# Toxicity-Competitiveness Tradeoff in Concentrated Liquidity Provision

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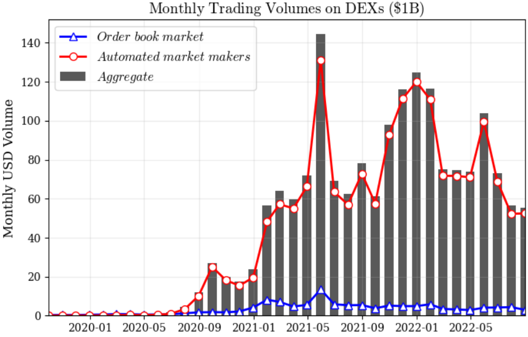
# Background

- Decentralized exchanges (DEX) and Automated Market Makers (AMM) on blockchain
  - a new market-making technology to attract resources and make it available to traders (c.f., limit-order book)



# What We Do

- Uniswap v3 introduced **Concentrated-Liquidity** AMM
  - Are they aligned with economic incentive? Are we using them in an efficient way? Need a closer look
- This paper proposes
  - Return factors in liquidity provider (LP) profits that emanate from different sources
    - Competitive vs. Non-competitive profits
    - Toxic vs. Non-toxic order flow
  - Tradeoff between competitiveness and order toxicity; and
  - Improved guideline for optimal liquidity provision

# What is AMM?

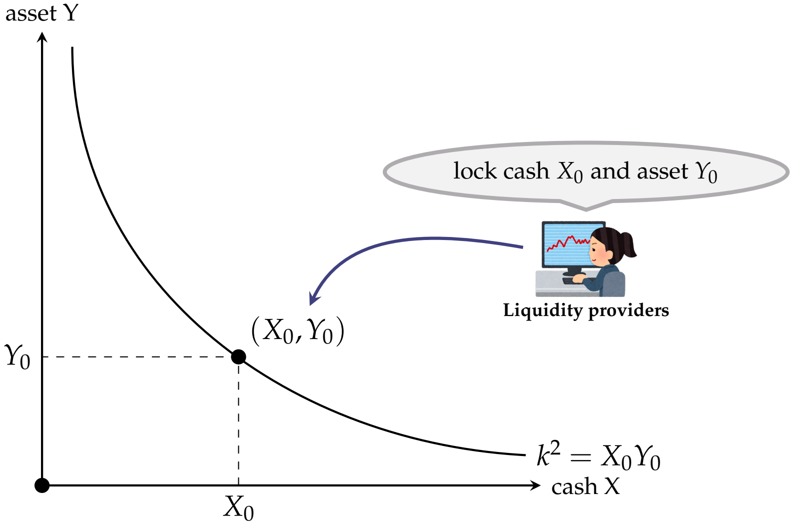
- A single-function algorithm that governs matching and pricing

$$k^2 = XY$$

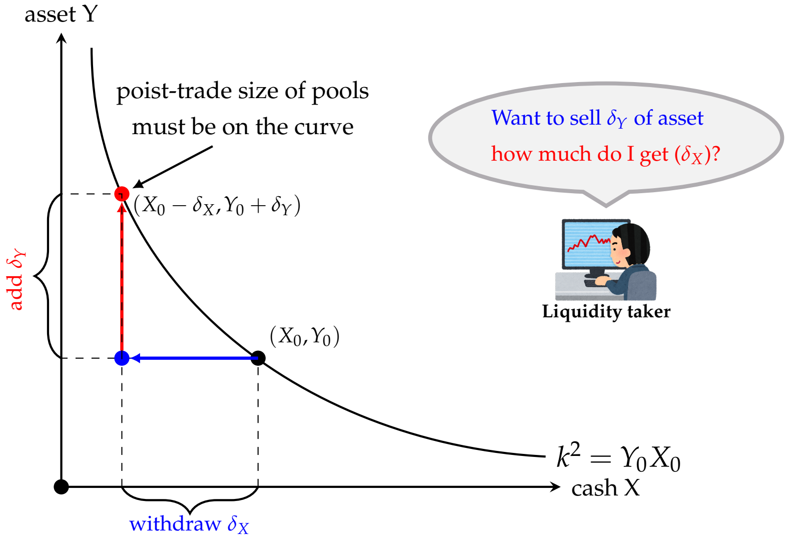
  - $Y$: asset, $X$: numeraire
- Liquidity providers stake $(X, Y)$ into liquidity pools
- Liquidity takers trade against the pool

# How does it work? $k^2 = XY$

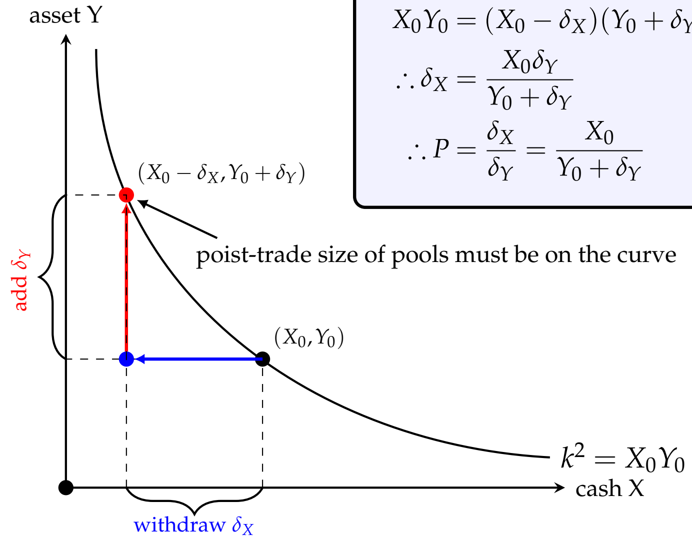
## Inject cash and asset into the pool



# Trade Execution and Price



# Execution Price




**AMM pricing**

$$X_0 Y_0 = (X_0 - \delta_X)(Y_0 + \delta_Y)$$
$$\therefore \delta_X = \frac{X_0 \delta_Y}{Y_0 + \delta_Y}$$
$$\therefore P = \frac{\delta_X}{\delta_Y} = \frac{X_0}{Y_0 + \delta_Y}$$

# Liquidity Provider Profit

When a trader sells $\delta_Y$ and pulls out $\delta_X$, what is LP return?

1. Liquidity re-valuation:

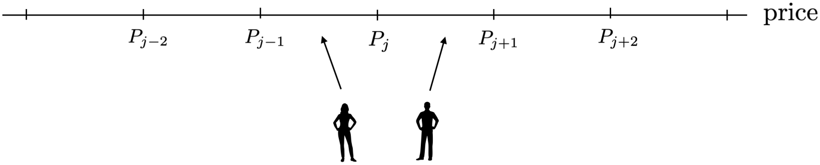
$$\pi = \underbrace{X_0 - \delta_X + P(Y_0 + \delta_Y)}_{\text{post-trade liquidity}} - \underbrace{(X_0 + P_0 Y_0)}_{\text{investment cost}}$$

   - $\pi < 0$ due to adverse selection
   - Trades in the right direction are **toxic**

2. Fee income: $100 \times \theta\%$ of fee in terms of payment
   - Buying $Y$ and paying $X \Rightarrow \theta\delta_X$
   - Selling $Y$ and receiving $X \Rightarrow P\theta\delta_Y$
   - Each LP earns the share

# Concentrated Liquidity AMM

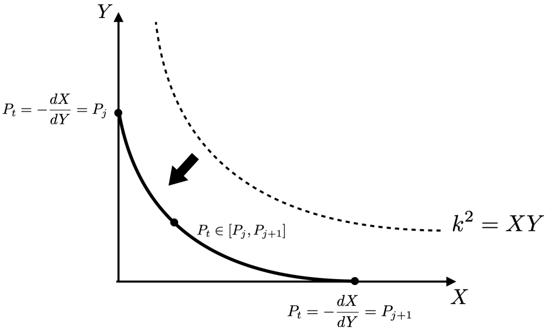
- Discretized price grid: $\{s_j\}_{j=-n}^{n}$ with $s_j = [P_j, P_{j+1}]$




- LP allocates liquidity by choosing $s_j$
  - enables concentrated bet
- Each range involves truncated AMM curve

# LP $i$ in range $s_j$ follows truncated curve

$$\left(x_i + k_i^j \sqrt{P_j}\right)\left(y_i + \frac{k_i^j}{\sqrt{P_{j+1}}}\right) = k_i^{j2} \qquad (1)$$




- Individual liquidity $k_i^j = k(s_j)$
- Aggregate liquidity $K_j = \int k_i^j di$

# LP Return with Concentrated Liquidity

- LP $i$ earns return whenever liquidity is "active"

$$P_t \in s_j = [P_j, P_{j+1}]$$

- Return from revaluation:

$$\pi_{val}(s_j) = \mathbb{E}[\Delta x_i^j + P_t \Delta y_i^j]$$

- Fee income: proportional to her liquidity share

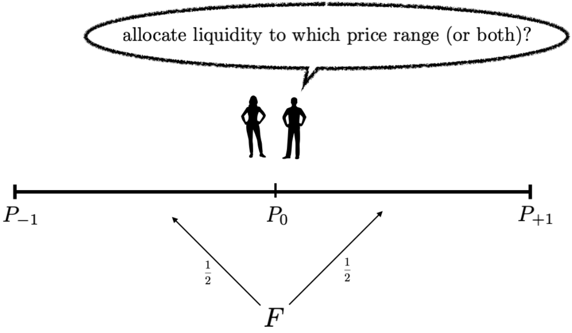
$$\pi_{fee}(s_j, K_j) = \mathbb{E}\left[\frac{k_i^j}{K_j}(\theta \times \text{volume})\right]$$

# Literature

- **Toxic** orders as a source of impermanent loss
  - ETH price $\uparrow$ in Binance $\Rightarrow$ arbitrageurs buy ETH in Uniswap
  - LPs in Uniswap incur adverse selection cost
  - Angeris et al. (2020), Aoyagi (2020), Angeris and Chitra (2020), Barbon and Ranaldo (2021), Capponi and Jia (2021), Lehar and Parlour (forthcoming)
- **Competitiveness** of LP profits
  - Fee income is proportional to $k/K$, generating competition
- Milionis, Wan, and Adams (2023) propose FLAIR
  - LPs should take positions with low toxicity and low competitiveness
- We show these two factors are negatively related (i.e., tradeoff)

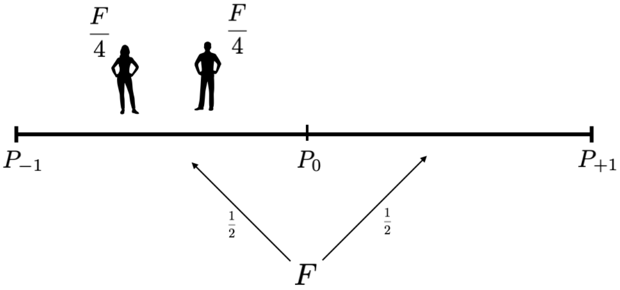
# Liquidity provision is like a Hoteling's game

- Two price ranges relative to current price $P_0$
  - Fee income $F = \theta V$ arises in each range with equal prob.
  - Individual LP earns $\theta V \times \frac{k_i^j}{K^j}$



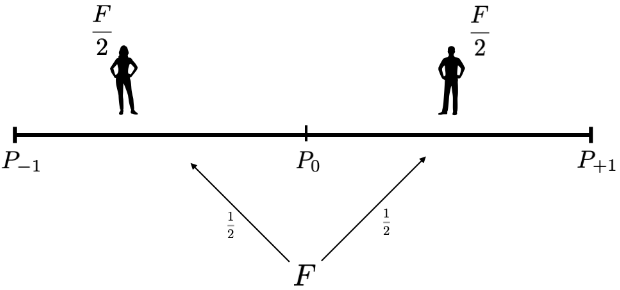
# Fee Income is Competitive Profit

- Taking the same strategy as your rival...need to share the pie



# Fee Income is Competitive Profit

- Taking a different strategy from your rival ⇒ higher expected profit




- Implication: want to avoid a tick with large standing liquidity
- Fee income is competitive profit with strategic substitution

# Or is it?

- If $V = vK$, competition is irrelevant.
  - But when?

- Two possible swap types:
  1. **Toxic arb** motivated by changes in the reference prices (e.g., Binance)
     - Volume is prop to $K$, and so is fee
     - Individual LP earns $\theta v K \times \frac{k_i}{K} = \theta v k_i$
     - No competition for profits
  2. **Noise trading** (non-toxic swap)
     - Motivated by exogenous reasons, e.g., hedge
     - Order size is independent of $K$, and so is fee
     - Competition for profits

- With these events randomly happening, what is optimal liquidity provision?

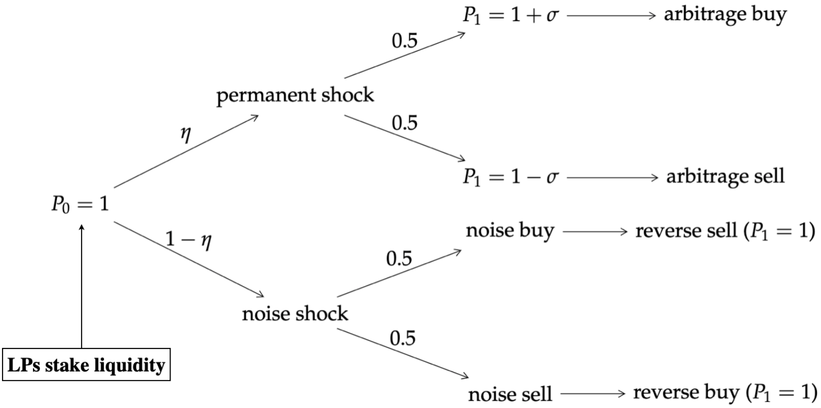
# One-Shot Trading Model

- Asset ($Y$) and cash ($X$) are traded in Uniswap at $P_t$
  - $P_t$ follows $\tilde{P}$ in Binance (price oracle)
  - Initial price: $P_0 = \tilde{P} = 1$
- Two types of events trigger a trade
  1. Permanent price change in Binance ($\tilde{P}$) with prob $\eta$

$$P_0 \rightarrow P_1 = \begin{cases} P_u = 1 + \sigma & \text{w.p. } 1/2 \\ P_l = 1 - \sigma & \text{w.p. } 1/2 \end{cases}$$

  2. Noise trading with prob $1 - \eta$ (no change in $\tilde{P}$)

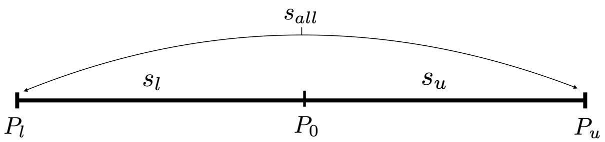
# Evolution of the Game



# Liquidity Provision

- Price grid and LPs' strategy space:

$$\mathcal{S} = \{s_u, s_l, s_{all}\}$$




- Associated liquidity parameter with constant $\alpha_j$

$$(k_i^u, k_i^l) = \begin{cases} (\alpha_u W_i, 0) & \text{if } s_i = s_u \\ (0, \alpha_l W_i) & \text{if } s_i = s_l \\ (\alpha_{all} W_i, \alpha_{all} W_i) & \text{if } s_i = s_{all} \end{cases}$$

  - $W_i$: initial wealth (but take $W_i \to 0$ in eqm)

# Expected Return of an LP

- Expected LP return:

$$\Pi(s_i) = \overbrace{\mathbb{E}\left[\sum_{j=u,l}(\Delta x_i^j + P_1 \Delta y_i^j)\right]}^{\pi_{val}:\text{revaluation}} + \overbrace{\mathbb{E}\left[\sum_{j=u,l}\frac{k_i^j}{K_j}\theta V_j\right]}^{\pi_{fee}:\text{ fee}}$$

  - By construction, initial liquidity is prop to $k_i^j$

$$\begin{cases} x_i^j = \beta_x^j k_i^j \\ y_i^j = \beta_y^j k_i^j \end{cases}$$

  - $V_j$: volume of payment (either in cash or asset)

# Case 1: Permanent Price Change

- $P_0 \neq \tilde{P} = 1 \pm \sigma \Rightarrow$ arbitrageur trades to ensure non-arb.
  - e.g., $P_0 \to P_u$: she buys asset and pays in cash
  - How much?
- Must consume all liquidity in $s_u$ to trigger $P_0 \to P_u$:

$$\begin{aligned}\Delta Y &= \int y_i^u \mathbb{I}_{\{s_i = s_u, s_{all}\}} di = \overbrace{\beta^u}^{\text{constant}} \int k_i^u \mathbb{I}_{\{s_i = s_u, s_{all}\}} di \\ &= \beta^u K_u\end{aligned}$$

  - same for $\Delta X$
- The total payment volume $\propto K$

$$V_j = \overbrace{v_j}^{\text{constant}} \times K_j$$

# Case 1: Permanent Price Change

- The expected profit from permanent shock:

$$\pi_T(s_i) = \mathbb{E}\left[\sum_{j=u,l} \psi_j k_i^j\right] + \mathbb{E}\left[\sum_{j=u,l} \frac{k_i^j}{K_j} \theta v_j K_j\right]$$

- **Toxic** order: the impermanent loss arises ($\pi_{val} < 0$)
  - arbitrageur withdraws more valuable asset by injecting less valuable one
- $\pi_T$ is proportional to $k_i$ and independent of $K$
  - **no competition** for profit

# Case 1: Permanent Price Change

**Proposition 1.**

(i) Toxic order flow generates non-competitive profit.
(ii) Comparing non-competitive profits with $s_i \in \mathcal{S}$, it holds that

$$\pi_T(s_u) > \pi_T(s_{all}) > \pi_T(s_l)$$

with

$$\pi_T(s_{all}) = q\pi_T(s_u) + (1-q)\pi_T(s_l).$$

- Betting in the upper range is optimal due to convex pricing convex
  - Traders POV: "buy" orders take larger cost than "sell"
  - LPs POV: "buy" orders are more profitable than "sell"
- Leads to a **concentration** of liquidity to $s_u$

# Case 2: Noise Trading

- Random traders may trade due to exogenous needs (e.g., hedge)
  - Exogenous volume $\omega$, buy and sell with equal prob
- Reverse trading follows the noise trading
- E.g., random buy of $\omega$ units of asset reverse
  1. Noise trading causes $(\Delta X(\omega), -\omega)$ and $P_0 \to P(\omega)$; but $\tilde{P} = 1$
  2. Arbitrageur conducts opposite trade $(-\Delta X(\omega), \omega)$ to drive the price back $P(\omega) \to P_0$
- No changes in liquidity value; only fee income arises

# Case 2: Noise Trading

- Expected profit from noise trading

$$\pi_{NT}(s_i, K) = \overbrace{\mathbb{E}\left[\sum_{j=u,l}(\Delta x_i^j + P_1 \Delta y_i^j)\right]}^{=0} + \overbrace{\mathbb{E}\left[\sum_{j=u,l}\frac{k_i^j}{K_j}\theta z \omega\right]}^{\pi_{fee}}$$

- No impermanent loss, i.e., **non-toxic** order flow
- **Competition** for fee income
  - $\pi_{NT}$ is affected by the share, $k_i/K$

# Case 2: Noise Trading

**Proposition 2.**

(i) Non-toxic order generates competitive profits
(ii) LP competition exhibits strategic substitution
(iii) Competitive profit from $s_{all}$ is a convex combination of profits from $s_u$ and $s_l$:

$$\pi_{NT}(s_{all}) = q\pi_{NT}(s_u) + (1-q)\pi_{NT}(s_l).$$

- LP $i$ avoids $s_j$ if $K_j$ is large
- Leads to a **dispersion** of liquidity

# Best Response Correspondence

Integrating both profit types:

$$\Pi(s_i, K) = \eta \pi_T(s_i) + (1-\eta)\pi_{NT}(s_i, K)$$

- depends on aggregate (rivals) liquidity allocation, $K = (K_l, K_u)$

**Proposition 3.**

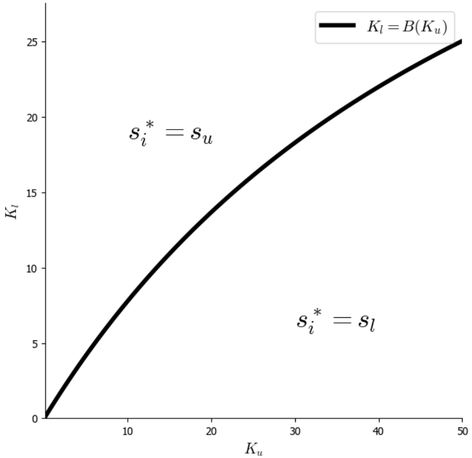
There exists a unique boundary $K_l = F(K_u)$, such that;

(i) If $K$ is on the boundary, LP $i$ is indifferent between all strategies in $\mathcal{S}$.

(ii) If $K$ lies in the upper (resp. lower) contour set, $s_u$ (resp. $s_l$) is the optimal strategy

(iii) $F$ is monotonically increasing in $K_u$ with $F(0) = 0$.

# Best Response




- Unique boundary due to $\Pi(s_{all}) = q\Pi(s_u) + (1-q)\Pi(s_l)$
- Strategic substitution: $K_u \uparrow \Rightarrow$ switch from $s_u$ to $s_l$

# Equilibrium Liquidity Distribution

**Proposition 4.**

There exists no equilibrium in pure strategies

- Intuition is drawn from Hotelling (1929)
  - If rivals take $s_l$ for sure, taking $s_u$ is optimal
- Consider mixed-strategy equilibrium

# Equilibrium Liquidity Distribution

- $K = (K_u, K_l)$ must be on the boundary to ensure indifference

$$K_l = F(K_u) \tag{2}$$

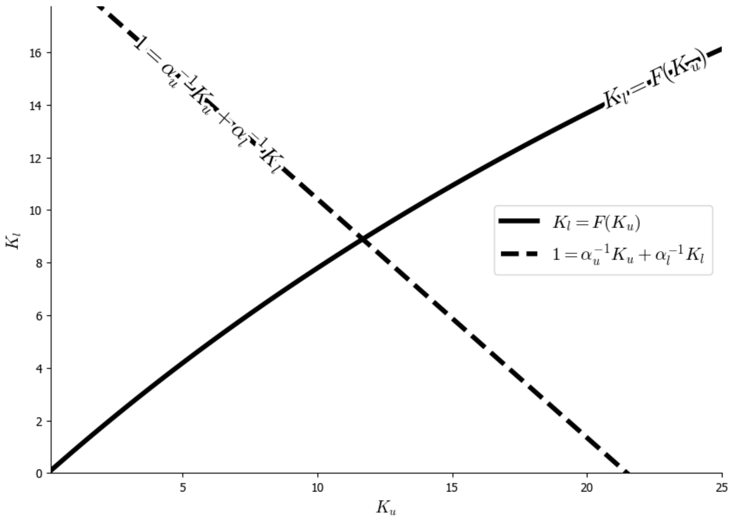
- LP randomizes her strategies with $\gamma_j = \Pr(s_i = s_j)$;
- With $W \equiv \int W_i di$, LLN implies[4]

$$\begin{cases} K_u = (\alpha_u \gamma_u + \alpha_{all} \gamma_{all}) W \\ K_l = (\alpha_l \gamma_l + \alpha_{all} \gamma_{all}) W \end{cases} \Rightarrow W = \frac{1}{\alpha_u} K_u + \frac{1}{\alpha_l} K_l \tag{3}$$

- As $F' > 0$, (2) and (3) pin down a unique $K^* = (K_u^*, K_l^*)$

[4] $\alpha_{all}^{-1} = \alpha_u^{-1} + \alpha_l^{-1}$.

# Equilibrium Liquidity Distribution

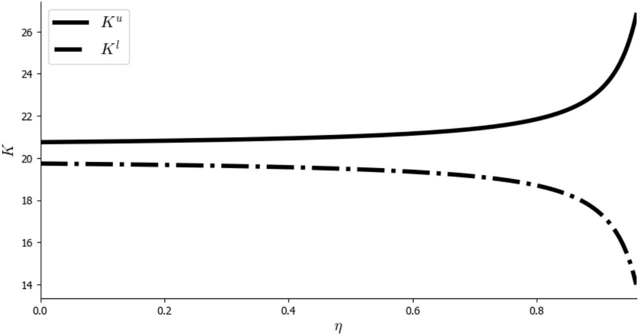


# Comparative Statics

- Distribution of $K$ is determined by the relative toxicity
- Toxic vs non-toxic $\Rightarrow$ non-competitive vs competitive profits
  $\Rightarrow$ concentration vs dispersion effects

# Comparative Statics

- High toxicity ($\eta \uparrow$) $\Rightarrow$ concentrated liquidity
- At $\eta \to 1$, $K$ does not react to parameters

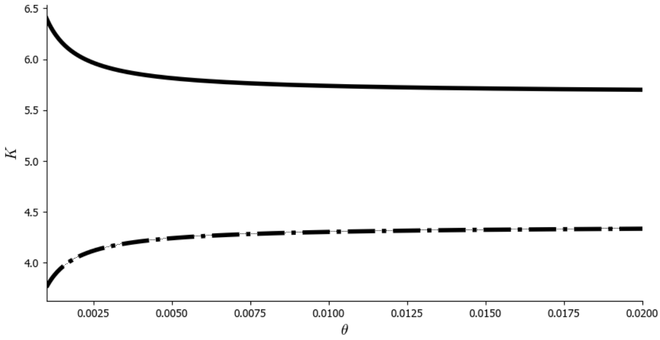


# Comparative Statics

- High fee rate ($\theta \uparrow$) $\Rightarrow$ dispersed liquidity
  - $\theta$ does not affect the impermanent loss of non-competitive profit

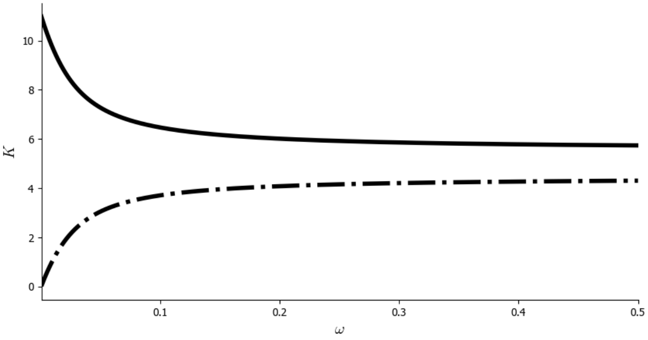
$$\Pi = \eta(\pi_{T,val} + \overbrace{\pi_{T,fee}}^{\propto\theta}) + (1-\eta)\overbrace{\pi_{NT,fee}}^{\propto\theta}$$

  - Relative presence of $\pi_{NT}$ increases



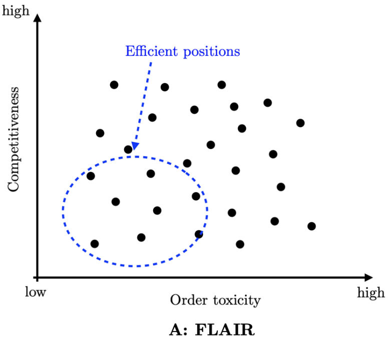
# Comparative Statics

- Large noise trading ($\omega \uparrow$) $\Rightarrow$ dispersed liquidity



# Toxicity-Competitiveness Tradeoff

- Toxicity and competitiveness are inter-dependent
  - more toxic orders ($\eta \uparrow$) $\Rightarrow$ LPs are less concerned about competition
- Literature takes them as independent factors



# Summary

- Concentrated liquidity mechanism may not result in a concentration of liquidity
  - Competitive and non-competitive profits bring about different forces
  - Observable indicators of their sources: toxicity
- Do LPs incorporate this tradeoff in reality? How much additional profits can they make?
  - Testing our model with Uniswap data

# Road Ahead

- Measuring toxicity of order flow
  1. Identify trades in relation to $\Delta P$
     1.1 orders triggered by $\Delta\tilde{P}$ in Binance and trading in right direction are toxic
     1.2 orders unrelated to $\Delta\tilde{P}$ in Binance or trading in wrong direction are non-toxic
  2. Directly observe panel data with trader (wallet) ID and fee bids
     - pattern analyses by Next Finance Tech Inc., classify bots into arb and hedge
- Implication for TradFi
  - Structure of LP profits resembles TradFi (LOB)
  - TradFi cannot directly identify toxic orders due to data limitation, but DeFi overcomes this issue

# Conclusion

- Tradeoff between competition and toxicity
  - Important implication for individual optimality and aggregate liquidity distribution
- Next step: data work and testing the model
  - If LPs in the real market follows the theory, it validates our paper
  - If not, we propose potential source of additional profit and show guideline for better liquidity allocation