EPFL
Geometrie, MATH-120

Automne 2016

Série 4

Exercice 1.

1. Montrer que tout sous-groupe d'un groupe commutatif est distingue.
2. Soit $\phi : G \to H$ un morphisme montrer que si G est commutatif alors $\mathrm{Im}(G)$ est commutatif.

Exercice 2. soit G un groupe et $A \subset G$ un sous-ensemble.

1. Montrer que l'intersection de tous les sous-groupes de G contenant A est un sous-groupe de G (pourquoi cette intersection n'est pas faite sur un ensemble vide.) On note ce sous-groupe $\langle A \rangle$ et on l'appelle sous-groupe engendre par A.
2. Montrer que $\langle A \rangle$ est le plus petit sous-groupe contenant A.
3. On suppose que $G = \langle A \rangle$. Soit $\phi : G \to H$ un morphisme de groupes. Montrer que si
$$A \subset \ker \phi$$
alors ϕ est le morphisme trivial $\phi \equiv e_H$ (ie. $\forall g \in G$, $\phi(g) = e_H$).
4. Montrer qu'un morphisme de groupe $\phi : G \to H$ est completement determine par ses valeurs prises au differentes elements de A (si deux morphismes ϕ, ϕ' coincidents pour tous les elements de A alors $\phi = \phi'$).

Démonstration.

1. First, we prove a more general fact. Let $\{H_i\}_{i\in I}$ be any family of subgroups of a given group $(G, \cdot, (-)^{-1})$. Then $H = \cap_{i\in I} H_i$ is a subgroup. It suffices to show that for any $h, h' \in H$ we have $h' \cdot h^{-1} \in H$; this condition implies easily that the neutral element, inverses and products of elements of H belong H. By the definition of H, we see that the elements h and h' belong to H_i for every $i \in I$. The sets H_i are subgroups of H and therefore $h' \cdot h^{-1} \in H_i$ for every $i \in I$. This means that $h' \cdot h^{-1} \in \cap_{i\in I} H_i = H$ which finishes the proof. To conclude, we apply the fact to the family of all subgroups containing A.
2. We remark that the set of subgroups is ordered by inclusion, that is, a subgroup $H \subset G$ is smaller than a subgroup $H' \subset G$ if and only if $H \subset H'$. We need to prove that $\langle A \rangle$ is contained in every subgroup of G containing A. Let H be a such subgroup. By definition, the group $\langle A \rangle$ is the intersection $\cap_{i\in I} H_i$ of all subgroups H_i containing A. This means that there exists an $i \in I$ such that $H = H_i$. Then clearly $\langle A \rangle = \cap_{i\in I} H_i \subset H_i = H$.

3. First, we observe that the homomorphism $\phi\colon G \to H$ is trivial if and only if $G = \ker\phi$. By assumptions, we know that $A \subset \ker\phi$ which by the previous part of the exercise implies that $G = \langle A\rangle \subset \ker\phi$. Of course $\ker\phi \subset G$ and therefore $\ker\phi = G$.
4. We consider the set $\mathrm{Eq}(\phi,\phi') = \{g \in G : \phi(g) = \phi'(g)\}$. For every $g, g' \in \mathrm{Eq}(\phi,\phi')$ we see that

$$\phi(g'\cdot g^{-1}) = \phi(g')\cdot\phi(g^{-1}) = \phi(g')\cdot\phi(g)^{-1} = \phi'(g')\cdot\phi'(g)^{-1} = \phi'(g'\cdot g^{-1}),$$

since ϕ and ϕ' are group homomorphisms. This means that $\mathrm{Eq}(\phi,\phi')$ is a subgroup of G containing A, and therefore $G = \mathrm{Eq}(\phi,\phi')$ which proves the claim.

□

Exercice 3. soit G un groupe et $g \in G$. Pour $n \in \mathbb{Z}$ on definit

$$g^n = \begin{cases} e_G & si\ n = 0 \\ g \star \cdots \star g \text{ (n fois)}, & si\ n \geqslant 1 \\ g^{-1} \star \cdots \star g^{-1} \text{($|n|$ fois)}, & si\ n \leqslant -1 \end{cases}$$

et on pose

$$g^{\mathbb{Z}} = \{g^n,\ n \in \mathbb{Z}\}.$$

1. Montrer que l'application

$$g^{\cdot} : \begin{array}{ccc} \mathbb{Z} & \mapsto & G \\ n & \mapsto & g^n \end{array}$$

est un morphisme de groupe.
2. Montrer que $g^{\mathbb{Z}} = \langle\{g\}\rangle$.
3. On suppose que G est fini de cardinal $|G| = p$ premier. Montrer que pour tout $g \neq e_G$ on a

$$G = g^{\mathbb{Z}}.$$

(On pourra considerer le morphisme d'inclusion $g^{\mathbb{Z}} \hookrightarrow G$.)
4. Montrer que G est commutatif.

Exercice 4. On considere l'application exponentielle (reelle)

$$\exp : x \in \mathbb{R} \mapsto \exp(x) = e^x.$$

1. Montrer que exp un isomorphisme du groupe additif $(\mathbb{R}, +)$ vers le groupe multiplicatif $(\mathbb{R}_{>0}, \times)$. Quel est l'isomorphisme inverse ?
2. Soit $\phi : (\mathbb{R}, +) \mapsto (\mathbb{R}_{>0}, \times)$ un morphisme de groupes. On suppose de plus que l'application $x \mapsto \phi(x)$ est continue et on pose $a = \phi(1)$. Soit $\lambda = \log a$, on va demontrer que $\phi(x) = \exp(\lambda x)$

— Montrer que pour tout $n \in \mathbb{Z}$, on a $\phi(n) = \exp(\lambda n)$.
— Montrer que pour tout $x \in \mathbb{R}$ et $n \in \mathbb{Z}$, $n \neq 0$, on a $\phi\left(\frac{x}{n}\right) = \phi(x)^{1/n}$. En deduire que pour tout $q \in \mathbb{Q}$, on a $\phi(q) = \exp(\lambda q)$
— Conclure (utiliser le fait que tout nombre reel est la limite d'une suite de nombres rationnels).

Démonstration. 1. In order to prove that $\exp\colon (\mathbb{R}, +) \to (\mathbb{R}_{>0}, \times)$ is an isomorphism it suffices to prove that $\exp(a + b) = \exp(a) \times \exp(b)$ which is clear.

2. We divide our proof in three steps suggested in the exercise.
— First, we prove $\phi(n) = \exp(\lambda n)$ for $n \in \mathbb{Z}_{\geqslant 0}$. This follows by induction with respect to the parameter n using the relation

$$\phi(n + 1) = \phi(n) \times \phi(1) = \exp(\lambda n) \times \exp(\lambda) = \exp(\lambda(n + 1)).$$

To extend the relation for all elements $n \in \mathbb{Z}$ we just observe that $1 = \phi(0) = \phi(n) \times \phi(-n)$.
— Again using induction, we observe that $\phi(ny) = \phi(y)^n$ for every $n \in \mathbb{Z}$. Setting $y = \frac{x}{n}$ and taking the n-th root, we obtain

$$\phi(\frac{x}{n}) = \left(\phi(\frac{x}{n})^n\right)^{1/n} = \phi(n \cdot \frac{x}{n})^{1/n} = \phi(x)^{1/n}.$$

For every $q \in \mathbb{Q}$, such that $q = \frac{m}{n}$ for $m, n \in \mathbb{Z}$, we take $x = m$ to see that $\phi(\frac{m}{n}) = \phi(m)^{1/n} = \exp(\lambda m)^{1/n} = \exp(\lambda q)$.
— To conclude, we take a number $x \in \mathbb{R}$ and a sequence of rational numbers $q_n \in \mathbb{Q}$ such that $\lim_{n\to\infty} q_n = x$. Since continuity allows us to exchange taking limits and operations ϕ and exp, we may compute as follows

$$\phi(x) = \phi(\lim_{n\to\infty} q_n) = \lim_{n\to\infty} \phi(q_n) = \lim_{n\to\infty} \exp(\lambda q_n) = \exp(\lambda \lim_{n\to\infty} q_n) = \exp(\lambda x).$$

□