Estimating Rigid Body Motion from a Variety of Geometric Data in 3D Conformal Geometric Algebra

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Abstract

The motion rotors, or *motors*, are used to model rigid body motion in 3D conformal geometric algebra. In this paper we present a technique for estimating the motor which best transforms one set of perturbed geometric objects onto another. The technique reduces to an eigenrotator problem and has advantages over matrix formulations. It allows motors to be estimated from a variety of geometric data such as points, spheres, circles, lines, planes, directions, and tangents; and the different types of geometric data are combined naturally in a single framework. Also, it excludes the possibility of a reflection unlike some matrix formulations. It returns the motor with the smallest translation and rotation angle when the optimal motor is not unique.

Keywords: rigid body motion, conformal geometric algebra, estimation

1 Introduction

The motion rotors or *motors*, denoted $\mathcal{M}$, are used to model rigid body motion in 3D conformal geometric algebra (CGA). It is often useful to be able to estimate a motor which best maps one data set onto another in some sense. The canonical problem involves two sets of noisy points where one set is nominally a rotated and translated version of the other. This situation arises frequently, for example when two sets of reconstructed 3D points need to be merged and they share some common points. Several solutions exist to minimise the squared distance between the points, using matrix techniques based on SVD, polar decomposition, and quaternions [1]. In addition to points, many other geometric objects such as lines, directions and planes provide useful information which can be used to help estimate the rigid body relationship between the data sets. In this paper we present a technique for estimating a motor from a wide variety of geometric data including points, rounds (point pairs, circles, spheres), flats (lines, planes), tangents, and directions; which can all be combined naturally in a single framework while excluding reflection. Some aspects of the work are expanded in [2].

The following conventions are used in this paper. The geometric algebra over $\mathbb{R}$ with p positive and q negative basis elements is denoted $\mathcal{G}_{p,q}$. When $q=0$ we write $\mathcal{G}_p$. A pure Euclidean multivector in $\mathcal{G}_3$ is usually represented in boldface. The grade-r, even, and odd elements of $\mathcal{G}_{p,q}$ are denoted $\mathcal{G}^r_{p,q}$, $\mathcal{G}^+_{p,q}$ and $\mathcal{G}^-_{p,q}$ respectively. We refer to $\mathcal{G}_{4,1}$ as the *conformal geometric algebra* (CGA) of 3D space. The dual of X is denoted $X^* = X \cdot I^{-1}$. The CGA vector n_o represents the origin and the CGA vector n_∞ represents the point at infinity, with $n_o \cdot n_\infty = -1$. The notation $\langle X\rangle_{i,j,\ldots,k}$ is used as an abbreviation for $\langle X\rangle_i + \langle X\rangle_j + \cdots + \langle X\rangle_k$.

2 The linear spaces $\mathbb{M}$, $\mathbb{B}$ and $\mathbb{S}$

The 8D linear space $\mathbb{M} = \mathcal{G}^+_3 \cup \mathcal{G}^-_3 n_\infty = \mathrm{sp}\{1, e_{12}, e_{13}, e_{23}, e_1 n_\infty, e_2 n_\infty, e_3 n_\infty, I_3 n_\infty\}$ is the smallest linear space in which motors reside. It is convenient to restrict most of the analysis to elements in $\mathbb{M}$ because many simplifications arise. Most of these are consequences of the following split: if $X \in \mathbb{M}$ then $X = R + Q$ where $R \in \mathcal{G}^+_3$ and $Q \in \mathcal{G}^-_3 n_\infty = \{\mathbf{V} n_\infty : \mathbf{V} \in \mathcal{G}^-_3\}$. $\mathbb{M}$ is closed under multiplication so $X, Y \in \mathbb{M} \Rightarrow XY \in \mathbb{M}$. It is also convenient to split $X \in \mathbb{M}$ into symmetric and antisymmetric parts $X = S + B$ where $S = \frac{1}{2}(X + \widetilde{X}) = \langle X\rangle_{0,4}$ and $B = \frac{1}{2}(X - \widetilde{X}) = \langle X\rangle_2$. The antisymmetric grade-2 elements of $\mathbb{M}$ will be denoted $\mathbb{B} = \mathrm{sp}\{e_{12}, e_{13}, e_{23}, e_1 n_\infty, e_2 n_\infty, e_3 n_\infty\}$ and the symmetric grade 0 and 4 elements denoted $\mathbb{S} = \mathrm{sp}\{1, I_3 n_\infty\}$. $\mathbb{S}$ is closed under multiplication so $S_1, S_2 \in \mathbb{S} \Rightarrow S_1 S_2 \in \mathbb{S}$. Note that if $X \in \mathbb{M}$ then $X\widetilde{X} = \langle X\widetilde{X}\rangle + \langle X\widetilde{X}\rangle_4$ so the condition $X\widetilde{X} = 1$ encodes two constraints: $\langle X\widetilde{X}\rangle = 1$ and $\langle X\widetilde{X}\rangle_4 = 0$ (there is only one grade-4 basis element in $\mathbb{M}$). The following lemma uses these constraints to characterise how the 6D motor manifold $\mathcal{M}$ sits in the 8D linear space $\mathbb{M}$.

Lemma 1. $X \in \mathcal{M} \Leftrightarrow X \in \mathbb{M}$ *and* $X\widetilde{X} = 1$.

Proof. Let $X = R + Q \in \mathbb{M}$ where $R \in \mathcal{G}_3^+$ and $Q \in \mathcal{G}_3^- n_\infty$. $X\widetilde{X} = 1$ implies $R\widetilde{R} = 1$ and $Q\widetilde{R} = \langle Q\widetilde{R}\rangle_2$. Thus R is a rotator and $X = R + Q\widetilde{R}R = \left(1 + \langle Q\widetilde{R}\rangle_2\right) R = TR$ where $T = 1 + \langle Q\widetilde{R}\rangle_2$ is a translator. □

The space $\mathbb{M}$ is incomplete in the sense that, given a basis of $\mathbb{M}$, we can not find a reciprocal basis that also lies in $\mathbb{M}$. We can enlarge $\mathbb{M}$ to a complete space such as $\mathbb{M} \cup \mathcal{G}_3^- n_o$ or $\mathcal{G}_{4,1}^+$ and then construct a reciprocal basis. The subspace spanned by reciprocal vectors associated with elements in $\mathbb{M}$ is denoted $\overline{\mathbb{M}} = \mathcal{G}_3^+ \cup \mathcal{G}_3^- n_o = \mathrm{sp}\{1, \widetilde{e}_{12}, \widetilde{e}_{13}, \widetilde{e}_{23}, e_1 n_o, e_2 n_o, e_3 n_o, \widetilde{I}_3 n_o\}$. We will sometimes want to project an element $X \in \mathcal{G}_{4,1}$ on $\mathbb{M}$ or $\overline{\mathbb{M}}$. Let $\{e_J\}$ be a basis for $\mathbb{M}$ and $\{e^J\}$ be the associated reciprocal basis in $\overline{\mathbb{M}}$. A projection on $\mathbb{M}$ is defined by $\mathrm{P}_{\mathbb{M}}(X) = \sum_J \langle e^J X\rangle e_J$. As $\langle \mathrm{P}_{\mathbb{M}}(X) Y\rangle = \sum_J \langle e^J X\rangle \langle e_J Y\rangle = \langle X \bar{\mathrm{P}}_{\mathbb{M}}(Y)\rangle$ the adjoint is the projection onto $\overline{\mathbb{M}}$ given by $\bar{\mathrm{P}}_{\mathbb{M}}(Y) = \mathrm{P}_{\overline{\mathbb{M}}}(Y) = \sum_J e^J \langle e_J Y\rangle$. When no ambiguity arises, it is convenient to use the terse notation P_X for the projection onto the basis of the linear space in which the element X resides. For example, if $X = R + Q \in \mathbb{M}$, where $R \in \mathcal{G}_3^+$ and $Q \in \mathcal{G}_3^- n_\infty$, then $\bar{\mathrm{P}}_{\mathbb{M}} = \bar{\mathrm{P}}_X = \mathrm{P}_R + \bar{\mathrm{P}}_Q$ because $\mathrm{P}_R = \bar{\mathrm{P}}_R$.

3 Geometry of the Motors

Refer to Fig. 1 which illustrates some of the concepts introduced in this section. Consider the curve $\psi(t) \in \mathcal{M}$ with $M = \psi(0)$ and $\Delta = \psi'(0)$. Differentiating the constraint $\widetilde{\psi}(t)\psi(t) = 1$ and evaluating at $t = 0$ gives $\widetilde{M}\Delta = -\widetilde{\Delta}M$. As $\Delta \in \mathbb{M}$, $\widetilde{M}\Delta \in \mathbb{M}$ and it follows that $\Delta = MB$ where $B \in \mathbb{B}$. We define the *tangent space* of $\mathcal{M}$ at $M \in \mathcal{M}$ by $\mathcal{T}_M = M\mathbb{B} = \{MB : B \in \mathbb{B}\} \subset \mathbb{M}$. Any element $X \in \mathbb{M}$ can be split $X = M(\widetilde{M}X) = M\langle \widetilde{M}X\rangle_2 + M\langle \widetilde{M}X\rangle_{0,4}$. The first term in this split is in $\mathcal{T}_M$ while the second term is of the form MS where $S \in \mathbb{S}$. We define the *normal space* of $\mathcal{M}$ at $M \in \mathcal{M}$ (restricted to $\mathbb{M}$) by $\mathcal{N}_M = M\mathbb{S} = \{MS : S \in \mathbb{S}\}$. If $X = MB \in \mathcal{T}_M$ and $Y = MS \in \mathcal{N}_M$ then $\langle X\widetilde{Y}\rangle = \langle MBS\widetilde{M}\rangle = \langle BS\rangle = 0$ and $\mathcal{T}_M$ is orthogonal to $\mathcal{N}_M$ so $\mathbb{M} = \mathcal{T}_M \oplus \mathcal{N}_M$. From the split, for $X \in \mathbb{M}$ we can define the projection on $\mathcal{T}_M$ along $\mathcal{N}_M$ by $\mathrm{P}_{\mathcal{T}_M}(X) = M\langle \widetilde{M}X\rangle_2$. It is clear that $\mathrm{P}_{\mathcal{T}_M}$ is idempotent, onto $\mathcal{T}_M$ and has null-space $\mathcal{N}_M$. Similarly, for $X \in \mathbb{M}$

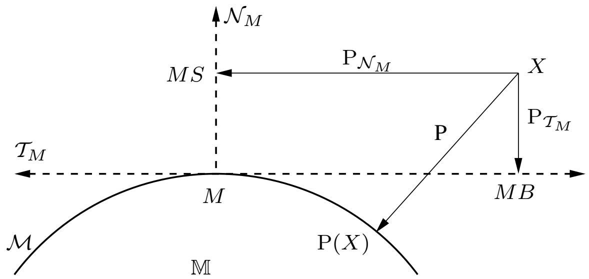

Figure 1: An intuitive sketch of the geometry of motors $\mathcal{M}$ in $\mathbb{M}$ showing the tangent space $\mathcal{T}_M$ and the normal space $\mathcal{N}_M$ at M, and the projections onto $\mathcal{T}_M$, $\mathcal{N}_M$, and $\mathcal{M}$.

the projection on $\mathcal{N}_M$, along $\mathcal{T}_M$ is defined by $\mathrm{P}_{\mathcal{N}_M}(X) = M\langle \widetilde{M}X\rangle_{0,4}$. It is also clear that $\mathrm{P}_{\mathcal{N}_M}$ is idempotent, along $\mathcal{T}_M$ and onto $\mathcal{N}_M$. Closely related to $\mathcal{N}_M$, we can define a polar decomposition for an element in $\mathbb{M}$.

Lemma 2. *An element* $X \in \mathbb{M}$ *with* $|X| \neq 0$ *has a unique polar decomposition* $X = MS = SM$ *where* $M \in \mathcal{M}$*,* $S \in \mathbb{S}$ *and* $\langle S\rangle > 0$.

Proof. A polar decomposition is given by $S = |X|\left(1 + \frac{\langle X\widetilde{X}\rangle_4}{2\langle X\widetilde{X}\rangle}\right)$ and $M = XS^{-1} = \frac{X}{|X|}\left(1 - \frac{\langle X\widetilde{X}\rangle_4}{2\langle X\widetilde{X}\rangle}\right)$. Suppose $MS = M'S'$ are two such decompositions, then $N = \widetilde{M}'M = S'S^{-1}$ is a symmetric motor ($N = \widetilde{N}$). Hence $N = \alpha + \beta I_3 n_\infty$ and $1 = N^2 = \alpha^2 + 2\alpha\beta I_3 n_\infty$ so $\beta = 0$ and $\alpha = 1$ because $\langle S\rangle > 0$ and $\langle S'\rangle > 0$. As $MI_3 n_\infty \widetilde{M} = I_3 n_\infty$ we have $MS = SM$. □

As shown, given $M \in \mathcal{M}$ any $X \in \mathbb{M}$ can be decomposed into components in $\mathcal{T}_M$ and $\mathcal{N}_M$ giving $X = MS + MB$. The polar decomposition can be interpreted as simply choosing M appropriately so that the component in $\mathcal{T}_M$ vanishes leaving $X = MS \in \mathcal{N}_M$. The polar decomposition of $X \in \mathbb{M}$ provides a natural way to define the operation of projection onto $\mathcal{M}$ given by

$$\mathrm{P}(X) = \frac{X}{|X|}\left(1 - \frac{\langle X\widetilde{X}\rangle_4}{2\langle X\widetilde{X}\rangle}\right) \in \mathcal{M}\,.$$

4 Estimating Motors

We have two sets of noisy geometric data and wish to estimate the motor that optimally maps one data set onto the other. To solve this problem we need to be precise about what optimal means so we will define a measure that is used to determine if two geometric objects are similar.

4.1 Similarity Measures in CGA

In order to set the problem up as an eigenrotator problem we need to restrict the similarity measure between objects P and Q to the simple form $\langle P\check{Q}\rangle$ where the *check* operator $\check{Q}$ is a grade dependent sign change defined by $\check{Q} = \langle Q\rangle_{0,1,3} - \langle Q\rangle_{2,4,5}$. This operation is motivated by the requirements (i) $\langle p\check{q}\rangle = \langle P\check{Q}\rangle$ where $p = P^*$ and $q = Q^*$ and (ii) $\langle P\check{Q}\rangle = \cos(\theta)$ when P, Q are flats (see below). This simple form is not as much of a restriction as it first seems. Consider the following examples:

Flats: Flats are objects like planes and lines. A flat can be modelled $P = p \wedge \mathbf{V} \wedge n_\infty$ where p is a point on the flat and $\mathbf{V}$ is a Euclidean blade representing the direction of the flat. If $\mathbf{V}$ is a Euclidean vector then $p \wedge \mathbf{V} \wedge n_\infty$ is a line and if $\mathbf{V}$ is a Euclidean bivector then $p \wedge \mathbf{V} \wedge n_\infty$ is a plane. The other cases are of no practical interest here. If P and Q are normalised flats so $|P| = 1$ and $|Q| = 1$ then

$$\langle P\check{Q}\rangle = \cos(\theta)$$

where θ is the dihedral angle between them. Two flats are considered similar if the angle between them is small. Note that for small θ, $\cos(\theta) \approx -\frac{\theta^2}{2!} + 1$ so in practice there is often no disadvantage in maximising $\cos(\theta)$ instead of $-\frac{\theta^2}{2!}$. Because $\cos(\theta) \geq -1$ it has an added benefit of restricting the influence of outliers. One potential concern with the measure is that it does not capture the distance between lines, only the angle. The distance is usually regarded as the closest distance that the lines pass. It is a simple matter to determine this distance, for example, by forming the motor $P\widetilde{Q}$ and making use of Chasles's decomposition. It is not clear how to do this while keeping the simple form of a scalar product $\langle P\check{Q}\rangle$. This is not usually a concern with planes as they will always intersect unless they are exactly parallel, so we are normally only interested in the angle between them.

Directions: Directions are used to model 1D direction and attitude and can be represented in CGA in the form $\Delta = \mathbf{V}n_\infty$ where $\mathbf{V}$ is a Euclidean blade. They are translation invariant so for translator T we have $T\Delta\widetilde{T} = \Delta$. The case where $\mathbf{V}$ is grade-1 gives a 1D or line direction, and the case where $\mathbf{V}$ is grade-2 gives a 2D or plane direction. The other cases (scalar and grade-3) are of no practical interest here. If Δ_p and Δ_q are two directions then $\langle \Delta_p\check{\Delta}_q\rangle = 0$ so we cannot use the directions directly. We can construct a meaningful quantity by representing the directions as flats $n_o \wedge \Delta$, dual flats $n_o \cdot \Delta^*$, or Euclidean directions $n_o \cdot \Delta$. If P and Q are two normalised directions represented in one of the above three forms then

$$\langle P\check{Q}\rangle = \cos(\theta)$$

where θ is the dihedral angle between them. Two directions are considered similar if the angle between them is small.

Rounds and Tangents: Rounds are objects like spheres, circles, and points pairs and have a radius, location, and direction. A round with squared radius ρ^2 at location p with normalised direction $\Delta = \mathbf{V}n_\infty$ can be represented in CGA as a blade $R' = \bar{s}\wedge(s\cdot\hat{\Delta})$ where $s = p-\frac{1}{2}\rho^2 n_\infty$ is an associated dual sphere and $\bar{s} = p + \frac{1}{2}\rho^2 n_\infty$ is the complementary dual sphere. If Δ is a bivector then R' is a point pair and if Δ is a trivector then R' is a circle. When $\Delta = n_\infty$ then $R' = \bar{s}$ and when $\Delta = I_3 n_\infty$ then $R' = S = -s^*$. Unfortunately, except in the case of spheres, taking the inner product between two rounds in this form does not give a particularly meaningful quantity. If we are prepared to consider a broader range of representations than blades then we can construct a meaningful quantity using the measure. Let $R = s + F$ be a flag (nested sequence of linear spaces) representation of the round with where s is as above and $F = R' \wedge n_\infty$ is the carrier flat so $s \wedge F = 0$. The representations R and R' are equivalent and we can easily calculate one from the other. If $P = s_p + F_p$ and $Q = s_q + F_q$ are two rounds represented as flags then $\langle P\check{Q}\rangle = \langle s_p s_q\rangle + \langle F_p\check{F}_q\rangle$ giving

$$\langle P\check{Q}\rangle = -\frac{1}{2}d^2 + \cos(\theta) + \frac{1}{2}(\rho_p^2 + \rho_q^2)$$

where d is the distance between the round locations, θ is the dihedral angle between the round carriers, and ρ_p, ρ_q are the round radii. As the radii are constant under rigid body motion, two rounds are considered similar if their locations are close and the angle between them is small. We can adjust ratio of the locational and angular parts by encoding a weight in the flag. For example if $w = |F|$ then $\langle P\check{Q}\rangle = -\frac{1}{2}d^2 + w^2\cos(\theta) + \frac{1}{2}(\rho_p^2 + \rho_q^2)$. While the flag form is completely general spheres are handled more easily in their blade form giving $\langle P\check{Q}\rangle = -\frac{1}{2}d^2 + \frac{1}{2}(\rho_p^2 + \rho_q^2)$. Tangents can be regarded as rounds with zero radius and can be used to model various objects such as tangent planes on a surface, tangent lines on a curve, and rays leaving a camera where the optical centre is the location. Tangents can be handled by simply replacing s_p and s_q by p and q giving $\langle P\check{Q}\rangle = -\frac{1}{2}d^2 + \cos(\theta)$. Points are classified as tangents with a null direction and as with spheres are more simply handled in blade form giving $\langle P\check{Q}\rangle = -\frac{1}{2}d^2$.

We have associated a physically meaningful measure with the basic objects available in CGA. Other ways of representing the objects P and

Q can be designed to give different measures. The only structural requirement is that they are expressed in the form $\langle P\check{Q}\rangle$.

4.2 Motor Estimation Problem Formulation

We are now in a position to formulate the estimation problem. Let P_k, $k = 1 \ldots n$ be a set of normalised CGA objects before motion, and Q_k, $k = 1 \ldots n$ be the set of objects after motion, $w_k \in \mathbb{R}$ be scalar weights, and $M \in \mathcal{M}$. The total similarity is given by the weighted sum of the symmetrised similarity between $MP_k\widetilde{M}$ and Q_k as follows

$$E = \langle \widetilde{M}\mathcal{L}M\rangle \qquad \text{where} \tag{1}$$

$$\mathcal{L}X = \frac{1}{2}\sum_{k=1}^{n} w_k(\check{Q}_k X P_k + \widetilde{\check{Q}}_k X \widetilde{P}_k)\ . \tag{2}$$

Note that $\mathcal{L}$ satisfies the useful symmetry property $\langle \widetilde{A}\mathcal{L}B\rangle = \langle \widetilde{B}\mathcal{L}A\rangle$ for all $A, B \in \mathcal{G}_{4,1}$. The data P_k, $k = 1 \ldots n$, need not all be of the same object type, but could contain a variety of geometric objects such as points, spheres, flats and directions. Clearly, for a given k, P_k and Q_k represent the same object type as one is simply a rotated and translated version of the other. The magnitude of the weights w_k can be used to adjust the contribution a data element makes based on its reliability, or to introduce attractive and repulsive contributions. We can now couch the problem of finding an optimal motor more precisely as maximising $\langle \widetilde{X}\mathcal{L}X\rangle$ subject to $X \in \mathcal{M}$. Using Lemma 1 we can rewrite this as

$$\max_{X\in\mathbb{M}} \langle \widetilde{X}\mathcal{L}X\rangle \text{ s.t. } \langle X\widetilde{X}\rangle = 1 \text{ and } \langle X\widetilde{X}\rangle_4 = 0\ . \tag{3}$$

The constraint $\langle X\widetilde{X}\rangle_4 = 0$ is awkward and makes solving problem (3) in closed form difficult. It transpires that for many object representations it is passive and can be dropped completely without affecting the solution. We will examine the conditions under which this is possible. The following lemma characterises those elements which are nearly motors, where we have not enforced the constraint $\langle X\widetilde{X}\rangle_4 = 0$.

Lemma 3. *$X \in \mathbb{M}$ and $\langle X\widetilde{X}\rangle = 1 \Leftrightarrow X = M + \beta M I_3 n_\infty$, $M \in \mathcal{M}$, and $\beta \in \mathbb{R}$.*

Proof. Let $X = MS$ be the polar decomposition of $X \in \mathbb{M}$ with $S = \alpha + \beta I_3 n_\infty$. Because $1 = \langle \widetilde{X}X\rangle = \langle S^2\rangle = \alpha^2$ and $\alpha \geq 0$ we have $\alpha = 1$ and $X = M + \beta M I_3 n_\infty$. Let $M \in \mathcal{M} \subset \mathbb{M}$. Note $I_3 n_\infty \in \mathbb{M}$ so $X = M + \beta M I_3 n_\infty \in \mathbb{M}$ and $\langle X\widetilde{X}\rangle = \langle M\widetilde{M}\rangle = 1$. ☐

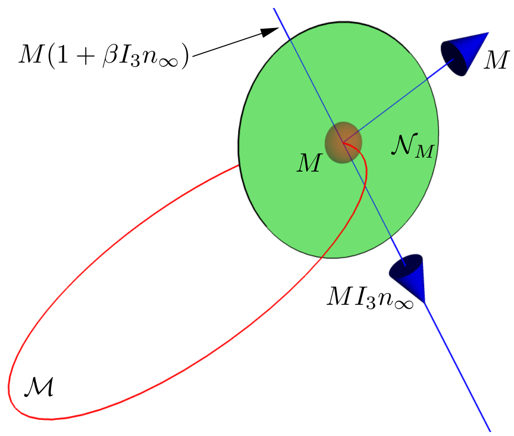

Figure 2: Sketch showing the 2D normal space $\mathcal{N}_M$ of $\mathcal{M}$ at M (restricted to $\mathbb{M}$). Imposing the constraint $\langle X\widetilde{X}\rangle = 1$ restricts us to the 1D subspace of $\mathcal{N}_M$ consisting of elements of the form $M(1 + \beta I_3 n_\infty)$.

On the LHS of Lemma 3 we have the 8D space $\mathbb{M}$ with one constraint imposed, on the RHS we have the 6D motor manifold $\mathcal{M}$ with an extra degree of freedom added through β. Let $\Psi = I_3 n_\infty = \widetilde{\Psi}$ and denote the set defined in Lemma 3 by $\mathcal{M}' = \{M + \beta M\Psi : M \in \mathcal{M}, \beta \in \mathbb{R}\}$. Note that for $X \in \mathcal{M}'$ we have $\langle X\widetilde{X}\rangle_4 = 2\beta\Psi$. When $\beta = 0$, $\langle X\widetilde{X}\rangle_4 = 0$ and $X \in \mathcal{M}$, as $|\beta|$ increases X leaves $\mathcal{M}$ along a 1D subspace of $\mathcal{N}_M$ and $\langle X\widetilde{X}\rangle_4 \neq 0$. A sketch of the situation is shown in Fig. 2. Note that M is both a point on $\mathcal{M}$ and a direction vector in $\mathcal{N}_M$. We will examine the behaviour of $\langle \widetilde{X}\mathcal{L}X\rangle$ with elements $X \in \mathcal{M}'$ which allows us to separate out the terms which result from relaxing the constraint $\langle X\widetilde{X}\rangle_4 = 0$. Expanding the objective function at $X = M + \beta M\Psi \in \mathcal{M}'$ gives

$$\langle \widetilde{X}\mathcal{L}X\rangle = C_0 + 2C_1\beta + C_2\beta^2\ . \tag{4}$$

where $C_0 = \langle \widetilde{M}\mathcal{L}M\rangle$, $C_1 = \langle \widetilde{M}\mathcal{L}(M\Psi)\rangle$, and $C_2 = \langle \widetilde{\Psi}\widetilde{M}\mathcal{L}(M\Psi)\rangle$. For a given $M \in \mathcal{M}$ this is a quadratic in β. We are interested in the cases where C_1 (the coefficient of β) vanishes and C_2 (coefficient of β^2) is not positive, independently of M. When C_2 is negative, leaving $\mathcal{M}$ decreases the objective function and maximising $\langle \widetilde{X}\mathcal{L}X\rangle$ subject to $X \in \mathcal{M}'$ will give us the optimal motor which solves problem (3). If C_2 vanishes then the solution is not unique and if $M \in \mathcal{M}$ is a solution then so is $M(1 + \beta\Psi)$. In such situations we can maximise $\langle \widetilde{X}\mathcal{L}X\rangle$ to give a solution, and then project the resulting X onto $\mathcal{M}$ to get the optimal motor. The object representations where C_1 vanishes and C_2 is not positive (so we can ignore the constraint $\langle \widetilde{X}X\rangle_4 = 0$) are referred to as *admissible*.

If $P'_k = \widetilde{M}Q_kM$ then C_1 is made up of terms $\langle\Psi(\check{P}'_kP_k + P_k\check{P}'_k)\rangle$ and C_2 is made up of terms $\langle\Psi\check{P}'_k\Psi P_k\rangle = -\langle n_\infty\check{P}'_kn_\infty P_k\rangle$. Because $\langle\widetilde{X}\check{P}_kYP_k\rangle = \langle\widetilde{X}\check{p}_kYp_k\rangle$ where $p_k = P^*_k$ and $q_k = Q^*_k$ we do not need to consider all cases. If P_k and P'_k are scalars (or pseudoscalars) then they are invariant under rigid body motion, and $\langle\Psi\check{P}'_kP_k\rangle = \check{P}'_kP_k\langle\Psi\rangle = 0$ and $\langle\Psi\check{P}'_k\Psi P_k\rangle = \check{P}'_kP_k\langle\Psi^2\rangle = 0$ so $C_1 = C_2 = 0$. If P_k and P'_k are vectors $\check{P}'_kP_k$ has no grade-4 part so $C_1 = 0$. As $n_\infty Q_kn_\infty = n_\infty\widetilde{M}Q_kMn_\infty$, C_2 is independent of M. If P_k and P'_k are points or dual spheres then $\langle n_\infty\check{P}'_kn_\infty P_k\rangle = 2$ so $C_2 < 0$. If P_k and P'_k dual planes then $n_\infty\check{P}'_kn_\infty P_k = 0$ so $C_2 = 0$. If P_k and P'_k are lines then $\langle\Psi\check{P}'_kP_k\rangle = 0$ and $n_\infty\check{P}'_kn_\infty P_k = 0$ so $C_1 = C_2 = 0$. We see that points, spheres, lines, and planes are all admissible. In the same spirit we can show flags of the form $X = \langle X\rangle_1 + \langle X\rangle_r$ where $r \neq 5$ are also admissible [2] so all the objects represented in Sect. 4.1 are admissible.

We wish to maximise $\langle\widetilde{X}\mathcal{L}X\rangle$ where $X \in \mathcal{M}$ as stated in problem (3). For admissible objects we can neglect the awkward condition $\langle\widetilde{X}X\rangle_4 = 0$ and solve the simpler problem:

$$\max_{X\in\mathbb{M}} \langle\widetilde{X}\mathcal{L}X\rangle \text{ subject to } \langle X\widetilde{X}\rangle = 1\,. \qquad (5)$$

Theorem 1. *Let P_k and Q_k, $k = 1\ldots n$ be two sets of admissible normalised conformal objects in $\mathcal{G}_{4,1}$, $w_k \in \mathbb{R}$ be scalar weights, and $\mathcal{L}$ be defined by $\mathcal{L}X = \frac{1}{2}\sum_{k=1}^n w_k(\check{Q}_kXP_k + \widetilde{\check{Q}}_kX\widetilde{P}_k)$. Then the maximiser of $\langle\widetilde{M}\mathcal{L}M\rangle$ subject to $M \in \mathcal{M}$ is given by $M = R+Q$ where R is an eigenrotator of $P_R\mathcal{L}'$ associated with the largest eigenvalue, $\mathcal{L}' = \mathcal{L} - \mathcal{L}(\bar{\mathrm{P}}_Q\mathcal{L}\mathrm{P}_Q)^+\mathcal{L}$, and $Q = -(\bar{\mathrm{P}}_Q\mathcal{L}\mathrm{P}_Q)^+\mathcal{L}R$.*

Proof. The Lagrange function associated with problem (5) is given by $L(X) = \frac{1}{2}\langle\widetilde{X}\mathcal{L}X\rangle - \frac{\alpha}{2}(\langle\widetilde{X}X\rangle - 1)$ for $X \in \mathbb{M}$. The first order optimality condition $\partial_{\widetilde{X}}L = 0$ gives $\bar{\mathrm{P}}_{\mathbb{M}}\mathcal{L}X = \alpha\bar{\mathrm{P}}_{\mathbb{M}}X$. Let $X = R+Q \in \mathbb{M}$ where $R \in \mathcal{G}^+_3$ and $Q \in \mathcal{G}^-_3n_\infty$. Using $\bar{\mathrm{P}}_{\mathbb{M}} = \mathrm{P}_R + \bar{\mathrm{P}}_Q$ we can separate $\partial_{\widetilde{X}}L = 0$ into R and Q components as follows:

$$\mathrm{P}_R\mathcal{L}R + \mathrm{P}_R\mathcal{L}Q = \alpha R, \quad \bar{\mathrm{P}}_Q\mathcal{L}R + \bar{\mathrm{P}}_Q\mathcal{L}Q = 0\,.$$

This is a standard form for quadratic minimisation with a homogeneous quadratic constraint and we can calculate Q from the second equation and then eliminate Q from the first equation in the usual way. This gives $\mathrm{P}_R\mathcal{L}'R = \alpha R$ where $\mathcal{L}' = \mathcal{L} - \mathcal{L}(\bar{\mathrm{P}}_Q\mathcal{L}\mathrm{P}_Q)^+\mathcal{L}$ and $Q = -(\bar{\mathrm{P}}_Q\mathcal{L}\mathrm{P}_Q)^+\mathcal{L}R$. At the maximum α equals $\langle\widetilde{X}\mathcal{L}X\rangle$ therefore R is the eigenrotator of $\mathrm{P}_R\mathcal{L}'$ associated with the largest eigenvalue. □

We see that the problem reduces to a small eigenrotator problem. This motor estimation method is easily implemented by forming the matrix representative of $\bar{\mathrm{P}}_{\mathbb{M}}\mathcal{L}$ as outlined in the following procedure:

1. Form a basis $e_k, k = 1, \cdots, 8$ of $\mathbb{M}$ where the first four basis vectors are associated with R and orthonormal, and the last four are associated with Q in the split $X = R + Q$ (e.g. $\{1, e_{12}, e_{13}, e_{23}, e_1n_\infty, e_2n_\infty, e_3n_\infty, I_3n_\infty\}$).
2. Form the 8×8 symmetric matrix $L_{ij} = \langle\widetilde{e}_i\bar{\mathrm{P}}_{\mathbb{M}}\mathcal{L}e_j\rangle = \langle\widetilde{e}_i\mathcal{L}e_j\rangle$ and partition into 4×4 sub-matrices $L = \begin{pmatrix} L_{rr} & L_{rq} \\ L_{qr} & L_{qq}\end{pmatrix}$.
3. Form the 4×4 matrix $L' = L_{rr} - L_{rq}(L^+_{qq}L_{qr})$.
4. Calculate $r \in \mathbb{R}^4$, a unit eigenvector of L' associated with the largest eigenvalue.
5. Calculate $q = -(L^+_{qq}L_{qr})r \in \mathbb{R}^4$.
6. Form the coefficient vector $m = \begin{pmatrix} r \\ q\end{pmatrix} \in \mathbb{R}^8$.
7. Calculate optimal motor $M = \sum_k m_ke_k \in \mathcal{M}$.

If the dimension d of the eigenspace of L' associated with the largest eigenvalue is greater than one then the optimal motor is not unique. This will happen in degenerate situations such as estimating a motor from only two pairs of points. A specific solution can be returned at the expense of a small increase in complexity, as follows. Let $V \in \mathbb{R}^{4\times d}$, $d \leq 4$ be an orthogonal matrix whose range is the eigenspace of L' associated with the largest eigenvalue. Any maximum unit eigenvector can be expressed as $r = Vx$ for unit vector $x \in \mathbb{R}^d$. Note $\cos(\frac{\theta}{2}) = \langle R\rangle = \sum_k r_k\langle e_k\rangle = r^Tz$ where $z \in \mathbb{R}^4$ with $z_k = \langle e_k\rangle$, and θ is the angle of rotation. With the natural basis above we get $z = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}^T$. Hence x^TV^Tz can be identified with $\cos(\frac{\theta}{2})$. Maximising x^TV^Tz subject to $x^Tx = 1$ gives the following enhancement to step 4 above:

4.′ Calculate $r = \mathrm{unit}(VV^Tz) \in \mathbb{R}^4$, the unit eigenvector of L' associated with the largest eigenvalue and the smallest angle of rotation, where $z \in \mathbb{R}^4$ with $z_k = \langle e_k\rangle, k = 1\ldots4$.

If $d = 1$ then there is no choice and $r = V$ or $r = -V$ as expected. Let $M = TR$ where $T = 1 - \frac{1}{2}\mathbf{t}n_\infty$ is a translator and $\mathbf{t}$ is the Euclidean translation vector. Note that $Q = -\frac{1}{2}\mathbf{t}Rn_\infty$ so we have $|q| = |n_o \cdot Q| = \frac{1}{2}|\mathbf{t}|$. The use of the Moore-Penrose pseudo-inverse in step 5 will ensure that the smallest translation $\mathbf{t}$ is returned when there is not a unique maximiser. This will happen in degenerate situations such as estimating a motor from a single pair of planes. Thus the estimated motor will maximise the measure and provide the motor with the smallest translation and rotation angle when there is not a unique maximiser.

5 Examples

In this section we provide some illustrations of the algorithm. The data is generated as follows. A random geometric object P_k is generated, such as a point, sphere, line, circle, or tangent. Noise is added to the data P_k by perturbing it with a small random motor $M_k \approx 1$ before applying the general fixed rigid body transformation M_o to give $Q_k = M_o M_k P_k \widetilde{M}_k \widetilde{M}_o$. The noise is sufficient to provide clear delineation between the objects in the figures presented. The motor estimation procedure (using step 4.$'$) is then applied to the data pairs (P_k, Q_k), $k = 1, \ldots, K$, to obtain an optimal estimate M of M_o. In the figures presented the dark data is the source data after the action of the estimated motor $Q'_k = M P_k \widetilde{M}$, and the light data is the target data Q_k. The difference between the sets is the error remaining after applying the estimation procedure and is due to the noise on the data. If no noise is present the fit is perfect and the data sits exactly on top of each other.

First consider the problem of fitting spheres. As discussed earlier the radius of the spheres plays no role, as it is invariant to rigid body transformations, and the situation is identical to the case of noisy points. With just one pair of spheres the fit is perfect and the centres of the spheres coincide. The rotational part vanishes because the smallest angle of rotation is zero and the estimated motor is a pure translator. With two pairs of spheres the optimal motor makes their centres lie on the same axis with equal separation. This example is a typical situation where there is insufficient information to get a unique maximiser of $\langle \widetilde{M} \mathcal{L} M \rangle$. For the estimated motor, the rotation about the axis is zero and the rotation is in a plane parallel with the axis through the points before and after motion. A more general situation is shown in Fig. 3 where there are five pairs of noisy spheres. The more complex example in Fig. 4 shows the algorithm being applied to five pairs of different objects. We use spheres, lines, circles and 1D and 2D tangents. We have not included planes simply because, unlike lines, it is hard to visualise the separation between planes in a figure.

6 Discussion

We have presented a technique for estimating motors from noisy geometric data. The data may comprise a variety of objects including points, rounds (point pairs, circles, spheres), flats (lines, planes), tangents, and directions; which can all be combined naturally in a single framework while excluding reflection. In order to formulate the problem we restricted the similarity measure to the simple form $\langle P\check{Q} \rangle$. In addition we restricted to *admissible* objects which allowed us to ignore the motor constraint $\langle M\widetilde{M} \rangle_4 = 0$ during optimisation, and which include all major classes of geometric primatives. The estimation procedure reduced to an eigenrotator problem.

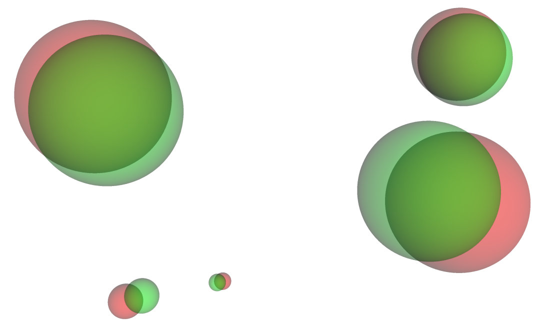

Figure 3: Five pairs of spheres used to estimate the rigid body motion.

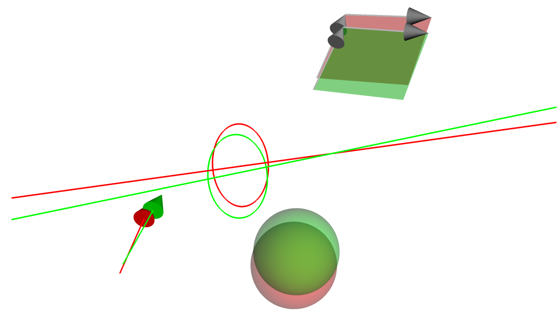

Figure 4: Five pairs of different objects (spheres, lines, circles, 1D and 2D tangents) used to estimate the rigid body motion.

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