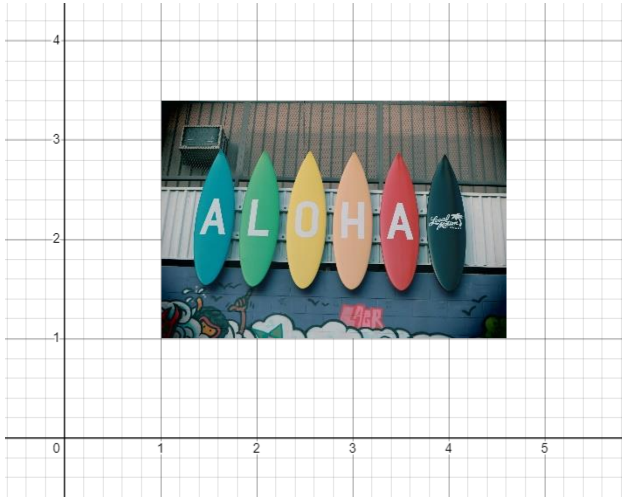
2D Projective Transformation

Sung Soo Hwang

Introduction

- Similarity transformation
  - Shape of an object is preserved(rotation, translation, scaling, etc.)

original



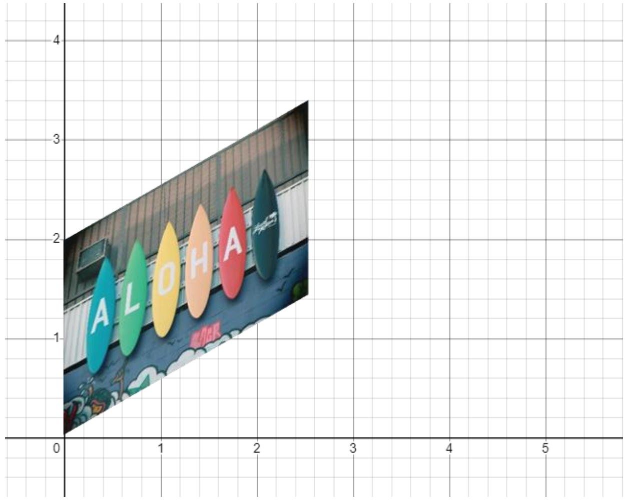
translation

scaling

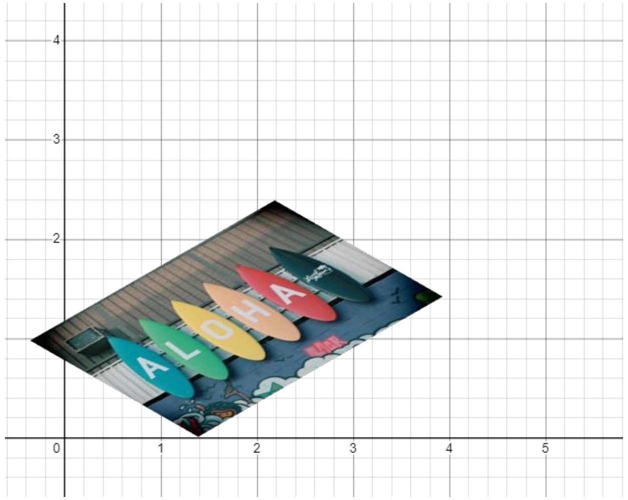
Introduction

- Affine transformation
  - Parallel lines are parallel after transformation

original



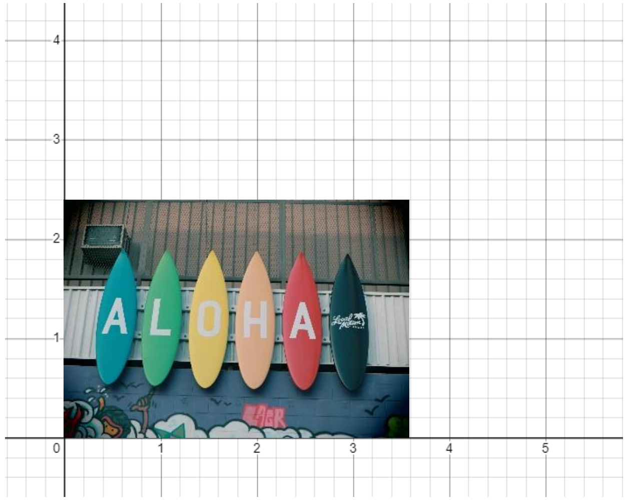
Affine



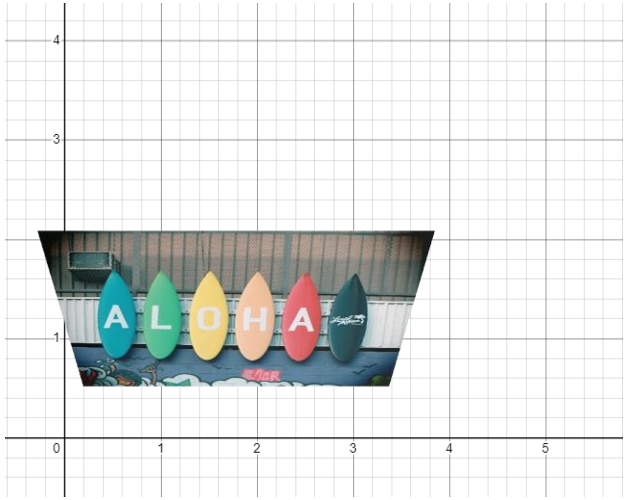
Affine

Introduction

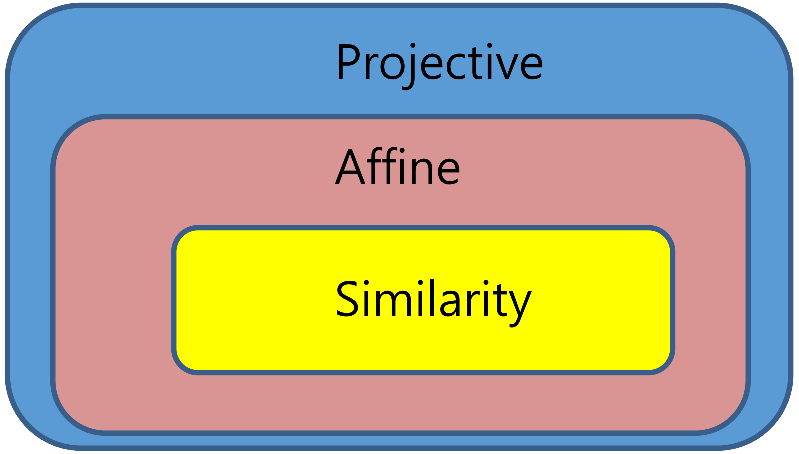
- Projective transformation
  - Lines are lines after projective transformation



original

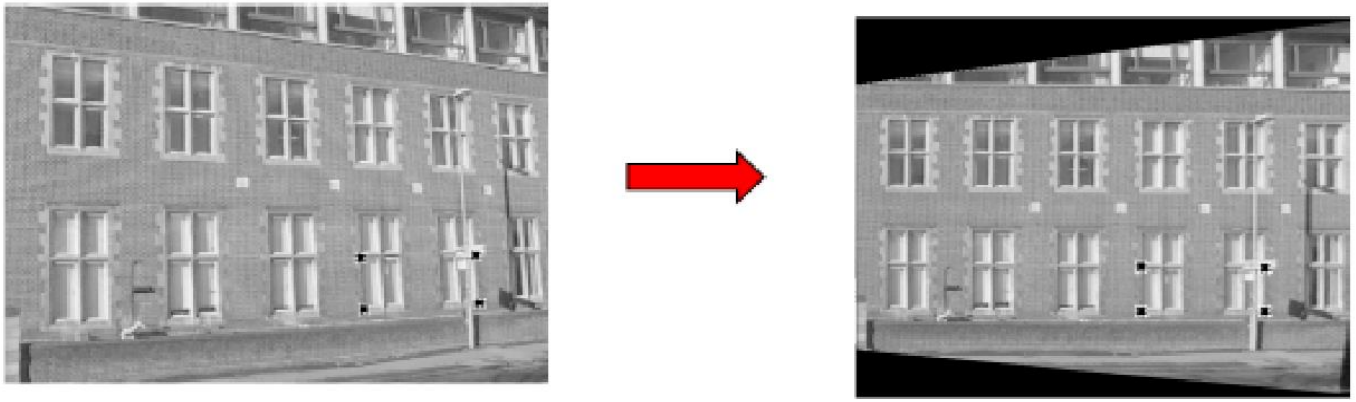


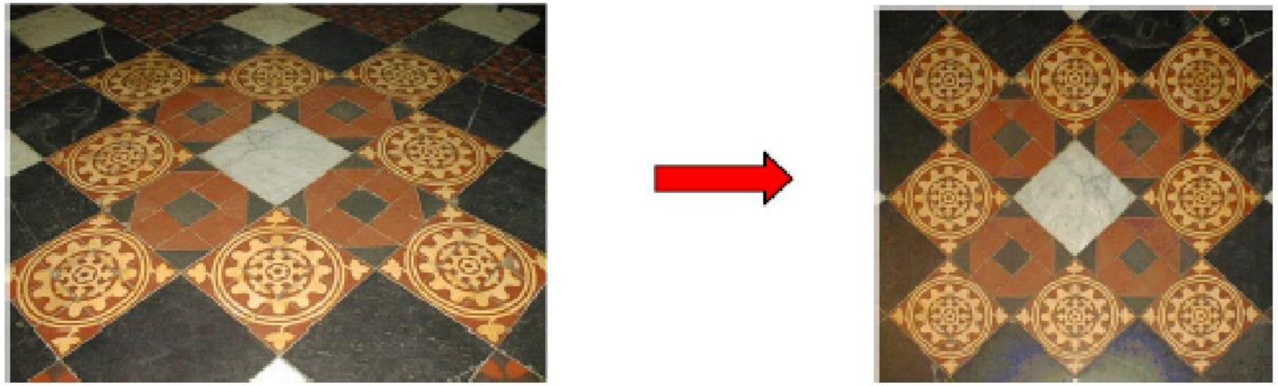
projective



Introduction

- Projective transformation = Perspective transformation = Homography





unesco
University Network Member

Ministry of Education

academic impact

HANDONG GLOBAL UNIVERSITY WORLD CHANGING

2D projective geometry

- 2D homogeneous coordinates and projective space
  - Inhomogeneous representation

Point

$$(x, y)^T \in \boldsymbol{R}^2 \qquad \boldsymbol{2D - vector\ space}$$

Line

$$\{(x, y)^T \mid ax + by + c = 0\}$$

$$\boldsymbol{I} = (a, b, c)^T \quad or \quad \boldsymbol{I} = (ka, kb, kc)^T \quad (k \neq 0)$$

$$\{(x, y)^T \mid \boldsymbol{I}^T \boldsymbol{x} = 0\} \quad \boldsymbol{where} \quad \boldsymbol{x} = (x, y, 1)^T$$

Conic

$$\{(x, y)^T \mid ax^2 + bxy + cy^2 + dx + ey + f = 0\}$$

$$\boldsymbol{C} = \begin{pmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix}$$

$$\{(x, y)^T \mid \boldsymbol{x}^T \boldsymbol{C} \boldsymbol{x} = 0\} \quad \boldsymbol{where} \quad \boldsymbol{x} = (x, y, 1)^T$$

2D projective geometry

- 2D homogeneous coordinates and projective space
  - homogeneous representation

Point

$(x,y)^T \in \boldsymbol{R}^2$ $\quad$ $\boldsymbol{2D - vector\ space}$

$\rightarrow \boldsymbol{x} = (x,y,1)^T \quad or \quad (kx,ky,k)^T \qquad (k \neq 0)$

$\boldsymbol{x} = (x,y,1)^T = (kx,ky,k)^T$

$\downarrow$

$(x,y,1) = (kx,ky,k)^T \quad where \quad k \neq 0$

$P^2 \equiv R^3 - \{(0,0,0)\}$ **2D-projective space**

$(0,0,0)^T \notin P^2$

$(x,y)^T \in \boldsymbol{R}^2$

$\downarrow$ $(x,y)^T = (x_1/x_3, x_2/x_3)^T \in R^2 \;\; where \;\; x_3 \neq 0$

$\boldsymbol{x} = (x_1,x_2,x_3)^T$ **Homogeneous coordinate system**

$\boldsymbol{x} = (x_1,x_2,0)^T$ **Point-at-infinity (ideal point)**

$(x_1/0, x_2/0)^T \notin R^2 \qquad (x_1,x_2,0) = (0,0,0)$

2D projective geometry

- Hierarchy of transformations
  - Isometries

Projective transformation

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \varepsilon\cos\theta & -\sin\theta & t_x \\ \varepsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \quad \varepsilon = \pm 1$$

$\varepsilon = 1\ orientation\ preserving$

$\varepsilon = -1\ orientation\ reversing$ (Horizontal Flip)

$$x' = H_E x = \begin{pmatrix} R & t \\ 0^T & 1 \end{pmatrix} x \qquad R^T R = I$$

3-DOF (1 rotation, 2 translation)

Special cases: pure rotation + pure translation

Invariants – angle, length, area

2D projective geometry

- Hierarchy of transformations
  - Similarities (isometry + scale)

Projective transformation

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$\longrightarrow$ $x' = H_S x = \begin{pmatrix} sR & t \\ 0^T & 1 \end{pmatrix} x$ $\qquad R^T R = I$

4-DOF (1 scale, 1 rotation, 2 translation)
also know as equi-form (shape preserving)

Invariants – ratio of lengths, ratio of area $\longleftarrow$ Invariants of isometries
- angle, length, area

unesco University Network Member

2D projective geometry

- Hierarchy of transformations
  - Affine transformations

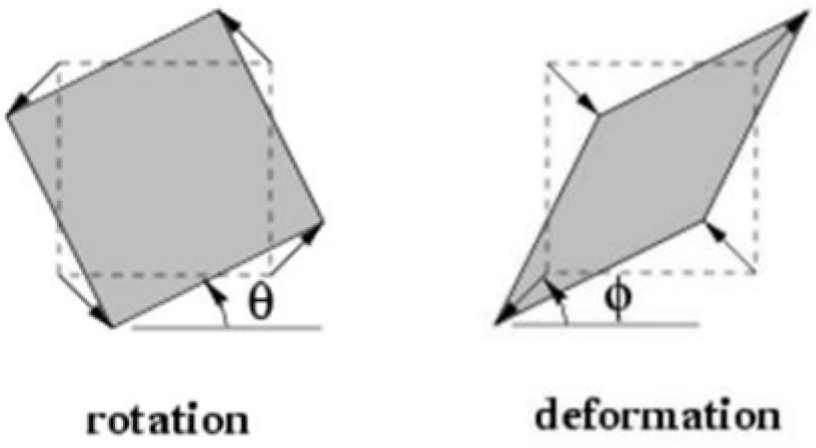
$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$x' = H_A x = \begin{pmatrix} A & t \\ 0^T & 1 \end{pmatrix} x$$

Where $A = R(\theta)R(-\emptyset)DR(\emptyset)$ $\quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

6-DOF (2 scale, 2 rotation, 2 translation)
non-isotropic scaling! (2-DOF: scale ratio and orientation)

Projective transformation



Invariants – ratio of lengths, ratio of area ← Invariants of similarities
- ratio of length, ratio of areas, angle, parallel lines

2D projective geometry

- Hierarchy of transformations
  - Projective transformations

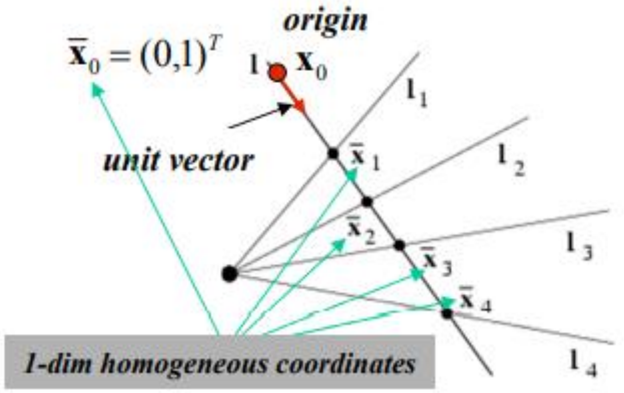
Projective transformation

$$\boldsymbol{x}' = \boldsymbol{H}_p \boldsymbol{x} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{t} \\ \mathrm{v}^T & \boldsymbol{v} \end{pmatrix} \boldsymbol{x} \qquad \mathrm{v} = (v_1, v_2)^T$$

8-DOF (2 scale, 2 rotation, 2 translation, 2 line-at-infinity)

Invariants – cross-ratio of four points on a line ←

Invariants of affine transformation
- parallel lines, ratio of parallel lengths, ratio of area



$$Cross(\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2, \bar{\mathbf{x}}_3, \bar{\mathbf{x}}_4) = \frac{|\bar{\mathbf{x}}_1 \bar{\mathbf{x}}_2||\bar{\mathbf{x}}_3 \bar{\mathbf{x}}_4|}{|\bar{\mathbf{x}}_1 \bar{\mathbf{x}}_3||\bar{\mathbf{x}}_2 \bar{\mathbf{x}}_4|}$$

where

$$|\bar{\mathbf{x}}_i \bar{\mathbf{x}}_j| = \det \begin{pmatrix} x_{i1} & x_{j1} \\ x_{i2} & x_{j2} \end{pmatrix} = \begin{vmatrix} x_{i1} & x_{j1} \\ x_{i2} & x_{j2} \end{vmatrix}$$

2D projective geometry

Projective transformation

- Hierarchy of transformations
  - Comparison
    - Affine transformations

$$\begin{pmatrix} A & t \\ 0^T & v \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}$$

The line at infinity stays at infinity, but points move along line.

- Projective transformations

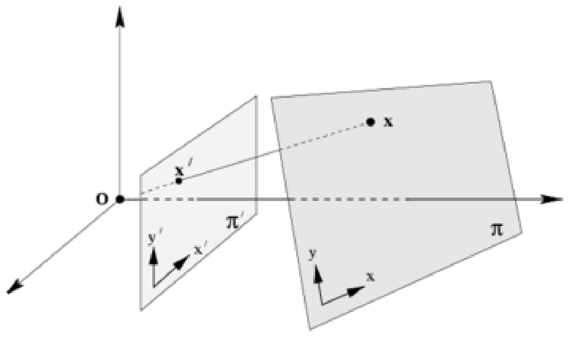
$$\begin{pmatrix} A & t \\ \mathrm{v}^T & v \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v_1 x_1 + v_2 x_2 \end{pmatrix}$$

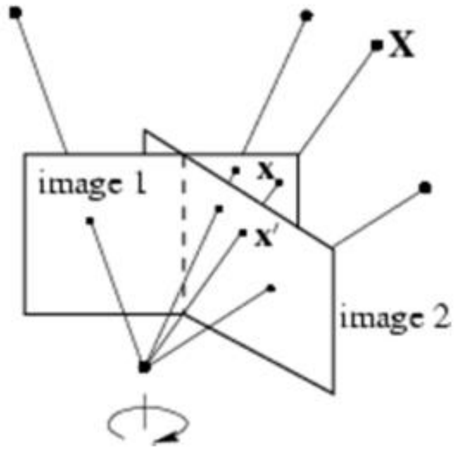
$(\mathrm{v} \neq 0)$

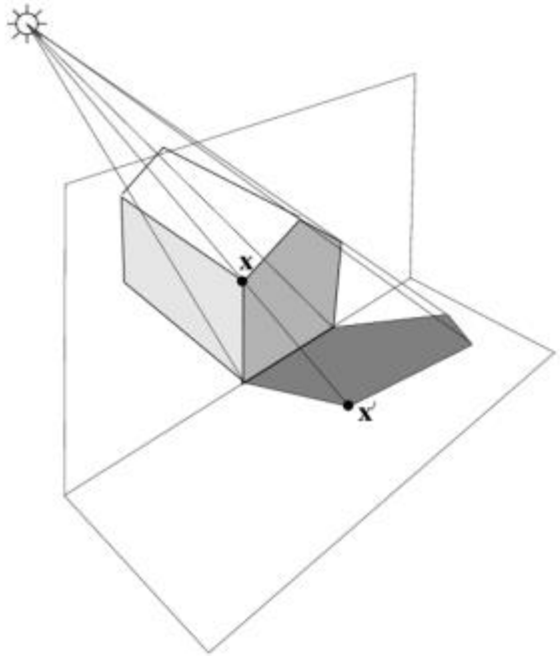
The line at infinity moves to a finite line (called vanishing line).

2D projective geometry

- Projective transformation (Homography or Projectivity or Collineation)
  - Definition

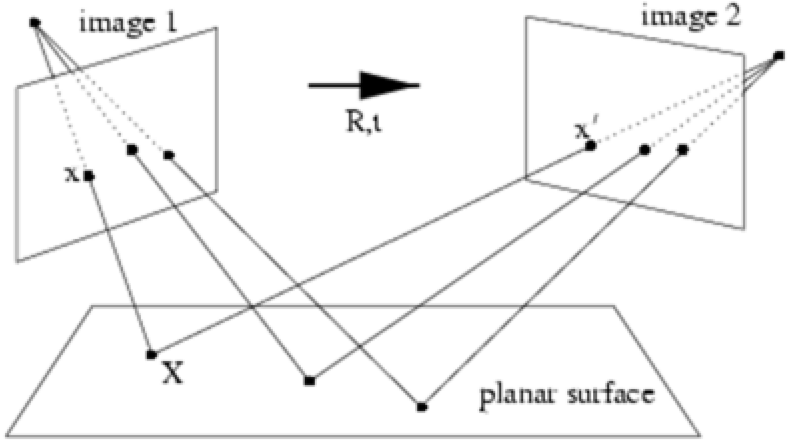






2D projective geometry

- Projective transformation (Homography or Projectivity or Collineation)
  - Definition



2D projective geometry

- Projective transformation (Homography or Projectivity or Collineation)
  - Definition

a mapping $h(\cdot): P^2 \rightarrow P^2$

such that

x_1, x_2, x_3 Lie on the same line $\longleftrightarrow$ (if and only if) $h(x_1), h(x_2), h(x_3)$ Lie on the same line

For $\forall\{x_1, x_2, x_3\}$

  - Theorem

a mapping $h(\cdot): P^2 \rightarrow P^2$ is a homography (or projectivity)

$\updownarrow$ if and only if

There exists a non-singular 3x3 matrix H
Such that for $\forall\, x \in P^2$
it is true that $h(x) = Hx$

How to apply projective transformation

- In order to apply perspective transformation, a matrix which explains the relationship between two images should be calculated
  - The dimension of the matrix is 3X3
  - However, only 8 elements in the matrix should be known

→ At least 4 corresponding pair should be given!