

```

In[278]:= FourStepCircleStatesToPositionProbability [State0_] :=
Module[{State = State0},
  ProbabilityAll = Simplify[Conjugate[State] * State];
  ProbabilaityMixed = Transpose[{{
    ProbabilityAll[[1, 1]] + ProbabilityAll[[2, 1]] + ProbabilityAll[[3, 1]],
    ProbabilityAll[[4, 1]] + ProbabilityAll[[5, 1]] + ProbabilityAll[[6, 1]],
    ProbabilityAll[[7, 1]] + ProbabilityAll[[8, 1]] + ProbabilityAll[[9, 1]],
    ProbabilityAll[[10, 1]] + ProbabilityAll[[11, 1]] + ProbabilityAll[[12, 1]]
  }}];
  For[k = 1, k ≤  $\frac{\text{Dimensions}[\text{ProbabilityAll}][[1]]}{\text{BitOrder}}$ , k++,  $\sum_{j=(\text{BitOrder}(k-1))+1}^{\text{BitOrder } k} \text{ProbabilityAll}[[j, 1]]$ 
  ];
  Return[ProbabilaityMixed];
]

```

```

In[279]:= LazyQuantumRandomWalkHistory[State0_, Steps0_, Type0_, History0_, ReturnType0_] :=
Module[
{
  State = State0, (* 12x1 matrix with input state *)
  Steps = Steps0, (* Number of iterations to perform *)
  Type = Type0, (* 0:Standard, 1:Odd's, 2:Even's, 3:Average's *)
  HistoryArray = History0, (* 0:Final State, 1:History *)
  ReturnType = ReturnType0 (* 0:States, 1:Position Probabilities *)
},
History = {};
BitOrder = 3;

$$\sigma = e^{\frac{\pi 2 i}{\text{BitOrder}}};$$


$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = H = \frac{\begin{pmatrix} 1 & 1 & 1 \\ 1 & \sigma^{\text{BitOrder}-1} & \sigma \\ 1 & \sigma & \sigma^{\text{BitOrder}-1} \end{pmatrix}}{\sqrt{\text{BitOrder}}};$$


$$M = \begin{pmatrix} a & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & c \\ d & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & f \\ g & 0 & 0 & 0 & h & 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & c & a & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & f & d & 0 & 0 & 0 & e & 0 & 0 & 0 & 0 \\ 0 & 0 & i & g & 0 & 0 & 0 & h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & a & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & f & d & 0 & 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 & 0 & i & g & 0 & 0 & 0 & h & 0 \\ 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & c & a & 0 & 0 \\ 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & f & d & 0 & 0 \\ 0 & h & 0 & 0 & 0 & 0 & 0 & 0 & i & g & 0 & 0 \end{pmatrix};$$

AppendTo[History, State];
For[j = 0, j < Steps, j++,
  State = Simplify[M.State];
  If[HistoryArray == 1 || Type == 3, AppendTo[History, State]]
];
If[HistoryArray == 1,
  If[ReturnType == 0,
    Return[History],
    Return[
      Map[FourStepCircleStatesToPositionProbability,
        History]]],
  If[ReturnType == 0,
    Return[State],
    Return[
      FourStepCircleStatesToPositionProbability[State]]];
]
]

LQRWHistogramPositionProbabilities[State0_, Steps0_] :=
Module[{State = State0, Steps = Steps0},
Return[
  Map[Flatten,
    Map[FourStepCircleStatesToPositionProbability,
      LazyQuantumRandomWalkHistory[State, Steps, 0, 1]]]]]

```

```

In[307]:= CumulativeMean[List0_] :=
Module[{List = List0},
Return[
(Accumulate[#] / Range[Length[#]])
& /@List]]

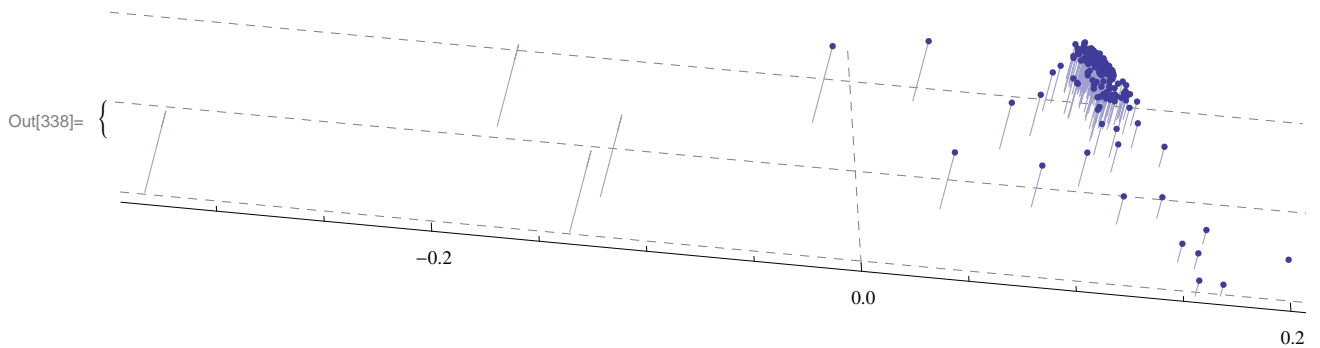
In[337]:= Table[ListPointPlot3D[v,
Filling -> Bottom,
BoxRatios -> {2.5, 10, 1},
ViewPoint -> {50, 0, 70},
Boxed -> False,
Axes -> {False, True, False},
FaceGrids -> {
{{0, 0, -1},
{Automatic, Range[0, 200, 10]}}},
FaceGridsStyle -> Directive[Gray, Dashed],
ImageSize -> Large
],
{v, Map[
Transpose[CumulativeMean[Transpose[Rest[LQRWHistogramPositionProbabilities [
SparseArray[
{{12, 1} -> 0,
{#, 1} -> 1}], 200]]]]] &,
Range[3]]]}

```

```

In[338]:= Table[ListPointPlot3D[v,
  Filling -> Bottom,
  BoxRatios -> {2.5, 10, 1},
  ViewPoint -> {50, 0, 70},
  Boxed -> False,
  Axes -> {False, True, False},
  FaceGrids -> {
    {{0, 0, -1},
     {Automatic, Range[0, 200, 10]}}},
  FaceGridsStyle -> Directive[Gray, Dashed],
  ImageSize -> Large
],
{v, Map[
  Transpose[
    Differences[CumulativeMean[
      Transpose[
        Rest[LQRWHistogramPositionProbabilities[
          SparseArray[
            {{12, 1} -> 0,
             {#, 1} -> 1}], 200]]]]]] &,
    Range[3]]]]]

```



```

In[362]:= (ListConvolve[{-1, 1}, #]) & /@ CumulativeMean[
  Transpose[
    LQRWHistogramPositionProbabilities[
      SparseArray[
        {{12, 1} -> 0,
         {1, 1} -> 1}], 10]]]

```

Out[362]=

$\frac{1}{3}$	$\frac{1}{9}$	$\frac{5}{54}$	$\frac{7}{270}$	$\frac{43}{3645}$	$\frac{79}{5103}$	$\frac{1087}{30618}$	$\frac{127}{39366}$	$\frac{1177}{98415}$	$\frac{29773}{1082565}$
$-\frac{1}{6}$	$-\frac{1}{54}$	$-\frac{1}{36}$	$-\frac{11}{1620}$	$\frac{167}{7290}$	$\frac{85}{10206}$	$-\frac{13}{61236}$	$\frac{2137}{236196}$	$-\frac{5407}{590490}$	$-\frac{145373}{6495390}$
0	$-\frac{2}{27}$	$-\frac{1}{27}$	$-\frac{1}{81}$	$-\frac{14}{243}$	$-\frac{164}{5103}$	$-\frac{179}{5103}$	$-\frac{1259}{59049}$	$\frac{1876}{295245}$	$\frac{56054}{3247695}$
$-\frac{1}{6}$	$-\frac{1}{54}$	$-\frac{1}{36}$	$-\frac{11}{1620}$	$\frac{167}{7290}$	$\frac{85}{10206}$	$-\frac{13}{61236}$	$\frac{2137}{236196}$	$-\frac{5407}{590490}$	$-\frac{145373}{6495390}$

```
In[373]:= N[Transpose[CumulativeMean[
  Transpose[
    LQRWHistogramPositionProbabilities[
      SparseArray[
        {{12, 1} → 0,
         {3, 1} → 1}], 10]]], 5]
```

```
Out[373]= 
$$\begin{pmatrix} 1.0000 & 0 & 0 & 0 \\ 0.66667 & 0.16667 & 0 & 0.16667 \\ 0.55556 & 0.18519 & 0.074074 & 0.18519 \\ 0.49074 & 0.18519 & 0.11111 & 0.21296 \\ 0.41481 & 0.18272 & 0.15309 & 0.24938 \\ 0.37174 & 0.19684 & 0.16872 & 0.26269 \\ 0.36038 & 0.18852 & 0.19616 & 0.25495 \\ 0.34459 & 0.20125 & 0.19222 & 0.26195 \\ 0.32637 & 0.21926 & 0.20839 & 0.24598 \\ 0.33476 & 0.20483 & 0.21108 & 0.24933 \\ 0.32000 & 0.19881 & 0.22333 & 0.25786 \end{pmatrix}$$

```

```
In[674]:= ListConvolve[{-1, 1}, {0,  $\frac{2}{3}$ ,  $\frac{5}{9}$ ,  $\frac{25}{54}$ ,  $\frac{59}{135}$ ,  $\frac{610}{729}$ ,  $\frac{697}{1701}$ ,  $\frac{1637}{4374}$ ,  $\frac{7303}{19683}$ ,  $\frac{35338}{98415}$ ,  $\frac{71789}{216513}$ }]
```

```
Out[674]=  $\left\{-\frac{2}{3}, -\frac{1}{9}, \frac{5}{54}, \frac{7}{270}, -\frac{1457}{3645}, \frac{2179}{5103}, \frac{1087}{30618}, \frac{127}{39366}, \frac{1177}{98415}, \frac{29773}{1082565}\right\}$ 
```

```
In[308]:= Transpose[CumulativeMean[Transpose[Rest[LQRWHistogramPositionProbabilities[
  SparseArray[
    {{12, 1} → 0,
     {1, 1} → 1}], 10]]]]]
```

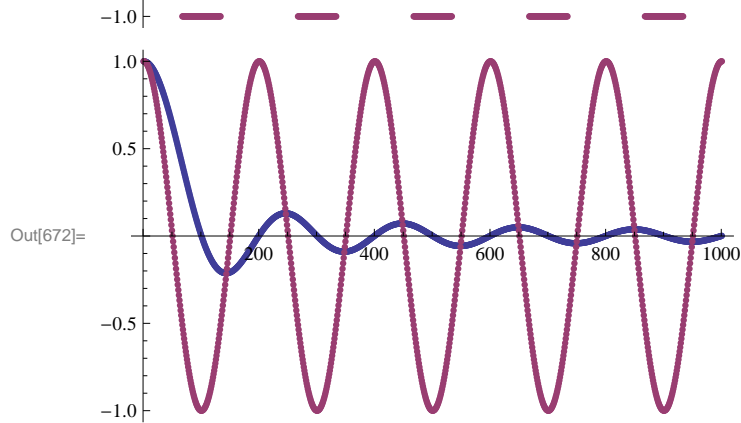
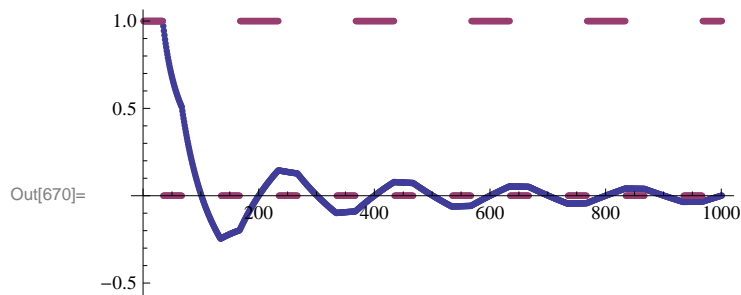
```
Out[308]= 
$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{5}{18} & \frac{1}{9} & \frac{5}{18} \\ \frac{23}{81} & \frac{23}{81} & \frac{4}{27} & \frac{23}{81} \\ \frac{8}{27} & \frac{89}{324} & \frac{25}{162} & \frac{89}{324} \\ \frac{377}{1215} & \frac{287}{1215} & \frac{88}{405} & \frac{287}{1215} \\ \frac{227}{729} & \frac{481}{2187} & \frac{544}{2187} & \frac{481}{2187} \\ \frac{623}{2187} & \frac{3302}{15309} & \frac{1448}{5103} & \frac{3302}{15309} \\ \frac{1279}{1279} & \frac{5305}{5305} & \frac{1990}{1990} & \frac{5305}{5305} \\ \frac{4374}{50993} & \frac{26244}{37169} & \frac{6561}{17272} & \frac{26244}{37169} \\ \frac{177147}{177147} & \frac{177147}{177147} & \frac{59049}{59049} & \frac{177147}{177147} \\ \frac{26053}{98415} & \frac{27439}{118098} & \frac{79891}{295245} & \frac{27439}{118098} \end{pmatrix}$$

```

```

In[669]:= RandomRange = Table[Round[Cos[i]], {i, 0, 10 Pi, 2 Pi / 200}];
ListPlot[
  {Accumulate[RandomRange] /
   Range[Length[RandomRange]], RandomRange}
]
RandomRange = Table[Cos[i], {i, 0, 10 Pi, 2 Pi / 200}];
ListPlot[
  {Accumulate[RandomRange] /
   Range[Length[RandomRange]], RandomRange}
]

```



```

In[284]:= CumulativeAverage[LQRWHistogramPositionProbabilities [
  SparseArray [
    {{12, 1} → 0,
     {1, 1} → 1}], 10]]

```

Out[284]=

1	0	0	0
$\frac{2}{3}$	$\frac{1}{6}$	0	$\frac{1}{6}$
$\frac{5}{9}$	$\frac{5}{27}$	$\frac{2}{27}$	$\frac{5}{27}$
$\frac{25}{54}$	$\frac{23}{108}$	$\frac{1}{9}$	$\frac{23}{108}$
$\frac{59}{135}$	$\frac{89}{405}$	$\frac{10}{81}$	$\frac{89}{405}$
$\frac{310}{729}$	$\frac{287}{1458}$	$\frac{44}{243}$	$\frac{287}{1458}$
$\frac{697}{1701}$	$\frac{962}{5103}$	$\frac{1088}{5103}$	$\frac{962}{5103}$
$\frac{1637}{1637}$	$\frac{1651}{1651}$	$\frac{181}{181}$	$\frac{1651}{1651}$
$\frac{4374}{7303}$	$\frac{8748}{10610}$	$\frac{729}{15920}$	$\frac{8748}{10610}$
$\frac{19683}{35338}$	$\frac{59049}{37169}$	$\frac{59049}{8636}$	$\frac{59049}{37169}$
$\frac{98415}{71789}$	$\frac{196830}{137195}$	$\frac{32805}{159782}$	$\frac{196830}{137195}$
$\frac{216513}{216513}$	$\frac{649539}{649539}$	$\frac{649539}{649539}$	$\frac{649539}{649539}$

In[285]:= **Transpose**[(**Accumulate**[**Rest**[#]] / **Range**[**Length**[**Rest**[#]]]) & /@**Transpose**[**OutputX**]]

Out[285]=
$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{5}{18} & \frac{1}{9} & \frac{5}{18} \\ \frac{23}{81} & \frac{23}{81} & \frac{4}{27} & \frac{23}{81} \\ \frac{8}{27} & \frac{89}{324} & \frac{25}{162} & \frac{89}{324} \\ \frac{377}{1215} & \frac{287}{1215} & \frac{88}{405} & \frac{287}{1215} \\ \frac{227}{729} & \frac{481}{2187} & \frac{544}{2187} & \frac{481}{2187} \\ \frac{623}{2187} & \frac{3302}{15309} & \frac{1448}{5103} & \frac{3302}{15309} \\ \frac{1279}{4374} & \frac{5305}{26244} & \frac{1990}{6561} & \frac{5305}{26244} \\ \frac{50993}{50993} & \frac{37169}{37169} & \frac{17272}{17272} & \frac{37169}{37169} \\ \frac{177147}{26053} & \frac{177147}{27439} & \frac{59049}{79891} & \frac{177147}{27439} \\ \frac{98415}{118098} & \frac{118098}{118098} & \frac{295245}{118098} & \frac{118098}{118098} \end{pmatrix}$$

In[286]:= **Rest**[**FoldList**[**Plus**, 0, {1, $\frac{1.}{3}$, $\frac{1.}{3}$, $\frac{5.}{27}$, $\frac{1.}{3}$, $\frac{89.}{243}$, $\frac{77.}{243}$, $\frac{275.}{2187}$, $\frac{755.}{2187}$, $\frac{4949.}{19683}$, $\frac{371.}{6561}$ }]] / **Range**[11.]

Out[286]= {1, 0.666667, 0.555556, 0.462963, 0.437037, 0.42524, 0.409759, 0.374257, 0.371031, 0.359071, 0.331569}

In[287]:= **Length**[{1, $\frac{1}{3}$, $\frac{1}{3}$, $\frac{5}{27}$, $\frac{1}{3}$, $\frac{89}{243}$, $\frac{77}{243}$, $\frac{275}{2187}$, $\frac{755}{2187}$, $\frac{4949}{19683}$, $\frac{371}{6561}$ }]

Out[287]= 11

In[288]:= **Accumulate**[{1, $\frac{1}{3}$, $\frac{1}{3}$, $\frac{5}{27}$, $\frac{1}{3}$, $\frac{89}{243}$, $\frac{77}{243}$, $\frac{275}{2187}$, $\frac{755}{2187}$, $\frac{4949}{19683}$, $\frac{371}{6561}$ }]

Out[288]= $\left\{1, \frac{4}{3}, \frac{5}{3}, \frac{50}{27}, \frac{59}{27}, \frac{620}{243}, \frac{697}{243}, \frac{6548}{2187}, \frac{7303}{2187}, \frac{70676}{19683}, \frac{71789}{19683}\right\}$

In[289]:= $\left\{1, \frac{4}{3}, \frac{5}{3}, \frac{50}{27}, \frac{59}{27}, \frac{620}{243}, \frac{697}{243}, \frac{6548}{2187}, \frac{7303}{2187}, \frac{70676}{19683}, \frac{71789}{19683}\right\}$

Out[289]= $\left\{1, \frac{4}{3}, \frac{5}{3}, \frac{50}{27}, \frac{59}{27}, \frac{620}{243}, \frac{697}{243}, \frac{6548}{2187}, \frac{7303}{2187}, \frac{70676}{19683}, \frac{71789}{19683}\right\}$

In[290]:= **Accumulate**[**Range**[10]] / **Range**[10]

Out[290]= $\left\{1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, 5, \frac{11}{2}\right\}$

In[291]:= **Range**[10]

Out[291]= {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

```
In[292]:= LQRWHistogramPositionProbabilities [
  SparseArray [
    {{12, 1} → 0,
     {1, 1} → 1}], 5]
```

$$\text{Out[292]} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{5}{27} & \frac{8}{27} & \frac{2}{9} & \frac{8}{27} \\ \frac{1}{3} & \frac{20}{81} & \frac{14}{81} & \frac{20}{81} \\ \frac{89}{243} & \frac{20}{243} & \frac{38}{81} & \frac{20}{243} \end{pmatrix}$$

```
In[293]:= FoutStepCircleStatesToPositionProbability [
  LazyQuantumRandomWalkHistory [
    SparseArray [
      {{12, 1} → 0,
       {1, 1} → 1}],
    5, 0, 0]]
```

$$\text{Out[293]} = \begin{pmatrix} \frac{89}{243} \\ \frac{20}{243} \\ \frac{38}{81} \\ \frac{20}{243} \end{pmatrix}$$

```
In[294]:= FoutStepCircleStatesToPositionProbability [
  LazyQuantumRandomWalkHistory [
    SparseArray [
      {{12, 1} → 0,
       {1, 1} → 1}],
    5, 0, 0]]
```

$$\text{Out[294]} = \begin{pmatrix} \frac{89}{243} \\ \frac{20}{243} \\ \frac{38}{81} \\ \frac{20}{243} \end{pmatrix}$$