CSE 321 - HOMEWORK 1

1) a) $\log_2 n^2+1 \in O(n)$

$$\lim_{n\to\infty} \frac{\log_2 n^2+1}{n} = \lim_{n\to\infty} \frac{2\log_2 n+1}{n} \xrightarrow{\text{L'Hospital}} \lim_{n\to\infty} \frac{2}{n\ln 2} = \frac{2}{n} = 0$$

$\log_2 n^2+1 \in O(n)$ This statement is $\boxed{\text{true.}}$

b) $\sqrt{n(n+1)} \in \Omega(n)$ $\quad (n+1) = n(1+\frac{1}{n})$

$$\lim_{n\to\infty} \frac{\sqrt{n(n+1)}}{n} = \lim_{n\to\infty} \frac{\sqrt{\cancel{n}\, n\,(1+\frac{1}{n})}}{\cancel{n}} = \lim_{n\to\infty}\sqrt{\frac{n+1}{n}} = \sqrt{\lim_{n\to\infty}\frac{n+1}{n}} = 1$$

$\sqrt{n(n+1)} \in \Theta(n)$ so $\sqrt{n(n+1)} \in O(n)$
$\sqrt{n(n+1)} \in \Omega(n)$ ✓

This statement is true.

c) $n^{n-1} \in \Theta(n^n)$

$$\lim_{n\to\infty} \frac{n^{n-1}}{n^n} = \lim_{n\to\infty} \frac{1}{n} = 0$$ So $n^{n-1} \in O(n^n)$

This statement is $\boxed{\text{false}}$

d) If $f(n) \in O(g(n))$, then $O(f(n)) \subseteq O(g(n))$

$$\lim_{n\to\infty} \frac{2^n+n^3}{4^n} \to \underbrace{\lim_{n\to\infty} \left(\frac{1}{2}\right)^n}_{0} + \underbrace{\lim_{n\to\infty} \frac{n^3}{4^n}}_{0} = 0$$ or

$2^n+n^3 \in O(4^n)$ so $O(2^n+n^3) \subset O(4^n)$

or

$O(n^9) \subset O(2^n) \Rightarrow O(n^3) \subset O(4^n)$.

This statement is $\boxed{\text{true}}$

e) $O(2\log_3^3 \sqrt{n}) \subset O(8\log_2 n^2)$ ignore constant

$O(\log_3 n) \subset O(\log_2 n) \Rightarrow O(\log_3^2 \log_2 n) \subset O(\log_2 n)$

$\hookrightarrow \log_3^2 \cdot \log_2^n$ $\log_3^2 \log_2 n \in \Theta(\log_2 n)$

$\lim_{n\to\infty} \frac{\log_3^2 \cancel{\log_2^n}}{\cancel{\log_2^n}} = \log_3^2$ This statement is $\boxed{\text{false.}}$

f) $\log_2 \sqrt{n}$ and $(\log_2 n)^2$

$\lim_{n\to\infty} \frac{\log_2 \sqrt{n}}{(\log_2 n)^2} = \lim_{n\to\infty} \frac{\frac{1}{2}\cancel{\log_2 n}}{(\log_2 n)^{\cancel{2}}} = \lim_{n\to\infty} \frac{1}{2\log_2 n} = 0$ $\log_2 \sqrt{n} \in O[(\log_2 n)^2]$

Statement is $\boxed{\text{false}}$

2) $10^n > 2^n > n^3 = 8^{\log n} > n^2 \log n > n^2 > \sqrt{n} > \log n$

$\underline{10^n > 2^n}$

$\lim_{n\to\infty} \frac{2^n}{10^n} = \lim_{n\to\infty} \frac{1}{5^n} = 0$ So $\boxed{10^n > 2^n}$

$\underline{2^n > n^3}$

$\lim_{n\to\infty} \frac{n^3}{2^n} = \lim_{n\to\infty} \frac{\cancel{3}n^2}{\cancel{\ln 2}\, 2^n}$ ignore constant $\rightarrow \lim_{n\to\infty} \frac{n^2}{2^n} \Rightarrow$ Continuing with the L'Hospital rule until the numerator is 1 and ignore constants, denominator always remains 2^n. The result would be 0. Therefore $2^n > 3^n$

$\underline{n^3 = 8^{\log n}}$

$8^{\log n} = n^{\log 8} = n^3$

$\underline{8^{\log n} > n^2 \log n}$

$\lim_{n\to\infty} \frac{n^2 \log n}{n^3} = \lim_{n\to\infty} \frac{\log n}{n} \xrightarrow{\text{L'Hospital}} \lim_{n\to\infty} \frac{1}{\ln . n} = 0$ So $n^3 > n^2 \log n$

$n^2 \log n > n^2$

$\lim_{n\to\infty} \frac{\cancel{n^2}}{\log n \cdot \cancel{n^2}} = \lim_{n\to\infty} \frac{1}{\log n} = 0$, So that, $n^2 \log n > n^2$ //

$n^2 > \sqrt{n}$

$\lim_{n\to\infty} \frac{\sqrt{n}}{n^2} = \lim_{n\to\infty} \frac{1}{2\sqrt{n} \cdot 2n} = 0$, So that, $n^2 > \sqrt{n}$

$\sqrt{n} > \log n$

$n \to x^2$

$\lim_{n\to\infty} \frac{\log n}{\sqrt{n}} = \lim_{x^2\to\infty} \frac{\log x^2}{\sqrt{x^2}} = \lim_{x^2\to\infty} \frac{2\log x}{x} = \lim_{x^2\to\infty} \frac{2}{\ln x.} = 0$

So that $\sqrt{n} > \log n$ //

3)

a) There is no such thing as a nested loop or an operation that changes the value of in this function to time complexity. The only thing that affects complexity is the loop. worst case and best case is length of array. Therefore, $O(n) = \Omega(n) = \Theta(n)$

b) Time complexity of a loop is considered as $O(\log \log n)$ if the loop variables is reduced / increased exponentially by a constant amount. (i) increases expenontial do it $i^2 * i$ on each step.
i takes value $2, 2^k, (2^k)^k = 2^{k^k}, \ldots 2^{k^{\log\log n}}$

So, there are in total $\log_k(\log n)$ many iterations, and each iteration takes a constant amount of time to run. So,

Time Complexity $= O(\log \log n)$

4) a) $\sum_{i=1}^{n} i^2 . \log i = \log 1 + 4\log 2 + 9\log 3 \ldots + n^2 \log n$

↳ Non-decreasing

$$\int_0^n (x^2 . \log x)\,dx \leq \sum_{i=1}^{n} i^2 \log i \leq \int_1^{n+1} (x^2 . \log x)\,dx$$

Upper bound:

$u = \log x \qquad v' = x^2$

$$\sum_{i=1}^{n} i^2 \log i \leq \int_1^{n+1} (x^2 . \log x)\,dx$$

$$\leq \frac{x^3}{3} \log x \Big|_1^{n+1} - \int_1^{n+1} \frac{x^2}{3\ln 10}\,dx$$

$$\leq \frac{x^3}{3} \log x \Big|_1^{n+1} - \frac{x^3}{9\ln 10}\Big|_1^{n+1}$$

$$\leq \frac{(n+1)^3}{3} \log(n+1) - \Big(\frac{1}{3}\underbrace{\cancel{\log 1}}_{0}\Big) - \frac{(n+1)^3}{9\ln 10} + \frac{1}{9\ln 10} \quad \text{(Ignore constant)}$$

$$\boxed{\sum_{i=1}^{n} i^2 \log i \in O(n^3 \log n)}$$

Lower bound:

$$\int_0^n (x^2 \log x)\,dx \leq \sum_{i=1}^{n} i^2 \log i \rightarrow f(x)$$

$$\frac{x^3}{3} \log x \Big|_0^n - \frac{x^3}{9\ln 10}\Big|_0^n \leq \sum_{i=1}^{n} i^2 \log i \xrightarrow{\text{modified}} 1 + \int_1^n (x^2 \log x)\,dx \leq f(x)$$

$\log 0 \rightarrow$ undefined

$$1 + \frac{x^3}{3} \log x \Big|_1^n - \frac{x^3}{9\ln 10}\Big|_1^n \leq f(x)$$

$$\cancel{1} + \frac{n^3}{3} \log n - \frac{1}{3}\underbrace{\cancel{\log 1}}_{0} - \frac{n^3}{9\ln 10} + \frac{1}{9\ln 10} \leq f(x)$$

Ignore constant

$$f(x) \in \Omega(n^3 \log n)$$

Therefore $\boxed{f(x) \in \Theta(n^3 \log n)}$

b) $\sum_{i=1}^{n} i^3$

$$\int_0^n x^3 dx \leq \sum_{i=1}^{n} i^3 \leq \int_1^{n+1} x^3 dx$$

Upper bound:

$$\sum_{i=1}^{n} i^3 \leq \int_1^{n+1} x^3 dx$$

$$f(x) \leq \frac{x^4}{4}\Big|_1^{n+1} \Rightarrow f(x) \leq \frac{(n+1)^4}{4} - \frac{1}{4}$$

$$f(x) \in O(n^4)$$

Lower bound:

$$\int_0^n x^3 dx \leq \sum_{i=1}^{n} i^3$$

$$\frac{x^4}{4}\Big|_0^n \leq f(x) \Rightarrow \frac{n^4}{4} - \frac{0}{4} \leq f(x)$$

$$f(x) \in \Omega(n^4)$$

$\boxed{\text{Therefore } f(x) \in \Theta(n^4)}$

c) $\sum_{i=1}^{n} 1/2\sqrt{i}$ $\quad \int_1^{n+1} \frac{1}{2\sqrt{x}} dx \leq f(x) \leq \int_0^n \frac{1}{2\sqrt{x}} dx$

Upper bound:

$$f(x) \leq \int_0^n \frac{1}{2\sqrt{x}} dx \Rightarrow f(x) \leq \sqrt{x}\Big|_0^n \rightarrow f(x) \leq \sqrt{n} - \sqrt{0}$$

$$f(x) \in O(\sqrt{n})$$

Lower bound:

$$\int_1^{n+1} \frac{1}{2\sqrt{x}} dx \leq f(x) \rightarrow \sqrt{x}\Big|_1^{n+1} \leq f(x) \rightarrow \sqrt{n+1} - 1 \leq f(x)$$

$$f(x) \in \Omega(\sqrt{n})$$

$\boxed{\text{Therefore } f(x) \in \Theta(\sqrt{n})}$

d) $\sum_{i=1}^{n} 1/i \quad \int_{1}^{n+1} \frac{1}{x} dx \leq f(x) \leq \int_{0}^{n} \frac{1}{x} dx$

Upper bound:

$$f(x) \leq \int_{0}^{n} 1/x \, dx \Rightarrow f(x) \leq \ln x \Big|_{0}^{n} \Rightarrow f(x) = \infty$$

$$f(x) \leq 1 + \int_{1}^{n} 1/x \, dx \Rightarrow f(x) \leq \ln x \Big|_{1}^{n} \Rightarrow f(x) \leq \ln n + 1$$

$$f(x) \in \Omega(\ln n)$$

Lower bound:

$$\int_{1}^{n+1} \frac{1}{x} dx \leq f(x) \Rightarrow \ln x \Big|_{1}^{n+1} \leq f(x) \Rightarrow \ln(n+1) + \ln(1)$$

$$f(x) \in O(\ln n)$$

$\boxed{\text{Therefore } f(x) \in \Theta(\log n)}$

5) Best Case: If $x = L[1]$, then the best case occurs

$$B(n) = 1 \in \Theta(1)$$

Worst Case: If $x = L[n]$ or x does not exist in L then the worst case occurs.

$$W(n) = n \in \Theta(n)$$