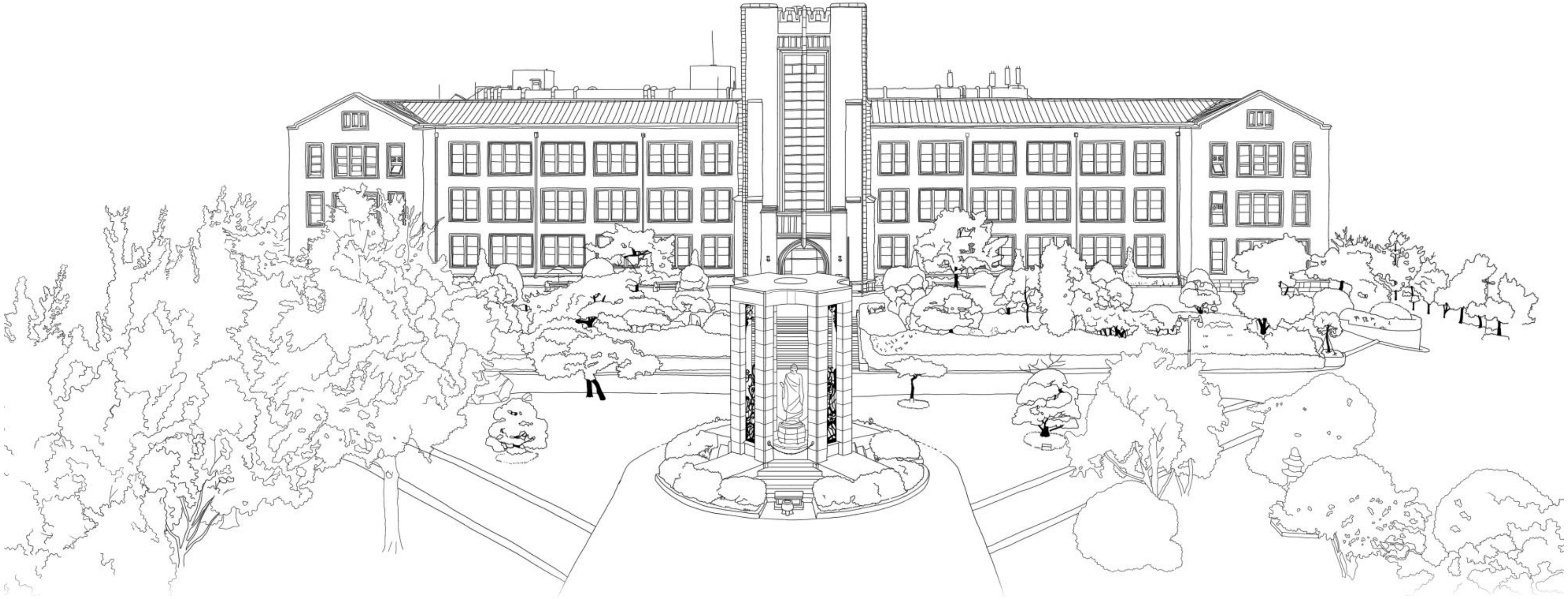

Introduction to Algorithms

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❖ Introduction

- Order of growth
  - ➢ **The order of growth of the running time of an algorithm**
    - ✓ Gives a simple characterization of the algorithm's efficiency
    - ✓ For large enough inputs, the multiplicative constants and lower-order terms are dominated by the effects of the input size itself

      Ex) $an^2 + bn + c \rightarrow \Theta(n^2)$
    - ✓ Enables to compare the relative performance of alternative algorithms

      Ex) Merge sort : $\Theta(\mathrm{n}\log n) <$ Insertion sort : $\Theta(n^2)$

  - ➢ **Asymptotic efficiency of algorithm**
    - ✓ How the running time of an algorithm increases with the size of the input in the limit
    - ✓ Several standard methods for simplifying the asymptotic analysis of algorithms

❖ Asymptotic notation

- **Asymptotic notation, functions, and running times**
  - ➢ Asymptotic notation
    - ✓ Describe the running times of algorithms
    - ✓ Applies to functions

      Ex) What we were writing as $\Theta(n^2)$ was the function about $an^2 + bn + c$

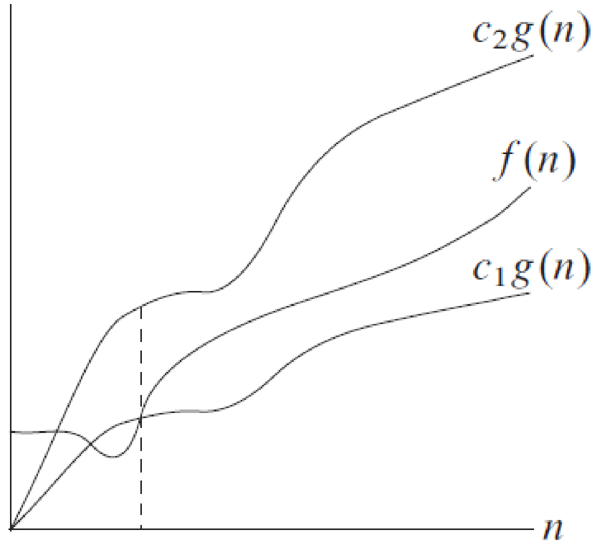
  - ➢ Functions
    - ✓ The functions to which we apply asymptotic notation will usually characterize the running times of algorithms
    - ✓ Characterize other aspect of algorithms – amount of space they use

  - ➢ Running time
    - ✓ Commonly interested in the worst case
    - ✓ Characterize the running time no matter what the input

❖ Asymptotic notation

- **Θ** notation (Big-**Θ** notation)
  - ➢ Definition

$$\Theta(g(n)) = \{ f(n) : \text{there exist positive constants } c_1, c_2, \text{ and } n_0 \text{ such that } 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \text{ for all } n \ge n_0 \} .^1$$



$f(n) = \Theta(g(n))$

- ✓ $\Theta(g(n))$ is the set of functions
- ✓ Function $f(n)$ is running time of algorithm
- ✓ n_0 is the minimum possible value
- ✓ $f(n) = \Theta(g(n))$ means $f(n)$ belongs to the set $\Theta(g(n))$
  ($f(n) \in \Theta(g(n))$)
- ✓ For all values of n to the right of n_0, the value of $f(n)$ lies at or above $c_1 g(n)$ and at or below $c_2 g(n)$
- ✓ $g(n)$ is asymptotically tight bound for $f(n)$

❖ Asymptotic notation

- $\boldsymbol{\Theta}$ notation
  - Example : Prove $\frac{1}{2}n^2 - 3n = \Theta(n^2)$
    - ✓ Let $f(n) = \frac{1}{2}n^2 - 3n,\ g(n) = \theta(n^2)$
    - ✓ According to definition

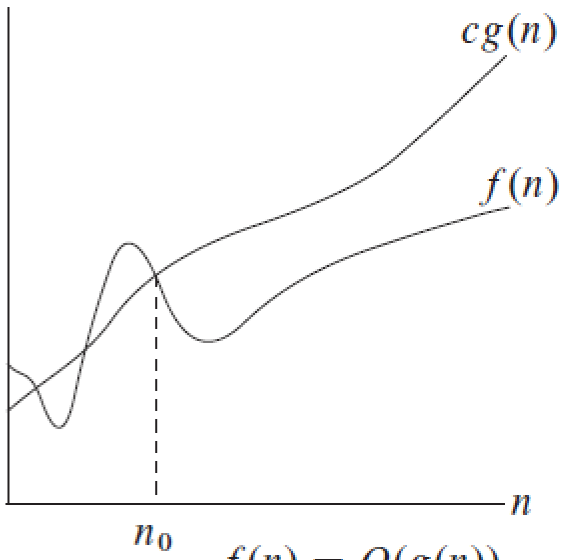
      $c_1 n^2 \le \frac{1}{2}n^2 - 3n \le c_2 n^2$
    - ✓ For all $n \ge n_0$, dividing by n^2

      $c_1 \le \frac{1}{2} - \frac{3}{n} \le c_2$
    - ✓ First, if $c_1 \le \frac{1}{14}$ & $n \ge 7 \rightarrow c_1 \le \frac{1}{2} - \frac{3}{n}$ is always true
    - ✓ Second, if $c_2 \ge \frac{1}{2}$ & $n \ge 1 \rightarrow c_2 \ge \frac{1}{2} - \frac{3}{n}$ is always true
    - ✓ So there exist c_1, c_2, n_0 so we can verify that $\frac{1}{2}n^2 - 3n = \theta(n^2)$
    - ✓ The important thing is that there exist some set about c_1, c_2, n_0, not which number of c_1, c_2, n_0 choosen

Asymptotic notation

- O notation (Big-O notation)
  - Definition

$$O(g(n)) = \{ f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0 \} .$$



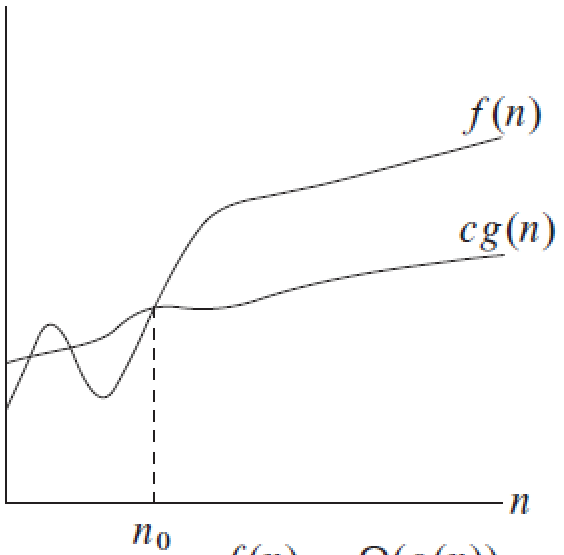
$f(n) = O(g(n))$

- ✓ Only have an asymptotic upper bound
- ✓ $f(n) = O(g(n))$ means $f(n)$ belongs to the set $O(g(n))$
  ($f(n) \in O(g(n))$)
- ✓ $f(n) = \Theta(g(n))$ implies $f(n) = O(g(n))$
  ($\Theta(g(n)) \subseteq O(g(n))$)
- ✓ For all values of n to the right of n_0, the value of $f(n)$ lies at or below $cg(n)$
- ✓ $g(n)$ is an asymptotic upper bound on $f(n)$

❖ Asymptotic notation

- Ω notation (Big−Ω notation)
  - ➢ Definition

$$\Omega(g(n)) = \{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le cg(n) \le f(n) \text{ for all } n \ge n_0\} .$$



$f(n) = \Omega(g(n))$

✓ Only have an asymptotic lower bound

✓ $f(n) = \Omega(g(n))$ means $f(n)$ belongs to the set $\Omega(g(n))$ ($f(n) \in \Omega(g(n))$)

✓ For all values of n to the right of n_0, the value of $f(n)$ lies at or above $cg(n)$ and

Theorem 3.1

For any two functions $f(n)$ and $g(n)$, we have $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. ■

Asymptotic notation

- o notation (Little-o notation)
  - Definition

$$o(g(n)) = \{ f(n) : \text{ for any positive constant } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \le f(n) < cg(n) \text{ for all } n \ge n_0 \} .$$

- ✓ Denote an upper bound that is not asymptotically tight (Relaxed upper bound)
- ✓ Big-O notation, the bound $0 \le f(n) \le cg(n)$ holds for some constant $c > 0$
- ✓ Little-o notation, the bound $0 \le f(n) < cg(n)$ holds for all constants $c > 0$
- ✓ Ex) $2n = o(n^2),\ but\ 2n^2 \ne o(n^2)$
- ✓ o-notation, function $f(n)$ becomes insignificant relative to $g(n)$ as n approaches infinity

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

❖ Asymptotic notation

- $\boldsymbol{\omega}$ notation (Little–$\boldsymbol{\omega}$ notation)
  - ➢ Definition

$$\omega(g(n)) = \{f(n) : \text{for any positive constant } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \le cg(n) < f(n) \text{ for all } n \ge n_0\}.$$

- ✓ Denote a lower bound that is not asymptotically tight (Relaxed lower bound)
- ✓ Big–Ω notation, the bound $0 \le cg(n) \le f(n)$ holds for some constant $c > 0$
- ✓ Little–ω notation, the bound $0 \le cg(n) < f(n)$holds for all constants $c > 0$
- ✓ Ex) $\frac{n^2}{2} = \omega(n),\ but\ \frac{n^2}{2} \ne \omega(n^2)$
- ✓ o–notation, function $f(n)$ becomes insignificant relative to $g(n)$ as n approaches infinity

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$$

❖ Introduction

- Divide and Conquer
  - ➢ Solve a problem recursively, applying three steps
    - ✓ Divide the problem into some subproblems that are smaller instances of the same problem
    - ✓ Conquer the subproblems by solving them recursively.
      1. If the subproblem sizes are small enough, just solve the subproblems in a straightforward manner
      2. Recursive case: subproblems are large enough to solve recursively
      3. Base case: subproblems become small enough that no longer recurse
    - ✓ Combine the solutions to the subproblems into the original problem
  - ➢ Sometimes, we solve subproblems that are not the same as the original problem
    - ✓ consider solving such subproblems as combine step
  - ➢ We will solve two Divide and Conquer problem
    - ✓ maximum-subarray problem
    - ✓ Multiplying N x N matrices

Divide and Conquer

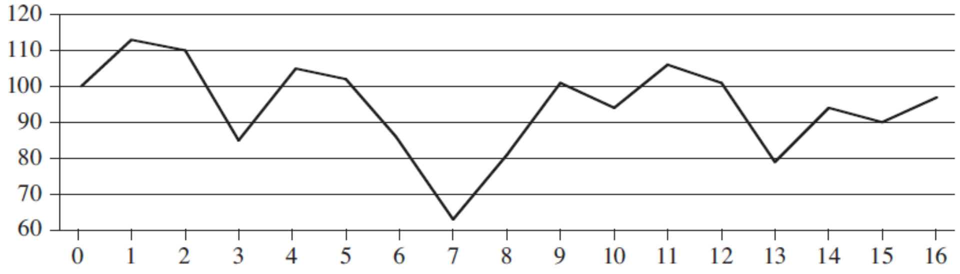
❖ The maximum subarray problem

- Definition
  - ➢ Task to find the series of continuous elements with the maximum sum in any given array
  - ➢ It is trivial if all elements in the array are non-negative

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
A	13	−3	−25	20	−3	−16	−23	18	20	−7	12	−5	−22	15	−4	7

maximum subarray

- Price of stock problem



Day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Price	100	113	110	85	105	102	86	63	81	101	94	106	101	79	94	90	97
Change		13	−3	−25	20	−3	−16	−23	18	20	−7	12	−5	−22	15	−4	7

- ✓ X axis: date Y axis: stock price
- ✓ The bottom row of the table gives the change in price from the previous day
- ✓ Have to find maximum profit → find maximum subarray in bottom row of the table

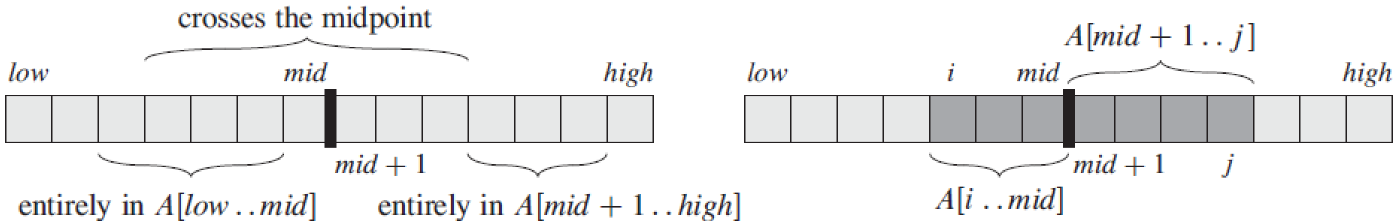
❖ The maximum subarray problem

- Brute Force solution
  - ➢ Just try every possible pair of buy and sell dates.
  - ➢ $\binom{n}{2} = \Theta(n^2)$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
A	13	–3	–25	20	–3	–16	–23	18	20	–7	12	–5	–22	15	–4	7

maximum subarray

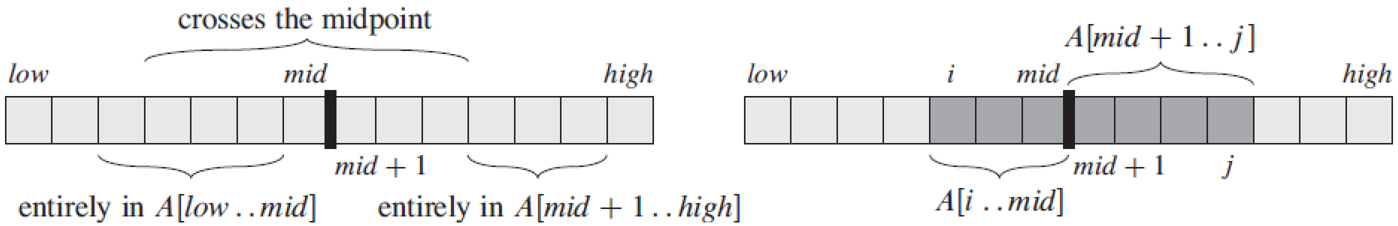
- Divide and Conquer solution



  - ➢ We want to find maximum subarray of the subarray $\boldsymbol{A[low..high]}$
  - ➢ Maximum subarray : $\boldsymbol{A[i..j]}$
  - ➢ We will find maximum subarray in three cases

❖ The maximum subarray problem

- Divide and Conquer Solution



➢ We will find maximum subarray in three cases

- entirely in the subarray $A[low..mid]$, so that $low \leq i \leq j \leq mid$,
- entirely in the subarray $A[mid+1..high]$, so that $mid < i \leq j \leq high$, or
- crossing the midpoint, so that $low \leq i \leq mid < j \leq high$.

✓ Any contiguous subarray $A[i..j]$ of $A[low..high]$ must lie in exactly one of the following places

✓ We can find maximum subarrays of $A[low..mid]$ and $A[mid+1..high]$ recursively

i. Two subproblems are smaller instances of the Original problem

✓ all that left to do is find a maximum subarray that crosses the midpoint

❖ The maximum subarray problem

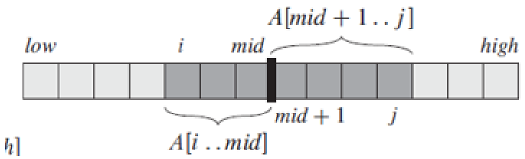
- Divide and Conquer Solution
  - ➢ Find a maximum subarray that crosses the midpoint

FIND-MAX-CROSSING-SUBARRAY$(A, low, mid, high)$

```
1   left-sum = -∞
2   sum = 0
3   for i = mid downto low
4       sum = sum + A[i]
5       if sum > left-sum
6           left-sum = sum
7           max-left = i
8   right-sum = -∞
9   sum = 0
10  for j = mid + 1 to high
11      sum = sum + A[j]
12      if sum > right-sum
13          right-sum = sum
14          max-right = j
15  return (max-left, max-right, left-sum + right-sum)
```

✓ This problem is not a smaller instance of original problem → it has the added restriction that the subarray must cross the midpoint

✓ Find maximum subarrays of the form $A[i..mid]$ and $A[mid+1..j]$



✓ Line 1 ~ 7 : get max−left

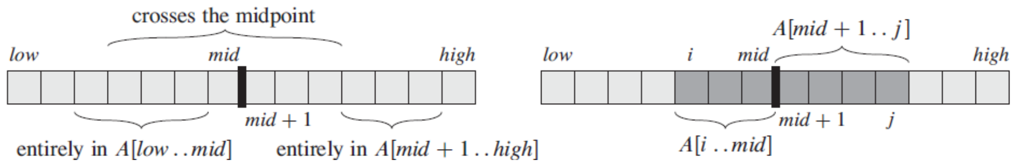
✓ Line 8 ~ 14 get max−right

✓ The total number of iteration : $high - low + 1 = n$

✓ It takes $\Theta(n)$

❖ The maximum subarray problem

- Divide and Conquer Solution
  - ➢ FIND–MAXIMUM–SUBARRAY$(\boldsymbol{A, low, high})$




```
FIND-MAXIMUM-SUBARRAY(A, low, high)
1   if high == low
2       return (low, high, A[low])              // base case: only one element
3   else mid = ⌊(low + high)/2⌋
4       (left-low, left-high, left-sum) =
            FIND-MAXIMUM-SUBARRAY(A, low, mid)
5       (right-low, right-high, right-sum) =
            FIND-MAXIMUM-SUBARRAY(A, mid + 1, high)
6       (cross-low, cross-high, cross-sum) =
            FIND-MAX-CROSSING-SUBARRAY(A, low, mid, high)
7       if left-sum ≥ right-sum and left-sum ≥ cross-sum
8           return (left-low, left-high, left-sum)
9       elseif right-sum ≥ left-sum and right-sum ≥ cross-sum
10          return (right-low, right-high, right-sum)
11      else return (cross-low, cross-high, cross-sum)
```

- ✓ Line 4~5: recursively solve the problem (left & right)
- ✓ Line 6: find max crossing subarray
- ✓ Line 7~11: fine maximum subarray out of the three cases

❖ The maximum subarray problem

- Analyzing Divide and Conquer algorithm
  - ➢ Set up a recurrence of the recursive FIND-MAXIMUM-SUBARRAY procedure
    - ✓ $T(n)$: the running time of FIND-MAXIMUM-SUBARRAY on a subarray of n elements
    - ✓ When $n = 1, T(1) = \theta(1)$
    - ✓ When $\mathrm{n} > 1$
      i. Recursively solve left & right subarray spend $T(n/2)$ time solving each of them
      ii. As we already seen, FIND-MAX-CROSSING-SUBARRAY takes $\theta(n)$ time
      iii. The left line take only $\theta(1)$ time
    - ✓ Therefore, we have

$$\begin{aligned} T(n) &= \Theta(1) + 2T(n/2) + \Theta(n) + \Theta(1) \\ &= 2T(n/2) + \Theta(n) . \end{aligned} \Rightarrow T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 , \\ 2T(n/2) + \Theta(n) & \text{if } n > 1 . \end{cases}$$

    - ✓ According to Master method $T(n) = \theta(n \log n) \rightarrow$ asymptotically faster than the brute-force

❖ Strassen's algorithm for matrix multiplication

- Matrix multiplication
  - ➢ Definition
    - ✓ Input: $A = (a_{ij})$ and B $= (b_{ij})$ are square $n \times n$ matrices
    - ✓ Output: $C = A \times B$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj} \qquad \text{for } i, j = 1, 2, \ldots, n, \text{ by}$$

  - ➢ pseudocode

```
SQUARE-MATRIX-MULTIPLY(A, B)
n = A.rows
let C be a new n × n matrix
for i = 1 to n
    for j = 1 to n
        c_ij = 0
        for k = 1 to n
            c_ij = c_ij + a_ik · b_kj
return C
```

✓ The running time for matrix multiplication

$$T(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} c = cn^3 = \theta(n^3)$$

✓ We will study about simple divide-and-conquer algorithm and Strassen's algorithm which runs in $O(n^{2.81})$ time

❖ Strassen's algorithm for matrix multiplication

- A simple divide–and–conquer algorithm
  - ➢ Divide
    - ✓ Divide $n \times n$ matrices into four $n/2 \times n/2$ matrices
    - ✓ Suppose that we partition each of A, B and C into four $n/2 \times n/2$ matrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

    - ✓ We can rewrite the equation $C = A \times B$ as

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

  - ➢ Conquer & Combine
    - ✓ Perform 8 multiplication and 4 addition of $n/2 \times n/2$ submatrices recursively

$$\begin{aligned} C_{11} &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21}, & C_{21} &= A_{21} \cdot B_{11} + A_{22} \cdot B_{21}, \\ C_{12} &= A_{11} \cdot B_{12} + A_{12} \cdot B_{22}, & C_{22} &= A_{21} \cdot B_{12} + A_{22} \cdot B_{22}. \end{aligned}$$

❖ Strassen's algorithm for matrix multiplication

- A simple divide-and-conquer algorithm
  - ➢ Pseudocode & analyzing

SQUARE-MATRIX-MULTIPLY-RECURSIVE(A, B)

1 $n = A.rows$
2 let C be a new $n \times n$ matrix
3 **if** $n == 1$
4 $\quad c_{11} = a_{11} \cdot b_{11}$
5 **else** partition A, B, and C as in equations (4.9)
6 $\quad C_{11} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{11}, B_{11}) $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{12}, B_{21})
7 $\quad C_{12} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{11}, B_{12}) $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{12}, B_{22})
8 $\quad C_{21} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{21}, B_{11}) $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{22}, B_{21})
9 $\quad C_{22} =$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{21}, B_{12}) $+$ SQUARE-MATRIX-MULTIPLY-RECURSIVE(A_{22}, B_{22})
10 **return** C

✓ Line5: do not create 12 new $n/2 \times n/2$ matrices
→ would spend $\Theta(n^2)$ time to copy entries
→ use index calculation

✓ Index Calculation : identify submatrix by a range of row & column indices of the original matrix
→ spend $\Theta(1)$ time

✓ When $n = 1$, $T(1) = \Theta(1)$

✓ When $n > 1$,
→ call recursive function 8 times takes $8T\left(\frac{n}{2}\right)$
→ four matrix additions takes $\Theta(n^2)$ times

❖ Strassen's algorithm for matrix multiplication

- A simple divide-and-conquer algorithm
  - ➢ Pseudocode & analyzing

$$\begin{aligned} T(n) &= \underbrace{\Theta(1)}_{\text{Index Calculation}} + \underbrace{8T(n/2)}_{\text{8 Multiplications}} + \underbrace{\Theta(n^2)}_{\text{4 Additions}} \\ &= 8T(n/2) + \Theta(n^2)\,. \end{aligned}$$

$$\Rightarrow \quad T(n) = \begin{cases} \Theta(1) & \text{if } n = 1\,, \\ 8T(n/2) + \Theta(n^2) & \text{if } n > 1\,. \end{cases}$$

- ✓ According to master method $T(n) = \theta(n^3)$
- ✓ This simple divide- and-conquer approach is no faster than the straightforward SQUARE-MATRIX-MULTIPLY procedure
- ✓ So now introduce Strassen's method

❖ Strassen's algorithm for matrix multiplication

- Strassen's method
  - ➢ Differences from the previous method
    - ✓ The key to Strassen's method is to make the recursion tree less complicated
    - ✓ It performs only seven recursive multiplications (not eight)
  - ➢ How does it work?
    - ✓ Divide the input matrices A, B and C into $n/2 \times n/2$ submatrices
      i. takes $\Theta(1)$ time by index calculation
    - ✓ Create 10 matrices s_1, s_2, .. , s_{10}, each of which is $n/2 \times n/2$ submatrices
      i. Matrices are the sum or difference of two matrices created in step 1
      ii. Takes $\Theta(n^2)$ times

$$
\begin{aligned}
S_1 &= B_{12} - B_{22}, & S_6 &= B_{11} + B_{22}, \\
S_2 &= A_{11} + A_{12}, & S_7 &= A_{12} - A_{22}, \\
S_3 &= A_{21} + A_{22}, & S_8 &= B_{21} + B_{22}, \\
S_4 &= B_{21} - B_{11}, & S_9 &= A_{11} - A_{21}, \\
S_5 &= A_{11} + A_{22}, & S_{10} &= B_{11} + B_{12}.
\end{aligned}
$$

❖ Strassen's algorithm for matrix multiplication

- Strassen's method
  - How does it work?
    - ✓ Using the submatrices created in step 1 and the 10 matrices created in step 2, recursively compute seven matrix products P_1, P_2, .. , P_7 which are $n/2 \times n/2$ submatrices
      i. Require us to perform seven multiplications of $n/2 \times n/2$ matrices
      ii. Takes $7T(n/2)$ times

$$
\begin{aligned}
P_1 &= A_{11} \cdot S_1 &&= A_{11} \cdot B_{12} - A_{11} \cdot B_{22} , \\
P_2 &= S_2 \cdot B_{22} &&= A_{11} \cdot B_{22} + A_{12} \cdot B_{22} , \\
P_3 &= S_3 \cdot B_{11} &&= A_{21} \cdot B_{11} + A_{22} \cdot B_{11} , \\
P_4 &= A_{22} \cdot S_4 &&= A_{22} \cdot B_{21} - A_{22} \cdot B_{11} ,
\end{aligned}
$$

$$
\begin{aligned}
P_5 &= S_5 \cdot S_6 &&= A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{11} + A_{22} \cdot B_{22} , \\
P_6 &= S_7 \cdot S_8 &&= A_{12} \cdot B_{21} + A_{12} \cdot B_{22} - A_{22} \cdot B_{21} - A_{22} \cdot B_{22} , \\
P_7 &= S_9 \cdot S_{10} &&= A_{11} \cdot B_{11} + A_{11} \cdot B_{12} - A_{21} \cdot B_{11} - A_{21} \cdot B_{12} .
\end{aligned}
$$

    - ✓ Compute the desired submatrices C_{11}, C_{12}, C_{21}, C_{22} of the result matrix C by adding and subtracting carious combinations of the P_i matrices.
      i. Takes $\theta(n^2)$ times

$$C_{11} = P_5 + P_4 - P_2 + P_6 \qquad C_{21} = P_3 + P_4$$

$$C_{12} = P_1 + P_2 \qquad C_{22} = P_5 + P_1 - P_3 - P_7$$

❖ Strassen's algorithm for matrix multiplication

- Strassen's method
  - ➢ Analyzing algorithm

✓ T(n) = $\underbrace{\theta(1)}_{\text{Index Calculation}} + \underbrace{\theta(n^2)}_{\text{Create 10 } s_i} + \underbrace{7T\left(\frac{n}{2}\right)}_{\text{Calculate 7 } P_i} + \underbrace{\theta(n^2)}_{\text{Compute } c_i}$

$= 7T\left(\frac{n}{2}\right) + \theta(n^2)$

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1\,, \\ 7T(n/2) + \Theta(n^2) & \text{if } n > 1\,. \end{cases}$$

✓ According to master method $T(n) = \theta\left(n^{\log 7}\right) = \theta(n^{2.8})$

✓ Strassen's algorithm, comparing with previous methods, is asymptotically faster!!

Thank You!