1 2D DCT Matrix Implementation.

The factorization of a 2D DCT into two 1D DCTs can be implemented by two consecutive matrix multiplications, i.e.,

$$F(u,v) = \mathbf{T} \cdot f(i,j) \cdot \mathbf{T}^T. \tag{1}$$

We will name $\mathbf{T}$ the *DCT-matrix*.

$$\mathbf{T}[i,j] = \begin{cases} \frac{1}{\sqrt{N}}, & \text{if } \ i = 0 \\ \sqrt{\frac{2}{N}} \cdot \cos\frac{(2j+1)\cdot i\pi}{2N}, & \text{if } \ i > 0 \end{cases} \tag{2}$$

where $i = 0, ..., N-1$ and $j = 0, ..., N-1$ are the row and column indices, and the block size is $N \times N$.

When $N = 8$, we have:

$$\mathbf{T_8}[i,j] = \begin{cases} \frac{1}{2\sqrt{2}}, & \text{if } \ i = 0 \\ \frac{1}{2} \cdot \cos\frac{(2j+1)\cdot i\pi}{16}, & \text{if } \ i > 0 \end{cases} \tag{3}$$

Hence,

$$\mathbf{T_8} = \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \cdots & \frac{1}{2\sqrt{2}} \\ \frac{1}{2}\cdot\cos\frac{\pi}{16} & \frac{1}{2}\cdot\cos\frac{3\pi}{16} & \frac{1}{2}\cdot\cos\frac{5\pi}{16} & \cdots & \frac{1}{2}\cdot\cos\frac{15\pi}{16} \\ \frac{1}{2}\cdot\cos\frac{\pi}{8} & \frac{1}{2}\cdot\cos\frac{3\pi}{8} & \frac{1}{2}\cdot\cos\frac{5\pi}{8} & \cdots & \frac{1}{2}\cdot\cos\frac{15\pi}{8} \\ \frac{1}{2}\cdot\cos\frac{3\pi}{16} & \frac{1}{2}\cdot\cos\frac{9\pi}{16} & \frac{1}{2}\cdot\cos\frac{15\pi}{16} & \cdots & \frac{1}{2}\cdot\cos\frac{45\pi}{16} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}\cdot\cos\frac{7\pi}{16} & \frac{1}{2}\cdot\cos\frac{21\pi}{16} & \frac{1}{2}\cdot\cos\frac{35\pi}{16} & \cdots & \frac{1}{2}\cdot\cos\frac{105\pi}{16} \end{bmatrix} \tag{4}$$

A closer look at the DCT-matrix will reveal that each row of the matrix is basically a 1D DCT basis function, ranging from DC to AC1, AC2, ..., AC7. The only difference is that we have added some constants and taken care of the orthonormal aspect of the DCT basis functions. (We will leave it as an exercise to verify that the rows and columns of $\mathbf{T}_8$ are orthonormal vectors, i.e., $\mathbf{T}_8$ is an Orthogonal Matrix.)

In summary, the implementation of the 2D DCT is now a simple matter of applying two matrix multiplications as in Eq.1. The first multiplication applies 1D DCT vertically (for each column), and the second applies 1D DCT horizontally (for each row). What has been achieved is exactly the two steps as indicated in the DCT factorization.

2 2D IDCT Matrix Implementation.

In this section, we will show how to reconstruct $f(i,j)$ from $F(u,v)$ losslessly by matrix multiplications. It turns out that the 2D IDCT matrix implementation is simply:

$$f(i,j) = \mathbf{T}^T \cdot F(u,v) \cdot \mathbf{T}. \tag{5}$$

Its derivation is as follows:

First, because $\mathbf{T} \cdot \mathbf{T}^{-1} = \mathbf{T}^{-1} \cdot \mathbf{T} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix, we can simply rewrite $f(i,j)$ as:

$$f(i,j) = \mathbf{T}^{-1} \cdot \mathbf{T} \cdot f(i,j) \cdot \mathbf{T}^T \cdot (\mathbf{T}^T)^{-1}.$$

According to Eq. 1,

$$F(u,v) = \mathbf{T} \cdot f(i,j) \cdot \mathbf{T}^T.$$

Hence,

$$f(i,j) = \mathbf{T}^{-1} \cdot F(u,v) \cdot (\mathbf{T}^T)^{-1}.$$

As stated above, the DCT-matrix $\mathbf{T}$ is orthogonal, therefore,

$$\mathbf{T}^T = \mathbf{T}^{-1}.$$

It follows,

$$f(i,j) = \mathbf{T}^T \cdot F(u,v) \cdot \mathbf{T}.$$

$$F(u,v) = T \cdot f(i,j) \cdot T^T$$

<u>Example :</u>

$$\begin{array}{c|c|}
 & j \\
\hline
i & f(i,j) \\
\hline
\end{array}
\qquad
\begin{bmatrix}
100 & 100 & \cdots & 100 \\
100 & & & \\
100 & & & \\
\vdots & & & \\
100 & 100 & \cdots & 100
\end{bmatrix}$$

$G(u,j) =$

$T \cdot f(i,j)$

$$\begin{array}{c|c}
 & j \\
u & \begin{bmatrix}
\overline{DC}_0 & \overline{DC}_1 & \overline{DC}_2 & \cdots & \overline{DC}_7 \\
\overline{AC}_{10} & \overline{AC}_{11} & & \cdots & \overline{AC}_{17} \\
 & & & & \\
\overline{AC}_{70} & \overline{AC}_{71} & & & \overline{AC}_{77}
\end{bmatrix}
\end{array}
\qquad
\begin{bmatrix}
\frac{800}{2\sqrt{2}} & \frac{800}{2\sqrt{2}} & \cdots & \frac{800}{2\sqrt{2}} \\
0 & 0 & \cdots & 0 \\
 & & \phi & \\
 & & &
\end{bmatrix}$$

** $\overline{AC}$ is after 1D-DCT for each column of $f(i,j)$. Similarly, $\overline{DC}$.

$F(u,v) =$

$G(u,j) \cdot T^T$

$$\begin{array}{c|c}
 & v \\
u & \begin{bmatrix}
DC & AC_{01} & AC_{02} & \cdots & AC_{07} \\
AC_{10} & AC_{11} & AC_{12} & \cdots & AC_{17} \\
AC_{20} & AC_{21} & AC_{22} & \cdots & AC_{27} \\
\vdots & & & & \\
AC_{70} & AC_{71} & & & AC_{77}
\end{bmatrix}
\end{array}
\qquad
\begin{bmatrix}
800 & 0 & 0 & \cdots & 0 \\
0 & & & & \\
0 & & & & \\
\vdots & & \phi & & \\
 & & & &
\end{bmatrix}$$