Abelian Sandpile Basics

2018

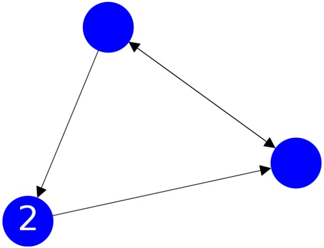
Abelian Sandpile Model

- Rough model of a pile of sand, based on a finite directed (multi-) graph G, with vertices V and edges E.
- Chips (grains of sand) are stacked on the vertices.
- Chips can flow to other vertices via edges of G.
- Originally studied on uniform grids.
- Also known as the chip-firing game.

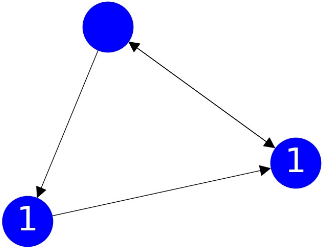
Chip-Firing

- Let n_i be the number of chips on vertex v_i
- Let $e_i = \{e \in E : e \text{ starts at } v_i\}$
- If a vertex $n_i \geq |e_{v_i}|$ it can **fire**.
- When a vertex fires, it transfers a chip down each of its **outbound** edges.

Chip-Firing Example



(a) Before firing: Lower left vertex has one outbound edge and two chips.

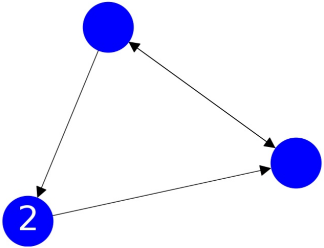


(b) After firing: One chip has been transferred to the far right vertex.

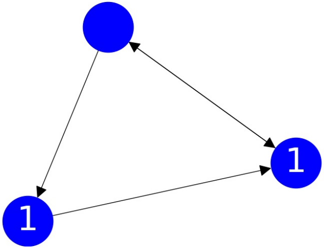
Figure: Firing example.

Chip Configuration

- A **configuration** is the set of pairs $\{(v_i, n_{i,t}) : v_i \in V\}$, where $n_{i,t}$ is the number of chips on v_i at time/iteration t.
- If we fix an ordering of V, use a vector: $c_t = (n_{0,t}, n_{1,t}, ..., n_{N,t})$



(a) $c_0 = (2, 0, 0)$



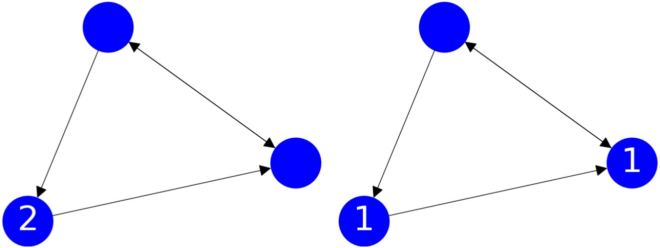
(b) $c_1 = (1, 1, 0)$

Figure: Vertices indexed counter-clockwise starting from lower left vertex

Does it Stabilize?

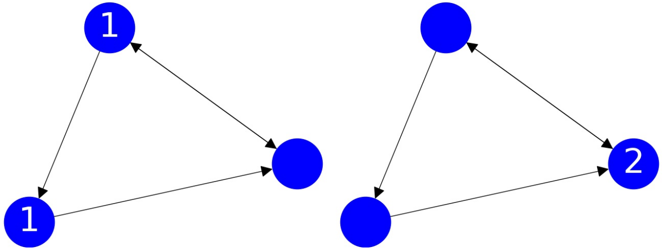
- A configuration is **stable** if none of the vertices can be fired, e.g. $n_i < |e_{v_i}| \ \forall \ i$.

- **Stabilization** is a sequence of vertex firings that result in a stable configuration.

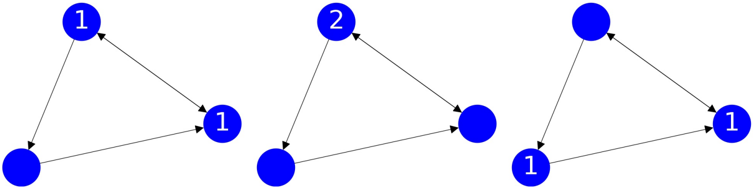


(a) $c_0 = (2, 0, 0)$ (b) $c_1 = (1, 1, 0)$

Is it stable?



(a) $c_2 = (1, 0, 1)$ (b) $c_{2'} = (0, 2, 0)$



(a) $c_3 = (0, 1, 1)$ (b) $c_4 = (0, 0, 2)$ (c) $c_5 = c_1 = (1, 1, 0)$

A Sink Vertex

- A vertex v of G with no outbound edges ($e_v = 0$) is known as a **sink**.
- Sink vertices **never** fire.
- If every vertex has a direct path to a sink it is the **universal** sink.
- **Every** configuration on a directed graph with a universal sink stabilizes.
- From now on, all graphs will have a universal sink.

Sink Example

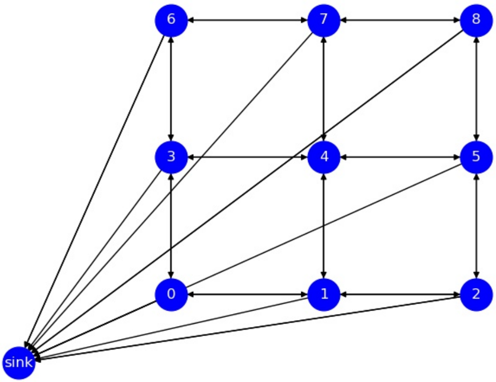


Figure: Directed graph with a sink. $|e_{v_i}| = 4$ for all non-sink vertices. Vertices are labeled with their index/ID.

Chip Configuration Revisited

- The sink vertex is excluded from the configuration vector

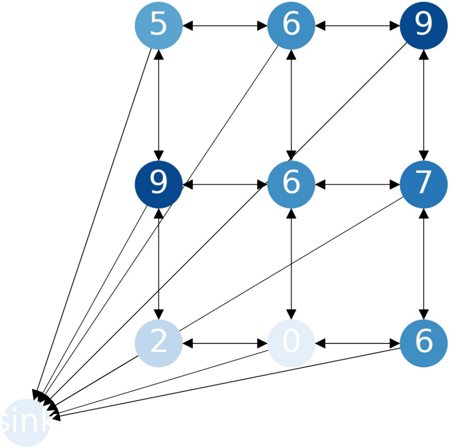


Figure: $c_t = (2, 0, 6, 9, 6, 7, 5, 6, 9)$, indexed as on previous slide.

Reduced Laplacian

- D is the out-degree matrix of G:

$$D_{i,j} = \begin{cases} |e_{v_i}| & : i = j \\ 0 & : i \neq j \end{cases}$$

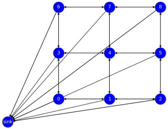
- e_{v_i,v_j} = edges from v_i to v_j

- A is the adjacency matrix of G:

$$A_{i,j} = \begin{cases} |e_{v_i,v_j}| & : i \neq j \\ 0 & : \text{otherwise} \end{cases}$$

- The **reduced Laplacian** is the matrix $L = D - A$ with rows/columns corresponding to the sink removed.

Reduced Laplacian Example



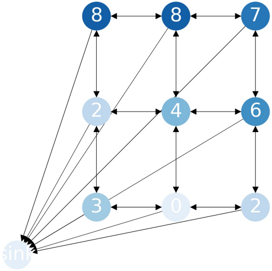
$$L = \begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix}$$

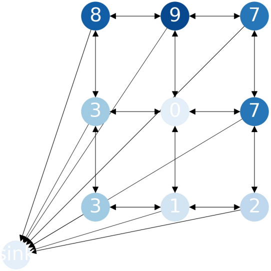
Firing and the Reduced Laplacian

- Let L_i be the row of the reduced Laplacian corresponding to v_i
- If vertex v_j fires at time/iteration t:

$$c_{t+1} = c_t - L_j$$

- $L_4 = (0, -1, 0, -1, 4, -1, 0, -1, 0)$





$c_t = (3, 0, 2, 2, 4, 6, 8, 8, 7)$ $\quad c_{t+1} = (3, 1, 2, 3, 0, 7, 8, 9, 7)$

Stabilizing Flipbook

Firing History

- The vertices fired were: [0, 1, 2, 1, 0, 2, 3, 3, 4, 4, 1, 5, 4, 5, 6, 3, 0, 6, 7, 7, 4, 5, 2, 1, 7, 6, 3, 4, 8, 8, 5, 7, 8]
- The **history** is the number of times each vertex fired in a stabilization:
  {0: 3, 1: 4, 2: 3, 3: 4, 4: 5, 5: 4, 6: 3, 7: 4, 8: 3}

An Alternate Stabilizing Flipbook

Firing History

- The vertices fired were: [8, 7, 6, 7, 8, 6, 5, 5, 4, 4, 7, 3, 4, 3, 2, 5, 8, 2, 1, 1, 4, 3, 6, 7, 1, 2, 5, 4, 0, 0, 3, 1, 0]
- **history**:
  {0: 3, 1: 4, 2: 3, 3: 4, 4: 5, 5: 4, 6: 3, 7: 4, 8: 3}
- !
- From a given starting configuration, all stabilizations reach the same **final** configuration
- Each vertex fires the **same number of times** in all stabilizations

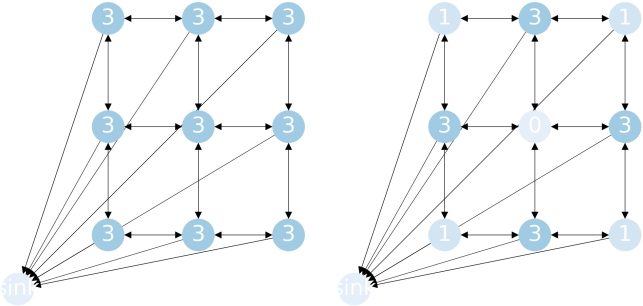
Chip Addition Operator

- If c is a configuration, let c^o denote its stabilization.
- Let $\mathbb{1}_{v_i}$ denote the configuration with 1 chip on v_i and 0 on all others.
- The **chip addition operator**, $E_{v_i}(c)$ adds one chip to c on the vertex v_i, then stabilizes:

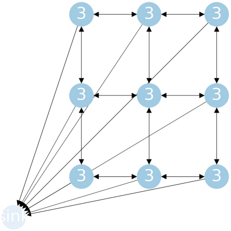
$$E_{v_i}(c) = (c + \mathbb{1}_{v_i})^o$$

Chip Addition Operator Example

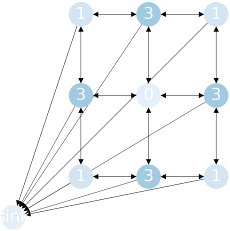
- $c = (3, 3, 3, 3, 3, 3, 3, 3, 3)$
- $E_4(c) = (1, 3, 1, 3, 0, 3, 1, 3, 1)$
- History: {0: 1, 1: 1, 2: 1, 3: 1, 4: 2, 5: 1, 6: 1, 7: 1, 8: 1}



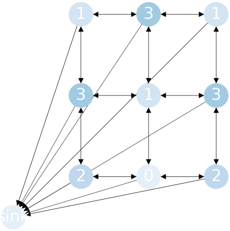
Chip Addition Operator Composition



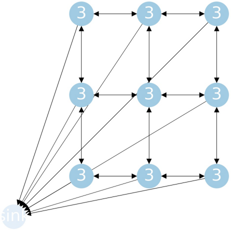
(a) c



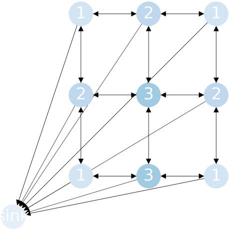
(b) $E_4(c)$



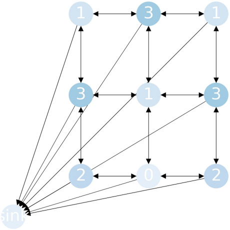
(c) $E_1(E_4(c))$



(a) c



(b) $E_1(c)$



(c) $E_4(E_1(c))$

Abelian Sandpile Model

- Move chips on a directed graph
- Stabilization is well defined
- Chip firing operators commute
- Sink vertex
- Next up, self-optimized criticality