a single point, known as the apex. Cones are important in geometry due to their connection to conic sections. The collection also includes **regular polyhedra**, such as pyramids, cubes and octahedra. Pyramids, seen in models 6, 11, 12, 18, 26, and 39, have sharp apexes and polygonal bases, with all non-base faces meeting at a single point. Cubes and other polyhedra with distinct symmetries are shown in models 17, 20, 36, 37, and 38. Additionally, the series features other elementary surfaces like **cylinders** (models 13, 27, 29, and 31), which consist of circular bases connected by a curved surface running parallel to the axis.

This varied collection provides tangible models of geometric shapes, which makes abstract mathematical concepts easier to understand. Historically, these models were available for circulation among professors and students, allowing them to investigate the geometric models with hands and eyes. Currently, during the renovation of Altgeld Hall, these models are being kept in the Mathematics Library office.

Figure 2: Photo of wooden models from the Mathematics Library.

3.2 Selected Models

We present here three examples from the wooden model series. A complete description of the entire collection will be made available through the online digital library.

3.2.1 Prolate Ellipsoid of Revolution (Wood 9)

The prolate ellipsoid of revolution is a fundamental geometric model represented by the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (a > 0,\ b > 0,\ c > 0)$$

This shape is characterized by two equal shorter axes and one elongated axis, creating a symmetry around its longest dimension. Its parametric equations:

$$
\begin{aligned}
x &= a \sin\theta \cos\phi, \\
y &= b \sin\theta \sin\phi, \\
z &= c \cos\theta
\end{aligned}
$$

provide a representation for both theoretical and practical applications. Prolate ellipsoids have their application in medical imaging, where their elongated form is ideal for modeling anatomical structures such as the human eye's lens or the kidney.

Photographer's number: wood 9

Tag on model:

Catalog:

Type of model: wood, from collection displayed in Math Library

Condition:

Other: Prolate Ellipsoid of Revolution

Location as of February 2023: Math Library in Altgeld Hall

Figure 3: Photo of Wood 9 with label from the Mathematics Library.

3.2.2 Right Cone with Conic Section (Wood 19)

The right cone is a classic geometric shape with an apex positioned directly above the center of its circular base. When the base is centered at the origin, the apex is located at $(0, 0, z)$, where $z \neq 0$. The cone is related to the geometric concept of conic sections–shapes formed when a plane intersects the surface of a cone.

This model is designed with removable pieces to illustrate the various conic sections. When a plane is parallel to the cone's circular base, the resulting conic section is a circle. If the plane is angled such that it intersects only the cone's lateral face, the section forms an ellipse. A plane parallel to one of the slant edges produces a parabola, while a plane cutting through both cones of a double right cone—two cones joined apex-to-apex with parallel bases—results in a hyperbola.

Figure 4: Photo of Wood 19 with label from the Mathematics Library.

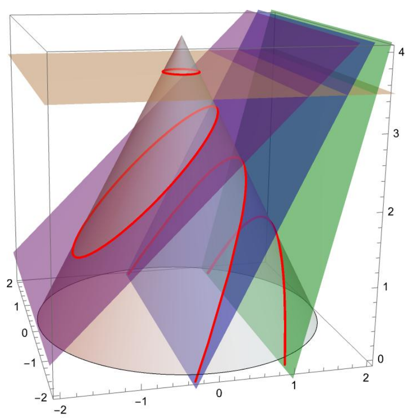

Figure 5: Illustration created with Mathematica.

3.2.3 Sphere with Parallels and Meridians (Wood 23)

A sphere with parallels and meridians is formed by revolving a circle about its diameter. The mathematical representation of the sphere is given by the equation:

$$x^2 + y^2 + z^2 = r^2,$$

where r denotes the radius of the sphere. This equation allows for straightforward calculations of volume and surface area, which are given by:

$$V = \frac{4}{3}\pi r^3, \quad A = 4\pi r^2.$$

Parallels are horizontal circles at fixed z-values, while meridians are vertical semicircles formed by fixing either the x- or y-coordinate. Together, parallels and meridians create a grid-like structure on the sphere's surface, akin to lines of latitude and longitude on Earth.

The sphere exhibits rotational symmetry about any axis through its center and reflectional symmetry across any plane that intersects its center. Cross-sections parallel to the equator form circles, with radii decreasing as the sections approach the poles.

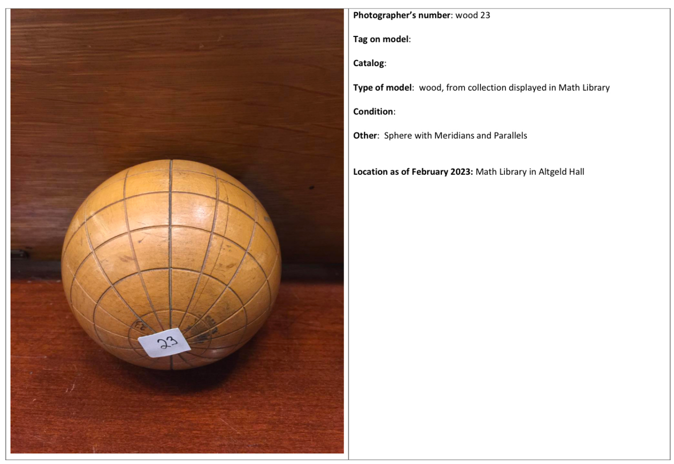

Figure 6: Photo of Wood 23 with label from the Mathematics Library.

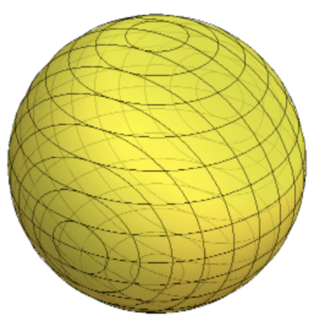

Figure 7: Illustration created with Mathematica.

3.2.4 Regular Dodecahedron (Wood 38)

The regular dodecahedron is one of the five Platonic solids, characterized by twelve congruent regular pentagonal faces, thirty equal edges, and twenty vertices.

Mathematically, the vertices of the dodecahedron can be expressed in terms of a scaling factor b and the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$. Assuming the center of the dodecahedron lies at the origin, the vertices are given by

$$b(\pm 1, \pm 1, \pm 1), \quad b(0, \pm\phi, \pm\frac{1}{\phi}), \quad b(\pm\frac{1}{\phi}, 0, \pm\phi), \quad \text{and} \quad b(\pm\phi, \pm\frac{1}{\phi}, 0).$$

The edge length be $a$is related to the scaling factor by $a = \frac{2b}{\phi}$.

The vertices $b(\pm 1, \pm 1, \pm 1)$ form a cube in centered at the origin. while the other vertices form rectangles on the coordinate planes: yz-, xz-, and xy-planes. The angle between two adjacent pentagonal faces is $2 \arctan \phi$. The surface area and volume of the dodecahedron can be calculated respectively as

$$A = a^2 \frac{15\phi}{\sqrt{3-\phi}} \quad \text{and} \quad V = a^3 \frac{5\phi^3}{6-2\phi}.$$

The dodecahedron is closely related to three spheres. The circumscribed sphere, which contains all the vertices, has a radius of $\frac{\phi\sqrt{3}}{2}a$. The inscribed sphere, which is tangent to all the faces, has a radius of $\frac{\phi^2}{2\sqrt{3-\phi}}a$. The midsphere, tangent to the midpoints of the edges, has a radius of $\frac{\phi^2}{2}a$. As part of the family of Platonic solids, the dodecahedron is related to the tetrahedron, cube, octahedron, and icosahedron. Its geometric elegance and intrinsic relationship to the golden ratio make it a central object of study in mathematics and related fields.

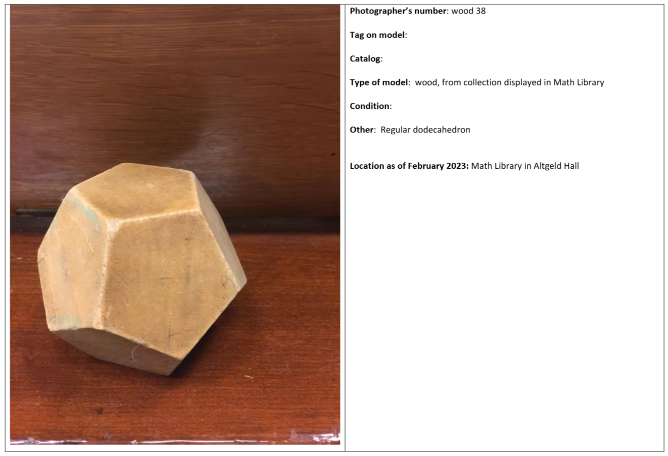

Figure 8: Photo of Wood 38 with label from the Mathematics Library.

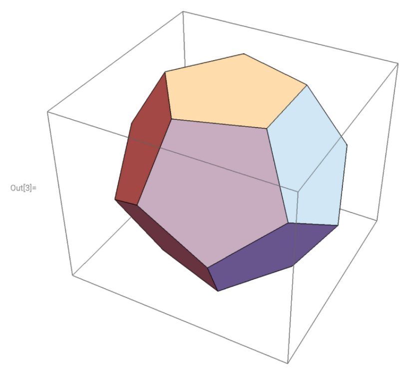

Figure 9: Illustration created with Mathematica.

4 Schilling Series 40 (Curves of Constant Width)

Photographer's number: 236

Tag on model:

Catalog: Schilling 1911 catalog ends with Series 40, Nr. 3, so this is not in it.

Type of model: The plates are metal.

Condition: good

Other: Tag translates as "Measuring apparatus for the models of Series 40, Nr. 4-19." These seem to be curves of constant width, and the base is simply for measuring. The figures drawn on them are related to how the curves (edges of the shapes) are defined/created. Series 40, Nr. 1-3 are 3-dimensional constant width bodies; these are the 2-dimensional versions.

Location as of February 2023: 1st floor Altgeld Hall, north side case

Figure 10: Photo of Schilling series 40 models.

4.1 Introduction

The models presented are based on the descriptions provided in the Schilling 1911 catalog, specifically Series 40 in [4, pp.106–107]. In the introductory section, we introduce three types of curves observed in the models: they are called C-curves, K-curves, and C'-curves [5].

4.1.1 Types of curves observed in the models

A ***C*-curve** is a *curve of constant width*, characterized by a constant distance between two parallel support lines regardless of rotation. In this collection, all the models have a constant width of 12 cm. Each C-curve fits between **parallel support lines**, with each line passing through a unique point on the curve. The distance between the lines equals the curve's width, and the collection contains a measuring frame that provides a physical representation of these lines. Additionally, C-curves possess infinitely many **double normals**, which are line segments perpendicular to the support lines and equal in length to the curve's width. In the models, the C-curve is depicted as the *outer curve*.

The ***K*-curve**, shown as the *black curve*, serves as the **evolute** of the C-curve which traces the path of its centers of curvature. Conversely, the C-curve is the **involute** of the K-curve. Tangents of fixed length (12 cm) centered on points of the K-curve trace the corresponding C-curve. When a tangent passes once around the K-curve, it undergoes a rotation of π radians.

Finally, the $\boldsymbol{C'}$**-curve**, represented by the *red curve*, is formed by tracing the centers of all double normals associated with a given C-curve. Together, these curves provide a comprehensive framework for understanding the geometric relationships in the models.

4.2 Constructions methods for curves of constant width

There are three ways to construct a C-curve (curve of constant width).

4.2.1 Construction using circles

To construct a normal Reuleaux triangle, start with a circle of any chosen radius and select a point on its circumference as the center of a second circle with the same radius. Repeat this process by choosing another point on the circumference of either circle to draw a third circle, also with the same radius. The key is that all three circles must have the same radius, but the triangle formed by connecting their centers does not need to be equilateral. This construction results in a Reuleaux triangle, a curve of constant width, defined by the arcs of these circles at their intersection points. This method allows for the creation of various shapes of Reuleaux triangles, depending on the placement of the centers.

4.2.2 Construction using tangent lines of the evolute

The construction of a curve of constant width involves using the involute of a deltoid (a type of tricuspoid) to generate an orbiform. The process begins with the deltoid, a curve with three cusps, which serves as the base for the construction. An involute is then created by unwinding a taut string along the deltoid's path. The path traced by the endpoint of the string forms the involute, which is central to the construction of the orbiform. Next, tangents are drawn at various points along the deltoid, as they define the boundaries of the constant-width curve. These tangential points generate the orbiform, a closed curve that maintains a constant width. This property is ensured because the sum of the radii from the tangential points to any given point on the curve remains the same. The distance between any two parallel tangents is consistently unchanged, confirming the constant width of the orbiform. [1]

4.2.3 Construction using a T-shape tool

To construct a C-curve from a convex arc with chord length b, first ensure that the arc lies between two support lines perpendicular to its chord. Confirm that for any circle of radius b, tangent to a support line and centered on the same side as the arc, the arc remains entirely within the circle. At each point on the arc, draw its support line and the perpendicular normal passing through that point. On the specified side of the support line, mark a point along the normal at a distance b. Repeat this process for all points along the arc. The resulting set of marked points, together with the original arc, forms a curve of constant width, satisfying the required geometric properties.

4.3 Selected Models

We present three examples from Schilling Series 40. A complete description of the entire collection will be made available through the online digital library. The original labels of the models are in German.

4.3.1 Model 4

This model is labeled as *a Reuleaux polygon composed of circular arcs.*

The Reuleaux triangle serves as the prototype for curves of constant width. It begins with an equilateral triangle, referred to as the K-curve, where the side length is 12 cm. To construct the Reuleaux triangle, fix one vertex of the equilateral triangle and create a circular arc between the extensions of the two sides adjacent to this vertex. This arc is centered at the chosen vertex, with a radius equal to the side length of the triangle. Repeat this process for the other two vertices, generating similar arcs centered at each vertex. The resulting closed curve is formed by three circular arc segments, collectively known as the C-curve.

Additionally, the C'-curve consists of three circular arcs centered at each vertex of the original K-curve.

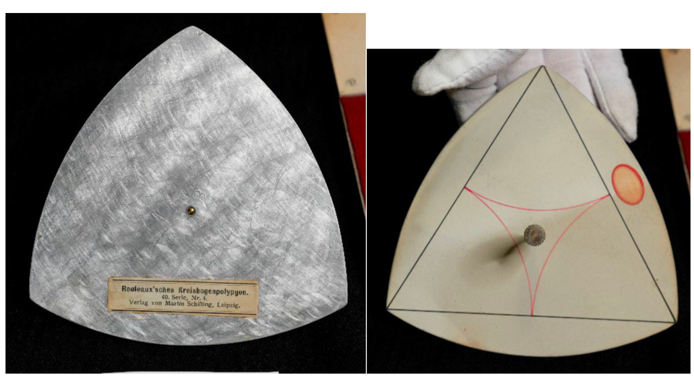

Figure 11: Photo of Schilling series 40 model 4.

4.3.2 Model 14

This model is labeled as *Outermost C-curves of an Isosceles Triangle with a Base Bent in the Middle, the Legs of Which are Equal to, Smaller Than, or Larger Than the Base.*

This curve of constant width is constructed from an isosceles triangle $U_1U_2U_3$ which we refer to as the K-curve. In this case, the equal legs U_1U_2 and U_1U_3 are longer than the base U_2U_3, which is bent at its midpoint M', creating a piecewise-linear segment. The vertices of the triangle are

$$U_1 = (0, h), U_2 = (-a, 0) \text{ and } U_3 = (a, 0), \text{ with } h > a$$

ensuring the legs are longer than the base. The midpoint $M' = (0, -b)$, where $b > 0$, introduces symmetry to the construction.

The C-curve is composed of seven smooth circular arcs, which are formed by creating sectors of varying radii starting at the vertices of the triangle either from the inside or outside of the triangle.

The C'-curve is composed of three circular arcs that intersect at tangent points. The center of each circle in the C'-curve is located at one of the triangle's vertices. The circles formed by the bottom two vertices have congruent radii, as M' is directly above the midpoint of the triangular base. The circle centered at the upper vertex has a larger radius than the other two, fulfilling the properties of an isosceles triangle. The radii of these arcs are chosen such that the resulting boundary maintains the constant-width property, meaning the distance between parallel tangents remains equal at any orientation. The final result is a closed, symmetric, and smooth shape with constant width, geometrically constructed from the given triangle.

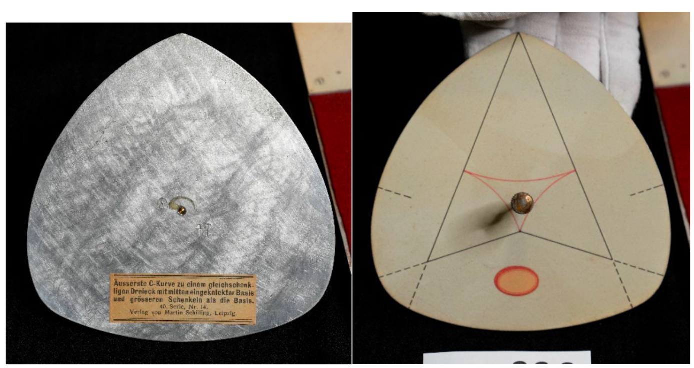

Figure 12: Photo of Schilling series 40 model 14.

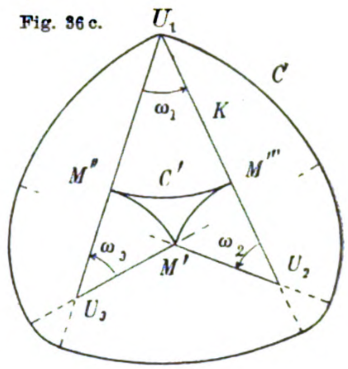

Figure 13: Model 14. Illustration given in [4]

4.3.3 Model 18

This model is labeled as *Outermost C-curve of a Special Equiangular Star.*

The K-curve for this model consists of a regular star with two intersecting edges extended to form the legs of an isosceles triangle. The triangle is closed by a base measuring 12 cm, and all edges of the star are extended to 12 cm with dashed lines, creating a cohesive framework for the design.

The C-curve is made up of eight circular arcs. These arcs are generated using the circular arc method, where each arc corresponds to a segment of a circle. The center of each circle is located at a corner of the K-curve, and the radius of each arc is determined by a related line segment on the curve. The C-curve inherits reflectional symmetry from the K-curve.

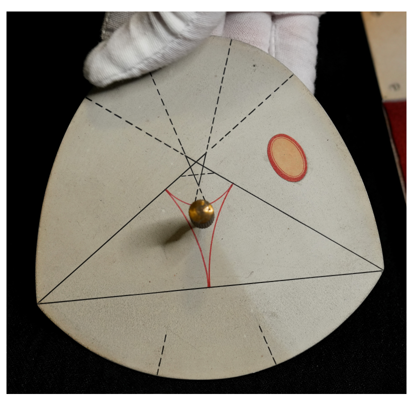

Figure 14: Photo of Schilling series 40 model 18.

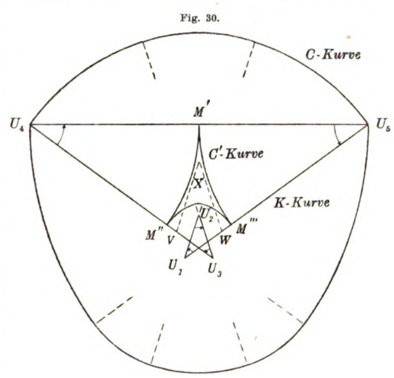

Figure 15: Model 18. Illustration given in [4]

4.3.4 Model 19

This model is originally labeled as *C-curve with two elliptical quadrants to a K-curve consisting of its evolute arcs and two segments.*

The model represents a curve of constant width constructed from two ellipses with equations given by

$$\begin{cases} x = a_1 \cos t \\ y = b_1 \sin t \end{cases} \text{and} \quad \begin{cases} x = a_2 \cos s \\ y = b_2 \sin s \end{cases},$$

where $t \in \left[\pi, \dfrac{3\pi}{2}\right]$, $s \in \left[\dfrac{3\pi}{2}, 2\pi\right]$, $a_1 > b_1$, $a_2 > b_2$, and $b_1 = b_2$. The sum of the semi-major axes of the two ellipses is 12 cm, that is, $a_1 + a_2 = 12$cm.

The C-curve is divided into three regions by taking the intersecting point of the horizontal line and the vertical line as the origin. Quadrant III is a quarter ellipse with major axis $2a_1$ and minor axis $2b_1$, and Quadrant IV is a quarter ellipse with major axis $2a_2$ and minor axis $2b_2$. The C-curve in Quadrants I and II is constructed by taking each point on the two quarter ellipses in Quadrants III and IV and extending it along the direction of the normal vector toward the concave side of the ellipse. Each point is extended by 12 cm along this normal direction to create a corresponding point in Quadrants I and II.

The *K-curve* consists of four parts: The curve in Quadrant II is the evolute of the quarter ellipse in Quadrant III. The curve in Quadrant I is the evolute of the quarter ellipse in Quadrant IV. Respectively, the equations are

$$\begin{cases} x = \dfrac{a_1^2 - b_1^2}{a_1} \cos^3 t \\ y = \dfrac{b_1^2 - a_1^2}{b_1} \sin^3 t \end{cases} \text{and} \quad \begin{cases} x = \dfrac{a_2^2 - b_2^2}{a_2} \cos^3 s \\ y = \dfrac{b_2^2 - a_2^2}{b_2} \sin^3 s \end{cases},$$

where $t \in \left[\pi, \frac{3\pi}{2}\right]$, $s \in \left[\frac{3\pi}{2}, 2\pi\right]$. The rest two parts of the K-*curve* are two line segments: a horizontal line segment connecting $\left(-\frac{a_1^2 - b_1^2}{a_1}, 0\right)$ and $\left(\frac{a_2^2 - b_2^2}{a_2}, 0\right)$, and a vertical line segment connecting $\left(0, \frac{a_1^2 - b_1^2}{b_1}\right)$ and $\left(0, \frac{a_2^2 - b_2^2}{b_2}\right)$.

In the figure below, $\widehat{P_1 P_1'}$ and $\widehat{P_2 P_1'}$ are two quarter ellipses from two distinct ellipses. These ellipses share semi-minor axes of equal length, and the sum of their semi-major axes is $12cm$, i.e., $|P_1 P_2| = |P_1 N| + |N P_2| = 12$cm.

To continue constructing the curve, we apply the "T-shape method". For each point on $\widehat{P_1 P_1' P_2}$, we draw the tangent line and the normal vector pointing toward the concave side of the two quarter ellipses. This arrangement creates the "T-shape." Each point is then extended by 12 cm along the normal vector, resulting in a new point. By repeating this process for all points on $\widehat{P_1 P_1' P_2}$, we generate the curve $\widehat{P_1 P_2' P_2}$. When combined with the two quarter ellipses, this process produces a closed, smooth curve of constant width.

Figure 16: Photo of Schilling series 40 model 19.

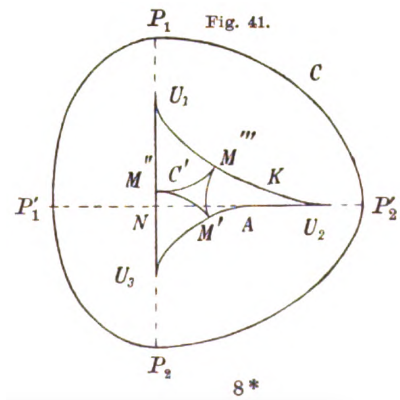

Figure 17: Model 19. Illustration given in [4]

References

[1] J. Havil. *Curves for the Mathematically Curious: An Anthology of the Unpredictable, Historical, Beautiful, and Romantic*. Princeton University Press, 2019.

[2] U. of Illinois Urbana-Champaign. The Altgeld math models. https://mathmodels.illinois.edu/cgi-bin/cview?SITEID=4&ID=342, n.d. Accessed: 2024-11-20.

[3] U. of Illinois Urbana-Champaign. The Altgeld math models collection. https://mathmodels.illinois.edu/cgi-bin/cview?SITEID=4, n.d. Accessed: 2024-11-20.

[4] F. Schilling. Die Theorie und Konstruktion der Kurven konstanter Breite. *Zeitschrift für Mathematik und Physik*, 63:67–136, 1914.

[5] M. Schilling. *Verlag von Modellen für den Höheren Mathematischen Unterricht*. 1913.