Introduction to Representation Theory of Finite Groups

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1 Introduction

Representation theory is used to study certain abstract algebraic structures, such as groups, using linear algebra. It has applications in combinatorics, number theory, geometry, and even quantum mechanics. But, for starters, these notes serve as an introduction to the topic, focused only on representations of finite groups.

2 Definitions and basic examples

Definition 2.1 (**Representation**). *Let G be a finite group. A representation of G is a homomorphism $\rho : G \to GL(V)$ for some finite-dimensional non-zero vector space V.*

This definition is equivalent to saying that an action of the group G on vector space V is a representation if any element of G induces a linear transformation on V.

Note: In terms of notations, we will write ρ_g instead of $\rho(g)$, so that when we apply the linear transformation on an element $v \in V$, we write $\rho_g(v)$ instead of $\rho(g)(v)$.

Definition 2.2 (**Degree**). *For the representation $\rho : G \to GL(V)$, the dimension of the vector space V is called the degree of ρ.*

Example 2.3. *Probably one of the simplest examples we can look at is the representation $\rho : G \to GL_1(\mathbb{C})$, where $G = \mathbb{Z}/2\mathbb{Z}$. "Identity maps to identity", therefore: $\rho_0 = 1$. Now, suppose $\rho_1 = x \in \mathbb{C}^*$. We know that $\rho_0 = \rho_{1+1} = x^2 \Rightarrow x \in \{-1, 1\}$.*

If $\rho_0 = \rho_1 = 1$, then we call this a trivial representation, let's call it ρ. The other representation that we can obtain has $\rho_0 = 1, \rho_1 = -1$. We will name it ϕ.

Example 2.4. *Let $G \cong \mathbb{Z}/4\mathbb{Z}$ be the cyclic group of order 4. We want to look at a representation on $GL_2(\mathbb{C})$, so we need to find a homomorphism $\rho : G \to GL_2(\mathbb{C})$. Writing the elements of G as*

$$G = \{e, r, r^2, r^3\},$$

we can make the following mapping:

$$e \mapsto I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$r \mapsto A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$r^2 \mapsto A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$r^3 \mapsto A^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since $A^4 = I_2$, we can see that $\{I_2, A, A^2, A^3\}$ forms a subgroup of $GL_2(\mathbb{C})$, isomorphic to $G = \mathbb{Z}/4\mathbb{Z}$.

Example 2.5 (**Standard representation of the symmetric group**). *Looking at S_n, we define its standard representation $\rho : S_n \to GL(V)$, where $V = \mathbb{C}^3$. Let the basis elements of $\mathbb{C}^3$ be $B = \{e_1, e_2, ..., e_n\}$ (the standard basis). Then we define the matrix $\rho_\sigma = \rho(\sigma)$ for a permutation $\sigma \in S_n$ by permuting the rows of the identity matrix according to σ. In other words, ρ_σ can also be defined as:*

$$\rho_\sigma(e_i) = e_{\sigma(i)}, \text{ for } e_i \in B.$$

If we were to look at a particular example, say for S_5, then

$$\rho(1\ 2) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \rho(1\ 2\ 3\ 4\ 5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \text{ and so on.}$$

Looking at the standard representation of S_n, one can notice that

$$\rho_\sigma(e_1 + e_2 + ... + e_n) = e_{\sigma(1)} + e_{\sigma(2)} + ... + e_{\sigma(n)} = e_1 + e_2 + ... + e_n,$$

which makes us wonder how we can define what "invariant" would mean here.

Definition 2.6 ($G-$**invariant subspaces**). *Let $\rho : G \to GL(V)$ be a representation and let $W \subseteq V$ be a subspace. We say that W is a $G-$invariant subspace if, $\forall g \in G$ and $w \in W$, we have*

$$\rho_g(w) \in W.$$

Definition 2.7 (**Subrepresentation**). *Let $\rho : G \to GL(V)$ be a representation and let $W \subseteq V$ be a $G-$invariant subspace. Then, a subrepresentation $\rho|_W : W \to GL(V)$ is defined by the map $(\rho|_W)_g(w) = \rho_g(w)$.*

A final definition for this section is for the equivalence of two representations.

Definition 2.8 (**Equivalence**). *Let $\rho : G \to GL(V)$ and $\phi : G \to GL(V')$ be two representations. They are said to be equivalent (notation: $\rho \sim \phi$) if there exists an isomorphism $T : V \to V'$ such that*

$$\phi_g = T\rho_g T^{-1}, \forall g \in G.$$

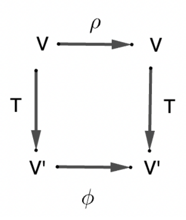

Figure 1: Equivalence diagram

3 Direct sum and irreducible representations

Recall that for two vector spaces V and V', with $V \bigcap V' = \{0\}$, the direct sum $V \oplus V' = \{v + v' \mid v \in V, v' \in V'\} \cong V \times V'$ has the dimension equal to $dim(V) + dim(V')$.

Definition 3.1 (**Direct sum of representations**). *Let us have $\rho : G \to GL(V)$ and $\phi : G \to GL(V')$ as representations of G. We define $\psi : G \to GL(V \oplus V')$ as the direct sum $\psi = \rho \oplus \phi$:*

$$\psi_g(v, v') = (\rho(v), \phi(v')).$$

Let's "translate" this definition in terms of matrices. So, let $V = \mathbb{C}^n$ and $V' = \mathbb{C}^m$. Therefore, we can say that we defined the representations as follows:

$$\rho : G \to GL_n(\mathbb{C}),$$

$$\phi : G \to GL_m(\mathbb{C}),$$

$$\psi : G \to GL_{n+m}(\mathbb{C}),$$

where ψ has the block diagonal matrix form

$$\psi_g = \left(\begin{array}{c|c} \rho_g & \mathbf{0} \\ \hline \mathbf{0} & \phi_g \end{array}\right).$$

Example 3.2. *Continuing on Example 2.3, where ρ was the trivial representation of $\mathbb{Z}_2$ on $\mathbb{C}^*$ and the representation ϕ on $\mathbb{Z}_2$ on $\mathbb{C}^*$ had $\phi_0 = 1$ and $\phi_1 = -1$, one can check that any $\psi : \mathbb{Z}_2 \to GL_n(\mathbb{C})$ with $\psi_0 = I_n$ and*

$$\psi_1 = \left(\begin{array}{cccc|cccc} \rho_1 & 0 & \dots & 0 & & & & \\ 0 & \rho_1 & \dots & 0 & & \multicolumn{2}{c}{\mathbf{0}} & \\ \vdots & \vdots & \ddots & \vdots & & & & \\ 0 & 0 & \dots & \rho_1 & & & & \\ \hline & & & & \phi_1 & 0 & \dots & 0 \\ & \multicolumn{2}{c}{\mathbf{0}} & & 0 & \phi_1 & \dots & 0 \\ & & & & \vdots & \vdots & \ddots & \vdots \\ & & & & 0 & 0 & \dots & \phi_1 \end{array}\right) = \left(\begin{array}{c|c} \rho_1 I_k & \mathbf{0} \\ \hline \mathbf{0} & \phi_1 I_j \end{array}\right) = \left(\begin{array}{c|c} I_k & \mathbf{0} \\ \hline \mathbf{0} & -I_j \end{array}\right),$$

where $k + j = n$, is a representation.

Proposition 3.3. *Let V be a vector space and let W_1, W_2 be two G-invariant subspaces such that $V = W_1 \oplus W_2$. Then, for representation $\rho : G \to GL(V)$,*

$$\rho \sim \rho|_{W_1} \oplus \rho|_{W_2}.$$

Note: When we have a $\rho : G \to GL(V)$ representation with a $\rho|_W : G \to GL(W)$ subrepresentation, we will simply write that V is a representation and W is a subrepresentation.

Definition 3.4 (**Irreducible representation**). *A representation $\rho : G \to GL(V)$ is irreducible if the only $G-$invariant subspaces of V are V and the null subspace* $\{0\}$.

This definition is equivalent to saying that a representation $\rho : G \to GL(V)$ is irreducible if it has no proper non-zero subrepresentations.

Example 3.5. *Since no one-dimensional vector space has non-zero proper subspaces, a one-dimensional representation will be irreducible.*

4 Complementary subrepresentations and projections

Since we discussed the direct sum of representations, we can now define complementary subrepresentations.

Definition 4.1 (**Complementary subrepresentation**). *Let $\rho : G \to GL(V)$ be a representation and $W \subseteq V$ be a subrepresentation. If we can find another subrepresentation $U \subseteq V$ such that $V = W \oplus U$, then U is called a complementary subrepresentation to W.*

Lemma 4.2. *Let V be a vector space and $W \subseteq V$ a subspace. If we define a surjective linear map $f : V \to W$*

$$f(w) = w, \ \forall w \in W,$$

then $ker(f) \subseteq V$ is a complementary subspace to W and

$$V = W \oplus ker(f).$$

Proof. f is surjective, hence $Im(f) = W$. Moreover, let $x \in ker(f) \bigcap W$. Then $f(x) = x = 0$, which implies that $W \bigcap ker(f) = \{0\}$.

From the Rank-Nullity Theorem, we know that

$$dim(W) + dim(ker(f)) = dim(Im(f)) + dim(ker(f)) = dim(V).$$

Therefore, $V = W \oplus ker(f)$.

QED

A linear map like this can be seen as a projection; let's introduce the actual definition.

Definition 4.3 (**Projection**). *Let $P : V \to V$ be a linear map from a vector space to itself. If $P \circ P = P$, then P is called a projection.*

So, looking at Lemma 4.2 and Definition 4.3, we can observe that for some $V = W \oplus U$, if $P : V \to W$ and $P' : V \to U$ are linear maps such that

$$P(w, u) = w, \ P'(w, u) = u.$$

Then $P|_W : W \to W$ and $P'|_U : U \to U$ are projections and

$$ker(P) = U, \ ker(P') = W.$$

Moreover, $(P + P')v = v, \ \forall v \in V$, i.e. $P + P'$ is the identity map.

Proposition 4.4. *Let $\rho : G \to GL(V)$ be a representation. Consider a subrepresentation $W \subseteq V$ and a G-linear projection $P : V \to W$. Then $ker(P)$ is a complementary subrepresentation to W.*

The proof is left as an exercise for the reader.

Note: A $G-$linear map is a linear map for which $f(g \cdot v) = g \cdot f(v), \ \forall g \in G, \ \forall v \in V$.

5 Maschke's theorem

Before introducing Maschke's Theorem, we need to state a few helpful propositions.

Proposition 5.1. *Every representation of a finite group is either irreducible or decomposable.*

Proposition 5.2. *Let $\rho : G \to GL(V)$ be a representation, $W \subseteq V$ be a subrepresentation. We write V as direct sum of the complementary subspaces $V = W \oplus U$.*
Let $P : V \to W$ be a projection with $ker(P) = U$ and $Pw = w$, $\forall w \in W$. Consider $f : V \to V$ such that

$$f(v) = \frac{1}{|G|} \sum_{g \in G} (\rho_g P \rho_{g^{-1}})(v).$$

Then f is a $G-$linear projection with image W.

Note: The trick we used to build f is called "averaging" of P over the group G.

Proof. First, let's check that $Im(f) \subseteq W$. Since $P\rho_{g^{-1}})(v) \in W$, and as W is a subrepresentation, we have $(\rho_g P \rho_{g^{-1}})(v) \in W$.

Secondly, we need to show that it is $G-$linear. To do so, we pick an arbitrary $x \in G$ and we will have that for any $v \in V$

$$\begin{aligned} f(\rho_x(v)) &= \frac{1}{|G|} \sum_{g \in G} (\rho_g P \rho_{g^{-1}})(\rho_x(v)) \\ &= \frac{1}{|G|} \sum_{g \in G} (\rho_g P \rho_{g^{-1}x})(v) \\ &= \frac{1}{|G|} \sum_{h \in G} (\rho_{xh} P \rho_h(v)) \\ &= \frac{1}{|G|} \sum_{h \in G} \rho_x(\rho_h P \rho_h(v)) \\ &= \rho_x f(v) \end{aligned}$$

Finally, we show that f is a projection. Since $\rho_g(w) \in W$, we have

$$\begin{aligned} f(w) &= \frac{1}{|G|} \sum_{g \in G} (\rho_g P \rho_{g^{-1}})(w) \\ &= \frac{1}{|G|} \sum_{g \in G} (\rho_g \rho_{g^{-1}})(w) \\ &= \frac{1}{|G|} \sum_{g \in G} w \\ &= w \end{aligned}$$

Now that we know that $f(w) = w$, that means that $W \subseteq Im(f)$. Hence, $Im(f) = W$.

With all of these, we managed to prove the proposition.

QED

Theorem 5.3 (**Maschke**). *Let $\rho : G \to GL(V)$ be a representation and W be a subrepresentation of V. Then there exists a complementary subrepresentation $W' \subseteq V$ of W.*

Proof. We know from linear algebra that we can find a complementary subspace U of W (extending the basis of W to a basis of V), so $V = W \oplus U$ and every vector $v \in V$ can be uniquely expressed as $v = w + u$, where $w \in W$ and $u \in U$.

We define the projection $P : V \to W$ as the coordinate map $Pv = w$, $\forall v = w + u \in V$. Therefore, $ker(P) = U$.

Now let us consider the map $f : V \to V$ such that

$$f(v) = \frac{1}{|G|} \sum_{g \in G} (\rho_g P \rho_{g^{-1}})(v).$$

By the "averaging" method and from Proposition 5.2, we know that $f(W)$ is a G-linear projection with image W. Let $W' = ker(f)$.

Therefore, by Proposition 4.4, W' is a complementary subrepresentation of W.

QED

Corollary 5.4. *Every representation can be written as a direct sum*

$$V = W_1 \oplus W_2 \oplus ... \oplus W_k$$

of irreducible subrepresentations.

6 References and notes

B. Steinberg, 2009, Representation Theory of Finite Groups
E. Segal, 2014, Group Representation Theory
T. Schedler, 2021, Group representation theory, Lecture Notes