Fairness in Lazy Quantum Random Walks

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Introduction

Random walks are a statistical tool, used to study patterns in randomness. They can be applied over a finite space (typically a graph) or an infinite continuum.

Quantum (random) walks are the quantum equivalent of classical random walks. They are studied to observe the statistical properties of quantum systems. These results can aid the design of randomized quantum algorithms, particularly efficiency concerns for those algorithms[1].

Much of the work concerning discrete quantum walks deals with a two state bit's, known as qubits. At each time step in a qubit system the particle must move. Our work looks at three state systems, who's particles are known as qutrits. The particle is not forced to move at each time step, there is a possibility that it can remain in the same location. This possibility to remain in place gives rise to the name "lazy" quantum walks.

Classical Walks

The most approachable application of a discrete classical random walk is a fair coin toss, the result of which moves a particle left or right on an infinite line. After this experiment has been run a number of times the distance from the origin is recorded. This series of experiments is then run a number of times, and the distances from the origin is recorded each time. When a histogram of these results is plotted we see that the distribution of distances from the origin is approximately normal (Figure 1). As the particle can only land on an odd numbered space after an odd number of steps and an even numbered space after an even number of steps, over a larger number of steps every second value will be zero.

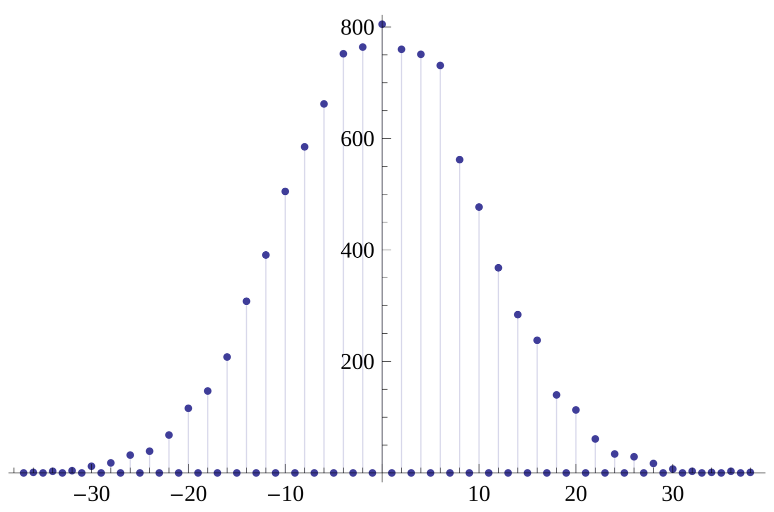

Figure 1: Histogram of final positions, 10,000 iterations of 100-step classical walk.

This can be generalized to show the probability of the particle being at a certain position after a certain number of steps[1].

	-5	-4	-3	-2	-1	0	1	2	3	4	5
0						1					
1					$\frac{1}{2}$		$\frac{1}{2}$				
2				$\frac{1}{4}$		$\frac{1}{2}$		$\frac{1}{4}$			
3			$\frac{1}{8}$		$\frac{3}{8}$		$\frac{3}{8}$		$\frac{1}{8}$		
4		$\frac{1}{16}$		$\frac{1}{4}$		$\frac{3}{8}$		$\frac{1}{4}$		$\frac{1}{16}$	
5	$\frac{1}{32}$		$\frac{5}{32}$		$\frac{5}{16}$		$\frac{5}{16}$		$\frac{5}{32}$		$\frac{1}{32}$

Figure 2: The probability of being at position i after T steps of the classical random walk on the line starting in 0.

However, on a closed graph the probabilities converge over time. On a 4 position circle, where the particle can move clockwise or counter-clockwise after each timestep, the probabilities converge to $\frac{1}{4}$ for each of the four positions.

We now consider a three state system on a graph. A particle can move left, right or remain stationary at each time step. Figure 3 shows the generalized probability distributions of a lazy classical walk.

	-5	-4	-3	-2	-1	0	1	2	3	4	5
0						1					
1					$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$				
2				$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{1}{9}$			
3			$\frac{1}{27}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{27}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{1}{27}$		
4		$\frac{1}{81}$	$\frac{4}{81}$	$\frac{10}{81}$	$\frac{16}{81}$	$\frac{19}{81}$	$\frac{16}{81}$	$\frac{10}{81}$	$\frac{4}{81}$	$\frac{1}{81}$	
5	$\frac{1}{243}$	$\frac{5}{243}$	$\frac{5}{81}$	$\frac{10}{81}$	$\frac{5}{27}$	$\frac{17}{81}$	$\frac{5}{27}$	$\frac{10}{81}$	$\frac{5}{81}$	$\frac{5}{243}$	$\frac{1}{243}$

Figure 3: The probability of being at position i after T steps of the lazy classical random walk on the line starting in 0.

The lazy classical walk has a normal distribution, the introduction of the lazy step removes the odd/even restriction. Like the standard classical walk, the lazy classical walk converges to equal values on a closed graph.

It is possible to vary the lazy bias of a classical walk. A common bias is to choose a probability per iteration of $\frac{1}{2}$ for the lazy move, $\frac{1}{4}$ for the clockwise move, and $\frac{1}{4}$ for the counter-clockwise move. This bias can be modeled as two coin tosses, where one coin signals movement or its absence, the second signals the direction of movement (if the first coin signals movement).

However, we have chosen to give each movement possibility a probability of $\frac{1}{3}$. This is for comparison with the quantum randomization function we will be examining.

Hadamard Gate

The quantum randomizing function we are examining is known as a Hadamard Gate (also known as a Hadamard Coin). A coin used for a two state (qubit) system is H_2

$$H_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{1}$$

It is unitary and has been shown[1] to be fair.

The Hadamard gate has been generalized by Marttala[2] to H_m, where m is the number of states in the system.

$$\sigma = e^{\frac{i2\pi}{m}} \tag{2}$$

$$H_m = \frac{1}{\sqrt{m}}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & \sigma^{(m-1)} & \sigma^{2(m-1)} & \cdots & \sigma^2 & \sigma \\ 1 & \sigma^{(m-2)} & \sigma^{2(m-2)} & \cdots & \sigma^4 & \sigma^2 \\ 1 & \sigma^{(m-3)} & \sigma^{2(m-3)} & \cdots & \sigma^6 & \sigma^3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \sigma & \sigma^2 & \cdots & \sigma^{(m-2)} & \sigma^{(m-1)} \end{pmatrix} \tag{3}$$

As we are interested in a lazy walk we want a three state gate H_3, where the three states represent a left movement, a right movement, and no movement.

$$H_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{-\frac{2}{3}i\pi} & e^{\frac{2}{3}i\pi} \\ 1 & e^{\frac{2}{3}i\pi} & e^{-\frac{2}{3}i\pi} \end{pmatrix} \tag{4}$$

$$H_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -(-1)^{\frac{1}{3}} & (-1)^{\frac{2}{3}} \\ 1 & (-1)^{\frac{2}{3}} & -(-1)^{\frac{1}{3}} \end{pmatrix} \tag{5}$$

We are satisfied that H_3 is unitary as

$$H_3 H_3^{\dagger} = I_3 \tag{6}$$

Quantum Walks on a Graph

A Lazy One Dimensional Discrete Quantum Walks takes place on the state space spanned by vectors

$$|n, p\rangle \tag{7}$$

where $n \in Z$ and $p \in \{0, 1, 2\}$ is a three-state variable. n represents the position of a particle on the walk and is the walks classical component. p is the quantum component, it is typically a two-state spin but we have added a third state to represent our lazy state.

One step of the walk is given by the transitions

$$|n, 0\rangle \longrightarrow a\,|n, 0\rangle + b\,|n+1, 1\rangle + c\,|n-1, 2\rangle \tag{8}$$

$$|n, 1\rangle \longrightarrow d\,|n, 0\rangle + e\,|n+1, 1\rangle + f\,|n-1, 2\rangle \tag{9}$$

$$|n, 2\rangle \longrightarrow g\,|n, 0\rangle + h\,|n+1, 1\rangle + i\,|n-1, 2\rangle \tag{10}$$

where

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = H_3 \tag{11}$$

This has been expanded from the two-state walk equations given elsewhere[1, 3].

Multiple randomizing iterations were performed for a four-position graph with no directional biasing. Varying input conditions were used to obtain a variety of results. A four position non-lazy circular quantum walk was also analyzed for comparison to the lazy results.

Results

The non-lazy graph converges to $\frac{1}{4}$ for each step. This is due to the fact that the positional probabilities repeat after 8 steps, caused by constructive and destructive interference on the graph.

	$\|0,0\rangle$	$\|0,1\rangle$	$\|1,0\rangle$	$\|1,1\rangle$	$\|2,0\rangle$	$\|2,1\rangle$	$\|3,0\rangle$	$\|3,1\rangle$
0	1							
1				$\frac{1}{\sqrt{2}}$			$\frac{1}{\sqrt{2}}$	
2	$\frac{1}{2}$	$\frac{1}{2}$			$\frac{1}{2}$	$-\frac{1}{2}$		
3							$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
4					1			

Figure 4: The probability of being at position i after T steps of the quantum random walk on a closed four step graph, starting at $|0, 0\rangle$.

As a result, the positional probabilities converge to equal values for all positions on a four step classical walk (lazy or not), and a non-lazy quantum walk. However, the positional probabilities do not converge to equal values for the lazy quantum walk on a four position circle.

We define our input condition as

$$x\,|0, 0\rangle + y\,|0, 1\rangle + y\,|0, 2\rangle \tag{12}$$

where

$$y = \sqrt{\frac{1 - x^2}{2}} \tag{13}$$

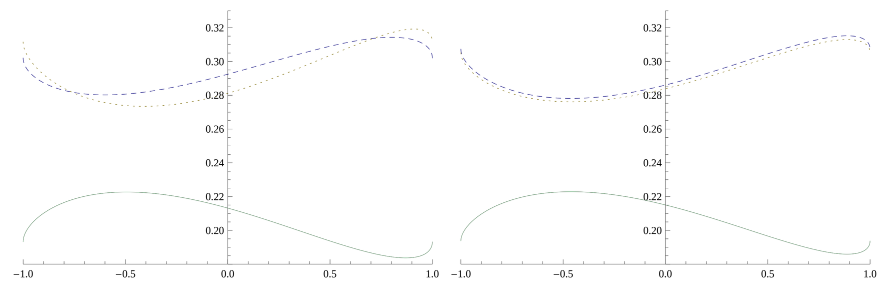

Figure 5: Trend Removal:(left) Obvious non-linear trend (right) In an area of activation

	AR(1) coefficient	$\widehat{\sigma^2}$	Numerator	Denominator	Ratio Statistic
Method I	−0.0027	15765	591.88	315.29	1.877
Method II	0.2839	11276	624.98	419.46	1.490
Method III	—	—	689.30	513.88	1.341

Conclusion and extensions

Non-parametric spectral estimation is shown to be an accurate and self-calibrating approach for analyzing periodic designs. The method makes few assumptions and is resistant to high-frequency artefacts whereas parametric time-domain approaches may be susceptible to these artefacts and biased by the assumptions they make on the form of the spectral density. The method can be easily extended to handle non-periodic event related designs and initial results are extremely promising.

Acknowledgements

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References

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[2] Peter Marttala. An extension of the Deutsch-Jozsa algorithm to arbitrary qudits. Master's thesis, University of Saskatchewan, 2007.

[3] Michael McGettrick. One Dimensional Quantum Walks with Memory. Apr 2010.