Abelian Sandpile Basics

2018

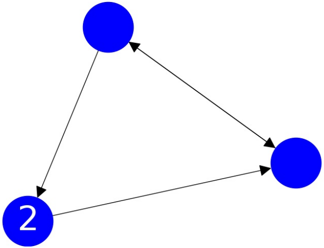
Abelian Sandpile Model

- Rough model of a pile of sand, based on a finite directed (multi-) graph G, with vertices V and edges E.
- Chips (grains of sand) are stacked on the vertices.
- Chips can flow to other vertices via edges of G.
- Originally studied on uniform grids.
- Also known as the chip-firing game.

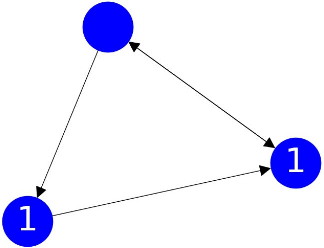
Chip-Firing

- Let n_i be the number of chips on vertex v_i
- Let $e_i = \{e \in E : e \text{ starts at } v_i\}$
- If a vertex $n_i \geq |e_{v_i}|$ it can **fire**.
- When a vertex fires, it transfers a chip down each of its **outbound** edges.

Chip-Firing Example



(a) Before firing: Lower left vertex has one outbound edge and two chips.

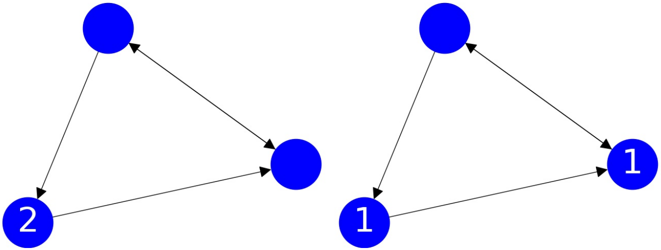


(b) After firing: One chip has been transferred to the far right vertex.

Figure: Firing example.

Chip Configuration

- A **configuration** is the set of pairs $\{(v_i, n_{i,t}) : v_i \in V\}$, where $n_{i,t}$ is the number of chips on v_i at time/iteration t.
- If we fix an ordering of V, use a vector: $c_t = (n_{0,t}, n_{1,t}, ..., n_{N,t})$

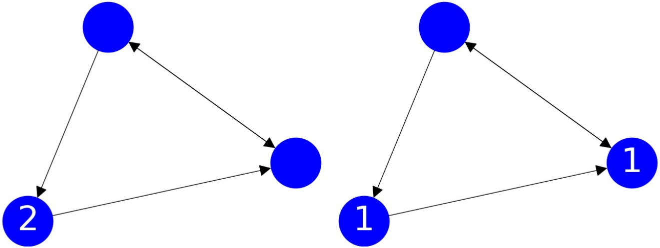


(a) $c_0 = (2, 0, 0)$ (b) $c_1 = (1, 1, 0)$

Figure: Vertices indexed counter-clockwise starting from lower left vertex

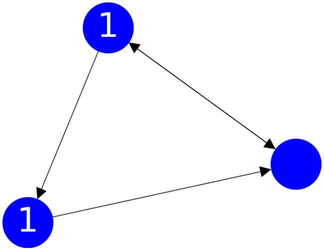
Does it Stabilize?

- A configuration is **stable** if none of the vertices can be fired, e.g. $n_i < |e_{v_i}| \ \forall \ i$.
- **Stabilization** is a sequence of vertex firings that result in a stable configuration.

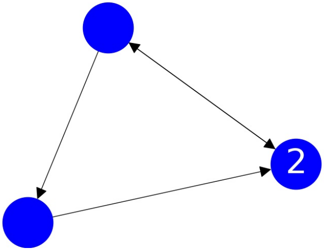


(a) $c_0 = (2, 0, 0)$ (b) $c_1 = (1, 1, 0)$

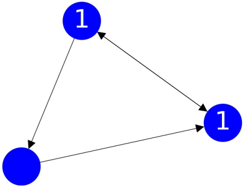
Is it stable?



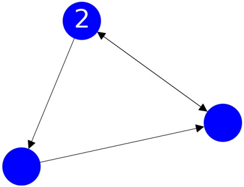
(a) $c_2 = (1, 0, 1)$



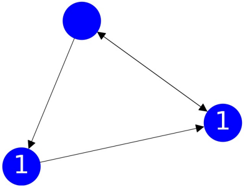
(b) $c_{2'} = (0, 2, 0)$



(a) $c_3 = (0, 1, 1)$



(b) $c_4 = (0, 0, 2)$



(c) $c_5 = c_1 = (1, 1, 0)$

A Sink Vertex

- A vertex v of G with no outbound edges ($e_v = $) is known as a **sink**.
- "Sink" vertices **never** fire.
- If every vertex has a direct path to a sink it is the **universal** sink.
- Every configuration on a directed graph with a universal sink stabilizes.
- From now on, all graphs will have a universal sink.

Sink Example

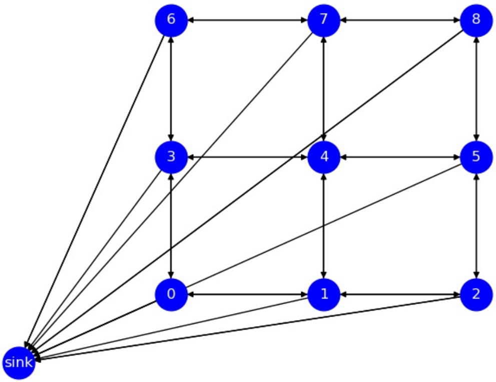


Figure: Directed graph with a sink. $|e_{v_i}| = 4$ for all non-sink vertices. Vertices are labeled with their index/ID.

Chip Configuration Revisited

- The sink vertex is excluded from the configuration vector

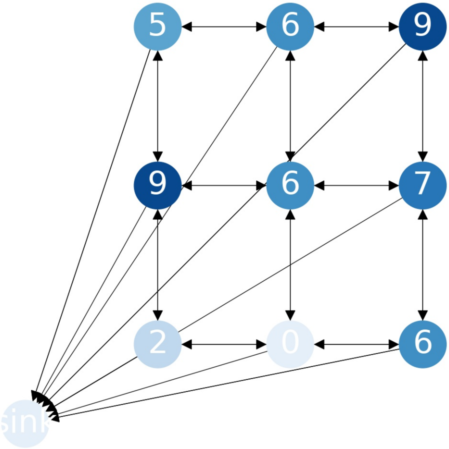


Figure: $c_t = (2, 0, 6, 9, 6, 7, 5, 6, 9)$, indexed as on previous slide.

Reduced Laplacian

- D is the out-degree matrix of G:

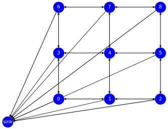
$$D_{i,j} = \begin{cases} |e_{v_i}| & : i = j \\ 0 & : i \neq j \end{cases}$$

- e_{v_i,v_j} = edges from v_i to v_j
- A is the adjacency matrix of G:

$$A_{i,j} = \begin{cases} |e_{v_i,v_j}| & : i \neq j \\ 0 & : \text{otherwise} \end{cases}$$

- The **reduced Laplacian** is the matrix $L = D - A$ with rows/columns corresponding to the sink removed.

Reduced Laplacian Example



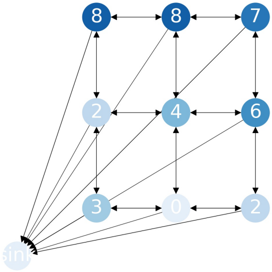
$$L = \begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix}$$

Firing and the Reduced Laplacian

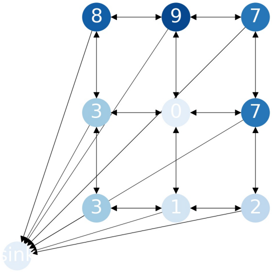
- Let L_i be the row of the reduced Laplacian corresponding to v_i
- If vertex v_j fires at time/iteration t:

$$c_{t+1} = c_t - v_j$$

- $L_4 = (0, -1, 0, -1, 4, -1, 0, -1, 0)$



$c_t = (3, 0, 2, 2, 4, 6, 8, 8, 7)$



$c_{t+1} = (3, 1, 2, 3, 0, 7, 8, 9, 7)$

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