EPFL
Geometrie, MATH-120

Automne 2016

Série 12

Here and throughout, by reflection we mean symmetry.

Exercice 1. 1. Calculer les parametres complexes des symetries axiales, R_1 et R_2 par rapport aux droites d'equation

$$3x + 4y = 2,\ -2x + 5y = 3.$$

2. A quoi est egale la composee

$$R_1 \circ R_2?$$

quels sont ses parametres complexes ?

3. Meme question pour les droites

$$3x + 4y = 2,\ 6x + 8y = 6.$$

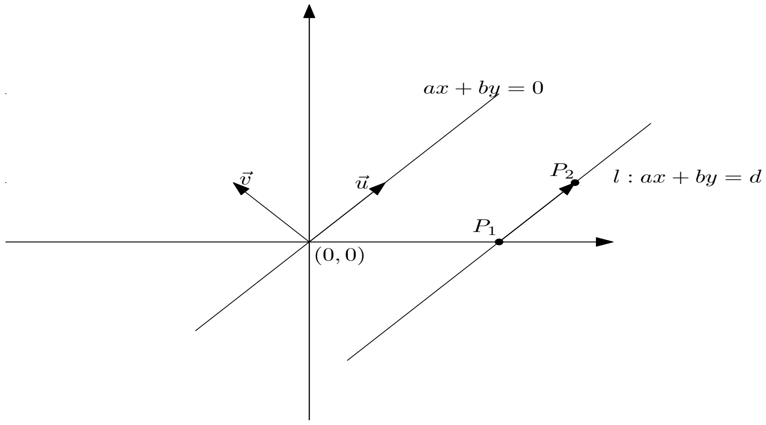
Solutions :

I. Find reflection matrices : Suppose l is a line defined by $ax + by = d$ with $b \neq 0$ (if $b = 0$ and $a \neq 0$, one can proceed in the same way). Assume that R is the reflection through the line l. This implies that R can be written as

$$R(\mathbf{x}) = A \cdot \mathbf{x} + v,\ \mathbf{x} = (x, y).$$

We now present two **correct** ways to find the matrix A.

1. Since l is defined by $ax+by = d$, we have the direction vector u of l is $\lambda(P_2-P_1)$ for some $\lambda \in \mathbb{R}$ and P_1, P_2 are distinct points on l. Here we can choose λ such that u is of good form. For instance, let $P_1 = (0, d/b)$ and $P_2 = (1, (d - a)/b)$, then we have $P_2 - P_1 = (1, -a/b)$. Therefore, we can choose $\lambda = b$, and as a result, we obtain $u = (b, -a)$.
   Let v be a perpendicular vector of u, for example, we can choose $v = (a, b)$.



In Serie 9, we have learned that the reflection through the line $ax + by = 0$ can be presented as follows :

$$Sym_u \colon w = (x, y) \to w - 2\frac{< w, v >}{< v, v >} \cdot v.$$

The map Sym_u can be understood as the linear part of the reflection R. So the matrix of R and Sym_u are the same. Since $< w, v >= ax + by$ and $< v, v >= a^2 + b^2$, we have

$$Sym_u(w) = \begin{pmatrix} \frac{x(b^2-a^2)-2aby}{a^2+b^2} \\ \frac{(a^2-b^2)y-2abx}{a^2+b^2} \end{pmatrix}$$

On the other hand, we know that $Sym_u(w) = Aw$. This implies that

$$A = \begin{pmatrix} \frac{b^2-a^2}{a^2+b^2} & \frac{-2ab}{a^2+b^2} \\ \frac{-2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} \end{pmatrix}$$

2. The second way is very short and practical. We know that the matrix A of R is of the form

$$A = \begin{pmatrix} c & s \\ s & -c \end{pmatrix}, \ c^2 + s^2 = 1.$$

To find c and s, we solve the following system $Av = -v$ with $v = (a, b)$. This is a system of two equations and two variables. After some calculations, we archive

$$c = \frac{b^2 - a^2}{a^2 + b^2}, \ s = \frac{-2ab}{a^2 + b^2}.$$

To find v, let P be an arbitrary point on l. Then we have $R(P) = P = AP + v$. This implies that

$$v = P - AP.$$

Applications :

1. If l is of the form $3x + 4y = 2$. Then we have

$$A = \begin{pmatrix} 7/25 & -24/25 \\ -24/25 & -7/25 \end{pmatrix}.$$

Let $P = (2, -1) \in l$, we have $v = P - A \cdot P = (12/25, 16/25)$. I other words,

$$R_1(\mathbf{x}) = \begin{pmatrix} 7/25 & -24/25 \\ -24/25 & -7/25 \end{pmatrix} \cdot \mathbf{x} + \begin{pmatrix} 12/25 \\ 16/25 \end{pmatrix}.$$

2. If l is of the form $-2x + 5y = 3$, by using the same way, we have

$$R_2(\mathbf{x}) = \begin{pmatrix} 21/29 & 20/29 \\ 20/29 & -21/29 \end{pmatrix} \cdot \mathbf{x} + \begin{pmatrix} -12/29 \\ 30/29 \end{pmatrix}.$$

II. Find $R_1 \circ R_2$: We have $R_1(\mathbf{x}) = A_1 \cdot \mathbf{x} + v_1$ and $R_2(\mathbf{x}) = A_2 \cdot \mathbf{x} + v_2$. This implies that

$$R_1 \circ R_2(\mathbf{x}) = R_1(R_2(\mathbf{x})) = R_1(A_2\mathbf{x} + v_2) = A_1 \cdot (A_2\mathbf{x} + v_2) + v_1 = A_1A_2\mathbf{x} + (A_1v_2 + v_1).$$

Hence,

$$R_1 \circ R_2(\mathbf{x}) = \begin{pmatrix} -333/725 & 644/725 \\ -644/725 & -333/725 \end{pmatrix} \cdot \mathbf{x} + \begin{pmatrix} -456/725 \\ 542/725 \end{pmatrix}.$$

III. Find complex parameters : Suppose that we have an isometry of the following form

$$R(x, y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Case 1 : If R is a reflection, then from the following system

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & -s \\ -s & -c \end{pmatrix}$$

we get $c = a$ and $s = -b$.

This implies that R can be presented in complex parameters as follows :

$$R = \overline{\rho z} + v,$$

where $\rho = a + i(-b)$ and $z = x + iy$ and $v = v_1 + iv_2$.

Case 2 : If R is a rotation, then from the following system

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$$

we get $c = a$ and $s = c$.

This implies that R can be presented in complex parameters as follows :

$$R = \rho z + v,$$

where $\rho = a + ic$ and $z = x + iy$ and $v = v_1 + iv_2$.

Applications : For this exercise, we apply for the case of rotations, i.e. $c = -333/725,\ s = -644/725,\ (v_1, v_2) = (-456/725, 542/725)$. In short,

$$R_1 \circ R_2(\mathbf{x}) = \rho z + v,$$

where $\rho = \frac{-333}{725} + i\frac{-644}{725}$, $z = x + iy$ and $v = \frac{-456}{725} + i\frac{542}{725}$.

Exercise 3 : Soit φ une isometrie et

$$\mathrm{Fix}(\varphi) = \{P \in \mathbb{R}^2,\ \varphi(P) = P\}$$

l'ensemble des points fixe de ϕ. Plus generalement pour $\Phi \subset \mathrm{Isom}(\mathbb{R}^2)$ un ensemble quelconque d'isometries, soit

$$\mathrm{Fix}(\Phi) = \{P \in \mathbb{R}^2,\ \forall \varphi \in \Phi,\ \varphi(P) = P\}$$

l'ensemble des points fixes de Φ.

1. Soit ψ une autre isometrie, et $\varphi' = \mathrm{Ad}(\psi)(\varphi) = \psi \circ \varphi \circ \psi^{-1}$ l'isometrie conjuguee ; que vaut $\mathrm{Fix}(\varphi')$ en fonction de $\mathrm{Fix}(\varphi)$. Meme question pour $\mathrm{Ad}(\psi)(\Phi)$.
2. Montrer (sans calcul) que le conjugue d'une symetrie axiale par une isometrie est une symetrie axiale ; meme question pour une symetrie glisssseeeee.
3. Montrer que toute droite (affine) peut etre envoyee sur toute autre droite par une rotation (affine). En deduire que toute symetrie axiale est conjuguee a la symetrie lineaire s_1.
4. Etant donne $s_{\beta,\nu}$ un symetrie axiale ou glissee donner une condition necessaire et suffisante sur (β, ν) pour que $s_{\beta,\nu}$ soit conjuguee a s_1 par un rotation ; quand c'est le cas quels sont les parametres de cette rotation et retrouver ainsi les formules qui donne l'axe d'une symetrie axiale en fonction de (β, ν).

Solution :

1. We have

$$\begin{aligned}\mathrm{Fix}(\varphi') &= \{P \in \mathbb{R}^2,\ \varphi'(P) = P\} \\ &= \{P \in \mathbb{R}^2,\ \psi \circ \varphi \circ \psi^{-1}(P) = P\} \\ &= \{P \in \mathbb{R}^2,\ \varphi\big(\psi^{-1}(P)\big) = \psi^{-1}(P)\} \\ &= \{P \in \mathbb{R}^2,\ \psi^{-1}(P) \in \mathrm{Fix}(\varphi)\} \\ &\overset{*}{=} \{P \in \mathbb{R}^2,\ P \in \psi\big(\mathrm{Fix}(\varphi)\big)\} \\ &= \psi\big(\mathrm{Fix}(\varphi)\big),\end{aligned}$$

where the equality with the asterisk holds because ψ is an injective mapping. For $\mathrm{Fix}\big(\mathrm{Ad}(\psi)(\Phi)\big)$ we have

$$\begin{aligned}\mathrm{Fix}\big(\mathrm{Ad}(\psi)(\Phi)\big) &= \bigcap_{\varphi\in\Phi} \mathrm{Fix}\big(\mathrm{Ad}(\psi)(\varphi)\big) = \bigcap_{\varphi\in\Phi} \psi\big(\mathrm{Fix}(\varphi)\big) \\ &= \psi\Big(\bigcap_{\varphi\in\Phi} \mathrm{Fix}(\varphi)\Big) = \psi\Big(\mathrm{Fix}(\Phi)\Big).\end{aligned}$$

2. Suppose φ is an axial symmetry and ψ is an arbitrary isometry. According to the lecture notes, we know the set $\mathrm{Fix}(\varphi)$ forms a line. Now by part 1 we have

$$\mathrm{Fix}\big(\mathrm{Ad}(\psi)(\varphi)\big) = \psi\big(\mathrm{Fix}(\varphi)\big),$$

which implies that $\mathrm{Fix}\big(\mathrm{Ad}(\psi)(\varphi)\big)$ also forms a line, and as a result, $\mathrm{Ad}(\psi)(\varphi)$ is again an axial symmetry, since axial symmetry is the only type of isometry whose set of fix points is a line.
Now suppose φ is a glide (glissee) symmetry. Then we know that $\mathrm{Fix}(\varphi) = \emptyset$. By the same reasoning as above, we get

$$\mathrm{Fix}\big(\mathrm{Ad}(\psi)(\varphi)\big) = \emptyset,$$

which according to the lecture notes, implies that $\mathrm{Ad}(\psi)(\varphi)$ is either a glide symmetry or a translation. But considering the complex form of the isometries, one can see that it cannot be a translation, since the complex variable z in $\mathrm{Ad}(\psi)(\varphi)$ appears in the conjugate form. So it is indeed a glide symmetry.

3. Consider two lines D, D'. Our goal is to find an affine rotation to map D to D'. We split the solution into the following two cases :

case i) $D \cap D' \neq \emptyset$: Suppose $P = D \cap D'$. We set up a complex coordinate system in such a way that P is the origin. Assume that the lines are determined by the following equations in this system

$$D = \{\lambda\beta : \lambda \in \mathbb{R}\}, D' = \{\lambda\beta' : \lambda \in \mathbb{R}\},$$

where $\beta, \beta' \in \mathbb{C} \setminus \{0\}$. Now define the rotation φ in the following way

$$\varphi(z) = \frac{\beta'}{\beta} \cdot z.$$

Then we get $\varphi(D) = D'$, since for an arbitrary $\lambda \in \mathbb{R}$, we have

$$\varphi(\lambda\beta) = \frac{\beta'}{\beta} \cdot \lambda\beta = \lambda\beta'.$$

case ii) $D \cap D' = \emptyset$: In this case, the two lines are parallel, so one can transform D to D' by using a translation which is perpendicular to the lines and its length is the distance between D and D'.

For the second part of the problem, suppose φ is an axial symmetry with axis line D and translation vector $\vec{v}$. Consider the line D' which is obtained by translating D in the direction $\frac{\vec{v}}{2}$ and let φ' be the linear axial symmetry with respect to D'. Once can check that $\varphi' = \varphi$. Therefore, we can work with φ' instead of φ. Now according to the first part, consider the rotation ψ which sends D' to the x-axis. We have

$$\psi \circ \varphi' \circ \psi^{-1} = s_1.$$

This holds since by applying the rotation ψ^{-1}, the x-axis is sent to D', and then by φ', we get the axial symmetry with respect to D', and at the end, the line D' is sent back to the x-axis by applying ψ.

4. For an arbitrary isometry ψ, we have

$$\psi \circ s_{\beta,\nu} \circ \psi^{-1} = s_1$$

if and only if we have

$$\psi^{-1} \circ s_1 \circ \psi = s_{\beta,\nu}.$$

Now suppose ψ is an arbitrary affine rotation with $\psi(z) = \alpha z + u$. We have

$$\begin{aligned}(\psi^{-1} \circ s_1 \circ \psi)(z) &= \psi^{-1}\big(s_1(\alpha z + u)\big) = \\ &= \psi^{-1}(\overline{\alpha z + u}) = \overline{\alpha}\big(\overline{\alpha z + u}\big) - \overline{\alpha} u \\ &= \overline{\alpha^2}\overline{z} + \overline{\alpha}(\overline{u} - u) = \overline{\alpha^2}\overline{z} - 2\overline{\alpha} \cdot \mathrm{Im}(u) i,\end{aligned}$$

where $\mathrm{Im}(u)$ is the imaginary part of u. Now we have

$$\psi^{-1} \circ s_1 \circ \psi = s_{\beta,\nu},$$

if and only if

$$\alpha^2 = \beta, \nu = -2\overline{\alpha} \cdot \mathrm{Im}(u) i,$$

or equivalently

$$\alpha^2 = \beta, \alpha\nu = -2 \cdot \mathrm{Im}(u)i.$$

So there exists such a rotation ψ iff there exists some $\alpha \in \mathbb{C}$ such that

$$\alpha^2 = \beta, \mathrm{Re}(\alpha) \cdot \mathrm{Re}(\nu) = \mathrm{Im}(\alpha) \cdot \mathrm{Im}(\nu).$$

On the other hand, by Proposition 3.8 of the lecture notes, the solutions of $\alpha^2 = \beta$ are

$$\alpha = \epsilon\sqrt{\frac{1}{2}(\mathrm{Re}(\beta) + |\beta|)} + \epsilon' i\sqrt{\frac{1}{2}(|\beta| - \mathrm{Re}(\beta))},$$

where $\epsilon, \epsilon' \in \{1, -1\}$ have to be chosen so that we have $2\mathrm{Re}(\alpha) \cdot \mathrm{Im}(\alpha) = \mathrm{Im}(\beta)$. So with this choice of α, the condition $\mathrm{Re}(\alpha) \cdot \mathrm{Re}(\nu) = \mathrm{Im}(\alpha) \cdot \mathrm{Im}(\nu)$ is equivalent to

$$\epsilon''\sqrt{(\mathrm{Re}(\beta) + |\beta|)} \cdot \mathrm{Re}(\nu) = \sqrt{(|\beta| - \mathrm{Re}(\beta))} \cdot \mathrm{Im}(\nu),$$

where the value of $\epsilon'' \in \{1, -1\}$ depends on $\mathrm{Im}(\beta)$ as discussed above. Also in this case we get that

$$u = \mathrm{Re}(u) - i\frac{\big(\mathrm{Re}(\alpha) \cdot \mathrm{Im}(\nu) + \mathrm{Im}(\alpha) \cdot \mathrm{Re}(\nu)\big)}{2},$$

where $\mathrm{Re}(u) \in \mathbb{R}$ can be arbitrarily chosen. (one can simply assume it to be zero)

For the formula defining the axis line D of $s_{\beta,\nu}$, first we find the line D', which passes through the origin and is the axis line of $s_{\beta,\mathbf{0}}$. For that, note that we have

$$D' = \{z \in \mathbb{C} : s_{\beta,\mathbf{0}}(z) = z\} = \{z \in \mathbb{C} : \overline{\beta z} = z\}$$

Suppose $\beta = a + ib$. The set of the solutions of $\overline{\beta z} = z$ is

$$\{\lambda(b + i(a - 1)) : \lambda \in \mathbb{R}\},$$

which implies that

$$D' = \{\lambda(b + i(a - 1)) : \lambda \in \mathbb{R}\}.$$

Now we find the formula of the axis line D of $s_{\beta,\nu}$. For that, note that according to what was mentioned earlier in part 3, we have

$$D = D' + \frac{\nu}{2}.$$

So we get

$$D = \{\lambda(b + i(a - 1)) + \frac{\nu}{2} : \lambda \in \mathbb{R}\}.$$

Exercise 5 : [$\star$] Le but de cet exercice est de montrer le resultat suivant :*Soit un morphisme de groupe continu*

$$\phi : (\mathbb{R}, +) \mapsto (\mathbb{C}^1, \times)$$

(la fonction $t \mapsto \phi(t) = x(t) + iy(t)$ est continue c'est a dire que $x(t)$ et $y(t)$ le sont) alors ϕ est derivable.

Pour demontrer ce resultat on procede comme suit : on pose

$$\Phi(u) = \int_0^u \phi(t)dt = \int_0^u x(t)dt + i\int_0^u y(t)dt.$$

Comme ϕ est continue sa primitive $\Phi(u)$ existe, est derivable de derivee

$$\Phi'(u) = \phi(u).$$

1. Montrer que
$$\Phi(u+1) = \Phi(u) + \phi(u)\Phi(1)$$
(on pourra ecrire $\int_0^{u+1} \cdots = \int_0^u \cdots + \int_u^{u+1} \cdots$, effectuer un changememtn de variable et utiliser la propriete principale de ϕ).
2. Montrer que ϕ est derivable.

Solution :
First we note that $\phi(t) = x(t) + iy(t)$ is derivative if $x(t)$ and $y(t)$ are derivative.

1. We have

$$\Phi(u) = \int_0^u \phi(t)dt = \int_0^u x(t)dt + \int_0^u y(t)dt.$$

For any $c > 0$, we have

$$\Phi(u+c) = \int_0^{u+c} \phi(t)dt = \int_0^u \phi(t)dt + \int_u^{u+c} \phi(t)dt = \Phi(u) + \int_u^{u+c} \phi(t)dt.$$

For the second integral, we set $x = t - u$, then $dx = dt$ and

$$\int_u^{u+c} \phi(t)dt = \int_0^c \phi(x+u)dx = \phi(u)\int_0^c \phi(x)dx = \phi(u)\Phi(c),$$

where we have used the property $\phi(x+u) = \phi(x)\phi(u)$.

In other words, if $c = 1$ then we have

$$\Phi(u+1) = \phi(u)\Phi(1).$$

2. Prove that $\phi(u)$ is derivative.

Before proving this statement, we note that if a function $f(x) = x(t) + iy(t)$ is derivative and $z = a + bi$ is a fixed complex number, then $zf(x)$ is also derivative.

To prove $\phi(u)$ is derivative, we will show that the following limitation exists :

$$\lim_{\delta \to 0} \frac{\phi(u+\delta) - \phi(u)}{\delta}$$

(note that here we mean that the limitations of the real part and the imaginary part exist.)

Since $x(t)$ and $y(t)$ are continuous functions, there always exits $c > 0$ such that $\Phi(c) \neq 0$. In this exercise, we can not make sure that $\Phi(1)$ is non-zero, for instance, one can take $x(t) = sin2\pi t$ and $y(t) = cos2\pi t$.

We have

$$\lim_{\delta \to 0} \frac{\phi(u+\delta) - \phi(u)}{\delta} = \lim_{\delta \to 0} \Phi(c)^{-1} \left(\frac{\Phi(u+\delta+1) - \Phi(u+1)}{\delta} - \frac{\Phi(u+\delta) - \Phi(u)}{\delta} \right).$$

This limitation exists since the following limitations exist

$$\lim_{\delta \to 0} \frac{\Phi(u+\delta+1) - \Phi(u+1)}{\delta} = \Phi(u+1)', \lim_{\delta \to 0} \frac{\Phi(u+\delta) - \Phi(u)}{\delta} = \Phi(u)',$$

which completes the solution of the exercise.