Five Lemma and the Homology Long Exact Sequence

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ABSTRACT. The topics related to exact sequences have broad applications in algebra and algebraic topology. Starting with the Five Lemma, in this paper we will apply the properties of exactness and use diagram chasing to complete detailed proofs. After that, the focus will shift towards showing that short exact sequences of chain complexes induce long exact sequences of homology groups. In the end, the theorem will be extended to relative homology long exact sequences, introduced after defining the concept of relative singular homology groups.

All sections will follow the notation and logic of [Hatcher(2002)], but other sources are also used for polishing explanations and understanding the topics better.

Contents

1 Overview of required concepts

For starters, we'll present the preliminary background necessary for understanding the future results and proofs. Let's start with the concepts of commutative diagrams and exact sequences, as they will be showcased in every section of this paper.

Definition 1.1 (Commutative diagram)**.** *A commutative diagram is a diagram of maps where any two compositions of maps starting at one point in the diagram and ending at another are equal.*

Definition 1.2 (Exact sequence)**.** *Let us have a sequence of objects A_n of an abelian category (in this paper, we will be working with abelian groups) together with homomorphisms $\alpha_n : A_n \longrightarrow A_{n-1}, \forall n$. Then*

$$\ldots \longrightarrow A_{n+1} \xrightarrow{\alpha_{n+1}} A_n \xrightarrow{\alpha_n} A_{n-1} \xrightarrow{\alpha_{n-1}} \ldots$$

is called an exact sequence if, for each n, we have the following equality of subgroups:

$$im\ \alpha_{n+1} = ker\ \alpha_n \subseteq A_n.$$

Now that exact sequences have been introduced, we can use them to express several algebraic concepts as in the lemma below.

Lemma 1.3. *The following equivalences hold:*

a) $0 \longrightarrow A \xrightarrow{\alpha} B$ *is exact if and only if α is injective ($ker\ \alpha = 0$).*

b) $A \xrightarrow{\alpha} B \longrightarrow 0$ *is exact if and only if α is surjective ($im\ \alpha = B$).*

c) $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$ *is exact if and only if α is an isomorphism.*

Our goal is to show that short exact sequences of chain complexes induce long exact sequences of homology groups, so it is customary to define these key terms, too.

Definition 1.4 (Short exact sequence)**.** *A sequence of abelian groups together with homomorphisms*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is a short exact sequence if and only if α is injective, β is surjective, $im\ \alpha = ker\ \beta$.

One can deduce that, in the case of a short exact sequence, β induces an isomorphism $C \approx B/\mathrm{im}\ \alpha$. Moreover, if α is an inclusion map, then $C \approx B/A$.

Definition 1.5 (Long exact sequence)**.** *A long exact sequence is an infinite exact sequence of homomorphisms.*

Definition 1.6 (Chain complex)**.** *A chain complex of abelian groups is a sequence of abelian groups C_i together with homomorphisms ∂_i*

$$\ldots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \ldots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

with $\partial_n \partial_{n+1} = 0, \forall n$.

As a remark, since $\partial_n \partial_{n+1} = 0$, then we can see that im $\partial_{n+1} \subset$ ker ∂_n. Moreover, these inclusions are actually equivalent to $\partial_n \partial_{n+1} = 0$. Therefore, for any exact sequence of α_n homomorphisms, $\alpha_n \alpha_{n+1} = 0$.

Further in the paper, we will use the notation $C_n(X)$ within our chain complexes to denote the free abelian group with the set of singular $n-$simplices in X as basis. The boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ is defined as

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma | [v_0, \dots, \hat{v_i}, \dots, v_n],$$

in the same way as we would define the boundary homomorphism for a general $\Delta-$complex.

Definition 1.7 (Homology group of the chain complex)**.** *The n-th homology group of the chain complex is the quotient group* $H_n = ker\ \partial_n / im\ \partial_{n+1}$.

Lastly, in our proofs, we'll use different words to describe the properties of elements, hence it's important to specify the meaning behind them.

Definition 1.8. *Given the chain complex defined previously, we say that the elements of ker* ∂_n *are **cycles** and the elements of im* ∂_{n+1}*, **boundaries.** The elements of* H_n *are **homology classes** and they are cosets of im* ∂_{n+1}*. Two cycles representing the same homology class are said to be **homologous** - their difference is a boundary.*

With all these in mind, we can move on to the first major theorem of this paper.

2 The Five Lemma

The Five Lemma is a result of commutative diagrams, which has applications in homological algebra. There are several versions of it, hence we chose to present the most general one and then state the more particular cases as corollaries.

Theorem 2.1 (The Five Lemma)**.** *In a commutative diagram of abelian groups as the one below, if the two rows are exact and*

a) *if* β *and* δ *are surjective and* ϵ *is injective, then* γ *is surjective.*

b) *if* β *and* δ *are injective and* α *is surjective, then* γ *is injective.*

$$\begin{array}{ccccccccc}
A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\
\alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \delta\downarrow & & \downarrow\epsilon \\
A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E'
\end{array}$$

Proof. The method that we are going to use for this proof is diagram chasing. Let us start with the first case:

a) Taking an arbitrary element $c' \in C'$, we know that, since δ is surjective, there exists some $d \in D$ such that $\delta(d) = k'(c') \in D'$. Since our diagram is commutative, we have that $\epsilon l(d) = l'\delta(d) = l'k'(c) = 0$. As ϵ is injective, we deduce that $l(d) = 0$, i.e. $d \in$ ker l. The rows of our diagram are exact, thus ker $l =$ im k and $d = k(c)$ for some $c \in C$.

Using the commutative diagram property and the results from above we get that $k'\gamma(c) = \delta k(c) = \delta(d) = k'(c')$ holds. This means that, under k', $c' - \gamma(c)$ maps to 0, i.e. $c' - \gamma(c) \in \ker k'$ and, by exactness, $c' - \gamma(c) \in \operatorname{im} j'$. Let $c' - \gamma(c) = j'(b')$ for some $b' \in B'$. There also exists some $b \in B$ such that $\beta(b) = b'$, since β is surjective.

As for the last steps in proving that γ is surjective, we will show that there exists a preimage in C of the arbitrary element $c' \in C'$ fixed at the beginning. From $c' - \gamma(c) = j'(b')$ and the commutative diagram property, we deduce that the following series of equalities holds:

$$c' = \gamma(c) + j'(b') = \gamma(c) + j'(\beta(b)) = \gamma(c) + j'(\beta(b)) = \gamma(c) + \gamma(j(b)) = \gamma(c + j(b)).$$

Hence, γ is surjective.

b) Take some $c \in C$ such that $\gamma(c) = 0$. Since δ is injective, from following the commutative diagram rules in $\delta k(c) = k'\gamma(c) = 0$, we can see that $k(c) = 0$, i.e. $c \in \ker k$. From exactness, $\ker k = \operatorname{im} j$ and $c = j(b)$ for some b. We then obtain that $j'\beta(b) = \gamma j(b) = \gamma(c) = 0$, so $\beta(b) \in \ker j'$. By exactness, $\beta(b) \in \operatorname{im} i'$, which means that there exists some $a' \in A'$ such that $\beta(b) = i'(a')$.

Given that α is surjective, $a' = \alpha(a)$ for some $a \in A$. As shown above, $\beta(b) = i'(a')$, which implies the equalities:

$$0 = i'(a') - \beta(b) = i'\alpha(a) - \beta(b) = \beta i(a) - \beta(b) = \beta(i(a) - b).$$

Since β is injective, its kernel is trivial, hence $i(a) - b = 0$, i.e. $b = i(a)$. Therefore, by exactness, $ji(a) = 0$, so $c = j(b) = ji(a) = 0$. Because $\gamma(c) = 0$ ends up implying that $c = 0$, we managed to show that γ is injective as its kernel is trivial. QED

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\alpha"'] & B \arrow[r, "j"] \arrow[d, "\beta"'] & C \arrow[r, "k"] \arrow[d, "\gamma"'] & D \arrow[r, "l"] \arrow[d, "\delta"'] & E \arrow[d, "\epsilon"] \\
A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "l'"] & E'
\end{tikzcd}$$

Corollary 2.2. *In a commutative diagram like the one above, if the two rows are exact and β, δ are bijective, α is surjective and ϵ is injective, then γ is bijective.*

Proof. β and δ are injective and surjective, and thus both *a)* and *b)* cases of the theorem will be satisfied, implying the bijectivity of γ. QED

Corollary 2.3. *In a commutative diagram like the one above, if the two rows are exact and $\alpha, \beta, \delta, \epsilon$ are isomorphisms, then γ is an isomorphism.*

Proof. Again, both cases of the theorem are satisfied as every map is an isomorphism. Hence, so is γ. QED

3 The Homology Long Exact Sequence

In the previous section, we familiarized ourselves with diagram chasing and using the exactness argument. These methods will come in handy when constructing the long exact sequences on homology groups.

First, let's talk about maps between chain complexes and between homology groups and how they are connected.

Proposition 3.1. *A chain map between chain complexes induces homomorphisms between the homology groups of the two complexes.*

Proof. Given a map $f : X \longrightarrow Y$, it induces a homomorphism $f_\# : C_n(X) \longrightarrow C_n(Y)$. This is defined by composing each singular $n-$simplex $\sigma : \Delta^n \longrightarrow X$ with f to get a singular $n-$simplex $f_\#(\sigma) = f\sigma : \Delta^n \longrightarrow Y$ and then extending $f_\#$ linearly via $f_\#(\sum_i n_i\sigma_i) = \sum_i n_i f_\#(\sigma_i) = \sum_i n_i f\sigma_i$.

Now, using the boundary definition, we have

$$f_\#\partial(\sigma) = f_\#(\sum_i (-1)^i \sigma|[v_0, ..., \hat{v_i}, ..., v_n]) = \sum_i (-1)^i f\sigma|[v_0, ..., \hat{v_i}, ..., v_n] = \partial f_\#(\sigma),$$

which means that $f_\#\partial = \partial f_\#$. The following commutative diagram is valid and the $f_\#$'s define a chain map from the singular chain complex of X to that of Y.

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & C_{n+1}(X) & \xrightarrow{\partial} & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle f_\#} & & \\
\dots & \longrightarrow & C_{n+1}(Y) & \xrightarrow{\partial} & C_n(Y) & \xrightarrow{\partial} & C_{n-1}(Y) & \longrightarrow & \dots
\end{array}$$

From $f_\#\partial = \partial f_\#$, it follows that $f_\#$ takes cycles to cycles because, for some $\partial\alpha = 0$, $0 = f_\#\partial(\alpha) = \partial f_\#(\alpha)$. Similarly, $f_\#$ takes boundaries to boundaries. (Here, recall **Definition 1.8** for cycles and boundaries.)

Therefore, we conclude that $f_\#$ induces a homomorphism $f_* : H_n(X) \longrightarrow H_n(Y)$. QED

We will keep this construction of f_* and use it in the proof of the theorem below.

Theorem 3.2. *A short exact sequence of chain complexes induces a long exact sequence of homology groups.*

To understand what the theorem states, let us consider the commutative diagram below, representing a short exact sequence of chain complexes.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & A_{n+1} & \xrightarrow{\partial} & A_n & \xrightarrow{\partial} & A_{n-1} & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \\
\dots & \longrightarrow & B_{n+1} & \xrightarrow{\partial} & B_n & \xrightarrow{\partial} & B_{n-1} & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle j} & & \\
\dots & \longrightarrow & C_{n+1} & \xrightarrow{\partial} & C_n & \xrightarrow{\partial} & C_{n-1} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Here, the columns are short exact sequences and the rows represent the chain complexes denoted as A, B, C. This short exact sequence of chain complexes will induce a long exact sequence of homology groups:

$$\dots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \longrightarrow \dots ,$$

where $H_n(A)$ is the homology group at A_n in the chain complex A ($\ker\partial/\mathrm{im}\partial$ at A_n) and analogously, $H_n(B)$ and $H_n(C)$ are defined.

After this clarification of the theorem, we can move on to the proof.

Proof. This proof is divided into two major parts: (1) defining the boundary map from $H_n(C)$ to $H_{n-1}(A)$ and (2) showing that the sequence of homology groups is, indeed, exact.

Part 1:

For this, we will use the large diagram from the explanation for our short exact sequence of chain complexes. It's clear that the commutative diagram claim refers to each "square" of the diagram and ultimately means that i, j are chain maps. From **Proposition 3.1**, we deduce that i, j induce the i_*, j_* maps on the homology.

Now it is important to understand how we can reach A_{n-1} from C_n, and for that we have to make use of the properties of i and j and work on defining the boundary map $\partial : H_n(C) \longrightarrow H_{n-1}(A)$. Our construction of the boundary will end up following the following schematic diagram:

$$\begin{tikzcd}
 & a \arrow[d, mapsto] & & A_{n-1} \arrow[d, "i"] \\
b \arrow[r, mapsto] \arrow[d, mapsto] & \partial b & B_n \arrow[r, "\partial"] \arrow[d, "j"'] & B_{n-1} \\
c & & C_n &
\end{tikzcd}$$

Take a cycle $c \in C_n$. Given that j is a surjection, there exists a $b \in B_n$ such that $c = j(b)$. Via the boundary map, we can obtain the element $\partial b \in B_{n-1}$. Let us remember that the columns in the commutative diagram represent short exact sequences, therefore, if $\partial b \in \ker j$, then $\partial b \in \operatorname{im} i$, i.e. there exists some $a \in A_{n-1}$ such that $\partial b = i(a)$.

To prove that $\partial b \in \ker j$ is straight-forward, as $j(\partial(b)) = \partial(j(b)) = \partial c = 0$. By the argument above, $\partial b = i(a)$, for some $a \in A_{n-1}$. Again, under the boundary map, we can obtain the following:

$$0 = \partial\partial b = \partial i(a) = i(\partial a) \Longrightarrow \partial a = 0,$$

since i is injective.

Hence, the boundary map $\partial : H_n(C) \longrightarrow H_{n-1}(A)$ sends the homology class of c to the homology class of a, i.e. $\partial[c] = [a]$. Seeing how this is constructed is not enough, though; we still need to prove that the boundary is well-defined and a homomorphism.

Let's assume that the map is well-defined and show that it is a homomorphism. Let us have $\partial[c_1] = [a_1]$ and $\partial[c_2] = [a_2]$. We know elements b_1, b_2 exist such that $c_1 = j(b_1), c_2 = j(b_2)$ and then $\partial b_1 = i(a_1), \partial b_2 = i(a_2)$.

Since j is a homomorphism, $j(b_1 + b_2) = j(b_1) + j(b_2) = c_1 + c_2$. Moreover, $\partial(b_1 + b_2) = \partial b_1 + \partial b_2 = i(a_1) + i(a_2) = i(a_1 + a_2)$. Therefore $\partial([c_1] + [c_2]) = [a_1] + [a_2]$ and the boundary $\partial : H_n(C) \longrightarrow H_{n-1}(A)$ is a homomorphism.

Now back to proving that the boundary map is well-defined. The three arguments below suffice to show that this definition is valid:

1. With i injective, we ensure that a is uniquely determined by ∂b.

2. Say that another b' is also chosen such that $c = j(b) = j(b')$. This implies that $j(b' - b) = 0$, i.e. $b' - b \in \ker j$, which by exactness, further implies that there exists some a' such that

$b' - b = i(a')$. We can now work with $b' = b + i(a')$ and follow the same steps with the boundary as before:

$$\partial b' = \partial(b + i(a')) = \partial b + \partial(i(a')) = i(a) + i(\partial a') = i(a + \partial a'),$$

meaning that a is replaced by $a + \partial a'$, which is homologous with a.

3. Now we choose a homologous element instead of c. Its form is $c + \partial c'$. With j surjective, we have that, for some b', $c' = j(b')$. Thus, $c + \partial c' = j(b) + \partial(j(b')) = j(b) + j(\partial b') = j(b + \partial b')$. In this case, b will be replaced by $b + \partial b'$ and applying the boundary map again, we obtain ∂b, which means that there is no change in the choice of a.

With these, we conclude the first part of the proof.

Part 2:

The remaining arguments that we have to make are going to be based on the following sequence:

$$\ldots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \longrightarrow \ldots,$$

where $\partial : H_n(C) \longrightarrow H_{n-1}(A)$ is defined as in Part 1. To prove the sequence is exact, we need to show that $\ker j_* = \operatorname{im} i_*$, $\ker i_* = \operatorname{im} \partial$ and $\ker \partial = \operatorname{im} j_*$. We will show this by inclusions in 6 steps.

1. We start by showing that $\operatorname{im} i_* \subseteq \ker j_*$. Since

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

is a short exact sequence, we have that $ji = 0$, which implies $j_* i_* = 0$ and therefore $\operatorname{im} i_* \subseteq \ker j_*$.

2. For showing $\operatorname{im} j_* \subseteq \ker \partial$, we use the definition of ∂ and we have that $\partial b = 0$ which means $\partial j = 0$. Thus $\operatorname{im} j_* \subseteq \ker \partial$.

3. To show $\operatorname{im} \partial \subseteq \ker i_*$ is straight-forward considering that $i_* \partial [c] = [\partial b] = 0$ for any c. Hence $i_* \partial = 0$, which is equivalent to the inclusion $\operatorname{im} \partial \subseteq \ker i_*$.

4. We have to carefully prove that $\ker j_* \subseteq \operatorname{im} i_*$. If we take a homology class $[b] \in \ker j_*$, it is actually represented by a cycle $b \in B_n$. From the commutative diagram, we deduce that $j(b) = \partial c'$ for some $c' \in C_{n+1}$ but also $c' = j(b')$ for some $b' \in B_{n+1}$ by the surjectivity of j.
Next, $j(b - \partial b') = j(b) - j(\partial b') = \partial c' - \partial(j(b')) = \partial c' - \partial c' = 0$, which implies that $b - \partial b' \in \ker j$. By exactness, $\ker j = \operatorname{im} i$, hence the existence of some $a \in A_n$ such that $b - \partial b' = i(a)$. Using that i is an injective homomorphism, one can observe how $i(\partial a) = \partial(i(a)) = \partial(b - \partial b') = \partial b = 0$, which implies that $\partial a = 0$.
We proved that a is a cycle and thus $i_*[a] = [b - \partial b'] = [b]$, i.e. $\ker j_* \subseteq \operatorname{im} i_*$.

5. Let's show that $\ker \partial \subseteq \operatorname{im} j_*$. Going back to the definition of ∂, which maps $[c]$ to $[a]$, by taking a c to represent a homology class in $\ker \partial$ we can determine some $b \in B_n$ such that $c = j(b)$ and then the $a \in A_{n-1}$ for which $\partial b = i(a)$. By the commutative diagram, there is some $a' \in A_n$ such that $a = \partial a'$.
Given that $\partial(b - i(a')) = \partial b - \partial i(a') = i(a) - i(\partial a') = i(a) - i(a) = 0$ (i.e. $b - i(a')$ is a cycle) and that $j(b - i(a')) = j(b) - ji(a') = c - 0 = c$, we can deduce that $j_*[b - i(a')] = [c]$. Hence, $\ker \partial \subseteq \operatorname{im} j_*$.

6. The last inclusion we have to prove is $\ker i_* \subseteq \operatorname{im} \partial$. For a cycle $a \in A_{n-1}$ with $i(a) = \partial b$, with $b \in B_n$, it is enough to show that $j(b)$ is a cycle. This is true because $\partial j(b) = j(\partial b) = ji(a) = 0$. Thus, $\partial[j(b)] = [a]$ and we conclude that $\ker i_* \subseteq \operatorname{im} \partial$.

So, we managed to show that $\ker j_* = \operatorname{im} i_*$, $\ker i_* = \operatorname{im} \partial$ and $\ker \partial = \operatorname{im} j_*$ which makes our sequence an exact one. With Part 1 and Part 2, the proof of the theorem is done. QED

4 The Relative Homology Long Exact Sequence

The theorem we proved earlier has another relevant consequence in topology: it yields the existence of a long exact sequence of homology groups associated with the pair of spaces (X, A). For this, some additional definitions are needed. Let's start with the construction of so-called **relative homology groups**.

Let X be a topological space and $A \subset X$ a subspace. Take the quotient group $C_n(X)/C_n(A)$ denoted by $C_n(X, A)$, where chains in A are trivial. The boundary map $\partial : C_n(X) \longrightarrow C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$ and hence it induces a quotient boundary map $\partial : C_n(X, A) \longrightarrow C_{n-1}(X, A)$.

$$\dots \longrightarrow C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \longrightarrow \dots$$

The above sequence of boundary maps is a chain complex, as the $\partial^2 = 0$ property carries over before passing to quotient groups, and we can define the relative homology groups $H_n(X, A)$ as $\ker \partial / \operatorname{im} \partial$. Now, we should explain what the terms "cycle" and "boundary" refer to, in this situation.

Definition 4.1 (Relative cycles)**.** *The elements of $H_n(X, A)$ are relative cycles, $n-$chains $\alpha \in C_n(X)$ such that $\partial\alpha \in C_{n-1}(A)$.*

Definition 4.2 (Relative boundary)**.** *A relative cycle α is trivial in $H_n(X, A)$ if and only if it is a relative boundary, i.e. $\alpha = \partial\beta + \gamma$ for some $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.*

The new goal is to apply **Theorem 3.2** to build long exact sequences that include relative homology groups.

Theorem 4.3. *The relative homology groups $H_n(X, A)$ for any pair (X, A) fit into a long exact sequence as the one below.*

$$\dots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X, A) \longrightarrow H_{n-1}(A) \longrightarrow H_{n-1}(X) \longrightarrow \dots$$

$$\dots \longrightarrow H_0(X, A) \longrightarrow 0$$

Proof. The proof is pretty straightforward. We take the short exact sequence of chain complexes

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\dots \arrow[r] & C_{n+1}(A) \arrow[r, "\partial"] \arrow[d, "i"'] & C_n(A) \arrow[r, "\partial"] \arrow[d, "i"'] & C_{n-1}(A) \arrow[r] \arrow[d, "i"'] & \dots \\
\dots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "j"'] & C_n(X) \arrow[r, "\partial"] \arrow[d, "j"'] & C_{n-1}(X) \arrow[r] \arrow[d, "j"'] & \dots \\
\dots \arrow[r] & C_{n+1}(X, A) \arrow[r, "\partial"] \arrow[d] & C_n(X, A) \arrow[r, "\partial"] \arrow[d] & C_{n-1}(X, A) \arrow[r] \arrow[d] & \dots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

This, by **Theorem 3.2**, yields a long exact sequence on homology groups:

$$\ldots \longrightarrow H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \longrightarrow \ldots$$

$$\ldots \longrightarrow H_0(X,A) \longrightarrow 0$$

QED

5 Conclusion

To conclude, in this paper, we were introduced to algebraic methods like diagram chasing, alongside the exactness property of some sequences. These helped with constructing a proof for the Five Lemma, but also further in the paper, when we explored long exact sequences on homology and relative homology groups. In the future, one can use the concepts in this paper to further study other results from algebra and algebraic topology, such as the Snake Lemma, excision, naturality, computational applications, Mayer-Vietoris sequences, and many others.

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